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Global dynamics of a delayed SIRS epidemic model with a wide class of nonlinear incidence rates

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Abstract In this paper, by constructing Lyapunov functionals, we consider the global dynamics of an SIRS epidemic model with a wide class of nonlinear incidence rates and distributed delays $\int_0^h p(\tau)f(S(t), I(t-\tau))d\tau$ under the condition that the total population $N(t)$ converges to 1. By using a technical lemma which is derived from strong condition of strict monotonicity of functions $f(S, I)$ and $f(S, I)/I$ with respect to $S \geq 0$ and $I > 0$, we extend the global stability result for an SIR epidemic model if $R_0 > 1$, where R_0 is the basic reproduction number. By using a limit system of the model, we also give a proof that the disease-free equilibrium is globally asymptotically stable for $R_0 = 1$.

Keywords SIRS epidemic model · nonlinear incidence rate · global asymptotic stability · Lyapunov functional

Mathematics Subject Classification (2000) 34K20 · 34K25 · 92D30

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1 Introduction

In order to understand the mechanism of disease transmission, many authors have paid attention to the stability analysis of the equilibria for various kinds of epidemic models (see [1–24] and the references therein).

To investigate the disease spreading effect transmitted by a vector (e.g. mosquitoes, rats, etc.) after an incubation time denoting the time during which the infectious agents develop in the vector, Takeuchi et al. [21] formulated an SIR (Susceptible-Infected-Recovered) epidemic model with distributed delays of the form $\beta S(t) \int_0^h p(\tau) I(t-\tau) d\tau$. However, when the reproduction number is larger than 1, it was shown that the endemic equilibrium is globally asymptotically stable only for the case that the delay h is small enough, that is, an open problem for the global stability of the endemic equilibrium for the case h is sufficiently large was still left. Later, by focusing on the equation deformation in time derivative of a Lyapunov functional, McCluskey [14] proved that the endemic equilibrium is globally asymptotically stable for any length of delay h if the basic reproduction number is larger than 1. Recently, Enatsu et al. [5] and McCluskey [16] considered the following SIR epidemic model with a wide class of nonlinear incidence rates and distributed delays:

$$\begin{cases} \frac{dS(t)}{dt} = \mu - \mu S(t) - \int_0^h p(\tau) f(S(t), I(t-\tau)) d\tau, \\ \frac{dI(t)}{dt} = \int_0^h p(\tau) f(S(t), I(t-\tau)) d\tau - (\mu + \gamma) I(t), \\ \frac{dR(t)}{dt} = \gamma I(t) - \mu R(t). \end{cases} \quad (1.1)$$

$S(t)$, $I(t)$ and $R(t)$ denote the fractions of susceptible, infective and recovered individuals at time t , respectively. $\mu > 0$ is the death rate of the population, $\gamma > 0$ is the recovery rate of the infective individuals. $h > 0$ is a maximum time taken to become infectious and the transmission of the infection is governed by an incidence rate $\int_0^h p(\tau) f(S(t), I(t-\tau)) d\tau$. $p(\tau)$ denotes the fraction of vector population in which the time taken to become infectious is $\tau \in [0, h]$ satisfying $p \in C([0, h], \mathbb{R}_{+0})$ and $\int_0^h p(\tau) d\tau = 1$.

By constructing suitable Lyapunov functionals, they showed that the global stability of equilibria of (1.1) is fully determined by the basic reproduction number when the functions $f(S, I)$ and $f(S, I)/I$ has monotone properties with respect to $S \geq 0$ and $I > 0$. The similar results have now been obtained in Huang and Takeuchi [8] for an SIR epidemic model with the wide class of nonlinear incidence rates and a discrete delay.

On the other hand, Mena-Lorcat and Hethcote [17] considered several SIRS (Susceptible-Infected-Recovered-Susceptible) epidemic models with a bilinear incidence rate and a standard incidence rate. The SIRS models describe the phenomena that susceptible individuals become infectious, then removed with immunity after recovery from infection and then susceptible again when the temporary immunity fades away. In order to investigate the effect of the immunity loss of diseases, various kinds of SIRS epidemic models and a significant body of work concerning the stability analysis of the steady states of the models have now been carried out (see, for example, [1, 9–12, 22] and the references therein).

In addition, many authors have suggested that transmission of the infection shall have a nonlinear incidence rate. Capasso and Serio [3] studied the cholera epidemic spread in Bari in 1973 and have given an assumption that the incidence rate may take the nonlinear form $\frac{\beta S(t)I(t)}{1+\alpha I(t)}$, which has been interpreted as saturated incidence rate measuring the crowding effect of the infective individuals (see also Xu and Ma [23]). Based on their idea, Ko-

robeinikov and Maini [10] and Korobeinikov [11, 12] obtained the global properties of SIR and SEIR models with a nonlinear incidence rate $h(S(t))g(I(t))$ and SIR, SEIR and SIRS epidemic models with a more general framework of the incidence rate $f(S(t), I(t))$, respectively. Thereafter, Zhou and Cui [24] have introduced a nonlinear incidence rate of the form $\beta S(t)I(t)(1 + \alpha I(t)^{k-1})$ with $k = 2$ for an SEIV epidemic model.

However, for global stability conditions of the endemic equilibrium of the delayed SIRS epidemic model, only restricted sufficient conditions are known by literatures. For the model with a bilinear incidence rate, Nakata et al. [20] obtained a sufficient condition which ensure the global stability of the endemic equilibrium. Vargas-De-León and Gómez-Alcaraz [22] constructed an another Lyapunov functional for the same global stability result. For the model with nonlinear incidence rates, Enatsu et al. [6] also obtained that the endemic equilibrium for an SIRS epidemic model with an incidence rate of the form $\int_0^h \beta S(t)G(I(t-\tau))d\tau$ is globally stable for a small rate of immunity loss by constructing a Lyapunov functional. By improving monotone iterative techniques in Xu and Ma [23], Muroya et al. [19] obtained sufficient conditions which ensure the global asymptotic stability of an endemic equilibrium for an SIRS epidemic model with an incidence rate $\beta S(t)I(t-\tau)/(1 + \alpha I(t-\tau)^p)$, where $p \geq 1$ (see also Muroya et al. [18] for $p = 2$). In fact, by introducing a generalized nonlinear incidence as a function of the number of infected individuals, Alexander and Moghadas [1] showed that stability of the endemic equilibrium can change through Hopf, saddle-node and Bogdanov-Takens bifurcations.

Motivated by the above facts, in this paper, we extend global stability results in [5, 6, 8, 16] to the following SIRS epidemic model with a wide class of the nonlinear incidence rates and distributed delays:

$$\begin{cases} \frac{dS(t)}{dt} = \mu - \mu S(t) - \int_0^h p(\tau)f(S(t), I(t-\tau))d\tau + \delta R(t), \\ \frac{dI(t)}{dt} = \int_0^h p(\tau)f(S(t), I(t-\tau))d\tau - (\mu + \gamma)I(t), \\ \frac{dR(t)}{dt} = \gamma I(t) - (\mu + \delta)R(t) \end{cases} \quad (1.2)$$

with the initial condition

$$\begin{cases} S(\theta) = \varphi_1(\theta), I(\theta) = \varphi_2(\theta), R(\theta) = \varphi_3(\theta), \\ \varphi_i(\theta) \geq 0, \theta \in [-h, 0], \varphi_i(0) > 0, \varphi_i \in C([-h, 0], \mathbb{R}^+), i = 1, 2, 3. \end{cases} \quad (1.3)$$

$\delta \geq 0$ is the rate at which recovered individuals lose immunity and return to the susceptible class. We here assume that $f: \mathbb{R}_{+0}^2 \rightarrow \mathbb{R}_{+0}$ is a locally Lipschitz continuous function on \mathbb{R}_{+0}^2 satisfying $f(0, I) = f(S, 0) = 0$ for $S, I \geq 0$ and the followings hold.

$$(H1) \begin{cases} i) f(S, I) \text{ is a strictly monotone increasing function of } S \geq 0, \\ \text{for any fixed } I > 0, \\ ii) \text{ if } R_0 > 1, \text{ then } f(S, I) \text{ is a monotone increasing function of } I \geq 0, \\ \text{for any fixed } S \geq 0, \end{cases}$$

and

$$(H2) \begin{cases} i) \phi(S, I) = \frac{f(S, I)}{I} \text{ is a bounded and monotone decreasing function of } \\ I > 0, \text{ for any fixed } S \geq 0, \\ ii) K(S) \equiv \lim_{I \rightarrow +0} \phi(S, I) \text{ is a continuous and monotone increasing} \\ \text{function on } S \geq 0, \end{cases}$$

where

$$R_0 = \frac{K(S_0)}{\mu + \gamma}, \quad S_0 = 1 \quad (1.4)$$

is the basic reproduction number of system (1.2). R_0 denotes the expected number of secondary infectious cases generated by one typical primary case in an entirely susceptible and sufficiently large population. We note that $0 < \phi(S, I) \leq K(S)$ holds for any $S, I > 0$.

It is well known by the fundamental theory of functional differential equations that the solution $(S(t), I(t), R(t))$ of system (1.2) is unique and positive for all $t \geq 0$.

System (1.2) always has a disease-free equilibrium $E_0 = (S_0, 0, 0)$. On the other hand, under the hypotheses (H1) and (H2), if $R_0 > 1$, then system (1.2) also admits a unique positive equilibrium $E_* = (S^*, I^*, R^*)$, where $S^*, I^*, R^* > 0$ (see Korobeinikov [11, 12] and Lemma 2.2 below).

The main results are as follows:

Theorem 1.1 *If $R_0 \leq 1$, then the disease-free equilibrium E_0 of system (1.2) is globally asymptotically stable.*

By applying a key property of strict monotonicity of functions $f(S, I)$ and $f(S, I)/I$ with respect to $S \geq 0$ and $I > 0$, we obtain sufficient conditions which ensure the global asymptotic stability of the endemic equilibrium E_* of system (1.2) for $R_0 > 1$.

Theorem 1.2 *If $R_0 > 1$, then the endemic equilibrium E_* of system (1.2) exists uniquely and system (1.2) is permanent. Moreover, if the following conditions hold:*

$$\left\{ \begin{array}{l} \text{(I) there exist positive constants } C_0, C_1 \text{ and } C_2 \text{ such that} \\ \text{for any } v_1 \leq S \leq S_0, 0 \leq I \leq S_0, S \leq S^* \text{ and } I \neq I^*, \\ \frac{f(S, I^*) - f(S^*, I^*)}{(S - S^*)I^*} \leq C_0, \\ \frac{f(S, I) - f(S, I^*)}{I - I^*} \geq C_1 > 0, \\ \frac{\frac{I}{f(S, I)} - \frac{I^*}{f(S, I^*)}}{I - I^*} \geq C_2 > 0, \\ \text{(II) } \delta^2 < 4C_0C_1C_2(\mu + \delta)(\mu + \gamma)\frac{f(v_1, I^*)}{I^*}, \end{array} \right. \quad (1.5)$$

then the endemic equilibrium E_ of system (1.2) is globally asymptotically stable, where $v = v_1 > 0$ is a unique positive solution of $\mu - K(v) - \mu v = 0$.*

Note that if $f(S, I) = \beta SI/(1 + \alpha I)$, then the incidence rate becomes saturated-type, which is of the form used in Xu and Ma [23].

Theorem 1.2 indicates that the endemic equilibrium of system (1.2) is globally asymptotically stable when $R_0 > 1$ for a small rate of immunity loss δ as long as the infection rate has suitable monotone properties of function f characterized by (H1), (H2) and the condition (1.5).

The organization of this paper is as follows. In Section 2, we offer basic results for system (1.2). In Section 3, we establish global asymptotic stability of the disease-free equilibrium for $R_0 \leq 1$ and prove Theorem 1.1. In Section 4, we establish global asymptotic stability of the endemic equilibrium for $R_0 > 1$ and prove Theorem 1.2 by means of Lyapunov functionals to the reduced system which is derived from system (1.2). Finally, a discussion is offered in Section 6.

2 Basic results

We offer some basic results of system (1.2). The following lemmas will be used in the proofs of Theorems 1.1 and 1.2.

Lemma 2.1 *The plane $S(t) + I(t) + R(t) = S_0$ is an invariant manifold of system (1.2), which is globally attractive in the first octant of \mathbb{R}^3 , that is,*

$$\lim_{t \rightarrow +\infty} (S(t) + I(t) + R(t)) = S_0. \quad (2.1)$$

Proof Let $N(t) = S(t) + I(t) + R(t)$. Then it follows from system (1.2) that

$$\begin{aligned} \frac{dN(t)}{dt} &= \mu - \mu S(t) - \mu I(t) - \mu R(t) \\ &= \mu - \mu N(t). \end{aligned}$$

Hence, we obtain that $\lim_{t \rightarrow +\infty} N(t) = S_0$. This completes the proof. \square

Lemma 2.2 *System (1.2) always has a disease-free equilibrium $E_0 = (S_0, 0, 0)$. Moreover, if $R_0 > 1$, then system (1.2) has a unique endemic equilibrium $E_* = (S^*, I^*, R^*)$ satisfying the following equations:*

$$\begin{cases} \mu - \mu S^* - f(S^*, I^*) + \delta R^* = 0, \\ f(S^*, I^*) - (\mu + \gamma) I^* = 0, \\ \gamma I^* - (\mu + \delta) R^* = 0. \end{cases} \quad (2.2)$$

Proof First, it is evident that there always exists a disease-free equilibrium E_0 . Second, we now show that system (1.2) has a unique endemic equilibrium $E_* = (S^*, I^*, R^*)$ if $R_0 > 1$. By (2.2), at a fixed point of the system, the following equalities hold.

$$\mu - \mu S - \left\{ (\mu + \gamma) - \frac{\gamma \delta}{\mu + \delta} \right\} I = 0, \quad f(S, I) - (\mu + \gamma) I = 0. \quad (2.3)$$

By the implicit function theorem and the hypotheses (H1), we see that $f(S, I) - (\mu + \gamma) I = 0$ defines a function $S = \xi(I)$ on neighborhood around $I = 0$. Then, it follows from (2.3) that

$$\lim_{I \rightarrow +0} \frac{f(\xi(I), I)}{I} = \mu + \gamma < K(S_0) = \lim_{I \rightarrow +0} \frac{f(S_0, I)}{I}, \quad (2.4)$$

if $R_0 = K(S_0)/(\mu + \gamma) > 1$. Therefore, by the hypotheses (H2), we obtain that

$$\lim_{I \rightarrow +0} \xi(I) < S_0. \quad (2.5)$$

By the hypothesis (H2), the function $\xi(I)$ is a monotone increasing function and either exists and is continuous for $I \in (0, \mu / \{(\mu + \gamma) - \frac{\gamma \delta}{\mu + \delta}\}]$, or reaches infinity in this interval. After substituting the relations $S = \xi(I)$ and $f(S, I) - (\mu + \gamma) I = 0$ into the first equation of (2.3), we consider the following equation.

$$H(I) := \mu - \mu \xi(I) - \left\{ (\mu + \gamma) - \frac{\gamma \delta}{\mu + \delta} \right\} I = 0,$$

from which we obtain that $H(I)$ is a strictly monotone decreasing function. By (2.5), we have that

$$\lim_{I \rightarrow +0} H(I) = \mu - \mu \lim_{I \rightarrow +0} \xi(I) > \mu - \mu S_0 = 0,$$

which implies that there exists a unique positive solution $0 < I^* < \mu / \{(\mu + \gamma) - \frac{\gamma\delta}{\mu + \delta}\} < S_0 = 1$ such that $H(I^*) = 0$. Therefore, there exists a unique positive solution $R^* > 0$ such that $\gamma I^* - (\mu + \delta)R^* = 0$. Hence, the proof is complete. \square

3 Global stability of the disease-free equilibrium E_0

In this section, we assume that $R_0 \leq 1$ and show the global asymptotic stability of the disease-free equilibrium E_0 of system (1.2) constructing a Lyapunov functional. From Lemma 2.1, the limit set of system (1.2) in the first octant of \mathbb{R}^3 locates on the plane $S + I + R = S_0$. Hence, the dynamics of system (1.2) in the first octant of \mathbb{R}^3 is equivalent to the following system:

$$\begin{cases} \frac{dS(t)}{dt} = \mu - \mu S(t) - \int_0^h p(\tau) f(S(t), I(t - \tau)) d\tau + \delta R(t), \\ \frac{dI(t)}{dt} = \int_0^h p(\tau) f(S(t), I(t - \tau)) d\tau - (\mu + \gamma) I(t), \\ \frac{dR(t)}{dt} = \gamma(S_0 - S(t) - R(t)) - (\mu + \delta) R(t). \end{cases} \quad (3.1)$$

Theorem 3.1 *If $R_0 \leq 1$, then the disease-free equilibrium E_0 of system (3.1) is globally asymptotically stable on $\{(S, I, R) \in \mathbb{R}_+^3 | S + I + R = S_0\}$.*

Proof We consider the following Lyapunov functional:

$$W(t) = \int_{S_0}^{S(t)} \left(1 - \frac{K(S_0)}{K(s)}\right) ds + I(t) + \int_0^h p(\tau) \int_{t-\tau}^t \frac{K(S_0)}{K(S(u + \tau))} f(S(u + \tau), I(u)) du d\tau.$$

We then obtain

$$\begin{aligned}
\frac{dW(t)}{dt} &= \left(1 - \frac{K(S_0)}{K(S(t))}\right) \left(\mu - \mu S(t) - \int_0^h p(\tau) f(S(t), I(t-\tau)) d\tau + \delta R(t)\right) \\
&\quad + \int_0^h p(\tau) f(S(t), I(t-\tau)) d\tau - (\mu + \gamma) I(t) \\
&\quad + \int_0^h p(\tau) \left\{ \frac{K(S_0)}{K(S(t+\tau))} f(S(t+\tau), I(t)) - \frac{K(S_0)}{K(S(t))} f(S(t), I(t-\tau)) \right\} d\tau \\
&= \left(1 - \frac{K(S_0)}{K(S(t))}\right) (-\mu(S(t) - S_0) + \delta R(t)) \\
&\quad - \int_0^h p(\tau) \left(1 - \frac{K(S_0)}{K(S(t))}\right) f(S(t), I(t-\tau)) d\tau \\
&\quad + \int_0^h p(\tau) f(S(t), I(t-\tau)) d\tau - (\mu + \gamma) I(t) \\
&\quad + \int_0^h p(\tau) \left\{ \frac{K(S_0)}{K(S(t+\tau))} f(S(t+\tau), I(t)) - \frac{K(S_0)}{K(S(t))} f(S(t), I(t-\tau)) \right\} d\tau \\
&= \left(1 - \frac{K(S_0)}{K(S(t))}\right) (-\mu(S(t) - S_0) + \delta R(t)) \\
&\quad + \int_0^h p(\tau) \left\{ \frac{K(S_0)}{K(S(t+\tau))} \frac{f(S(t+\tau), I(t))}{I(t)} - (\mu + \gamma) \right\} I(t) d\tau.
\end{aligned}$$

Noting that $\delta R(t) \left(1 - \frac{K(S_0)}{K(S(t))}\right) \leq 0$ on the plane $\{(S, I, R) \in \mathbb{R}_+^3 \mid S + I + R = S_0\}$, we obtain

$$\begin{aligned}
\frac{dW(t)}{dt} &\leq -\mu \left(1 - \frac{K(S_0)}{K(S(t))}\right) (S(t) - S_0) \\
&\quad + \int_0^h p(\tau) \left\{ \frac{K(S_0)}{K(S(t+\tau))} K(S(t+\tau)) - (\mu + \gamma) \right\} I(t) d\tau \\
&= -\mu \left(1 - \frac{K(S_0)}{K(S(t))}\right) (S(t) - S_0) + (\mu + \gamma)(R_0 - 1)I(t).
\end{aligned}$$

Thus, it holds that $\frac{dW(t)}{dt} \leq 0$ with equality if $S(t) = S_0$. Hence, we have $\lim_{t \rightarrow +\infty} S(t) = S_0$, which implies from system (3.1) that $\lim_{t \rightarrow +\infty} I(t) = 0$ and $\lim_{t \rightarrow +\infty} R(t) = 0$ hold. By an extension of Lyapunov-LaSalle asymptotic stability theorem (see also Kuang [13, Theorem 5.3]), the disease-free equilibrium E_0 of system (3.1) is globally asymptotically stable. This completes the proof. \square

Proof of Theorem 1.1. From Theorem 3.1, we immediately obtain the conclusion of this theorem. \square

4 Global stability of the endemic equilibrium E_* for $R_0 > 1$

In this section, we establish the global asymptotic stability of the endemic equilibrium E_* of system (1.2) for $R_0 > 1$ by using a lower bound of the susceptible individuals $S(t)$ for large t .

First, we obtain the following theorem, which indicates that the disease eventually persists in the host population when $R_0 > 1$.

Theorem 4.1 *If $R_0 > 1$, then for any solution of system (1.2), it holds that*

$$\begin{cases} \liminf_{t \rightarrow +\infty} S(t) \geq v_1, \liminf_{t \rightarrow +\infty} I(t) \geq v_2 := qI^* \exp(-(\mu + \gamma)\rho h), \\ \liminf_{t \rightarrow +\infty} R(t) \geq v_3 := \frac{\gamma v_2}{\mu + \delta}, \end{cases}$$

where $v_1 > 0$ satisfies $\mu - K(v_1) - \mu v_1 = 0$ and $0 < q < 1$ and $\rho \geq 1$ satisfy

$$S^* < \frac{\mu - (K(S_0) + \varepsilon_S)qI^*}{\mu} (1 - e^{-\mu\rho h}), \quad 0 < q < \frac{\mu}{(K(S_0) + \varepsilon_S)I^*}. \quad (4.1)$$

Proof Let $(S(t), I(t), R(t))$ be a solution of system (1.2) with initial condition (1.3). By Lemma 2.1, it follows that $\limsup_{t \rightarrow +\infty} I(t) \leq 1$, which implies from the first equation of system (1.2) and the hypothesis (H2) that, for any $\varepsilon_I > 0$, there is an integer $T_I \geq 0$ such that

$$\begin{aligned} \frac{dS(t)}{dt} &\geq \mu - \int_0^h p(\tau) \frac{f(S(t), I(t-\tau))}{I(t-\tau)} I(t-\tau) d\tau - \mu S(t) \\ &\geq \mu - K(S(t)) \int_0^h p(\tau) I(t-\tau) d\tau - \mu S(t) \\ &= \mu - K(S(t))(1 + \varepsilon_I) - \mu S(t), \end{aligned} \quad (4.2)$$

for $t \geq T_I + h$. Let us now consider the following auxiliary equation:

$$\frac{dS(t)}{dt} = \mu - K(S(t)) - \mu S(t).$$

Then one can obtain that $\lim_{t \rightarrow +\infty} S(t) = v_1 > 0$. Since (4.2) holds for arbitrary $\varepsilon_I > 0$ sufficiently small, it follows that $\liminf_{t \rightarrow +\infty} S(t) \geq v_1 > 0$.

We now prove that it is impossible that $I(t) \leq qI^*$ for all sufficiently large t . Suppose to the contrary that there exists a sufficiently large $t_1 \geq T_S$ such that $I(t) \leq qI^*$ holds for all $t \geq t_1$. Then, similar to the above discussion, we have that for any $t \geq t_1 + h$,

$$\begin{aligned} \frac{dS(t)}{dt} &\geq \mu - \int_0^h p(\tau) \phi(S(t), I(t-\tau)) I(t-\tau) d\tau - \mu S(t) \\ &\geq \mu - (K(S_0) + \varepsilon_S)qI^* - \mu S(t), \end{aligned}$$

which yields for $t \geq t_1 + h$,

$$\begin{aligned} S(t) &\geq S(t_1 + h)e^{-\mu(t-t_1-h)} + e^{-\mu t} \int_{t_1+h}^t e^{\mu s} (\mu - (K(S_0) + \varepsilon_S)qI^*) ds \\ &= S(t_1 + h)e^{-\mu(t-t_1-h)} + \frac{\mu - (K(S_0) + \varepsilon_S)qI^*}{\mu} (1 - e^{-\mu(t-t_1-h)}). \end{aligned} \quad (4.3)$$

Hence, it follows from (4.3) that for $t \geq t_1 + h + \rho h$,

$$\begin{aligned} S(t) &> \frac{\mu - (K(S_0) + \varepsilon_S)qI^*}{\mu} (1 - e^{-\mu\rho h}) \\ &= S^\Delta > S^*. \end{aligned} \quad (4.4)$$

Now, we define the following function:

$$V(t) = I(t) + \int_0^h p(\tau) \int_t^{t+\tau} f(S(u), I(u-\tau)) du d\tau. \quad (4.5)$$

Calculating the derivative of $V(t)$ along solutions of system (1.2) gives as follows:

$$\begin{aligned}\frac{dV(t)}{dt} &= \int_0^h p(\tau) f(S(t), I(t-\tau)) d\tau - (\mu + \gamma) I(t) \\ &\quad + \int_0^h p(\tau) \{f(S(t+\tau), I(t)) - f(S(t), I(t-\tau))\} d\tau \\ &= \int_0^h p(\tau) f(S(t+\tau), I(t)) d\tau - (\mu + \gamma) I(t).\end{aligned}$$

For $t \geq t_1 + h + \rho h$, it follows from (4.4) and the relation $\mu + \gamma = \phi(S^*, I^*)$ that

$$\begin{aligned}\frac{dV(t)}{dt} &= \int_0^h p(\tau) \{\phi(S(t+\tau), I(t)) - (\mu + \gamma)\} I(t) d\tau \\ &> \int_0^h p(\tau) \{\phi(S(t+\tau), I^*) - \phi(S^*, I^*) + \phi(S^*, I^*) - (\mu + \gamma)\} I(t) d\tau \\ &= \int_0^h p(\tau) \{\phi(S(t+\tau), I^*) - \phi(S^*, I^*)\} I(t) d\tau \\ &\geq \{\phi(S^\Delta, I^*) - \phi(S^*, I^*)\} I(t).\end{aligned}\tag{4.6}$$

Setting $\underline{i} = \min_{\theta \in [-h, 0]} I(\theta + t_1 + \rho h + 2h)$, we claim that $I(t) \geq \underline{i}$ for all $t \geq t_1 + h + \rho h$. Otherwise, if there is a $T \geq 0$ such that $I(t) \geq \underline{i}$ for $t_1 + h + \rho h \leq t \leq t_1 + 2h + \rho h + T$, $I(t_1 + 2h + \rho h + T) = \underline{i}$ and $\frac{dI(t)}{dt}|_{t=t_1+2h+\rho h+T} \leq 0$, it follows from the second equation of system (1.2), the conditions (H1) and (H2) that for $t_2 = t_1 + 2h + \rho h + T$,

$$\begin{aligned}\frac{dI(t)}{dt}\Big|_{t=t_2} &= \int_0^h p(\tau) f(S(t_2), I(t_2-\tau)) d\tau - (\mu + \gamma) I(t_2) \\ &= \int_0^h p(\tau) \phi(S(t_2), I(t_2-\tau)) I(t_2-\tau) d\tau - (\mu + \gamma) I(t_2) \\ &> \int_0^h p(\tau) \phi(S(t_2), I^*) I(t_2-\tau) d\tau - (\mu + \gamma) I(t_2) \\ &\geq \{\phi(S(t_2), I^*) - (\mu + \gamma)\} I(t_2) \\ &\geq \{\phi(S^\Delta, I^*) - (\mu + \gamma)\} \underline{i} \\ &> \{\phi(S^*, I^*) - (\mu + \gamma)\} \underline{i} = 0.\end{aligned}$$

This is a contradiction. Therefore $I(t) \geq \underline{i}$ for all $t \geq t_1 + h + \rho h$. It follows from (4.6) that

$$\frac{dV(t)}{dt} > \{\phi(S^\Delta, I^*) - \phi(S^*, I^*)\} \underline{i} > 0, \text{ for } t \geq t_1 + 2h + \rho h,$$

which implies that $\lim_{t \rightarrow +\infty} V(t) = +\infty$. However, it holds from (2.1) and (4.5) that $\limsup_{t \rightarrow +\infty} V(t) < +\infty$. Hence the claim holds.

Thus, we proved that it is impossible that $I(t) \leq qI^*$ for all sufficiently large t . Now, we are left to consider the following two possibilities:

- (i) $I(t) \geq qI^*$ for all t sufficiently large,
- (ii) $I(t)$ oscillates about qI^* for all t sufficiently large.

If the first case holds, then we immediately get the conclusion of the proof. If the second case holds, we show that $I(t) \geq qI^* \exp(-(\mu + \gamma)\rho h)$ for all t sufficiently large. Let $t_3 < t_4$ be sufficiently large such that

$$I(t_3) = I(t_4) = qI^*, I(t) < qI^*, t_3 < t < t_4.$$

If $t_4 - t_3 \leq \rho h$, then it follows from the second equation of system (1.2) that

$$\frac{dI(t)}{dt} > -(\mu + \gamma)I(t),$$

that is,

$$\begin{aligned} I(t) &> I(t_3) \exp(-(\mu + \gamma)(t - t_3)) \\ &\geq qI^* \exp(-(\mu + \gamma)\rho h) = v_2. \end{aligned}$$

If $t_4 - t_3 > \rho h$, we obtain from the second equation of system (1.2) that $I(t) \geq v_2$ for $t_3 \leq t \leq t_3 + \rho h$. We now claim that $I(t) \geq v_2$ for all $t_3 + \rho h \leq t \leq t_4$. Otherwise, there is a $T^* > 0$ such that $I(t) \geq v_2$ for $t_3 \leq t \leq t_3 + \rho h + T^* < t_4$, $I(t_3 + \rho h + T^*) = v_2$ and $\frac{dI(t)}{dt}|_{t=t_3+\rho h+T^*} \leq 0$. On the other hand, for $t_0 = t_3 + \rho h + T^*$, it follows from the second equation of system (1.2) and the relation $\phi(S(t_0), I(t_0)) > \phi(S(t_0), I^*) \geq \phi(S^\Delta, I^*) > \phi(S^*, I^*)$ that

$$\begin{aligned} \left. \frac{dI(t)}{dt} \right|_{t=t_0} &= \int_0^h p(\tau) f(S(t_0), I(t_0 - \tau)) d\tau - (\mu + \gamma)I(t_0) \\ &= \int_0^h p(\tau) \phi(S(t_0), I(t_0 - \tau)) I(t_0 - \tau) d\tau - (\mu + \gamma)I(t_0) \\ &> \{\phi(S(t_0), I^*) - (\mu + \gamma)\} I(t_0) \\ &\geq \{\phi(S^\Delta, I^*) - (\mu + \gamma)\} I(t_0) \\ &> \{\phi(S^*, I^*) - (\mu + \gamma)\} I(t_0) = 0, \end{aligned}$$

which is a contradiction. Hence $I(t) \geq qI^* \exp(-(\mu + \gamma)\rho h) = v_2$ for $t_3 \leq t \leq t_4$. Since the interval $[t_3, t_4]$ is arbitrarily chosen, we conclude that $I(t) \geq v_2$ for all t sufficiently large for the second case. Thus, we obtain that $\liminf_{t \rightarrow +\infty} I(t) \geq v_2$. From the above discussion, one can see that $\liminf_{t \rightarrow +\infty} R(t) \geq v_3$. Hence, this completes the proof. \square

Recalling by Lemma 2.2 that the existence of the endemic equilibrium $\tilde{E}_* \equiv (S^*, I^*)$ of system (4.10) is guaranteed for the case $R_0 > 1$, for a fixed $0 \leq \tau \leq h$, we put

$$\begin{cases} x_t = \frac{S(t)}{S^*}, \tilde{x}_t = \frac{f(S(t), I^*)}{f(S^*, I^*)}, y_t = \frac{I(t)}{I^*}, \tilde{y}_{t,\tau} = \frac{f(S(t+\tau), I(t))}{f(S(t+\tau), I^*)}, \\ g(x) = x - 1 - \ln x \geq g(1) = 0, \text{ for } x > 0. \end{cases} \quad (4.7)$$

The following lemma plays a key role to obtain Theorem 1.2.

Lemma 4.1 *For all $t \geq 0$ and $0 \leq \tau \leq h$, it holds that*

$$(1 - x_t) \left(1 - \frac{1}{\tilde{x}_t}\right) \leq -C_0 \frac{f(S(t), I^*)}{S^* I^*} \left(1 - \frac{1}{\tilde{x}_t}\right)^2 \quad (4.8)$$

and

$$g(y_t) - g(\tilde{y}_{t,\tau}) \geq C_1 C_2 I^* (y_t - 1)^2, \quad (4.9)$$

with equality if and only if $x_t = 1$ and $y_t = \tilde{y}_{t,\tau} = 1$, respectively.

Proof First, by the condition (I) of (1.5), we obtain that

$$\begin{aligned}
(1-x_t) \left(1 - \frac{1}{\tilde{x}_t}\right) &= \left(1 - \frac{S(t)}{S^*}\right) \left(1 - \frac{f(S^*, I^*)}{f(S(t), I^*)}\right) \\
&= -\frac{1}{S^*} (S(t) - S^*) \left(1 - \frac{f(S^*, I^*)}{f(S(t), I^*)}\right) \\
&\leq -\frac{C_0}{S^* I^*} (f(S(t), I^*) - f(S^*, I^*)) \left(1 - \frac{f(S^*, I^*)}{f(S(t), I^*)}\right) \\
&= -C_0 \frac{f(S(t), I^*)}{S^* I^*} \left(1 - \frac{1}{\tilde{x}_t}\right)^2.
\end{aligned}$$

Second, we have that

$$\tilde{y}_{t,\tau} - 1 = \frac{f(S(t+\tau), I(t)) - f(S(t+\tau), I^*)}{f(S(t+\tau), I^*)}$$

and

$$\begin{aligned}
y_t - \tilde{y}_{t,\tau} &= \frac{I(t)}{I^*} - \frac{f(S(t+\tau), I(t))}{f(S(t+\tau), I^*)} \\
&= \frac{f(S(t+\tau), I(t))}{I^*} \left(\frac{I(t)}{f(S(t+\tau), I(t))} - \frac{I^*}{f(S(t+\tau), I^*)} \right).
\end{aligned}$$

Then, by the hypotheses (H1) and (H2), we immediately obtain

$$\begin{aligned}
(y_t - \tilde{y}_{t,\tau})(\tilde{y}_{t,\tau} - 1) &= \frac{f(S(t+\tau), I(t))}{I^* f(S(t+\tau), I^*)} \left(\frac{I(t)}{f(S(t+\tau), I(t))} - \frac{I^*}{f(S(t+\tau), I^*)} \right) \\
&\quad \times (f(S(t+\tau), I(t)) - f(S(t+\tau), I^*)) \\
&\geq 0,
\end{aligned}$$

with equality if and only if $y_t = \tilde{y}_{t,\tau} = 1$. Moreover, since $g'(x) = 1 - \frac{1}{x}$ and $g''(x) = \frac{1}{x^2} \geq 0$ for all $x > 0$, it holds that

$$\begin{aligned}
g(y_t) - g(\tilde{y}_{t,\tau}) &\geq g'(\tilde{y}_{t,\tau})(y_t - \tilde{y}_{t,\tau}) \\
&\geq \frac{\tilde{y}_{t,\tau} - 1}{\tilde{y}_{t,\tau}} (y_t - \tilde{y}_{t,\tau}) \\
&= \frac{1}{I^*} \left(\frac{I(t)}{f(S(t+\tau), I(t))} - \frac{I^*}{f(S(t+\tau), I^*)} \right) \\
&\quad \times (f(S(t+\tau), I(t)) - f(S(t+\tau), I^*)) \\
&\geq \frac{C_1 C_2}{I^*} (I(t) - I^*)^2 \\
&= C_1 C_2 I^* (y_t - 1)^2,
\end{aligned}$$

with equality if and only if $y_t = \tilde{y}_{t,\tau} = 1$. Hence, we get the conclusion. \square

Now, we are in a position to prove the global asymptotic stability of the endemic equilibrium E_* of system (1.2) for $R_0 > 1$.

Proof of Theorem 1.2. From Lemma 2.1, the limit set of system (1.2) in the first octant of \mathbb{R}^3 locates on the plane $S + I + R = S_0$. Hence, the dynamics of system (1.2) in the first octant of \mathbb{R}^3 is equivalent to the following system:

$$\begin{cases} \frac{dS(t)}{dt} = (\mu + \delta) - (\mu + \delta)S(t) - \int_0^h p(\tau)f(S(t), I(t - \tau))d\tau - \delta I(t), \\ \frac{dI(t)}{dt} = \int_0^h p(\tau)f(S(t), I(t - \tau))d\tau - (\mu + \gamma)I(t). \end{cases} \quad (4.10)$$

We consider the following Lyapunov functional (cf. [5, 7, 14, 15]):

$$V(t) = U(t) + U_+(t), \quad (4.11)$$

where

$$\begin{cases} U(t) = \int_{S^*}^{S(t)} \left(1 - \frac{f(S^*, I^*)}{f(\tau, I^*)} \right) d\tau + \left(I(t) - I^* - \ln \frac{I(t)}{I^*} \right), \\ U_+(t) = f(S^*, I^*) \int_0^h p(\tau) \int_{t-\tau}^t g \left(\frac{f(S(u+\tau), I(u))}{f(S(u+\tau), I^*)} \right) du d\tau. \end{cases}$$

The time derivative of $\frac{dU(t)}{dt}$ along the solution of system (4.10) satisfies as follows:

$$\begin{aligned} \frac{dU(t)}{dt} &= \left(1 - \frac{f(S^*, I^*)}{f(S(t), I^*)} \right) \left\{ (\mu + \delta) - \int_0^h p(\tau)f(S(t), I(t - \tau))d\tau - (\mu + \delta)S(t) - \delta I(t) \right\} \\ &\quad + \left(1 - \frac{I^*}{I(t)} \right) \left(\int_0^h p(\tau)f(S(t), I(t - \tau))d\tau - (\mu + \gamma)I(t) \right). \end{aligned}$$

Rearranging $\mu + \delta = (\mu + \delta)S^* + f(S^*, I^*) + \delta I^*$ and $\mu + \gamma = \frac{f(S^*, I^*)}{I^*}$ gives

$$\begin{aligned}
\frac{dU(t)}{dt} &= \left(1 - \frac{f(S^*, I^*)}{f(S(t), I^*)}\right) \\
&\quad \times \left\{ (\mu + \delta)(S^* - S(t)) + \left(f(S^*, I^*) - \int_0^h p(\tau) f(S(t), I(t - \tau)) d\tau\right) \right\} \\
&\quad + \left(1 - \frac{I^*}{I(t)}\right) \left(\int_0^h p(\tau) f(S(t), I(t - \tau)) d\tau - f(S^*, I^*) \frac{I(t)}{I^*}\right) \\
&\quad + \delta \left(1 - \frac{f(S^*, I^*)}{f(S(t), I^*)}\right) (I^* - I(t)) \\
&= (\mu + \delta) S^* \left(1 - \frac{S(t)}{S^*}\right) \left(1 - \frac{f(S^*, I^*)}{f(S(t), I^*)}\right) \\
&\quad + f(S^*, I^*) \left(1 - \frac{f(S^*, I^*)}{f(S(t), I^*)}\right) \int_0^h p(\tau) \left(1 - \frac{f(S(t), I(t - \tau))}{f(S^*, I^*)}\right) d\tau \\
&\quad + f(S^*, I^*) \left(1 - \frac{I^*}{I(t)}\right) \int_0^h p(\tau) \left(\frac{f(S(t), I(t - \tau))}{f(S^*, I^*)} - \frac{I(t)}{I^*}\right) d\tau \\
&\quad + \delta \left(1 - \frac{f(S^*, I^*)}{f(S(t), I^*)}\right) (I^* - I(t)) \\
&= (\mu + \delta) S^* \left(1 - \frac{S(t)}{S^*}\right) \left(1 - \frac{f(S^*, I^*)}{f(S(t), I^*)}\right) \\
&\quad + f(S^*, I^*) \int_0^h p(\tau) \left\{ \left(2 - \frac{f(S^*, I^*)}{f(S(t), I^*)} - \frac{I^*}{I(t)} \frac{f(S(t), I(t - \tau))}{f(S^*, I^*)}\right) \right. \\
&\quad \left. + \left(\frac{f(S(t), I(t - \tau))}{f(S(t), I^*)} - \frac{I(t)}{I^*}\right) \right\} d\tau + \delta \left(1 - \frac{f(S^*, I^*)}{f(S(t), I^*)}\right) (I^* - I(t)) \\
&= (\mu + \delta) S^* \left(1 - \frac{S(t)}{S^*}\right) \left(1 - \frac{f(S^*, I^*)}{f(S(t), I^*)}\right) \\
&\quad + f(S^*, I^*) \int_0^h p(\tau) \left\{ -g\left(\frac{f(S^*, I^*)}{f(S(t), I^*)}\right) - g\left(\frac{I^*}{I(t)} \frac{f(S(t), I(t - \tau))}{f(S^*, I^*)}\right) \right. \\
&\quad \left. + g\left(\frac{f(S(t), I(t - \tau))}{f(S(t), I^*)}\right) - g\left(\frac{I(t)}{I^*}\right) \right\} d\tau + \delta \left(1 - \frac{f(S^*, I^*)}{f(S(t), I^*)}\right) (I^* - I(t)).
\end{aligned}$$

Second, calculating $\frac{dU_+(t)}{dt}$ gives as follows.

$$\frac{dU_+(t)}{dt} = f(S^*, I^*) \int_0^h p(\tau) \left\{ g\left(\frac{f(S(t + \tau), I(t))}{f(S(t + \tau), I^*)}\right) - g\left(\frac{f(S(t), I(t - \tau))}{f(S(t), I^*)}\right) \right\} d\tau.$$

Finally, we obtain that

$$\begin{aligned}
\frac{dV(t)}{dt} &= (\mu + \delta)S^* \left(1 - \frac{S(t)}{S^*}\right) \left(1 - \frac{f(S^*, I^*)}{f(S(t), I^*)}\right) \\
&\quad + f(S^*, I^*) \int_0^h p(\tau) \left\{ -g\left(\frac{f(S^*, I^*)}{f(S(t), I^*)}\right) - g\left(\frac{I^*}{I(t)} \frac{f(S(t), I(t-\tau))}{f(S^*, I^*)}\right) \right. \\
&\quad \left. + g\left(\frac{f(S(t+\tau), I(t))}{f(S(t+\tau), I^*)}\right) - g\left(\frac{I(t)}{I^*}\right) \right\} d\tau \\
&\quad + \delta \left(1 - \frac{f(S^*, I^*)}{f(S(t), I^*)}\right) (I^* - I(t)) \\
&= (\mu + \delta)S^* (1 - x_t) \left(1 - \frac{1}{\bar{x}_t}\right) - \delta I^* \left(1 - \frac{1}{\bar{x}_t}\right) (y_t - 1) \\
&\quad - f(S^*, I^*) \int_0^h p(\tau) \left\{ g\left(\frac{1}{\bar{x}_t}\right) + g\left(\frac{\bar{x}_t \bar{y}_{t-\tau, \tau}}{y_t}\right) + g(y_t) - g(\bar{y}_{t, \tau}) \right\} d\tau.
\end{aligned} \tag{4.12}$$

By applying (4.8), (4.9) and the relation $g\left(\frac{1}{\bar{x}_t}\right) + g\left(\frac{\bar{x}_t \bar{y}_{t-\tau, \tau}}{y_t}\right) \geq 0$ to (4.12), it holds that

$$\begin{aligned}
\frac{dV(t)}{dt} &\leq -C_0 \frac{f(S(t), I^*)}{I^*} (\mu + \delta) \left(1 - \frac{1}{\bar{x}_t}\right)^2 \\
&\quad - \delta I^* \left(1 - \frac{1}{\bar{x}_t}\right) (y_t - 1) - C_1 C_2 f(S^*, I^*) I^* (y_t - 1)^2.
\end{aligned}$$

Using $f(S^*, I^*) = (\mu + \gamma)I^*$ gives as follows. we have that

$$\begin{aligned}
\frac{dV(t)}{dt} &\leq -\left\{ C_0 \frac{f(S(t), I^*)}{I^*} (\mu + \delta) \left(1 - \frac{1}{\bar{x}_t}\right)^2 + \delta I^* \left(1 - \frac{1}{\bar{x}_t}\right) (y_t - 1) \right. \\
&\quad \left. + C_1 C_2 (\mu + \gamma) (I^*)^2 (y_t - 1)^2 \right\}.
\end{aligned}$$

By Theorem 4.1, for any $0 < \varepsilon < v_1$, there exists a $T_\varepsilon > 0$ such that $S(t) > v_1 - \varepsilon$ for any $t > T_\varepsilon$. From the condition (II) of (1.5), we may restrict this $\varepsilon > 0$ sufficiently small such that

$$\delta^2 - 4C_0 C_1 C_2 (\mu + \delta) (\mu + \gamma) \frac{f(v_1 - \varepsilon, I^*)}{I^*} < 0.$$

Then, we have that

$$\begin{aligned}
&(\delta I^*)^2 - 4 \left\{ C_0 \frac{f(S(t), I^*)}{I^*} (\mu + \delta) \right\} \left\{ C_1 C_2 (\mu + \gamma) (I^*)^2 \right\} \\
&< (I^*)^2 \left\{ \delta^2 - 4C_0 C_1 C_2 (\mu + \delta) (\mu + \gamma) \frac{f(v_1 - \varepsilon, I^*)}{I^*} \right\} < 0,
\end{aligned}$$

from which we obtain that $\frac{dV(t)}{dt} \leq 0$ holds for all $t > T_\varepsilon$ with equality if and only if $S(t) = S^*$ and $I(t) = I^*$. Thus, by an extension of LaSalle's invariant principle (see also Kuang [13, Corollary 5.2]), E_* is globally asymptotically stable. Hence, the proof is complete. \square

5 Applications

In this section, we illustrate some examples in order to validate the feasibility of our global stability results with respect to the rate of immunity lost δ . We consider the following SIRS epidemic model with a discrete delay:

$$\begin{cases} \frac{dS(t)}{dt} = \mu - \mu S(t) - \beta \frac{S(t)}{1 + \alpha_S S(t)} \frac{I(t - \tau)}{1 + \alpha_I I(t - \tau)} + \delta R(t), \\ \frac{dI(t)}{dt} = \beta \frac{S(t)}{1 + \alpha_S S(t)} \frac{I(t - \tau)}{1 + \alpha_I I(t - \tau)} - (\mu + \gamma)I(t), \\ \frac{dR(t)}{dt} = \gamma I(t) - (\mu + \delta)R(t), \quad \tau > 0. \end{cases} \quad (5.1)$$

$\beta > 0$ denotes the infection force of disease. Here, $\frac{1}{1 + \alpha_S S(t)}$ and $\frac{1}{1 + \alpha_I I(t - \tau)}$ measures the inhibition effects from the behavioral change of the susceptible individuals and infective individuals, respectively. For the case $\alpha_S = \alpha_I = 0$, the incidence rate becomes a form which is proposed in Vargas-De-León and Gómez-Alcaraz [22] and for the case $\alpha_S = 0$, the incidence rate becomes a form which is proposed in Xu and Ma [23]. From (1.4), the basic reproduction number of system (5.1) becomes $R_0 = \frac{\beta}{(\mu + \gamma)(1 + \alpha_S)}$.

Using similar techniques in the proofs of Theorems 1.1 and 1.2, we establish the following result (cf. McCluskey [14, Section 5]):

Corollary 5.1 *If $R_0 \leq 1$, then the disease-free equilibrium E_0 of system (1.2) is globally asymptotically stable. If $R_0 > 1$, then the endemic equilibrium E_* of system (5.1) exists uniquely and system (5.1) is permanent. Moreover, the endemic equilibrium E_* of system (5.1) is globally asymptotically stable if*

$$\begin{cases} \delta^2 < 4\tilde{C}_0\tilde{C}_1\tilde{C}_2(\mu + \delta)(\mu + \gamma) \frac{\beta\tilde{v}_1}{(1 + \alpha_S\tilde{v}_1)(1 + \alpha_I I^*)}, \\ \tilde{C}_0 = \frac{(1 + \alpha_S\tilde{v}_1)(1 + \alpha_I I^*)}{\beta} \leq \frac{(1 + \alpha_S S^*)(1 + \alpha_I I^*)}{\beta}, \\ \tilde{C}_1 = \frac{\beta}{(1 + \alpha_S\tilde{v}_1)(1 + \alpha_I)^2} \leq \frac{\beta}{(1 + \alpha_S\tilde{v}_1)(1 + \alpha_I)(1 + \alpha_I I^*)}, \\ \tilde{C}_2 = \frac{(1 + \alpha_S)\alpha_I}{\beta}, \end{cases} \quad (5.2)$$

where $v = \tilde{v}_1$ is a unique positive solution of $\mu - \frac{\beta v}{(1 + \alpha_S v)} - \mu v = 0$.

For system (5.1), under the conditions

$$\tau = 0.1, \quad \alpha_S = \alpha_I = 0.1, \quad \gamma = 0.01 \quad \text{and} \quad \mu = 0.02, \quad (5.3)$$

we consider two cases of $\beta = 0.02$ and $\beta = 0.06$. First, we consider the case $\beta = 0.02$. Then, we obtain $R_0 = 0.606 \dots \leq 1$. By Theorem 1.1, the disease-free equilibrium E_0 of system (5.1) is globally asymptotically stable for any $\delta \geq 0$.

Second, we consider the case $\beta = 0.06$. Then, we obtain $R_0 = 1.818 \dots > 1$ and $\tilde{v}_1 = 0.512 \dots$. For this case, since the condition (5.2) becomes $0 < \delta < \delta^* := 0.013 \dots$, the endemic equilibrium E_* of system (5.1) is globally asymptotically stable for any $0 < \delta < \delta^*$. From a biological point of view, for a small loss of immunity rate, the prevalence of the disease can settle to an endemic steady state independently of the initial conditions.

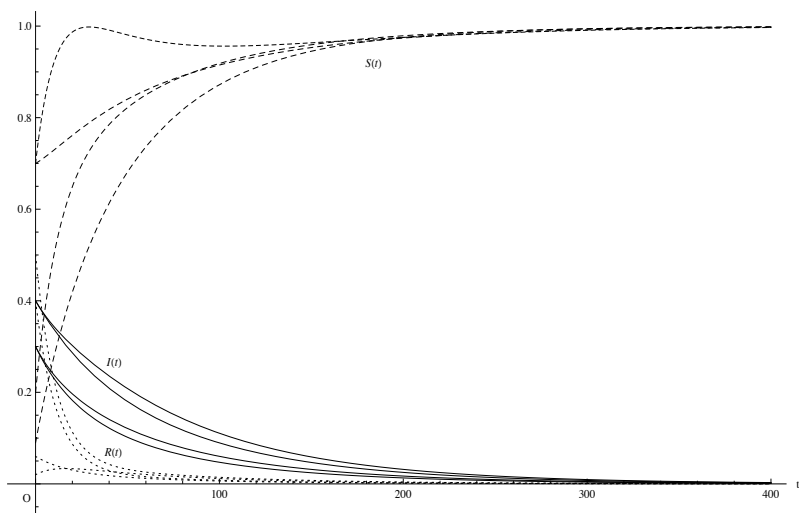


Fig. 1 The graph trajectory of $S(t)$, $I(t)$ and $R(t)$ of system (5.1). For the case (5.3) with $\beta = 0.02$ and $\delta = 0.07$, we have $R_0 = 0.606\cdots < 1$ and $E_0 = (1, 0, 0)$.

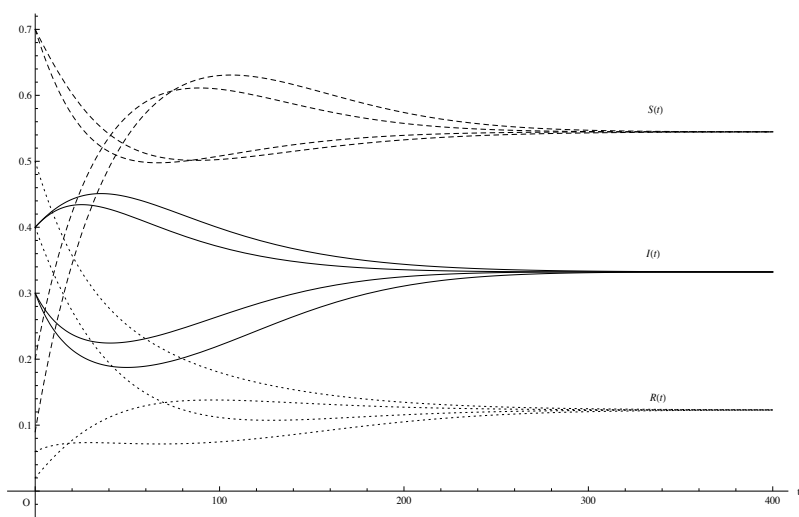


Fig. 2 The graph trajectory of $S(t)$, $I(t)$ and $R(t)$ of system (5.1). For the case (5.3) with $\beta = 0.06$ and $\delta = 0.007 < \delta^*$, we have $R_0 = 1.818\cdots > 1$ and $E^* = (0.544\cdots, 0.332\cdots, 0.123\cdots)$.

Figures 1 and 2 indicate that the disease-free equilibrium E_0 and the endemic equilibrium E_* of system (5.1) are globally asymptotically stable for the first and the second cases with $\delta = 0.07$ and $\delta = 0.007$, respectively.

On the other hand, Figure 3 indicates that the endemic equilibrium E_* of system (5.1) is also globally asymptotically stable even if the condition (5.2) in Corollary 5.1 fail for the second case with $\delta = 0.07 \geq \delta^*$ for $R_0 > 1$. There is still an open problem to determine the global asymptotic stability of the endemic equilibrium of system (5.1) for $\delta \geq \delta^*$.

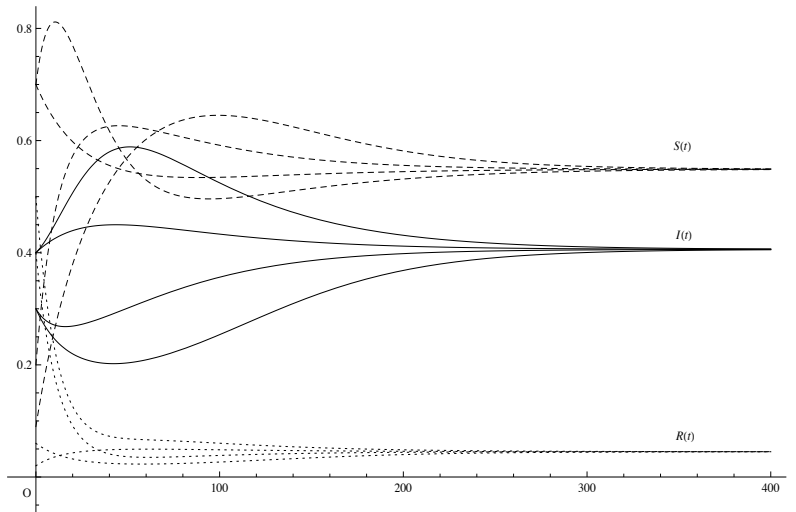


Fig. 3 The graph trajectory of $S(t)$, $I(t)$ and $R(t)$ of system (5.1). For the case (5.3) with $\beta = 0.06$ and $\delta = 0.07 \geq \delta^*$, we have $R_0 = 1.818 \dots > 1$ and $E^* = (0.548 \dots, 0.406 \dots, 0.045 \dots)$.

6 Discussion

In this paper, for an SIRS epidemic model with a wide class of nonlinear incidence rates and distributed delays $\int_0^h p(\tau) f(S(t), I(t-\tau)) d\tau$, we established the global asymptotic stability of the disease-free equilibrium E_0 for $R_0 \leq 1$ and the endemic equilibrium E_* of system (1.2) for $R_0 > 1$. By using a limit system of the model, a proof that the disease-free equilibrium is globally asymptotically stable for $R_0 = 1$ is also given. In particular, without imposing any restriction on the size of a maximum latent period h , the global asymptotic stability of the endemic equilibrium E_* of system (1.2) is established for a small loss of immunity rate δ . By means of strict monotonicity of functions $f(S, I)$ and $f(S, I)/I$ with respect to $S \geq 0$ and $I > 0$, we obtain Lemma 4.1 which plays an important role to establish the condition (1.5) such that V is a Lyapunov functional. Hence, our result is an extension to the global stability result for an SIR epidemic model.

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