

On a generalization of the global attractivity for a periodically forced Pielou's equation

Keigo Ishihara^a Yukihiro Nakata^{b,*}

^aDepartment of Pure and Applied Mathematics, Waseda University, 3-4-1 Ohkubo, Shinjuku-ku, Tokyo 169-8555, Japan

^bBasque Center For Applied Mathematics, Bizkaia Technology Park, Building 500 E-48160 Derio, Spain

Abstract

In this paper, we study the global attractivity for a class of periodic difference equation with delay which has a generalized form of Pielou's difference equation. The global dynamics of the equation is characterized by using a relation between the upper and lower limit of the solution. There are two possible global dynamics: zero solution is globally attractive or there exists a periodic solution which is globally attractive. Recent results in [E. Camouzis, G. Ladas, Periodically forced Pielou's equation, J. Math. Anal. Appl. 333 (1) (2007) 117-127] is generalized. Two examples are given to illustrate our results.

Key words: Periodically forced difference equation; global attractivity; Pielou's equation

1. Introduction

Several authors have studied difference equations for mathematical models in population biology (see [1–11] and the references therein). Pielou's equation was proposed by Pielou in [11] as a discrete analogue of the logistic equation with delay.

Camouzis and Ladas [3] studied the following Pielou's equation with a periodic coefficient,

$$x_{n+1} = \frac{\beta_n x_n}{1 + x_{n-1}}, n = 0, 1, 2, \dots, \quad (1.1)$$

where $\beta_n, n = 0, 1, 2, \dots$ is a periodic sequence with an arbitrary positive integer k . Initial conditions are given by $x_0 > 0, x_{-1} \geq 0$. They proved that every solution converges to 0, if $\prod_{n=1}^k \beta_n \leq 1$, while there exists a k -periodic solution which is globally attractive, if $\prod_{n=1}^k \beta_n > 1$ by using an interesting relation between the upper and lower limit of the solution. Recently, Nyerges [10]

* Corresponding author

Email addresses: keigo.i.123@gmail.com (Keigo Ishihara), yunayuna.na@gmail.com (Yukihiro Nakata).

studied the global dynamics of a general autonomous difference equation by extending their idea. Let us introduce the main result in Camouzis and Ladas [3].

Theorem A (see Camouzis and Ladas [3, Theorems 3.2, 3.3 and 3.4]) *If*

$$\prod_{n=1}^k \beta_n > 1,$$

then there exists a k -periodic solution x_n^ such that $x_n^* = x_{n+k}^*$ which is globally attractive, that is, for any solution of (1.1), it holds that*

$$\lim_{n \rightarrow +\infty} (x_n - x_n^*) = 0.$$

In this paper, we further generalize Theorem A. The present paper is focused on nonautonomous difference equations, different from Nyerges [10]. We shall study the following difference equation,

$$x_{n+1} = x_n f_n(x_{n-1}), n = 0, 1, 2, \dots, \quad (1.2)$$

where $f_n(x), n = 0, 1, 2, \dots$, is continuous, bounded and positive function on $x \in [0, +\infty)$ and k -periodic on n , that is,

$$f_n(x) = f_{n+k}(x) \text{ where } k \text{ is an arbitrary positive integer.}$$

It is assumed that the initial conditions are given by $x_0 = x^0 > 0, x_{-1} = x^{-1} \geq 0$. For the function $f_n(x), n = 1, 2, \dots, k$, we impose the following monotonicity property

$$f_n(x) \text{ is strictly decreasing on } x \in [0, +\infty) \text{ for } n = 1, 2, \dots, k, \quad (1.3)$$

and

$$x f_n(x) \text{ is strictly increasing on } x \in [0, +\infty) \text{ for } n = 1, 2, \dots, k. \quad (1.4)$$

Under the assumption (1.3), $\prod_{n=1}^k f_n(x)$ is also strictly decreasing on $x \in [0, +\infty)$ and hence, there exists a some constant $c \geq 0$ such that

$$c = \lim_{x \rightarrow +\infty} \prod_{n=1}^k f_n(x).$$

The following theorem is a generalized version of Theorem A.

Theorem 1.1 *Assume that (1.3) and (1.4). If*

$$\prod_{n=1}^k f_n(0) > 1, \text{ and } \lim_{x \rightarrow +\infty} \prod_{n=1}^k f_n(x) < 1, \quad (1.5)$$

then there exists a k -periodic solution x_n^ such that $x_n^* = x_{n+k}^*$ which is globally attractive, that is, for any solution of (1.2), it holds that*

$$\lim_{n \rightarrow +\infty} (x_n - x_n^*) = 0.$$

One can see that the assumptions (1.3) and (1.4) are nice properties to obtain the global character of the periodic difference equation which has a form (1.2). Motivated by Camouzis and Ladas [3], we also find a useful relation between the upper and lower limit of the solution. Then we can obtain the existence of k -periodic solution which is globally attractive if (1.5) holds.

The paper is organized as follows. At first, we consider the case where every solution approaches to zero solution in Section 2. In Section 3, we show that every solution is bounded above and below by a positive constant, respectively. This makes possible to consider a set of the upper and lower limit of the solutions which are positive constants (see (3.5)). In Sections 4 and 5, we consider the existence of a k -periodic solution which is globally attractive. We divide the discussion in two cases, k is an even integer in Section 4 and k is an odd integer in Section 5. It is important to establish the relation (see Lemmas 4.3 and 5.3) between the upper and lower limit

of the solution in these sections. In Section 6, we apply Theorems 1.1 to two nonautonomous difference equations. The global attractivity for a delayed Beverton-Holt equation with a periodic coefficient is established.

2. Global attractivity of zero solution

First of all, we consider the case where every solution approaches to zero. Let us introduce the following result which generalizes Theorem 3.1 in Camouzis and Ladas [3].

Theorem 2.1 *Assume that (1.3). If*

$$\prod_{n=1}^k f_n(0) \leq 1, \quad (2.1)$$

then, for any solution of (1.2), it holds that

$$\lim_{n \rightarrow +\infty} x_n = 0.$$

PROOF. We have

$$\prod_{n=1}^k f_n(x_n) \begin{cases} < \prod_{n=1}^k f_n(0) & \text{if there exists } n \in \{1, 2, \dots, k\} \text{ such that } x_n > 0, \\ = \prod_{n=1}^k f_n(0) & \text{for } x_n = 0, n = 1, 2, \dots, k, \end{cases}$$

from (1.3). Since $x_n > 0$ for $n = 1, 2, \dots$, for given the initial conditions $x_0 = x^0 > 0, x_{-1} = x^{-1} \geq 0$, it holds

$$x_n = x_{n-k} \prod_{j=1}^k f_{n-j}(x_{n-j-1}) < x_{n-k}, \text{ for } n = k+1, k+2, \dots$$

Thus, we obtain the conclusion and the proof is complete. \square

3. Permanence

In this section, we show that every solution is bounded above and below by a positive constant, respectively, if (1.5) holds. Based on the following result, we investigate the existence of the periodic solution which is globally attractive for any solution in Sections 4 and 5.

Theorem 3.1 *Assume that (1.3). If (1.5), then, for any solution of (1.2), it holds that*

$$0 < \bar{x} \prod_{n=1}^k f_n(\bar{x}) \leq \liminf_{n \rightarrow +\infty} x_n \leq \limsup_{n \rightarrow +\infty} x_n \leq \bar{x} \prod_{n=1}^k f_n(0) < +\infty,$$

where $\bar{x} = \bar{x} \prod_{n=1}^k f_n(0)$ and \bar{x} is a unique positive solution of $\prod_{n=1}^k f_n(x) = 1$.

PROOF. We see that there exists $\bar{x} < +\infty$ such that $\prod_{n=1}^k f_n(\bar{x}) = 1$ by (1.5).

Suppose that $\limsup_{n \rightarrow +\infty} x_n = +\infty$. Then there exists a subsequence $\{\bar{n}_m\}_{m=1}^{+\infty}$ such that

$$x_{\bar{n}_m} = \max_{0 \leq n \leq \bar{n}_m} x_n \text{ and } \lim_{m \rightarrow +\infty} x_{\bar{n}_m} = \limsup_{n \rightarrow +\infty} x_n = +\infty. \quad (3.1)$$

From (1.2), it holds that

$$\begin{aligned} x_{\bar{n}_m} &= x_{\bar{n}_m-1} f_{\bar{n}_m-1}(x_{\bar{n}_m-2}) \\ &= x_{\bar{n}_m-2} f_{\bar{n}_m-2}(x_{\bar{n}_m-3}) f_{\bar{n}_m-1}(x_{\bar{n}_m-2}) \\ &= \dots \\ &= x_{\bar{n}_m-k} \prod_{j=1}^k f_{\bar{n}_m-j}(x_{\bar{n}_m-j-1}). \end{aligned} \quad (3.2)$$

From (3.1) and (3.2), we see

$$\prod_{j=1}^k f_{\bar{n}_m-j}(x_{\bar{n}_m-j-1}) \geq 1,$$

and, by (1.3), it follows

$$\prod_{j=1}^k f_{\bar{n}_m-j}(x_{\bar{n}_m-j-1}) \geq \prod_{j=1}^k f_{\bar{n}_m-j}(x_{\bar{n}_m-j-1}) \geq 1,$$

where

$$x_{\bar{n}_m-j-1} = \min_{1 \leq j \leq k} x_{\bar{n}_m-j-1}.$$

Then, we see $x_{\bar{n}_m-j-1} \leq \bar{x}$ and hence, from (3.2), we obtain

$$\begin{aligned} x_{\bar{n}_m} &= x_{\bar{n}_m-j-1} \prod_{j=1}^{j+1} f_{\bar{n}_m-j}(x_{\bar{n}_m-j-1}) \\ &\leq \bar{x} \prod_{n=1}^k f_n(0) < +\infty, \end{aligned}$$

which leads a contradiction to our assumption. Thus

$$\limsup_{n \rightarrow +\infty} x_n < +\infty.$$

Moreover, similar to the above discussion, we obtain that

$$\limsup_{n \rightarrow +\infty} x_n \leq \bar{x} \prod_{n=1}^k f_n(0) < +\infty.$$

Next, suppose that $\liminf_{n \rightarrow +\infty} x_n = 0$. Then there exists a subsequence $\{\underline{n}_m\}_{m=1}^{+\infty}$ such that

$$x_{\underline{n}_m} = \min_{0 \leq n \leq \underline{n}_m} x_n \text{ and } \lim_{m \rightarrow +\infty} x_{\underline{n}_m} = \liminf_{n \rightarrow +\infty} x_n = 0. \quad (3.3)$$

From (1.2), it holds that

$$\begin{aligned} x_{\underline{n}_m} &= x_{\underline{n}_m-1} f_{\underline{n}_m-1}(x_{\underline{n}_m-2}) \\ &= x_{\underline{n}_m-2} f_{\underline{n}_m-2}(x_{\underline{n}_m-3}) f_{\underline{n}_m-1}(x_{\underline{n}_m-2}) \\ &= \dots \\ &= x_{\underline{n}_m-k} \prod_{j=1}^k f_{\underline{n}_m-j}(x_{\underline{n}_m-j-1}). \end{aligned} \quad (3.4)$$

From (3.3) and (3.4), we see

$$\prod_{j=1}^k f_{\underline{n}_m-j}(x_{\underline{n}_m-j-1}) \leq 1,$$

and, by (1.3), it follows

$$\prod_{j=1}^k f_{\underline{n}_m-j}(x_{\underline{n}_m-j-1}) \leq \prod_{j=1}^k f_{\underline{n}_m-j}(x_{\underline{n}_m-j-1}) \leq 1,$$

where

$$x_{\underline{n}_m-j-1} = \max_{1 \leq j \leq k} x_{\underline{n}_m-j-1}.$$

Then, we see $x_{\underline{n}_m-j-1} \geq \bar{x}$. Hence, from (3.4), we obtain

$$x_{\underline{n}_m} \geq \bar{x} \prod_{n=1}^k f_n(\bar{x}) > 0,$$

where $\bar{x} = \bar{x} \prod_{n=1}^k f_n(0)$. This leads a contradiction to our assumption. Thus, we obtain

$$\liminf_{n \rightarrow +\infty} x_n > 0.$$

Moreover, similar to the above discussion, we obtain

$$\liminf_{n \rightarrow +\infty} x_n \geq \bar{x} \prod_{n=1}^k f_n(\bar{x}) > 0.$$

Hence, the proof is complete. \square

Remark 3.2 *The assumption (1.4) is not needed for the permanence of the solution.*

Hereafter, we assume (1.3) and (1.4) hold. We are interested in the existence of k -periodic solution which is globally attractive. Let us introduce some notations which are used throughout the paper. At first, we set

$$\begin{cases} S_h &= \limsup_{n \rightarrow +\infty} x_{kn+h}, \\ I_h &= \liminf_{n \rightarrow +\infty} x_{kn+h}, \end{cases} \text{ for } h = \dots, -1, 0, 1, \dots, \quad (3.5)$$

and will show

$$S_h = I_h, \text{ for } h = 1, 2, \dots, k,$$

holds in order to establish the existence of the periodic solution which is globally attractive. Obviously, by Theorem 3.1, we see $(S_h, I_h) \in (0, \infty) \times (0, \infty)$ if (1.5) holds. Further, from (3.5), it follows

$$\begin{cases} S_h = \limsup_{n \rightarrow +\infty} x_{kn+h} &= \limsup_{n \rightarrow +\infty} x_{k(n+j)+h} = S_{k+j+h}, \\ I_h = \liminf_{n \rightarrow +\infty} x_{kn+h} &= \liminf_{n \rightarrow +\infty} x_{k(n+j)+h} = I_{k+j+h}, \end{cases} \quad (3.6)$$

where j is a some integer and the relation (3.6) will be used if necessary. We, further, introduce two sets of subsequences $\{\bar{n}_m^h\}_{m=0}^{+\infty}$ and $\{\underline{n}_m^h\}_{m=0}^{+\infty}$ for $h = 1, 2, \dots, k$ such that

$$\begin{cases} \lim_{m \rightarrow +\infty} x_{k\bar{n}_m^h+h} &= \limsup_{n \rightarrow +\infty} x_{kn+h} = S_h, \\ \lim_{m \rightarrow +\infty} x_{k\underline{n}_m^h+h} &= \liminf_{n \rightarrow +\infty} x_{kn+h} = I_h. \end{cases} \text{ for } h = 1, 2, \dots, k.$$

Finally, for simplicity of the proof, we also define $f_h(x)$ as

$$f_h(x) = f_{k+h}(x) \text{ for } h = -1, -2, \dots, -k.$$

4. Global attractivity for the case where k is an even integer

In this section, we show that Theorem 1.1 holds when k is an even integer. For the reader, we first consider the case $k = 2$ in Section 4.1. (4.1) in Lemma 4.1 has an important role in this subsection. Then, we give Theorem 4.2 which shows that there exists a 2-periodic solution which is globally attractive. In Section 4.2, we generalize these results to the case where k is an arbitrary even integer.

4.1. Case: $k = 2$

We introduce the following lemma which plays a crucial role in the proof of Theorem 4.2.

Lemma 4.1 *Let $k = 2$. Assume that (1.3) and (1.4). If (1.5), then it holds that*

$$f_h(I_{h-1})f_{h-1}(S_h) = 1, \quad (4.1)$$

for $h = 1, 2$.

PROOF. At first, we see that

$$\begin{aligned} x_{k\bar{n}_m^h+h} &= x_{k\bar{n}_m^h+h-1} f_{k\bar{n}_m^h+h-1}(x_{k\bar{n}_m^h+h-2}) \\ &= x_{k\bar{n}_m^h+h-2} f_{k\bar{n}_m^h+h-2}(x_{k\bar{n}_m^h+h-3}) f_{k\bar{n}_m^h+h-1}(x_{k\bar{n}_m^h+h-2}) \\ &= x_{k\bar{n}_m^h+h-2} f_{h-2}(x_{k\bar{n}_m^h+h-3}) f_{h-1}(x_{k\bar{n}_m^h+h-2}), \end{aligned} \quad (4.2)$$

for $h = 1, 2$, from (1.2) and by considering the limiting equation of (4.2) (letting $m \rightarrow +\infty$) and using (1.3) and (1.4), it follows

$$S_h \leq S_{h-2}f_{h-2}(I_{h-3})f_{h-1}(S_{h-2}) = S_h f_h(I_{h-1})f_{h-1}(S_h). \quad (4.3)$$

Similarly, (by considering the limiting equation of (4.2) for the subsequences $\{\underline{n}_m^h\}_{m=0}^{+\infty}$, $h = 1, 2$ and using (1.3) and (1.4)), we can obtain

$$I_h \geq I_{h-2}f_{h-2}(S_{h-3})f_{h-1}(I_{h-2}) = I_h f_h(S_{h-1})f_{h-1}(I_h). \quad (4.4)$$

Consequently, by (4.3) and (4.4), the following holds

$$\begin{cases} 1 & \leq f_h(I_{h-1})f_{h-1}(S_h), \\ 1 & \geq f_h(S_{h-1})f_{h-1}(I_h), \end{cases} \text{ for } h = 1, 2,$$

from which, we obtain (4.1) and the proof is complete. \square

Then, we obtain the following result.

Theorem 4.2 *Let $k = 2$. Assume that (1.3) and (1.4). If (1.5), then there exists a 2-periodic solution x_n^* such that $x_n^* = x_{n+2}^*$ which is globally attractive, that is, for any solution of (1.2), it holds that*

$$\lim_{n \rightarrow +\infty} (x_n - x_n^*) = 0.$$

PROOF. In order to obtain the conclusion, we will show

$$S_h = I_h, h = 1, 2. \quad (4.5)$$

Let

$$U_{h,h-j} = \lim_{m \rightarrow +\infty} x_{k\bar{n}_m^h+h-j} \text{ for } j = 1, 2, \dots, 5,$$

and we claim

$$U_{h,h-j} = \begin{cases} S_{h-j} & \text{for } j = 2, 4, \\ I_{h-j} & \text{for } j = 1, 3, 5, \end{cases} \quad (4.6)$$

for $h = 1, 2$.

At first, we see that it holds

$$\begin{aligned} x_{k\bar{n}_m^h+h-j} &= x_{k\bar{n}_m^h+h-j-2}f_{h-2}(x_{k\bar{n}_m^h+h-j-2})f_{h-1}(x_{k\bar{n}_m^h+h-j-2}) \\ &= x_{k\bar{n}_m^h+h-j-2}f_{h-2}(x_{k\bar{n}_m^h+h-j-3})f_{h-1}(x_{k\bar{n}_m^h+h-j-2}), \end{aligned} \quad (4.7)$$

for $h = 1, 2$ and $j = 0, 1, 2, \dots$ from (1.2).

Firstly, we show

$$U_{h,h-j} = \begin{cases} S_{h-j} = S_h & \text{for } j = 2, \\ I_{h-j} = I_{h-1} & \text{for } j = 3, \end{cases} \quad (4.8)$$

for $h = 1, 2$. Suppose that there exists some $\bar{h} \in \{1, 2\}$ such that $U_{\bar{h},\bar{h}-2} < S_{\bar{h}}$ or $U_{\bar{h},\bar{h}-3} > I_{\bar{h}-1}$. By considering the limiting equation of (4.7) with $j = 0$, it follows

$$S_h = U_{h,h-2}f_h(U_{h,h-3})f_{h-1}(U_{h,h-2}).$$

Then, by (1.3) and (1.4), it follows $S_{\bar{h}} < S_{\bar{h}}f_{\bar{h}}(I_{\bar{h}-1})f_{\bar{h}-1}(S_{\bar{h}})$, which implies

$$1 < f_h(I_{h-1})f_{h-1}(S_h), \text{ for } h = \bar{h}.$$

This leads a contradiction to (4.1) in Lemma 4.1. Thus, (4.8) holds.

Next, we show

$$U_{h,h-j} = \begin{cases} S_{h-j} = S_h & \text{for } j = 4, \\ I_{h-j} = I_{h-1} & \text{for } j = 5, \end{cases} \quad (4.9)$$

for $h = 1, 2$. Suppose that there exists some $\bar{h} \in \{1, 2\}$ such that $U_{\bar{h}, \bar{h}-4} < S_{\bar{h}}$ or $U_{\bar{h}, \bar{h}-5} > I_{\bar{h}-1}$. By considering the limiting equation of (4.7) with $j = 2$ and substituting (4.8), it follows

$$U_{h,h-2} = S_h = U_{h,h-4} f_h(U_{h,h-5}) f_{h-1}(U_{h,h-4}),$$

for $h = 1, 2$. Then, by (1.3) and (1.4), it follows $S_{\bar{h}} < S_{\bar{h}} f_{\bar{h}}(I_{\bar{h}-1}) f_{\bar{h}-1}(S_{\bar{h}})$ which implies

$$1 < f_h(I_{h-1}) f_{h-1}(S_h), \text{ for } h = \bar{h}.$$

This gives a contradiction to (4.1) in Lemma 4.1. Thus, (4.9) holds.

By considering the limiting equation of (4.7) with $j = 1$ and using (4.8)-(4.9), we see

$$U_{h,h-1} = U_{h,h-3} f_{h-1}(U_{h,h-4}) f_h(U_{h,h-3}) = I_{h-1} f_{h-1}(S_h) f_h(I_{h-1}).$$

Hence, it follows

$$U_{h,h-1} = I_{h-1}, \quad (4.10)$$

for $h = 1, 2$, by (4.1) in Lemma 4.1. Consequently, (4.6) holds from (4.8), (4.9) and (4.10).

From (1.2), it holds

$$x_{k\bar{n}_m^h+h} = x_{k\bar{n}_m^h+h-1} f_{k\bar{n}_m^h+h-1}(x_{k\bar{n}_m^h+h-2}) = x_{k\bar{n}_m^h+h-1} f_{h-1}(x_{k\bar{n}_m^h+h-2}),$$

and by considering the limiting equation and using (4.6), we obtain

$$S_h = U_{h,h-1} f_{h-1}(U_{h,h-2}) = I_{h-1} f_{h-1}(S_h).$$

By (4.1) in Lemma 4.1, it holds

$$S_h f_h(I_{h-1}) = I_{h-1} f_{h-1}(S_h) f_h(I_{h-1}) = I_{h-1}, \quad (4.11)$$

for $h = 1, 2$. On the other hand, similar to the above discussion, it also holds

$$I_h f_h(S_{h-1}) = S_{h-1} f_{h-1}(I_h) f_h(S_{h-1}) = S_{h-1}, \quad (4.12)$$

for $h = 1, 2$. Consequently, it holds

$$I_{h-1} = S_h f_h(I_{h-1}) \leq I_h f_h(S_{h-1}) = S_{h-1},$$

by (4.11) and (4.12), and hence, (4.5) holds. Then, from (3.5), we see

$$\liminf_{n \rightarrow +\infty} x_{kn+h} = \limsup_{n \rightarrow +\infty} x_{kn+h} = \lim_{n \rightarrow +\infty} x_{kn+h},$$

for $h = 1, 2$ and there exist two positive constants $x_1^* = S_1 = I_1$ and $x_2^* = S_2 = I_2$ such that

$$x_2^* = \lim_{n \rightarrow +\infty} x_{2n},$$

and

$$x_1^* = \lim_{n \rightarrow +\infty} x_{2n+1}.$$

The proof is complete. \square

4.2. Case: k is an even integer

In this subsection, we generalize results in Section 4.1 to the case where k is an arbitrary even integer.

Lemma 4.3 *Let k be an even integer. Assume that (1.3) and (1.4). If (1.5), then it holds that*

$$(f_{h-k}(I_{h-k-1})f_{h+1-k}(S_{h-k})) \cdots (f_{h-4}(I_{h-5})f_{h-3}(S_{h-4})) (f_{h-2}(I_{h-3})f_{h-1}(S_{h-2})) = 1, \quad (4.13)$$

for $h = 1, 2, \dots, k$.

PROOF. At first, from (1.2), we see that it holds

$$\begin{aligned} x_{k\bar{n}_m^h+h} &= x_{k\bar{n}_m^h+h-2} f_{k\bar{n}_m^h+h-2}(x_{k\bar{n}_m^h+h-3}) f_{k\bar{n}_m^h+h-1}(x_{k\bar{n}_m^h+h-2}) \\ &= x_{k\bar{n}_m^h+h-2} f_{h-2}(x_{k\bar{n}_m^h+h-3}) f_{h-1}(x_{k\bar{n}_m^h+h-2}), \end{aligned} \quad (4.14)$$

for $h = 1, 2, \dots, k$, and by considering the limiting equation of (4.14) and using (1.3) and (1.4), it follows

$$S_h \leq S_{h-2} f_{h-2}(I_{h-3}) f_{h-1}(S_{h-2}), \quad (4.15)$$

for $h = 1, 2, \dots, k$. From (4.15) we obtain

$$\begin{aligned} S_h &\leq S_{h-2} (f_{h-2}(I_{h-3}) f_{h-1}(S_{h-2})) \\ &\leq S_{h-4} (f_{h-4}(I_{h-5}) f_{h-3}(S_{h-4})) (f_{h-2}(I_{h-3}) f_{h-1}(S_{h-2})) \\ &\leq \dots \\ &\leq S_{h-k} \bar{G}_h, \end{aligned}$$

where

$$\bar{G}_h = (f_{h-k}(I_{h-k-1}) f_{h+1-k}(S_{h-k})) \cdots (f_{h-4}(I_{h-5}) f_{h-3}(S_{h-4})) (f_{h-2}(I_{h-3}) f_{h-1}(S_{h-2})). \quad (4.16)$$

Then

$$1 \leq \bar{G}_h,$$

holds for $h = 1, 2, \dots, k$, because we have $S_h = S_{h-k}$ from (3.6).

Similarly (by considering the limiting equation of (4.14) for the subsequences $\{n_m^h\}_{m=0}^{+\infty}$, $h = 1, 2, \dots, k$ and using (1.3) and (1.4)), it follows

$$I_h \geq I_{h-2} f_{h-2}(S_{h-3}) f_{h-1}(I_{h-2}), \quad (4.17)$$

for $h = 1, 2, \dots, k$. From (4.17) we obtain

$$\begin{aligned} I_h &\geq I_{h-2} (f_{h-2}(S_{h-3}) f_{h-1}(I_{h-2})) \\ &\geq I_{h-4} (f_{h-4}(S_{h-5}) f_{h-3}(I_{h-4})) (f_{h-2}(S_{h-3}) f_{h-1}(I_{h-2})) \\ &\geq \dots \\ &\geq I_{h-k} \underline{G}_h, \end{aligned}$$

where

$$\underline{G}_h = (f_{h-k}(S_{h-k-1}) f_{h+1-k}(I_{h-k})) \cdots (f_{h-4}(S_{h-5}) f_{h-3}(I_{h-4})) (f_{h-2}(S_{h-3}) f_{h-1}(I_{h-2})). \quad (4.18)$$

Then

$$1 \geq \underline{G}_h,$$

holds for $h = 1, 2, \dots, k$, because we have $I_h = I_{h-k}$ from (3.6). Consequently, it holds that

$$\underline{G}_h \leq 1 \leq \bar{G}_h, \text{ for } h = 1, 2, \dots, k. \quad (4.19)$$

By (4.16) and (4.18), we see

$$\bar{G}_1 = \underline{G}_k \text{ and } \bar{G}_h = \underline{G}_{h-1} \text{ for } h \in \{2, 3, \dots, k\}. \quad (4.20)$$

Therefore, we obtain (4.13) from (4.19) and (4.20). The proof is complete. \square

We show that there exists a k -periodic solution which is globally attractive for any solution for the case where k is an even integer.

Theorem 4.4 *Let k be an even integer. Assume that (1.3) and (1.4). If (1.5), then there exists a k -periodic solution x_n^* such that $x_n^* = x_{n+k}^*$ which is globally attractive, that is, for any solution of (1.2), it holds that*

$$\lim_{n \rightarrow +\infty} (x_n - x_n^*) = 0.$$

PROOF. To obtain the conclusion, we will show

$$S_h = I_h \text{ for } h = 1, 2, \dots, k. \quad (4.21)$$

Let

$$U_{h,h-j} = \lim_{m \rightarrow +\infty} x_{k\bar{m}^h+h-j} \text{ for } j = 2, 3, \dots, k+1,$$

and we claim that it holds

$$U_{h,h-j} = \begin{cases} S_{h-j} & \text{for } j = 2, 4, \dots, k, \\ I_{h-j} & \text{for } j = 3, 5, \dots, k+1, \end{cases} \quad (4.22)$$

for $h = 1, 2, \dots, k$.

From (1.2), it holds

$$\begin{aligned} x_{k\bar{m}^h+h-j} &= x_{k\bar{m}^h+h-j-2} f_{k\bar{m}^h+h-j-2}(x_{k\bar{m}^h+h-j-2}) f_{k\bar{m}^h+h-j-1}(x_{k\bar{m}^h+h-j-2}) \\ &= x_{k\bar{m}^h+h-j-2} f_{h-j-2}(x_{k\bar{m}^h+h-j-2}) f_{h-j-1}(x_{k\bar{m}^h+h-j-2}). \end{aligned} \quad (4.23)$$

Firstly, we show

$$U_{h,h-j} = \begin{cases} S_{h-j} & \text{for } j = 2, \\ I_{h-j} & \text{for } j = 3, \end{cases} \quad (4.24)$$

for $h = 1, 2, \dots, k$. Suppose that there exists \bar{h} such that $U_{\bar{h},\bar{h}-2} < S_{\bar{h}-2}$, or $U_{\bar{h},\bar{h}-3} > I_{\bar{h}-3}$. By letting $m \rightarrow +\infty$ and considering the limiting equation of (4.23) with $j = 0$, we obtain

$$S_h = U_{h,h-2} f_{h-2}(U_{h,h-3}) f_{h-1}(U_{h,h-2}), \quad (4.25)$$

for $h = 1, 2, \dots, k$. Then, we see that one of the following holds

$$\prod_{i=1}^p S_{2i} < \prod_{i=1}^p [S_{2i-2} f_{2i-2}(I_{2i-3}) f_{2i-1}(S_{2i-2})],$$

or

$$\prod_{i=0}^{p-1} S_{2i+1} < \prod_{i=0}^{p-1} [S_{2i-1} f_{2i-1}(I_{2i-2}) f_{2i}(S_{2i-1})],$$

where $p = \frac{k}{2}$, since

$$\begin{cases} \prod_{i=1}^p S_{2i} &= \prod_{i=1}^p [U_{2i,2i-2} f_{2i-2}(U_{2i,2i-3}) f_{2i-1}(U_{2i,2i-2})], \\ \prod_{i=0}^{p-1} S_{2i+1} &= \prod_{i=0}^{p-1} [U_{2i+1,2i-1} f_{2i-1}(U_{2i+1,2i-2}) f_{2i}(U_{2i+1,2i-1})], \end{cases}$$

follows from (4.25). Then, we obtain

$$1 < \prod_{i=1}^p [f_{2i-2}(I_{2i-3}) f_{2i-1}(S_{2i-2})],$$

or

$$1 < \prod_{i=0}^{p-1} [f_{2i-1}(I_{2i-2})f_{2i}(S_{2i-1})],$$

and this gives a contradiction to (4.13) with $h = k$ and $h = k - 1$, respectively, in Lemma 4.3. Thus (4.24) holds.

Next, we assume that

$$U_{h,h-j} = \begin{cases} S_{h-j} & \text{for } j = j_1, \\ I_{h-j} & \text{for } j = j_1 + 1, \end{cases} \quad (4.26)$$

for $h = 1, 2, \dots, k$ where j_1 is a positive even number. Under the assumption (4.26), we show

$$U_{h,h-j} = \begin{cases} S_{h-j} & \text{for } j = j_1 + 2, \\ I_{h-j} & \text{for } j = j_1 + 3, \end{cases} \quad (4.27)$$

for $h = 1, 2, \dots, k$. Suppose that there exists \bar{h} such that $U_{\bar{h},\bar{h}-j_1-2} < S_{\bar{h}-j_1-2}$, or $U_{\bar{h},\bar{h}-j_1-3} > I_{\bar{h}-j_1-3}$. By considering the limiting equation of (4.23) with $j = j_1$ and substituting (4.26),

$$U_{h,h-j_1} = S_{h-j_1} = U_{h,h-j_1-2}f_{h-j_1-2}(U_{h,h-j_1-3})f_{h-j_1-1}(U_{h,h-j_1-2}),$$

for $h = 1, 2, \dots, k$. Then, similar to the above discussion, we can show that (4.27) holds. Thus, (4.22) holds by mathematical induction.

From (1.2), it holds

$$\begin{aligned} x_{k\bar{n}_m+h} &= x_{k\bar{n}_m+h-(k-1)}f_{k\bar{n}_m+h-(k-1)}(x_{k\bar{n}_m+h-k}) \cdots f_{k\bar{n}_m+h-2}(x_{k\bar{n}_m+h-3})f_{k\bar{n}_m+h-1}(x_{k\bar{n}_m+h-2}) \\ &= x_{k\bar{n}_m+h-(k-1)}f_{h-(k-1)}(x_{k\bar{n}_m+h-k}) \cdots f_{h-2}(x_{k\bar{n}_m+h-3})f_{h-1}(x_{k\bar{n}_m+h-2}), \end{aligned}$$

and by considering the limiting equation and using (4.22), we obtain

$$\begin{aligned} S_h &= U_{h,h-(k-1)}f_{h-(k-1)}(U_{h,h-k}) \cdots f_{h-2}(U_{h,h-3})f_{h-1}(U_{h,h-2}) \\ &= I_{h-(k-1)}f_{h-(k-1)}(S_{h-k}) \cdots f_{h-2}(I_{h-3})f_{h-1}(S_{h-2}). \end{aligned}$$

By (4.13) in Lemma 4.3, we see

$$f_{h-(k-1)}(S_{h-k}) \cdots f_{h-2}(I_{h-3})f_{h-1}(S_{h-2}) = \frac{1}{f_{h-k}(I_{h-k-1})},$$

hence, it holds that

$$S_h f_{h-k}(I_{h-k-1}) = I_{h+1}, \quad (4.28)$$

for $h = 1, 2, \dots, k$. Similar to the above discussion, it also holds

$$I_h f_{h-k}(S_{h-k-1}) = S_{h+1}, \quad (4.29)$$

for $h = 1, 2, \dots, k$. Consequently, it holds

$$I_{h+1} = S_h f_{h-k}(I_{h-k-1}) \leq I_h f_{h-k}(S_{h-k-1}) = S_{h+1},$$

by (4.28) and (4.29), and hence, (4.21) holds. Then, from (3.5), we see

$$\liminf_{n \rightarrow +\infty} x_{kn+h} = \limsup_{n \rightarrow +\infty} x_{kn+h} = \lim_{n \rightarrow +\infty} x_{kn+h},$$

for $h = 1, 2, \dots, k$, and there exist k positive constants $x_h^* = S_h = I_h, h = 1, 2, \dots, k$ such that

$$x_h^* = \lim_{n \rightarrow +\infty} x_{kn+h}.$$

The proof is complete. \square

5. Global attractivity for the case where k is an arbitrary odd integer

In this section, we show that Theorem 1.1 holds when k is an odd integer. For the reader, we first consider the case $k = 3$ in Section 5.1. (5.1) in Lemma 5.1 has an important role in this subsection. Then, we give Theorem 5.2 which states that there exists a 3-periodic solution which is globally attractive. In Section 5.2, we generalize these results to the case where k is an arbitrary odd integer.

5.1. Case: $k = 3$

First, we introduce the following lemma which plays a crucial role in the proof of Theorem 5.2.

Lemma 5.1 *Let $k = 3$. Assume that (1.3) and (1.4). If (1.5), then it holds that*

$$\begin{cases} S_h &= S_{h-2}f_{h-2}(I_{h-3})f_{h-1}(S_{h-2}), \\ I_h &= I_{h-2}f_{h-2}(S_{h-3})f_{h-1}(I_{h-2}), \end{cases} \quad (5.1)$$

for $h = 1, 2, 3$.

PROOF. From (1.2), it holds

$$\begin{aligned} x_{kn+h} &= x_{kn+h-2}f_{kn+h-2}(x_{kn+h-3})f_{kn+h-1}(x_{kn+h-2}) \\ &= x_{kn+h-2}f_{h-2}(x_{kn+h-3})f_{h-1}(x_{kn+h-2}), \end{aligned}$$

for $h = 1, 2, 3$. Then it follows that

$$\begin{cases} x_{k\bar{n}_m^h+h} &= x_{k\bar{n}_m^h+h-2}f_{h-2}(x_{k\bar{n}_m^h+h-3})f_{h-1}(x_{k\bar{n}_m^h+h-2}), \\ x_{k\underline{n}_m^h+h} &= x_{k\underline{n}_m^h+h-2}f_{h-2}(x_{k\underline{n}_m^h+h-3})f_{h-1}(x_{k\underline{n}_m^h+h-2}), \end{cases}$$

for $h = 1, 2, 3$. By considering the limiting equation and using (1.3) and (1.4), we get

$$\begin{cases} S_h &\leq S_{h-2}f_{h-2}(I_{h-3})f_{h-1}(S_{h-2}), \\ I_h &\geq I_{h-2}f_{h-2}(S_{h-3})f_{h-1}(I_{h-2}), \end{cases} \quad (5.2)$$

for $h = 1, 2, 3$. In order to obtain the conclusion, we show that (5.2) holds with equality.

From (5.2), we see that

$$\begin{cases} \frac{S_h f_{h-1}(I_{h-2})}{I_h f_{h-1}(S_{h-2})} &\leq \frac{S_{h-2}}{I_h} f_{h-2}(I_{h-3})f_{h-1}(I_{h-2}), \\ \frac{S_{h-2} f_{h-2}(I_{h-3})}{I_{h-2} f_{h-2}(S_{h-3})} &\geq \frac{S_{h-2}}{I_h} f_{h-2}(I_{h-3})f_{h-1}(I_{h-2}), \end{cases} \quad (5.3)$$

for $h = 1, 2, 3$. It then follows

$$\frac{S_h f_{h-1}(I_{h-2})}{I_h f_{h-1}(S_{h-2})} \leq \frac{S_{h-2} f_{h-2}(I_{h-3})}{I_{h-2} f_{h-2}(S_{h-3})}, \quad (5.4)$$

for $h = 1, 2, 3$. By multiplying (5.4), we obtain

$$\left(\frac{S_h f_{h-1}(I_{h-2})}{I_h f_{h-1}(S_{h-2})} \right) \left(\frac{S_{h+1} f_h(I_{h-1})}{I_{h+1} f_h(S_{h-1})} \right) \leq \left(\frac{S_{h-2} f_{h-2}(I_{h-3})}{I_{h-2} f_{h-2}(S_{h-3})} \right) \left(\frac{S_{h-1} f_{h-1}(I_{h-2})}{I_{h-1} f_{h-1}(S_{h-2})} \right),$$

and hence, it holds that

$$\frac{S_h S_{h+1} f_h(I_{h-1})}{I_h I_{h+1} f_h(S_{h-1})} \leq \frac{S_{h-2} S_{h-1} f_{h-2}(I_{h-3})}{I_{h-2} I_{h-1} f_{h-2}(S_{h-3})},$$

for $h = 1, 2, 3$. Therefore, the following inequalities hold

$$\begin{cases} \frac{S_1 S_2 f_1(I_3)}{I_1 I_2 f_1(S_3)} \leq \frac{S_2 S_3 f_2(I_1)}{I_2 I_3 f_2(S_1)}, \\ \frac{S_2 S_3 f_2(I_1)}{I_2 I_3 f_2(S_1)} \leq \frac{S_3 S_1 f_3(I_2)}{I_3 I_1 f_3(S_2)}, \\ \frac{S_3 S_1 f_3(I_2)}{I_3 I_1 f_3(S_2)} \leq \frac{S_1 S_2 f_1(I_3)}{I_1 I_2 f_1(S_3)}, \end{cases}$$

and it follows

$$\frac{S_1 S_2 f_1(I_3)}{I_1 I_2 f_1(S_3)} \leq \frac{S_2 S_3 f_2(I_1)}{I_2 I_3 f_2(S_1)} \leq \frac{S_3 S_1 f_3(I_2)}{I_3 I_1 f_3(S_2)} \leq \frac{S_1 S_2 f_1(I_3)}{I_1 I_2 f_1(S_3)}.$$

Hence, we see that

$$\frac{S_1 S_2 f_1(I_3)}{I_1 I_2 f_1(S_3)} = \frac{S_2 S_3 f_2(I_1)}{I_2 I_3 f_2(S_1)} = \frac{S_3 S_1 f_3(I_2)}{I_3 I_1 f_3(S_2)} = \frac{S_1 S_2 f_1(I_3)}{I_1 I_2 f_1(S_3)}. \quad (5.5)$$

(5.5) implies that (5.3) holds with equality and thus, (5.2) also holds with equality for $h = 1, 2, 3$. Therefore, we obtain the conclusion. The proof is complete. \square

Then, we obtain the following result.

Theorem 5.2 *Let $k = 3$. Assume that (1.3) and (1.4). If (1.5), then there exists a 3-periodic solution x_n^* such that $x_n^* = x_{n+3}^*$ which is globally attractive, that is, for any solution of (1.2), it holds that*

$$\lim_{n \rightarrow +\infty} (x_n - x_n^*) = 0.$$

PROOF. In order to obtain the conclusion, we will show

$$S_h = I_h, h = 1, 2, 3. \quad (5.6)$$

Let

$$U_{k,k-j} = \lim_{m \rightarrow +\infty} x_{k\bar{n}_m^k + k - j} \text{ for } j = 1, 2, \dots, 5,$$

and we claim

$$U_{k,k-j} = \begin{cases} S_{k-j} & \text{for } j = 2, 4, \\ I_{k-j} & \text{for } j = 1, 3, 5. \end{cases} \quad (5.7)$$

At first, we see that it holds

$$\begin{aligned} x_{k\bar{n}_m^k + k - j} &= x_{k\bar{n}_m^k + k - j - 2} f_{k-j-2}(x_{k\bar{n}_m^k + k - j - 2}) f_{k-j-1}(x_{k\bar{n}_m^k + k - j - 2}) \\ &= x_{k\bar{n}_m^k + k - j - 2} f_{k-j-2}(x_{k\bar{n}_m^k + k - j - 2}) f_{k-j-1}(x_{k\bar{n}_m^k + k - j - 2}), \end{aligned} \quad (5.8)$$

for $j = 0, 1, 2, \dots$, from (1.2).

Firstly, we show

$$U_{k,k-j} = \begin{cases} S_{k-j} & \text{for } j = 2, \\ I_{k-j} & \text{for } j = 3. \end{cases} \quad (5.9)$$

Suppose that $U_{k,k-2} < S_{k-2}$ or $U_{k,k-3} > I_{k-3}$. By considering the limiting equation of (5.8) with $j = 0$, it follows

$$S_k = U_{k,k-2} f_{k-2}(U_{k,k-3}) f_{k-1}(U_{k,k-2}).$$

Then, by (1.3) and (1.4), it follows

$$S_k < S_{k-2} f_{k-2}(I_{k-3}) f_{k-1}(S_{k-2}).$$

This gives a contradiction to (5.1) with $h = k$ in Lemma 5.1. Thus, (5.9) holds.

Next, we show

$$U_{k,k-j} = \begin{cases} S_{k-j} & \text{for } j = 4, \\ I_{k-j} & \text{for } j = 5. \end{cases} \quad (5.10)$$

Suppose that $U_{k,k-4} < S_{k-4}$ or $U_{k,k-5} > I_{k-5}$. By considering the limiting equation of (5.8) with $j = 2$ and substituting (5.9), it follows

$$U_{k,k-2} = S_{k-2} = U_{k,k-4} f_{k-4}(U_{k,k-5}) f_{k-3}(U_{k,k-4}).$$

Then, by (1.3) and (1.4), it follows

$$S_{k-2} < S_{k-4} f_{k-4}(I_{k-5}) f_{k-3}(S_{k-4}).$$

This gives a contradiction to (5.1) with $h = k - 2$ in Lemma 5.1. Thus, (5.10) holds.

By considering the limiting equation of (5.8) with $j = 1$ and using (5.9)-(5.10), it follows

$$\begin{aligned} U_{k,k-1} &= U_{k,k-3} f_{k-3}(U_{k,k-4}) f_{k-2}(U_{k,k-3}) \\ &= I_{k-3} f_{k-3}(S_{k-4}) f_{k-2}(I_{k-3}). \end{aligned}$$

Hence, it holds

$$U_{k,k-1} = I_{k-1}, \quad (5.11)$$

by (5.1) in Lemma 5.1. Consequently, (5.7) holds from (5.9), (5.10) and (5.11).

From (1.2), it holds

$$x_{k\bar{n}_m^k+k} = x_{k\bar{n}_m^k+k-1} f_{k\bar{n}_m^k+k-1}(x_{k\bar{n}_m^k+k-2}) = x_{k\bar{n}_m^k+k-1} f_{k-1}(x_{k\bar{n}_m^k+k-2}),$$

and by considering the limiting equation and using (5.7), we obtain

$$S_k = I_{k-1} f_{k-1}(S_{k-2}). \quad (5.12)$$

Similar to the above discussion, it also holds

$$I_k = S_{k-1} f_{k-1}(I_{k-2}). \quad (5.13)$$

Consequently, it holds

$$I_k = S_{k-1} f_{k-1}(I_{k-2}) \leq I_{k-1} f_{k-1}(S_{k-2}) = S_k,$$

by (5.12) and (5.13), and hence (5.6) holds. Then,

$$\liminf_{n \rightarrow +\infty} x_{kn+h} = \limsup_{n \rightarrow +\infty} x_{kn+h} = \lim_{n \rightarrow +\infty} x_{kn+h},$$

for $h = 1, 2, 3$ and there exist 3-positive constants $x_h^* = S_h = I_h, h = 1, 2, 3$ such that

$$x_h^* = \lim_{n \rightarrow +\infty} x_{kn+h}.$$

The proof is complete. \square

5.2. Case: k is an odd integer

Lemma 5.3 Let k be an odd integer. Assume that (1.3) and (1.4). If (1.5), then it holds that

$$\begin{cases} S_h &= S_{h-2}f_{h-2}(I_{h-3})f_{h-1}(S_{h-2}), \\ I_h &= I_{h-2}f_{h-2}(S_{h-3})f_{h-1}(I_{h-2}), \end{cases} \quad (5.14)$$

for $h = 1, 2, \dots, k$.

PROOF. From (1.2), it holds

$$\begin{aligned} x_{kn+h} &= x_{kn+h-2}f_{kn+h-2}(x_{kn+h-3})f_{kn+h-1}(x_{kn+h-2}) \\ &= x_{kn+h-2}f_{h-2}(x_{kn+h-3})f_{h-1}(x_{kn+h-2}), \end{aligned}$$

for $h = 1, 2, \dots, k$. Then, it follows that

$$\begin{cases} x_{k\bar{n}_m^h+h} &= x_{k\bar{n}_m^h+h-2}f_{h-2}(x_{k\bar{n}_m^h+h-3})f_{h-1}(x_{k\bar{n}_m^h+h-2}), \\ x_{k\underline{n}_m^h+h} &= x_{k\underline{n}_m^h+h-2}f_{h-2}(x_{k\underline{n}_m^h+h-3})f_{h-1}(x_{k\underline{n}_m^h+h-2}), \end{cases}$$

for $h = 1, 2, \dots, k$. By considering the limiting equation and using (1.3) and (1.4), we get

$$\begin{cases} S_h &\leq S_{h-2}f_{h-2}(I_{h-3})f_{h-1}(S_{h-2}), \\ I_h &\geq I_{h-2}f_{h-2}(S_{h-3})f_{h-1}(I_{h-2}), \end{cases} \quad (5.15)$$

for $h = 1, 2, \dots, k$. In order to obtain the conclusion, we show that (5.15) holds with equality.

From (5.15), we see that

$$\begin{cases} \frac{S_h f_{h-1}(I_{h-2})}{I_h f_{h-1}(S_{h-2})} &\leq \frac{S_{h-2} f_{h-2}(I_{h-3}) f_{h-1}(I_{h-2})}{I_h} \\ \frac{S_{h-2} f_{h-2}(I_{h-3})}{I_{h-2} f_{h-2}(S_{h-3})} &\geq \frac{S_{h-2} f_{h-2}(I_{h-3}) f_{h-1}(I_{h-2})}{I_h} \end{cases} \quad (5.16)$$

for $h = 1, 2, \dots, k$. Thus, it follows that

$$\frac{S_h f_{h-1}(I_{h-2})}{I_h f_{h-1}(S_{h-2})} \leq \frac{S_{h-2} f_{h-2}(I_{h-3})}{I_{h-2} f_{h-2}(S_{h-3})}, \quad (5.17)$$

for $h = 1, 2, \dots, k$. By multiplying (5.17), we obtain

$$\left(\frac{S_h f_{h-1}(I_{h-2})}{I_h f_{h-1}(S_{h-2})} \right) \left(\frac{S_{h+1} f_h(I_{h-1})}{I_{h+1} f_h(S_{h-1})} \right) \leq \left(\frac{S_{h-2} f_{h-2}(I_{h-3})}{I_{h-2} f_{h-2}(S_{h-3})} \right) \left(\frac{S_{h-1} f_{h-1}(I_{h-2})}{I_{h-1} f_{h-1}(S_{h-2})} \right),$$

and hence, it holds that

$$\frac{S_h S_{h+1} f_h(I_{h-1})}{I_h I_{h+1} f_h(S_{h-1})} \leq \frac{S_{h-2} S_{h-1} f_{h-2}(I_{h-3})}{I_{h-2} I_{h-1} f_{h-2}(S_{h-3})},$$

for $h = 1, 2, \dots, k$. Therefore, the following inequalities hold

$$\left\{ \begin{array}{l} \frac{S_1 S_2 f_1(I_k)}{I_1 I_2 f_1(S_k)} \leq \frac{S_{k-1} S_k f_{k-1}(I_{k-2})}{I_{k-1} I_k f_{k-1}(S_{k-2})}, \\ \frac{S_{k-1} S_k f_{k-1}(I_{k-2})}{I_{k-1} I_k f_{k-1}(S_{k-2})} \leq \frac{S_{k-3} S_{k-2} f_{k-3}(I_{k-4})}{I_{k-3} I_{k-2} f_{k-3}(S_{k-4})}, \\ \dots \\ \frac{S_k S_{k+1} f_k(I_{k-1})}{I_k I_{k+1} f_k(S_{k-1})} \leq \frac{S_{k-2} S_{k-1} f_{k-2}(I_{k-3})}{I_{k-2} I_{k-1} f_{k-2}(S_{k-3})}, \\ \dots \\ \frac{S_3 S_4 f_3(I_2)}{I_3 I_4 f_3(S_2)} \leq \frac{S_1 S_2 f_1(I_k)}{I_1 I_2 f_1(S_k)}. \end{array} \right.$$

Thus, we obtain

$$\frac{S_1 S_2 f_1(I_k)}{I_1 I_2 f_1(S_k)} = \dots = \frac{S_{k-1} S_k f_{k-1}(I_{k-2})}{I_{k-1} I_k f_{k-1}(S_{k-2})} = \frac{S_k S_{k+1} f_k(I_{k-1})}{I_k I_{k+1} f_k(S_{k-1})}. \quad (5.18)$$

(5.18) implies that (5.16) holds with equality and thus, (5.15) also holds with equality for $h = 1, 2, \dots, k$. Then, we obtain the conclusion. Hence, the proof is complete. \square

Theorem 5.4 *Let k be an odd integer. Assume that (1.3) and (1.4). If (1.5), then there exists a k -periodic solution x_n^* such that $x_n^* = x_{n+k}^*$ which is globally attractive, that is, for any solution of (1.2), it holds that*

$$\lim_{n \rightarrow +\infty} (x_n - x_n^*) = 0.$$

PROOF. In order to obtain the conclusion, we will show

$$S_h = I_h, h = 1, 2, \dots, k. \quad (5.19)$$

Let

$$U_{k,k-j} = \lim_{m \rightarrow +\infty} x_{k\bar{n}_m^k + k - j} \text{ for } j = 1, 2, \dots, k,$$

and we claim

$$U_{k,k-j} = \begin{cases} S_{k-j}, & \text{for } j = 2, 4, \dots, k-1, \\ I_{k-j}, & \text{for } j = 1, 3, 5, \dots, k. \end{cases} \quad (5.20)$$

At first, we see that it holds

$$\begin{aligned} x_{k\bar{n}_m^k + k - j} &= x_{k\bar{n}_m^k + k - j - 2} f_{k-j-2}(x_{k\bar{n}_m^k + k - j - 2}) f_{k-j-1}(x_{k\bar{n}_m^k + k - j - 2}) \\ &= x_{k\bar{n}_m^k + k - j - 2} f_{k-j-2}(x_{k\bar{n}_m^k + k - j - 2}) f_{k-j-1}(x_{k\bar{n}_m^k + k - j - 2}), \end{aligned} \quad (5.21)$$

for $j = 0, 1, 2, \dots$

Firstly, we show

$$U_{k,k-j} = \begin{cases} S_{k-j} & \text{for } j = 2, \\ I_{k-j} & \text{for } j = 3. \end{cases} \quad (5.22)$$

Suppose that $U_{k,k-2} < S_{k-2}$ or $U_{k,k-3} > I_{k-3}$. By considering the limiting equation of (5.21) with $j = 0$, it follows

$$S_k = U_{k,k-2} f_{k-2}(U_{k,k-3}) f_{k-1}(U_{k,k-2}).$$

Then, by (1.3) and (1.4), it follows

$$S_k < S_{k-2} f_{k-2}(I_{k-3}) f_{k-1}(S_{k-2}).$$

This gives a contradiction to (5.14) with $h = k$ in Lemma 5.3. Thus, (5.22) holds.

Next we assume that

$$U_{k,k-j} = \begin{cases} S_{k-j} & \text{for } j = j_1, \\ I_{k-j} & \text{for } j = j_1 + 1, \end{cases} \quad (5.23)$$

where j_1 is a positive even integer. Under the assumption (5.23), we show

$$U_{k,k-j} = \begin{cases} S_{k-j} & \text{for } j = j_1 + 2, \\ I_{k-j} & \text{for } j = j_1 + 3. \end{cases} \quad (5.24)$$

Suppose that $U_{k,k-j_1-2} < S_{k-j_1-2}$ or $U_{k,k-j_1-3} > I_{k-j_1-3}$. By considering the limiting equation of (5.21) with $j = j_1$, it follows

$$U_{k,k-j_1} = S_{k-j_1} = U_{k,k-j_1-2} f_{k-j_1-2}(U_{k,k-j_1-3}) f_{k-j_1-1}(U_{k,k-j_1-2}).$$

Then, by (1.3) and (1.4), it follows

$$S_{k-j_1} < S_{k-j_1-2} f_{k-j_1-2}(I_{k-j_1-3}) f_{k-1}(S_{k-j_1-2}).$$

This gives a contradiction to (5.14) with $h = k - j_1$ in Lemma 5.3. Thus, (5.24) holds.

By considering the limiting equation of (5.21) with $j = 1$ and substituting (5.22) and (5.24) with $j_1 = 2$, it follows

$$\begin{aligned} U_{k,k-1} &= U_{k,k-3} f_{k-3}(U_{k,k-4}) f_{k-2}(U_{k,k-3}) \\ &= I_{k-3} f_{k-3}(S_{k-4}) f_{k-2}(I_{k-3}). \end{aligned}$$

Hence,

$$U_{k,k-1} = I_{k-1}, \quad (5.25)$$

by (5.14) with $h = k - 1$ in Lemma 5.3. Hence, (5.20) holds by (5.25) and mathematical induction.

From (1.2), it holds

$$x_{k\bar{n}_m^k+k-j} = x_{k\bar{n}_m^k+k-j-1} f_{k\bar{n}_m^k+k-j-1}(x_{k\bar{n}_m^k+k-j-2}) = x_{k\bar{n}_m^k+k-j-1} f_{k-j-1}(x_{k\bar{n}_m^k+k-j-2}),$$

for $j = 0, 1, 2, \dots, k-1$, and by considering the limiting equation and using (5.20), we obtain

$$\begin{cases} I_{k-j} &= S_{k-j-1} f_{k-j-1}(I_{k-j-2}), j = 1, 3, \dots, k, \\ S_{k-j} &= I_{k-j-1} f_{k-j-1}(S_{k-j-2}), j = 2, 4, \dots, k-1. \end{cases} \quad (5.26)$$

Similar to the above discussion, it also holds

$$\begin{cases} S_{k-j} &= I_{k-j-1} f_{k-j-1}(S_{k-j-2}), j = 1, 3, \dots, k, \\ I_{k-j} &= S_{k-j-1} f_{k-j-1}(I_{k-j-2}), j = 2, 4, \dots, k-1. \end{cases} \quad (5.27)$$

Consequently, it holds

$$I_{k-j} = S_{k-j-1} f_{k-j-1}(I_{k-j-2}) \leq I_{k-j-1} f_{k-j-1}(S_{k-j-2}) = S_{k-j},$$

for $j = 1, 2, \dots, k$, by (5.26) and (5.27), and hence (5.19) holds. Then,

$$\liminf_{n \rightarrow +\infty} x_{kn+h} = \limsup_{n \rightarrow +\infty} x_{kn+h} = \lim_{n \rightarrow +\infty} x_{kn+h},$$

for $h = 1, 2, \dots, k$ and there exist k positive constants $x_h^* = S_h = I_h, h = 1, 2, \dots, k$ such that

$$x_h^* = \lim_{n \rightarrow +\infty} x_{kn+h}.$$

The proof is complete. \square

Finally, we establish Theorem 1.1 from Theorems 4.2, 4.4, 5.2 and 5.4.

6. Applications

In this section, we give two examples to demonstrate our result. At first, we introduce the following example.

$$f_n(x) = \frac{\lambda}{1 + (\lambda - 1) \frac{x}{K_n}}, n = 0, 1, 2, \dots, x \in [0, +\infty), \quad (6.1)$$

where $\lambda > 1$ and $K_n = K_{n+k} > 0$ for $n = 0, 1, 2, \dots$, (k is an arbitrary positive integer). (1.2) with (6.1) is called a delayed Beverton-Holt equation (see also [1, 4]). Since $f_n(x)$ is periodic on n and satisfies (1.3) and (1.4), the following result is derived from Theorem 1.1.

Corollary 6.1 *There exists a k -periodic solution x_n^* such that $x_n^* = x_{n+k}^*$ which is globally attractive, that is, for any solution of (1.2) with (6.1), it holds that*

$$\lim_{n \rightarrow +\infty} (x_n - x_n^*) = 0.$$

One can see (1.2) with (6.1) is equivalent to the Pielou's equation by a simple transformation. Therefore, Corollary 6.1 is also derived from Theorem A.

Secondly, we introduce the following example.

$$f_n(x) = \frac{\beta_n}{1 + \alpha_n^1 \frac{x}{1 + \alpha_n^2 x}}, n = 0, 1, 2, \dots, x \in [0, +\infty), \quad (6.2)$$

where $\beta_n, \alpha_n^1, \alpha_n^2$ are positive periodic sequences with a period k . It is not necessary that $\beta_n, \alpha_n^1, \alpha_n^2$ share the same period and, in this case, we can also easily find such a k . We obtain the following results by applying Theorems 2.1 and 1.1, respectively.

Corollary 6.2 *If $\prod_{n=1}^k \beta_n \leq 1$, then, for any solution of (1.2) with (6.2), it holds that*

$$\lim_{n \rightarrow +\infty} x_n = 0.$$

On the other hand, for the case $\prod_{n=1}^k \beta_n > 1$, we establish the following result.

Corollary 6.3 *If*

$$\prod_{n=1}^k \beta_n > 1, \text{ and } \prod_{n=1}^k \frac{\beta_n}{1 + \frac{\alpha_n^1}{\alpha_n^2}} < 1,$$

then, there exists a k -periodic solution x_n^ such that $x_n^* = x_{n+k}^*$ which is globally attractive, that is, for any solution of (1.2) with (6.2), it holds that*

$$\lim_{n \rightarrow +\infty} (x_n - x_n^*) = 0.$$

Acknowledgements

The authors are very grateful to the anonymous referee for carefully reading and valuable comments which led to a significant improvement of the original manuscript. This work was partially done during the second author's stay as a member of International Research Training Group (IGK 1529) at the Technical University Darmstadt. The authors would like to express their gratitude to Professor Yoshiaki Muroya for his valuable comments to this paper.

References

- [1] E. Braverman, S. H. Saker, On the Cushing-Henson conjecture, delay difference equations and attenuant cycles, *J. Difference Equ. Appl.* 14 (3) (2008) 275–286.
- [2] E. Camouzis, Global convergence in periodically forced rational equations, *J. Difference Equ. Appl.* 14 (10-11) (2008) 1011–1033.
- [3] E. Camouzis, G. Ladas, Periodically forced Pielou’s equation, *J. Math. Anal. Appl.* 333 (1) (2007) 117–127.
- [4] V. L. Kocic, A note on the nonautonomous Beverton-Holt model, *J. Difference Equ. Appl.* 11 (4-5) (2005) 415–422.
- [5] V. L. Kocić, G. Ladas, Global behavior of nonlinear difference equations of higher order with applications, vol. 256 of *Mathematics and its Applications*, Kluwer Academic Publishers Group, Dordrecht, 1993.
- [6] V. L. Kocic, D. Stutson, G. Arora, Global behavior of solutions of a nonautonomous delay logistic difference equation, *J. Difference Equ. Appl.* 10 (13-15) (2004) 1267–1279.
- [7] M. R. S. Kulenović, O. Merino, Stability analysis of Pielou’s equation with period-two coefficient, *J. Difference Equ. Appl.* 13 (5) (2007) 383–406.
- [8] S. A. Kuruklis, G. Ladas, Oscillations and global attractivity in a discrete delay logistic model, *Quart. Appl. Math.* 50 (2) (1992) 227–233.
- [9] E. Liz, V. Tkachenko, S. Trofimchuk, Global stability in discrete population models with delayed-density dependence, *Mathematical Biosciences* 199 (1) (2006) 26 – 37.
- [10] G. Nyerges, A note on a generalization of Pielou’s equation, *J. Difference Equ. Appl.* 14 (5) (2008) 563–565.
- [11] E. C. Pielou, *An introduction to mathematical ecology*, Wiley-Interscience A Division of John Wiley & Sons, Inc., New York-London-Sydney, 1969.