

# Global stability of SIRS epidemic models with a class of nonlinear incidence rates and distributed delays

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## Abstract

In this paper, we establish the global asymptotic stability of a disease-free equilibrium and an endemic equilibrium of an SIRS epidemic model with a class of nonlinear incidence rates and distributed delays. By using strict monotonicity of the incidence function and constructing a Lyapunov functional, we obtain sufficient conditions under which the endemic equilibrium is globally asymptotically stable. When the nonlinear incidence rate is a saturated incidence rate, our result provides a new global stability condition for a small rate of immunity loss.

*Keywords:* SIRS epidemic model; nonlinear incidence rate; global asymptotic stability; distributed delays; Lyapunov functional

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## 1. Introduction

Mathematical models which describe the dynamics of infectious diseases have played a crucial role in the disease control in epidemiological aspects. In order to understand the mechanism of disease transmission, many authors have proposed various kinds of epidemic models (see [1]-[19] and the references therein).

Much attention has been paid to the analysis of the stability of the disease-free equilibrium and the endemic equilibrium of the epidemic models. Mena-Lorca and Hethcote [15] considered several SIRS epidemic models with a bilinear incidence rate which takes the form  $\beta S(t)I(t)$  and a standard incidence rate which takes the form  $\beta S(t)I(t)/N(t)$ , where  $N(t) = S(t) + I(t) + R(t)$ . A threshold parameter of the models was also found in Mena-Lorca and Hethcote

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[15] to determine whether the disease dies out or approaches to an endemic equilibrium. Later, in order to investigate the effect of an immunity loss of diseases, various kinds of SIRS epidemic models and a significant body of work concerning the stability analysis of the steady states of the models have been carried out (see, for example, [7, 8, 9, 10, 11, 12, 16, 17, 19] and the references therein).

In modeling of those communicable diseases, an incidence rate has played a vital role in ensuring that the model can give a reasonable qualitative description for the disease dynamics. A bilinear incidence rate and a standard incidence rate were frequently used in the literature of mathematical modeling. On the other hand, many authors have suggested that transmission of the infection shall have a nonlinear incidence rate. For example, Capasso and Serio [2] studied the cholera epidemic spread in Bari in 1973 and have given an assumption that the incidence rate takes the nonlinear form  $\frac{\beta S(t)I(t)}{1+\alpha I(t)}$ , which has been interpreted as a saturated incidence rate. The saturation effect has been originally introduced for the Holling functional response of the predator in a prey-predator system. This incidence rate includes the behavioral change and crowding effect of the infective individuals and prevents the unboundedness of the contact rate. Korobeinikov and Maini [9] thereafter formulated a variety of models with an incidence rate of the form  $F(S(t))G(I(t))$  and Korobeinikov [10, 11] obtained the global properties of basic SIR and SIRS epidemic models with a more general framework of the incidence rate  $F(S(t), I(t))$  but no delays.

Recently, Xu and Ma [18] investigated the spread of an infectious disease transmitted by a vector (e.g. mosquitoes, rats, etc.) after an incubation time  $\tau$  denoting the time during which the infectious agents develop in the vector, that is, they incorporated time delay effects and formulated an SIRS epidemic model with the saturated incidence rate  $\beta S(t) \frac{I(t-h)}{1+\alpha I(t-h)}$  (see also Section 6). More authors have now proposed several reasons for the nonlinearity of the incidence rates and introduced various nonlinear incidence functions with delays (see also [7, 19]).

In this paper, in order to study the impact of nonlinearity of those incidence rates and time delay effects, we consider the global dynamics of the following SIRS epidemic model with a class of nonlinear incidence rates and distributed delays:

$$\begin{cases} \frac{dS(t)}{dt} = B - \mu S(t) - \beta S(t) \int_0^h f(\tau)G(I(t-\tau))d\tau + \delta R(t), \\ \frac{dI(t)}{dt} = \beta S(t) \int_0^h f(\tau)G(I(t-\tau))d\tau - (\mu + \gamma)I(t), \\ \frac{dR(t)}{dt} = \gamma I(t) - (\mu + \delta)R(t). \end{cases} \quad (1.1)$$

$S(t)$ ,  $I(t)$  and  $R(t)$  denote the numbers of susceptible, infective and recovered individuals at time  $t$ , respectively.  $B > 0$  is the recruitment rate of the population,  $\mu > 0$  is the natural death rate of the population,  $\beta > 0$  is the

proportionality constant,  $\gamma > 0$  is the natural recovery rate of the infective individuals,  $\delta \geq 0$  is the rate at which recovered individuals lose immunity and return to the susceptible class.  $h > 0$  is a maximum time taken to become infectious and the transmission of the infection is governed by an incidence rate  $\beta S(t) \int_0^h f(\tau) G(I(t-\tau)) d\tau$ .  $f(\tau)$  denotes the fraction of vector population in which the time taken to become infectious is  $\tau$ . Here,  $f : [0, h] \rightarrow [0, +\infty)$  is continuous on  $[0, h]$  satisfying  $\int_0^h f(\tau) d\tau = 1$ . The initial conditions for system (1.1) take the form

$$S(\theta) = \varphi_1(\theta), \quad I(\theta) = \varphi_2(\theta), \quad R(\theta) = \varphi_3(\theta), \quad -h \leq \theta \leq 0, \quad (1.2)$$

where  $\varphi = (\varphi_1, \varphi_2, \varphi_3)^T \in C$  such that  $\varphi_i(\theta) = \varphi_i(0) \geq 0$  ( $-h \leq \theta \leq 0$ ,  $i = 1, 3$ ),  $\varphi_2(\theta) \geq 0$  ( $-h \leq \theta \leq 0$ ).  $C$  denotes the Banach space  $C([-h, 0], \mathbb{R}_{+0}^3)$  of continuous functions mapping the interval  $[-h, 0]$  into  $\mathbb{R}_{+0}^3$  with the supremum norm, where  $\mathbb{R}_{+0}^n = \{(x_1, \dots, x_n) | x_i \geq 0, i = 1, \dots, n\}$  for  $n \geq 1$ . From a biological meaning, we assume that  $\varphi_i(0) > 0$  for  $i = 1, 2, 3$ .

Throughout the paper, we further assume that

- (H1)  $G(I)$  is continuous and monotone increasing on  $[0, +\infty)$  with  $G(0) = 0$ .  
(H2)  $I/G(I)$  is monotone increasing on  $(0, +\infty)$  with  $\lim_{I \rightarrow +0} (I/G(I)) = 1$ .

We here note that  $G$  is a Lipschitz continuous function on  $[0, +\infty)$ .

We define the basic reproduction number of system (1.1) by

$$R_0 = \frac{B\beta}{\mu(\mu + \gamma)}. \quad (1.3)$$

$R_0$  denotes the expected number of secondary infectious cases generated by one typical primary case in an entirely susceptible and sufficiently large population.

It is well known by the fundamental theory of functional differential equations that the solution  $(S(t), I(t), R(t))$  of system (1.1) is unique and positive for all  $t \geq 0$ . System (1.1) always has a disease-free equilibrium  $E_0 = (B/\mu, 0, 0)$  and if  $R_0 > 1$ , then system (1.1) allows a unique endemic equilibrium  $E_* = (S^*, I^*, R^*)$ ,  $S^* > 0$ ,  $I^* > 0$  and  $R^* > 0$  (see Lemma 4.1).

In the present paper, applying a Lyapunov functional technique for a delayed SIR epidemic model in McCluskey [13] and the property that the total population  $N(t)$  of system (1.1) converges to a positive constant  $B/\mu$ , we obtain a sufficient condition which ensures the global asymptotic stability of the endemic equilibrium  $E_*$  of system (1.1) for  $R_0 > 1$  (see Lemma 5.1). The main results are as follows.

**Theorem 1.1** *If  $R_0 < 1$ , then the disease-free equilibrium  $E_0$  of system (1.1) is globally asymptotically stable.*

**Theorem 1.2** *If  $R_0 > 1$ , then the endemic equilibrium  $E_*$  of system (1.1) exists uniquely and system (1.1) is permanent. Moreover, assume that the following conditions hold.*

(I) There exist positive constants  $C_1$  and  $C_2$  such that

$$\inf_{0 \leq I \leq B/\mu} \frac{G(I) - G(I^*)}{I - I^*} \geq C_1 > 0 \text{ and } \inf_{0 < I \leq B/\mu} \frac{\frac{I}{G(I)} - \frac{I^*}{G(I^*)}}{I - I^*} \geq C_2 > 0,$$

$$\text{(II) } \delta^2 - 4C_1C_2(\mu + \gamma)(\mu + \delta) \frac{B}{\mu + \beta G(B/\mu)} < 0.$$

Then the endemic equilibrium  $E_*$  of system (1.1) is globally asymptotically stable.

We show the global asymptotic stability of the endemic equilibrium for a small rate of immunity loss  $\delta$  as long as the infection rate has suitable properties concerning the concavity of function  $G(I)$  characterized by the hypotheses (H1) and (H2).

The organization of this paper is the following. In Section 2, we offer a basic result for system (1.1). In Section 3, we prove Theorem 1.1. In Section 4, we establish the permanence and global asymptotic stability of the endemic equilibrium  $E_*$  of system (1.1) for  $R_0 > 1$ . In Section 5, we prove Theorem 1.2 by means of Lyapunov functionals on a reduced system which is derived from system (1.1) with a key lemma (see Lemma 5.1). To show the feasibility of our global stability conditions of the endemic equilibrium for  $R_0 > 1$ , we offer numerical examples in Section 6. Finally, a discussion is offered in Section 7.

## 2. A basic result

We now state a basic result of system (1.1). The following lemma will be used in the proofs of Theorem 1.1, Theorem 1.2 and Lemma 4.2.

**Lemma 2.1** *The plane  $S(t) + I(t) + R(t) = B/\mu$  is an invariant manifold of system (1.1), which is globally attractive in the first octant of  $\mathbb{R}^3$ , that is,*

$$\lim_{t \rightarrow +\infty} (S(t) + I(t) + R(t)) = \frac{B}{\mu}. \quad (2.1)$$

**Proof.** Let  $N(t) = S(t) + I(t) + R(t)$ . Then it follows from system (1.1) that

$$\begin{aligned} \frac{dN(t)}{dt} &= B - \mu S(t) - \mu I(t) - \mu R(t) \\ &= B - \mu N(t). \end{aligned}$$

Hence, we obtain that  $\lim_{t \rightarrow +\infty} N(t) = B/\mu$ . This completes the proof.  $\square$

## 3. Stability of the disease-free equilibrium $E_0$ for $R_0 < 1$

### 3.1. Local stability of the disease-free equilibrium $E_0$

We now give the following theorem concerning the local asymptotic stability of the disease-free equilibrium  $E_0$  of system (1.1).

**Lemma 3.1** *If  $R_0 < 1$ , then the disease free equilibrium  $E_0$  of system (1.1) is locally asymptotically stable. Furthermore, the disease-free equilibrium  $E_0$  of system (1.1) is unstable if  $R_0 > 1$ .*

**Proof.** By the hypothesis (H2), the characteristic equation of system (1.1) at the disease-free equilibrium  $E_0$  is of the form

$$(\lambda + \mu) \left\{ \lambda + (\mu + \gamma) \left( 1 - R_0 \int_0^h f(\tau) e^{-\lambda\tau} d\tau \right) \right\} (\lambda + \mu + \delta) = 0. \quad (3.1)$$

Clearly,  $\lambda = -\mu, -(\mu + \delta)$  are always roots of (3.1). All other roots of (3.1) are determined by the following equation:

$$\lambda + (\mu + \gamma) \left( 1 - R_0 \int_0^h f(\tau) e^{-\lambda\tau} d\tau \right) = 0. \quad (3.2)$$

For the case  $R_0 < 1$ , we suppose to the contrary that the disease-free equilibrium  $E_0$  is not locally asymptotically stable. Then, there exists a root  $\lambda = \tilde{\lambda}$  such that  $\text{Re}\tilde{\lambda} \geq 0$ . However, from (3.2), we obtain

$$\begin{aligned} \text{Re}\tilde{\lambda} &= (\mu + \gamma) \left\{ R_0 \exp(-\text{Re}\tilde{\lambda}\tau) \int_0^h f(\tau) \cos(\text{Im}\tilde{\lambda}\tau) d\tau - 1 \right\} \\ &\leq (\mu + \gamma)(R_0 - 1) < 0, \end{aligned}$$

which is a contradiction. Hence, if  $R_0 < 1$ , then the disease-free equilibrium  $E_0$  of system (1.1) is locally asymptotically stable. Now, we put

$$T(\lambda) := \lambda + (\mu + \gamma) \left( 1 - R_0 \int_0^h f(\tau) e^{-\lambda\tau} d\tau \right). \quad (3.3)$$

For the case  $R_0 > 1$ , it is directly seen from (3.3) that, for  $\lambda \in \mathbb{R}$ ,

$$T(0) = (\mu + \gamma)(1 - R_0) < 0, \quad \lim_{\lambda \rightarrow +\infty} T(\lambda) = +\infty.$$

Therefore, (3.1) has at least one positive real root. Hence, if  $R_0 > 1$ , then the disease-free equilibrium  $E_0$  is unstable. This completes the proof.  $\square$

### 3.2. Global stability of the disease-free equilibrium $E_0$

In this subsection, by constructing a Lyapunov functional, we prove Theorem 1.1.

**Proof of Theorem 1.1.** First, by (1.3) and  $R_0 < 1$ , we choose  $\varepsilon_s > 0$  sufficiently small such that

$$\beta \left( \frac{B}{\mu} + \varepsilon_s \right) < \mu + \gamma. \quad (3.4)$$

We consider the following Lyapunov functional:

$$W(t) = I(t) + (\mu + \gamma) \int_0^h f(\tau) \int_{t-\tau}^t I(u) du d\tau.$$

From Lemma 2.1, one can see that for  $\varepsilon_s > 0$ , there is a  $T > 0$  such that  $S(t) \leq \frac{B}{\mu} + \varepsilon_s$ , for  $t > T$ . Then we obtain

$$\begin{aligned} \frac{dW(t)}{dt} &= \beta S(t) \int_0^h f(\tau) G(I(t-\tau)) d\tau - (\mu + \gamma) I(t) \\ &\quad + (\mu + \gamma) \int_0^h f(\tau) (I(t) - I(t-\tau)) d\tau \\ &\leq \beta \left( \frac{B}{\mu} + \varepsilon_s \right) \int_0^h f(\tau) G(I(t-\tau)) d\tau - (\mu + \gamma) \int_0^h f(\tau) I(t-\tau) d\tau \\ &= \beta \left( \frac{B}{\mu} + \varepsilon_s \right) \int_0^h f(\tau) I(t-\tau) d\tau - (\mu + \gamma) \int_0^h f(\tau) I(t-\tau) d\tau \\ &= \int_0^h f(\tau) \left\{ \beta \left( \frac{B}{\mu} + \varepsilon_s \right) - (\mu + \gamma) \right\} I(t-\tau) d\tau, \end{aligned}$$

for  $t > T + h$ . From (3.4) and arbitrariness of  $\varepsilon_s > 0$ , we obtain that  $\frac{dW(t)}{dt} \leq 0$  holds for  $t > T + h$ . Thus, it holds that  $\lim_{t \rightarrow +\infty} W(t) = 0$ , which implies that  $\lim_{t \rightarrow +\infty} I(t) = 0$ . It follows that  $\lim_{t \rightarrow +\infty} R(t) = 0$  and  $\lim_{t \rightarrow +\infty} S(t) = B/\mu$  hold. By Lemma 3.1 and Lyapunov-LaSalle asymptotic stability theorem, the disease-free equilibrium  $E_0$  of system (1.1) is globally asymptotically stable. This completes the proof.  $\square$

**Remark 3.1** To establish the global asymptotic stability of the disease-free equilibrium  $E_0$  of system (1.1) for  $R_0 < 1$ , the hypothesis of the monotonicity of  $G(I)$  of  $I \geq 0$  in (H1) is not necessary for our analysis.

#### 4. Permanence for $R_0 > 1$

##### 4.1. Existence and uniqueness of the endemic equilibrium $E_*$

In this subsection, by the hypothesis (H2), we give a lemma of the unique existence of the endemic equilibrium  $E_*$  of system (1.1) for  $R_0 > 1$ :

**Lemma 4.1** *If  $R_0 > 1$ , then system (1.1) has a unique endemic equilibrium  $E_*$  satisfying*

$$\begin{cases} B - \mu S^* - \beta S^* G(I^*) + \delta R^* = 0, \\ \beta S^* G(I^*) - (\mu + \gamma) I^* = 0, \\ \gamma I^* - (\mu + \delta) R^* = 0. \end{cases} \quad (4.1)$$

**Proof.** Assume that  $R_0 > 1$ . From the second and the third equations of (4.1), it holds that

$$S^* = \frac{(\mu + \gamma)I^*}{\beta G(I^*)} \quad (4.2)$$

and

$$R^* = \frac{\gamma I^*}{\mu + \delta}. \quad (4.3)$$

After substituting (4.2) into the first equation of (4.1), we consider the following equation:

$$H(I) := B - \frac{\mu(\mu + \gamma)I}{\beta G(I)} - \frac{\mu(\mu + \gamma + \delta)I}{\mu + \delta} = 0.$$

By the hypothesis (H2), one can see that  $H(I)$  is a strictly monotone decreasing function of  $I \in (0, +\infty)$  satisfying

$$\lim_{I \rightarrow 0} H(I) = B - \frac{\mu(\mu + \gamma)}{\beta} = B \left(1 - \frac{1}{R_0}\right) > 0$$

and  $H(I) < 0$  holds for any  $I > \frac{B(\mu + \delta)}{\mu(\mu + \gamma + \delta)}$ . Hence, there exists a unique positive  $I^* > 0$  such that  $H(I^*) = 0$ . By (4.2) and (4.3), there exists a unique endemic equilibrium  $E_* = (S^*, I^*, R^*)$ . Hence, the proof is complete.  $\square$

#### 4.2. Permanence for $R_0 > 1$

In this subsection, we show the permanence of system (1.1). The following lemma indicates that the disease eventually persists in the host population if  $R_0 > 1$ .

**Lemma 4.2** *If  $R_0 > 1$ , then for any solutions of system (1.1), it holds that*

$$\begin{cases} \liminf_{t \rightarrow +\infty} S(t) \geq v_1 := \frac{B}{\mu + \beta G(B/\mu)}, \\ \liminf_{t \rightarrow +\infty} I(t) \geq v_2 := qI^* e^{-(\mu + \gamma)(h + \rho h)}, \\ \liminf_{t \rightarrow +\infty} R(t) \geq v_3 := \frac{\gamma v_2}{\mu + \delta}, \end{cases}$$

where  $q > 0$  and  $\rho > 0$  satisfy  $S^* < S^\Delta := \frac{B}{k}(1 - e^{-k\rho h})$ ,  $k = \mu + \beta G(qI^*)$ .

**Proof.** Let  $(S(t), I(t), R(t))$  be a solution of system (1.1) with initial condition (1.2). By Lemma 2.1, it holds that  $\limsup_{t \rightarrow +\infty} I(t) \leq \frac{B}{\mu}$ . For  $\varepsilon_I > 0$  sufficiently small, there is a  $T_1 > 0$  such that  $I(t) \leq \frac{B}{\mu} + \varepsilon_I$  for  $t > T_1$ . From the hypothesis (H1), we therefore derive from the first equation of system (1.1)

$$\frac{dS(t)}{dt} \geq B - \left\{ \mu + \beta G\left(\frac{B}{\mu} + \varepsilon_I\right) \right\} S(t),$$

which implies that

$$\liminf_{t \rightarrow +\infty} S(t) \geq \frac{B}{\mu + \beta G(B/\mu + \varepsilon_I)}.$$

Since the above inequality holds for arbitrary  $\varepsilon_I > 0$  sufficiently small, it follows that  $\liminf_{t \rightarrow +\infty} S(t) \geq v_1$ .

We now show that  $\liminf_{t \rightarrow +\infty} I(t) \geq v_2$ . First, we prove that it is impossible that  $I(t) \leq qI^*$  for all  $t \geq \rho h$ . Suppose to the contrary that  $I(t) \leq qI^*$  for all  $t \geq \rho h$ . From the first equation of system (1.1), one can obtain that

$$\frac{dS(t)}{dt} \geq B - (\mu + \beta G(qI^*))S(t), \text{ for } t \geq \rho h + h,$$

which yields

$$\begin{aligned} S(t) &\geq e^{-k(t-\rho h-h)} \left\{ S(\rho h + h) + B \int_{\rho h+h}^t e^{k(\theta-\rho h-h)} d\theta \right\} \\ &> \frac{B}{k} (1 - e^{-k(t-\rho h-h)}), \text{ for } t \geq \rho h + h. \end{aligned} \quad (4.4)$$

Hence it follows from (4.4) that

$$S(t) > \frac{B}{k} (1 - e^{-k\rho h}) = S^\Delta > S^*, \text{ for } t \geq 2\rho h + h. \quad (4.5)$$

For  $t \geq 0$ , we define

$$V(t) = I(t) + \beta S^* \int_0^h f(\tau) \int_{t-\tau}^t G(I(u)) du d\tau. \quad (4.6)$$

Noting that  $I(t) \leq qI^* < I^*$ , calculating the derivative of  $V(t)$  along solutions of system (1.1) gives as follows.

$$\begin{aligned} &\frac{dV(t)}{dt} \\ &= \beta \int_0^h f(\tau) G(I(t-\tau))(S(t) - S^*) d\tau + \beta S^* G(I(t)) - (\mu + \gamma)I(t) \\ &= \beta \int_0^h f(\tau) G(I(t-\tau))(S(t) - S^*) d\tau + \left\{ \beta S^* \frac{G(I(t))}{I(t)} - (\mu + \gamma) \right\} I(t) \\ &\geq \beta \int_0^h f(\tau) G(I(t-\tau))(S(t) - S^*) d\tau + \left\{ \beta S^* \frac{G(I^*)}{I^*} - (\mu + \gamma) \right\} I(t) \\ &= \beta \int_0^h f(\tau) G(I(t-\tau))(S(t) - S^*) d\tau \\ &> \beta \int_0^h f(\tau) G(I(t-\tau))(S^\Delta - S^*) d\tau, \text{ for } t \geq 2\rho h + h. \end{aligned} \quad (4.7)$$



Setting  $\underline{i} = \min_{\theta \in [-h, 0]} I(\theta + 2\rho h + 2h)$ , we claim that  $I(t) \geq \underline{i}$  for all  $t \geq 2\rho h + h$ . Otherwise, if there is a  $T \geq 0$  such that  $I(t) \geq \underline{i}$  for  $2\rho h + h \leq t \leq 2\rho h + 2h + T$ ,  $I(2\rho h + 2h + T) = \underline{i}$  and  $\frac{dI(t)}{dt}|_{t=2\rho h+2h+T} \leq 0$ , then it follows from (4.5), the second equation of (4.1) that

$$\begin{aligned} \frac{dI(t)}{dt} \Big|_{t=2\rho h+2h+T} &= \beta S(t) \int_0^h f(\tau) G(I(t-\tau)) d\tau - (\mu + \gamma) I(t) \\ &\geq \beta S(t) G(I(t)) - (\mu + \gamma) I(t) \\ &\geq \left\{ \beta S(t) \frac{G(I^*)}{I^*} - (\mu + \gamma) \right\} \underline{i} \\ &> \left\{ \beta S^\Delta \frac{G(I^*)}{I^*} - (\mu + \gamma) \right\} \underline{i} \\ &> \left\{ \beta S^* \frac{G(I^*)}{I^*} - (\mu + \gamma) \right\} \underline{i} = 0. \end{aligned}$$

This is a contradiction. Therefore  $I(t) \geq \underline{i}$  for all  $t \geq 2\rho h + h$ . By the hypothesis (H1), it follows from (4.7) that

$$\frac{dV(t)}{dt} > \beta G(\underline{i})(S^\Delta - S^*) > 0, \text{ for } t \geq 2\rho h + 2h,$$

which implies that  $\lim_{t \rightarrow +\infty} V(t) = +\infty$ . However, from Lemma 2.1, it holds that  $\limsup_{t \rightarrow +\infty} V(t) \leq \frac{B}{\mu} + \beta S^* G(\frac{B}{\mu}) < +\infty$ . This leads to a contradiction. Hence the claim is proved.

Second, since the above claim holds, we are left to consider two possibilities:

- (i)  $I(t) \geq qI^*$  for all  $t$  sufficiently large,
- (ii)  $I(t)$  oscillates about  $qI^*$  for all  $t$  sufficiently large.

If the first case holds, then we immediately get the conclusion. If the second case holds, we show that  $I(t) \geq v_2$  for all  $t$  sufficiently large. Let  $t_1 < t_2$  be sufficiently large such that

$$I(t_1) = I(t_2) = qI^*, \quad I(t) < qI^*, \quad t_1 < t < t_2.$$

If  $t_2 - t_1 \leq h + \rho h$ , then it follows from the second equation of system (1.1) that

$$\frac{dI(t)}{dt} \geq -(\mu + \gamma) I(t),$$

that is,

$$I(t) \geq I(t_1) e^{-(\mu+\gamma)(t-t_1)} = qI^* e^{-(\mu+\gamma)(h+\rho h)} = v_2.$$

If  $t_2 - t_1 > h + \rho h$ , we similarly obtain from the second equation of system (1.1) that  $I(t) \geq v_2$  for  $t_1 \leq t \leq t_1 + h + \rho h$ . We now claim that  $I(t) \geq v_2$  for all  $t_1 + h + \rho h \leq t \leq t_2$ . Otherwise, there is a  $T^* > 0$  such that  $I(t) \geq v_2$  for

$t_1 \leq t \leq t_1 + h + \rho h + T^*$ ,  $I(t_1 + h + \rho h + T^*) = v_2$  and  $\frac{dI(t)}{dt}|_{t=t_1+h+\rho h+T^*} \leq 0$ . On the other hand, it follows from the second equation of system (1.1) that

$$\begin{aligned} \frac{dI(t)}{dt} \Big|_{t=t_1+h+\rho h+T^*} &= \beta S(t) \int_0^h f(\tau) G(I(t-\tau)) d\tau - (\mu + \gamma) I(t) \\ &\geq \beta S^\Delta G(I(t)) - (\mu + \gamma) I(t) \\ &\geq \left\{ \beta S^\Delta - (\mu + \gamma) \frac{v_2}{G(v_2)} \right\} G(v_2). \end{aligned}$$

By (4.1), (4.5) and the hypothesis (H2), it holds that

$$\frac{dI(t)}{dt} \Big|_{t=t_1+h+\rho h+T^*} \geq \left\{ \beta S^\Delta - (\mu + \gamma) \frac{I^*}{G(I^*)} \right\} G(v_2) > 0,$$

which is a contradiction. Hence  $I(t) \geq v_2$  for  $t_1 \leq t \leq t_2$ . Since the interval  $[t_1, t_2]$  is arbitrarily chosen, we conclude that  $I(t) \geq v_2$  holds for all  $t$  sufficiently large for the second case. Thus, we obtain  $\liminf_{t \rightarrow +\infty} I(t) \geq v_2$ , from which we have  $\liminf_{t \rightarrow +\infty} R(t) \geq v_3$ . Hence, this completes the proof.  $\square$

## 5. Global stability of the endemic equilibrium $E_*$ for $R_0 > 1$

From Lemma 2.1, we see that the limit set of system (1.1) in the first octant of  $\mathbb{R}^3$  locates on the plane  $S + I + R = B/\mu$ . Hence, the dynamics of system (1.1) in the first octant of  $\mathbb{R}^3$  is equivalent to the following system:

$$\begin{cases} \frac{dS(t)}{dt} = \frac{B(\mu + \delta)}{\mu} - (\mu + \delta)S(t) - \beta S(t) \int_0^h f(\tau) G(I(t-\tau)) d\tau - \delta I(t), \\ \frac{dI(t)}{dt} = \beta S(t) \int_0^h f(\tau) G(I(t-\tau)) d\tau - (\mu + \gamma) I(t). \end{cases} \quad (5.1)$$

We now discuss the global asymptotic stability of the endemic equilibrium  $\tilde{E}_*$  of system (5.1) for  $R_0 > 1$ . By Lemma 4.1, we recall that the existence of the endemic equilibrium  $\tilde{E}_* := (S^*, I^*)$  of system (5.1) is guaranteed for the case  $R_0 > 1$ . Put

$$x_t = \frac{S(t)}{S^*}, \quad y_t = \frac{I(t)}{I^*}, \quad \tilde{y}_t = \frac{G(I(t))}{G(I^*)}. \quad (5.2)$$

The following key lemma plays an important role to obtain Theorem 1.2.

**Lemma 5.1** *For all  $t \geq 0$ , under the condition (I), it holds that*

$$g(y_t) - g(\tilde{y}_t) \geq C_1 C_2 I^* (y_t - 1)^2, \quad (5.3)$$

where  $g(x) = x - 1 - \ln x \geq g(1) = 0$  defined for all  $x > 0$ .

**Proof.** First, we obtain that

$$\tilde{y}_t - 1 = \frac{G(I(t)) - G(I^*)}{G(I^*)}$$

and

$$y_t - \tilde{y}_t = \frac{I(t)}{I^*} - \frac{G(I(t))}{G(I^*)} = \frac{G(I(t))}{I^*} \left( \frac{I(t)}{G(I(t))} - \frac{I^*}{G(I^*)} \right).$$

Then it follows from the condition (I) that

$$\begin{aligned} (\tilde{y}_t - 1)(y_t - \tilde{y}_t) &= \frac{G(I(t))}{I^*G(I^*)} (G(I(t)) - G(I^*)) \left( \frac{I(t)}{G(I(t))} - \frac{I^*}{G(I^*)} \right) \\ &\geq \frac{C_1 C_2 G(I(t))}{I^* G(I^*)} (I(t) - I^*)^2 \\ &= \frac{C_1 C_2 I^* G(I(t))}{G(I^*)} (y_t - 1)^2 \\ &= C_1 C_2 I^* \tilde{y}_t (y_t - 1)^2. \end{aligned} \quad (5.4)$$

By  $g''(x) = \frac{1}{x^2} > 0$  for  $x > 0$  and (5.4), we obtain that

$$\begin{aligned} g(y_t) - g(\tilde{y}_t) &\geq g'(\tilde{y}_t)(y_t - \tilde{y}_t) \\ &= \left( 1 - \frac{1}{\tilde{y}_t} \right) (y_t - \tilde{y}_t) \\ &= \frac{1}{\tilde{y}_t} (\tilde{y}_t - 1)(y_t - \tilde{y}_t) \\ &\geq C_1 C_2 I^* (y_t - 1)^2. \end{aligned}$$

Hence, we get the conclusion of this lemma.  $\square$

Now, we are in a position to prove the global asymptotic stability of the endemic equilibrium  $\tilde{E}_*$  for  $R_0 > 1$  by applying techniques in McCluskey [13, Proof of Theorem 4.1].

**Theorem 5.1** *If  $R_0 > 1$  and the conditions (I) and (II) hold, then the endemic equilibrium  $\tilde{E}_*$  of system (5.1) is globally asymptotically stable.*

**Proof.** Let us consider the following Lyapunov functional (see McCluskey [13, Proof of Theorem 4.1]):

$$U(t) = \frac{1}{\beta G(I^*)} U_S(t) + \frac{I^*}{\beta S^* G(I^*)} U_I(t) + U_+(t), \quad (5.5)$$

where

$$U_S(t) = g\left(\frac{S(t)}{S^*}\right), \quad U_I(t) = g\left(\frac{I(t)}{I^*}\right), \quad U_+(t) = \int_0^h f(\tau) \int_{t-\tau}^t g\left(\frac{G(I(s))}{G(I^*)}\right) ds d\tau. \quad (5.6)$$

We now show that  $\frac{dU(t)}{dt} \leq 0$ . First, we calculate  $\frac{dU_S(t)}{dt}$ .

$$\frac{dU_S(t)}{dt} = \frac{S(t) - S^*}{S^*S(t)} \left\{ \frac{B(\mu + \delta)}{\mu} - (\mu + \delta)S(t) - \beta S(t) \int_0^h f(\tau)G(I(t - \tau))d\tau - \delta I(t) \right\}.$$

By the relation  $\frac{B(\mu + \delta)}{\mu} = (\mu + \delta)S^* + \beta S^*G(I^*) + \delta I^*$ , we have

$$\begin{aligned} & \frac{dU_S(t)}{dt} \\ &= \frac{S(t) - S^*}{S^*S(t)} \left\{ (\mu + \delta)S^* + \beta S^*G(I^*) + \delta I^* \right. \\ & \quad \left. - (\mu + \delta)S(t) - \beta S(t) \int_0^h f(\tau)G(I(t - \tau))d\tau - \delta I(t) \right\} \\ &= \frac{S(t) - S^*}{S^*S(t)} \left[ -(\mu + \delta)(S(t) - S^*) \right. \\ & \quad \left. + \beta \int_0^h f(\tau) \{S^*G(I^*) - S(t)G(I(t - \tau))\} d\tau - \delta(I(t) - I^*) \right] \\ &= -\frac{(\mu + \delta)(S(t) - S^*)^2}{S^*S(t)} + \beta G(I^*) \int_0^h f(\tau) \left(1 - \frac{S^*}{S(t)}\right) \left(1 - \frac{S(t)}{S^*} \frac{G(I(t - \tau))}{G(I^*)}\right) d\tau \\ & \quad - \frac{\delta(S(t) - S^*)(I(t) - I^*)}{S^*S(t)} \\ &= -\frac{S^*(\mu + \delta)(x_t - 1)^2}{S(t)} + \beta G(I^*) \int_0^h f(\tau) \left(1 - \frac{1}{x_t}\right) (1 - x_t \tilde{y}_{t-\tau}) d\tau \\ & \quad - \frac{\delta I^*}{S(t)} (x_t - 1)(y_t - 1) \\ &= -\frac{S^*(\mu + \delta)(x_t - 1)^2}{S(t)} + \beta G(I^*) \int_0^h f(\tau) \left(1 - \frac{1}{x_t} - x_t \tilde{y}_{t-\tau} + \tilde{y}_{t-\tau}\right) d\tau \\ & \quad - \frac{\delta I^*}{S(t)} (x_t - 1)(y_t - 1). \end{aligned} \tag{5.7}$$

We secondly calculate  $\frac{dU_I(t)}{dt}$ .

$$\frac{dU_I(t)}{dt} = \frac{I(t) - I^*}{I^*I(t)} \left\{ \beta S(t) \int_0^h f(\tau)G(I(t - \tau))d\tau - (\mu + \gamma)I(t) \right\}.$$

By the relation  $(\mu + \gamma)I^* = \beta S^* G(I^*)$ , we have

$$\begin{aligned}
\frac{dU_I(t)}{dt} &= \frac{I(t) - I^*}{I^* I(t)} \left( \beta S(t) \int_0^h f(\tau) G(I(t - \tau)) d\tau - \beta S^* \frac{G(I^*)}{I^*} I(t) \right) \\
&= \beta S^* \frac{G(I^*)}{I^*} \int_0^h f(\tau) \left( 1 - \frac{I^*}{I(t)} \right) \left( \frac{S(t)}{S^*} \frac{G(I(t - \tau))}{G(I^*)} - \frac{I(t)}{I^*} \right) d\tau \\
&= \beta S^* \frac{G(I^*)}{I^*} \int_0^h f(\tau) \left( 1 - \frac{1}{y_t} \right) (x_t \tilde{y}_{t-\tau} - y_t) d\tau \\
&= \beta S^* \frac{G(I^*)}{I^*} \int_0^h f(\tau) \left( x_t \tilde{y}_{t-\tau} - \frac{x_t \tilde{y}_{t-\tau}}{y_t} - y_t + 1 \right) d\tau. \tag{5.8}
\end{aligned}$$

Finally, calculating  $\frac{dU_+(t)}{dt}$  gives as follows.

$$\begin{aligned}
\frac{dU_+(t)}{dt} &= \int_0^h f(\tau) \left( g\left(\frac{G(I(t))}{G(I^*)}\right) - g\left(\frac{G(I(t - \tau))}{G(I^*)}\right) \right) d\tau \\
&= \int_0^h f(\tau) (g(\tilde{y}_t) - g(\tilde{y}_{t-\tau})) d\tau \\
&= \int_0^h f(\tau) (\tilde{y}_t - \ln \tilde{y}_t - \tilde{y}_{t-\tau} + \ln \tilde{y}_{t-\tau}) d\tau. \tag{5.9}
\end{aligned}$$

Combining (5.3), (5.7), (5.8) and (5.9), it follows from Lemma 5.1 that

$$\begin{aligned}
\frac{dU(t)}{dt} &= \frac{1}{\beta G(I^*)} \left\{ -\frac{S^*(\mu + \delta)(x_t - 1)^2}{S(t)} \right. \\
&\quad \left. + \beta G(I^*) \int_0^h f(\tau) \left( 1 - \frac{1}{x_t} - x_t \tilde{y}_{t-\tau} + \tilde{y}_{t-\tau} \right) d\tau - \frac{\delta I^*}{S(t)} (x_t - 1)(y_t - 1) \right\} \\
&\quad \left\{ \beta S^* \frac{G(I^*)}{I^*} \int_0^h f(\tau) \left( x_t \tilde{y}_{t-\tau} - \frac{x_t \tilde{y}_{t-\tau}}{y_t} - y_t + 1 \right) d\tau \right\} \\
&\quad + \int_0^h f(\tau) (\tilde{y}_t - \ln \tilde{y}_t - \tilde{y}_{t-\tau} + \ln \tilde{y}_{t-\tau}) d\tau \\
&= -\frac{S^*(\mu + \delta)(x_t - 1)^2}{\beta G(I^*) S(t)} - \frac{\delta I^*}{\beta G(I^*) S(t)} (x_t - 1)(y_t - 1) - (g(y_t) - g(\tilde{y}_t)) \\
&\quad - \int_0^h f(\tau) \left\{ g\left(\frac{1}{x_t}\right) + g\left(\frac{x_t \tilde{y}_{t-\tau}}{y_t}\right) \right\} d\tau \\
&\leq -\frac{S^*(\mu + \delta)(x_t - 1)^2 + \delta I^* (x_t - 1)(y_t - 1) + \beta G(I^*) S(t) C_1 C_2 I^* (y_t - 1)^2}{\beta G(I^*) S(t)}. \tag{5.10}
\end{aligned}$$

By Lemma 4.2, for any  $0 < \varepsilon < v_1$ , there exists a  $T_\varepsilon > h$  such that  $S(t) > v_1 - \varepsilon$  for any  $t > T_\varepsilon$ . From the condition (II), we may restrict this  $\varepsilon > 0$  sufficiently small such that

$$\delta^2 - 4C_1 C_2 (\mu + \gamma) (\mu + \delta) (v_1 - \varepsilon) < 0.$$

Then, for all  $t > T_\varepsilon$ , it follows that

$$\begin{aligned} & (\delta I^*)^2 - 4C_1C_2\beta S^*G(I^*)(\mu + \delta)I^*S(t) \\ &= (I^*)^2 \{ \delta^2 - 4C_1C_2(\mu + \gamma)(\mu + \delta)S(t) \} \\ &< (I^*)^2 \{ \delta^2 - 4C_1C_2(\mu + \gamma)(\mu + \delta)(v_1 - \varepsilon) \} < 0, \end{aligned}$$

from which we obtain that  $\frac{dU(t)}{dt} \leq 0$  for all  $t > T_\varepsilon$ . We recall that by Lemma 5.1,  $\frac{dU(t)}{dt} = 0$  if  $x_t = 1$  and  $y_t = 1$ , or equivalently, if  $S(t) = S^*$  and  $I(t) = I^*$  for all  $t > T_\varepsilon$ . It follows from Lemma 2.1, 4.2 and LaSalle's invariant principle that  $\tilde{E}_*$  of system (5.1) is globally asymptotically stable. This completes the proof.  $\square$

**Proof of Theorem 1.2.** Summarizing results of Lemmas 2.1, 4.1, 4.2 and Theorem 5.1, we obtain the conclusion of this theorem.  $\square$

## 6. Applications

In this section, we illustrate some examples in order to validate the feasibility of our analytical results for  $R_0 > 1$  for the following SIRS epidemic model proposed in Xu and Ma [18]:

$$\begin{cases} \frac{dS(t)}{dt} = B - \mu S(t) - \beta S(t)G(I(t-h)) + \delta R(t), \\ \frac{dI(t)}{dt} = \beta S(t)G(I(t-h)) - (\mu + \gamma)I(t), \\ \frac{dR(t)}{dt} = \gamma I(t) - (\mu + \delta)R(t), \quad h \geq 0, \end{cases} \quad (6.1)$$

where  $G(I) = \frac{I}{1+\alpha I}$ ,  $\alpha > 0$ .

If  $R_0 < 1$ , then the disease-free equilibrium  $E_0$  of system (6.1) is globally asymptotically stable. If  $R_0 > 1$ , then system (6.1) is permanent. Concerning the global stability of a unique endemic equilibrium  $E_*$  for  $R_0 > 1$ , Muroya *et al.* [17, Corollary 4.1] established the following theorem:

**Theorem A** *If  $R_0 > 1$  and*

$$\delta > \bar{\delta}_1(\alpha) := \frac{\beta\gamma}{\alpha(\mu + \gamma) + \beta} - \mu, \quad (6.2)$$

*then the endemic equilibrium  $E_*$  of system (6.1) is globally asymptotically stable.*

Their result for  $p = 1$  improved the global stability condition of the endemic equilibrium in Xu and Ma [18, Theorem 3.1]. On the other hand, for  $R_0 > 1$ , similar to Theorem 1.2, we establish the following result:

Parameter	Description	Value	Reference
$\beta$	Transmission rate	0.05 per day per individual	Assumed
$B$	Recruitment rate	30 individuals per day	Assumed
$\mu$	Natural death rate	0.02 per day	[4]
$\gamma$	Recovery rate of infectives	0.077 per day	[4]
$h$	Latency period	0.1 days	Assumed

Table 1: Parameters of system (6.1) and their values used in Figure 2. For the above parameter values, we have  $R_0 = 257.732 \dots > 1$ .

**Corollary 6.1** *If  $R_0 > 1$  and*

$$0 \leq \delta < \underline{\delta}_2(\alpha), \quad (6.3)$$

where

$$\underline{\delta}_2(\alpha) := \frac{2C_1C_2(\mu + \gamma)B}{\mu + \beta G(B/\mu)} + \sqrt{\left\{ \frac{2C_1C_2(\mu + \gamma)B}{\mu + \beta G(B/\mu)} \right\}^2 + \frac{4\mu C_1C_2(\mu + \gamma)B}{\mu + \beta G(B/\mu)}},$$

then the endemic equilibrium  $E_*$  of system (6.1) is globally asymptotically stable.

Next, for system (6.1), using parameter values given in Table 1, we carry out some computational experiments to investigate the feasibility of our global stability condition (6.3) with respect to the rate of immunity loss  $\delta \geq 0$ . For the parameter values, we obtain  $R_0 = 257.732 \dots > 1$  and the endemic equilibrium  $E_*$  exists. In Figure 1, we show regions of global and local stability of the endemic equilibrium  $E_*$ , which are obtained by Theorem A, Corollary 6.1 and Xu and Ma [18, Section 2] in the parameter space  $(\alpha, \delta)$ .

First, we consider the case  $\alpha = 1.4$ . Then we obtain that  $\bar{\delta}_1(\alpha) = 0.0007 \dots$  by (6.2) and  $\underline{\delta}_2(\alpha) = 0.0021 \dots$  by (6.3). Hence, the endemic equilibrium  $E_*$  of system (6.1) is globally asymptotically stable for any  $\delta \geq 0$ . From a biological point of view, the prevalence of the disease settles to an endemic steady state independently of initial conditions concerning the fractions of a host population for any rate of immunity loss.

Second, we consider the case  $\alpha = 1.1$ . Then we obtain  $\bar{\delta}_1(\alpha) = 0.0045 \dots$  by (6.2) and  $\underline{\delta}_2(\alpha) = 0.0021 \dots$  by (6.3). Therefore, the endemic equilibrium  $E_*$  of system (6.1) is globally asymptotically stable for  $0 \leq \delta < 0.0021 \dots$  or  $\delta > 0.0045 \dots$ . Thus, the global stability of the endemic equilibrium  $E_*$  is guaranteed for a small rate of immunity loss  $\delta$  even if the condition (6.2) in Theorem A does not hold.

On the other hand, Figure 2 indicates that the endemic equilibrium  $E_*$  of system (6.1) is also globally asymptotically stable for the case  $\alpha = 1.1$  and  $\delta = 0.003 \in [\underline{\delta}_2(\alpha), \bar{\delta}_1(\alpha)]$  with the parameter values in Table 1. There is still an open problem to determine the global asymptotic stability of the endemic equilibrium  $E_*$  of system (6.1) when both of the conditions (6.2) and (6.3) fail.

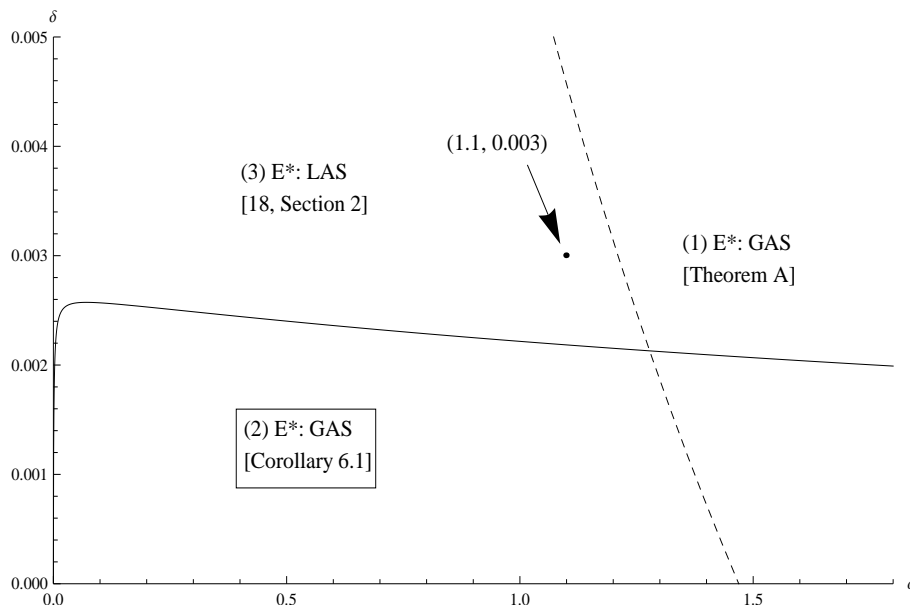


Figure 1: Curves of  $\bar{\delta}_1(\alpha)$  (dotted line) and  $\underline{\delta}_2(\alpha)$  (dashed line) for the parameter set in Table 1 [(1):  $\delta > \bar{\delta}_1(\alpha)$ , (2):  $0 \leq \delta \leq \bar{\delta}_1(\alpha)$  and  $0 \leq \delta < \underline{\delta}_2(\alpha)$ , (3):  $0 \leq \delta \leq \bar{\delta}_1(\alpha)$  and  $\delta \geq \underline{\delta}_2(\alpha)$ ]. Here, GAS and LAS denote globally asymptotically stable and locally asymptotically stable, respectively.

## 7. Discussion

In this paper, we consider delayed SIRS epidemic models with a class of nonlinear incidence rates. For the nonlinear incidence rate, we put the hypotheses (H1) and (H2), which describe the crowding (saturation) effects observed in the literature of epidemiology. For  $R_0 < 1$ , we establish the global asymptotic stability of the disease-free equilibrium in Theorem 1.1 and for  $R_0 > 1$ , we obtain sufficient conditions of the global asymptotic stability of the endemic equilibrium in Theorem 1.2.

In particular, for  $R_0 > 1$ , by using strict monotonicity of the functions  $G(I)$  and  $I/G(I)$  on a neighborhood of  $I^*$ , we establish Lemma 5.1, which plays a key role to construct a Lyapunov functional for the reduced limit system (5.1).

For the special case  $\delta = 0$  (a delayed SIR epidemic model), Beretta and Takeuchi [1] obtained the global stability of a disease-free equilibrium and the local stability of an endemic equilibrium of the model with a bilinear incidence rate (*i.e.*  $G(I) = I$ ). However, on their global stability analysis of the endemic equilibrium, they required a condition that the size of time delay  $h$  should be small enough. The global stability of the endemic equilibrium for a sufficiently large  $h$  remained unsolved for a long time. Later, by applying techniques of equation deformation on a calculation of the time derivative of a Lyapunov



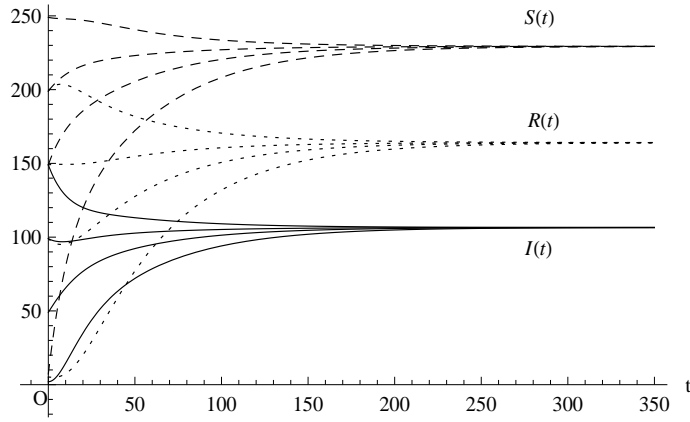


Figure 2: The graph trajectory of  $S(t)$ ,  $I(t)$  and  $R(t)$  of system (6.1). For the parameter values in Table 1 with  $\alpha = 1.1$  and  $\delta = 0.003$ , we have  $R_0 = 257.732 \dots > 1$  and  $E^* = (229.338 \dots, 106.56 \dots, 164.102 \dots)$ .

functional, McCluskey [13] solved the problem and established that the endemic equilibrium of the model is globally asymptotically stable whenever it exists. The similar global stability results for delayed SIR epidemic models with a wide class of incidence rates are now obtained in [3, 6, 14].

On the other hand, for the case  $\delta > 0$ , there are few global stability results concerning an endemic equilibrium of delayed SIRS epidemic models. For a delayed SIRS epidemic model with the saturated incidence rate, by applying new monotone iterative techniques in [16], Muroya *et al.* [17] recently obtained sufficient conditions which ensure the global asymptotic stability of an endemic equilibrium of the model with an incidence rate  $\beta S(t) \frac{I(t-h)}{1+\alpha I^p(t-h)}$  with  $p > 0$  for large  $\delta$ . Their result for  $p = 1$  improved the global stability condition of the endemic equilibrium in Xu and Ma [18] for (6.1) (see Theorem A). For the system (6.1), we derive Corollary 6.1 from Theorem 1.2 and find the new global stability region of the endemic equilibrium in the parameter space  $(\alpha, \delta)$ , compared with that of Theorem A (see Figure 1). This illustrates that the global stability of the endemic equilibrium still holds for small  $\delta$  as well as the case  $\delta = 0$ .

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