Permanence and extinction for a nonautonomous SEIRS epidemic model

Toshikazu Kuniya [a,*] Yukihiko Nakata [b]

[a] *Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba Meguro-ku, Tokyo 153-8914, Japan*

[b] *Basque Center for Applied Mathematics, Mazarredo, 14, E-48009 Bilbao, Spain*

Abstract

In this paper, we study the long-time behavior of a nonautonomous SEIRS epidemic model. We obtain new sufficient conditions for the permanence (uniform persistence) and extinction of infectious population of the model. By numerical examples we show that there are cases such that our results improve the previous results obtained in [T. Zhang and Z. Teng, On a nonautonomous SEIRS model in epidemiology, Bull. Math. Bio., (2007) 69, 2537-2559]. We discuss a relation between our results and open questions proposed in the paper.

Key words: SEIRS epidemic model, Nonautonomous system, Permanence, Extinction, Basic reproduction number

1. Introduction

In this paper, we consider the following nonautonomous SEIRS epidemic model.

* Corresponding author

Email addresses: tkuniya@ms.u-tokyo.ac.jp (Toshikazu Kuniya), nakata@bcamath.org (Yukihiko Nakata).

$$
\begin{cases}
\dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda(t) - \beta(t)S(t)I(t) - \mu(t)S(t) + \delta(t)R(t), \\
\dfrac{\mathrm{d}E(t)}{\mathrm{d}t} = \beta(t)S(t)I(t) - (\mu(t) + \varepsilon(t))\,E(t), \\
\dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = \varepsilon(t)E(t) - (\mu(t) + \gamma(t))\,I(t), \\
\dfrac{\mathrm{d}R(t)}{\mathrm{d}t} = \gamma(t)I(t) - (\mu(t) + \delta(t))\,R(t)
\end{cases}
\tag{1.1}
$$

with initial value

$$S(0) > 0, \quad E(0) \geq 0, \quad I(0) > 0, \quad R(0) \geq 0. \tag{1.2}$$

Here $S(t)$, $E(t)$, $I(t)$ and $R(t)$ denote the size of susceptible, exposed (not infectious but infected), infectious and recovered population at time $t \geq 0$, respectively. $\Lambda(t)$ denotes the birth rate, $\beta(t)$ denotes the disease transmission coefficient, $\mu(t)$ denotes the mortality, $\varepsilon(t)$ denotes the rate of developing infectivity, $\gamma(t)$ denotes the recovery rate and $\delta(t)$ denotes the rate of losing immunity at time t.

In the field of mathematical epidemiology, the qualitative analysis of mathematical epidemic models has been carried out by many authors (see [1-22] and references therein). One of the main streams of the field is the analysis of autonomous models (see for instance [8, 9, 13, 14, 18, 19] and references therein). For instance, in case where system (1.1) is autonomous (that is, all parameters are given by time-independent functions $\Lambda(t) = \Lambda$, $\beta(t) = \beta$, $\mu(t) = \mu$, $\varepsilon(t) = \varepsilon$, $\gamma(t) = \gamma$ and $\delta(t) = \delta$), we obtain the basic reproduction number (see e.g., [5]) as

$$R_0 = \frac{\varepsilon\beta}{(\mu + \varepsilon)(\mu + \gamma)} \frac{\Lambda}{\mu}. \tag{1.3}$$

It is well known that the infectious disease dies out if $R_0 \leq 1$ and the disease persists if $R_0 > 1$ (see [10, 14]).

On the other hand, in the real world, quite a few infectious diseases spread seasonally (one of the reasons of such a phenomenon is, for instance, the seasonal change of the number of infectious vectors [3]). Therefore, the study of periodic epidemic models has recently been carried out enthusiastically (see e.g., [2–4, 11, 12, 15, 16, 20–23] and references therein). The definition of the basic reproduction number R_0 for periodic epidemic models was firstly given by Bacaër and Guernaoui [3]. For system (1.1) with periodic parameters, Nakata and Kuniya [12] proved that R_0 plays the role as a threshold parameter for determining the global dynamics of solutions, that is, the disease-free periodic solution is globally asymptotically stable if $R_0 < 1$ and the disease persists if $R_0 > 1$.

The nonautonomous case is an extension of the periodic case. The study of the basic reproduction number R_0 for general time-heterogeneous epidemic models has recently been carried out by Inaba [7] and Thieme [17].

Zhang and Teng [22] analyzed the dynamics of nonautonomous SEIRS epidemic model (1.1) and obtained some sufficient conditions for the permanence and extinction of the infectious population. One can notice that results obtained in Theorems 4.1 and 5.1 in their paper do not determine the disease dynamics completely, since those conditions do not give a threshold-type condition even in the autonomous case.

In this paper we obtain new sufficient conditions for the permanence and extinction of system (1.1). We prove that our condtions gives the threshold-type result by the basic reproduction number given as in (1.3) when every parameter is given as a constant parameter. Thus our result is an extension result of the threshold-type result in the autonomous system. Our results may contribute to predict the disease dynamics, such as permanence and extinction of the infectious population, when the phenomena is modeld as a nonautonomous system.

This paper is organized as follows. In Section 2 we present preliminary setting and propositions, which we use to analyze the long-time behavior of system (1.1) in the following sections. In Sections 3 and 4 we prove our main theorems on the extinction and permanence of infectious population of system (1.1). In Section 5, we derive explicit conditions for the existence and permanence of infectious population of system (1.1) for some special cases. We prove that when every parameter is given as a constant parameter our conditions for the permanence and extinction becomes the threshold condition by the basic reproduction number. In Section 6 we provide numerical examples to illustrate the validity of our results. Moreover, those examples illustrate the cases where our theoretical result can determine the dynamics even conditions proposed in [22] are not satisfied.

2. Preliminaries

As in [22] we put the following assumptions for system (1.1).

Assumption 2.1 ***(i)*** *Functions Λ, β, μ, δ, ε and γ are positive, bounded and continuous on $[0,+\infty)$ and $\beta(0)>0$.*

(ii) *There exist constants $\omega_i>0$ $(i=1,2,3)$ such that*

$$\liminf_{t\to+\infty}\int_t^{t+\omega_1}\beta(s)\mathrm{d}s>0,\quad \liminf_{t\to+\infty}\int_t^{t+\omega_2}\mu(s)\mathrm{d}s>0,\quad \liminf_{t\to+\infty}\int_t^{t+\omega_3}\Lambda(s)\mathrm{d}s>0.$$

In what follows, we denote by $N^*(t)$ the solution of

$$\frac{\mathrm{d}N^*(t)}{\mathrm{d}t}=\Lambda(t)-\mu(t)N^*(t) \tag{2.1}$$

with initial value $N^*(0)=S(0)+E(0)+I(0)+R(0)>0$. By adding equations of (1.1), we easily see that $N^*(t)=S(t)+E(t)+I(t)+R(t)$ means the size of total population at time t. From Lemma 2.1, Theorem 3.1 and Remark 3.2 in [22], we have the following results.

Proposition 2.2 ***(i)*** *There exist positive constants $m>0$ and $M>0$, which are independent from the choice of initial value $N^*(0)>0$, such that*

$$0<m\le\liminf_{t\to+\infty}N^*(t)\le\limsup_{t\to+\infty}N^*(t)\le M<+\infty. \tag{2.2}$$

(ii) *The solution $(S(t),E(t),I(t),R(t))$ of system (1.1) with initial value (1.2) exists, uniformly bounded and*

$$S(t)>0,\quad E(t)>0,\quad I(t)>0,\quad R(t)\ge 0$$

for all $t>0$.

For $p>0$ and $t>0$ we define

$$G(p,t):=\beta(t)N^*(t)p+\gamma(t)-\left(1+\frac{1}{p}\right)\varepsilon(t)$$

and

$$W(p,t) := pE(t) - I(t), \tag{2.3}$$

where E and I are solutions of system (1.1). In Sections 3 and 4 we use the following lemma in order to investigate the long-time behavior of system (1.1).

Lemma 2.3 *If there exist positive constants $p > 0$ and $T_1 > 0$ such that $G(p,t) < 0$ for all $t \geq T_1$, then there exists $T_2 \geq T_1$ such that either $W(p,t) > 0$ for all $t \geq T_2$ or $W(p,t) \leq 0$ for all $t \geq T_2$.*

PROOF. Suppose that there does not exist $T_2 \geq T_1$ such that either $W(p,t) > 0$ for all $t \geq T_2$ or $W(p,t) \leq 0$ for all $t \geq T_2$ hold. Then, there necessarily exists $s \geq T_1$ such that $W(p,s) = 0$ and $\mathrm{d}W(p,s)/\mathrm{d}t > 0$. Hence we have

$$pE(s) = I(s) \tag{2.4}$$

and

$$\begin{aligned} &p\{\beta(s)S(s)I(s) - (\mu(s)+\varepsilon(s))E(s)\} - \{\varepsilon(s)E(s) - (\mu(s)+\gamma(s))I(s)\} \\ &= I(s)\{\beta(s)S(s)p + (\mu(s)+\gamma(s))\} - pE(s)\left\{(\mu(s)+\varepsilon(s)) + \frac{1}{p}\varepsilon(s)\right\} > 0. \end{aligned} \tag{2.5}$$

Substituting (2.4) into (2.5) we have

$$0 < pE(s)\left\{\beta(s)S(s)p + \gamma(s) - \left(1+\frac{1}{p}\right)\varepsilon(s)\right\} \leq pE(s)G(p,s).$$

From (ii) of Proposition 2.2, we have $G(p,s) > 0$, which is a contradiction. □

3. Extinction of infectious population

In this section, we obtain sufficient conditions for the extinction of infectious population of system (1.1). The definition of the extinction is as follows:

Definition 3.1 *We say that the infectious population of system (1.1) is extinct if*

$$\lim_{t\to+\infty} I(t) = 0.$$

We give one of the main results of this paper.

Theorem 3.2 *If there exist positive constants $\lambda > 0$, $p > 0$ and $T_1 > 0$ such that*

$$R_1(\lambda,p) := \limsup_{t\to+\infty} \int_t^{t+\lambda} \{\beta(s)N^*(s)p - (\mu(s)+\varepsilon(s))\}\,\mathrm{d}s < 0, \tag{3.1}$$

$$R_1^*(\lambda,p) := \limsup_{t\to+\infty} \int_t^{t+\lambda} \left\{\varepsilon(s)\frac{1}{p} - (\mu(s)+\gamma(s))\right\}\mathrm{d}s < 0 \tag{3.2}$$

and $G(p,t) < 0$ for all $t \geq T_1$, then the infectious population of system (1.1) is extinct.

PROOF. From Lemma 2.3, we only have to consider the following two cases.

(i) $pE(t) > I(t)$ for all $t \geq T_2$.

(ii) $pE(t) \leq I(t)$ for all $t \geq T_2$.

First we consider the case (i). From the second equation of system (1.1), we have

$$\begin{aligned}\frac{\mathrm{d}E(t)}{\mathrm{d}t} &= \beta(t)\left(N^*(t) - E(t) - I(t) - R(t)\right) I(t) - \left(\mu(t) + \varepsilon(t)\right) E(t) \\ &< \beta(t)\left(N^*(t) - E(t) - I(t) - R(t)\right) pE(t) - \left(\mu(t) + \varepsilon(t)\right) E(t) \\ &< E(t)\left\{\beta(t)N^*(t)p - \left(\mu(t) + \varepsilon(t)\right)\right\}.\end{aligned}$$

Hence, we obtain

$$E(t) < E(T_2)\exp\left(\int_{T_2}^{t} \left\{\beta(s)N^*(s)p - \left(\mu(s) + \varepsilon(s)\right)\right\} \mathrm{d}s\right) \tag{3.3}$$

for all $t \geq T_2$. From (3.1) we see that there exist constants $\delta_1 > 0$ and $T_3 > T_2$ such that

$$\int_{t}^{t+\lambda} \left\{\beta(s)N^*(s)p - \left(\mu(s) + \varepsilon(s)\right)\right\} \mathrm{d}s < -\delta_1 \tag{3.4}$$

for all $t \geq T_3$. From (3.3) and (3.4) we have $\lim_{t\to+\infty} E(t) = 0$. Then it follows from $pE(t) > I(t)$ for all $t \geq T_2$ that $\lim_{t\to+\infty} I(t) = 0$.

Next we consider the case (ii). Since we have $E(t) \leq I(t)/p$ for all $t \geq T_2$, it follows from the third equation of system (1.1) that

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} \leq I(t)\left\{\varepsilon(t)\frac{1}{p} - \left(\mu(t) + \gamma(t)\right)\right\}.$$

Hence we have

$$I(t) \leq I(T_2)\exp\left(\int_{T_2}^{t} \left\{\varepsilon(s)\frac{1}{p} - \left(\mu(s) + \gamma(s)\right)\right\} \mathrm{d}s\right) \tag{3.5}$$

for all $t \geq T_2$. Now it follows from (3.2) that there exist constants $\delta_2 > 0$ and $T_4 > T_2$ such that

$$\int_{t}^{t+\lambda} \left\{\varepsilon(s)\frac{1}{p} - \left(\mu(s) + \gamma(s)\right)\right\} \mathrm{d}s < -\delta_2 \tag{3.6}$$

for all $t \geq T_4$. From (3.5) and (3.6) we have $\lim_{t\to+\infty} I(t) = 0$. □

4. Permanence of infectious population

In this section, we obtain sufficient conditions for the permanence of infectious population of system (1.1). The definition of the permanence is as follows:

Definition 4.1 *We say that the infectious population of system (1.1) is permanent if there exist positive constants $I_1 > 0$ and $I_2 > 0$, which are independent from the choice of initial value satisfying (1.2), such that*

$$0 < I_1 \leq \liminf_{t\to+\infty} I(t) \leq \limsup_{t\to+\infty} I(t) \leq I_2 < +\infty.$$

We give one of the main results of this paper.

Theorem 4.2 *If there exist positive constants $\lambda > 0$, $p > 0$ and $T_1 > 0$ such that*

$$R_2(\lambda, p) := \liminf_{t\to+\infty} \int_{t}^{t+\lambda} \left\{\beta(s)N^*(s)p - \left(\mu(s) + \varepsilon(s)\right)\right\} \mathrm{d}s > 0, \tag{4.1}$$

$$R_2^*(\lambda, p) := \liminf_{t\to+\infty} \int_{t}^{t+\lambda} \left\{\varepsilon(s)\frac{1}{p} - \left(\mu(s) + \gamma(s)\right)\right\} \mathrm{d}s > 0 \tag{4.2}$$

and $G(p,t)<0$ for all $t \geq T_1$, then the infectious population of system (1.1) is permanent.

Before we give the proof of Theorem 4.2, we introduce the following lemma.

Lemma 4.3 *If there exist positive constants $\lambda > 0$, $p > 0$ and $T_1 > 0$ such that (4.1), (4.2) and $G(p,t) < 0$ hold for all $t \geq T_1$, then $W(p,t) \leq 0$ for all $t \geq T_2 \geq T_1$, where T_2 is given as in Lemma 2.3.*

PROOF. From Lemma 2.3 we have only two cases, $W(p,t) > 0$ for all $t \geq T_2$ or $W(p,t) \leq 0$ for all $t \geq T_2$. Suppose that $W(p,t) > 0$ for all $t \geq T_2$. Then, we have $E(t) > I(t)/p$ for all $t \geq T_2$. It follows from the third equation of system (1.1) that

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} > \varepsilon(t)\frac{1}{p}I(t) - (\mu(t)+\gamma(t))\,I(t) = I(t)\left\{\varepsilon(t)\frac{1}{p} - (\mu(t)+\gamma(t))\right\}$$

for all $t \geq T_2$. Hence, we obtain

$$I(t) > I(T_2)\exp\left(\int_{T_2}^{t}\left\{\varepsilon(s)\frac{1}{p} - (\mu(s)+\gamma(s))\right\}\mathrm{d}s\right) \tag{4.3}$$

for all $t \geq T_2$. From the inequality (4.2), we see that there exist positive constants $\eta > 0$ and $T > 0$ such that

$$\int_{t}^{t+\lambda}\left\{\varepsilon(s)\frac{1}{p} - (\mu(s)+\gamma(s))\right\}\mathrm{d}s > \eta \tag{4.4}$$

for all $t \geq T$. Since the inequality (4.3) holds for all $t \geq \max(T_2, T)$, it follows from (4.4) that $\lim_{t\to+\infty} I(t) = +\infty$. This contradicts with the boundedness of I, stated in (ii) of Proposition 2.2. □

Using Lemma 4.3 we prove Theorem 4.2.

PROOF. (Proof of Theorem 4.2.) For simplicity, let $m_\epsilon := m - \epsilon$ and $M_\epsilon := M + \epsilon$, where $\epsilon > 0$ is a constant. From the inequality (2.2) of (i) of Proposition 2.2, we see that for any $\epsilon > 0$, there exists $T > 0$ such that

$$m_\epsilon < N^*(t) < M_\epsilon \tag{4.5}$$

for all $t \geq T$. The inequality (4.1) implies that for sufficiently small $\eta > 0$ there exists $T_1 \geq T$ such that

$$\int_{t}^{t+\lambda}\{\beta(s)N^*(s)p - (\mu(s)+\varepsilon(s))\}\,\mathrm{d}s > \eta \tag{4.6}$$

for all $t \geq T_1$. We define

$$\beta^+ := \sup_{t\geq 0}\beta(t), \quad \mu^+ := \sup_{t\geq 0}\mu(t), \quad \varepsilon^+ := \sup_{t\geq 0}\varepsilon(t), \quad \gamma^+ := \sup_{t\geq 0}\gamma(t).$$

From (4.5) and (4.6) we see that for positive constants $\eta_1 < \eta$ and $T_2 \geq T_1$ there exist small $\epsilon_i > 0,\ i \in \{1,2,3\}$ such that

$$\int_{t}^{t+\lambda}\{\beta(s)\,(N^*(s) - \epsilon_1 - k\epsilon_2 - \epsilon_3)\,p - (\mu(s)+\varepsilon(s))\}\,\mathrm{d}s > \eta_1, \tag{4.7}$$

$$N^*(t) - \epsilon_1 - k\epsilon_2 - \epsilon_3 > m_\epsilon \tag{4.8}$$

hold for all $t \geq T_2$, where $k := 1 + (\beta^+ M_\epsilon + \gamma^+)\,\omega_2$. From (ii) of Assumption 2.1, ϵ_2 can be chosen sufficiently small so that

$$\int_t^{t+\omega_2} \{\beta(s) M_\epsilon \epsilon_2 - (\mu(s) + \varepsilon(s))\,\epsilon_1\}\,\mathrm{d}s < -\eta_1, \tag{4.9}$$

$$\int_t^{t+\omega_2} \{\gamma(s) \epsilon_2 - (\mu(s) + \delta(s))\,\epsilon_3\}\,\mathrm{d}s < -\eta_1 \tag{4.10}$$

hold for all $t \geq T_2$.

First we claim that

$$\limsup_{t \to +\infty} I(t) > \epsilon_2.$$

In fact, if it is not true, then there exists $T_3 \geq T_2$ such that

$$I(t) \leq \epsilon_2 \tag{4.11}$$

for all $t \geq T_3$. Suppose that $E(t) \geq \epsilon_1$ for all $t \geq T_3$. Then, from (4.5) and (4.11), we have

$$\begin{aligned} E(t) &= E(T_3) + \int_{T_3}^t \{\beta(s)\,(N^*(s) - E(s) - I(s) - R(s))\,I(s) - (\mu(s) + \varepsilon(s))\,E(s)\}\,\mathrm{d}s \\ &\leq E(T_3) + \int_{T_3}^t \{\beta(s) M_\epsilon \epsilon_2 - (\mu(s) + \varepsilon(s))\,\epsilon_1\}\,\mathrm{d}s \end{aligned}$$

for all $t \geq T_3$. Thus, from (4.9), we have $\lim_{t \to +\infty} E(t) = -\infty$, which contradicts with (ii) of Proposition 2.2. Therefore, we see that there exists an $s_1 \geq T_3$ such that $E(s_1) < \epsilon_1$. Suppose that there exists an $s_2 > s_1$ such that $E(s_2) > \epsilon_1 + \beta^+ M_\epsilon \omega_2 \epsilon_2$. Then, we see that there necessarily exists an $s_3 \in (s_1, s_2)$ such that $E(s_3) = \epsilon_1$ and $E(t) > \epsilon_1$ for all $t \in (s_3, s_2]$. Let n be an integer such that $s_2 \in [s_3 + n\omega_2, s_3 + (n+1)\,\omega_2]$. Then, from (4.9), we have

$$\begin{aligned} &\epsilon_1 + \beta^+ M_\epsilon \omega_2 \epsilon_2 \\ &\quad < E(s_2) \\ &\quad = E(s_3) + \int_{s_3}^{s_2} \{\beta(s)\,(N^*(s) - E(s) - I(s) - R(s))\,I(s) - (\mu(s) + \varepsilon(s))\,E(s)\}\,\mathrm{d}s \\ &\quad < \epsilon_1 + \left\{\int_{s_3}^{s_3+n\omega_2} + \int_{s_3+n\omega_2}^{s_2}\right\} \{\beta(s) M_\epsilon \epsilon_2 - (\mu(s) + \varepsilon(s))\,\epsilon_1\}\,\mathrm{d}s \\ &\quad < \epsilon_1 + \int_{s_3+n\omega_2}^{s_2} \beta(s) M_\epsilon \epsilon_2 \ \mathrm{d}s \\ &\quad < \epsilon_1 + \beta^+ M_\epsilon \omega_2 \epsilon_2, \end{aligned}$$

which is a contradiction. Therefore, we see that

$$E(t) \leq \epsilon_1 + \beta^+ M_\epsilon \omega_2 \epsilon_2 \tag{4.12}$$

for all $t \geq s_1$. In a similar way, from (4.10), we can show that there exists $\tilde{s}_1 \geq T_3$ such that

$$R(t) \leq \epsilon_3 + \gamma^+ \omega_2 \epsilon_2 \tag{4.13}$$

for all $t \geq \tilde{s}_1$. Now, from Lemma 4.3, there exists $T_4 \geq \max(s_1, \tilde{s}_1)$ such that $W(p,t) = pE(t) - I(t) \leq 0$ for all $t \geq T_4$. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(t) &= \{\beta(t)\left(N^*(t) - E(t) - I(t) - R(t)\right)I(t) - \left(\mu\left(t\right) + \varepsilon\left(t\right)\right)E(t)\} \\ &\geq E(t)\left\{\beta(t)\left(N^*(t) - E(t) - I(t) - R(t)\right)p - \left(\mu\left(t\right) + \varepsilon\left(t\right)\right)\right\} \\ &\geq E(t)\left\{\beta(t)\left(N^*(t) - \epsilon_1 - k\epsilon_2 - \epsilon_3\right)p - \left(\mu\left(t\right) + \varepsilon\left(t\right)\right)\right\}\end{aligned}$$

since it follows from (4.11)-(4.13) that $E(t) + I(t) + R(t) \leq \epsilon_1 + k\epsilon_2 + \epsilon_3$ for all $t \geq T_4$. Hence, we have

$$E(t) \geq E(T_4)\exp\left(\int_{T_4}^{t}\left\{\beta(s)\left(N^*(s) - \epsilon_1 - k\epsilon_2 - \epsilon_3\right)p - \left(\mu\left(s\right) + \varepsilon\left(s\right)\right)\right\}\mathrm{d}s\right).$$

It follows from (4.7) that $\lim_{t\to+\infty} E(t) = +\infty$ and this contradicts with the boundedness of E, stated in (ii) of Proposition 2.2. Thus, we see that our claim $\limsup_{t\to+\infty} I(t) > \epsilon_2$ is true.

Next, we prove

$$\liminf_{t\to+\infty} I(t) \geq I_1,$$

where $I_1 > 0$ is a constant given in the following lines. From inequalities (4.7)-(4.9) and (ii) of Assumption 2.1, we see that there exist constants $\tilde{T}_3\,(\geq T_2)$, $\lambda_2 > 0$ and $\eta_2 > 0$ such that

$$\int_t^{t+\lambda_3}\left\{\beta(s)M_\epsilon\epsilon_2 - \left(\mu(s) + \varepsilon(s)\right)\epsilon_1\right\}\mathrm{d}s < -M_\epsilon, \tag{4.14}$$

$$\int_t^{t+\lambda_3}\left\{\gamma(s)\epsilon_2 - \left(\mu(s) + \delta(s)\right)\epsilon_3\right\}\mathrm{d}s < -M_\epsilon, \tag{4.15}$$

$$\int_t^{t+\lambda_3}\left\{\beta(s)\left(N^*(s) - \epsilon_1 - k\epsilon_2 - \epsilon_3\right)p - \left(\mu(s) + \varepsilon(s)\right)\right\}\mathrm{d}s > \eta_2, \tag{4.16}$$

$$\int_t^{t+\lambda_3}\beta(s)\mathrm{d}s > \eta_2 \tag{4.17}$$

for all $\lambda_3 \geq \lambda_2$ and $t \geq \tilde{T}_3$. Let $C > 0$ be a constant satisfying

$$e^{-(\mu^+ + \varepsilon^+)\lambda_2}m_\epsilon v_2\eta_2 e^{C\eta_2} > \epsilon_1 + \beta^+ M_\epsilon\omega_2\epsilon_2, \tag{4.18}$$

where $v_2 = \epsilon_2 e^{-(\gamma^+ + \mu^+)2\lambda_2}$. Since we proved $\limsup_{t\to+\infty} I(t) > \epsilon_2$, there are only two possibilities as follows:

(i) $I(t) \geq \epsilon_2$ for all $t \geq \exists\tilde{T}_4 \geq \tilde{T}_3$.

(ii) $I(t)$ oscillates about ϵ_2 for large $t \geq \tilde{T}_3$.

In case (i), we have $\liminf_{t\to+\infty} I(t) \geq \epsilon_2 =: I_1$. In case (ii), there necessarily exist two constants $t_1,\ t_2 \geq \tilde{T}_3$ $(t_2 \geq t_1)$ such that

$$\begin{cases} I(t_1) = I(t_2) = \epsilon_2, \\ I(t) < \epsilon_2 \quad \text{for all } t \in (t_1, t_2). \end{cases}$$

Suppose that $t_2 - t_1 \leq C + 2\lambda_2$. Then, from (1.1) we have

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} \geq -\left(\mu^+ + \gamma^+\right)I(t). \tag{4.19}$$

Hence, we obtain

$$I(t) \geq \ I(t_1)\exp\left(\int_{t_1}^{t} -\left(\mu^+ + \gamma^+\right)\mathrm{d}s\right) \geq \ \epsilon_2 e^{-(\mu^+ + \gamma^+)(C+2\lambda_2)} \ := \ I_1 \tag{4.20}$$

for all $t \in (t_1, t_2)$. Suppose that $t_2 - t_1 > C + 2\lambda_2$. Then, from (4.19), we have

$$I(t) \geq \epsilon_2 e^{-(\mu^+ + \gamma^+)(C+2\lambda_2)} = I_1$$

for all $t \in (t_1, t_1 + C + 2\lambda_2)$. Now, we are in a position to show that $I(t) \geq I_1$ for all $t \in [t_1 + C + 2\lambda_2, t_2)$. Suppose that $E(t) \geq \epsilon_1$ for all $t \in [t_1, t_1 + \lambda_2]$. Then, from (4.14), we have

$$\begin{aligned} E(t_1 + \lambda_2) &\leq E(t_1) + \int_{t_1}^{t_1+\lambda_2} \{\beta(s) M_\epsilon \epsilon_2 - (\mu(s) + \varepsilon(s))\,\epsilon_1\}\, ds \\ &< M_\epsilon - M_\epsilon = 0, \end{aligned}$$

which is a contradiction. Therefore, there exists an $s_4 \in [t_1, t_1 + \lambda_2]$ such that $E(s_4) < \epsilon_1$. Then, as in the proof of $\limsup_{t\to+\infty} I(t) > \epsilon_2$, we can show that $E(t) \leq \epsilon_1 + \beta^+ M_\epsilon \omega_2 \epsilon_2$ for all $t \geq s_4$. Similarly, from (4.15), we can show that there exists an $\tilde{s}_4 \in [t_1, t_1 + \lambda_2]$ such that $R(t) \leq \epsilon_3 + \gamma^+ \omega_2 \epsilon_2$ for all $t \geq \tilde{s}_4$. Thus, we have

$$E(t) \leq \epsilon_1 + \beta^+ M_\epsilon \omega_2 \epsilon_2 \quad \text{and} \quad R(t) \leq \epsilon_3 + \gamma^+ \omega_2 \epsilon_2 \tag{4.21}$$

for all $t \geq t_1 + \lambda_2 \geq \max(s_4, \tilde{s}_4)$. From (4.19), we have

$$I(t) \geq v_2 = \epsilon_2 e^{-(\mu^+ + \gamma^+)2\lambda_2} \tag{4.22}$$

for all $t \in [t_1, t_1 + 2\lambda_2]$. Thus, from (4.8), (4.21) and (4.22), we have

$$\begin{aligned} \frac{\mathrm{d}E(t)}{\mathrm{d}t} &= \beta(t)\,(N^*(t) - E(t) - I(t) - R(t))\,I(t) - (\mu(t) + \varepsilon(t))\,E(t) \\ &\geq \beta(t) m_\epsilon v_2 - \left(\mu^+ + \varepsilon^+\right) E(t) \end{aligned}$$

for all $t \in [t_1 + \lambda_2, t_1 + 2\lambda_2]$. Hence, from (4.17),

$$\begin{aligned} &E(t_1 + 2\lambda_2) \\ &\geq e^{-(\mu^+ + \varepsilon^+)(t_1+2\lambda_2)} \left\{ E(t_1 + \lambda_2) e^{(\mu^+ + \varepsilon^+)(t_1+\lambda_2)} + \int_{t_1+\lambda_2}^{t_1+2\lambda_2} \beta(s) m_\epsilon v_2 e^{(\mu^+ + \varepsilon^+)s} \mathrm{d}s \right\} \\ &\geq e^{-(\mu^+ + \varepsilon^+)(t_1+2\lambda_2)} \int_{t_1+\lambda_2}^{t_1+2\lambda_2} \beta(s) m_\epsilon v_2 e^{(\mu^+ + \varepsilon^+)s} \mathrm{d}s \\ &\geq e^{-(\mu^+ + \varepsilon^+)\lambda_2} \eta_2 m_\epsilon v_2. \end{aligned} \tag{4.23}$$

Now we suppose that there exists a $t_0 > 0$ such that $t_0 \in (t_1 + C + 2\lambda_2, t_2)$, $I(t_0) = I_1$ and $I(t) \geq I_1$ for all $t \in [t_1, t_0]$. Note that from Lemma 4.3, without loss of generality, we can assume that t_1 is so large that $W(p,t) = pE(t) - I(t) \leq 0$ for all $t \geq t_1 + 2\lambda_2$. Then, from (4.21), we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} E(t) &= \{\beta(t)\,(N^*(t) - E(t) - I(t) - R(t))\,I(t) - (\mu(t) + \varepsilon(t))\,E(t)\} \\ &\geq E(t)\,\{\beta(t)\,(N^*(t) - E(t) - I(t) - R(t))\,p - (\mu(t) + \varepsilon(t))\} \\ &\geq E(t)\,\{\beta(t)\,(N^*(t) - \epsilon_1 - k\epsilon_2 - \epsilon_3)\,p - (\mu(t) + \varepsilon(t))\} \end{aligned}$$

for all $t \in (t_1 + 2\lambda_2, t_2)$. Thus, from (4.16) and (4.23), we have

$$\begin{aligned} &E(t_0) \\ &\geq E(t_1+2\lambda_2)\exp\left(\int_{t_1+2\lambda_2}^{t_0}\{\beta(s)\left(N^*(s)-\epsilon_1-k\epsilon_2-\epsilon_3\right)p-(\mu(s)+\varepsilon(s))\}\,\mathrm{d}s\right) \\ &\geq e^{-(\mu^+ + \varepsilon^+)\lambda_2}\eta_2 m_\epsilon v_2 e^{C\eta_2}. \end{aligned}$$

Thus, from (4.21), we have

$$\epsilon_1 + \omega_2\beta^+ M_\epsilon \epsilon_2 \geq e^{-(\mu^+ + \varepsilon^+)\lambda_2}\eta_2 m_\epsilon v_2 e^{C\eta_2},$$

which contradicts with (4.18). Therefore, $I(t) \geq I_1$ for all $t \in [t_1 + C + 2\lambda_2, t_2)$, which implies $\liminf_{t\to+\infty} I(t) \geq I_1$.

Since $\limsup_{t\to+\infty} I(t) \leq \limsup_{t\to+\infty} N^*(t) < M < +\infty$, the infectious population of system (1.1) is permanent. □

5. Applications

In this section, we consider some special cases of system (1.1). Applying Theorems 3.2 and 4.2, we derive explicit conditions for the extinction and permanence of infectious population of system (1.1).

First, we assume that all coefficients of system (1.1) are given by identically constant functions. Then, (1.1) becomes an autonomous system. We show that, in this case, our results obtained in Sections 3 and 4 become a well-known threshold-type result formulated by the basic reproduction number R_0 given as in (1.3).

For $p > 0$ we define

$$R(p) := \beta\frac{\Lambda}{\mu}p - (\mu + \varepsilon), \qquad R^*(p) := \varepsilon\frac{1}{p} - (\mu + \gamma)$$

and

$$G(p) := \beta\frac{\Lambda}{\mu}p + \gamma - \left(1 + \frac{1}{p}\right)\varepsilon.$$

Then, one can see that $R_i(\lambda, p) = R(p)$, $R_i^*(\lambda, p) = R^*(p)$ $(i = 1, 2)$ and $G(p, t) = G(p)$ in the autonomous case.

Proposition 5.1 *Suppose that functions Λ, β, μ, ε, γ and δ of system (1.1) are identically positive constant functions. Then we have*

(i) *There exists $p > 0$ such that $R(p) < 0$, $R^*(p) < 0$ and $G(p) < 0$ if and only if $R_0 < 1$.*

(ii) *There exists $p > 0$ such that $R(p) > 0$, $R^*(p) > 0$ and $G(p) < 0$ if and only if $R_0 > 1$.*

Here R_0 is defined as in (1.3).

PROOF. We only prove (i) because (ii) is proved in a similar manner. Suppose that there exists $p > 0$ such that $R(p) < 0$, $R^*(p) < 0$ and $G(p) < 0$ hold. Then, it follows from $R(p) < 0$ and $R^*(p) < 0$ that

$$\frac{\varepsilon}{\mu + \gamma} < p < \frac{(\mu + \varepsilon)\mu}{\beta\Lambda}. \tag{5.1}$$

Hence we obtain $R_0 < 1$. Suppose, on the contrary, that $R_0 < 1$. Then, it is obvious that there exists $p > 0$ such that (5.1) holds. Since we have

$$G\left(\frac{\varepsilon}{\mu+\gamma}\right) = \frac{\varepsilon\beta\Lambda}{(\mu+\gamma)\,\mu} + \gamma - \left(1 + \frac{\mu+\gamma}{\varepsilon}\right)\varepsilon = (\mu+\varepsilon)\,(R_0 - 1) < 0,$$

there exists $p > 0$ being close enough to $\varepsilon/\,(\mu+\gamma)$ so that both (5.1) and $G(p) < 0$ hold. For such p we have $R(p) < 0$, $R^*(p) < 0$ and $G(p) < 0$. □

Proposition 5.1 implies that our conditions for the extinction and permanence for the nonautonomous system (1.1) cover the threshold-type result in the autonomous case.

Next we focus on the case where only μ, ε and γ are constant functions. We have the following threshold-type results.

Corollary 5.2 *Suppose that μ, ε and γ of system (1.1) are identically positive constant functions. Then, we have*

(i) *The infectious population of system (1.1) is extinct if there exists $T_1 > 0$ such that*

$$\frac{\varepsilon\beta(t)N^*(t)}{(\mu+\varepsilon)\,(\mu+\gamma)} < 1 \tag{5.2}$$

for all $t \geq T_1$.

(ii) *The infectious population of system (1.1) is permanent if there exists $T_1 > 0$ such that*

$$\frac{\varepsilon\beta(t)N^*(t)}{(\mu+\varepsilon)\,(\mu+\gamma)} > 1 \tag{5.3}$$

for all $t \geq T_1$.

PROOF. We only prove (i) because (ii) is proved in a similar manner. For the proof of (i), it suffices to show that there exist constants $p > 0$ and $\lambda > 0$ such that (3.1) and (3.2) hold and $G\,(p,t) < 0$ for all $t \geq T_1$. From (5.2), we have

$$\frac{\varepsilon}{\mu+\gamma} < \frac{\mu+\varepsilon}{\limsup_{t\to+\infty}\int_t^{t+1}\beta(s)N^*(s)\mathrm{d}s}.$$

We choose $p > 0$ such that

$$\frac{\varepsilon}{\mu+\gamma} < p < \frac{\mu+\varepsilon}{\limsup_{t\to+\infty}\int_t^{t+1}\beta(s)N^*(s)\mathrm{d}s}. \tag{5.4}$$

Then one can see that (3.1) and (3.2) with $\lambda = 1$ hold. Next we show that for such p, we have $G\,(p,t) < 0$ for all $t \geq T_1$. In fact, from (5.2), we have

$$\beta(t)N^*(t)\frac{\varepsilon}{\mu+\gamma} - (\mu+\varepsilon) = \beta(t)N^*(t)\frac{\varepsilon}{\mu+\gamma} + \gamma - \left(1 + \frac{\mu+\gamma}{\varepsilon}\right)\varepsilon < 0$$

for all $t \geq T_1$. Thus, one can find small enough $\bar{\varepsilon} > 0$ such that $G\,(p,t) < 0$ holds for

$$p \in \left(\frac{\varepsilon}{\mu+\gamma}, \frac{\varepsilon}{\mu+\gamma} + \bar{\varepsilon}\right) \subset \left(\frac{\varepsilon}{\mu+\gamma},\ \frac{\mu+\varepsilon}{\limsup_{t\to+\infty}\int_t^{t+1}\beta(s)N^*(s)\mathrm{d}s}\right)$$

and $t \geq T_1$, due to the continuity of G with respect to p. □

It is easily seen that the existence of $T_1 > 0$ such that (5.2) or (5.3) hold for all $t \geq T_1$ is a sufficient condition for

$$\limsup_{t\to+\infty} \int_t^{t+\lambda} \{\varepsilon\beta(s)N^*(s) - (\mu+\varepsilon)(\mu+\gamma)\}\,\mathrm{d}s < 0 \tag{5.5}$$

or

$$\liminf_{t\to+\infty} \int_t^{t+\lambda} \{\varepsilon\beta(s)N^*(s) - (\mu+\varepsilon)(\mu+\gamma)\}\,\mathrm{d}s > 0 \tag{5.6}$$

with $\lambda = 1$, respectively, where (5.5) and (5.6) are conditions proposed in Questions 1 and 2 in [22] for the extinction and permanence of infectious population of system (1.1), respectively. However, one can see that those conditions do not imply the conditions given in Corollary 5.2. In fact, conditions (5.5) and (5.6) are not suitable as a threshold condition for the global dynamics of system (1.1) because they overestimate the value of the basic reproduction number R_0 even in the situation where only function $\beta(t)$ is periodic and other coefficients are constant functions (see Section 5.1.2 of [2]) and it was shown in [12] that whether the infectious population of system (1.1) is extinct or permanent is perfectly determined by R_0 in the periodic case.

6. Numerical examples

In this section we perform numerical simulations in order to verify the validity of Theorems 3.2 and 4.2 and to show that in some special cases, our results can improve the previous results for the permanence and extinction of system (1.1) obtained by Zhang and Teng [22].

Fix

$$\Lambda(t) \equiv 1, \quad \mu(t) \equiv 1, \quad \varepsilon(t) = 0.3\,(1 + 0.5\cos(2\pi t))\,, \quad \gamma(t) = 0.5\,(1 + 0.5\cos(2\pi t))$$

and $\delta(t) \equiv 0.1$. Then, from (2.1), we have $\lim_{t\to+\infty} N^*(t) = 1$. Here we assume $N^*(0) = 1$ and thus $N^*(t) \equiv 1$.

Let $\beta(t) = 6.49\,(1 + 0.5\cos(2\pi t))$. Then, system (1.1) becomes periodic with period 1. We choose $\lambda = 1$ and $p = 0.20011$. Then we have

$$\begin{aligned} R_1(\lambda, p) &= \int_0^1 \{6.49\,(1 + 0.5\cos(2\pi s)) \times 0.20011 - (1 + 0.3\,(1 + 0.5\cos(2\pi s)))\}\,\mathrm{d}s \\ &\backsimeq -0.0012861\cdots < 0, \end{aligned}$$

$$\begin{aligned} R_1^*(\lambda, p) &= \int_0^1 \left\{0.3\,(1 + 0.5\cos(2\pi s)) \times \frac{1}{0.20011} - (1 + 0.5\,(1 + 0.5\cos(2\pi t)))\right\}\mathrm{d}s \\ &\backsimeq -0.000824546\cdots < 0 \end{aligned}$$

and

$$\begin{aligned} G(p,t) = &\, 6.49\,(1 + 0.5\cos(2\pi t)) \times 0.20011 + 0.5\,(1 + 0.5\cos(2\pi t)) \\ &- \left(1 + \frac{1}{0.20011}\right) \times 0.3\,(1 + 0.5\cos(2\pi t)) \\ \backsimeq &\, -0.000461554\,(1 + 0.5\cos(2\pi t)) < 0 \end{aligned}$$

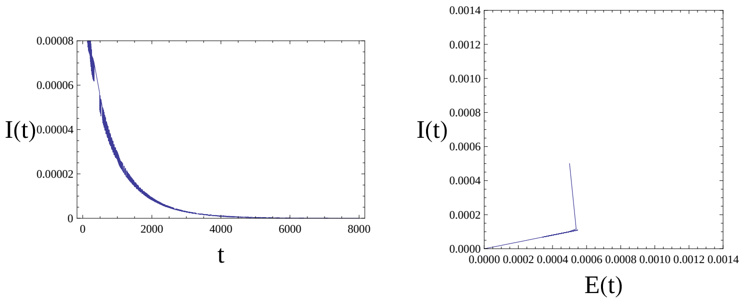

Figure 1. The first example of dynamics of $I(t)$ and $E(t)$ of system (1.1) ($E(0) = I(0) = 0.0005$). $\beta(t) = 6.49\,(1 + 0.5\cos(2\pi t))$ and $p = 0.20011$. The infectious population is extinct.

for all $t > 0$. From (i) of Theorem 3.2, we see that the infectious population of system (1.1) is extinct, see Figure 1 for a numerical simulation of solution behavior. In this example we have

$$\frac{\int_t^{t+\lambda} \beta(s)N^*(s)\mathrm{d}s}{\int_t^{t+\lambda} \mu(s)\mathrm{d}s} = \frac{\int_0^1 6.49\,(1+0.5(\cos(2\pi s)))\,\mathrm{d}s}{1} = 6.49 > 1.$$

This implies that a sufficient condition proposed in Theorem 5.1 in [22] for the extinction of infectious population does not hold. Thus their criterion can not determine the extinction of infectious population in this example.

Next we set $\beta(t) = 6.51\,(1 + 0.5\cos(2\pi t))$. We choose $\lambda = 1$ and $p = 0.1997$. Then,

$$\begin{aligned} R_2(\lambda, p) &= \int_0^1 \{6.51\,(1+0.5\cos(2\pi s)) \times 0.1997 - (1 + 0.3\,(1+0.5\cos(2\pi s)))\}\,\mathrm{d}s \\ &\simeq 0.000047\cdots > 0, \end{aligned}$$

$$\begin{aligned} R_2^*(\lambda, p) &= \int_0^1 \left\{0.3\,(1+0.5\cos(2\pi s)) \times \frac{1}{0.1997} - (1 + 0.5\,(1+0.5\cos(2\pi t)))\right\}\mathrm{d}s \\ &\simeq 0.00225338\cdots > 0 \end{aligned}$$

and

$$\begin{aligned} G(p, t) &= 6.51\,(1+0.5\cos(2\pi t)) \times 0.1997 + 0.5\,(1+0.5\cos(2\pi t)) \\ &\quad - \left(1 + \frac{1}{0.1997}\right) \times 0.3\,(1+0.5\cos(2\pi t)) \\ &\simeq -0.00220638\,(1+0.5\cos(2\pi t)) < 0 \end{aligned}$$

for all $t > 0$. Thus, from (ii) of Theorem 4.2, we see that the infectious population of system (1.1) is permanent, see Figure 2 for a numerical simulation of solution behavior. On the other hand, one can compute

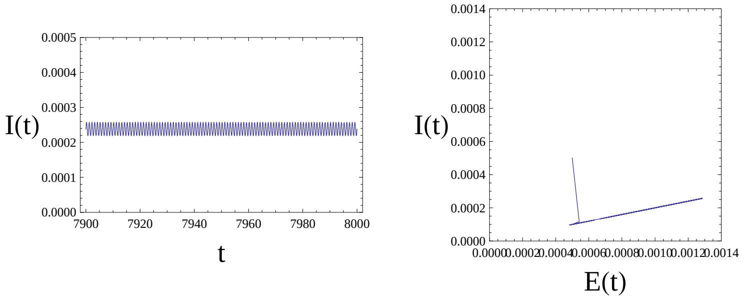

Figure 2. The second example of dynamics of $I(t)$ and $E(t)$ of system (1.1) ($E(0) = I(0) = 0.0005$). $\beta(t) = 6.51\,(1 + 0.5\cos(2\pi t))$ and $p = 0.1997$. The infectious population is permanent.

$$\frac{\int_t^{t+\lambda} 2\sqrt{\beta(s)\varepsilon(s)N^*(s)}\mathrm{d}u}{\int_t^{t+\lambda} (\mu(s) + \varepsilon(s) + \mu(s) + \gamma(s))\,\mathrm{d}u}$$
$$= \frac{\int_0^1 2\sqrt{6.51\,(1 + 0.5\cos(2\pi s)) \times 0.3\,(1 + 0.5\cos(2\pi s))}\mathrm{d}s}{\int_0^1 (2 + 0.8\,(1 + 0.5\cos(2\pi s)))\,\mathrm{d}s} \backsimeq 0.99821 < 1.$$

This implies, similar to the previous example, that a sufficient condition proposed in Theorem 4.1 in [22] for the permanence of infectious population fails in this example.

7. Discussion

In this paper, we have investigated the global dynamics of a nonautonomous SEIRS epidemic model (1.1). We obtain new sufficient conditions for the extinction and permanence of infectious population of system (1.1) in Theorems 3.2 and 4.2, respectively. We analyze the dynamics of system (1.1) via considering the behavior of a function defined as in (2.3), see Lemmas 2.3 and 4.3.

In Section 5, we prove that when every parameter of system (1.1) is given as a constant parameter, our conditions in Theorems 3.2 and 4.2 become the threshold condition by the basic reproduction number R_0, We remark that conditions given in Theorems 4.1 and 5.1 in [22] for the permanence and extinction do not give a threshold-type condition even in the autonomous case. In the same section we also discuss a relation between our results and open problems proposed in [22]. For a special case, we show that our conditions are sufficient, but not necessary for (5.5) and (5.6), which were conjectured as conditions for the permanence and extinction of infectious population. For the case in which every parameter is given as a periodic function, in [12] it was proved that the basic reproduction number R_0 works as a threshold parameter to determine the global stability of the disease-free equilibrium and permanence of infectious population. An approximation method for the basic reproduction number R_0 in [3] shows that the conjectured condition does not determine the permanence and extinction completely, see Section 5 in [12] for the detail.

In Section 6 we provide numerical examples to illustrate the validity of our results. In those examples we show that conditions in Theorems 4.1 and 5.1 in [22] for the permanence and extinction of infectious population of system (1.1) are not satisfied.

One may argue that our conditions for the permanence and extinction may not sharp. It is still an open problem that if the basic reproduction number R_0 for (1.1) works as a threshold parameter to determine the permanence and extinction of infectious population like in the autonomous system.

Acknowledgements

The authors would like to thank Prof. Yoshiaki Muroya and the reviewers very much for their valuable comments and suggestions on an earlier version of this paper, and Prof. Nicolas Bacaër for having a discussion with us at the conference "R0 and related concepts: methods and illustrations" held at Paris on 29-31, October, 2008. TK was supported by Japan Society for the Promotion of Science (JSPS), No. 222176. YN was supported by Spanish Ministry of Science and Innovation (MICINN), MTM2010-18318.

References

[1] R. Anderson, R. May, Infectious diseases of humans: dynamics and control, Oxford University Press, USA, 1991.

[2] N. Bacaër, Approximation of the basic reproduction number R_0 for vector-borne diseases with a periodic vector population, Bull. Math. Biol. 69 (3) (2007) 1067–1091.

[3] N. Bacaër, S. Guernaoui, The epidemic threshold of vector-borne diseases with seasonality. The case of cutaneous leishmaniasis in Chichaoua, Morocco, J. Math. Biol. 53 (3) (2006) 421–436.

[4] N. Bacaër, R. Ouifki, Growth rate and basic reproduction number for population models with a simple periodic factor, Math. Biosci. 210 (2) (2007) 647–658.

[5] O. Diekmann, J. A. P. Heesterbeek, Mathematical epidemiology of infectious diseases, Wiley Series in Mathematical and Computational Biology, John Wiley & Sons Ltd., Chichester, 2000, model building, analysis and interpretation.

[6] M. W. Hirsch, H. L. Smith, X.-Q. Zhao, Chain transitivity, attractivity, and strong repellors for semidynamical systems, J. Dynam. Differential Equations 13 (1) (2001) 107–131.

[7] H. Inaba, On a new perspective of the basic reproduction number in heterogeneous environments, J. Math. Biol.DOI 10.1007/s00285-011-0463-z.

[8] M. Y. Li, J. R. Graef, L. Wang, J. Karsai, Global dynamics of a SEIR model with varying total population size, Math. Biosci. 160 (2) (1999) 191–213.

[9] M. Y. Li, J. S. Muldowney, Global stability for the SEIR model in epidemiology, Math. Biosci. 125 (2) (1995) 155–164.

[10] W. M. Liu, H. W. Hethcote, S. A. Levin, Dynamical behavior of epidemiological models with nonlinear incidence rates, J. Math. Biol. 25 (4) (1987) 359–380.

[11] J. Ma, Z. Ma, Epidemic threshold conditions for seasonally forced SEIR models, Math. Biosci. Eng. 3 (1) (2006) 161–172 (electronic).

[12] Y. Nakata, T. Kuniya, Global dynamics of a class of seirs epidemic models in a periodic environment, J. Math. Anal. Appl. 363 (2010) 230–237.

[13] M. Song, W. Ma, Y. Takeuchi, Permanence of a delayed SIR epidemic model with density dependent birth rate, J. Comput. Appl. Math. 201 (2) (2007) 389–394.

[14] C. Sun, Y. Lin, S. Tang, Global stability for an special SEIR epidemic model with nonlinear incidence rates, Chaos Solitons Fractals 33 (1) (2007) 290–297.

[15] H. R. Thieme, Uniform weak implies uniform strong persistence for non-autonomous semiflows, Proc. Amer. Math. Soc. 127 (8) (1999) 2395–2403.

[16] H. R. Thieme, Uniform persistence and permanence for non-autonomous semiflows in population biology, Math. Biosci. 166 (2) (2000) 173–201.

[17] H. R. Thieme, Spectral bound and reproduction number for infinite-dimensional population structure and time heterogeneity, SIAM J. Appl. Math. 70 (1) (2009) 188–211.

[18] P. van den Driessche, J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, Math. Biosci. 180 (2002) 29–48, john A. Jacquez memorial volume.
[19] W. Wang, X.-Q. Zhao, An epidemic model in a patchy environment, Math. Biosci. 190 (1) (2004) 97–112.
[20] W. Wang, X.-Q. Zhao, Threshold dynamics for compartmental epidemic models in periodic environments, J. Dynam. Differential Equations 20 (3) (2008) 699–717.
[21] F. Zhang, X.-Q. Zhao, A periodic epidemic model in a patchy environment, J. Math. Anal. Appl. 325 (1) (2007) 496–516.
[22] T. Zhang, Z. Teng, On a nonautonomous SEIRS model in epidemiology, Bull. Math. Biol. 69 (8) (2007) 2537–2559.
[23] T. Zhang, Z. Teng, S. Gao, Threshold conditions for a non-autonomous epidemic model with vaccination, Appl. Anal. 87 (2) (2008) 181–199.
[24] X.-Q. Zhao, Dynamical systems in population biology, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 16, Springer-Verlag, New York, 2003.