HOMOGENEOUS SINGULARITY AND THE ALEXANDER POLYNOMIAL OF A PROJECTIVE PLANE CURVE

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Abstract. The Alexander polynomial of a plane curve is an important invariant in global theories on curves. However, it seems that this invariant and even a much stronger one the fundamental group of the complement of a plane curve may not distinguish non-reduced curves. In this article, we consider a general problem which concerns a hypersurface of the complex projective space $\mathbb{P}^n$ defined by an arbitrary homogeneous polynomial f. The singularity of f at the origin of $\mathbb{C}^{n+1}$ is studied, by means of the characteristic polynomials $\Delta_l(t)$ of the monodromy, and via the relation between the monodromy zeta function and the Hodge spectrum. Especially, we go further with $\Delta_1(t)$ in the case $n = 2$ and aim to regard it as an alternative object of the Alexander polynomial for f non-reduced. This work is based on knowledge of multiplier ideals and local systems.

1. Introduction

Let f be a homogeneous polynomial of degree d in $n+1$ variables with coefficients in $\mathbb{C}$, which defines a holomorphic function germ at the origin O of $\mathbb{C}^{n+1}$. In general, according to [22] and [18], the Milnor fiber of (f, O) is up to diffeomorphism a manifold $M = f^{-1}(\delta) \cap B_\varepsilon$, for $B_\varepsilon \subset \mathbb{C}^{n+1}$ a ball of radius ε around O and $0 < \delta \ll \varepsilon \ll 1$, which has the homotopy type of a bouquet of μ spheres of dimension n. Since here f is a homogeneous polynomial, however, $f^{-1}(\delta) \cap B_\varepsilon$ is a deformation retract of $f^{-1}(\delta) \cong f^{-1}(1)$, thus we may consider M as $f^{-1}(1)$. The monodromy $T : H^*(M, \mathbb{C}) \to H^*(M, \mathbb{C})$ of the singularity may be given explicitly to be the $\mathbb{C}$-linear endomorphism induced by the map $(x_0, \ldots, x_n) \mapsto (e^{\frac{2\pi i}{d}} x_0, \ldots, e^{\frac{2\pi i}{d}} x_n)$. It becomes classical for f being an isolated homogeneous singularity at O where many important invariants such as the Milnor number μ, the characteristic polynomials of T, the signature and Hodge numbers of M are computed completely in topological and algebraic methods as well as via mixed Hodge structures (cf. [23], [29]).

In the case where f is a reduced homogeneous polynomial, Esnault [12] introduced a method to compute the Betti numbers, the rank and the signature of the intersection matrices of the singularity (f, O), using mixed Hodge structures on cohomology groups of the Milnor fiber M and the existence of spectral sequences converging to the cohomology groups, together with resolution of singularity. The work by Esnault definitely inspired the study by Loeser-Vaquié [21] of the Alexander polynomial of a reduced complex projective plane curve, where they provided a formula for the Alexander polynomial of such a curve which generalizes the previous one by Libgober [19, 20]. It is likely that the approaches of Libgober in [20] and Loeser-Vaquié in [21], as well as the work by Nadel in [25], are also starting points of the studies on multiplier ideals and local systems, which were thereafter studied strongly by Esnault-Viehweg [13], Ein-Lazarsfeld [11], Demailly [8], Kollar [16], Budur [3, 5], Budur-Saito [7].

Due to the development of the theory of multiplier ideals and local systems, Budur [6] gives an explicit description of the local system of the complement in $\mathbb{P}^n$ of the divisor defined by a homogeneous polynomial f *without the condition of reducedness*. In the present work, we use Budur's article [6] to study the characteristic polynomials, the Hodge spectrum and the monodromy zeta function of an

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arbitrary homogeneous hypersurface singularity. Let us now review in a few words what we shall do in the rest. We denote by D the closed subscheme of $\mathbb{P}^n$ defined by the zero locus of a degree d homogeneous polynomial f and by U the complement of D in $\mathbb{P}^n$. Then, as shown in [5, 6], there is an eigensheaf decomposition of the $\mathcal{O}_U$-module sheaf $\sigma_*\mathbb{C}_M$ into the unitary local systems $\mathcal{V}_k$ on U given by the eigensheaf of T with respect to the eigenvalue $e^{-\frac{2\pi i k}{d}}$, $0 \leq k \leq d-1$, where σ is the canonical projection $M \to U$. On cohomology level, using the Leray spectral sequence, one gets $H^l(U, \mathcal{V}_k)$ to be the eigenspace of the monodromy T on $H^l(M, \mathbb{C})$ with respect to the eigenvalue $e^{-\frac{2\pi i k}{d}}$, for any l in $\mathbb{N}$ (cf. [6]). Assume that D has r distinct irreducible components D_i and that m_i is the multiplicity of D_i in D. By [6, Lemma 4.2], for each k, modulo the identification RH in [5, Theorem 1.2], the local system $\mathcal{V}_k$ is nothing but the element $(\mathcal{O}_{\mathbb{P}^n}(\sum_{j=1}^r \{\frac{km_j}{d}\} d_j), (\{\frac{km_1}{d}\}, \dots, \{\frac{km_r}{d}\}))$ in the group $\mathrm{Pic}^\tau(\mathbb{P}^n, D)$ of *realizations of boundaries of* $\mathbb{P}^n$ *on* D (cf. [5, Definition 1.1]).

The problem of computing the complex dimension of $H^l(U, \mathcal{V}_k)$ can be solved completely under the works by Budur [3, 4, 5, 6] in terms of resolution of singularity. Let $\pi : Y \to \mathbb{P}^n$ be a log-resolution of the family $\{D_1, \dots, D_r\}$, with exceptional divisor $E = \pi^*(\bigcup_{j=1}^r D_j) = \sum_{j \in A} N_j E_j$, E_j being irreducible. Denote by $\mathcal{L}^{(k)}$ the invertible sheaf $\pi^*\mathcal{O}_{\mathbb{P}^n}\left(\sum_{j=1}^r \left\{\frac{km_j}{d}\right\} d_j\right) \otimes \mathcal{O}_Y\left(-\left\lfloor \sum_{j=1}^r \left\{\frac{km_j}{d}\right\} \pi^* D_j \right\rfloor\right)$ on Y. As proved in Lemma 3.10, we get

$$\dim_{\mathbb{C}} H^l(U, \mathcal{V}_{d-k}) = \sum_{p \geq 0} \dim_{\mathbb{C}} H^{l-p}(Y, \Omega_Y^p(\log E) \otimes {\mathcal{L}^{(k)}}^{-1}),$$

for $l \geq 0$ and $1 \leq k \leq d$, from which the characteristic polynomial $\Delta_l(t)$ of T on $H^l(M, \mathbb{C})$ follows. Observe that this description is not really useful in practice since it is too difficult to compute the number on the right hand side of the previous equality. However, in the special case where $n = 2$ and $l = 1$, we obtain in Theorem 4.3 an explicit formula for $\Delta_1(t)$ in terms of the multiplier ideal of $\sum_{j=1}^r \{\frac{km_j}{d}\} C_j$, where we write C_j instead of D_j when D is a curve C. This is the most important result of the article.

There is another result in the present article, Theorem 4.4, which discusses the relation between the Hodge spectrum and the monodromy zeta function of a homogeneous singularity. This can be realized directly from [6, Proposition 4.3] and Proposition 3.8. In order to come back to the Alexander polynomial of a complex projective plane curve, we mention in Section 5 the case f reduced, where Loeser-Vaquié's formula [21] is recovered and Artal Bartolo's method to compute the Alexander polynomial of the curve [2] is recalled with some small remarks.

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2. Multiplier ideals and Hodge spectrum

2.1. **Multiplier ideals.** Let X be a smooth complex algebraic variety and let $D = \{D_1, \dots, D_r\}$ be a family of closed subschemes of X. A *log-resolution* of the family D is a proper birational morphism $\pi : Y \to X$, where Y is a smooth complex algebraic variety, such that the exceptional set $\mathrm{Ex}(\pi) := \{y \in Y \mid \pi \text{ is not biregular at } y\}$, the support $\mathrm{Supp}(\det \mathrm{Jac}_\pi)$ of the determinant of the Jacobian of π, the preimages $\pi^{-1}(D_j)$, $1 \leq j \leq r$, and the union $\mathrm{Ex}(\pi) \cup \mathrm{Supp}(\det \mathrm{Jac}_\pi) \cup \bigcup_{j=1}^r \pi^{-1}(D_j)$ are simple normal crossing divisors. The existence of such a log-resolution is proved by Hironaka. Let K_X (resp. K_Y) denote the canonical divisor of X (resp. Y). Then $K_{Y/X} := K_Y - \pi^* K_X$ is the divisor defined by $\det \mathrm{Jac}_\pi$, which is known as the canonical divisor of π. For any $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{Q}_{>0}^r$, we set

$$\mathcal{J}(X, \alpha D) := \pi_* \mathcal{O}_Y(K_{Y/X} - \lfloor \pi^*(\alpha D) \rfloor), \tag{2.1}$$

where $\alpha D := \sum_{j=1}^r \alpha_j D_j$, and $\lfloor \pi^*(\alpha D) \rfloor$ is the round-down of the coefficients of the irreducible components of the divisor $\pi^*(\alpha D)$. It is obvious that $\mathcal{J}(X, \alpha D)$ is a sheaf of ideals on X, which is an ideal of $\pi_* \mathcal{O}_Y(K_Y) = \mathcal{O}_X$.

Theorem 2.1 (Lazarsfeld [17]). *For any $\alpha \in \mathbb{Q}^r_{>0}$, the sheaf of ideals $\mathcal{J}(X, \alpha D)$ is independent of the choice of π, and $R^i \mathcal{J}(X, \alpha D) = 0$ for $i \geq 1$. The sheaf of ideals $\mathcal{J}(X, \alpha D)$ is called the* multiplier ideal *of αD.*

For instance, when $X = \mathbb{C}^n$ and D is defined by a monomial ideal I, by Howald's computation [14], the multiplier ideal $\mathcal{J}(X, \alpha D)$ is a monomial ideal generated by $x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ for all $(r_1, \ldots, r_n) \in \mathbb{N}^n$ such that $(\gamma_1 + 1, \ldots, \gamma_n + 1)$ is in the interior $\mathrm{Int}(\alpha\Gamma(I))$ of $\alpha\Gamma(I)$, where $\Gamma(I)$ is the Newton polyhedron of I.

Now let D be a closed subscheme of X. A *jumping number* of D in X is a number $\alpha \in \mathbb{Q}_{>0}$ such that $\mathcal{J}(X, \alpha D) \neq \mathcal{J}(X, (\alpha - \varepsilon)D)$ for all $\varepsilon > 0$. The *log canonical threshold* $\mathrm{lct}(X, D)$ of (X, D) is the smallest jumping number of D in X. In [24], Mustata proves that, with such a log-resolution π as above, $\mathrm{lct}(X, D) = \min_l\{(a_l + 1)/N_l\}$. To determine how a singular point affects a jumping number, Budur [3] introduces a notion of *inner jumping multiplicity.* By definition, the inner jumping multiplicity $m_{\alpha,\mathbf{p}}(D)$ of α at a closed point $\mathbf{p} \in D$ is the dimension of the complex vector space

$$\mathcal{K}_{\mathbf{p}}(X, \alpha D) := \mathcal{J}(X, (\alpha - \varepsilon)D)/\mathcal{J}(X, (\alpha - \varepsilon)D + \delta\{\mathbf{p}\}),$$

for $0 < \varepsilon \ll \delta \ll 1$. If $m_{\alpha,\mathbf{p}}(D) \neq 0$, the number α is called an *inner jumping number* of (X, D) at $\mathbf{p}$. It is proved by Budur in [3, Proposition 2.8] that if α is an inner jumping number of (X, D) at $\mathbf{p}$, for some $\mathbf{p} \in D$, then α is a jumping number of (X, D). Furthermore, Budur can provide an explicit formula computing the number $m_{\alpha,\mathbf{p}}(D)$, which we recall as follows. Let $\pi : Y \to X$ be a log-resolution of the family $\{D, \{\mathbf{p}\}\}$, with $E = \pi^*(D) = \sum_{i\in A} N_i E_i$, E_i irreducible components, and, for $d \in \mathbb{N}_{>0}$, let $J_{d,\mathbf{p}} := \{i \in A \mid N_i \neq 0,\ d|N_i,\ \pi(E_i) = \mathbf{p}\}$ and $E_{d,\mathbf{p}} := \bigcup_{i\in J_{d,\mathbf{p}}} E_i$.

Proposition 2.2 (Budur [3], Proposition 2.7). *Assume $\alpha = \frac{k}{d}$, with k and d coprime positive integers. Then $m_{\alpha,\mathbf{p}}(D) = \chi(Y, \mathcal{O}_{E_{d,\mathbf{p}}}(K_{Y/X} - \lfloor(1 - \varepsilon)\alpha\pi^*D\rfloor))$, where χ is the sheaf Euler characteristic and $0 < \varepsilon \ll 1$.*

2.2. **Hodge spectrum.** Let X be a smooth complex variety of pure dimension n, let f be a regular function on X with zero locus $D \neq \emptyset$, and let $\mathbf{p}$ be a closed point in D_{red}. Fixing a smooth metric on X we may define a closed ball $B(\mathbf{p}, \varepsilon)$ around $\mathbf{p}$ in X and a punctured closed disc D^*_δ around the origin of $\mathbb{A}^1_{\mathbb{C}}$. It is well known (cf. [22]) that, for $0 < \delta \ll \varepsilon \ll 1$, the map

$$f : B(\mathbf{p}, \varepsilon) \cap f^{-1}(D^*_\delta) \to D^*_\delta$$

is a smooth locally trivial fibration, called Milnor fibration, whose diffeomorphism type is independent of such ε and δ. Denote the Milnor fiber $B(\mathbf{p}, \varepsilon) \cap f^{-1}(\delta)$ by $M_{\mathbf{p}}$, the geometric monodromy $M_{\mathbf{p}} \to M_{\mathbf{p}}$ and its cohomology level $H^*(M_{\mathbf{p}}, \mathbb{C}) \to H^*(M_{\mathbf{p}}, \mathbb{C})$ by the same symbol T.

Let $\mathrm{MHS}^{\mathrm{mon}}_{\mathbb{C}}$ be the abelian category of complex mixed Hodge structures endowed with an automorphism of finite order. For an object (H, T_H) of $\mathrm{MHS}^{\mathrm{mon}}_{\mathbb{C}}$, one defines its Hodge spectrum as a fractional Laurent polynomial

$$\mathrm{Hsp}(H, T_H) := \sum_{\alpha\in\mathbb{Q}} n_\alpha t^\alpha,$$

where $n_\alpha := \dim_{\mathbb{C}} Gr_F^{\lfloor\alpha\rfloor} H_{e^{2\pi i\alpha}}$, $H_{e^{2\pi i\alpha}}$ is the eigenspace of T_H with respect to the eigenvalue $e^{2\pi i\alpha}$, and F is the Hodge filtration. By [28] and [27], for any l, $H^l(M_{\mathbf{p}}, \mathbb{C})$ carries a canonical mixed Hodge structure, which is compatible with the semisimple part T_s of T so that $(H^l(M_{\mathbf{p}}, \mathbb{C}), T_s)$ is an object of $\mathrm{MHS}^{\mathrm{mon}}_{\mathbb{C}}$. As in [9, Section 4.3] and [3, Section 3], we set

$$\mathrm{Hsp}'(f, \mathbf{p}) := \sum_{j\in\mathbb{Z}} (-1)^j \mathrm{Hsp}(\widetilde{H}^{n-1+j}(M_{\mathbf{p}}, \mathbb{C}), T_s),$$

where we use the reduced cohomology $\widetilde{H}$ to present the vanishing cycle sheaf cohomology, since $\widetilde{H}^l(M_{\mathbf{p}}, \mathbb{C})_{e^{2\pi i\alpha}} = H^l(M_{\mathbf{p}}, \mathbb{C})_{e^{2\pi i\alpha}}$ if $l \neq 0$ or $\alpha \notin \mathbb{Z}$, $\widetilde{H}^0(M_{\mathbf{p}}, \mathbb{C})_1 = \mathrm{coker}(H^0(*, \mathbb{C}) \to H^0(M_{\mathbf{p}}, \mathbb{C})_1)$ (cf. also [7, Section 5.1]). Then the Hodge spectrum of f at $\mathbf{p}$, denoted by $\mathrm{Sp}(f, \mathbf{p})$, is the following

$$\mathrm{Sp}(f, \mathbf{p}) = t^n \iota(\mathrm{Hsp}'(f, \mathbf{p})),$$

where ι is given by $\iota(t^\alpha) = t^{-\alpha}$. Writing $\mathrm{Sp}(f, \mathbf{p}) = \sum_{\alpha\in\mathbb{Q}} n_{\alpha,\mathbf{p}}(f)t^\alpha$ one calls the coefficients $n_{\alpha,\mathbf{p}}(f)$ the *spectrum multiplicities* of f at $\mathbf{p}$. By [7, Proposition 5.2], $n_{\alpha,\mathbf{p}}(f) = 0$ if α is a rational number with $\alpha \leq 0$ or $\alpha \geq n$. Moreover, it implies from [6, Corollary 2.3] that, for $\alpha \in (0, n) \cap \mathbb{Q}$,

$$n_{\alpha,\mathbf{p}}(f) = \sum_{j\in\mathbb{Z}} (-1)^j \dim_{\mathbb{C}} Gr_F^{\lfloor n-\alpha\rfloor} H^{n-1+j}(M_{\mathbf{p}}, \mathbb{C})_{e^{-2\pi i\alpha}}. \tag{2.2}$$

Specially, using [9, Corollary 4.3.1] and important computations on multiplier ideals, Budur [3] proved the following result, which provides an effective way to compute $n_{\alpha,\mathbf{p}}(f)$, for $\alpha \in (0, 1] \cap \mathbb{Q}$.

Theorem 2.3 (Budur [3])**.** *Let X be a smooth quasi-projective complex variety, and D an effective integral divisor on X. Assume that $\mathbf{p}$ is a closed point of D_{red} and f is any local equation of D at $\mathbf{p}$. Then, for any $\alpha \in (0, 1] \cap \mathbb{Q}$, $n_{\alpha,\mathbf{p}}(f) = m_{\alpha,\mathbf{p}}(D)$.*

Remark from Theorem 2.3 that, for $\alpha \in (0, 1]$, t^α appears in $\mathrm{Sp}(f, \mathbf{p})$ if and only if α is an inner jumping number of (X, D) at $\mathbf{p}$. If $\mathbf{p}$ is an isolated singularity of D, Theorem 2.3 may be even applied to the previous remark when replacing X by an open neighborhood of X to obtain Varchenko's result [30] (see Corollary in [3, Section 1]).

3. Local systems and Milnor fibers of homegeneous singularities

3.1. **Local systems and normal G-covers.** Let us recall some basic notions of local systems and cyclic covers in [13] and [5]. A complex *local system* $\mathcal{V}$ on a complex manifold is a locally constant sheaf of finite dimensional complex vector spaces. The rank of a locally constant sheaf is the dimension of a stalk as a complex vector space. As mentioned in Budur [5], local systems of rank one on a complex manifold U correspond to morphisms of groups $H_1(U) \to \mathbb{C}^*$. In this correspondence, a local system is called *unitary* if it is sent to a morphism of groups $H_1(U) \to S^1 = \{\eta \in \mathbb{C}^* \mid |\eta| = 1\}$. The constant sheaf $\mathbb{C}_U$ and any local system of rank one of finite order are simple examples of unitary local systems.

Let X be a smooth complex projective variety of dimension n, and f a regular function on X with zero divisor $D := f^{-1}(0)$. Denote $U := X \setminus D$ and write $D_{\mathrm{red}} = \bigcup_{j=1}^r D_j$, where D_j are distinct irreducible reduced subvarieties of D. We may use D as the family $\{D_1, \dots, D_r\}$ by abuse of notation (and the following definition will be in this sense), we write $c_1(\mathcal{L})$ for the first Chern class of a line bundle $\mathcal{L}$ and consider the group

$$\mathrm{Pic}^\tau(X, D) := \big\{(\mathcal{L}, \alpha) \in \mathrm{Pic}(X) \times [0, 1)^r \mid c_1(\mathcal{L}) = \alpha[D] \in H^2(X, \mathbb{R})\big\}.$$

with the following operation

$$(\mathcal{L}, \alpha) \cdot (\mathcal{L}', \alpha') := (\mathcal{L} \otimes \mathcal{L}' \otimes \mathcal{O}_X(-\lfloor(\alpha + \alpha')D\rfloor)), \{\alpha + \alpha'\}), \tag{3.1}$$

where $\alpha[D] := \sum_{j=1}^r \alpha_j[D_j]$, an $\mathbb{R}$-linear combination of the cohomology classes $[D_j]$ in $H^2(X, \mathbb{R})$, and as above $\alpha D := \sum_{j=1}^r \alpha_j D_j$, $\lfloor\alpha\rfloor := (\lfloor\alpha_1\rfloor, \dots, \lfloor\alpha_r\rfloor)$ and $\{\alpha\} := \alpha - \lfloor\alpha\rfloor$.

Theorem 3.1 (Budur [5], Theorem 1.2)**.** *There is a canonical isomorphism of groups*

$$RH : \mathrm{Pic}^\tau(X, D) \cong \mathrm{Hom}(H_1(U), S^1),$$

By this theorem, one may identify a unitary local system of rank one on U with an element of $\mathrm{Pic}^\tau(X, D)$.

Let $\pi : Y \to X$ be a log resolution of the family $\{D_1, \dots, D_r\}$, and $E := Y \setminus \pi^{-1}(U) = \sum_{j\in A} N_j E_j$, with E_j irreducible. We shall use the following two important results.

Proposition 3.2 (Budur [5], Proposition 3.3)**.** *The map $\pi^*_{par} : \mathrm{Pic}^\tau(X, D) \to \mathrm{Pic}^\tau(Y, E)$ which sends $(\mathcal{L}, \alpha)$ to $(\pi^*\mathcal{L} - \lfloor\beta E\rfloor, \{\beta\})$ with β defined by $\pi^*(\alpha D) = \beta E$ is an isomorphism of groups.*

Theorem 3.3 (Budur [4], Theorem 4.6)**.** *Let $\mathcal{V}$ be a rank one unitary local system on U which corresponds to $(\mathcal{L}, \alpha) \in \mathrm{Pic}^\tau(X, D)$. Then, for all $p, q \in \mathbb{N}$, we have*

$$Gr_F^p H^{p+q}(U, \mathcal{V}^\vee) = H^{n-q}(Y, \Omega_Y^p(\log E)^\vee \otimes \omega_Y \otimes \pi^*\mathcal{L} \otimes \mathcal{O}_Y(-\lfloor\pi^*(\alpha D)\rfloor))^\vee.$$

In particular,

$$Gr_F^0 H^q(U, \mathcal{V}^\vee) = H^{n-q}(X, \omega_X \otimes \mathcal{L} \otimes \mathcal{J}(X, \alpha D))^\vee.$$

Let G be a finite group. By [5, Corollary 1.10], the dual group $G^* = \mathrm{Hom}(G, \mathbb{C}^*)$ of G gives rise to a normal G-cover of X unramified above U. Namely, the normal G-cover of X is the morphism

$$\phi : \widetilde{X} = \mathrm{Spec}_{\mathcal{O}_X}\left(\bigoplus_{\eta \in G^*} \mathcal{L}_\eta^{-1}\right) \to X$$

induced by the $\mathcal{O}_X$-module structural morphisms $\mathcal{O}_X \to \mathcal{L}_\eta$, for all $\eta \in G^*$, where we identify G^* with the subgroup $\{(L_\eta, \alpha_\eta) \mid \eta \in G^*\}$ of $\mathrm{Pic}^\tau(X, D)$. The group G acts on $\mathcal{L}_\eta^{-1}$ via the character η, hence acts on the $\mathcal{O}_X$-module sheaf $\phi_*\mathcal{O}_{\widetilde{X}}$. By [5, Corollary 1.11], $\phi_*\mathcal{O}_{\widetilde{X}}$ admits an eigensheaf decomposition

$$\phi_*\mathcal{O}_{\widetilde{X}} = \bigoplus_{\eta \in G^*} \mathcal{L}_\eta^{-1}, \tag{3.2}$$

where the eigensheaf $\mathcal{L}_\eta^{-1}$ is with respect to the eigenvalue η of the action of G on $\phi_*\mathcal{O}_{\widetilde{X}}$.

Now we consider the log-resolution π. By Proposition 3.2, since $\{(\mathcal{L}_\eta, \alpha_\eta) \mid \eta \in G^*\}$ is a finite subgroup of $\mathrm{Pic}^\tau(X, D)$, $\{(\pi^*\mathcal{L}_\eta - \lfloor \beta_\eta E \rfloor, \beta_\eta) \mid \eta \in G^*\}$, with β_η defined by $\pi^*(\alpha_\eta D) = \beta_\eta E$, is a finite subgroup of $\mathrm{Pic}^\tau(Y, E)$. By the same way as previous we can construct the corresponding normal G-cover of Y unramified above $\pi^{-1}(U) \cong U$ as follows

$$\rho : \widetilde{Y} = \mathrm{Spec}_{\mathcal{O}_Y}\left(\bigoplus_{\eta \in G^*} \pi^*\mathcal{L}_\eta^{-1} \otimes \mathcal{O}_Y\left(\lfloor \beta_\eta E \rfloor\right)\right) \to Y,$$

where the group G of acts on $\widetilde{Y}$ and on $\rho_*\mathcal{O}_{\widetilde{Y}}$. Moreover, similarly as (3.2), we have

Proposition 3.4 (Budur [5], Corollary 1.12)**.** *There is an eigensheaf decomposition*

$$\rho_*\mathcal{O}_{\widetilde{Y}} = \bigoplus_{\eta \in G^*} \pi^*\mathcal{L}_\eta^{-1} \otimes \mathcal{O}_Y\left(\lfloor \beta_\eta E \rfloor\right),$$

the eigensheaf $\pi^*\mathcal{L}_\eta^{-1} \otimes \mathcal{O}_Y(\lfloor \beta_\eta E \rfloor)$ is with respect to the eigenvalue η of the action of G on $\rho_*\mathcal{O}_{\widetilde{Y}}$.

3.2. **Milnor fibers of homegeneous singularity.** Let $f(x_0, \ldots, x_n) \in \mathbb{C}[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree d. We shall take f into two closely interactive entities, a Milnor fiber at the origin of $\mathbb{C}^{n+1}$ and a complex projective hypersurface of $\mathbb{P}^n$. By [22, Lemma 9.4], the Minor fiber M of f at the origin of $\mathbb{C}^{n+1}$ is diffeomorphic to $\{(x_0, \ldots, x_n) \in \mathbb{C}^{n+1} \mid f(x_0, \ldots, x_n) = 1\}$. The geometric monodromy $M \to M$ is given by multiplication of elements of M by $e^{\frac{2\pi i}{d}}$, which induces an endomorphism T of the complex vector space $H^*(M, \mathbb{C})$.

Following [6, Section 4], we consider the smooth complex projective variety $X = \mathbb{P}^n$ and the closed subscheme D of X defined by the zero locus of f. Put $U := X \setminus D$. Since the action of $\mathbb{Z}/d\mathbb{Z}$ on M is free, we have a natural isomorphism $M/(\mathbb{Z}/d\mathbb{Z}) \cong U$. Denote by σ the quotient map $M \to U$, which is the cyclic cover of degree d of U. Then there is an eigensheaf decomposition of the $\mathcal{O}_U$-module sheaf $\sigma_*\mathbb{C}_M$ as follows

$$\sigma_*\mathbb{C}_M = \bigoplus_{k=0}^{d-1} \mathcal{V}_k,$$

where $\mathcal{V}_k$ is the unitary local system on U given by the eigensheaf of T with respect to the eigenvalue $e^{-\frac{2\pi i k}{d}}$. This implies that

$$H^l(U, \sigma_*\mathbb{C}_M) = \bigoplus_{k=0}^{d-1} H^l(U, \mathcal{V}_k).$$

Let us consider the Leray spectral sequence

$$E_2^{p,q} = H^q(U, R^p\sigma_*\mathbb{C}_M) \Rightarrow H^{p+q}(M, \mathbb{C}_M).$$

Since σ is a finite morphism of schemes, $R^p\sigma_*\mathbb{C}_M = 0$ for all $p \geq 1$, hence, by this spectral sequence, we have $H^l(U, \sigma_*\mathbb{C}_M) = H^l(M, \mathbb{C}_M) = H^l(M, \mathbb{C})$, for $l \in \mathbb{N}$. This implies the following important lemma (cf. [6, Section 4]).

Lemma 3.5 (Budur [6])**.** *The complex vector space $H^l(U,\mathcal{V}_k)$ if nontrivial is the eigenspace of the monodromy action T on $H^l(M,\mathbb{C})$ with respect to the eigenvalue $e^{-\frac{2\pi ik}{d}}$, that is,*

$$H^l(M,\mathbb{C})_{e^{-\frac{2\pi ik}{d}}} = H^l(U,\mathcal{V}_k),$$

for $0\le k\le d-1$ and $l\ge 0$.

In fact, there are two commuting monodromy actions on $H^l(M,\mathbb{C})$, where the endomorphism T is the first one. The second one is, for each k, the monodromy of $\mathcal{V}_k$ around a generic point of D_j, which, by [6, Lemma 4.1], is given by multiplication by $e^{\frac{2\pi ikm_j}{d}}$. Together with [5, Proposition 3.3], it proves the following important lemma.

Lemma 3.6 (Budur [6], Lemma 4.2)**.** *Assume $D=\sum_{j=1}^r m_jD_j$, with D_j irreducible of degree d_j. Then the element in $\operatorname{Pic}^\tau(X,D)$ corresponding via the isomorphism RH in Theorem 3.1 to the unitary local system $\mathcal{V}_k$ is $(\mathcal{O}_{\mathbb{P}^n}(\sum_{j=1}^r\{\frac{km_j}{d}\}d_j),(\{\frac{km_1}{d}\},\ldots,\{\frac{km_r}{d}\}))$.*

Notice that $\sum_{j=1}^r\{\frac{km_j}{d}\}d_j$ is an integer, because, if for every $1\le j\le r$ we write $km_j=dn_j+s_j$, with $n_j,s_j\in\mathbb{N}$, $0\le s_j<d$, we have

$$\sum_{j=1}^r\left\{\frac{km_j}{d}\right\}d_j=\sum_{j=1}^r\frac{s_jd_j}{d}=\sum_{j=1}^r\frac{km_jd_j-dn_jd_j}{d}=k-\sum_{j=1}^r n_jd_j.$$

Fix a log-resolution $\pi:Y\to\mathbb{P}^n$ of the family of closed subschemes $\{D_1,\ldots,D_r\}$ of $\mathbb{P}^n$, and, as previous, denote $E=\pi^*(\bigcup_{j=1}^r D_j)=\sum_{j\in A}N_jE_j$, with E_j irreducible components of $\pi^{-1}(D)$. Let

$$\mathcal{L}^{(k)}:=\pi^*\mathcal{O}_{\mathbb{P}^n}\left(\sum_{j=1}^r\left\{\frac{km_j}{d}\right\}d_j\right)\otimes\mathcal{O}_Y\left(-\left\lfloor\sum_{j=1}^r\left\{\frac{km_j}{d}\right\}\pi^*D_j\right\rfloor\right). \tag{3.3}$$

Denote by B the set of integers k such that $0\le k\le d-1$ and d divides km_j for all $1\le j\le r$, and by $\overline{B}$ the complement of B in $[0,d-1]\cap\mathbb{Z}$.

Remark 3.7. If k is in B, then $\mathcal{L}^{(k)}=\mathcal{O}_Y$. Furthermore, if k is in B and $k\neq 0$, so is $d-k$; if k and k' are in B, so is either $k+k'$ or $k+k'-d$; hence we can consider B as a subgroup of $\mathbb{Z}/d\mathbb{Z}$. Let $m=\gcd(m_1,\ldots,m_r)$, and $u_j\in\mathbb{N}_{>0}$ with $m_j=mu_j$ for $1\le j\le r$. Then $k\in B$ if and only if $0\le k\le d-1$ and ku_s is divisible by $\sum_{j=1}^r d_ju_j$ for any $1\le s\le r$. Since $u_1,\ldots,u_r$ are coprime, the latter means that k is divisible by $\sum_{j=1}^r d_ju_j$, hence the cardinal $|B|$ of B equals m.

For simplicity of notation, from now on, if $\mathcal{A}$ is a sheaf on $\mathbb{P}^n$, and $l\in\mathbb{Z}$, we shall write $\mathcal{A}(l)$ in stead of $\mathcal{A}\otimes\mathcal{O}_{\mathbb{P}^n}(l)$.

Proposition 3.8. *With the notation as in Lemma 3.6 we have*

(i) $\dim_{\mathbb{C}} Gr_F^pH^{p+q}(U,\mathcal{V}_k)=\dim_{\mathbb{C}}H^q(Y,\Omega_Y^p(\log E))$, *for $k\in B$;*
(ii) $\dim_{\mathbb{C}} Gr_F^pH^{p+q}(U,\mathcal{V}_{d-k})=\dim_{\mathbb{C}}H^q(Y,\Omega_Y^p(\log E)\otimes{\mathcal{L}^{(k)}}^{-1})$, *for $k\in\overline{B}$.*

In particular, for $k\in\overline{B}$,

$$\dim_{\mathbb{C}} Gr_F^0H^q(U,\mathcal{V}_{d-k})=\dim_{\mathbb{C}}H^{n-q}\left(\mathbb{P}^n,\mathcal{J}\left(\mathbb{P}^n,\sum_{j=1}^r\left\{\frac{km_j}{d}\right\}D_j\right)\left(\sum_{j=1}^r\left\{\frac{km_j}{d}\right\}d_j-n-1\right)\right).$$

Proof. Due to the group law (3.1) of $\operatorname{Pic}^\tau(X,D)$ and definition of $\mathcal{V}_k$, it is obvious that $\mathcal{V}_k=\mathcal{V}_k^\vee=\mathcal{V}_0$ for $k\in B$, and that $\mathcal{V}_{d-k}=\mathcal{V}_k^\vee$ for $k\in\overline{B}$. Then, by Lemma 3.6 and Theorem 3.3, we have

$$Gr_F^pH^{p+q}(U,\mathcal{V}_k)=H^{n-q}(Y,\Omega_Y^p(\log E)^\vee\otimes\omega_Y)^\vee$$

for $k\in B$, and

$$\begin{aligned} Gr_F^pH^{p+q}(U,\mathcal{V}_{d-k}) &= H^{n-q}(Y,\Omega_Y^p(\log E)^\vee\otimes\omega_Y\otimes\mathcal{L}^{(k)})^\vee \\ &= H^{n-q}(Y,(\Omega_Y^p(\log E)\otimes{\mathcal{L}^{(k)}}^{-1})^\vee\otimes\omega_Y)^\vee, \end{aligned}$$

for $k\in\overline{B}$. Applying the Serre duality we obtain (i) and (ii).

For the rest statement, we again apply Lemma 3.6 and the particular case in Theorem 3.3, together with the definition of multiplier ideal. □

Denote $\mathcal{L}^{(k)}_{\mathrm{red}} := \pi^*\mathcal{O}_{\mathbb{P}^n}(k) \otimes \mathcal{O}_Y(-\lfloor \frac{k}{d} E \rfloor)$, for $0 \le k \le d-1$.

Corollary 3.9. *With the notation as in Lemma 3.6 and D being reduced, for $1 \le k \le d$,*

(i) $\dim_{\mathbb{C}} Gr_F^p H^{p+q}(U, \mathcal{V}_{d-k}) = \dim_{\mathbb{C}} H^q(Y, \Omega^p_Y(\log E) \otimes {\mathcal{L}^{(k)}_{\mathrm{red}}}^{-1})$;
(ii) $\dim_{\mathbb{C}} Gr_F^0 H^q(U, \mathcal{V}_{d-k}) = \dim_{\mathbb{C}} H^{n-q}(\mathbb{P}^n, \mathcal{J}(\mathbb{P}^n, \frac{k}{d} D)(k-n-1))$.

Proof. Applying Proposition 3.8 to the special case $m_1 = \cdots = m_r = 1$ we obtain the statements. Note that, in this case, $B = \{0\}$ and $\overline{B} = \{1, \ldots, d-1\}$. □

Lemma 3.10. *With the notation as in Lemma 3.6, and by observation $\mathcal{L}^{(d)} = \mathcal{L}^{(0)}$, we have*

(i) $\dim_{\mathbb{C}} H^1(U, \mathcal{V}_k) = r - 1$, *if $n = 2$ and $k \in B$;*
(ii) $\dim_{\mathbb{C}} H^l(U, \mathcal{V}_{d-k}) = \sum_{p \ge 0} \dim_{\mathbb{C}} H^{j-p}(Y, \Omega^p_Y(\log E) \otimes {\mathcal{L}^{(k)}}^{-1})$, *if $j \ge 0$ and $1 \le k \le d$.*

Proof. By Proposition 3.8 (i), if $k \in B$, we have $\dim_{\mathbb{C}} Gr_F^p H^{p+q}(U, \mathcal{V}_k) = \dim_{\mathbb{C}} H^q(Y, \Omega^p_Y(\log E))$, thus

$$\dim_{\mathbb{C}} H^1(U, \mathcal{V}_0) = \dim_{\mathbb{C}} H^1(Y, \mathcal{O}_Y) + \dim_{\mathbb{C}} H^0(Y, \Omega^1_Y(\log E)).$$

Assume that $n = 2$. It is a fact that $\dim_{\mathbb{C}} H^1(Y, \mathcal{O}_Y) = 0$, because Y is birationally equivalent to $\mathbb{P}^2$, and that $\dim_{\mathbb{C}} H^0(Y, \Omega^1_Y(\log E)) = r - 1$, due to the proof of Théorème 6 in [12]. This proves (i).

The statement (ii) of this lemma is a consequence of Proposition 3.8 (ii). □

4. Characteristic polynomials and zeta functions of homogeneous singularities

As in Section 3.2, we shall work with a homogeneous polynomial $f(x_0, \ldots, x_n) \in \mathbb{C}[x_0, \ldots, x_n]$ of degree d. By considering its germ at the origin of $\mathbb{C}^{n+1}$ we study some singularity invariants, including the characteristic polynomials and the zeta function of the monodromy.

4.1. **Characteristic polynomials.** Recall that the Milnor fiber M of the singularity $f(x_0, \ldots, x_n)$ at the origin of $\mathbb{C}^{n+1}$ is diffeomorphic to $\{(x_0, \ldots, x_n) \in \mathbb{C}^{n+1} \mid f(x_0, \ldots, x_n) = 1\}$, and the monodromy T is induced by $e^{\frac{2\pi i}{d}} \cdot (x_0, \ldots, x_n) = (e^{\frac{2\pi i}{d}} x_0, \ldots, e^{\frac{2\pi i}{d}} x_n)$. By definition, the (monodromy) characteristic polynomial $\Delta_l(t)$ of the endomorphism $T|_{H^l(M,\mathbb{C})}$ of $H^l(M, \mathbb{C})$ is the monic polynomial

$$\Delta_l(t) = \det(t\mathrm{Id} - T|_{H^l(M,\mathbb{C})}).$$

Assume that

$$f(x_0, \ldots, x_n) = \prod_{j=1}^{r} f_j(x_0, \ldots, x_n)^{m_j},$$

where $f_j(x_0, \ldots, x_n)$ are distinct irreducible homogeneous polynomials of degree d_j, $1 \le j \le r$. As above, we denote by D_j the complex projective plane curve $\{(x_0 : \cdots : x_n) \in \mathbb{P}^2 \mid f_j(x_0, \ldots, x_n) = 0\}$, for $1 \le j \le r$.

Fix a log-resolution $\pi : Y \to \mathbb{P}^n$ of the family $D = \{D_1, \ldots, D_r\}$, with normal crossing divisor $E = \pi^{-1}(\bigcup_{j=1}^r D_j)$. As mentioned in Section 3, there is an isomorphism $M/(\mathbb{Z}/d\mathbb{Z}) \cong U = \mathbb{P}^2 \setminus C$ so that the canonical projection $\sigma : M \to U$ induces a eigensheaf decomposition $\sigma_* \mathbb{C}_M = \bigoplus_{k=0}^{d-1} \mathcal{V}_k$, where $\mathcal{V}_k$ are the unitary local systems on U given in Lemma 3.6. By Lemma 3.5, for $1 \le k \le d$ and $l \in \mathbb{N}$, the vector space $H^l(U, \mathcal{V}_{d-k})$ if nontrivial is the eigenspace of $T|_{H^l(M,\mathbb{C})}$ with respect to the eigenvalue $e^{\frac{2\pi i k}{d}}$. This together with Lemma 3.10 and Remark 3.7 proves the following lemma.

Lemma 4.1. *Let $\Delta_l(t)$ be the characteristic polynomial of the endomorphism $T|_{H^l(M,\mathbb{C})}$ of $H^l(M, \mathbb{C})$. Then, with the previous notation and $l \in \mathbb{N}$, one has*

$$\triangle_l(t) = \prod_{k=0}^{d-1} (t - e^{\frac{2\pi i k}{d}})^{h_l^{(k)}},$$

where,

$$h_l^{(k)} := \dim_{\mathbb{C}} H^l(U, \mathcal{V}_{d-k}) = \sum_{p+q=l} h^q(\Omega_Y^p(\log E) \otimes {\mathcal{L}^{(k)}}^{-1}),$$

with $h^q(\Omega_Y^p(\log E) \otimes {\mathcal{L}^{(k)}}^{-1}) = \dim_{\mathbb{C}} H^q(Y, \Omega_Y^p(\log E) \otimes {\mathcal{L}^{(k)}}^{-1})$, *and*

$$\mathcal{L}^{(k)} = \pi^* \mathcal{O}_{\mathbb{P}^n}\left(\sum_{j=1}^r \left\{\frac{km_j}{d}\right\} d_j\right) \otimes \mathcal{O}_Y\left(-\left\lfloor \sum_{j=1}^r \left\{\frac{km_j}{d}\right\} \pi^* D_j \right\rfloor\right).$$

As above, we denote by B the set of k in $\mathbb{Z}$ such that $0 \le k \le d-1$ and d divides km_j for all $1 \le j \le r$, by $\overline{B}$ the complement of B in $[0, d-1] \cap \mathbb{Z}$, and $m = \gcd(m_1, \ldots, m_r)$. Due to Remark 3.7, B may be considered as a subgroup of $\mathbb{Z}/d\mathbb{Z}$. Let G be the quotient group $(\mathbb{Z}/d\mathbb{Z})/B$. For convenience, we shall identify $k \in [0, d-1] \cap \mathbb{Z}$ with its class in G.

Lemma 4.2. *With the notation as in Lemma 4.1, one has*

$$\Delta_l(t) = \prod_{k \in G} (t^m - e^{\frac{2\pi i k m}{d}})^{h_l^{(k)}}$$

for $l \in \mathbb{N}$. In particular, $\Delta_0(t) = t^m - 1$.

Proof. If k and k' belong to the same class in G, we have $h_l^{(k)} = h_l^{(k')}$. This together with Lemma 4.1 implies the first statement. Since $h^0(\mathcal{O}_Y) = 1$, it remains to check that $h^0({\mathcal{L}^{(k)}}^{-1}) = 0$ for $k \in G \setminus \{0\}$. By Lemmas 3.5 and 3.10, we have

$$\dim_{\mathbb{C}} H^0(M, \mathbb{C}) = \sum_{k \in B} h^0({\mathcal{L}^{(k)}}^{-1}) + \sum_{k \in \overline{B}} h^0({\mathcal{L}^{(k)}}^{-1}). \tag{4.1}$$

It is known that $\dim_{\mathbb{C}} H^0(M, \mathbb{C}) = m$ (cf. [10, Proposition 2.3]). Note that $|B| = m$ (cf. Remark 3.7), and that, for $k \in B$, $\mathcal{L}^{(k)} = \mathcal{O}_Y$ and $h^0(\mathcal{O}_Y) = 1$. Then (4.1) is equivalent to $\sum_{k \in \overline{B}} h^0({\mathcal{L}^{(k)}}^{-1}) = 0$, which implies that $h^0({\mathcal{L}^{(k)}}^{-1}) = 0$ for $k \in \overline{B}$; in particular, $h^0({\mathcal{L}^{(k)}}^{-1}) = 0$ for $k \in G \setminus \{0\}$. □

Let us now consider the case where $n = 2$. In this case, we shall denote C (resp. C_j) instead of D (resp. D_j). Then the characteristic polynomial $\Delta_1(t)$ is an important invariant of the homogeneous surface singularity. The following theorem is a main result in the present article.

Theorem 4.3. *With the previous notation and $n = 2$, one has*

$$\Delta_1(t) = (t^m - 1)^{r-1} \prod_{k \in G \setminus \{0\}} \left(t^{2m} - 2t^m \cos\frac{2km\pi}{d} + 1\right)^{\ell_k},$$

where

$$\ell_k := \dim_{\mathbb{C}} H^1\left(\mathbb{P}^2, \mathcal{J}\left(\mathbb{P}^2, \sum_{j=1}^r \left\{\frac{km_j}{d}\right\} C_j\right)\left(\sum_{j=1}^r \left\{\frac{km_j}{d}\right\} d_j - 3\right)\right).$$

Proof. According to Lemma 4.2, it suffices to prove that

$$h^1({\mathcal{L}^{(k)}}^{-1}) = \ell_k \tag{4.2}$$

and

$$h^0(\Omega_Y^1(\log E) \otimes {\mathcal{L}^{(k)}}^{-1}) = \ell_{d-k}, \tag{4.3}$$

for $k \in G \setminus \{0\}$. The equality (4.2) is a direct corollary of Proposition 3.8 and Lemma 3.10.

To prove (4.3) we consider a common G-equivariant desingularization of $\widetilde{X}$ and $\widetilde{Y}$, say, $\theta : Z \to \widetilde{X}$ and $\nu : Z \to \widetilde{Y}$, in the sense of [1], such that $\pi \circ \rho \circ \nu = \phi \circ \theta =: u$. Here, we use the notation in Section 3.1 and work with the case $X = \mathbb{P}^2$. Note that $G^* = \{(\mathcal{O}_{\mathbb{P}^2}(\sum_{j=1}^r \{\frac{km_j}{d}\} d_j), (\{\frac{km_1}{d}\}, \ldots, \{\frac{km_r}{d}\}))\}_{0 \le k \le d-1}$, which is by Remark 3.7 a subgroup of order $\frac{d}{m}$ of the group $\operatorname{Pic}^\tau(\mathbb{P}^2, C)$. We may choose Z such that

$\Delta := Z \setminus u^{-1}(U)$ is a normal crossing divisor. An analogue of [12, Corollaire 4] shows that, for any $q \in \mathbb{N}$,

$$\begin{aligned} (\rho\circ\nu)_*\Omega^q_Z(\log\Delta) &\cong \Omega^q_Y(\log E)\otimes(\rho\circ\nu)_*\mathcal{O}_Z, \\ R^p(\rho\circ\nu)_*\Omega^q_Z(\log\Delta) &= 0 \quad \text{if } p>0 \end{aligned} \tag{4.4}$$

(see also [13, Lemma 3.22]). By the Leray spectral sequence

$$E_2^{p,q} = H^q(Y, R^p(\rho\circ\nu)_*\Omega^1_Z(\log\Delta)) \Rightarrow H^{p+q}(Z, \Omega^1_Z(\log\Delta))$$

and by (4.4), we get, in particular,

$$H^0(Y, \Omega^1_Y(\log E)\otimes(\rho\circ\nu)_*\mathcal{O}_Z) = H^0(Z, \Omega^1_Z(\log\Delta)). \tag{4.5}$$

Since G^* is a finite subgroup of $\mathrm{Pic}^\tau(\mathbb{P}^2, C)$, we deduce from Proposition 3.4 that

$$(\rho\circ\nu)_*\mathcal{O}_Z = \rho_*\mathcal{O}_{\widetilde{Y}} = \bigoplus_{k\in G} {\mathcal{L}^{(k)}}^{-1},$$

where as mentioned previously we identify $k \in [0, d-1]\cap\mathbb{Z}$ with its class in G. This yields the following decomposition

$$H^0(Y, \Omega^1_Y(\log E)\otimes(\rho\circ\nu)_*\mathcal{O}_Z) = \bigoplus_{k\in G} H^0(Y, \Omega^1_Y(\log E)\otimes{\mathcal{L}^{(k)}}^{-1}). \tag{4.6}$$

Note that, due to the proof of Lemma 3.10, the direct summand of (4.6) corresponding to $k = 0$ has complex dimension $r-1$.

Now we compute the dimension of complex vector space on the right hand side of (4.5). Similarly as in the proof of Lemma 7 of [12], one may point out that

$$\dim_{\mathbb{C}} H^0(Z, \Omega^1_Z(\log\Delta)) = \dim_{\mathbb{C}} H^0(Z, \Omega^1_Z) + (r-1). \tag{4.7}$$

On the other hand, by [5, Corollary 1.13], we have

$$H^0(Z, \Omega^1_Z) \cong \bigoplus_{k\in G} H^1\left(\mathbb{P}^2, \mathcal{J}\left(\mathbb{P}^2, \sum_{j=1}^r \left\{\frac{km_j}{d}\right\} C_j\right)\left(\sum_{j=1}^r \left\{\frac{km_j}{d}\right\} d_j - 3\right)\right). \tag{4.8}$$

In the decomposition (4.8), look at the direct summand corresponding to $k = 0$. They are nothing but $H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3)) = H^1(\mathbb{P}^2, \omega_{\mathbb{P}^2})$. By the Serre duality, $\dim_{\mathbb{C}} H^1(\mathbb{P}^2, \omega_{\mathbb{P}^2}) = \dim_{\mathbb{C}} H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 0$. Therefore, from (4.5), (4.6), (4.7) and (4.8), we get

$$\sum_{k\in G\setminus\{0\}} h^0(\Omega^1_Y(\log E)\otimes{\mathcal{L}^{(k)}}^{-1}) = \sum_{k\in G\setminus\{0\}} \ell_k. \tag{4.9}$$

Repeating the proof of [21, Proposition 4.6] and using (4.2) we obtain that

$$h^0(\Omega^1_Y(\log E)\otimes{\mathcal{L}^{(k)}}^{-1}) \geq \ell_{d-k},$$

for $k \in G\setminus\{0\}$. This together with (4.9) means the equality $h^0(\Omega^1_Y(\log E)\otimes{\mathcal{L}^{(k)}}^{-1}) = \ell_{d-k}$, thus (4.3) is proved. □

4.2. **A formula for the monodromy zeta function.** By definition, the monodromy zeta function the homogeneous singularity $f(x_0, \ldots, x_n)$ at the origin O of $\mathbb{C}^{n+1}$ is the function

$$\zeta_{f,O}(t) = \prod_{l\geq 0} \det(\mathrm{Id} - tT|_{H^l(M,\mathbb{C})})^{(-1)^{l+1}}.$$

This function may be expressed via the polynomials $\Delta_l(t)$ as $\zeta_{f,O}(t) = \prod_{l\geq 0}(t^{\dim_{\mathbb{C}} H^l(M,\mathbb{C})}\Delta_l(\frac{1}{t}))^{(-1)^{l+1}}$, from which, by Lemma 4.2,

$$\zeta_{f,O}(t) = \prod_{k\in G}\left(1 - e^{\frac{2\pi ikm}{d}} t^m\right)^{\sum_{l\geq 0}(-1)^{l+1} h_l^{(k)}}. \tag{4.10}$$

As explained in [6], the only numbers $\alpha \in (0, n+1) \cap \mathbb{Q}$ such that $n_{\alpha,O}(f)$, the coefficients of t^α in $\mathrm{Sp}(f, O)$, can be nonzero are of the form $\frac{k}{d} + p$, with $k, p \in \mathbb{Z}$, $1 \leq k \leq d$ and $0 \leq p \leq n+1$. Then it implies from (2.2) and Lemma 3.5 that

$$n_{\frac{k}{d}+p,O}(f) = \sum_{j\in\mathbb{Z}} (-1)^j \dim_{\mathbb{C}} Gr_F^{n-p} H^{n+j}(U, \mathcal{V}_k), \tag{4.11}$$

for integers $1 \leq k \leq d$ and $0 \leq p \leq n+1$, where $\mathcal{V}_k$ is the local system corresponding to the element $(\mathcal{O}_{\mathbb{P}^2}(\sum_{j=1}^r \{\frac{km_j}{d}\} d_j), (\{\frac{km_1}{d}\}, \ldots, \{\frac{km_r}{d}\}))$ in $\mathrm{Pic}^\tau(X, D)$ via the isomorphism RH in Theorem 3.1 (cf. Lemma 3.6). Note that $\mathcal{V}_d = \mathcal{V}_0$. By Proposition 3.8 and (4.11), we have

$$n_{\frac{d-k}{d}+p,O}(f) = \sum_{j\in\mathbb{Z}} (-1)^j h^{p+j}(\Omega_Y^{n-p}(\log E) \otimes {\mathcal{L}^{(k)}}^{-1}), \tag{4.12}$$

for $k \in G$ when $p < n$, and $k \in G \setminus \{0\}$ when $p = n$, where $\mathcal{L}^{(k)}$ and $h^q(\Omega_Y^p(\log E) \otimes {\mathcal{L}^{(k)}}^{-1})$ are as in Lemma 4.1 (see also (3.3)).

Theorem 4.4. *The monodromy zeta function and the Hodge spectrum of the singularity f are fit into a relation as follows*

$$\zeta_{f,O}(t)^{(-1)^{n+1}} = (1 - t^m)^{1+\sum_{p=1}^n n_{p,O}(f)} \prod_{k\in G\setminus\{0\}} \left(1 - e^{\frac{2\pi i k m}{d}} t^m\right)^{\sum_{p=0}^n n_{\frac{d-k}{d}+p,O}(f)}.$$

Proof. Recall from Lemma 4.1 that $h_l^{(k)} = \sum_{p+q=l} h^q(\Omega_Y^p(\log E) \otimes {\mathcal{L}^{(k)}}^{-1})$. Since $h^0(\mathcal{O}_Y) = 1$ and $h^q(\mathcal{O}_Y) = 0$ for all $q \geq 1$, the formula (4.12) gives

$$(-1)^{n+1} + (-1)^{n+1} \sum_{p=0}^{n-1} n_{p+1,O}(f) = \sum_{j\in\mathbb{Z}} (-1)^{n+j+1} h_{n+j}^{(0)}.$$

As in the proof of Lemma 4.2, if $k \in G \setminus \{0\}$, then $h^0({\mathcal{L}^{(k)}}^{-1}) = 0$, thus by (4.12) we have

$$(-1)^{n+1} \sum_{p=0}^{n} n_{\frac{d-k}{d}+p,O}(f) = \sum_{j\in\mathbb{Z}} (-1)^{n+j+1} h_{n+j}^{(k)}.$$

Now applying (4.10) we obtain the statement of the theorem. □

Remark 4.5. The formula (4.12) has the following interesting consequence. Assume that $f(x_0, \ldots, x_n)$ is a homogeneous polynomial and has isolated singularity at the origin O of $\mathbb{C}^{n+1}$. Then the non-trivial characteristic polynomials of the singularity only appear in the degrees 0 and n. This means that $h_l^{(k)} = 0$ for all $l \notin \{0, n\}$ and $0 \leq k \leq d-1$. Using the proof of Theorem 4.4, we obtain the identities

$$h_n^{(0)} = \sum_{p=1}^{n} n_{p,O}(f) \quad \text{and} \quad h_n^{(k)} = \sum_{p=0}^{n} n_{\frac{d-k}{d}+p,O}(f) \text{ for } 1 \leq k \leq d-1,$$

which prove the below result. By convention, we may consider the zero space $\{0\}$ as an eigenspace of the monodromy of the singularity with dimension zero.

Corollary 4.6. *Let $f(x_0, \ldots, x_n) \in \mathbb{C}[x_0, \ldots, x_n]$ be a homogeneous polynomial defining an isolated singularity at the origin O of $\mathbb{C}^{n+1}$. Then the complex dimension of the eigenspace of the monodromy of the singularity with respect to the eigenvalue 1 (resp. $e^{\frac{2\pi i k}{d}}$, for $1 \leq k \leq d-1$) is $\sum_{p=1}^n n_{p,O}(f)$ (resp. $\sum_{p=0}^n n_{\frac{d-k}{d}+p,O}(f)$, for $1 \leq k \leq d-1$).*

5. The Alexander polynomial of a complex projective plane curve

5.1. **Alexander polynomials.** We start this section with the definition of Alexander polynomial of a projective curve. Let C be a reduced complex projective plane curve of degree d with r distinct irreducible components. Let L be a line in $\mathbb{P}^2$ which is general with respect to C, that is, L intersects with C at exactly d distinct points. Such a line L exists since C is reduced. Then the manifold $W := \mathbb{P}^2 \setminus (C \cup L)$ has a homotopy type of a finite CW-complex. By van Kampen's theorem [15], the natural map

$$\pi_1(L \setminus (L \cap C)) \to \pi_1(W)$$

is an surjective homomorphism and the group $\pi_1(W)$ is generated by the images of all the d standard generators of the free group $\pi_1(L \setminus (L \cap C))$. The generators of $\pi_1(W)$ are loops in L going once around a point of $L \cap C$, and if two loops respectively go around two points of $L \cap C$ belonging to the same irreducible component of C they give rise to two conjugate elements in $\pi_1(W)$. This explains that

$$H_1(W, \mathbb{Z}) \cong \pi_1(W)/[\pi_1(W), \pi_1(W)] \cong \mathbb{Z}^r,$$

and that the Hurewicz morphism

$$\pi_1(W) \to H_1(W, \mathbb{Z})$$

is nothing but the canonical projection

$$\pi_1(W) \to \pi_1(W)/[\pi_1(W), \pi_1(W)],$$

with $[\pi_1(W), \pi_1(W)]$ being the commutator subgroup of $\pi_1(W)$.

We consider the surjective homomorphism $\varphi : \pi_1(W) \to \mathbb{Z}$ which is the composition of the Hurewicz morphism and the sum function. Then there exists an infinity cyclic cover $\widetilde{W}_\varphi \to W$ with respect to φ such that $\pi_1(\widetilde{W}_\varphi) = \ker \varphi$. Let $t : \widetilde{W}_\varphi \to \widetilde{W}_\varphi$ be the canonical generator of the group of cover transformations $\mathrm{Desk}(\widetilde{W}_\varphi/W) \cong \mathbb{Z}$. By this, $\mathbb{Z}$ acts naturally on $H^1(\widetilde{W}_\varphi, \mathbb{C})$ in the way so that $t \cdot c := t^*(c)$ for any class c in $H^1(\widetilde{W}_\varphi, \mathbb{C})$, from which $H^1(\widetilde{W}_\varphi, \mathbb{C})$ has a structure of $\mathbb{C}[t, t^{-1}]$-module. Since $\mathbb{C}[t, t^{-1}]$ is a principal ideal domain the torsion $\mathbb{C}[t, t^{-1}]$-module $H^1(\widetilde{W}_\varphi, \mathbb{C})$ admits up to order of summands a unique decomposition via monic polynomials $\Delta_l(t) \in \mathbb{C}[t] \subset \mathbb{C}[t, t^{-1}]$ with $\Delta_l(0) \neq 0$, $1 \leq l \leq m$, for some $m \in \mathbb{N}_{>0}$, namely,

$$H^1(\widetilde{W}_\varphi, \mathbb{C}) = \bigoplus_{l=1}^{m} \mathbb{C}[t, t^{-1}]/(\Delta_l(t)).$$

Then the *(global) Alexander polynomial* $\Delta_C(t)$ of the curve C is defined to be $\Delta_C(t) = \prod_{l=1}^m \Delta_l(t)$. One can prove that $\Delta_C(t)$ is independent of L general with respect to C (cf. [26]).

It is known that if C is irreducible and the fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ is either abelian or finite then the Alexander polynomial is trivial. One may prove easily that the multiplicity of the factor $t-1$ in $\Delta_C(t)$ is exactly $r-1$ as r is the number of irreducible component of C.

Assume that $\mathbf{p}$ is a singular point of C. We may consider the Milnor fiber $M_{\mathbf{p}}$ and the monodromy $T_{\mathbf{p}} : H^1(M_{\mathbf{p}}, \mathbb{C}) \to H^1(M_{\mathbf{p}}, \mathbb{C})$ of $(C, \mathbf{p})$. Denote by $\Delta_{C,\mathbf{p}}(t)$ the characteristic polynomial of the endomorphism $T_{\mathbf{p}}$. Let $\mathrm{Sing}(C)$ be the set of singular points of the curve C. Then, by Libgober [19], the Alexander polynomial $\Delta_C(t)$ divides the product $\prod_{\mathbf{p} \in \mathrm{Sing}(C)} \Delta_{C,\mathbf{p}}(t)$, and it also divides the *Alexander polynomial at infinity* $(t^d - 1)^{d-2}(t-1)$.

5.2. **Loeser-Vaquié's formula.** According to [21], to compute the Alexander polynomial $\Delta_C(t)$ it is useful to apply Randell's result [26]. In fact, viewing the homogeneous polynomial defining C as a germ of a singularity at the origin of $\mathbb{C}^3$ we may consider its Milnor fiber M and the monodromy T induced by

$$(x, y, z) \mapsto (e^{\frac{2\pi i}{d}} x, e^{\frac{2\pi i}{d}} y, e^{\frac{2\pi i}{d}} z).$$

Randell [26] shows that $\Delta_C(t)$ equals the characteristic polynomial $\Delta_1(t)$ of $T|_{H^1(M,\mathbb{C})}$. Applying that result of Randell and Theorem 4.3 to the case $m_1 = \cdots = m_r = 1$ we recover the following, which was proved by Loeser-Vaquié in [21]. For simplicity of notation, we write $\mathcal{J}_\alpha$ for $\mathcal{J}(\mathbb{P}^2, \alpha C)$, for any $\alpha \in (0, 1] \cap \mathbb{Q}$.

Theorem 5.1 (Loeser-Vaquié [21])**.** *If C be a reduced complex projective plane curve of degree d with r irreducible components, then*

$$\triangle_C(t) = (t-1)^{r-1} \prod_{k=1}^{d-1} \left(t^2 - 2t\cos\frac{2k\pi}{d} + 1\right)^{\dim_{\mathbb{C}} H^1(\mathbb{P}^2, \mathcal{J}_{\frac{k}{d}}(k-3))}.$$

Remark 5.2. We do not need to compute dimension of all the cohomology groups of the sheaves $\mathcal{J}_{\frac{k}{d}}(k-3)$. Indeed, by [12] and [21], if the coefficient of $t^{\frac{k}{d}}$ appears in the Hodge spectrum $\mathrm{Sp}(f, \mathbf{p})$ is zero, for every singular point $\mathbf{p}$ of C, then $H^1(\mathbb{P}^2, \mathcal{J}_{\frac{k}{d}}(k-3)) = 0$. By Theorem 2.3, $t^{\frac{k}{d}}$ appears in $\mathrm{Sp}(f, \mathbf{p})$ if and only if $\frac{k}{d}$ is an inner jumping number of $(\mathbb{P}^2, C)$ at $\mathbf{p}$.

5.3. **Computation of** $\dim_{\mathbb{C}} H^1(\mathbb{P}^2, \mathcal{J}_{\frac{k}{d}}(k-3))$**.** Let us use this paragraph to review the work by Artal Bartolo [2] in computing $\dim_{\mathbb{C}} H^1(\mathbb{P}^2, \mathcal{J}_{\frac{k}{d}}(k-3))$. For $1 \leq k \leq d-1$, we denote as usual by $\mathcal{J}_{\frac{k}{d},\mathbf{p}}$ the stalk at $\mathbf{p} \in C$ of the sheaf $\mathcal{J}_{\frac{k}{d}}$. It may be easily checked that, if $\mathbf{p}$ is non-singular, $\mathcal{J}_{\frac{k}{d},\mathbf{p}} = \mathcal{O}_{\mathbb{P}^2,\mathbf{p}}$. Consider a map

$$\psi_k : H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k-3)) \to \bigoplus_{\mathbf{p}\in C} \mathcal{O}_{\mathbb{P}^2,\mathbf{p}}/\mathcal{J}_{\frac{k}{d},\mathbf{p}}$$

defined as follows. We may identify the vector space $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k-3))$ with the space of polynomials in $\mathbb{C}[x,y]$ of degree $\leq k-3$, and that using the Taylor expansion at $\mathbf{p}$ each element g of $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k-3))$ induces a holomorphic function germ $g_{\mathbf{p}}$ at $\mathbf{p} \in C$. Then ψ_k is given by

$$\psi_k(g) = (g_{\mathbf{p}} + \mathcal{J}_{\frac{k}{d},\mathbf{p}})_{\mathbf{p}\in \mathrm{Sing}(C)},$$

which is a complex linear map.

Lemma 5.3 (Artal Bartolo [2])**.** $\dim_{\mathbb{C}} H^1(\mathbb{P}^2, \mathcal{J}_{\frac{k}{d}}(k-3)) = \dim_{\mathbb{C}} \mathrm{coker}(\psi_k)$.

We may also obtain a proof of this lemma using [12, Remarque 11] and Proposition 3.8 (for $n = 2$). By a simple computation, we have $\dim_{\mathbb{C}} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k-3)) = \frac{1}{2}(k-2)(k-1)$. To compute the dimension of the target space of ψ_k we follow [12] using a log-resolution of $(\mathbb{P}^2, C)$. Let $\pi : Y \to \mathbb{P}^2$ be a log-resolution of C, with numerical data given as follows

$$E = \pi^{-1}(C) = \sum_{i\in A} N_i E_i, \quad E_{\mathbf{p}} := \sum_{i\in A, \pi(E_i)=\mathbf{p}} N_i E_i$$

for $\mathbf{p} \in C$, and $K_{Y/\mathbb{P}^2} = \sum_{i\in A} a_i E_i$, where E_i are irreducible component of $\pi^{-1}(C)$.

Proposition 5.4. *With π as previous, one has*

$$\dim_{\mathbb{C}} H^1(\mathbb{P}^2, \mathcal{J}_{\frac{k}{d}}(k-3)) = \frac{1}{2}\sum_{\mathbf{p}\in C} \lfloor \frac{k}{d} E_{\mathbf{p}} \rfloor \cdot (K_{Y/\mathbb{P}^2} - \lfloor \frac{k}{d} E_{\mathbf{p}} \rfloor) - \frac{1}{2}(k-2)(k-1) + \dim_{\mathbb{C}} \ker(\psi_k).$$

Proof. From Lemma 5.3 we deduce that

$$\dim_{\mathbb{C}} H^1(\mathbb{P}^2, \mathcal{J}_{\frac{k}{d}}(k-3)) = \sum_{\mathbf{p}\in C} \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^2,\mathbf{p}}/\mathcal{J}_{\frac{k}{d},\mathbf{p}} - \dim_{\mathbb{C}} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k-3)) + \dim_{\mathbb{C}} \ker(\psi_k).$$

By [12, Remarque 11], $\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^2,\mathbf{p}}/\mathcal{J}_{\frac{k}{d},\mathbf{p}} = \frac{1}{2}\lfloor \frac{k}{d} E_{\mathbf{p}} \rfloor \cdot (K_{Y/\mathbb{P}^2} - \lfloor \frac{k}{d} E_{\mathbf{p}} \rfloor)$, and the proposition is proved. □

Due to Proposition 5.4, computing $\dim H^1(\mathbb{P}^2, \mathcal{J}_{\frac{k}{d}}(d-3-k))$ reduces to computing $\dim \ker(\psi_k)$, the dimension of the vector space of complex projective plane curves of degree $k-3$ passing through all the singular points $\mathbf{p}$ of C with germ contained in $\mathcal{J}_{\frac{k}{d},\mathbf{p}}$. Note that, if $\mathbf{p}$ is a singular point of C of type A_1, the last formula in the proof of Lemma 5.4 shows that $\mathcal{J}_{\frac{k}{d},\mathbf{p}} = \mathcal{O}_{\mathbb{P}^2,\mathbf{p}}$; therefore, like the case of a non-singular point, an A_1-singularity does not contribute to $\Delta_C(t)$.

Example 5.5. Let us consider an irreducible curve C of degree d whose singular points are either of type A_1 or of type $B_{a,b}$ (i.e., the local equation is $x^a + y^b = 0$), where a and b are positive integers such that ab divides d. The Hodge spectrum of each singularity of C of type $B_{a,b}$ equals $\sum_{i=1}^{a-1} \sum_{j=1}^{b-1} t^{\frac{i}{a}+\frac{j}{b}}$. Therefore, we get that, if $\mathbf{p}$ is a singular point of type $B_{a,b}$ of C, $J_{\frac{1}{a}+\frac{1}{b},\mathbf{p}}$ is the maximal ideal of $\mathcal{O}_{\mathbb{P}^2,\mathbf{p}}$, and in the case, $\dim_{\mathbb{C}} H^1(\mathbb{P}^2, \mathcal{J}_{\frac{1}{a}+\frac{1}{b}}(\frac{d}{a} + \frac{d}{b} - 3))$ can be easily computed.

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