A specialization property of index

Ananyo Dan Inder Kaur

December 5, 2016

Abstract

In [Kol13] Kollár defined i-th index of a proper scheme over a field. In this note we study how index behaves under specialization, in any characteristic.

1 Introduction

In [Kol13], Kollár generalized the classical definition of index of a variety. For a proper scheme X defined over a field k (of any characteristic), he defines $\mathrm{elw}_i(X)$ to be the ideal in $\mathbb{Z}$ generated by $\chi(\mathcal{F})$ as $\mathcal{F}$ ranges over all coherent sheaves on X supported in a subscheme of dimension at most i. One can easily notice,

$$\mathrm{elw}_0(X) \subset \mathrm{elw}_1(X) \subset ... \subset \mathrm{elw}_{\dim X}(X).$$

Among other important properties, it is known that $\mathrm{elw}_i(-)$ is birational invariant. In this note we ask whether this property is locally constructible i.e., given a flat family $\pi : X \to B$ of proper k-schemes and any closed irreducible subset $Y \subset B$, does there exist a non-empty open subset U of Y on which the function $u \mapsto \mathrm{elw}_i(\pi^{-1}(u))$ is constant? We observe that this is not the case in general (see Example 3.1). However, we prove that the function $\mathrm{elw}_i(-)$ is invariant under specialization. In particular, we prove:

Theorem 1.1. Let R be a discrete valuation ring, $\pi : X_R \to \mathrm{Spec}(R)$ be a flat, projective morphism and X_R is an integral scheme. Suppose that the dimension of the fibers is equal to n. Denote by K (resp. k) the fraction field (resp. residue field) of R. Then, for any $0 \leq i \leq n$, $\mathrm{elw}_i(X_K) = \mathrm{elw}_i(X_k)$.

We note that there is no assumption on the characteristic of K or k in this article.

We now discuss the strategy of the proof. The inequality $\mathrm{elw}_i(X_K) \geq \mathrm{elw}_i(X_k)$ is a direct consequence of [Kol13, Proposition 4]. To prove the reverse inequality we introduce a concept of semistable elw-indices. The key observation is that this coincides with Kollár's definition of the elw-index (see Proposition 2.4). We then check that any

Mathematics Subject Classification(2010):Primary 14G05, 14C25, Secondary 13H15, 14D06
Keywords: Index of varieties, Semistable sheaves, Specialization

semistable sheaf on the special fiber can be lifted to the generic fiber (see Proposition 3.3). This gives the reverse inequality, hence proves the theorem.

Acknowledgements This work was done when the first author was a post-doctoral fellow and the second author was a visiting researcher at the Tata Institute of Fundamental Research, Mumbai. The second author thanks TIFR for their hospitality. The first author is currently supported by ERCEA Consolidator Grant 615655-NMST and also by the Basque Government through the BERC $2014-2017$ program and by Spanish Ministry of Economy and Competitiveness MINECO: BCAM Severo Ochoa excellence accreditation SEV-2013 $-$ 0323

2 Semi-stable Esnault-Levine-Wittenberg index

Recall, the definition of the i-th index as given in [Kol13]:

Definition 2.1. Let Y be a projective scheme over a field k. For $0 \leq i \leq \dim Y$, the i-th *Esnault-Levine-Wittenberg index*, denoted by $\mathrm{elw}_i(Y)$ is defined as the ideal of $\mathbb{Z}$ generated by $\chi(\mathcal{F})$ as where $\mathcal{F}$ runs through all coherent sheaves on Y of dimension $\dim \mathcal{F} := \dim \mathrm{Supp}\mathcal{F} \leq i$.

We now define a new semi-stable version of the above index:

Definition 2.2. Let Y be as before. For $0 \leq i \leq \dim Y$, the *semi-stable Esnault-Levine-Wittenberg index*, denoted by $\mathrm{elw}_i^{ss}(Y)$ is defined as the ideal of $\mathbb{Z}$ generated by $\chi(\mathcal{F})$ as $\mathcal{F}$ runs through all coherent semi-stable sheaves over Y of dimension $\dim \mathcal{F} \leq i$.

Abuse of Notations 2.3. We will sometimes abuse notation $\mathrm{elw}_i(Y)$ to mean the generator of the ideal. This usage will be clear from the context.

Proposition 2.4. Let Y be a projective scheme of pure dimension n. Then, for all $0 \leq i \leq \dim Y$, $\mathrm{elw}_i^{ss}(Y) = \mathrm{elw}_i(Y)$.

Proof. Clearly, $\mathrm{elw}_m^{ss}(Y) \subset \mathrm{elw}_m(Y)$ for all $0 \leq m \leq n$. We now prove the converse. Let $\mathcal{F}$ be a coherent sheaf of dimension m. There exists an unique torsion filtration:

$$0 \subset T_0(\mathcal{F}) \subset ... \subset T_m(\mathcal{F}) = \mathcal{F}$$

where $T_i(\mathcal{F})$ is the maximal subsheaf of $\mathcal{F}$ of dimension at most equal to i. Note by definition, $T_i(\mathcal{F})/T_{i-1}(\mathcal{F})$ is zero or pure of dimension i. If $T_i(\mathcal{F})/T_{i-1}(\mathcal{F})$ is non-zero then, by [HL10, Theorem 1.3.4], there exists an unique Harder-Narasimahan filtration

$$0 = E_{-1} \subsetneq E_0 \subsetneq E_1 \subsetneq ... \subsetneq E_l = T_i(\mathcal{F})/T_{i-1}(\mathcal{F})$$

of the pure sheaf $T_i(\mathcal{F})/T_{i-1}(\mathcal{F})$ such that E_j/E_{j-1} is semistable of dimension i, for every $j = 1, ..., l$. As Euler characteristic is additive,

$$\chi(T_i(\mathcal{F})/T_{i-1}(\mathcal{F})) = \sum_j \chi(E_j/E_{j-1}) \text{ and } \chi(\mathcal{F}) = \sum_i \chi(T_i(\mathcal{F})/T_{i-1}(\mathcal{F})).$$

Since $i \leq m$, this implies $\mathrm{elw}_m(Y) \subset \mathrm{elw}_m^{ss}(Y)$. Therefore, $\mathrm{elw}_m^{ss}(Y) = \mathrm{elw}_m(Y)$, for all $0 \leq m \leq n$. This proves the proposition. □

3 Specialization properties of the index function

We first observe (Example 3.1) that the Esnault-Levine-Wittenberg index does not vary in an algebraic way in flat families of projective varieties (in particular, not semi-continuous). But $\mathrm{elw}_i(-)$ is invariant under specialization (Theorem 3.4).

Example 3.1. Denote by $Y := \mathbb{A}^1_{\mathbb{R}}$ and X the hypersurface in $\mathbb{P}^2 \times Y$ defined by the polynomial $X^2 + Y^2 = t$, where t is the coordinate corresponding to $\mathbb{A}^1$. Denote by $\pi : X \to Y$ the natural projection map. Observe that π is a flat morphism.

Denote by $I^i_\pi : Y \to \mathbb{Z}$ the function that associates to any $y \in Y$, the number $\mathrm{elw}_i(X_y)$. We prove that I^0_π is not semi-continuous. Observe that for $t > 0$, there always exists rational point on the fiber X_t, hence $\mathrm{elw}_0(X_t) = 1$. For $t \leq 0$, there does not exist a rational point but the resulting curve X_t contains a $\mathbb{C}$-point. But $\mathbb{C}$ is an extension of degree 2 over $\mathbb{R}$. Hence, $\mathrm{elw}_0(X_t) = 2$ for $t \leq 0$. This shows that I^0_π is not semi-continuous (in the Zariski topology).

Notation 3.2. Let R be a discrete valuation ring, $\pi : X_R \to \mathrm{Spec}(R)$ be a flat, projective morphism and X_R is an integral scheme. Suppose that the dimension of the fibers is equal to n. Denote by K (resp. k) the fraction field (resp. residue field) of R. Denote by X_k the special fiber and by X_K the generic fiber. There is no assumption on the characteristic of k and K.

Proposition 3.3. Suppose $\mathcal{F}_k$ is a semistable coherent sheaf on X_k. Then, there exists a semistable coherent sheaf $\mathcal{F}_R$ on X_R such that $\mathcal{F}_R \otimes_R k \cong \mathcal{F}_k$.

Proof. As $\mathcal{F}_k$ is semistable there exists a free $\mathcal{O}_{X_k}$-module $\mathcal{H}_k$ such that $\mathcal{F}_k$ is a quotient of $\mathcal{H}_k$. Let the resulting short exact sequence be of the form:

$$0 \to \mathcal{G}_k \to \mathcal{H}_k \to \mathcal{F}_k \to 0.$$

As $\mathcal{H}_k$ is a free $\mathcal{O}_{X_k}$-module, there exists an unique free $\mathcal{O}_{X_R}$-module $\mathcal{H}_R$ such that $\mathcal{H}_R \otimes_R k$. Denote by $\mathcal{G}_R$ the fiber product $\mathcal{H}_R \times_{\mathcal{H}_k} \mathcal{G}_k$ i.e., the set of sections of $\mathcal{H}_R$, which when restricted to $\mathcal{H}_k$ gives us $\mathcal{G}_k$. Denote by $\mathcal{F}_R$ the quotient of $\mathcal{H}_R$ by $\mathcal{G}_R$. As $\mathcal{G}_R \otimes_R k \cong \mathcal{G}_k$, the universal property of cokernel implies $\mathcal{F}_R \otimes_R k \cong \mathcal{F}_k$. Since $\mathcal{G}_R$ is a subsheaf of a free sheaf it is torsion free and as R is principal ideal domain, $\mathcal{G}_R$ is R-flat (see [Har77, Example III.9.1.3]). Using [HL10, Lemma 2.1.4], we have $\mathcal{F}_R$ is R-flat. By the open nature of semi-stability [HL10, Proposition 2.3.1], this implies $\mathcal{F}_R$ is semi-stable. This proves the proposition. □

We finally prove the main theorem of this article.

Theorem 3.4. The following is true: for any $0 \leq i \leq n$, $\mathrm{elw}_i(X_K) = \mathrm{elw}_i(X_k)$.

Proof. By [Kol13, Proposition 4]

$$\mathrm{elw}_i(X_K) = (\chi(\mathcal{O}_Z) | Z \subset X, \text{ of dimension } \leq i).$$

Given, any Z on X_K, the scheme-theorectic closure $\bar{Z}$ is flat over $\mathrm{Spec}(R)$ (see [Har77, Proposition III.9.8]). As Euler characteristic is invariant in flat families, $\chi(\mathcal{O}_Z) = \chi(\mathcal{O}_{\bar{Z}_k})$, where $\bar{Z}_k$ is the special fiber for the flat morphism from $\bar{Z}$ to $\mathrm{Spec}(R)$. Hence, for $0 \leq i < n$, $\mathrm{elw}_i(X_K) \geq \mathrm{elw}_i(X_k)$.

Using Proposition 3.3, we conclude that $\mathrm{elw}_i^{ss}(X_K) \leq \mathrm{elw}_i^{ss}(X_k)$, for all $0 \leq i \leq n$. By Proposition 2.4, we get

$$\mathrm{elw}_i(X_K) = \mathrm{elw}_i^{ss}(X_K) \leq \mathrm{elw}_i^{ss}(X_k) = \mathrm{elw}_i(X_k).$$

This implies $\mathrm{elw}_i(X_K) = \mathrm{elw}_i(X_k)$.

This completes the proof of the theorem. □

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BCAM - Basque Centre for Applied Mathematics, Alameda de Mazarredo 14, 48009 Bilbao, Spain.
E-mail address: adan@bcamath.org

Freie Universität Berlin, FB Mathematik und Informatik, Arnimallee 3, 14195 Berlin, Germany.
E-mail address: kaur@math.fu-berlin.de