A NOTE ON THE DETERMINANT MAP

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ABSTRACT. Classically, there exists a determinant map from the moduli space of semi-stable sheaves on a smooth, projective variety to the Picard scheme. Unfortunately, if the underlying variety is singular, then such a map does not exist. In the case the underlying variety is a nodal curve, a similar map was produced by Bhosle on a stratification of the moduli space of semi-stable sheaves. In this note, we generalize this result to the higher dimension case.

1. INTRODUCTION

Recall, the classical notion of determinant of a coherent sheaf. Given a projective scheme X and a coherent sheaf $\mathcal{F}$ with a finite locally free resolution,

$$0 \to \mathcal{L}_r \to \mathcal{L}_{r-1} \to ... \to \mathcal{L}_0 \to \mathcal{F} \to 0,$$

the determinant of $\mathcal{F}$, $\det(\mathcal{F})$ is defined to be $\otimes \det(\mathcal{L}_i)^{(-1)^i}$. If X is not regular, one cannot guarantee the existence of such a finite locally free resolution. So, the classical definition of determinant cannot be extended to the general case.

One of the first results in this direction was due to Bhosle (see [Bho92, Proposition 4.7]), where she considers moduli of semi-stable sheaves on nodal curves. She introduces the theory of parabolic bundles and their moduli spaces. Using this, she defines a determinant map from a stratification of a given moduli space of semi-stable sheaves on the nodal curve to certain moduli spaces of parabolic line bundles. The stratification of the moduli space arises from an explicit description of the stalk of a torsion-free sheaf at a node on a curve. Unfortunately, such a description does not exist for higher dimensional projective varieties. Hence, her techniques cannot be generalized to higher dimension. In this article, we use completely different techniques to obtain a similar result without any restriction on the dimension of the underlying scheme.

In this article we introduce the notion of the *alternating determinant* of a rank n coherent sheaf $\mathcal{F}$, denoted $\mathcal{A}\mathrm{lt}^n(\mathcal{F})$. This sheaf is a semi-stable rank one sheaf (see Proposition 3.5). We prove that,

Theorem 1.1 (see Theorems 4.14 and 5.1). There exists a stratification

$$M_X(P) = \coprod_{i=1}^{\infty} V_{R_i},$$

by locally closed subscheme V_{R_i} satisfying: $\dim V_{R_i} > V_{R_{i+1}}$ unless $V_{R_i} = \emptyset$ and for each i, there exists a Hilbert polynomial L_i such that there exists an alternating determinant map

$$\text{Adet} : \coprod V_{R_i} \to \coprod M_X(L_i)$$

with $\text{Adet}|_{V_{R_i}}$ taking values in $M_X(L_i)$.

In the case, the underlying variety is a nodal curve, our stratification agrees with the one given in [Bho92]. Moreover, there is a natural map from $\mathcal{A}\mathrm{lt}^n(\mathcal{F})$ to the determinant of $\mathcal{F}$, defined by Bhosle in terms of parabolic line bundles. The two sheaves differ only at the nodes by certain mixed terms described in [Bho92, §4.6].

Furthermore, if the underlying variety X is smooth, we have the following:

Theorem 1.2. Let X be a smooth, projective variety and P the Hilbert polynomial of a torsion-free semi-stable rank n sheaf on X with degree coprime to n. Denote by $M_X(P)$ the moduli space of torsion-free semi-stable sheaves on X of rank n. Then,

(1) there exists a Hilbert polynomial L of a rank 1 torsion-free sheaf on X such that for any closed point $s \in M_X(P)$, the corresponding coherent sheaf $\mathcal{F}_s$ satisfies the condition: $\mathcal{A}\mathrm{lt}^n(\mathcal{F}_s)$ has Hilbert polynomial L. Furthermore, $\mathcal{A}\mathrm{lt}^n(\mathcal{F}_s)$ is an invertible sheaf.

(2) there exists a natural map $\text{Adet} : M_X(P) \to M_X(L)$ which maps a closed point $s \in M_X(P)$ to the point in $M_X(L)$ corresponding to $\mathcal{A}\mathrm{lt}^n(\mathcal{F}_s)$.

See Proposition 3.7 and Corollary 5.2 for a proof of the statement.

We now discuss our strategy. We define the alternating determinant $\mathcal{A}\mathrm{lt}^n(\mathcal{F})$ as the sheaf of alternating multilinear n-forms on $\mathcal{F}$ (see Definiton 3.3). Ofcourse, if a sheaf is locally free then its alternating determinant is the same as the dual of its determinant. One can check that in the case X is non-singular and $\mathcal{F}$ is torsion-free, the alternating determinant of $\mathcal{F}$ is also isomorphic to the dual of the determinant i.e., $\mathcal{A}\mathrm{lt}^n(\mathcal{F}) = \det(\mathcal{F})^\vee$ (see Proposition 3.7). In general (when X is just a projective variety), $\mathcal{A}\mathrm{lt}^n(\mathcal{F})$ is a rank one semi-stable sheaf. Therefore, a map on a moduli space of semi-stable sheaves, induced by taking alternating determinant, must have image in a moduli space of rank one semi-stable sheaves. Using Yoneda embedding one observes that such a map must be induced by a natural transformation between the corresponding moduli functors. This is where the problem lies. The obstruction to defining such a natural tranformation is the fact that alternating determinant of a sheaf need not commute with pullback. However, we can stratify the moduli space such that there exists a well-defined alternating determinant map on each strata.

We first prove that given any locally closed subscheme of the Quot-scheme parametrizing semi-stable quotient sheaves, the locus of points where the $\mathcal{H}$om-functor commute with pull-back, is open (see Theorem 2.2 and Proposition 4.6). This gives rise to the required stratification on the semi-stable locus of the Quot-scheme (see Notation 4.13). One can then observe that this stratification induces a similar stratification on the moduli space (see Corollary 4.12 and Theorem 4.14). The remaining statements of Theorem 1.1 is not hard. The proof of Theorem 1.2 is a direct application of reflexive sheaves and basic properties of locally free sheaves.

Notation 1.3. We fix some notations that will be used throughout this article. Denote by k, an algebraically closed field of *any* characteristic. Given a projective k-variety X and a k-algebra A, denote by $X \times A$ the scheme $X \times \mathrm{Spec}(A)$. Given a sheaf $\mathcal{F}$ on $X \times A$ and an A-module M, denote by $\mathcal{F} \otimes_A M$, the sheaf associated to the presheaf which to an open set $U \subset X \times A$ associates the $\mathcal{O}_{X \times A}(U)$-module $\mathcal{F}(U) \otimes_A M$.

2. Hom functor on families of schemes

The aim of this section is to study variation of dual of a coherent sheaf in flat families of noetherian schemes. We prove that given a flat family $f : X \to Y$ and a coherent sheaf $\mathcal{F}$ on X, the locus of points $u \in Y$ for which the natural map $(\mathcal{F}^\vee)|_{X_u} \to (\mathcal{F}|_{X_u})^\vee$ is an isomorphism, is open in Y (see Theorem 2.2). This is a generalization of [Har77, Theorem III.12.11] with two major differences: the sheaf $\mathcal{F}$ need not be flat, contrary to the assumption in the reference. Furthermore, in the case $Y = \mathrm{Spec}(A)$ for a noetherian ring A, the functor T^i from the category of A-modules to itself (see [Har77, III. §12]) does not map a finitely generated A-module M to another finitely generated A-module, hence differs from the setup in the reference. Due to these two properties, several conclusions in the reference fail. We circumvent these problems to prove Theorem 2.2. The theorem plays an important role in the remaining part of the article.

Setup 2.1. Let $f : X \to Y$ be a proper, flat surjective morphism between noetherian schemes. Assume $Y = \mathrm{Spec}(A)$ for some noetherian ring A. Let $\mathcal{F}$ be a coherent sheaf on X (not necessarily flat over Y).

The main result of this section is the following:

Theorem 2.2. Suppose there exists $t \in Y$ such that the natural morphism

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)|_{X_t} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_{X_t})$$

is isomorphic. Then, there exists an open neighbourhood U in Y containing t such that for all $u \in U$, the morphism

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)|_{X_u} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_{X_u})$$

is isomorphic.

Definition 2.3. Denote by $\mathfrak{M}_A$ the category of A-modules and Coh_X the category of coherent sheaves on X. Define the functor $T^i : \mathfrak{M}_A \to \mathrm{Coh}_X$ which associates to an A-module M, the sheaf $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X \otimes_A M)$, where $\mathcal{O}_X \otimes_A M$ is the sheaf defined by $\mathcal{O}_X \otimes_A M(U) = \mathcal{O}_X(U) \otimes_A M$ for any open set $U \subset X$.

Definition 2.4. For a given $x \in X$, denote by T_x^i the functor from the category of $\mathcal{O}_{Y,f(x)}$-modules to finitely generated $\mathcal{O}_{X,x}$-modules, which takes an $\mathcal{O}_{Y,f(x)}$-module M to $\text{Ext}^i_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,f(x)}} M)$. We say that T^i is *left exact* (resp. *right exact, exact*) at *some point* $y_0 \in Y$ if for all points $x \in f^{-1}(y_0)$, T_x^i is left exact (resp. right exact, exact) on the category of $\mathcal{O}_{Y,y_0}$-modules.

Given a complex $N^\bullet$,

$$0 \to N^0 \xrightarrow{d_0} N^1 \xrightarrow{d_1} N^2 \xrightarrow{d_3} ...$$

denote by $W^i(N^\bullet) := \text{coker}(d_{i-1} : N^{i-1} \to N^i)$. As $d_i^2 = 0$, we have a natural morphism $W^i(N^\bullet) \to N^{i+1}$. The kernel of this morphism is $H^i(N^\bullet)$. Fix a locally free resolution of $\mathcal{F}$,

$$... \to \mathcal{L}_2 \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{F} \to 0$$

Denote by $\mathcal{H}\text{om}^\bullet$ the complex

$$0 \to \mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{L}_0, \mathcal{O}_X) \xrightarrow{d_0} \mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{L}_1, \mathcal{O}_X) \xrightarrow{d_1} ...$$

Although the following Proposition is similar to [Har77, Proposition III.12.7], the proof in the reference does not hold in our setup as $W^i(\mathcal{H}\text{om}^\bullet)_x$ is not a finitely generated A-module for any $x \in X$, which is used in an important step in the reference.

Proposition 2.5. If T^i is left exact (resp. right exact) at some point $y_0 \in Y$, then the same is true for all points y in a suitable open neighbourhood U of y_0.

Proof. By Proposition A.3, T^i is left exact at y_0 if and only if $W^i(\mathcal{H}\text{om}^\bullet)_x$ is $\mathcal{O}_{Y,y_o}$-flat for all $x \in f^{-1}(y_0)$. By the open nature of flatness, there exists an open neighbourhood of U containing $f^{-1}(y_0)$ such that $W^i(\mathcal{H}\text{om}^\bullet)_u$ is $\mathcal{O}_{Y,f(u)}$-flat for all $u \in U$. As f is proper and $X \backslash U$ is closed, so is $f(X \backslash U)$. Denote by $V := Y \backslash f(X \backslash U)$. Then, for all $v \in V$ and $v_x \in f^{-1}(v)$, $W^i(\mathcal{H}\text{om}^\bullet)_{v_x}$ is $\mathcal{O}_{Y,v}$-flat. Applying Proposition A.3 once again, we conclude that T^i is left-exact at v for all $v \in V$.

Given a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

there exists an exact sequence by Lemma A.2,

$$T^i(M') \to T^i(M) \to T^i(M'') \to T^{i+1}(M') \to T^{+1}(M) \to T^{i+1}(M'').$$

Then, T^i is right exact at a point y if and only if T^{i+1} is left exact at y. So, the second statement follows from the first applied to T^{i+1}. This proves the proposition. $\square$

The following statement is similar to [Har77, Proposition III.12.10], but several steps in the proof given in the reference fails in our setup. As above this is because $T^0(M)_x$ is not a finitely generated A-module in our case (a fact used extensively in the reference).

Proposition 2.6. Assume that for some $y \in Y$, the map

$$\phi : T^0(A) \otimes k(y) \to T^0(k(y))$$

is surjective. Then, T^0 is right exact at y.

Proof. By making a flat base extension, $\operatorname{Spec} \mathcal{O}_{Y,y} \to Y$ if necessary, we may assume that y is a closed point of Y, A is a local ring with maximal ideal m and $k(y) = A/m$. By Proposition A.4, it is sufficient to show that $\phi(M): T^0(A) \otimes M \to T^0(M)$ is surjective for all A-modules M. Since T^0 and tensor product commute with direct limits, it is sufficient to consider finitely generated M.

First, we consider A-modules M of finite length and we show that $\phi(M)$ is surjective, by induction on the length of M. If the length is 1 then $M = k$ and $\phi(k)$ is surjective by hypothesis. Given a short exact sequence,

$$0 \to M' \to M \to M'' \to 0,$$

of A-modules of finite length, we have M' and M'' have length less than the length of M. As T^0 is exact in the middle (Lemma A.2), we have a commutative diagram with exact rows:

$$
\begin{array}{ccccccc}
T^0(A) \otimes M' & \longrightarrow & T^0(A) \otimes M & \longrightarrow & T^0(A) \otimes M'' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
T^0(M') & \longrightarrow & T^0(M) & \longrightarrow & T^0(M'') & &
\end{array}
$$

The two outside vertical arrows are surjective by the induction hypothesis, so the middle one is surjective also.

Now let M be any finitely generated A-module. For each n, $M/m^n M$ is a module of finte length, so by the previous case, $\phi_n : T^0(A) \otimes M/m^n M \to T^0(M/m^n M)$ and $T^0(A) \otimes (m^n/m^{n+1})M \to T^0((m^n/m^{n+1})M)$ are surjective. Denote by $\mathcal{K}_n$ the kernel of the induced morphism $\psi_n : T^0((m^n/m^{n+1})M) \to T^0(M/m^{n+1}M)$. Then, the morphism ψ_n factors through $T^0((m^n/m^{n+1})M)/\mathcal{K}_n$. As T^0 is exact in the middle by Lemma A.2, we have the following commutative diagram of exact sequences:

$$
\begin{array}{ccccccc}
T^0(A) \otimes m^n/m^{n+1}M & \longrightarrow & T^0(A) \otimes M/m^{n+1}M & \longrightarrow & T^0(A) \otimes M/m^n M & \longrightarrow & 0 \\
\downarrow & & \phi_{n+1}\downarrow & & \phi_n\downarrow & & \\
0 \longrightarrow \dfrac{T^0(m^n/m^{n+1}M)}{\mathcal{K}_n} & \xrightarrow{\ \psi_n\ } & T^0(M/m^{n+1}M) & \longrightarrow & T^0(M/m^n M) & &
\end{array}
$$

Finally, using Snake lemma, this implies the natural morphism from $\ker(\phi_{n+1})$ to $\ker(\phi_n)$ is surjective. By [Har77, Example II.9.1.1] this implies $\ker(\phi_n)$ satisfies the Mittag-Leffler condition. Then, by [Har77, Proposition II.9.1], the map

$$\varprojlim \phi_n : T^0(A) \otimes M^\wedge \to \varprojlim T^0(M/m^n M)$$

is surjective. Denote by $i : \hat{X} \to X$ the completion of X with respect to m (see [Har77, §II.9]). Now, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, -)$ commutes with inverse limit i.e.,

$$\varprojlim T^0(M/m^n M) = \varprojlim \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X \otimes_A M/m^n M) \cong$$

$$\cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \varprojlim \mathcal{O}_X \otimes_A M/m^n M) \cong$$

$$\cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, i_*(i^*(\mathcal{O}_X \otimes_A M))).$$

By the adjunction property,

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, i_*(i^*(\mathcal{O}_X \otimes_A M))) \cong \mathcal{H}om_{\mathcal{O}_{\hat{X}}}(i^*\mathcal{F}, i^*(\mathcal{O}_X \otimes_A M)) = \mathcal{H}om_{\mathcal{O}_{\hat{X}}}(\hat{\mathcal{F}}, (\mathcal{O}_X \otimes_A M)^\wedge)$$

which by EGA-III$_\mathrm{I}$ Proposition 12.3.5, is isomorphic to $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X \otimes_A M)^\wedge$ i.e.,

$$\varprojlim T^0(M/m^n M) \cong T^0(M)^\wedge.$$

In particular, the map

$$\varprojlim \phi_n : \left(T^0(A) \otimes M\right)^\wedge \to \left(T^0(M)\right)^\wedge \text{ is surjective.}$$

Since completion is an exact, faithful functor for finitely generated $\mathcal{O}_X$-modules (see [Har77, Theorem II.9, 3A]), it follows that

$$\phi(M) : T^i(A) \otimes M \to T^i(M)$$

is surjective and we are done. $\square$

Finally, using Propositions 2.5 and 2.6, we can prove Theorem 2.2.

Proof of Theorem 2.2. As there exists $t \in Y$ such that the natural morphism

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \twoheadrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)|_{X_t} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_{X_t})$$

is surjective, Proposition 2.6 implies T^0 is right exact at t. Then, Proposition 2.5 implies there exists an open neighbourhood U containing t such that for all $u \in U$, T^0 is right exact at u. By Proposition A.4, this is equivalent to the morphism

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)|_{X_u} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_{X_u})$$

being an isomorphism. This proves the theorem. $\square$

3. Alternating determinant of coherent sheaves

In this section, we give the definition of alternating determinant of coherent sheaves. This is the dual of the top wedge product of the coherent sheaf. We observe that the alternating determinant of a torsion-free sheaf on an integral scheme is a semi-stable rank one torsion-free sheaf (Proposition 3.5). Moreover, if the underlying scheme is non-singular then the alternating determinant of a torsion-free sheaf is isomorphic to the dual of the determinant (Proposition 3.7).

Recall, the following definition:

Definition 3.1. Let E be a coherent sheaf of dimension d on a projective scheme X. Write the Hilbert polynomial $P(E, m)$ of E as follows:

$$P(E, m) = \sum_{i=0}^{d} \alpha_i(E) \frac{m^i}{i!}.$$

If $\dim X = d$ then, the rank of E is defined as

$$\mathrm{rk}(E) := \frac{\alpha_d(E)}{\alpha_d(\mathcal{O}_X)}$$

and the degree of E is defined as

$$\deg(E) := \alpha_{d-1}(E) - \mathrm{rk}(E).\alpha_{d-1}(\mathcal{O}_X).$$

Setup 3.2. Let k be an algebraically closed field (of any characteristic), X an integral k-scheme. Let P be the Hilbert polynomial of a coherent sheaf on X of rank n with degree coprime to n.

Definition 3.3. Let R be a ring and M a finitely generated R-module. For any integer $n > 0$, denote by $\operatorname{Alt}_R^n(M, R)$, the R-submodule of $\operatorname{Hom}_R(M \otimes_R M \otimes_R ... \otimes_R M, R)$ consisting of alternating R-multilinear maps from n-copies of M to R.

Let X be a projective scheme and $\mathcal{F}$ a pure coherent sheaf on X of rank n. Let $\{U_i\}$ be an affine open covering of X. Define $\mathcal{A}\mathrm{lt}_{U_i}^n(\mathcal{F})$, the coherent sheaf associated to the finitely generated $\mathcal{O}_{U_i}(U_i)$-module $\operatorname{Alt}_{\mathcal{O}_{U_i}(U_i)}^n(\mathcal{F}(U_i))$. We call the the *alternating determinant* of $\mathcal{F}$, denoted $\mathcal{A}\mathrm{lt}^n(\mathcal{F})$, the sheaf obtained via glueing $\mathcal{A}\mathrm{lt}_{U_i}^n(\mathcal{F})$

Lemma 3.4. Let X be an projective scheme, $\mathcal{F}$ a coherent pure sheaf on X. Then, for any positive integer n, the sheaf $\mathcal{A}\mathrm{lt}^n(\mathcal{F})$ is isomorphic (as $\mathcal{O}_X$-modules) to $\mathcal{H}om_{\mathcal{O}_X}(\bigwedge^n \mathcal{F}, \mathcal{O}_X)$.

Proof. There is a natural surjective morphism

$$i : \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} ... \otimes_{\mathcal{O}_X} \mathcal{F}(n - \text{copies}) \to \bigwedge^n \mathcal{F}$$

defined on small enough open sets by $m_1 \otimes m_2 \otimes ... \otimes m_n \mapsto m_1 \wedge m_2 \wedge ... \wedge m_n$. This induces an injective morphism

$$i^* : \mathcal{H}om_{\mathcal{O}_X}(\bigwedge^n \mathcal{F}, \mathcal{O}_X) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F} \otimes ... \otimes \mathcal{F}, \mathcal{O}_X).$$

Clearly, the image of i^* is contained in $\mathcal{A}\mathrm{lt}^n(\mathcal{F})$. It remain to prove that the image of i^* coincides with $\mathcal{A}\mathrm{lt}^n(\mathcal{F})$. It suffices to prove this on the level of stalks, which follows from the universal property of exterior powers: Indeed, for any $x \in X$ and an alternating $\mathcal{O}_{X,x}$-multilinear n-form $\phi : \mathcal{F}_x \otimes \mathcal{F}_x \otimes ... \otimes \mathcal{F}_x \to \mathcal{O}_{X,x}$, there exists an unique morphism $\psi : \bigwedge^n \mathcal{F}_x \to \mathcal{O}_{X,x}$ such that $\psi \circ i_x = \phi$, where i_x is the localization of the map i at the point x. This proves the lemma. $\square$

We see now that the alternating determinant of a torsion-free sheaf is semi-stable.

Proposition 3.5. Let X be an integral projective scheme and $\mathcal{F}$ a torsion-free sheaf on X of rank n. Then, the alternating determinant $\mathcal{A}\mathrm{lt}^n(\mathcal{F})$ is a rank 1 semi-stable sheaf.

Proof. As $\mathcal{F}$ is torsion-free, there exists a non-empty open, dense subset $U \subset X$ such that $\mathcal{F}|_U$ is locally-free. Let $\mathcal{G}$ be a subsheaf of $\mathcal{A}\mathrm{lt}^n(\mathcal{F})$. By Lemma 3.4, $\mathcal{A}\mathrm{lt}^n(\mathcal{F})$ is torsion-free (dual of a coherent sheaf is torsion-free), hence so is $\mathcal{G}$. Denote by $\mathcal{G}'$ the cokernel $\mathcal{A}\mathrm{lt}^n(\mathcal{F})/\mathcal{G}$. As $\mathcal{F}|_U$ is locally free, $\mathcal{A}\mathrm{lt}^n(\mathcal{F})|_U$ is invertible, which means $\mathcal{G}'|_U$ is supported in a codimension 1 subscheme in U (quotient of two rank 1 torsion-free sheaves is torsion). Therefore, $\mathcal{G}'$ is supported in a codimension 1 subscheme in X. This means, the Hilbert polynomial corresponding to $\mathcal{G}'$ is of degree at most dimension $\dim X - 1$. Obviously, the leading coefficient of the Hilbert polynomial corresponding to $\mathcal{G}'$ is positive. It then follows from the definition of degree mentioned above that $\deg(\mathcal{G}) \leq \deg(\mathcal{A}\mathrm{lt}^n(\mathcal{F}))$. As rank of $\mathcal{A}\mathrm{lt}^n(\mathcal{F})$ and $\mathcal{G}$ are of rank 1, $\mathcal{A}\mathrm{lt}^n(\mathcal{F})$ is therefore semi-stable. This proves the proposition. $\square$

Definition 3.6. Let $f : Y \to S$ be a flat morphism of schemes and $\mathcal{E}$ a coherent sheaf on Y flat over S. We say that $\mathcal{E}$ is S_r *relative to* f if the following holds: for each $x \in Y$, $s = f(x)$, we have

$$\mathrm{depth}_{\mathcal{O}_{Y_s,x}}(\mathcal{E}|_{Y_s}) \geq \min(r, \dim \mathcal{O}_{Y,x} - \dim \mathcal{O}_{S,s}).$$

In other words, the restriction of $\mathcal{E}$ to each fiber is S_r.

The following proposition tells us that the alternating determinant of a torsion-free sheaf on a smooth scheme coincides with the dual of its determinant.

Proposition 3.7. Suppose that X is a non-singular variety and S is a k-scheme. For any coherent, torsion-free sheaf $\mathcal{F}_S$ of rank n on X_S, flat over S, its alternating determinant $\mathcal{A}\mathrm{lt}^n(\mathcal{F}_S)$ is isomorphic to $\det(\mathcal{F}_S)^\vee$ (see [HL10, §1.1.17] for the definition of determinant).

Proof. Denote by $\pi : X_S \to S$ the natural projection map. Since X is non-singular, π is smooth. Hence, there exists a finite, locally free resolution of $\mathcal{F}_S$. Choose one such resolution,

$$0 \to \mathcal{L}_m \xrightarrow{\phi_m} \mathcal{L}_{m-1} \xrightarrow{\phi_{m-1}} ... \xrightarrow{\phi_1} \mathcal{L}_0 \xrightarrow{\phi_0} \mathcal{F}_S \to 0.$$

Denote by n_i the rank of $\mathcal{L}_i$ for $i = 1, ..., m$. By definition, $\det(\mathcal{F}) = \otimes_i \left(\bigwedge^{n_i} \mathcal{L}_i \right)^{(-1)^i}$. Consider now the short exact sequences:

$$0 \to \ker \phi_i \to \mathcal{L}_i \xrightarrow{\phi_i} \ker \phi_{i-1} \to 0 \ \text{ for all } i \geq 1, \ \text{ and } 0 \to \ker \phi_0 \to \mathcal{L}_0 \to \mathcal{F}_S \to 0.$$

As $\mathcal{L}_i$ and $\mathcal{F}_S$ are flat over S, one can prove recursively that $\ker \phi_i$ is also flat over S for all i. Since $\mathcal{L}_i$ are locally-free and $\mathcal{F}_S$ is torsion-free, [Har80, Proposition 1.1] implies $\ker \phi_0$ is S_2 relative to π. Hence, by recursion $\ker \phi_i$ is also S_2 relative to π for all $i \geq 0$. By [HL10, Lemma 2.1.8], one can show that there exists an open set $U \subset X_S$ with $X_s \backslash U_s$ of codimension at least 2 for all $s \in S$ and $\ker \phi_i|_U$ and $\mathcal{F}_S|_U$ are locally free of rank, say n_i' and n, respectively. Then, restricting the above set of short exact sequences to U we get,

$$\left(\bigwedge^{n_i} \mathcal{L}_i|_U \right) \cong \left(\bigwedge^{n_i'} \ker \phi_i|_U \right) \otimes \left(\bigwedge^{n_{i-1}'} \ker \phi_{i-1}|_U \right) \quad \text{for all } i > 0 \text{ and}$$

$$\left(\bigwedge^{n_0} \mathcal{L}_0|_U \right) \cong \left(\bigwedge^{n_i'} \ker \phi_0|_U \right) \otimes \left(\bigwedge^n \mathcal{F}_S|_U \right).$$

As taking dual commutes with tensor product of locally free sheaves, we have

$$\left(\bigwedge^{n_i} \mathcal{L}_i|_U \right)^\vee \cong \left(\bigwedge^{n_i'} \ker \phi_i|_U \right)^\vee \otimes \left(\bigwedge^{n_{i-1}'} \ker \phi_{i-1}|_U \right)^\vee \quad \text{for all } i > 0 \text{ and}$$

$$\left(\bigwedge^{n_0} \mathcal{L}_0|_U \right)^\vee \cong \left(\bigwedge^{n_i'} \ker \phi_0|_U \right)^\vee \otimes \left(\bigwedge^n \mathcal{F}_S|_U \right)^\vee.$$

This implies

$$\left(\bigwedge^n \mathcal{F}_S|_U \right)^\vee \cong \bigotimes_{i=0}^m \left(\bigwedge^{n_i} \mathcal{L}_i|_U \right)^{(-1)^{i+1}}.$$

As $\mathcal{O}_X$ is S_2 relative to π, we have $j_* \mathcal{O}_U \cong \mathcal{O}_{X_S}$ (see [HK04, Proposition 3.5]) where $j : U \to X_S$ is the open immersion. Hence, by adjoint property of $\mathcal{H}om$ and local freeness of $\mathcal{F}|_U$, we have

$$\mathcal{H}om_{X_S}\left(\bigwedge^n \mathcal{F}_S, \mathcal{O}_{X_S}\right) \cong j_* \mathcal{H}om_U\left(\bigwedge^n \mathcal{F}_S|_U, \mathcal{O}_U\right) \cong j_*\left(\bigotimes_{i=0}^m \left(\bigwedge^{n_i} \mathcal{L}_i|_U\right)^{(-1)^{i+1}}\right).$$

The projection formula implies that,

$$j_*\left(\bigotimes_{i=0}^m \left(\bigwedge^{n_i} \mathcal{L}_i|_U\right)^{(-1)^{i+1}}\right) \cong \bigotimes_{i=0}^m \left(\bigwedge^{n_i} \mathcal{L}_i\right)^{(-1)^{i+1}}.$$

By Lemma 3.4, we finally conclude that $\mathcal{A}lt^n(\mathcal{F}_S) \cong \det(\mathcal{F}_S)^\vee$. This completes the proof of the proposition. $\qquad\square$

4. Alternating determinant stratification on the moduli space

In the introduction, we discussed the obstacle to defining an alternating determinant map on the entire moduli space of semi-stable sheaves. The aim of this section is to stratify the moduli space such that the alternating determinant map can be defined over each strata. This is done in the main theorem of this section (Theorem 4.14). We use this in the next section to define the expected alternating determinant map.

The key step is Proposition 4.6, which tells us that given any locally closed subscheme of the Quot-scheme, parametrizing semi-stable sheaves, we can find an open subscheme for which the alternating determinant functor commutes with pullback. We observe in Corollary 4.12 that under certain conditions, there exists universal geometric quotient of such an open subscheme. We combine this in Theorem 4.14 to give the required stratification and observe that on each strata the Hilbert polynomial of the alternating determinant, remains unchanged.

We first recall certain standard results on moduli spaces of semi-stable sheaves.

Definition 4.1. Let P be the Hilbert polynomial of a pure coherent sheaf on X whose rank is coprime to its degree. We define a functor $\mathcal{M}_X(P)$ as follows:

$$\mathcal{M}_X(P) : \mathrm{Sch}/k \to \mathrm{Sets}$$

such that for a k-scheme T,

$$\mathcal{M}_X(P)(T) := \left\{ \begin{array}{l} \text{isomorphism classes of pure, coherent sheaves } \mathcal{F} \text{ on } X \times_k T \\ \text{flat over } T \text{ and for every geometric point } t \in T, \mathcal{F}|_{X_t} \text{ is a} \\ \text{semi-stable sheaf with Hilbert Polynomial } P \text{ on } X_t. \end{array} \right\}$$

Remark 4.2. Since X is integral, [HL10, Lemma 1.2.13 and 1.2.14] implies that a semi-stable sheaf on X is also stable. Hence, the moduli functor $\mathcal{M}_X(P)$ coincides with the moduli functor defined by replacing in Defintion 4.1 the semi-stable condition by stable. Then, by [Lan04, Theorem 0.2] the functor $\mathcal{M}_X(P)$ is universally corepresentable by a projective k-scheme $M_X(P)$.

Notation 4.3. Let $\mathcal{F}$ be a Giesekar semi-stable sheaf on X with Hilbert polynomial P and rank n. By [HL10, Corollary 1.7.7] there exists an integer N_0 depending only on P such that $\mathcal{F}$ is e-regular for all $e \geq N_0$. Fix such an integer e. Denote by $V := H^0(\mathcal{F}(e))$ and $\mathcal{H} := \mathcal{O}_X(-e) \otimes_k V$. Denote by $\mathrm{Quot}_{\mathcal{H}/X/P}$ the scheme parametrizing all quotients of the form $\mathcal{H} \twoheadrightarrow \mathcal{Q}_0$, where $\mathcal{Q}_0$ has Hilbert polynomial P with $H^0(\mathcal{Q}_0(e))$ (non-canonically) isomorphic to V. Denote by $\mathcal{Q}$ the universal quotient on $X \times \mathrm{Quot}_{\mathcal{H}/X/P}$ associated to $\mathrm{Quot}_{\mathcal{H}/X/P}$.

Denote by R the subset of $\mathrm{Quot}_{\mathcal{H}/X/P}$ parametrizing coherent quotients of $\mathcal{H}$ which are semi-stable sheaves on X with Hilbert polynomial P. By [HL10, Proposition 2.3.1] one notices that R is an open subscheme in $\mathrm{Quot}_{\mathcal{H}/X/P}$. The group $\mathrm{GL}(V) = \mathrm{Aut}(\mathcal{H})$ acts on $\mathrm{Quot}_{\mathcal{H}/X/P}$ from the right by the composition $[\rho] \circ g = [\rho \circ g]$ for some $[\rho : \mathcal{H} \to \mathcal{F}]$ and $g \in \mathrm{GL}(V)$. By [HL10, Theorem 4.3.3] R is the set of semi-stable points of $\mathrm{Quot}_{\mathcal{H}/X/P}$ under this group action. This induces a group action of $\mathrm{GL}(V)$ on R. By [HL10, Lemma 4.3.1], $M_X(P)$ is the geometric quotient of R by this action. Denote by

$$\pi : R \to M_X(P)$$

the corresponding geometric quotients. By [HL10, Corollary 4.3.5], the quotient π is a $\mathrm{PGL}(V)$-bundle.

Remark 4.4. Given a morphism of finite type $f : Y \to Z$ between noetherian schemes and a coherent sheaf $\mathcal{F}$ on Z, there is a natural morphism from $f^* \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{F}, \mathcal{O}_Z)$ to $\mathcal{H}om_{\mathcal{O}_Y}(f^*\mathcal{F}, \mathcal{O}_Y)$. In particular, using Lemma 3.4, this induces a natural morphism from $f^* \mathcal{A}lt^n(\mathcal{F})$ to $\mathcal{A}lt^n(f^*\mathcal{F})$, where $n = \mathrm{rk}(\mathcal{F})$.

Notation 4.5. Denote by $\mathcal{H}_R$ the pullback of the sheaf $\mathcal{H}$ under the natural projection map $\mathrm{pr}_1 : X \times_k R \to X$. Recall, we have the universal quotient:

$$\mathcal{H}_R \to \mathcal{Q}|_{X \times R} \to 0.$$

Taking the wedge powers, we get the surjective morphism

$$\phi_n : \bigwedge^n \mathcal{H}_R \to \bigwedge^n \mathcal{Q}|_{X \times R}.$$

Denote by $\mathcal{G}_n$ the kernel of the morphism ϕ_n, by $Q_n := \bigwedge^n \mathcal{Q}|_{X \times R}$ and $\mathcal{H}_n := \bigwedge^n \mathcal{H}_R$.

The following proposition can be formulated more generally in terms of families of coherent sheaves parametrized by a noetherian scheme, however for simplicity, we restrict to the setup relevant for this article.

Proposition 4.6. Let B be a locally closed subscheme of R. Then, there exists an open dense subscheme U_B of B such that for all $u \in U_B$, the induced morphism

$$\mathcal{A}lt^n(\mathcal{Q}|_{X_B}) \otimes k(u) \to \mathcal{A}lt^n(\mathcal{Q}|_{X_u}) \text{ is an isomorphism.} \tag{4.1}$$

Proof. For simplicity, we abuse the notations to denote by $\mathcal{Q}_n, \mathcal{H}_n$ and $\mathcal{G}_n$ the restrictions $\mathcal{Q}_n|_{X_B}, \mathcal{H}_n|_{X_B}$ and $\mathcal{G}_n|_{X_B}$, respectively. Since $\mathcal{H}_n$ is a $\mathcal{O}_{X_B}$-free sheaf, we have the following $\mathcal{H}om$-exact sequence:

$$0 \to (\mathcal{Q}_n)^\vee \to \mathcal{H}_n^\vee \to (\mathcal{G}_n)^\vee \xrightarrow{\phi_B} \mathcal{E}xt^1_{X_B}(\mathcal{Q}_n, \mathcal{O}_{X_B}) \to 0. \tag{4.2}$$

Denote by U the open dense subscheme such that $(\ker \phi_B)|_{X_U}$ and $\mathcal{E}xt^1_{X_B}(\mathcal{Q}_n, \mathcal{O}_{X_B})|_{X_U}$ are flat over U. The existence of such an open set follows from [GW10, Theorem 10.84,

pp. 271]. Since $\mathcal{H}_n|_{X_U}$ is already flat over U, by [Har77, Proposition III.9.1A], we have $\mathcal{G}_n^\vee|_{X_U}$ and $\mathcal{Q}_n^\vee|_{X_U}$ are also flat over U. Therefore, for each $u \in U$, we have a long exact sequence:

$$0 \to (\mathcal{Q}_n)^\vee|_{X_u} \to \mathcal{H}_n^\vee|_{X_u} \to (\mathcal{G}_n)^\vee|_{X_u} \xrightarrow{\phi_B} \mathcal{E}xt^1_{X_B}(\mathcal{Q}_n, \mathcal{O}_{X_B})|_{X_u} \to 0.$$

Since $\mathcal{G}_n$ is a coherent sheaf, it can be written as a quotient of a locally free sheaf, say $\mathcal{L}_0$ on X_B (use [Har77, Corollary II.5.18]). Denote by $j : \mathcal{L}_0 \twoheadrightarrow \mathcal{G}_n$ the corresponding quotient map. Arguing as before, there exists an open dense subset V of B such that for each $v \in V$, we have the exact sequence:

$$0 \to (\mathcal{G}_n)^\vee|_{X_v} \to (\mathcal{L}_0^\vee)|_{X_v} \to (\ker j)^\vee|_{X_u} \to \mathcal{E}xt^1_{X_B}(\mathcal{G}_n, \mathcal{O}_{X_B})|_{X_v} \to 0.$$

Since $\mathcal{L}_0$ is locally-free, $(\mathcal{L}_0^\vee)|_{X_v} \cong (\mathcal{L}_0|_{X_v})^\vee$, which gives us the following commutative diagram of exact sequences:

$$
\begin{array}{ccc}
0 \longrightarrow (\mathcal{G}_n)^\vee|_{X_v} \longrightarrow (\mathcal{L}_0^\vee)|_{X_v} \\
\phi_3 \downarrow \qquad \circlearrowleft \qquad \cong \downarrow \\
0 \longrightarrow (\mathcal{G}_n|_{X_v})^\vee \longrightarrow (\mathcal{L}_0|_{X_v})^\vee
\end{array}
$$

This implies that the morphism ϕ_3 is injective. Therefore, for any point $u \in U \cap V$, we have the following diagram of exact sequences:

$$
\begin{array}{ccccccccc}
0 \longrightarrow & (\mathcal{Q}_n)^\vee|_{X_u} & \longrightarrow & (\mathcal{H}_n^\vee)|_{X_u} & \longrightarrow & (\mathcal{G}_n)^\vee|_{X_u} & \xrightarrow{\phi_B} & \mathcal{E}xt^1_{X_B}(\mathcal{Q}_n, \mathcal{O}_{X_B})|_{X_u} & \longrightarrow 0 \\
& \phi_1 \downarrow & \circlearrowleft & \phi_2 \downarrow & \circlearrowleft & \phi_3 \downarrow & \circlearrowleft & \downarrow & \\
0 \longrightarrow & (\mathcal{Q}_n|_{X_u})^\vee & \longrightarrow & (\mathcal{H}_n|_{X_u})^\vee & \longrightarrow & (\mathcal{G}_n|_{X_u})^\vee & \xrightarrow{\phi_u} & \mathcal{E}xt^1_{X_u}(\mathcal{Q}_n|_{X_u}, \mathcal{O}_{X_u}) & \longrightarrow 0
\end{array}
$$

Since $\mathcal{H}_n$ is $\mathcal{O}_{X_B}$-free, ϕ_2 is an isomorphism. Since ϕ_3 is injective, one uses the short five lemma or diagram chase to conclude that ϕ_1 is an isomorphism. Since $U \cap V$ is dense in B, we have the proposition. $\qquad \square$

Notation 4.7. Let B be as in Proposition 4.6. Using Theorem 2.2 and Proposition 4.6, there exists an open dense subscheme U_B of B such that $u \in U_B$ if and only if $u \in B$ satisfies the property (4.1) stated in Proposition 4.6.

Lemma 4.8. Suppose the $\mathrm{PGL}(V)$-bundle $\pi : R \to M_X(P)$ descends to B in the sense that there exists a locally closed subscheme $B' \subset M_X(P)$ such that $\pi|_B : B \to B'$ is a $\mathrm{PGL}(V)$-bundle. Then, for any geometric point $x_0 \in B', \pi^{-1}(x_0) \cap U_B \neq \emptyset$ if and only if $\pi^{-1}(x_0) \subset U_B$.

Proof. By the definiton of geometric good quotient, x_0 corresponds to a semi-stable coherent sheaf $\mathcal{F}_0$ on X with Hilbert polynomial P. Since $\pi|_B$ is a $\mathrm{PGL}(V)$-bundle, $\pi|_B^{-1}(x_0) = \pi^{-1}(x_0)$ which equals to quotients of the form $[\mathcal{H} \overset{\phi}{\cong} \mathcal{H} \twoheadrightarrow \mathcal{F}_0]$ as ϕ runs through all the automorphisms of $\mathcal{H}$. A quotient $[\mathcal{H} \twoheadrightarrow \mathcal{F}_0]$ is in U_B if and only if it satisfies the relation (4.1). It then follows directly that $\pi^{-1}(x_0) \cap U_B \neq \emptyset$ if and only if $\pi^{-1}(x_0) \subset U_B$. This proves the lemma. $\qquad \square$

Corollary 4.9. Hypothesis as in Lemma 4.8. For simplicity, we denote by π the restriction $\pi|_B : B \to B'$. The scheme $\pi^{-1}(\pi(B \backslash U_B))$ does not intersect U_B, where $\pi(B \backslash U_B)$ denotes the scheme-theoretic image of the closed subscheme $B \backslash U_B$.

Proof. Denote by $V := \pi^{-1}(\pi(B \backslash U_B))$. Suppose that $V \cap U_B \neq \emptyset$. Since $V \cap U_B$ is a non-empty quasi-compact scheme, it contains a closed point, say x_0. As the geometric quotient morphism π is of finite type, $\pi(x_0)$ is closed. This means in particular that there exists a closed point $\pi(x_0) \in B'$ such that $\pi^{-1}(\pi(x_0))$ intersects U_B but is not completely contained in U_B. But this contradicts Lemma 4.8. This proves the corollary. $\square$

Notation 4.10. Hypothesis as in Lemma 4.8. Denote by $V_B := B' \backslash (\pi|_B(B \backslash U_B))$. Observe that V_B is an open subscheme in B'.

This directly implies:

Corollary 4.11. The scheme U_B is isomorphic (as schemes) to $V_B \times_{B'} B \cong \pi^{-1}(V_B)$.

Proof. By Corollary 4.9, U_B is contained in $\pi^{-1}(V_B)$. We now prove that U_B coincides with $\pi^{-1}(V_B)$. Suppose this is not the case. This means that there exists a closed point $x_0 \in \pi^{-1}(V_B)$ not in U_B. Since the morphism π is of finite-type, $\pi(x_0)$ is closed. In other words, there exists a closed point $y_0 \in V_B$ such that the fiber $\pi^{-1}(y_0)$ is not entirely contained in U_B. But this contradicts Lemma 4.8, hence proves the corollary. $\square$

Corollary 4.12. The natural morphism $\pi|_{U_B}$ factors through V_B. Furthermore, the morphism $\pi : U_B \to V_B$, makes U_B into a $\mathrm{PGL}(V)$-bundle over V_B.

Proof. Since pullback of a $\mathrm{PGL}(V)$-bundle is again a $\mathrm{PGL}(V)$-bundle, Corollary 4.11 implies this corollary. $\square$

We can now define a stratification on R induced by the alternating determinant:

Notation 4.13. We follow notations as in Notations 4.7. Denote by $R_1 := U_R$, $R'_1 := R \backslash R_1$, $R_2 := U_{R'_1}$, $R'_2 := R'_1 \backslash R_2$ and inductively denote by $R_i := U_{R'_{i-1}}$ and $R'_i := R'_{i-1} \backslash R_i$. Set $R'_0 := R$.

Theorem 4.14 (Stratification). The following holds true:

   (1) there exists an integer i_0 such that for all $i \geq i_0$, $R_i = \emptyset$,
   (2) the set of locally closed subschemes, $\{R_i\}$ defines a stratification of R i.e., $R = \bigcup_{i=1}^{i_0} R_i$ and $R_i \cap R_j = \emptyset$ for $i \neq j$,
   (3) the stratification $\{R_i\}$ induces a stratification $\{V_{R_i}\}$ on $M_X(P)$ (notations as in Notation 4.10) and for each i, $\pi_i : R_i \to V_{R_i}$ is a $\mathrm{PGL}(V)$-bundle morphism.
   (4) for a fixed i, there exists a Hilbert polynomial L_i (depending only on i) such that for every $u \in R_i$ the corresponding quotient $[\mathcal{H}_u \twoheadrightarrow \mathcal{Q}_u]$ satisfies the property: $\mathcal{A}lt^n(\mathcal{Q}_u)$ has Hilbert polynomial L_i,

Proof. Using Proposition 4.6, observe that

$$\dim R = \dim R_1 > \dim R'_1 = \dim R_2, \text{ and } \dim R'_{i-1} = \dim R_i > \dim R'_i = \dim R_{i+1}.$$

Since R is finite dimensional there exists an integer i_0 such that for all $i \geq i_0$, $R_i = \emptyset$, which proves (1). Then, by construction, (2) follows. To prove (3), we need to prove that

for each i, π descends to R'_i in the sense of Lemma 4.8. Then, by Corollary 4.12, π induces a PGL(V)-bundle morphism $\pi_{i+1} : R_{i+1} \to V_{R'_i}$. Denoting $V_{R'_i}$ by $V_{R_{i+1}}$, we will have (3).

We prove that R'_i satisfies the hypothesis of Lemma 4.8, inductively. Trivially, $R'_0 = R$ satisfies the hypothesis. Suppose there exists m such that for all $i < m$, R'_i satisfies the hypothesis. This means in particular, by Corollaries 4.11 and 4.12 that for all $i \leq m$, π induces, by restriction, a PGL(V)-bundles $\pi_i : R_i \to V_{R_i}$ and $R_i = \pi^{-1}(V_{R_i})$. Denote by $V'_{R_m} := M_X(P)\backslash(V_{R_1} \coprod V_{R_2} \coprod ... \coprod V_{R_m})$. One can think of V'_{R_m} as the locally closed subscheme obtained by recursiverly removing from $M_X(P)\backslash \left(V_{R_1} \coprod ... \coprod V_{R_{i-1}}\right)$ the open subscheme V_{R_i}, for $i \leq m$. We then have

$$\pi^{-1}(V'_{R_m}) = R\backslash \left(\pi^{-1}(V_{R_1}) \coprod ... \coprod \pi^{-1}(V_{R_m})\right) = R\backslash \left(R_1 \coprod ... \coprod R_m\right) = R'_m.$$

Since base change of PGL(V)-bundle is again a PGL(V)-bundle, we have $\pi : R'_m \to V'_{R_m}$ is a PGL(V)-bundle. This proves the induction step hence (3).

We now prove (4). Since Hilbert polynomial remains unchanged in flat families, we are going to prove that the sheaf $\mathcal{A}\mathrm{lt}^n(\mathcal{Q}|_{X_{R'_{i-1}}})|_{X_{R_i}}$ is flat over R_i, which will directly imply part (4) of the theorem.

As flatness is a local property, it suffices to prove the statement for any affine open subscheme $\mathrm{Spec}\, A$ of R_i. It then suffices to prove for any short exact sequence of A-modules,

$$0 \to M' \to M \to M'' \to 0 \tag{4.3}$$

the functor $- \otimes_A (\mathcal{A}\mathrm{lt}^n(\mathcal{Q}|_{X_{R'_{i-1}}})|_{X\times A})$ is exact.

Consider the functor

$$T : \mathfrak{M}_A \to \mathrm{Coh}_{X\times_k A} \text{ defined by } M \mapsto \mathcal{H}\mathrm{om}_{X\times A}\left(\bigwedge^n \mathcal{Q}|_{X\times A}, \mathcal{O}_{X\times A} \otimes_A M\right)$$

and the natural map $\phi_M : T(A) \otimes_A M \to T(M)$. As $\mathrm{Spec}\, A \subset R'_{i-1}$ is open,

$$T(A) = \mathcal{A}\mathrm{lt}^n(\mathcal{Q}|_{X\times A}) \cong \mathcal{A}\mathrm{lt}^n(\mathcal{Q}|_{X\times R'_{i-1}})|_{X\times A}.$$

Hence, we need to prove that $T(A)$ is A-flat. Since $\mathcal{H}\mathrm{om}_{X\times A}(\bigwedge^n \mathcal{Q}|_{X\times A}, \mathcal{O}_{X\times A} \otimes -)$ is left-exact, we have the following diagram:

$$\begin{array}{ccccccc}
T(A) \otimes_A M' & \xrightarrow{\rho} & T(A) \otimes_A M & \longrightarrow & T(A) \otimes M'' & \longrightarrow & 0 \\
\downarrow{\phi_{M'}} & \circlearrowleft & \downarrow{\phi_M} & \circlearrowleft & \downarrow{\phi_{M''}} & & \\
0 \longrightarrow T(M') & \longrightarrow & T(M) & \longrightarrow & T(M'') & &
\end{array}$$

By assumption, for all $u \in \mathrm{Spec}(A)$, $\phi_{k(u)}$ is surjective, where $k(u)$ is the residue field corresponding to u. Then, by Proposition 2.6, ϕ_L is surjective for any A-module L. Proposition A.4 then implies ϕ_L is an isomorphism for all A-modules L. The commutative diagram then implies that ρ is injective. Hence, $T(A)$ is A-flat. This completes the proof of (4). $\qquad\square$

5. Alternating determinant map on the stratification

In this section, we finally give the alternating determinant map on the stratification of the moduli space (Theorem 5.1). We observe that if the underlying scheme is smooth, then there exists no non-trivial strata and the alternating determinant map is simply the dual of the determinant map (Corollary 5.2).

Theorem 5.1. For the stratification $\{V_{R_i}\}$ of $M_X(P)$ as in Theorem 4.14, there exists a natural alternating determinant map:

$$\text{Adet} : \coprod_i V_{R_i} \to \coprod_i M_X(L_i)$$

which to a geometric point $s \in M_X(P)$ associates its alternating determinant, where L_i is as in Theorem 4.14(4).

Proof. We use the notations as in [HL10, §4.2]. By Theorem 4.14(3), R_i is a PGL(V)-bundle over V_{R_i}. Since PGL(V) bundles are universal categorical quotients, the functor $\underline{R_i/\text{GL}(V)}$ is corepresented by $\underline{V_{R_i}}$. By Theorem 4.14, there exists a natural tranformation from

$$\underline{R_i/\text{GL}(V)} \to \mathcal{M}_X(L_i) \text{ defined by } [\mathcal{H} \to \mathcal{E}] \mapsto \mathcal{A}\text{lt}^n(\mathcal{E}).$$

By the universal property of corepresentation, there exists a natural transformation F_i from $\underline{V_{R_i}}$ to $\underline{M_X(L_i)}$ such that the following diagram commute,

$$
\begin{array}{ccc}
\underline{R_i/\text{GL}(V)} & \longrightarrow & \underline{V_{R_i}} \\
\downarrow & \circlearrowleft \quad F_i & \downarrow \\
\mathcal{M}_X(L_i) & \longrightarrow & \underline{M_X(L_i)}
\end{array}
$$

By Yoneda lemma, there exists an unique morphism $\text{Adet}|_{V_{R_i}} : V_{R_i} \to M_X(L_i)$ which gives rise to the natural transformation F_i. This proves the theorem. $\square$

We finally observe that in the classical case, when the underlying scheme is smooth, we get an alternating determinant map on the entire moduli space and not just on its stratification:

Corollary 5.2 (Smooth case)**.** If X is a smooth, projective variety then there exists no non-trivial stratification of $M_X(P)$. In particular, there exists an alternating determinant map $\text{Adet} : M_X(P) \to M_X(L)$, which to a geometric point $s \in M_X(P)$ associates its alternating determinant.

Proof. Since dual of locally free sheaves commutes with pull-back, Proposition 3.7 implies that for all $s \in R$,

$$\mathcal{A}\text{lt}^n(\mathcal{Q})|_{X_s} \cong \det(\mathcal{Q})^\vee|_{X_s} \cong \det(\mathcal{Q}|_{X_s})^\vee \cong \mathcal{A}\text{lt}^n(\mathcal{Q}|_{X_s}).$$

This proves that $R_1 = R$, which is the first part of the corollary. The second part follows identically as in the proof of Theorem 5.1 above. $\square$

Appendix A. Generalities

Setup A.1. Let $f : X \to Y$ be a proper, flat surjective morphism between noetherian schemes. Assume $Y = \mathrm{Spec}(A)$ for some noetherian ring A. Let $\mathcal{F}$ be a coherent sheaf on X (not necessarily flat over Y).

We use the same notations as in Definitions 2.3 and 2.4.

Lemma A.2. Each T^i is an additive covariant functor from A-modules to itself which is exact in the middle. The collection $(T^i)_{i \geq 0}$ forms a δ-functor. Furthermore, T^i commutes with direct limit.

Proof. Clearly, T^i is an additive covariant δ-functor. Using [Har77, III. Proposition 6.4], we can conclude $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, - \otimes_A M)$ is exact in the middle. As $\mathcal{F}$ is a coherent sheaf, by [Har77, Corollary II.5.18], there exists a resolution of $\mathcal{F}$ by finite locally free sheaves $\mathcal{L}_i$:

$$... \to \mathcal{L}_2 \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{F} \to 0.$$

Hence, [Bre13, Theorem 1.1] implies $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, -)$ commutes with direct limit. This completes the proof of the lemma. $\square$

The proof of Propositions A.3 and and A.4 below are very similar to the proofs [Har77, Proposition III.12.4, 12.5] but we reproduce them as our initial hypothesis is different from the ones in the reference. They will be used in the proof of Theorem 2.2.

Proposition A.3. Let $x \in X$. The following are equivalent:

(1) T^i_x is left exact
(2) $W^i(\mathcal{H}om^\bullet)_x$ is $\mathcal{O}_{Y,f(x)}$-flat, where $W^i(\mathcal{H}om^\bullet)_x$ is the stalk of the sheaf at x.

Proof. Given any injective morphism $0 \to M \to M'$ of $\mathcal{O}_{Y,f(x)}$-modules, it suffices to prove that the induced morphism $T^i_x(M) \to T^i_x(M')$ is injective if and only if so is $W^i(\mathcal{H}om^\bullet)_x \otimes M \to W^i(\mathcal{H}om^\bullet)_x \otimes M'$. Indeed, this is due to the property of flatness that $W^i(\mathcal{H}om^\bullet)_x$ is $\mathcal{O}_{Y,f(x)}$-flat if and only if for all injective morphism $0 \to M \to M'$ of $\mathcal{O}_{Y,f(x)}$-modules, the induced morphism $W^i(\mathcal{H}om^\bullet)_x \otimes M \to W^i(\mathcal{H}om^\bullet)_x \otimes M'$ is injective.

Denote by $\mathcal{H}om^\bullet_M$ (resp. $\mathcal{H}om^\bullet_{M'}$) the complexes

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}_0, \mathcal{O}_X \otimes_A M) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}_1, \mathcal{O}_X \otimes_A M) \to ....$$

$$(\text{resp. } 0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}_0, \mathcal{O}_X \otimes_A M') \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}_1, \mathcal{O}_X \otimes_A M') \to ....)$$

Using Definition 2.4, we have an exact sequence:

$$0 \to H^i(\mathcal{H}om^\bullet_M) \to W^i(\mathcal{H}om^\bullet_M)_x \xrightarrow{d_i} \mathcal{H}om(\mathcal{L}_{i+1}, \mathcal{O}_X \otimes_A M)_x$$

and $H^i(\mathcal{H}om^\bullet_M) \cong mr Ext^i_{\mathcal{O}_{X,x}}(\mathcal{F}_x, M \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x})$. Since $- \otimes_{\mathcal{O}_{Y,f(x)}} M$ is right exact and $\mathcal{H}om(\mathcal{L}_j, M \otimes_A \mathcal{O}_X)_x \cong \mathcal{H}om(\mathcal{L}_j, \mathcal{O}_X)_x \otimes_{\mathcal{O}_{Y,f(x)}} M$ for all j, one can easily check that

$$W^i(\mathcal{H}om^\bullet_M)_x \cong W^i(\mathcal{H}om^\bullet)_x \otimes_A M.$$

Similarly for the $\mathcal{O}_{Y,f(x)}$-module M'. Therefore, we have the following diagram:

$$
\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \downarrow \\
0 & \longrightarrow & T_x^i(M') & \longrightarrow & W^i(\mathcal{H}om^\bullet)_x \otimes M' & \longrightarrow & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}_{i+1}, \mathcal{O}_X)_x \otimes M' \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow \\
0 & \longrightarrow & T_x^i(M) & \longrightarrow & W^i(\mathcal{H}om^\bullet)_x \otimes M & \longrightarrow & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}_{i+1}, \mathcal{O}_X)_x \otimes M
\end{array}
$$

The third vertical arrow is injective as $\mathcal{L}_{i+1}$ is a locally-free sheaf, which implies $\mathcal{L}_{i+1}^\vee$ is $\mathcal{O}_X$-flat and hence A-flat. A simple diagram chase shows us that α is injective if and only if so is β. This completes the proof of the proposition. $\qquad\square$

Proposition A.4. For any A-module M, there is a natural map

$$\phi : T^i(A) \otimes M \to T^i(M).$$

Furthermore, the following conditions are equivalent:

(1) T^i is right exact
(2) ϕ is an isomorphism for all M
(3) ϕ is surjective for all M

Proof. Since T^i is a functor, we have a natural map for any M:

$$M = \mathrm{Hom}(A, M) \xrightarrow{\psi} \mathrm{Hom}(T^i(A), T^i(M)).$$

This gives ϕ, by setting $\phi(\sum a_j \otimes m_j) = \sum \psi(m_j) a_j$ where $m_j \in M$ and $a_j \in T^i(A)$.

Since T^i and $\otimes$ commute with direct limits (see Lemma A.2), it will be sufficient to consider only finitely generated A-modules (every module can be written as a direct limit of finitely generated modules). Let $A^r \to A^s \to M \to 0$ be an exact sequence. Then we have a diagram:

$$
\begin{array}{ccccccc}
T^i(A) \otimes A^r & \longrightarrow & T^i(A) \otimes A^s & \longrightarrow & T^i(A) \otimes M & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \phi} & & \\
T^i(A^r) & \longrightarrow & T^i(A^s) & \longrightarrow & T^i(M) & &
\end{array}
$$

where the bottom row is not necessarily exact. The first two vertical arrows are isomorphisms. Thus, if T^i is right exact, then ϕ is an isomorphism. This proves (1) $\Rightarrow$ (2). The implication (2) $\Rightarrow$ (3) is obvious, so we have only to prove (3) $\Rightarrow$ (1). We must show if

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence of A-modules then

$$T^i(M') \to T^i(M) \to T^i(M'') \to 0$$

is exact. By Lemma A.2, it is exact in the middle, so we have only to show that $T^i(M) \to T^i(M'')$ is surjective. This follows from the diagram

$$
\begin{array}{ccc}
T^i(A) \otimes M \longrightarrow T^i(A) \otimes M'' \longrightarrow 0 \\
\phi(M) \downarrow \qquad\qquad \phi(M'') \downarrow \\
T^i(M) \longrightarrow T^i(M'')
\end{array}
$$

and the fact that $\phi(M'')$ is surjective. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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