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Universidad del País Vasco — Euskal Herriko Unibertsitatea

# TESIS DOCTORAL

# Sobre la controlabilidad de algunas ecuaciones en Cardiología, Biología, Mecánica de Fluidos y Viscoelasticidad

Presentada por:

Felipe Wallison Chaves-Silva

Dirigida por:

Enrique Zuazua Iriondo

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# DOCTORAL THESIS

# About the controlability of some equations in Cardiology, Biology, Fluid Mechanics, and Viscoelasticity


Author:

Felipe Wallison Chaves-Silva

Advisor:

Enrique Zuazua Iriondo



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*"Life can only be understood backwards; but it must be lived forwards".*
*Soren Kierkegaard*

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# Abstract


In this thesis we analyze the properties of controllability and observability for selected partial differential equations which model various phenomena in cardiology, biology, fluid mechanics and viscoelasticity.

We begin, in chapter 2, with the analysis of the uniform controllability of families of linear coupled parabolic systems approximating parabolic-elliptic systems. We prove, under appropriate assumptions on the coupling terms, the uniform, with respect to the degenerating parameter, null controllability of the family when only one control is acting on the system.

In chapter 3, we analyze the uniform null controllability of a family of nonlinear reaction-diffusion systems approximating a nonlinear parabolic-elliptic system modeling electrical activity in the cardiac tissue. Combining Carleman estimates and energy inequalities, we prove the uniform null controllability of the family by means of a single control.

Chapter 4 studies the controllability of the parabolic Keller-Segel system of chemotaxis which converges to its parabolic-elliptic version. We show that this nonlinear coupled parabolic system is locally uniformly controllable around a solution of the parabolic-elliptic system when the control is acting on the chemical component.

In chapter 5, we consider the wave equation with both a viscous Kelvin-Voigt and a frictional damping as a model of viscoelasticity. Decomposing the system in its parabolic and hyperbolic parts, we prove the null controllability of the system when the control region, driven by the flow of an ODE, covers the whole domain.

Finally, in chapter 6, we study the cost of controlling the Stokes system to zero. Using a new controllability result for a hyperbolic system with a pressure term and the control transmutation method, we show that the cost of driving the Stokes system to rest at a time $T > 0$ is of order $e^{C/T}$ when $T \longrightarrow 0^+$, as in the case of the heat equation.

# Resumen

En esta tesis analizamos las propiedades de controlabilidad y observabilidad de algunas ecuaciones en derivadas parciales que modelan diversos fenómenos en cardiología, biología, mecánica de fluidos y viscoelasticidad.

En el capítulo 2 comenzamos estudiando la controlabilidad uniforme de familias de sistemas parabólicos lineales acoplados y que aproximan sistemas parabólicos-elípticos. Mostramos, con hipóteses adecuadas sobre los términos de acoplamiento, la controlabilidad uniforme a cero con respecto al parámetro de degeneración, cuando sólo un control actúa sobre el sistema.

En el capítulo 3 analizamos la controlabilidad uniforme de una familia de sistemas de reacción-difusión no lineal que aproximan un sistema parabólico-elíptico no lineal que modela la actividad eléctrica en el tejido cardíaco; utilizando estimaciones de Carleman y desigualdades de energía, probamos la controlabilidad uniforme a cero de la familia por medio de un único control.

El capítulo 4 está dedicado al estudio de la controlabildad del sistema parabólico de Keller-Segel de la quimiotaxis que converge a su versión parabólica-elíptica. Mostramos que este sistema parabólico no lineal acoplado es localmente uniformemente controlable en torno a una solución del sistema parabólico-elíptico cuando el control está actuando en el componente químico.

En el capítulo 5 estudiamos la controlabilidad de la ecuación de ondas con una viscosidad del tipo Kelvin-Voigt y un amortiguamiento por fricción como un modelo de la viscoelasticidad. Descomponiendo el sistema en sus partes parabólica e hiperbólica, mostramos que el sistema es controlable a cero cuando la región de control se mueve, según el flujo de una EDO, de forma que cubra todo el dominio.

Por último, en el capítulo 6 estudiamos el coste de controlar el sistema de Stokes a cero. Probamos un nuevo resultado de controlabilidad para un sistema hiperbólico con un término de presión y, utilizando el método de transmutación de controles, mostramos que el coste de conducir el sistema de Stokes a cero en un tiempo $T > 0$ es del orden de $e^{C/T}$ cuando $T \longrightarrow 0^+$, es decir, el mismo orden que para la ecuación del calor.

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# Chapter 1

# Introduction

## 1.1 General introduction

Control theory is the area of mathematics concerning dynamical systems whose behavior can be changed by means of controls applied through actuators. The origins of control theory can be traced back to the $19^{\text{th}}$ century, with the application of the theory of differential equations to the study of the efficiency of mechanical systems in the industrial revolution. Nowadays, thanks to the works of mathematicians like R. Bellman, H. Fattorini, R. Kalman, J. -L. Lions, L. S. Potryagin and D. Russell, and many others, control theory is a rich interdisciplinary branch of mathematics, with applications in areas such as engineering, biology, economics and medicine. For more details see, for instance, [42, 118, 119] and the rich references therein.

Generally, a control system can be written under the following abstract form

$$\begin{cases} \frac{dy}{dt} = H(y,u), \; t>0, \; y \in Y, \;\; u \in \mathcal{U}_{ad}, \\ y(0) = y_0, \end{cases} \tag{1.1}$$

where $y$ is the *state*, the unknown of the system that we are willing to control, $y_0$ is the initial state, $u$ is the *control*, the variable that can be freely chosen to act on the system and $Y$ and $\mathcal{U}_{ad}$ are the *state space* and the set of *admissible controls*, respectively.

Given a control system, the aim is to find a control such that the associated state behaves in an appropriate manner in a given final time. This is the so-called controllability problem. We distinguish several different notions of controllability. We say that the system is *approximately controllable* if, for any initial state, it is possible to steer the solution to a state arbitrarily close (in some topology) to any target. On the other hand, the *exact controllability* means that the system can be driven from any initial data to any target. The system has the *null controllability* property if, for any initial data, the solution can be driven to zero. Finally, another interesting concept of controllability is the *exact controllability to trajectories*, which means that it is possible to make the state of the system to join any prescribed trajectory, i. e., a given solution of the system.

In this thesis we analyze the controllability problem for certain partial differential equations (PDE's) modeling several physical phenomena. We will also be concerned with the problem of *uniform controllability*. In this case, a family of dynamical systems depending on a parameter $\epsilon > 0$ and approximating (1.1), when $\epsilon \to 0^+$, is considered

$$\begin{cases} \frac{dy^\epsilon}{dt} = H^\epsilon(\epsilon, y^\epsilon, u^\epsilon), \ y^\epsilon \in Y^\epsilon \subset Y, \ \ u^\epsilon \in \mathcal{U}_{ad}, \\ y^\epsilon(0) = y_0^\epsilon. \end{cases} \tag{1.2}$$

First we prove that, for every $\epsilon > 0$, there exists a control $u^\epsilon \in \mathcal{U}_{ad}$ such that the associated solution $y^\epsilon$ of (1.2) satisfies $y^\epsilon(T) = y_T^\epsilon$, for given states $y_0^\epsilon$ and $y_T^\epsilon$ that converge to some $y_0$ and $y_T$ in $Y$, respectively. The goal is then to know if there exist $y \in Y$ and $u \in \mathcal{U}_{ad}$, solution of (1.1), such that $y^\epsilon \to y$ and $u^\epsilon \to u$, when $\epsilon \to 0^+$. Moreover, we want this convergence to be strong enough in such a way that the initial and final conditions are preserved: $y(0) = y_0$ and $y(T) = y_T$. If $y_T \equiv 0$, we say that system (1.2) has the *uniform null controllability* property.

There is by now a well established literature on controllability problems (see, for instance, [25, 86, 115, 125, 135]). The theory for finite dimensional systems was developed in the beginning of the 1960's (see [64, 76]) and, thanks to the famous Kalman rank condition, the problem is nowadays completely understood for the linear case (see [79, 122]). For nonlinear finite dimensional systems the problem is also fairly well understood, since there are many powerful sufficient conditions for local and global controllability (see [25]).

For PDE's the situation is a bit more delicate, even in the linear framework. One reason is that a linear PDE governing the evolution may be, for instance, of hyperbolic type (wave equation, Maxwell equations), of dispersive type (plate equation, Schrödinger equation, Korteweg-de Vries equation), or of parabolic type (heat equation, Stokes equation), inducing very specific properties on the flow such as: the Huygens principle and the property of propagation of singularities with finite velocity for hyperbolic equations, the infinite speed propagation property together with a weak (resp. strong) smoothing effect for dispersive (resp. parabolic) equations and the time irreversibility for parabolic equations. Accordingly, it is not possible to expect an exact controllability result to hold for the heat equation with a control localized in some small part of the domain (the solution will be smooth outside the control region) and, consequently, one cannot attain an arbitrary final state. Hence, it will be natural to look either at the properties of approximate/null controllability or at the controllability to trajectories as long as the heat equation is concerned. In constrast, due to its time reversibility, it is natural to seek for the property of exact controllability for the wave equation.

Let us recall some concepts and fundamental results on the controllability of linear equations. These results will be important even for the nonlinear control problems we treat, since in this case the first step is the analysis of the controllability of an appropriate linear system.

We restrict the presentation to the case of Hilbert spaces. Nevertheless, we point out that a similar theory can be developed in Banach spaces.

### 1.1.1 Controllability and Observability

In this section we essentially follow the presentations given in [25, 108, 125]. We consider two (real or complex) Hilbert spaces $(H, <,>_H)$ and $(U, <,>_U)$, a time $T > 0$, $y_0 \in H$ and a closed unbounded operator $A : D(A) \longrightarrow H$ which generates a strongly continuous semigroup $S(t)_{t\geq 0}$. We are interested in the following class of linear control problems

$$\begin{cases} \frac{dy}{dt} = Ay + Bu, \ t \in [0, T], \\ y(0) = y_0, \end{cases} \tag{1.3}$$

where $B \in \mathcal{L}(U; D(A)')$ is the operator describing the way the control $u$ acts on the system.

We assume that the operator $B$ satisfies the following *admissibility condition*:

$$\forall T > 0, \exists C_T > 0 \text{ such that } \int_0^T ||B^*S(t)^*z||_U dt \leq C_T||z||_H^2, \ \forall z \in D(A^*), \tag{1.4}$$

where $B^*$, $S(t)^*$ and $A^*$ are the adjoint operators of $B$, $S(t)$ and $A$, respectively.

Under the admissibility condition (1.4), the Cauchy problem (1.3) is well-posed in the sense of Hadarmad, i.e., for every $y_0 \in H$ and $u \in L^2(0, T; U)$ there exists a unique $y \in C([0, T]; H)$ satisfying (1.3). Moreover,

$$||y||_{C([0,T];H)} \leq C(||y_0||_H + ||u||_{L^2(0,T;U)}),$$

for a positive constant $C$ depending on $T$, $A$ and $B$.

We now summarize the different notions of controllability for system (1.3).

**Definition 1.1.** System (1.3) is exactly controllable at time $T$ if, for any $y_0, y_T \in H$, there exists $u \in L^2(0, T; U)$ such that the solution $y$ of (1.3) fulfills $y(T) = y_T$.

**Definition 1.2.** System (1.3) is null controllable at time $T$ if, for any $y_0 \in H$, there exists $u \in L^2(0, T; U)$ such that the solution $y$ of (1.3) fulfills $y(T) = 0$.

**Definition 1.3.** System (1.3) is approximately controllable at time $T$ if, for any $y_0, y_T \in H$ and any $\epsilon > 0$, there exists $u \in L^2(0,T;U)$ such that the solution $y$ of (1.3) fulfills $||y(T) - y_T||_H < \epsilon$.

It is clear from the definitions that exact controllability implies both null and approximate controllability. The reverse being, in general, not true.

Since the problem (1.3) is linear, it can be easily shown that null controllability is equivalent to the exact controllability to trajectories:

**Definition 1.4.** System (1.3) is exactly controllable to trajectories at time $T$ if, for any $y_0 \in H$ and any solution $\overline{y}$ of (1.3), i.e., a solution of (1.3) with $\overline{y}(0) = \overline{y}_0 \in H$ and some given $\bar{u}$, there exists a control $u$ such that the associated solution of (1.3) fulfills $y(T) = \overline{y}(T)$.

In the finite dimensional case (i.e., $A \in \mathbb{R}^{N\times N}, B \in \mathbb{R}^{N\times M}$, $N, M \in \mathbb{N}$), all four definitions above are equivalent to a purely algebraic condition, the Kalman condition: $rank(B, AB, A^2B, \ldots, A^{N-1}B) = N$. As a consequence, for finite dimensional systems, controllability at a time $T_0 > 0$ implies controllability at any time $T > 0$.

As noticed by D. Russell in [116], and formalized by J. L. Lions in the famous H.U.M. (Hilbert Uniqueness Method) (see [86, 87, 88]), the properties of controllability for system (1.3) are equivalent to certain measurements (observabilities) of its adjoint system (dual problem).

Indeed, let us consider the adjoint system of (1.3):

$$\begin{cases} -\frac{dz}{dt} = A^*z, \ t \in [0,T], \\ z(T) = z_T \in H. \end{cases} \tag{1.5}$$

The following results hold.

**Theorem 1.1.** *System (1.3) is exactly controllable at time $T$ if and only if there exists a constant $C > 0$ such that*

$$||z_T||_H^2 \leq C^2 \int_0^T ||B^*z(t)||_U^2 dt, \ \forall z_T \in H. \tag{1.6}$$

Inequality (1.6) is called *strong observability inequality*. It means that one can recover a complete information about the initial state $z_T$ from a measurement on $[0,T]$ of the output $B^*z(t)$.

**Theorem 1.2.** *System (1.3) is null controllable at time $T$ if and only if there exists a constant $C > 0$ such that*

$$||z(0)||_H^2 \leq C^2 \int_0^T ||B^*z(t)||_U^2 dt, \ \forall z_T \in H. \tag{1.7}$$

Inequality (1.7) is called *weak observability inequality*. Only $z(0)$ is recovered, not $z_T$. Notice, however, that when system (1.3) is reversible then null and exact controllability are equivalent, which is not the case if the system is not reversible.

**Theorem 1.3.** *System (1.3) is approximately controllable at time $T$ if and only if, for any $z_T \in H$,*

$$B^* z(t) = 0 \text{ on } [0,T] \Longrightarrow z_T = 0. \tag{1.8}$$

Property (1.8) is called the *unique continuation property* for the system (1.5).

From Theorem 1.2 we immediately infer that null controllability for (1.3) implies its approximate controllability by backward uniqueness.

We remark that there is no reason for the uniqueness of a control driving an initial state $y_0$ to a final state $y_T$. However, for the exact and null controllability problem, we can define in a natural way a distinguished control, the one of $L^2(0,T;U)$ minimal norm.

Let us assume that the system (1.3) is exactly controllable at time $T$. Then, for every $y_T \in H$, the set $U^T(y_T)$ of $u \in L^2(0,T;U)$ such that

$$[y_t = Ay + Bu, y(0) = 0] \Longrightarrow [y(T) = y_T]$$

is nonempty. The set $U^T(y_T)$ is clearly a closed affine subspace of $L^2(0,T;U)$. Let $\mathcal{U}^T(y_T)$ denote the element of $U^T(y_T)$ of smallest $L^2(0,T;U)$-norm. It immediate follows that the map

$$\begin{array}{rccc} \mathcal{U}^T(y_T): & H & \to & L^2(0,T;U) \\ & y_T & \mapsto & \mathcal{U}^T(y_T) \end{array}$$

is a linear map. Moreover, using the closed graph theorem, it can be shown that this linear map is continuous. The norm of $\mathcal{U}^T(y_T)$, denoted by $C^E_{opt}(T)$, is called the *cost of the exact controllability* of system (1.3). The following result holds.

**Proposition 1.1.** *$C^E_{opt}(T)$ is the infimum of the constants $C > 0$ for which the strong observability (1.6) holds, i.e.,*

$$C^E_{opt}(T) = ||\mathcal{U}^T(y_T)||_{\mathcal{L}(H;L^2(0,T;U))} = \inf_{C>0} \left\{ ||z_T||^2_H \leq C^2 \int_0^T ||B^* z(t)||^2_U dt, \ \forall z_T \in H \right\}.$$

In the case where (1.3) is null controllable at time $T$, for every $y_0 \in H$, the set $U^T(y_0)$ of $u \in L^2(0,T;U)$ such that

$$[y_t = Ay + Bu, y(0) = y_0] \Longrightarrow [y(T) = 0]$$

is nonempty. The set $U^T(y_0)$ is a closed affine subspace of $L^2(0,T;U)$. As before, let

$\mathcal{U}^T(y_0)$ denote the element of $U^T(y_0)$ of smallest $L^2(0,T;U)$-norm. Again, it is not hard to see that the map

$$\begin{array}{rccc} \mathcal{U}^T(y_0): & H & \to & L^2(0,T;U) \\ & y_0 & \mapsto & \mathcal{U}^T(y_0) \end{array}$$

is a continuous linear map. The norm of $\mathcal{U}^T(y_0)$, denoted by $C^N_{opt}(T)$, is called the *cost of the null controllability* of (1.3). One can obtain the following result.

**Proposition 1.2.** *$C^N_{opt}(T)$ is the infimum of the constants $C > 0$ for which the weak observability property (1.7) holds, i.e.,*

$$C^N_{opt}(T) = ||\mathcal{U}^T(y_0)||_{\mathcal{L}(H;L^2(0,T;U))} = \inf_{C>0}\left\{||z(0)||^2_H \leq C^2 \int_0^T ||B^*z(t)||^2_U dt,\ \forall z_T \in H\right\}.$$

From Propositions 1.1 and 1.2, we see that the cost of the exact/null controllability of (1.3) is the optimal constant for which the strong/weak observability for the adjoint system (1.5) holds .

If system (1.3) is exactly (resp. null) controllable, there is a constructive way to build the controls $\mathcal{U}^T(y_T)$ (resp. $\mathcal{U}^T(y_0)$) of $L^2(0,T;U)$ minimal norm described above. Let us explain this in the context of the exact controllability. For any $y_0 \in H$, the duality between (1.3) and (1.5) gives

$$< y(T), z_T >_H = \int_0^T < u(t), B^*z(t) >_U dt + < y_0, z(0) >_H .$$

We introduce the following functional $\mathcal{J} : H \to \mathbb{R}$ given by

$$\mathcal{J}(z_T) = \frac{1}{2}\int_0^T ||B^*z(t)||^2_U dt + < y_0, z(0) >_H - < y_T, z_T >_H . \tag{1.9}$$

It is not difficult to see that if $\mathcal{J}$ has a minimum $\hat{z}_T$ then, taking $\hat{u} = B^*\hat{z}$, where $\hat{z}$ is the solution of (1.5) associated to $\hat{z}_T$, the solution $y$ of (1.3) with the control $\hat{u}$ satisfies $y(T) = y_T$.

Indeed, the functional $\mathcal{J}$ is clearly strictly convex and, from the admissibility condition (1.4), continuous. Finally, using the strong observability inequality (1.6), one can easily show the coercivity of $\mathcal{J}$. Hence, $\mathcal{J}$ has a unique minimizer $\hat{z}_T$ and the control $\hat{u} = B^*\hat{z}$ is the one of $L^2(0,T;U)$ minimal norm. Moreover, the following estimate holds:

$$||\hat{u}||_{L^2(0,T;U)} \leq C^E_{opt}(T)||y_T||_H. \tag{1.10}$$

Similarly, in the null controllability case, we obtain the control $\hat{u}_N$ of $L^2(0,T;U)$

minimal norm as the minimizer of the following functional

$$\mathcal{J}_N(z_T) = \frac{1}{2}\int_0^T ||B^*z(t)||_U^2 dt + < y_0, z(0) >_H, \tag{1.11}$$

which is clearly strictly convex and continuous. The coercivity is not straightforward as in the exact controllability case. Nevertheless, we can show that $\mathcal{J}_N$ is coercive in the space $\overline{H}$, which is the completion of $H$ with the norm (which follows from the weak observability (1.7))

$$||z_T||_* = \left(\int_0^T ||B^*z(t)||_U dt\right)^{1/2}.$$

The control $\hat{u}_N$ obtained as the minimizer of $\mathcal{J}_N$ satisfies

$$||\hat{u}||_{L^2(0,T;U)} \leq C_{opt}^N(T)||y_0||_H. \tag{1.12}$$

From the above, we see that the study of the controllability of a given linear PDE is equivalent to the obtainment of a suitable observability inequality for the adjoint system, i.e., a full knowledge of the solution of the adjoint system at a given time using only local measurements of it. However, we point out that the proof of such inequalities require tools adapted to the PDE under investigation; e.g. *Ingham inequalities*, *multiplier methods*, *microlocal analysis* or *Carleman inequalities* ([4, 46, 73, 86, 113, 115, 134]).

### 1.1.2 Methodology

The crucial analytic tool we employ when trying to prove observability inequalities are the so-called Carleman inequalities. These are weighted energy estimates for the solutions of PDE's, with weights of exponential type. They were first introduced for the quantification of unique continuation, going back to the early work of Carleman [12]. Over the last few years, the field of applications of Carleman inequalities has gone beyond its original domain: nowadays they are also used in the study of inverse problems and control theory for PDE's (see, for instance, [46, 134, 135]).

For a space domain $\Omega$, a control region $\omega \subset \Omega$, and a time $T > 0$, the Carleman inequalities will obey the following basic structure:

$$\iint_{\Omega\times(0,T)} \beta_1^2|\varphi|^2 dxdt \leq C \iint_{\omega\times(0,T)} \beta_2^2|\varphi|^2 dxdt, \tag{1.13}$$

where $\varphi$ is the solution of the PDE (the adjoint system) and the constant $C > 0$ and the weight functions $\beta_1$ and $\beta_2$ depend on some parameters and are independent of the initial state.

The basic idea behind a Carleman type inequality like (1.13) can be already seen when dealing with the stability of ODE's (see [134]). Indeed, consider the linear ODE in $\mathbb{R}^N$:

$$\begin{cases} y_t(t) = a(t)y(t), \quad t \in [0,T], \\ y(0) = y_0, \end{cases} \tag{1.14}$$

where $a \in L^\infty(0,T)$. It is well-known that all solutions of (1.14) satisfy:

$$\max_{t\in[0,T]} |y(t)| \leq e^{||a||_\infty T}|y_0|, \quad \forall y_0 \in \mathbb{R}^N. \tag{1.15}$$

A Carleman type proof of (1.15) is performed as follows. For any $\lambda \in \mathbb{R}$, we have

$$\frac{d}{dt}\left(e^{-\lambda t}|y(t)|^2\right) = (2a(t) - \lambda)e^{-\lambda t}|y(t)|^2. \tag{1.16}$$

Choosing $\lambda$ large enough so that $2a(t) - \lambda \leq 0$ for almost every $t \in (0,T)$, we find that

$$|y(t)| \leq e^{\lambda T/2}|y_0|, \ t \in [0,T], \tag{1.17}$$

which proves, in particular, (1.15). The key point in the above proof, and in all Carleman type estimates, is that one must take parameters in the weight functions to be large enough in order to absorb the undesired terms.

Let us now describe some of the main difficulties one may encounter when dealing with the uniform controllability of a given family of PDE's. It is convenient to first have a quick look at the finite-dimensional context. In fact, even in this case, it may happen that the limit system is not controllable while the approximating family is controllable for every value of the parameter $\epsilon > 0$. To see this pathology, we consider the following two examples, due to Chow [22]:

$$\begin{cases} y_{1,t} = -5y_1 + \epsilon u, \\ y_{2,t} = -y_2 + u, \\ y_{3,t} = -(1+\epsilon)y_3 + u, \\ y_{4,t} = -(1+\epsilon)y_4 + 2u \end{cases} \tag{1.18}$$

and

$$\begin{cases} y_{1,t} = -y_1 + y_2 + u, \\ \epsilon y_{2,t} = -y_2 - u. \end{cases} \tag{1.19}$$

Using the Kalman rank condition, one can show that both systems (1.18) and (1.19) are controllable for each $\epsilon \in (0,1)$ but not for $\epsilon = 0$.

As we will see, the key point in the study of the uniform controllability for (1.2) is the construction, for each $\epsilon > 0$, of a control $u^\epsilon$ such that $||u^\epsilon||_{\mathcal{U}_{ad}} \leq C||y_0^\epsilon||_Y$ for a constant $C$ independent of $\epsilon$ and some appropriate norm $||.||_{\mathcal{U}_{ad}}$. In fact, if this is the case, since the state of a controllable system can be estimated in terms of the initial state and the control, we will be able to obtain a uniform bound for $y^\epsilon$ in terms of the initial data $y_0^\epsilon$. Notice that this problem is closely related to the study of the cost of the controllability of system (1.3), which is also tackled in this thesis.

To end this section, we summarize the main achievements of this thesis.

**Chapter 2:** We consider a family of linear coupled parabolic systems approximating a parabolic-elliptic system. Under appropriate assumptions on the coupling terms, the uniform null controllability of the family, when only one control is acting on the system, is proved. The results of this chapter are based on the article [20], in collaboration with S. Guerrero and J.-P. Puel.

**Chapter 3:** The uniform null controllability of a family of nonlinear reaction-diffusion systems approximating a nonlinear parabolic-elliptic system modeling electrical activity in the cardiac tissue is studied. Combining Carleman estimates and energy inequalities, the uniform null controllability of the family by means of a single control is obtained. The results of this chapter are based on the articles [5] and [6], in collaboration with M. Bendahmane.

**Chapter 4:** We analyze the controllability of the parabolic Keller-Segel system of chemotaxis which converges to its parabolic-elliptic version. Using Carleman estimates and an inverse mapping theorem, we show that this parabolic system is locally uniformly controllable around a solution of the parabolic-elliptic system when the control is acting on the chemical component. This chapter is based on the article [19], in collaboration with S. Guerrero.

**Chapter 5:** We consider the wave equation with both a viscous Kelvin-Voigt and frictional damping as a model of viscoelasticity. Decomposing the system in its parabolic and hyperbolic part, we prove the null controllability of the system when the control region, driven by the flow of an ODE, covers all the domain. This chapter is based on the article [21], in collaboration with L. Rosier and E. Zuazua.

**Chapter 6:** We study the cost of controlling the Stokes system to zero. Using a new controllability result for a hyperbolic system with a pressure term and the control transmutation method, we show that the cost of driving the Stokes system to rest at time $T$ is of order $e^{C/T}$ as $T \longrightarrow 0^+$, the same as for the heat equation. The results of this chapter is based on the article [18].

In the following section, the content of each chapter is discussed more specifically.

## 1.2 Main contents of the thesis

Given a domain $\Omega \subset \mathbb{R}^N$ and a time $T > 0$, we set $Q := \Omega \times (0,T)$, $\Sigma := \partial\Omega \times (0,T)$. We use $\omega$, $\omega_1$ and $\omega_2$ to denote (small) open regions of $\Omega$ in which the controls will act, and denote by $\beta = \beta(x,t)$ a weight in the form

$$\beta(x,t) = e^{\frac{\eta(x)}{t^k(T-t)^k}}, \tag{1.20}$$

for some constant $k \geq 1$ and an appropriate function $\eta$ that depends on the control region and such that $\eta(x) < 0$ for all $x \in \Omega$.

### Chapter 2: Controllability of fast diffusion coupled parabolic systems

The first chapter of this thesis is concerned with the analysis of the uniform null controllability of the following family of coupled parabolic systems:

$$\left| \begin{array}{ll} u_t^\epsilon - \Delta u^\epsilon = au^\epsilon + bv^\epsilon + f^\epsilon 1_{\omega_1} & \text{in } Q, \\ \epsilon v_t^\epsilon - \Delta v^\epsilon = cu^\epsilon + dv^\epsilon + g^\epsilon 1_{\omega_2} & \text{in } Q, \\ u^\epsilon = v^\epsilon = 0 & \text{on } \Sigma, \\ u^\epsilon(0) = u_0;\ v^\epsilon(0) = v_0 & \text{in } \Omega, \end{array} \right. \tag{1.21}$$

where $a = a(x,t)$, $b = b(x,t)$, $c = c(x,t)$ and $d = d(x,t)$, $f^\epsilon$ and $g^\epsilon$ are internal controls and $\epsilon$ is a small positive parameter, intended to tend to zero.

Systems like (1.21) are prototypes of biological systems modeling aggregation phenomena or chemical systems having two different concentrations (for more details, see references [11, 61, 62, 85] or chapters 3 and 4 of this thesis).

Our interest in the uniform null controllability of (1.21) comes from the fact that, in applications, this family of parabolic systems is usually approximated by the parabolic-elliptic system:

$$\left| \begin{array}{ll} u_t - \Delta u = au + bv + f 1_{\omega_1} & \text{in } Q, \\ -\Delta v = cu + dv + g 1_{\omega_2} & \text{in } Q, \\ u = v = 0 & \text{on } \Sigma, \\ u(0) = u_0 & \text{in } \Omega. \end{array} \right. \tag{1.22}$$

Notice that, even if this approximation is consistent with the existence and uniqueness point of view, it is not clear at all what can be done from a control theory point of view. The main reason for that arises from the fact that we are considering systems having different physical properties and therefore one could expect these two systems to have, at least a priori, different control properties. In chapter 2 it is shown that the properties of controllability and observability for (1.22) can be obtained as a limit of the respective properties for the family (1.21).

In fact, the following result is proved.

**Theorem 1.4.** *Let $T > 0$, $0 < \epsilon < 1/2$ and let $\Omega \subset \mathbb{R}^N (N \geq 1)$ be a bounded connected open set whose boundary is sufficiently regular, $(u_0, v_0) \in L^2(\Omega)^2$ and $a, b, c$ and $d$ be $C^3(\overline{Q})$ functions. It holds:*

*(i) If $c \neq 0$ in $\overline{\omega}$, for some $\omega \subset\subset \omega_1 \subset \Omega$, and $d < \mu_1$ ($\mu_1$ being the first eigenvalue of the Laplacian with Dirichlet boundary condition), system (1.21) is uniformly null controllable, with respect to $\epsilon$, with control only on the first equation. More precisely, for any $0 < \epsilon < 1/2$, there exists $f^\epsilon = f(\epsilon) \in L^2(Q)$ such that the associated solution $(u^\epsilon, v^\epsilon)$ of (1.21) ($g^\epsilon \equiv 0$) satisfies:*

$$\big(u^\epsilon(T), v^\epsilon(T)\big) = (0, 0). \tag{1.23}$$

*Moreover, the following estimate on the control can be obtained:*

$$||f^\epsilon 1_{\omega_1}||^2_{L^2(Q)} \leq C(||u_0||^2_{L^2(\Omega)} + \epsilon||v_0||^2_{L^2(\Omega)}), \tag{1.24}$$

*where $C$ is a constant that does not depend on $\epsilon$, $u_0$ and $v_0$.*

*(ii) If $b \neq 0$ in $\overline{\omega}$, for some $\omega \subset\subset \omega_2 \subset \Omega$, and $d < \mu_1$, then system (1.21) is uniformly null controllable, with respect to $\epsilon$, with control acting only on the second equation. More precisely, for any $0 < \epsilon < 1/2$, there exists $g^\epsilon = g(\epsilon) \in L^2(Q)$ such that the associated solution $(u^\epsilon, v^\epsilon)$ of (1.21) ($f^\epsilon \equiv 0$) satisfies:*

$$\big(u^\epsilon(T), v^\epsilon(T)\big) = (0, 0). \tag{1.25}$$

*Moreover, the following estimate on the control can be obtained:*

$$||g^\epsilon 1_{\omega_2}||^2_{L^2(Q)} \leq C(||u_0||^2_{L^2(\Omega)} + \epsilon||v_0||^2_{L^2(\Omega)}), \tag{1.26}$$

*where $C$ does not depend on $\epsilon$, $u_0$ and $v_0$.*

From Theorem 1.4, we obtain a control $f$ (resp. $g$), the weak limit of $\{f^\epsilon\}_{\epsilon>0}$ (resp. $\{g^\epsilon\}_{\epsilon>0}$), such that the solution $(u^\epsilon, v^\epsilon)$ of (1.21) converges weakly to $(u, v)$ solution of (1.21) associated to $f$ (resp. $g$) in the space $\big(L^2(0, T; H^1_0(\Omega)) \cap C([0, T]; L^2(\Omega))\big) \times L^2(0, T; H^1_0(\Omega))$.

The key point in the proof of Theorem 1.4 is the obtainment of a suitable uniform observability inequality for the adjoint system of (1.21) (to simplify the notation we omit the index $\epsilon$ on each term):

$$\left|\begin{array}{ll} -\varphi_t - \Delta\varphi = a\varphi + c\xi & \text{in } Q, \\ -\epsilon\xi_t - \Delta\xi = b\varphi + d\xi & \text{in } Q, \\ \varphi = \xi = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi_T;\ \xi(T) = \xi_T & \text{in } \Omega. \end{array}\right. \tag{1.27}$$

In fact, proving case 1 of Theorem 1.4 is equivalent to prove that the following observability estimate

$$||\varphi(0)||^2_{L^2(\Omega)} + \epsilon||\xi(0)||^2_{L^2(\Omega)} \leq C \iint_{\omega_1\times(0,T)} |\varphi|^2 dxdt \tag{1.28}$$

holds for all solutions $(\varphi, \xi)$ of (1.27), where $C$ is bounded with respect to $\epsilon$. Analogously, the proof of case 2 of Theorem 1.4 is equivalent to show that

$$||\varphi(0)||^2_{L^2(\Omega)} + \epsilon||\xi(0)||^2_{L^2(\Omega)} \leq C \iint_{\omega_2\times(0,T)} |\xi|^2 dxdt, \tag{1.29}$$

for all solutions $(\varphi, \xi)$ of (1.27).

In order to explain the methodology of the proof of Theorem 1.4, let us consider the case 1 and describe its main ideas.

The uniform observability inequality (1.28) is a consequence of the following Carleman inequality.

**Theorem 1.5.** *Assume $c \neq 0$ in $\overline{\omega}$, for some $\omega \subset\subset \omega_1 \subset \Omega$, and $d < \mu_1$. Then, for any $0 < \epsilon < 1/2$, the solution $(\varphi, \xi)$ of (1.27) satisfies:*

$$\iint_Q \beta_1^2|\varphi|^2 dxdt + \iint_Q \beta_2^2|\xi|^2 dxdt \leq C \iint_{\omega_1\times(0,T)} \beta_3^2|\varphi|^2 dxdt, \tag{1.30}$$

*where $C$ does not depend on $\epsilon$ and the weights $\beta_i$, $i = 1, 2, 3$ are of the form* (1.20).

In fact, combining Theorem 1.5 and energy estimates for the adjoint system (1.21), leads to inequality (1.28).

Notice that in (1.30), global estimates of $\varphi$ and $\xi$ are given in terms of a local estimate of $\varphi$. The main difficulty when trying to obtain an estimate like (1.30) for system (1.27) comes from the fact that the equations for $\varphi$ and $\xi$ have different diffusion rates. Actually, the key point in the proof is to write equations for $\varphi$ and $\xi$ with the same diffusion rate.

In order to do so, we consider the operator $\mathcal{L}_{\gamma,\theta}$ given by

$$\mathcal{L}_{\gamma,\theta} := \gamma\partial_t - \Delta - \theta, \text{ for } \gamma \in \mathbb{R} \text{ and } \theta \in L^\infty(Q)$$

and define a new function

$$w = \mathcal{L}_{-\epsilon,d}\varphi.$$

We extend the adjoint system (1.27) to a system of four equations, namely

$$\left|\begin{array}{ll}\mathcal{L}_{-1,a}w = \varphi(cb + \mathcal{L}_{-\epsilon,0}a - \mathcal{L}_{-1,0}d) + \xi\mathcal{L}_{-\epsilon,0}c - 2\nabla\xi\nabla c + 2\nabla\varphi(\nabla d - \nabla a) & \text{in } Q, \\ \mathcal{L}_{-\epsilon,d}\varphi = w & \text{in } Q, \\ \mathcal{L}_{-1,a}\varphi = c\xi & \text{in } Q, \\ \mathcal{L}_{-\epsilon,d}\xi = b\varphi & \text{in } Q, \\ \varphi = \xi = w = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi_T;\ \xi(T) = \xi_T;\ w(T) = -\epsilon\varphi_T - \Delta\varphi_T - d\varphi_T & \text{in } \Omega.\end{array}\right. \tag{1.31}$$

In (1.31), we simply have applied the operator in the second equation of (1.27) to $\varphi$, denoting this new unknown as $w$ and writing the equation satisfied by $w$. This gives a parabolic equation for $\varphi$ with the same diffusion rate as the one in the equation for $\xi$, i.e., equations $(1.31)_2$ and $(1.31)_4$, respectively.

We divide the proof into four steps.

*Step 1*: We view equations of (1.31) as heat equations and apply a suitable Carleman estimate for heat equations with a precise dependence with respect to the degenerating parameter multiplying the time derivative. Adding the inequalities obtained for each equation, we obtain global estimates of $\varphi$, $\xi$ and $w$ in terms of local estimates of $\varphi$, $\xi$ and $w$.

*Step 2*: Using the second equation in (1.31), we eliminate the local integral of $w$ appearing in the Carleman estimate obtained in *step 1*. This yields global estimates of $\varphi$, $\xi$ and $w$ in terms of local estimates of $\varphi$ and $\xi$.

*Step 3*: Using $(1.31)_3$, we estimate a local integral of $\xi$ in terms of a local integral of $\varphi$ and a local integral of $\varphi_t$.

*Step 4*: Finally, using $(1.31)_1$ and $(1.31)_2$, we estimate $\varphi_t$ locally in terms of a local integral of $\varphi$.

Using Theorem 1.5 and energy estimates for (1.27) we finish the proof of case 1 of Theorem 1.4.

The proof of case 1 of Theorem 1.4 described above also includes the second case of the same theorem. For that, one just needs to notice that the system formed by the first two equations in (1.31) has the same structure as the adjoint system (1.27) and advance similarly as in steps 1 and 2 given above in order to prove the following result.

**Theorem 1.6.** *Assume $b \neq 0$ in $\overline{\omega}$, for some $\omega \subset\subset \omega_2 \subset \Omega$, and $d < \mu_1$. Then, for any*

$0 < \epsilon < 1/2$, *the solution* $(\varphi, \xi)$ *of* (2.10) *satisfies:*

$$\iint_Q \beta_1^2 |\xi|^2 dxdt + \iint_Q \beta_2^2 |\varphi|^2 dxdt \leq C \iint_{\omega_2 \times (0,T)} \beta_3^2 |\xi|^2 dxdt, \tag{1.32}$$

*where* $C$ *does not depend on* $\epsilon$ *and the weights* $\beta_i$, $i = 1, 2, 3$ *are of the form* (1.20).

### Chapter 3: Uniform null controllability for a degenerating reaction-diffusion system approximating a simplified cardiac model

In chapter 3, we analyze the properties of controllability and observability for a family of reaction-diffusion systems which degenerates into a parabolic-elliptic system describing the cardiac electric activity on a domain $\Omega \subset \mathbb{R}^N (N = 1, 2, 3)$.

The *bidomain model* (see, for instance, [23, 59, 65]) governing the electrical activity in the cardiac tissue is given by the following nonlinear coupled parabolic system:

$$\left| \begin{array}{ll} c_m v_t - Div(\mathbf{M}_i(x)\nabla u_i) + h(v) = f 1_{\omega_1} & \text{in } Q, \\ c_m v_t + Div(\mathbf{M}_e(x)\nabla u_e) + h(v) = g 1_{\omega_2} & \text{in } Q, \\ u_i = u_e = 0 & \text{on } \Sigma, \\ v(0, x) = v_0(x), & \text{in } \Omega, \end{array} \right. \tag{1.33}$$

where $c_m > 0$ is the surface capacitance of the membrane, the nonlinear function $h : \mathbb{R} \to \mathbb{R}$ is the transmembrane ionic current and $f$ and $g$ are the stimulation currents applied to $\omega_1$ and $\omega_2$, respectively.

In (1.33), the functions $u_i = u_i(t, x)$ and $u_e = u_e(t, x)$ represent, respectively, the *intracellular* and *extracellular* electric potentials, and we call their difference $v = u_i - u_e$ the *transmembrane* potential. The intracellular and extracellular conductivity tensors $\mathbf{M}_i(x)$ and $\mathbf{M}_e(x)$, aiming to model the anisotropic properties of the media, are supposed to be $C^\infty$, bounded, symmetric and positive semidefinite.

Due to its difficulty to be implemented (see, for instance, [7, 23, 45]), in many applications the bidomain model is simplified into the *monodomain model*:

$$\left| \begin{array}{ll} c_m v_t - \frac{\mu}{\mu+1} Div\big(\mathbf{M}_e(x)\nabla v\big) + h(v) = f 1_{\omega_1} & \text{in } Q, \\ -Div\big(\mathbf{M}(x)\nabla u_e\big) = Div\big(\mathbf{M}_i(x)\nabla v\big) & \text{in } Q, \\ v = u_e = 0 & \text{on } \Sigma, \\ v(0) = v_0 & \text{in } \Omega, \end{array} \right. \tag{1.34}$$

where $M = M_i + M_e$.

Notice that the main difference between the bidomain model (1.33) and the monodomain model (1.34) is the fact that the first one is a system of two coupled parabolic equations whereas the second one is a system of a parabolic-elliptic type. In this part of the thesis the goal is to show that, actually, the properties of controllability and observability for the monodomain model can be obtained as a limit process of the controllability properties of a family of coupled parabolic systems related to the bidomain model.

More precisely, given $\epsilon > 0$, we approximate the monodomain model by the following family of parabolic systems:

$$\left| \begin{array}{ll} c_m v^\epsilon_t - \frac{\mu}{\mu+1} Div\big(\mathbf{M}_e(x)\nabla v^\epsilon\big) + h(v^\epsilon) = f^\epsilon 1_{\omega_1} & \text{in } Q, \\ \epsilon u^\epsilon_{e,t} - Div\big(\mathbf{M}(x)\nabla u^\epsilon_e\big) = Div\big(\mathbf{M}_i(x)\nabla v^\epsilon\big) & \text{in } Q, \\ v^\epsilon = u^\epsilon_e = 0 & \text{on } \Sigma, \\ v^\epsilon(0) = v_0;\ u^\epsilon_e(0) = u_{e,0} & \text{in } \Omega. \end{array} \right. \tag{1.35}$$

We prove the next theorem.

**Theorem 1.7.** *Assume $h$ is $C^1(\mathbb{R})$, globally Lipschitz and $h(0) = 0$. Given $(v_0, u_{e,0}) \in L^2(\Omega)^2$, there exists a control $f^\epsilon \in L^2(\omega_1 \times (0,T))$ such that, for any $0 < \epsilon < 1/2$, the solution $(v^\epsilon, u^\epsilon_e)$ of (1.35) satisfies:*

$$v^\epsilon(T) = u^\epsilon_e(T) = 0.$$

*Moreover, the control $f^\epsilon$ satisfies the estimate:*

$$||f^\epsilon 1_{\omega_1}||^2_{L^2(Q)} \le C\big(||v_0||^2_{L^2(\Omega)} + \epsilon||u_{e,0}||^2_{L^2(\Omega)}\big), \tag{1.36}$$

*for a constant $C = C(\Omega, \omega_1, ||a||_{L^\infty}, T) > 0$.*

As usual when dealing with the controllability of nonlinear systems, the key point for proving Theorem 1.7 is a uniform controllability result for the linearized system:

$$\left| \begin{array}{ll} c_m v^\epsilon_t - \frac{\mu}{\mu+1} Div\big(\mathbf{M}_e(x)\nabla v^\epsilon\big) + a(t,x) v^\epsilon = f^\epsilon 1_{\omega_1} & \text{in } Q, \\ \epsilon u^\epsilon_{e,t} - Div\big(\mathbf{M}(x)\nabla u^\epsilon_e\big) = Div\big(\mathbf{M}_i(x)\nabla v^\epsilon\big) & \text{in } Q, \\ v^\epsilon = u^\epsilon_e = 0 & \text{on } \Sigma, \\ v^\epsilon(0) = v_0;\ u^\epsilon_e(0) = u_{e,0} & \text{in } \Omega, \end{array} \right. \tag{1.37}$$

where $a$ is a bounded function.

The uniform null controllability result proved for (1.37) is the following.

**Theorem 1.8.** *Given $(v_0, u_{e,0}) \in L^2(\Omega)^2$, there exists a control $f^\epsilon \in L^2(\omega_1 \times (0,T))$ such that, for any $0 < \epsilon < 1/2$, the associated solution of (1.37) is driven to zero at time $T$. That is to* *say, the associated solution verifies:*

$$v^\epsilon(T) = 0,\ u_e^\epsilon(T) = 0.$$

*Moreover, the control $f^\epsilon$ satisfies the estimate:*

$$||f^\epsilon 1_{\omega_1}||^2_{L^2(Q)} \le C\big(||v_0||^2_{L^2(\Omega)} + \epsilon||u_{e,0}||^2_{L^2(\Omega)}\big), \tag{1.38}$$

*for a constant $C = C(\Omega, \omega_1, ||a||_{L^\infty}, T) > 0$.*

Once we have proved Theorem 1.8, we prove Theorem 1.7 using standard fixed point arguments.

The idea of the proof of Theorem 1.8 is to consider the adjoint system of (1.37) (to simplify the notation we omit the index $\epsilon$)

$$\left|\begin{array}{ll} -c_m\varphi_t - \frac{\mu}{\mu+1}Div(\mathbf{M}_e(x)\nabla\varphi) + a(t,x)\varphi = Div(\mathbf{M}_i(x)\nabla\varphi_e) & \text{in } Q, \\ -\epsilon\varphi_{e,t} - Div(\mathbf{M}(x)\nabla\varphi_e) = 0 & \text{in } Q, \\ \varphi = \varphi_e = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi_T;\ \varphi_e(T) = \varphi_{e,T} & \text{in } \Omega \end{array}\right. \tag{1.39}$$

and to show the following uniform Carleman inequality:

**Theorem 1.9.** *There exists a positive constant $C = C(\Omega, \omega_1, ||a||_{L^\infty}, T)$, such that, for every $(\varphi_T, \varphi_{e,T}) \in L^2(\Omega)^2$, every $0 < \epsilon < 1/2$ and every $a \in L^\infty(Q)$, the solution $(\varphi, \varphi_e)$ of (1.39) satisfies:*

$$\iint_Q \beta_1^2|\rho|^2 dxdt + \iint_Q \beta_2^2|\varphi|^2 dxdt \le C\iint_{\omega_1\times(0,T)} \beta_3^2|\varphi|^2 dxdt, \tag{1.40}$$

*where $\rho = Div(M(x)\nabla\varphi_e(x,t))$ and the weights $\beta_i$, $i = 1,2,3$ are of the form* (1.20).

Notice that, since the control is acting on the first equation of (1.37), we need a Carleman inequality giving global estimates of $\varphi$ and $\varphi_e$ in terms of a local integral of $\varphi$, uniform with respect to $\varepsilon$. Because the coupling term in the first equation of the adjoint system (1.39) is in $Div(\mathbf{M}_i(x)\nabla\varphi_e)$ and not in $\varphi_e$, the results of Chapter 2 cannot be applied directly. However, we can work with smooth solutions of (1.39) and then consider $\rho(x,t) = Div(\mathbf{M}(x)\nabla\varphi_e(x,t))$ as a new variable, and work with the system

$$\left|\begin{array}{ll} -\varphi_t - Div(\mathbf{M}_e(x)\nabla\varphi) + a(x,t)\varphi = \rho & \text{in } Q, \\ -\epsilon\rho_t - Div(\mathbf{M}(x)\nabla\rho) = 0 & \text{in } Q, \\ \varphi = \rho = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi_T; \rho(T) = \rho_T & \text{in } \Omega. \end{array}\right. \tag{1.41}$$

For system (1.41), the results of chapter 2 can be applied in order to obtain the inequality (1.40). Nevertheless, it is important to say that in chapter 3 we give another direct proof of this result, combining Carleman inequalities and weighted energy estimates, which, for this particular case, is simpler than the one obtained through the arguments of chapter 2.

Finally, using Theorem 1.9 and energy estimates, we deduce the observability inequality:

$$||\varphi(0)||^2_{L^2(\Omega)} + \varepsilon||\varphi_e(0)||^2_{L^2(\Omega)} \leq e^{C(1+1/T+||a||^{2/3}_{L^\infty}+||a||_{L^\infty}T)} \iint_{\omega_1\times(0,T)} |\varphi|^2 dxdt, \qquad (1.42)$$

for all solutions of (1.39), where $C = C(\Omega, \omega_1)$.

Theorem 1.8 then follows directly from the observability inequality (1.42).

### Chapter 4: A uniform controllability result for the Keller-Segel system

Chapter 4 of this thesis is concerned with the uniform null controllability of the Keller-Segel system:

$$\left|\begin{array}{ll} u_t - \Delta u = -\nabla\cdot(u\nabla v) & \text{in } Q,\\ \epsilon v_t - \Delta v = au - bv + g\chi & \text{in } Q,\\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \Sigma,\\ u(x,0) = u_0;\ v(x,0) = v_0 & \text{in } \Omega, \end{array}\right. \qquad (1.43)$$

where $a$ and $b$ are positive real constants, $g$ is an internal control and $\epsilon$ is a small positive parameter, which is intended to tend to zero. In (1.43), $\chi : \mathbb{R}^N \to \mathbb{R}$ is a $C^\infty$ function such that *supp* $\chi \subset\subset \omega$, $0 \leq \chi \leq 1$ and $\chi \equiv 1$ in $\omega'$.

Here, we assume that $\Omega \subset \mathbb{R}^N$ ($N = 2, 3$) is a bounded connected open set whose boundary $\partial\Omega$ is regular enough and $\omega' \subset\subset \omega$.

The Keller-Segel model (1.43) is an equation from mathematical biology which aims to explain the aggregation phenomena of organisms due to chemotaxis, i.e., the change of motion when a population formed of individuals (such as amoebae, bacteria, endothelial cells etc.) reacts in response (taxis) to an external chemical stimulus spread in the environment where they reside. Chemotaxis is a fundamental cellular process in the development of multicellular organisms and, particularly, it plays an essential role in embryonic development, tissue homeostasis, wound healing, immune response, progression of diseases, as well as finding food, repellent action and forming the multicellular body of protozoa (see [61, 62] and the references therein).

In many applications (see [82, 105] and references therein), system (1.43) is approximated by the following parabolic-elliptic system:

$$\left|\begin{array}{ll} u_t - \Delta u = -\nabla\cdot(u\nabla v) & \text{in } Q, \\ -\Delta v = au - bv + g\chi & \text{in } Q, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \Sigma, \\ u(x,0) = u_0 & \text{in } \Omega. \end{array}\right. \tag{1.44}$$

Therefore, as in the previous chapters, it is natural to seek for the uniform null controllability for the Keller-Segel system (1.43).

In chapter 4, we prove the uniform controllability of (1.43) around a particular trajectory of (1.44). More precisely, the following result is proved.

**Theorem 1.10.** *Let $0 < \epsilon \leq 1$ and let $(M_1, M_2) \in \mathbb{R}^2_+$ be such that $aM_1 - bM_2 = 0$. Then there exists $\delta > 0$ such that, for any $(u_0, v_0) \in H^1(\Omega) \times H^2(\Omega)$ satisfying $\int_\Omega u_0 dx = M_1$, $\frac{\partial v_0}{\partial \nu} = 0$ on $\partial\Omega$ and $||(u_0 - M_1, v_0 - M_2)||_{H^1(\Omega)\times H^2(\Omega)} \leq \delta$, one can find $g = g(\epsilon) \in L^2(0, T; H^1(\Omega))$, uniformly bounded with respect to $\epsilon$, such that the associated solution $(u, v)$ of (1.43) satisfies:*

$$(u(T), v(T)) = (M_1, M_2) \text{ in } \Omega.$$

In (1.43) (and (1.44)), $u = u(x,t) \geq 0$ and $v = v(x,t) \geq 0$ represent the concentrations of species (i.e, the population density) and the chemical (i.e., concentration of the chemical substance), respectively. Therefore, we are controlling the Keller-Segel system through the chemical concentration, which is reasonable from a biological point of view. Here, the condition $aM_1 - bM_2 = 0$ means that $(M_1, M_2)$ is a stationary solution of the parabolic-elliptic system (1.44).

In order to study controllability of (1.43) around $(M_1, M_2)$, we first analyze the uniform null controllability of its linearization around this trajectory, namely

$$\left|\begin{array}{ll} u_t - \Delta u = -M_1\Delta v + h_1 & \text{in } Q, \\ \epsilon v_t - \Delta v = au - bv + g\chi_\omega + h_2 & \text{in } Q, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \Sigma, \\ u(x,0) = u_0;\ v(x,0) = v_0 & \text{in } \Omega, \\ \int_\Omega u_0(x)dx = 0, & \end{array}\right. \tag{1.45}$$

where $h_1$ and $h_2$ are applied forces belonging to appropriate spaces $X_1$ and $X_2$, respectively, having exponential decay at $t = T$ and $\int_\Omega h_1(x,t)dx = 0$ for almost all $t \in [0, T]$. The idea is to prove that there exists $g$ so that the solution of (1.45) satisfies $(u(T), v(T)) = (0, 0)$ and so that the quantity $\nabla\cdot(u\nabla v)$ belongs to $X_1$. Then we employ an inverse mapping argument introduced in [129] and obtain the controllability of (1.43) around $(M_1, M_2)$.

As in chapters 2 and 3, we prove the null controllability of the linear system (1.45) through a *global Carleman inequality* for the solutions of its adjoint system, that is to say,

$$\left|\begin{array}{ll} -\varphi_t - \Delta\varphi = a\xi + f_1 & \text{in } Q, \\ -\epsilon\xi_t - \Delta\xi = -b\xi - M_1\Delta\varphi + f_2 & \text{in } Q, \\ \frac{\partial\varphi}{\partial\nu} = \frac{\partial\xi}{\partial\nu} = 0 & \text{on } \Sigma, \\ \varphi(x,T) = \varphi_T;\ \xi(x,T) = \xi_T & \text{in } \Omega, \\ \int_\Omega \varphi_T(x)dx = 0, & \end{array}\right. \tag{1.46}$$

where $f_1$ and $f_2$ are arbitrary $L^2$ functions.

Notice that, in the adjoint system (1.46), $\int_\Omega \varphi_T dx = 0$. This condition comes from the fact that $\int_\Omega u(x,t)dx = 0$ for all $t \in [0,T]$, where $(u,v)$ is the solution of (1.45).

Since the control is acting on the second equation of (1.45), we need to bound global integrals of $\varphi$ and $\xi$ in terms of a local integral of $\xi$ and global integrals of $f_1$ and $f_2$. The main difficulty when proving a Carleman inequality of this type for (1.46) comes from the fact that the coupling in the second equation is in $\Delta\varphi$ and not in $\varphi$, as in chapter 3. In fact, the inequality we prove contains global terms with the $L^2$-weighted norms of $\Delta\varphi$ and $\xi$ in the left hand side, no global terms in $\varphi$, while a local integral of $\xi$ and global integrals of $f_1$ and $f_2$ appear in its right-hand side.

The Carleman inequality proved for system (1.46) is the following.

**Theorem 1.11.** *There exists $C = C(\Omega,\omega,T) > 0$ such that, for any $0 < \epsilon \leq 1$ and any $f_1, f_2 \in L^2(Q)$, the solution $(\varphi,\xi)$ of (1.46) satisfies:*

$$\iint_Q \beta_1^2|\xi|^2 dxdt \iint_Q \beta_2^2|\Delta\varphi|^2 dxdt \leq C\Big(\iint_Q \beta_3^2|\chi|^2|\xi|^2 dxdt + \iint_Q \beta_4^2|f_1|^2 dxdt + \iint_Q \beta_5^2|f_2|^2 dxdt\Big), \tag{1.47}$$

*where the weights $\beta_i$, $i = 1, 2 \ldots, 5$ are of the form (1.20).*

The proof of Theorem 1.11 is divided into 3 steps.

*Step 1. Carleman Inequality for $\Delta\varphi$.*

We write $\beta_1\varphi = \eta + \psi$, where $\eta$ solves a heat equation with $f_1$ as a right-hand side term and $\psi$ solves a heat equation with a right-hand side in $H^1(0,T;H^2(\Omega))$. We consider the equation satisfied by $\psi$ and apply the usual Carleman inequality for the heat equation with a Neumann boundary for $\psi$ and combine it with the usual energy estimates for the equation satisfied by $\eta$. This gives a global estimate of $\Delta\varphi$ in terms of a local integral of $\Delta\psi$ and global integrals of $\Delta\xi$ and $f_1$.

*Step 2. Carleman inequality for $\xi$.*

In the second step we obtain a Carleman inequality for $\xi$, with a precise dependence of the degenerating parameter multiplying the time derivative and combine it with the Carleman inequality obtained in step 1. This gives a global estimate of $\xi$ and $\Delta\varphi$ in terms of local integrals of $\xi$ and $\Delta\psi$ and global integrals of $f_1$ and $f_2$.

*Step 3. Estimate of the local integral of $\Delta\psi$.*

In the last step, we estimate a local integral of $\Delta\psi$ in terms of a local integral of $\xi$ and global integrals of $f_1$ and $f_2$. Combining steps 2 and 3, the proof of Theorem 1.11 is completed.

Once we have proved Theorem 1.11, we concentrate on the null controllability problem for the linear system (1.45) with a right-hand side which decays exponentially as $t \to T^-$.

This result is crucial in order to prove the local controllability of (1.43). Indeed, we want to find $g \in L^2(0,T;H^1(\Omega))$ such that the solution of

$$\begin{cases} L(u,v) = (h_1, h_2 + g\chi) & \text{in } \ Q, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \ \Sigma, \\ u(x,0) = u_0; \ v(x,0) = v_0 & \text{in } \ \Omega, \end{cases} \tag{1.48}$$

where

$$L(u,v) = (u_t - \Delta u + M_1 \Delta v, \epsilon v_t - \Delta v + bv - au), \tag{1.49}$$

verifies

$$u(x,T) = v(x,T) = 0 \text{ in } \Omega. \tag{1.50}$$

Furthermore, we want to prove the existence of a solution of the previous problem in an appropriate weighted space. In order to do so, we consider weights similar to those in (1.20), but that do not degenerate at $t = 0$ and prove the following refined Carleman inequality.

**Lemma 1.1.** *There exists $C = C(\Omega, \omega, T) > 0$ such that, for any $0 < \epsilon \leq 1$, every solution of (1.46) satisfies:*

$$\begin{aligned} &\iint_Q \gamma_1^2 |\xi|^2 dx dt + \iint_Q \gamma_2^2 |\varphi - (\varphi)_\Omega|^2 dx dt + ||\varphi(0) - (\varphi(0))_\Omega||^2 + \epsilon ||\xi(0)||^2 \\ &\leq C\Big( \iint_Q \gamma_3^2 |\chi|^2 |\xi|^2 dx dt + \iint_Q \gamma_4^2 |f_1|^2 dx dt + \iint_Q \gamma_5^2 |f_2|^2 dx dt \Big), \end{aligned} \tag{1.51}$$

*where*

$$(\varphi)_\Omega(t) = \frac{1}{|\Omega|} \int_\Omega \varphi(x,t) dx$$

*and*

$$\gamma(x,t) = e^{\frac{\eta(x)}{(T-t)^k}}, \tag{1.52}$$

*for a constant* $k \geq 1$.

The proof of this lemma is standard, being just the combination of inequality (1.47) and an energy estimate for the adjoint system (1.46).

Once we have got (1.51), we solve (1.48)-(1.50). Actually, we prove two controllability results: first, we obtain a null controllability result for (1.48) with no supplementary regularity for the control and the state; second, we prove (1.48)-(1.50) by means of states and controls more regular.

The null controllability result for (1.48) with no supplementary regularity for the control and the state is the following.

**Proposition 1.3.** *Let* $0 < \epsilon \leq 1$ *and let* $(M_1, M_2) \in \mathbb{R}^2_+$ *satisfying* $aM_1 - bM_2$ *=0. Assume that* $(u_0, v_0) \in L^2_0(\Omega) \times L^2(\Omega)$, $\gamma_2^{-1}h_1 \in L^2_0(Q)$ *and that* $\gamma_1^{-1}h_2 \in L^2(Q)$*. Then there exists* $g = g(\epsilon) \in L^2(\omega \times (0,T))$*, bounded independently from* $\epsilon$*, such that* (1.48)-(1.50) *are satisfied.*

The proof of Proposition 1.3 follows from the fact that (1.51) implies the existence of a unique minimizer for the functional

$$\begin{aligned} J_\delta(\varphi_T, \xi_T) =& \frac{1}{2}\iint_{\omega\times(0,T)} |\xi|^2 dxdt + (u_0, (\varphi_0 - (\varphi(0))_\Omega)) + \epsilon(v_0, \xi(0)) \\ &+ \iint_Q h_1(\varphi - (\varphi)_\Omega) dxdt + \iint_Q h_2 \xi dxdt. \end{aligned}$$

The second main null controllability result for (1.48) is the key to deduce controllability properties for the nonlinear system (1.43).

The Banach space where (1.48)-(1.50) are solved is:

$$\begin{aligned} E = \{&(u,v,g) \in E_0 : e^{-2s\hat\beta}\hat\gamma^3 \Pi_1 L(u,v) \in L^2(Q), e^{-s\beta - s\hat\beta}\hat\gamma^2\big(\Pi_2 L(u,v) - g\chi\big) \in L^2(0,T;H^1(\Omega)), \\ &\int_\Omega \Pi_1 L(u,v)dx = 0 \text{ and } \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \Sigma\}, \end{aligned}$$

with

$$\begin{aligned} E_0 = \Big\{&(u,v,g) : ||\gamma_4^{-1}u||_{L^2(Q)} + ||\gamma_5^{-1}v||_{L^2(Q)} + ||\gamma_3^{-1}g||_{L^2(Q)} < \infty, \\ &\gamma_6^{-1}u \in L^2(0,T;H^2(\Omega)), \gamma_6^{-1}u \in L^\infty(0,T;H^1(\Omega)), \\ &\gamma_7^{-1}\Delta v \in L^2(0,T;H^1(\Omega)),\ \gamma_7^{-1}\nabla v \in L^2(0,T;H^2(\Omega))\Big\}, \end{aligned}$$

for some appropriate weights $\gamma_6$ and $\gamma_7$ of the form (1.52).

The null controllability result for (1.48) with regular state is the following.

**Proposition 1.4.** *Let $0 < \epsilon \leq 1$ and let $(M_1, M_2) \in \mathbb{R}^2_+$ be such that $aM_1 - bM_2 = 0$ and assume the following hypotheses on the initial condition and the right-hand side*

$$(u_0, v_0) \in H^1(\Omega) \times H^2(\Omega), \int_\Omega u_0 dx = 0, \frac{\partial v_0}{\partial \nu} = 0 \text{ on } \partial\Omega \tag{1.53}$$

*and*

$$(\gamma_2^{-1} h_1, \gamma_1^{-1} h_2) \in L^2(Q) \times L^2(0, T; H^1(\Omega)). \tag{1.54}$$

*Then there exists a control $g = g(\epsilon) \in L^2((0, T); H^1(\Omega))$, bounded independently from $\epsilon$, such that, if $(u, v)$ is the associated solution of (1.48), one has $(u, v) \in E$. In particular, (1.50) holds.*

In order to prove Proposition 1.4, we prove the existence of a unique solution $(\hat{u}, \hat{v}, \hat{g})$ for the extremal problem:

$$\begin{cases} \inf \frac{1}{2}\left(\iint_Q |\chi(x)|^2 \gamma_3^{-2} |g|^2 + \iint_Q \gamma_4^{-2} |u|^2 + \iint_Q \gamma_5^{-2} |v|^2\right) \\ \text{subject to } g \in L^2(0, T; H^1(\Omega)) \text{ and} \\ \begin{cases} u_t - \Delta u = -M_1 \Delta v + h_1 & \text{in } Q, \\ \epsilon v_t - \Delta v + bv = au + g\chi + h_2 & \text{in } Q, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \Sigma, \\ u(x, 0) = u_0; \ v(x, 0) = v_0 & \text{in } \Omega, \\ u(x, T) = 0; \ v(x, T) = 0 & \text{in } \Omega. \end{cases} \end{cases} \tag{1.55}$$

Using Lagrange's principle, there exist dual variables $(\hat{z}, \hat{w})$ such that

$$\begin{cases} (\hat{u}, \hat{v}) = (\gamma_4^2 \Pi_1 L^*(\hat{z}, \hat{w}), \gamma_5^2 \Pi_2 L^*(\hat{z}, \hat{w})) & \text{in } Q, \\ \hat{g} = -\gamma_3^2 \hat{w} \chi & \text{in } Q, \end{cases} \tag{1.56}$$

where $L^*$ is the adjoint operator of $L$, i.e.,

$$L^*(z, w) = (-z_t - \Delta z - aw, -\epsilon w_t - \Delta w + bw + M_1 \Delta z).$$

Finally, using the regularizing effect of parabolic equations, it is shown that $(\hat{u}, \hat{v}, \hat{g})$ has the desired regularity properties.

For proving Theorem 1.10, we first reduce the problem to a local null controllability result. We write $(z, w) = (u - M_1, v - M_2)$, where $(u, v)$ is the solution of (1.43). The pair

$(z,w)$ is then the solution of the nonlinear problem:

$$\left| \begin{array}{ll} L(z,w) = (-\nabla \cdot (z\nabla w), g\chi_\omega) & \text{in } Q, \\ \frac{\partial z}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 & \text{on } \Sigma, \\ z(x,0) = u_0 - M_1;\ w(x,0) = v_0 - M_2 & \text{in } \Omega. \end{array} \right. \tag{1.57}$$

We notice that $(z(T), w(T)) = (0,0)$ if and only if $(u(T), v(T)) = (M_1, M_2)$.

The end of the proof is to apply the following inverse mapping theorem (see [27]):

**Theorem 1.12.** *Let $E$ and $G$ be two Banach spaces and let $\mathcal{A} : E \to G$ be a continuous function from $E$ to $G$ defined in $B_\eta(0)$ for some $\eta > 0$ with $\mathcal{A}(0) = 0$. Let $\Lambda$ be a continuous and linear operator from $E$ onto $Y$ and suppose there exists $C_0 > 0$ such that*

$$||e||_E \leq C_0 ||\Lambda(e)||_G \tag{1.58}$$

*and that there exists $\delta < C_0^{-1}$ such that*

$$||\mathcal{A}(e_1) - \mathcal{A}(e_2) - \Lambda(e_1 - e_2)|| \leq \delta ||e_1 - e_2|| \tag{1.59}$$

*whenever $e_1, e_2 \in B_\eta(0)$. Then the equation $\mathcal{A}(e) = h$ has a solution $e \in B_\eta(0)$ whenever $||h||_G \leq c\eta$, where $c = C_0^{-1} - \delta$.*

In fact, we use this theorem with the space $E$ defined before and

$$G = X \times L_0^2(\Omega) \times L^2(\Omega),$$

where

$$X = \{(h_1,h_2) : \gamma_2^{-1} h_1 \in L^2(Q), \gamma_1^{-1} h_2 \in L^2(0,T;H^1(\Omega)), \int_\Omega h_1(x,t)dx = 0\}. \tag{1.60}$$

The operator $\mathcal{A}$ will be given by

$$\mathcal{A}(z,w,g) = (L(u,v) + ((\nabla \cdot (z\nabla w), -g\chi)), z(0), w(0))\ \forall (z,w,g) \in E.$$

Using the regularity of the functions within the space $E$, it is shown that $\mathcal{A} \in C^1(E;G)$ and that

$$\mathcal{A}(0,0,0) = (L(u,v) + (0, -g\chi)), z(0), w(0))\ \forall (z,w,g) \in E.$$

Using Proposition 1.4, we finish the proof of Theorem 1.10.

### Chapter 5: Null controllability of a system of viscoelasticity with a moving control

Chapter 5 of the present thesis is devoted to extension of the work of P. Martin *et al.* [95] to the multi-dimensional case. More precisely, we analyze the controllability of the following model of viscoelasticity:

$$\left| \begin{array}{ll} y_{tt} - \Delta y - \Delta y_t + b(x) y_t = 1_{\omega(t)} h, & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = y_0(x), \; y_t(x,0) = y_1(x) & \text{in } \Omega, \end{array} \right. \tag{1.61}$$

where $\Omega$ is a smooth, bounded open set in $\mathbb{R}^N$ $(N \geq 1)$ and $b \in L^\infty(\Omega)$.

In [95], in the $1-d$ setting, the authors show the null controllability of (1.61) if the control region moves so that its support covers the whole domain where the equation evolves in time.

For null controllability to hold in the multi-dimensional case, we will also assume that the control region moves in such a way that its support covers the whole domain. Actually, if $\omega(t) \equiv \omega$ for all $0 < t < T$, i.e., if the support of the control does not move in time (as it is often considered), system (1.61) is not controllable, except for the trivial case where $\omega = \Omega$.

The techniques used in [95] are based on Fourier analysis and can not be applied in the multi-dimensional case. Hence, we use a different strategy, based on the fact that system (1.61) can be rewritten as the following parabolic-ODE system:

$$\left\{ \begin{array}{l} y_t - \Delta y + (b-1) y = z, \\ z_t + z = 1_{\omega(t)} h + (b-1) y. \end{array} \right. \tag{1.62}$$

Our analysis of the controllability of (1.62) is done through Carleman inequalities for its adjoint system, also a parabolic-ODE system. The key is the use of the same weight function for both Carleman inequalities, the one for the heat and the one for the ODE. To our knowledge, all the Carleman inequalities for the heat equation available in the literature are proved in the case where the control region is fixed. In the problem we are dealing with, the control region is moving in time, so that a transport effect is added to the ODE (see system (5.97)). Hence, the proof of Carleman inequalities for the heat equation and ODE's when the control region is moving is also one of the novelties we present in this thesis.

Our null controllability result for (1.61) is as follows.

**Theorem 1.13.** *Under appropriate assumptions on the trajectory of the control region $\omega(t)$ (see conditions (5.9)-(5.13)), for any $T > 0$ and any $(y_0, y_1) \in L^2(\Omega)^2$ with $y_1 - \Delta y_0 \in L^2(\Omega)$,* *there exists a function* $h \in L^2(0,T;L^2(\Omega))$ *for which the solution of*

$$\left| \begin{array}{ll} y_{tt} - \Delta y - \Delta y_t + b(x)y_t = 1_{\omega(t)}h, & \textit{in } Q, \\ y(x,t) = 0, & \textit{on } \Sigma, \\ y(.,0) = y_0; y_t(.,0) = y_1 & \textit{in } \Omega \end{array} \right. \tag{1.63}$$

*fulfills* $y(.,T) = y_t(.,T) = 0$.

Conditions (5.9)-(5.13) of Theorem 1.13 basically mean that at the end of the evolution of $\omega(t)$ we have covered the whole domain $\Omega$.

The proof of Theorem 1.13 is performed by showing the null controllability of the system

$$\left| \begin{array}{ll} y_t - \Delta y + (b(x)-1)y = z & \text{in } Q, \\ z_t + z = 1_{\omega(t)}h + (b(x)-1)y & \text{in } Q, \\ y(x,t) = 0 & \text{on } \Sigma, \\ z(x,0) = z_0(x); y(x,0) = y_0(x) & \text{in } \Omega. \end{array} \right. \tag{1.64}$$

More precisely, we prove the following result.

**Theorem 1.14.** *Under appropriate assumptions on the trajectory of the control region* $\omega(t)$ *(see conditions* (5.9)-(5.13)*), for any* $T > 0$ *and* $(y_0, y_1) \in L^2(\Omega)^2$*, there exists a control function* $h \in L^2(0,T;L^2(\Omega))$ *for which the solution* $(y,z)$ *of* (1.64) *satisfies* $y(.,T) = z(.,T) = 0$.

In Theorem 1.14 we observe the need for conditions (5.9)-(5.13). Indeed, the fact that at the end of the evolution of $\omega(t)$ the whole domain $\Omega$ is covered, is a necessary condition to control the ODE $(1.64)_2$.

Let us now give the main idea of the proof of Theorem 1.14.

First, we consider the adjoint system of (1.64):

$$\left| \begin{array}{ll} -p_t - \Delta p + (b(x)-1)p = (b(x)-1)q & \text{in } Q, \\ -q_t + q = p & \text{in } Q, \\ p(x,t) = 0 & \text{on } \Sigma, \\ p(x,T) = p_0(x); q(x,T) = q_0(x) & \text{in } \Omega \end{array} \right. \tag{1.65}$$

and prove the following two lemmas.

**Lemma 1.2.** *There exists a constant* $C_0 > 0$ *such that, for all* $p \in C([0,T];L^2(\Omega))$ *with* $p_t + \Delta p \in L^2(0,T;L^2(\Omega))$*, the following holds:*

$$\int_0^T \int_\Omega \theta^2 |p|^2 dxdt \leq C_0 \left( \int_0^T \int_\Omega \theta_1^2 |p_t + \Delta p|^2 dxdt + \int_0^T \int_{\omega(t)} \theta^2 |p|^2 dxdt \right). \tag{1.66}$$

**Lemma 1.3.** *There exists a constant $C_1 > 0$, such that, for all $q \in H^1(0,T;L^2(\Omega))$, the following holds:*

$$\int_0^T\int_\Omega \theta_1^2|q|^2dxdt \leq C_1\Big(\int_0^T\int_\Omega \theta_2^2|q_t|^2dxdt + \int_0^T\int_{\omega(t)} \theta_1^2|q|^2dxdt\Big). \tag{1.67}$$

In Lemmas 1.2 and 1.3, the weights $\theta_i = \theta_i(x,t)$, $i = 1,2$ are similar to those defined in (1.20). The main difference is that $\eta = \eta(x,t)$, which is related to the fact that our control region is moving in time.

To finish the proof of Theorem 1.14, we proceed as follows.

*Step 1. Carleman estimates with the same weight.*

We apply the Carleman inequalities given in Lemmas 1.2 and 1.3 to $p$ and $q$, respectively. We combine these inequalities and obtain global estimates of $p$ and $q$ in terms of local integrals of $p$ and $q$.

*Step 2. Estimate of the local integral of $p$.*

Similarly as in chapter 2 of this thesis, we estimate a local integral of $p$ in terms of a local integral of $q$.

*Step 3. Observability Inequality*

Finally, using semigroup theory, we prove the following observability inequality:

$$\int_\Omega [|p(0)|^2 + |q(0)|^2]dx \leq C\int_0^T\int_{\omega(t)} |q(x,t)|^2\, dxdt, \tag{1.68}$$

for all $(p_0, q_0) \in L^2(\Omega)^2$.

From the observability inequality (1.68), we complete the proof of Theorem 1.14, and hence, the proof of Theorem 1.13.

### Chapter 6: A hyperbolic system and the cost of null controllability for the Stokes system

The last chapter of this thesis is devoted to the analysis of the cost of the null controllability as $T \longrightarrow 0^+$ of the Stokes system:

$$\left|\begin{array}{ll} y_t - \Delta y + \nabla p = g1_\omega & \text{in } Q,\\ div\ y = 0 & \text{in } Q,\\ y = 0 & \text{on } \Sigma,\\ y(0) = y_0 & \text{in } \Omega. \end{array}\right. \tag{1.69}$$

Our motivation for this chapter comes from the well-known fact that the cost of the null controllability for the single heat equation

$$\left| \begin{array}{ll} v_t - \Delta v = f 1_\omega & \text{in } Q, \\ v = 0 & \text{on } \Sigma, \\ v(0) = v_0 & \text{in } \Omega \end{array} \right. \tag{1.70}$$

is of the form $e^{C(\Omega,\omega)(1+1/T)}$ as $T \longrightarrow 0^+$ (see, for instance, [98, 99]). More precisely, one has

$$||f 1_\omega||_{L^2(Q)} \leq e^{C(\Omega,\omega)(1+1/T)} ||v_0||_{L^2(\Omega)}. \tag{1.71}$$

However, unlike the case of the heat equation, the known results in the literature about the null controllability of the Stokes system (see, for instance, [38]) give

$$||g 1_\omega||_{L^2(Q)} \leq e^{C(\Omega,\omega)(1+1/T^4)} ||y_0||_{L^2(\Omega)}. \tag{1.72}$$

As observed in [32], the main reason for the form of the cost in (1.71) is due to the fact that the fundamental solution of the heat equation in $\mathbb{R}^N$ is given by

$$\Phi(x,t) = \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x|^2}{4t}}. \tag{1.73}$$

Since the fundamental solutions of the heat equation and the Stokes system have, at least for $N = 2, 3$, the same behavior in time (see [56, 57, 121]), the following natural question arises:

**Question 1.1.** Do the costs of the controllability for the heat equation and the Stokes system have the same order in time as $T \longrightarrow 0^+$?

The aim of chapter 6 is to give a positive answer to Question 1.1. In order to do so, we assume that $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded connected open set and star shaped with respect to the origin, and that the boundary $\partial\Omega$ is sufficiently regular. The control region $\omega$ will be a nonempty subset of $\Omega$ satisfying:

$$\exists \mathcal{O} \subset \mathbb{R}^N, \mathcal{O} \text{ being a neighborhood of } \partial\Omega \text{ and } \omega = \Omega \cap \mathcal{O}. \tag{1.74}$$

We introduce the following spaces, usual in the context of fluid mechanics,

$$V = \{u \in H_0^1(\Omega)^N;\ div\ u = 0\},$$

$$H = \{u \in L^2(\Omega)^N;\ div\ u = 0, u \cdot \nu = 0 \text{ on } \partial\Omega\}.$$

The main result of chapter 6 is then stated as follows

**Theorem 1.15.** *Assume $\omega$ satisfies (1.74) and let $0 < T \leq 1$. For any $y_0 \in H$, there exists a control $g \in L^2(\omega \times (0,T))$ such that the solution $y$ of (1.69) satisfies:*

$$y(T) = 0.$$

*Moreover, there exist positive constants $C_1$ and $C_2$, depending only on $\Omega$ and $\omega$, such that*

$$\iint_{\omega \times (0,T)} |g|^2 dx dt \leq C_1 e^{C_2/T} |y_0|_H^2, \tag{1.75}$$

*for all $y_0 \in H$ and $0 < T \leq 1$.*

The first attempt to prove Theorem 1.15 is to analyze the different ways one can prove (1.71) and (1.72). In fact, there are at least two different methods to prove (1.71). The first one is based on spectral decompositions, it is the so-called Lebeau-Robbiano strategy (see [82]). The second one is based on the use of global Carleman inequalities (see [44, 46]). For the Stokes system, it seems that a Lebeau-Robbiano strategy is difficult to prove, since one must deal with the pressure and, to our knowledge, it has not been proved yet to hold. Consequently, the most known method used to prove (1.72) is based on Carleman inequalities (see [38]).

The main difference when proving (1.71) and (1.72) by means of Carleman inequalities are the weights one must use. Indeed, for the heat equation the weights used are of the form

$$\rho(t) = \frac{e^{C/(t(T-t))}}{t(T-t)}, \tag{1.76}$$

while for the Stokes system the weights take the form

$$\rho(t) = \frac{e^{C/(t^4(T-t)^4)}}{t^4(T-t)^4}. \tag{1.77}$$

If we were able to use weights as (1.76) for the Stokes system then these two equations would have costs of controllability of the same order. However, a careful analysis of both proofs indicates that the obstruction to have weights of the form (1.76) for the Stokes system is due to the pressure term and that, probably, it is of purely technical nature.

Hence, our strategy will not be based on the use of Carleman inequalities but rather on the application of the Control Transmutation Method (CTM).

In fact, we consider the following hyperbolic system with a pressure term:

$$\left|\begin{array}{ll} u_{tt} - \Delta u + \nabla p = h 1_{\omega}, & \text{in } Q, \\ div\ u = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(0) = u^0; u_t(0) = u^1 & \text{in } \Omega. \end{array}\right. \tag{1.78}$$

We show the null controllability of (1.78) and apply the CTM in order to guarantee the null controllability for the Stokes system (1.69). Moreover, since we know the cost of the controllability for (1.78), we use the transmutation formula for the control and show the estimate (1.75).

We prove the following controllability result for (1.78).

**Theorem 1.16.** *Assume $\omega$ satisfies (1.74) and let $T/2 > R_0 := \max\{|x|, x \in \overline{\Omega}\}$. Given $(u_0, u_1) \in V \times H$, there exists a control $h \in L^2(\omega \times (0,T))$ such that the associated solution $u$ of (1.78) satisfies:*

$$u(T) = u_t(T) = 0.$$

*Moreover, there exists a positive constant $C$, such that the following holds:*

$$\iint_{\omega \times (0,T)} |h|^2 dx dt \leq C\big(||u_0||_V^2 + |u^1|_H^2\big). \tag{1.79}$$

Notice that system (1.78) is not of Cauchy-Kowalewski type, which makes impossible the application of Holgrem's Theorem as in the case of the wave equation. As far as we know, Theorem 1.16 is a completely new result. The exact controllability of (1.78) when the control is acting on the boundary was shown in [107].

Let us now give the main ideas of the proof of Theorem 1.16.

We consider the adjoint system of (1.78):

$$\left|\begin{array}{ll} \phi_{tt} - \Delta \phi + \nabla q = 0 & \text{in } Q, \\ div\ \phi = 0 & \text{in } Q, \\ \phi = 0 & \text{on } \Sigma, \\ \phi(0) = \phi^0; \phi_t(0) = \phi^1 & \text{in } \Omega, \end{array}\right. \tag{1.80}$$

where $\phi^0 \in H$ and $\phi^1 \in V'$.

We need to show the existence of a positive constant $C$ such that the following observability inequality holds:

$$|\phi^0|_H^2 + ||\phi^1||_{V'}^2 \leq C \iint_{\omega \times (0,T)} |\phi|^2 dx dt, \tag{1.81}$$

for all solutions of (1.80).

First we prove the following two results.

**Theorem 1.17.** *If we take* $T/2 > R_0 := \max\{|x|, x \in \overline{\Omega}\}$ *then, for every solution of (1.80) with initial data* $(\phi^0, \phi^1) \in V \times H$*, the following estimate holds:*

$$|\phi^1|_H^2 + ||\phi^0||_V^2 \leq \frac{R_0}{2(T - 2R_0)} \iint_\Sigma \left(\frac{\partial \phi}{\partial \nu}\right)^2 d\Sigma. \tag{1.82}$$

The proof of Theorem 1.17 is performed through the multiplier method, exactly as in the case of the wave equation, by taking into account that, for regular solutions,

$$\iint_Q \frac{\partial p}{\partial x_i} x_k \frac{\partial \phi^i}{\partial x_k} dxdt = 0.$$

**Theorem 1.18.** *Assume* $\omega$ *satisfies* (1.74) *and that there exists a constant* $C > 0$ *such that, for every* $(\phi^0, \phi^1) \in V \times H$*, the weak solution* $\phi$ *of (1.80) satisfies:*

$$||\phi^0||_V^2 + |\phi^1|_H^2 \leq C \iint_{\omega \times (0,T)} |\phi_t|^2 dxdt. \tag{1.83}$$

*Then inequality (1.81) holds for all solutions of* (1.80) *with initial data* $(\phi^0, \phi^1)$ *in* $H \times V'$.

We also prove Theorem 1.18 borrowing ideas from the controllability theory for the wave equation.

Furthermore, using contradiction arguments and adapting the ideas of [136] to the present case, we prove (1.83).

Once the exact/null controllability of (1.78) is proved, we use the Control Transmutation Method as follows.

We introduce two different time intervals $(0, T)$ and $(0, L)$ and consider the Stokes system

$$\left| \begin{array}{ll} y_t - \Delta y + \nabla p = g1_\omega & \text{in } Q_t := \Omega \times (0, T), \\ div\ y = 0 & \text{in } Q_t, \\ y = 0 & \text{on } \Sigma_t := \partial\Omega \times (0, T), \\ y(0) = y_0 & \text{in } \ \Omega \end{array} \right. \tag{1.84}$$

and the associated hyperbolic system with a pressure term

$$\left| \begin{array}{ll} u_{ll} - \Delta u + \nabla q = h1_\omega & \text{in } Q_l := \Omega \times (0, L), \\ div\ u = 0 & \text{in } Q_l, \\ u = 0 & \text{on } \Sigma_l := \partial\Omega \times (0, L), \\ u(0) = y_0; u_l(0) = 0 & \text{in } \Omega \end{array} \right. \tag{1.85}$$

in $\Omega \times (0,T)$ and $\Omega \times (0,L)$, respectively.

We take $y_0 \in V$ and $L > 2R_0$. It follows from Theorem 1.16 that system (1.85) is null controllable, with a control $h \in L^2(\omega \times (0,L))$ satisfying (1.79).

We set

$$y(t) = \int k(t,s)u(s)ds$$

and

$$g(t) = \int k(t,s)h(s)ds,$$

where $k$ is the solution of

$$\left|\begin{array}{l} k_t = \partial_s^2 k \text{ in } \mathcal{D}'((0,T)\times(-L,L)), \\ k(0,x) = \delta(0), \\ k(T,x) = 0, \\ ||k||^2_{L^2((0,T)\times(-L,L))} \leq \gamma e^{\alpha L^2/T}, \end{array}\right. \tag{1.86}$$

for appropriate constants $\alpha$ and $\gamma$.

Since $u$ is the solution of (1.85) and $k$ is the soluion of (1.86), we can prove that $(y,g)$ solves, together with some $p$, the Stokes system (1.84). We also have that $y(T) = 0$, since $k$ is a controlled solution of the heat equation. From the definition of the control $g$, it follows that

$$\iint_{\omega\times(0,T)} |g|^2 dxdt \leq ||k||^2_{L^2((0,T)\times(-L,L))}||h||^2_{L^2(\omega\times(0,T))}.$$

Finally, using the properties of $k$ and Theorem 1.16, inequality (1.75) holds in the case of $y_0 \in V$.

Using the regularizing effect of the Stokes system, we finish the proof of Theorem 1.15.

# Capítulo 1

# Introducción

## 1.1 Introducción general

La teoría del control es la parte de las matemáticas relacionada con los sistemas dinámicos cuyo comportamiento se puede cambiar por medio de controles aplicados a través de actuadores. Los orígenes de la teoría del control se remontan al siglo XIX, con la aplicación de la teoría de ecuaciones diferenciales al estudio de la eficiencia de sistemas mecánicos en la revolución industrial. Actualmente, gracias al trabajo de matemáticos como R. Bellman, H. Fattorini, R. Kalman, J.-L. Lions, L.S. Potryagin, D. Russell, entre otros, la teoría del control es una rica rama interdisciplinaria de las matemáticas, con aplicaciones en áreas como ingeniería, biología, economía y medicina. Para más detalles véase, por ejemplo, [42, 118, 119] y las referencias allí citadas.

De manera general, un sistema de control se puede escribir en la forma abstracta

$$\begin{cases} \frac{dy}{dt} = H(y,u),\ t>0,\ y\in Y,\ \ u\in\mathcal{U}_{ad}, \\ y(0)=y_0, \end{cases} \tag{1.1}$$

donde $y$ es el *estado*, la variable que estamos dispuestos a controlar, $y_0$ es el estado inicial, $u$ es el *control*, la variable que podemos elegir libremente para que actúe en el sistema y $Y$ y $\mathcal{U}_{ad}$ son, respectivamente, el *espacio de los estados* y el conjunto de los *controles admisibles*.

Dado un sistema de control, el objetivo será encontrar un control tal que el estado asociado se comporte de una manera apropiada en un tiempo final dado. A este tipo de problemas se les llama problemas de controlabilidad. Se distinguen varias diferentes nociones de controlabilidad. Se dice que el sistema es *aproximadamente controlable* si, para cualquier estado inicial dado, se puede llevar la solución del sistema a un estado arbitrariamente cerca (en alguna topología) de cualquier otro estado dado. Por otro lado, la *controlabilidad exacta* significa que se puede llevar el sistema desde cualquier dato inicial a cualquier otro estado dado. Decimos que tenemos la propiedad de la *controlabilidad nula o a cero* si, para cualquier dato inicial, la solución puede ser llevada a cero. Por último, otro concepto interesante es el de la *controlabilidad exacta a trayectorias*, que significa que es posible llevar el estado del sistema a cualquier trayectoria prescrita, es decir, a una determinada solución del sistema.

En esta tesis analizamos la controlabilidad de algunas ecuaciones en derivadas parciales (EDPs) que modelan varios fenómenos físicos. También estudiaremos el problema de la *controlabilidad uniforme*. En este caso consideramos una familia de sistemas dinámicos,

$$\begin{cases} \frac{dy^\epsilon}{dt} = H^\epsilon(\epsilon, y^\epsilon, u^\epsilon), \ y^\epsilon \in Y^\epsilon \subset Y, \ \ u^\epsilon \in \mathcal{U}_{ad}, \\ y^\epsilon(0) = y_0^\epsilon, \end{cases} \tag{1.2}$$

dependiendo de un parámetro $\epsilon > 0$ y aproximando (1.1), cuando $\epsilon \to 0^+$.

Primero probamos que, para cada $\epsilon > 0$, existe un control $u^\epsilon \in \mathcal{U}_{ad}$ tal que la solución $y^\epsilon$ de (1.2) satisface $y^\epsilon(T) = y_T^\epsilon$, para ciertos estados dados $y_0^\epsilon$ y $y_T^\epsilon$ convergiendo a $y_0$ y $y_T$ en $Y$, respectivamente. El objetivo será entonces saber si existen $y \in Y$ y $u \in \mathcal{U}_{ad}$, solución de (1.1), tal que $y^\epsilon \to y$ y $u^\epsilon \to u$ cuando $\epsilon \to 0^+$. Por otra parte, queremos que esta convergencia sea lo suficientemente fuerte de manera que las condiciones iniciales y finales sean preservadas: $y(0) = y_0$ y $y(T) = y_T$. En el caso en que $y_T \equiv 0$, se dice que el sistema (1.2) tiene la propiedad de *controlabilidad uniforme a cero*.

Actualmente existe un vasta literatura relacionada a los problemas de control (véase, por ejemplo, [25, 86, 115, 125, 135]). La teoría para los sistemas de dimensión finita ha sido desarrollada a principios de la década de 1960 (véase [64, 76]) y, gracias a la famosa condición del rango de Kalman, el problema está completamente entendido en el caso lineal (véase [79, 122] ). Para los sistemas no lineales de dimensión finita el problema está también bastante bien entendido, ya que existen varios métodos conocidos de condiciones suficientes para el estudio de la controlabilidad local y global (véase [25]).

Para EDPs la situación es un poco más delicada, incluso en el marco lineal. Una de las razones es que una EDP lineal que gobierna la evolución puede ser, por ejemplo, del tipo hiperbólico (ecuación de ondas, las ecuaciones de Maxwell), del tipo dispersivo (ecuación de placas, ecuación de Schrödinger, ecuación de Korteweg-de Vries), o del tipo parabólico (ecuación del calor, ecuación de Stokes), que inducen propiedades muy específicas sobre el flujo: el principio de Huygens y la propiedad de la propagación de singularidades con velocidad finita para ecuaciones hiperbólicas, la velocidad de propagación infinita juntamente con un efecto regularizante débil (resp. fuerte) para ecuaciones dispersivas (resp. parabólicas) y la irreversibilidad en tiempo para ecuaciones parabólicas. En consecuencia, no es posible esperar un resultado de controlabilidad exacta para la ecuación del calor con un control localizado en una pequeña parte del dominio (la solución será regular fuera de la región de control), por lo que no se puede alcanzar un estado final arbitrario. Por eso, en lo que a la ecuación del calor se refiere, es natural estudiar las propiedades de controlabilidad aproximada, nula o a trayectorias. Por el contrario, debido a su reversibilidad en el tiempo, es natural estudiar la propiedad de la controlabilidad exacta para la ecuación de ondas.

Para un mejor desarrollo de este trabajo, en la siguiente sección recordaremos algunos conceptos y resultados fundamentales en el estudio de la controlabilidad de ecuaciones lineales. Estos resultados serán importantes incluso para los problemas de control no lineales que trataremos ya que, en este caso, el primer paso es el análisis de las propiedades de controlabilidad de un sistema lineal apropiado.

Restringiremos la presentación al caso de los espacios de Hilbert, sin embargo, señalamos que una teoría similar puede ser desarrollada en espacios de Banach.

### 1.1.1 Controlabilidad y Observabilidad

En esta sección seguimos esencialmente [25, 108, 125]. Consideramos dos espacios de Hilbert (reales o complejos) $(H, <,>_H)$ y $(U, <,>_U)$, un tiempo $T > 0$, $y_0 \in H$ y un operador no acotado $A : D(A) \longrightarrow H$ que genera un semigrupo fuertemente continuo $S(t)_{t\geq 0}$. Estamos interesados en la siguiente clase de problemas de control lineal

$$\begin{cases} \frac{dy}{dt} = Ay + Bu,\ t \in [0,T], \\ y(0) = y_0, \end{cases} \tag{1.3}$$

donde $B \in \mathcal{L}(U; D(A)')$ es un operador que describe la manera en que el control $u$ actúa en el sistema.

Asumimos que el operador $B$ satisface la *condición de admisibilidad*:

$$\forall T > 0, \exists C_T > 0 \text{ tal que } \int_0^T ||B^* S(t)^* z||_U dt \leq C_T ||z||_H^2,\ \forall z \in D(A^*), \tag{1.4}$$

donde $B^*$, $S(t)^*$ y $A^*$ son los operadores adjuntos de $B$, $S(t)$ y $A$, respectivamente.

Bajo la condición de admisibilidad (1.4) el problema de Cauchy (1.3) está bien planteado en el sentido de Hadarmad, es decir, para todo $y_0 \in H$ y $u \in L^2(0,T;U)$, existe una única $y \in C([0,T];H)$ solución de (1.3). Además, existe una constante $C > 0$, dependiendo solamente de $T$, $A$ y $B$, tal que

$$||y||_{C([0,T];H)} \leq C(||y_0||_H + ||u||_{L^2(0,T;U)}).$$

Recordamos de nuevo las diferentes nociones de controlabilidad para el sistema

(1.3).

**Definición 1.1.** Un sistema (1.3) es exactamente controlable en el tiempo $T$ si, para todo $y_0, y_T \in H$, existe un control $u \in L^2(0,T;U)$ tal que la solución $y$ de (1.3) satisface $y(T) = y_T$.

**Definición 1.2.** Un sistema (1.3) es controlable a cero en el tiempo $T$ si, para todo $y_0, y_T \in H$, existe un control $u \in L^2(0,T;U)$ tal que la solución $y$ de (1.3) satisface $y(T) = 0$.

**Definición 1.3.** Un sistema (1.3) es aproximadamente controlable en el tiempo $T$ si, para todo $y_0, y_T \in H$ y todo $\epsilon > 0$, existe un control $u \in L^2(0,T;U)$ tal que la solución $y$ de (1.3) satisface$||y(T) - y_T||_H < \epsilon$.

De la definición se deduce que la controlabilidad exacta implica tanto la controlabilidad nula como la controlabilidad aproximada. El recíproco, en general, falso.

Para sistemas lineales se demuestra que la controlabilidad nula es equivalente a la controlabilidad exacta a trayectorias:

**Definición 1.4.** Se dice que (1.3) es exactamente controlable a trayectorias en el tiempo $T$ si, para todo $y_0 \in H$ y cualquier solución $\overline{y}$ de (1.3), es decir, una solución de (1.3) con $\overline{y}(0) = \overline{y}_0 \in H$ y algún $\bar{u}$ dado, existe un control $u$ tal que la solución asociada de (1.3) satisface $y(T) = \overline{y}(T)$.

En el caso de dimensión finita, es decir: $A \in \mathbb{R}^{N\times N}, B \in \mathbb{R}^{N\times M}$, $N, M \in \mathbb{N}$, las cuatro definiciones anteriores son equivalentes a una condición puramente algebraica, la condición del rango de Kalman: $rank(B, AB, A^2B, \dots, A^{N-1}B) = N$. Como consecuencia, para sistemas de dimensión finita, controlabilidad en un tiempo $T_0 > 0$ implica controlabilidad en cualquier tiempo $T > 0$.

Como fue notado por D. Russell en [116], y formalizado por J.-L. Lions en el famoso HUM (Método de la Unicidad de Hilbert) (véase [86, 87, 88]), las propiedades de controlabilidad para el sistema (1.3) son equivalentes a ciertas mediciones (observabilidades) de su sistema adjunto (problema dual).

En efecto, consideremos el sistema adjunto de (1.3):

$$\begin{cases} -\frac{dz}{dt} = A^*z, \ t \in [0,T], \\ z(T) = z_T \in H. \end{cases} \tag{1.5}$$

Los siguientes resultados son ciertos.

**Teorema 1.1.** *El sistema* (1.3) *es exactamente controlable en el tiempo* $T$ *si y solamente si existe una constante* $C > 0$ *tal que*

$$||z_T||_H^2 \leq C^2 \int_0^T ||B^* z(t)||_U^2 dt, \ \forall z_T \in H. \tag{1.6}$$

A la desigualdad (1.6) se le llama *desigualdad de observabilidad fuerte*. Eso significa que podemos recuperar una información completa del estado inicial $z_T$ desde una medición en $[0, T]$ de $B^* z(t)$.

**Teorema 1.2.** *El sistema* (1.3) *es controlable a cero en el tiempo* $T$ *si y solamente si existe una constante* $C > 0$ *tal que*

$$||z(0)||_H^2 \leq C^2 \int_0^T ||B^* z(t)||_U^2 dt, \ \forall z_T \in H. \tag{1.7}$$

A la desigualdad (1.7) se le llama la *desigualdad de observabilidad débil*. Solamente $z(0)$ puede ser recuperado, pero no $z_T$.

**Teorema 1.3.** *El sistema* (1.3) *es aproximadamente controlable en el tiempo* $T$ *si y solamente si para todo* $z_T \in H$,

$$B^* z(t) = 0 \text{ on } [0, T] \Longrightarrow z_T = 0. \tag{1.8}$$

La propiedad (1.8) es conocida como la *propiedad de continuación única* para el sistema (1.5).

Del Teorema 1.2 se concluye, por la unicidad retrógada, que la controlabilidad a cero para (1.3) implica la controlabilidad aproximada del sistema.

Remarcamos que no hay razón para la unicidad del control que lleva un estado inicial $y_0$ a un estado final $y_T$. Entre tanto, entre todos los controles posibles se distingue el de norma mínima en $L^2(0, T; U)$.

En efecto, supongamos que el sistema (1.3) es exactamente controlable en el tiempo $T$. Entonces, para cada $y_T \in H$, el conjunto $U^T(y_T)$ de los $u \in L^2(0, T; U)$ tal que

$$[y_t = Ay + Bu, y(0) = 0] \Longrightarrow [y(T) = y_T]$$

es no vacío. El conjunto $U^T(y_T)$ es claramente un subespacio afín cerrado de $L^2(0, T; U)$. Denotemos por $\mathcal{U}^T(y_T)$ el elemento de $U^T(y_T)$ de norma mínima en $L^2(0, T; U)$. Es imediato que la aplicación

$$\begin{array}{rccc} \mathcal{U}^T(y_T): & H & \to & L^2(0, T; U) \\ & y_T & \mapsto & \mathcal{U}^T(y_T) \end{array}$$

es lineal. Además, utilizando el teorema de la gráfica cerrada, se demuestra que esta aplicación es continua. A la norma de $\mathcal{U}^T(y_T)$, denotada por $C^E_{opt}(T)$, se le llama *coste de la controlabilidad exacta* de (1.3). Se tiene el siguiente resultado.

**Proposición 1.1.** *$C^E_{opt}(T)$ es el ínfimo de las constantes $C > 0$ para las cuales se cumple la desigualdad de observabilidad fuerte* (1.6)*, es decir,*

$$C^E_{opt}(T) = ||\mathcal{U}^T(y_T)||_{\mathcal{L}(H;L^2(0,T;U))} = \inf_{C>0}\left\{||z_T||^2_H \le C^2\int_0^T ||B^*z(t)||^2_U dt,\ \forall z_T \in H\right\}.$$

En el caso en que (1.3) es controlable a cero en el tiempo $T$, para todo $y_0 \in H$, el conjunto $U^T(y_0)$ de los $u \in L^2(0,T;U)$ tal que

$$[y_t = Ay + Bu, y(0) = y_0] \Longrightarrow [y(T) = 0]$$

es no vacío. El conjunto $U^T(y_0)$ es un subespacio afín cerrado de $L^2(0,T;U)$. Como antes, denotemos por $\mathcal{U}^T(y_0)$ al elemento de $U^T(y_0)$ de norma mínima en $L^2(0,T;U)$. No es difícil ver que la aplicación

$$\begin{array}{rccc}\mathcal{U}^T(y_0): & H & \to & L^2(0,T;U)\\ & y_0 & \mapsto & \mathcal{U}^T(y_0)\end{array}$$

es lineal y continua. A la norma de $\mathcal{U}^T(y_0)$, denotada por $C^N_{opt}(T)$, se le llama el *coste de la controlabilidad a cero* de (1.3). Se tiene el siguiente resultado.

**Proposición 1.2.** *$C^N_{opt}(T)$ es el ínfimo de las constantes $C > 0$ para las cuales se cumple la desigualdad de observabilidad débil* (1.7)*, es decir,*

$$C^N_{opt}(T) = ||\mathcal{U}^T(y_0)||_{\mathcal{L}(H;L^2(0,T;U))} = \inf_{C>0}\left\{||z(0)||^2_H \le C^2\int_0^T ||B^*z(t)||^2_U dt,\ \forall z_T \in H\right\}.$$

De las proposiciones 1.1 y 1.2 se concluye que el coste de la controlabilidad exacta/ a cero para (1.3) es la mejor constante para la cual la desigualdad de observabilidad fuerte/débil para el sistema adjunto (1.5) es cierta.

Si el sistema (1.3) es exactamente controlable (resp. a cero), existe una manera constructiva de obtener el control $\mathcal{U}^T(y_T)$ (resp. a $\mathcal{U}^T(y_0)$) de norma mínima en $L^2(0,T;U)$. Explicaremos el argumento en el caso de la controlabilidad exacta. De hecho, sea $y_0 \in H$. La dualidad entre (1.3) y (1.5) nos da

$$< y(T), z_T >_H = \int_0^T < u(t), B^*z(t) >_U dt + < y_0, z(0) >_H .$$

Introducimos entonces el funcional $\mathcal{J} : H \to \mathbb{R}$ definido por

$$\mathcal{J}(z_T) = \frac{1}{2} \int_0^T ||B^* z(t)||_U^2 dt + < y_0, z(0) >_H - < y_T, z_T >_H . \quad (1.9)$$

No es muy difícil ver que si $\mathcal{J}$ tiene un mínimo $\hat{z}_T$ entonces, poniendo $\hat{u} = B^* \hat{z}$, donde $\hat{z}$ es la solución de (1.5) asociada a $\hat{z}_T$, la solución $y$ de (1.3) con este control $\hat{u}$ verifica $y(T) = y_T$.

Es fácil ver que el funcional $\mathcal{J}$ es estrictamente convexo. Utilizando la condición de admisibilidad (1.4), se puede mostrar que es continuo. Por último, de la desigualdad de observabilidad fuerte (1.6) uno obtiene imediatamente la coercividad del funcional $\mathcal{J}$. Por consecuencia, $\mathcal{J}$ tiene un único mínimo $\hat{z}_T$ y $\hat{u} = B^* \hat{z}$ es el control de norma mínima en $L^2(0, T; U)$. Además, tenemos la siguiente estimación para el control:

$$||\hat{u}||_{L^2(0,T;U)} \leq C^E_{opt}(T) ||y_T||_H . \quad (1.10)$$

De forma similar, en el caso de la controlabilidad a cero, obtenemos un control $\hat{u}_N$ de norma mínima en $L^2(0, T; U)$ como el mínimo del siguiente funcional

$$\mathcal{J}_N(z_T) = \frac{1}{2} \int_0^T ||B^* z(t)||_U^2 dt + < y_0, z(0) >_H , \quad (1.11)$$

que es, claramente, estrictamente convexo y continuo. La coercividad no es imediata como en el caso de la controlabilidad exacta, pero se puede mostrar que $\mathcal{J}_N$ es coercivo en el espacio $\overline{H}$, el completado de $H$ con la norma (debido a la desigualdad de observabilidad débil (1.7))

$$||z_T||_* = \left( \int_0^T ||B^* z(t)||_U dt \right)^{1/2} .$$

El control $\hat{u}_N$ obtenido como el mínimo de $\mathcal{J}_N$ satisface

$$||\hat{u}||_{L^2(0,T;U)} \leq C^N_{opt}(T) ||y_0||_H . \quad (1.12)$$

De lo expuesto anteriormente, vemos que el estudio de la controlabilidad de una EDP lineal dada es equivalente a la obtención de una desigualdad de observabilidad adecuada para su sistema adjunto, es decir, un conocimiento completo de la solución del sistema adjunto en un momento determinado utilizando unicamente mediciones locales del mismo. No obstante, señalamos que la prueba de estas desigualdades requiere herramientas adaptadas a la EPD en cuestión; por ejemplo, *desigualdades de Ingham*, *métodos de los multiplicadores*, *análisis microlocal* y *desigualdades de Carleman* ([4, 46, 73, 86, 113, 115, 134]).

### 1.1.2 Metodología

La herramienta analítica fundamental que emplearemos para demostrar las desigualdades de observabilidad serán las llamadas desigualdades de Carleman. Estas desigualdades son, básicamente, estimaciones de energía con peso exponencial para las soluciones de una EDP dada. Dichas desigualdades fueron introducidas por primera vez para estudiar la cuantificación de la continuación única, remontándose a los primeros trabajos de Carleman [12]. Entre tanto, en los últimos años, el espectro de aplicaciones de las desigualdades de Carleman ha aumentado más allá de su motivación original: hoy en día también se utilizan en el estudio de problemas inversos y en la teoría del control de EDPs (para más detalles ver [46, 134, 135]).

Dado un dominio espacial $\Omega$, una región de control $\omega \subset \Omega$ y un tiempo $T > 0$, las desigualdades de Carleman obedecerán la siguiente estructura básica:

$$\iint_{\Omega\times(0,T)} \beta_1^2|\varphi|^2 dxdt \leq C \iint_{\omega\times(0,T)} \beta_2^2|\varphi|^2 dxdt, \tag{1.13}$$

donde $\varphi$ es la solución de la EDP (el sistema adjunto) y la constante $C > 0$ y las funciones peso $\beta_1$ and $\beta_2$ dependen de ciertos parámetros pero son independientes del estado inicial.

La idea básica que motiva las desigualdades de Carleman como (1.13) puede presentarse incluso en la siguiente EDO en $\mathbb{R}^N$:

$$\left| \begin{array}{l} y_t(t) = a(t)y(t), \quad t \in [0,T], \\ y(0) = y_0, \end{array} \right. \tag{1.14}$$

donde $a \in L^\infty(0,T)$ (see [134]). Es bien sabido que las soluciones de (1.14) satisfacen:

$$\max_{t\in[0,T]} |y(t)| \leq e^{||a||_\infty T}|y_0|, \quad \forall y_0 \in \mathbb{R}^N. \tag{1.15}$$

Una demostración del tipo Carleman para (1.15) es la siguiente. Para todo $\lambda \in \mathbb{R}$, tenemos

$$\frac{d}{dt}\left(e^{-\lambda t}|y(t)|^2\right) = (2a(t) - \lambda)e^{-\lambda t}|y(t)|^2. \tag{1.16}$$

Eligiendo $\lambda$ suficientemente grande de forma que $2a(t) - \lambda \leq 0$ para casi todo $t \in (0,T)$, se obtiene

$$|y(t)| \leq e^{\lambda T/2}|y_0|, \; t \in [0,T], \tag{1.17}$$

que prueba, en particular, (1.15). La clave de esta demostración, y de todas las estimaciones del tipo Carleman, es que uno debe elegir los parámetros en las funciones peso lo suficientemente grandes de forma que los términos no deseados puedan ser absorbidos.

Ahora describiremos algunas de las principales dificultades que uno puede encontrar cuando se intenta analizar la controlabilidad uniforme de una determinada familia de EDPs. Es conveniente primero analizar el caso de dimensión finita. De hecho, incluso en este caso, puede suceder que el sistema límite no sea controlable mientras que la familia que lo aproxima lo sea, para cada valor del parámetro $\epsilon > 0$. Para ver esta patología, consideramos los siguientes dos ejemplos, debidos a Chow [22]:

$$\left| \begin{array}{l} y_{1,t} = -5y_1 + \epsilon u, \\ y_{2,t} = -y_2 + u, \\ y_{3,t} = -(1+\epsilon)y_3 + u, \\ y_{4,t} = -(1+\epsilon)y_4 + 2u \end{array} \right. \tag{1.18}$$

y

$$\left| \begin{array}{l} y_{1,t} = -y_1 + y_2 + u, \\ \epsilon y_{2,t} = -y_2 - u. \end{array} \right. \tag{1.19}$$

Utilizando la condición del rango de Kalman, se demuestra que los sistemas (1.18) y (1.19) son controlables para cada $\epsilon \in (0,1)$ pero no para $\epsilon = 0$.

Como veremos más adelante, la clave en el estudio de la controlabilidad uniforme para (1.2) es la construcción, para cada $\epsilon > 0$, de un control $u^\epsilon$ tal que $||u^\epsilon||_{\mathcal{U}_{ad}} \leq C||y_0^\epsilon||_Y$ para una constante $C > 0$ independiente de $\epsilon$ y alguna norma apropiada $||.||_{\mathcal{U}_{ad}}$. De hecho, si este es el caso, ya que el estado de un sistema controlable se puede estimar en términos del estado inicial y el control, seremos capaces de obtener una estimación uniforme para $y^\epsilon$ en términos de los datos iniciales $y_0^\epsilon$. Obsérvese también que este problema está directamente relacionado con el estudio del coste de la controlabilidad de (1.2), que también será abordado en esta tesis.

Para terminar esta seción, describiremos de manera resumida los principales logros de esta tesis.

**Capítulo 2:** Analizamos la controlabilidad uniforme de familias de sistemas parabólicos lineales acoplados que aproximan sistemas parabólico-elípticos y mostramos, con hipóteses adecuadas en los téminos de acoplamiento, la controlabilidad uniforme a cero de las familias cuando sólo un control está actuando en el sistema. Los resultados de este capítulo 2 están basados en los resultados del artículo [20], en colaboración con S. Guerrero y J.-P. Puel.

**Capítulo 3:** Se analiza la controlabilidad uniforme a cero de una familia de sistemas de reacción-difusión no lineales que aproximan a un sistema parabólico-elíptico no lineal que modela la actividad eléctrica en el tejido cardíaco, probamos la controlabilidad uniforme a cero de la familia por medio de un único control en el sistema. Este capítulo está basado en los artículos [5] y [6], en colaboración con M. Bendahmane.

**Capítulo 4:** Estudiamos la controlabilidad del sistema parabólico de Keller-Segel de la quimiotaxis que converge a su versión parabólico-elíptico. Mostramos que este sistema parabólico es localmente uniformemente controlable a cero en torno a una solución del sistema parabólico-elíptico cuando el control está actuando en la ecuación del componente químico. Este capítulo está basado en el artículo [19], en colaboración con S. Guerrero.

**Capítulo 5:** Analizamos la controlabilidad de la ecuación de ondas con viscosidad del tipo Kelvin-Voigt y un amortiguamiento por fricción como un modelo de la viscoelasticidad. Probamos la controlabilidad a cero del sistema cuando la región de control se mueve, según el flujo de una EDO, de forma que cubra todo el dominio. Este capítulo está basado en el artículo [21], en colaboración con L. Rosier y E. Zuazua.

**Capítulo 6:** Estudiamos el coste de la controlabilidad a cero para el sistema de Stokes. Utilizando el método de transmutación de controles, probamos que el coste para llevar el sistema de Stokes a cero en un tiempo $T > 0$, cuando $T \longrightarrow 0^+$, es del orden de $e^{C/T}$, como en el caso de la ecuación del calor. Este capítulo está basado en el artículo [18].

En lo que sigue describiremos en más detalles el contenido de las diferentes partes de esta tesis.

## 1.2 Contenido de la tesis

Dado un dominio $\Omega \subset \mathbb{R}^N$ y un tiempo $T > 0$, fijamos la notación $Q := \Omega \times (0, T)$, $\Sigma := \partial\Omega \times (0, T)$. Además, usaremos $\omega$, $\omega_1$ y $\omega_2$ para denotar (pequeños) conjuntos abiertos de $\Omega$ en los cuales el control acutará y usaremos $\beta = \beta(x, t)$ para denotar una función peso que tiene la siguiente forma

$$\beta(x,t) = e^{\frac{\eta(x)}{t^k(T-t)^k}}, \tag{1.20}$$

para algún $k \geq 1$ y un función $\eta$ apropiada, dependiendo de la región de control y tal que $\eta(x) < 0$ para todo $x \in \Omega$.

### Capítulo 2: Controlabilidad de sistemas parabólicos acoplados y con difusión rápida

El capítulo 2 de esta tesis está dedicado al análisis de la controlabilidad uniforme de la siguiente familia de sistemas parabólicos acoplados:

$$\left| \begin{array}{ll} u^\epsilon_t - \Delta u^\epsilon = a u^\epsilon + b v^\epsilon + f^\epsilon 1_{\omega_1} & \text{en } Q, \\ \epsilon v^\epsilon_t - \Delta v^\epsilon = c u^\epsilon + d v^\epsilon + g^\epsilon 1_{\omega_2} & \text{en } Q, \\ u^\epsilon = v^\epsilon = 0 & \text{sobre } \Sigma, \\ u^\epsilon(0) = u_0;\ v^\epsilon(0) = v_0 & \text{en } \Omega, \end{array} \right. \tag{1.21}$$

donde $a = a(x,t)$, $b = b(x,t)$, $c = c(x,t)$ y $d = d(x,t)$, $f^\epsilon$ y $g^\epsilon$ son controles y $\epsilon$ es un parámetro positivo, que convergerá a cero.

Los sistemas de la forma (1.21) son prototipos de sistemas biológicos que modelan fenómenos de agregación o de sistemas químicos con dos diferentes concentraciones (para más detalles, ver las referencias [11, 61, 62, 85] o los capítulos 3 y 4 de esta tesis).

Nuestro interés en la controlabilidad uniforme de (1.21) es debido a que, en la práctica, esta familia de sistemas parabólicos usualmente se aproxima mediante el sistema
parabólico-elíptico:

$$\left| \begin{array}{ll} u_t - \Delta u = a u + b v + f 1_{\omega_1} & \text{en } Q, \\ -\Delta v = c u + d v + g 1_{\omega_2} & \text{en } Q, \\ u = v = 0 & \text{sobre } \Sigma, \\ u(0) = u_0 & \text{en } \Omega. \end{array} \right. \tag{1.22}$$

Obsérvese que, incluso si esta aproximación es consistente con la teoría de existencia y unicidad, no está del todo claro qué es lo que se puede hacer desde el punto de vista de la teoría de control. La razón principal se debe a que estamos considerando sistemas que tienen diferentes propiedades físicas y, por tanto, uno podría esperar que estos sistemas tengan, al menos a priori, diferentes propiedades de control. En el capítulo 2 mostramos que las propiedades de controlabilidad y observabilidad para el sistema (1.22) pueden ser obtenidas como límite de las respectivas propiedades de controlabilidad y observabilidad para la familia (1.21).

En efecto, probamos el siguiente resultado:

**Teorema 1.4.** *Sean $T > 0$, $\Omega \subset \mathbb{R}^N (N \geq 1)$ un conjunto abierto conexo y limitado cuyo borde es regular, $(u_0, v_0) \in L^2(\Omega)^2$ y $a, b, c$ y $d$ funciones en $C^3(\overline{Q})$. Entonces:*

*(i) Si $c \neq 0$ en $\overline{\omega}$ para algun $\omega \subset\subset \omega_1 \subset \Omega$ y $d < \mu_1$ ($\mu_1$ es el primer autovalor del Laplaciano*

*con condición de Dirichlet en el borde), el sistema (1.21) es uniformemente controlable a cero, con respecto a $\epsilon$, con un control actuando solamente en la primera ecuación. Más precisamente, para cada $\epsilon > 0$, existe $f^\epsilon = f(\epsilon) \in L^2(Q)$ tal que la solución $(u^\epsilon, v^\epsilon)$ de (1.21) ($g^\epsilon \equiv 0$) satisface:*

$$\big(u^\epsilon(T), v^\epsilon(T)\big) = (0, 0). \tag{1.23}$$

*Además, tenemos la siguiente estimación para el control:*

$$||f^\epsilon 1_{\omega_1}||^2_{L^2(Q)} \leq C(||u_0||^2_{L^2(\Omega)} + \epsilon||v_0||^2_{L^2(\Omega)}), \tag{1.24}$$

*donde $C > 0$ es una constante que no depende de $\epsilon$, $u_0$ y $v_0$.*

*(ii) Si $b \neq 0$ en $\overline{\omega}$ para algún $\omega \subset\subset \omega_2 \subset \Omega$ y $d < \mu_1$, entonces el sistema (1.21) es controlable uniformemente a cero, con respecto a $\epsilon$, con un control actuando solamente en la segunda ecuación. Más precisamente, para cada $\epsilon > 0$ existe $g^\epsilon = g(\epsilon) \in L^2(Q)$ tal que la solución $(u^\epsilon, v^\epsilon)$ de (1.21) ($f^\epsilon \equiv 0$) satisface:*

$$\big(u^\epsilon(T), v^\epsilon(T)\big) = (0, 0). \tag{1.25}$$

*Además, tenemos la siguiente estimación para el control:*

$$||g^\epsilon 1_{\omega_2}||^2_{L^2(Q)} \leq C(||u_0||^2_{L^2(\Omega)} + \epsilon||v_0||^2_{L^2(\Omega)}), \tag{1.26}$$

*donde $C > 0$ es una constante que no depende de $\epsilon$, $u_0$ y $v_0$.*

Obtenemos imediatamente del Teorema 1.4 un control $f$ (resp. $g$), el límite débil de $\{f^\epsilon\}_{\epsilon>0}$ (resp. $\{g^\epsilon\}_{\epsilon>0}$), tal que la solución $(u^\epsilon, v^\epsilon)$ de (1.21) converge débilmente a la solución $(u, v)$ de (1.21) asociada a $f$ (resp. $g$) en el espacio $\big(L^2(0,T; H^1_0(\Omega)) \cap C([0,T]; L^2(\Omega))\big) \times L^2(0,T; H^1_0(\Omega))$.

La clave en la demostración del Teorema 1.4 es la obtención de una desigualdad de observabilidad uniforme para el sistema adjunto de (1.21) (para simplificar la notación no escribiremos el índice $\epsilon$ en cada termino):

$$\left|\begin{array}{ll} -\varphi_t - \Delta\varphi = a\varphi + c\xi & \text{en } Q, \\ -\epsilon\xi_t - \Delta\xi = b\varphi + d\xi & \text{en } Q, \\ \varphi = \xi = 0 & \text{sobre } \Sigma, \\ \varphi(T) = \varphi_T;\ \xi(T) = \xi_T & \text{en } \Omega. \end{array}\right. \tag{1.27}$$

De esta manera, el caso 1 del Teorema 1.4 es equivalente a mostrar que la siguiente estimación de observabilidad

$$||\varphi(0)||^2_{L^2(\Omega)} + \epsilon||\xi(0)||^2_{L^2(\Omega)} \leq C \iint_{\omega_1\times(0,T)} |\varphi|^2 dxdt, \tag{1.28}$$

es cierta para todas las soluciones $(\varphi, \xi)$ de (1.27), donde $C > 0$ está acotada con respecto a $\epsilon$. De manera análoga, el caso 2 del Teorema 1.4 es equivalente a mostrar que

$$||\varphi(0)||^2_{L^2(\Omega)} + \epsilon||\xi(0)||^2_{L^2(\Omega)} \leq C \iint_{\omega_2\times(0,T)} |\xi|^2 dxdt, \tag{1.29}$$

para todas las soluciones de $(\varphi, \xi)$ de (1.27).

Ahora explicaremos la metodología de la prueba del Teorema 1.4. Consideramos el caso 1 y daremos las ideas principales.

Probaremos la desigualdad de observalidad uniforme (1.28) como una consecuencia de la siguiente desigualdad de Carleman.

**Teorema 1.5.** *Sea $c \neq 0$ en $\overline{\omega}$ para algún $\omega \subset\subset \omega_1 \subset \Omega$ y $d < \mu_1$, la solución $(\varphi, \xi)$ del sistema (1.27) satisface:*

$$\iint_Q \beta_1^2|\varphi|^2 dxdt + \iint_Q \beta_2^2|\xi|^2 dxdt \leq C \iint_{\omega_1\times(0,T)} \beta_3^2|\varphi|^2 dxdt, \tag{1.30}$$

*donde $C > 0$ no dependende de $\epsilon$ y las funciones peso $\beta_i$, $i = 1, 2, 3$ son de la forma* (1.20).

Combinando el Teorema 1.5 y estimaciones de energía para el sistema adjunto (1.21), probamos (1.28).

Obsérvese que en (1.30) tenemos estimaciones globales de $\varphi$ y $\xi$ en términos de estimaciones locales de $\varphi$. La principal dificultad cuando uno intenta probar una estimación de la forma (1.30) para el sistema (1.27) se debe a que las ecuaciones para $\varphi$ y $\xi$ tienen diferentes tasas de difusión. De hecho, la clave de la prueba es obtener ecuaciones para $\varphi$ y $\xi$ con la misma tasa de difusión.

Para obtener ecuaciones para $\varphi$ y $\xi$ con la misma tasa de difusión, consideramos el operador $\mathcal{L}_{\gamma,\theta}$ dado por

$$\mathcal{L}_{\gamma,\theta} := \gamma\partial_t - \Delta - \theta, \text{ para } \gamma \in \mathbb{R} \text{ y } \theta \in L^\infty(Q)$$

y definimos la nueva función

$$w = \mathcal{L}_{-\epsilon,d}\varphi.$$

Extendemos el sistema adjunto (1.27) a un sistema de cuatro ecuaciones de la forma

$$\left|\begin{array}{ll}\mathcal{L}_{-1,a}w=\varphi(cb+\mathcal{L}_{-\epsilon,0}a-\mathcal{L}_{-1,0}d)+\xi\mathcal{L}_{-\epsilon,0}c-2\nabla\xi\nabla c+2\nabla\varphi(\nabla d-\nabla a) & \text{en } Q,\\ \mathcal{L}_{-\epsilon,d}\varphi=w & \text{en } Q,\\ \mathcal{L}_{-1,a}\varphi=c\xi & \text{en } Q,\\ \mathcal{L}_{-\epsilon,d}\xi=b\varphi & \text{en } Q,\\ \varphi=\xi=w=0 & \text{sobre } \Sigma,\\ \varphi(T)=\varphi_T;\ \xi(T)=\xi_T;\ w(T)=-\epsilon\varphi_T-\Delta\varphi_T-d\varphi_T & \text{en } \Omega.\end{array}\right. \tag{1.31}$$

En (1.31) hemos aplicado a $\varphi$ el operador de la segunda ecuación de (1.27), llamando a esa nueva incógnita $w$ y escribiendo la ecuación satisfecha por $w$. Eso nos da una nueva ecuación parabólica para $\varphi$ con la misma tasa de difusión que la de la ecuación de $\xi$, es decir, las ecuaciones $(1.31)_2$ y $(1.31)_4$, respectivamente.

Procedemos en cuatro etapas.

*Etapa 1*: Vemos las ecuaciones de (1.31) como ecuaciones del calor y aplicamos desigualdades de Carleman adecuadas para ecuaciones del calor con una dependencia precisa con respecto al parámetro de degeneración multiplicando la derivada en el tiempo. Sumando las desigualdades obtenidas para cada ecuación, obtenemos estimaciones globales de $\varphi$ , $\xi$ y $w$ en términos de estimaciones locales de $\varphi$ , $\xi$ y $w$.

*Etapa 2*: Utlilizando la segunda ecuación en (1.31), eliminamos la integral local de $w$ que aparece en la desigualdad de Carleman obtenida en la etapa 1. Eso nos da estimaciones globales de $\varphi$, $\xi$ y $w$ en términos de estimaciones locales de $\varphi$ y $\xi$.

*Etapa 3*: Utilizando $(1.31)_3$ estimamos una integral local de $\xi$ en términos de una integral local del $\varphi$ y una integral local de $\varphi_t$.

*Etapa 4*: Por último, utlizando $(1.31)_1$ y $(1.31)_2$, estimamos $\varphi_t$ localmente en términos de una integral local de $\varphi$.

Usando Teorema 1.5 y estimaciones de energía para (1.27) terminamos la prueba del caso 1 del Teorema 1.4.

La prueba del caso 1 del Teorema 1.5 también contiene la demostración del caso 2 del mismo teorema. Para ello utilizamos el hecho que el sistema formado por las dos primeras ecuaciones en (1.31) poseen la misma estructura que el sistema adjunto (1.27) y entonces procedemos como en las etapas 1 y 2 para probar el siguiente resultado.

**Teorema 1.6.** *Si $b\neq 0$ en $\overline{\omega}$ para algún $\omega\subset\subset\omega_2\subset\Omega$ y $d<\mu_1$, la solución $(\varphi,\xi)$ del sistema* (1.27) *satisface:*

$$\iint_Q \beta_1^2|\xi|^2dxdt+\iint_Q \beta_2^2|\varphi|^2dxdt\leq C\iint_{\omega_2\times(0,T)}\beta_3^2|\xi|^2dxdt, \tag{1.32}$$

*donde $C > 0$ no depende de $\epsilon$ y las funciones peso $\beta_i$, $i = 1, 2, 3$, son de la forma* (1.20).

### Capítulo 3: Controlabilidad uniforme a cero para un sistema de reación-difusión aproximando un modelo cardíaco simplificado

En el capítulo 3 estudiamos las propiedades de controlabilidad y observabilidad de una familia de sistemas de reación-difusión que se degenera en un sistema parabólico-elíptico describiendo la actividad eléctrica en un dominio $\Omega \subset \mathbb{R}^N (N = 1, 2, 3)$.

El *modelo de bidominio* (véase, por ejemplo, [23, 59, 65]) que gobierna la actividad eléctrica en el tejido cardíaco es dado por el siguiente sistema parabólico acoplado no lineal:

$$\left|\begin{array}{ll} c_m v_t - Div(\mathbf{M}_i(x)\nabla u_i) + h(v) = f 1_{\omega_1} & \text{en } Q, \\ c_m v_t + Div(\mathbf{M}_e(x)\nabla u_e) + h(v) = g 1_{\omega_2} & \text{en } Q, \\ u_i = u_e = 0 & \text{sobre } \Sigma, \\ v(0,x) = v_0(x), & \text{en } \Omega, \end{array}\right. \tag{1.33}$$

donde $c_m > 0$ es la superficie capacitante de la membrana, la función no lineal $h : \mathbb{R} \to \mathbb{R}$ es la corriente iónica de la transmembrana, y $f$ y $g$ son las corrientes de estimulación aplicadas a $\omega_1$ y $\omega_2$, respectivamente.

En (1.33), las funciones $u_i = u_i(t, x)$ y $u_e = u_e(t, x)$ representan los potenciales eléctricos *intracelular* y *extracelular*, respectivamente. A la diferencia $v = u_i - u_e$ la llamamos el *potencial de la transmembrana*. A los tensores de conductividad intracelular y extracelular $\mathbf{M}_i(x)$ y $\mathbf{M}_e(x)$, que modelan las propiedades anisotrópicas del medio, se les pide que sean $C^\infty$, acotados, simétricos y semidefinidos positivos.

Debido a su dificultad de implementación (véase, por ejemplo, [7, 23, 45]), en muchas aplicaciones el modelo de bidominio es simplificado en el *modelo de monodominio*:

$$\left|\begin{array}{ll} c_m v_t - \frac{\mu}{\mu+1} Div\big(\mathbf{M}_e(x)\nabla v\big) + h(v) = f 1_{\omega_1} & \text{en } Q, \\ -Div\big(\mathbf{M}(x)\nabla u_e\big) = Div\big(\mathbf{M}_i(x)\nabla v\big) & \text{en } Q, \\ v = u_e = 0 & \text{sobre } \Sigma, \\ v(0) = v_0 & \text{en } \Omega, \end{array}\right. \tag{1.34}$$

donde $M = M_i + M_e$.

Obsérvese que la principal diferencia entre el modelo de bidominio (1.33) y el modelo de monodominio (1.34) es que el primero es un sistema de dos ecuaciones parabólicas acopladas mientras que el segundo es un sistema del tipo parabólico-elíptico. En esta parte de la tesis nuestro objetivo será mostrar que, en realidad, las propiedades de controlabilidad y observabilidad para el modelo de monodominio pueden se obtenidos como un proceso límite de las propiedades de controlabilidad y observabilidad para una familia de sistemas parabólicos acoplados relacionados con el modelo de bidominio.

Más precisamente, dado $\epsilon > 0$, aproximamos el modelo de monodominio por la siguiente familia de sistemas parabólicos:

$$\left|\begin{array}{ll} c_m v_t^\epsilon - \frac{\mu}{\mu+1} Div\big(\mathbf{M}_e(x)\nabla v^\epsilon\big) + h(v^\epsilon) = f^\epsilon 1_{\omega_1} & \text{en } Q, \\ \epsilon u_{e,t}^\epsilon - Div\big(\mathbf{M}(x)\nabla u_e^\epsilon\big) = Div\big(\mathbf{M}_i(x)\nabla v^\epsilon\big) & \text{en } Q, \\ v^\epsilon = u_e^\epsilon = 0 & \text{sobre } \Sigma, \\ v^\epsilon(0) = v_0;\ u_e^\epsilon(0) = u_{e,0} & \text{en } \Omega. \end{array}\right. \tag{1.35}$$

Probamos:

**Teorema 1.7.** *Supongamos que $h$ es una función en $C^1(\mathbb{R})$, globalmente Lipschitz y que $h(0) = 0$. Dados $v_0$ y $u_{e,0}$ en $L^2(\Omega)$, existe un control $f^\epsilon \in L^2(\omega_1 \times (0,T))$ tal que la solución $(v^\epsilon, u_e^\epsilon)$ de (1.35) satisface:*

$$v^\epsilon(T) = u_e^\epsilon(T) = 0.$$

*Además, el control $f^\epsilon$ tiene la siguiente estimación:*

$$||f^\epsilon 1_{\omega_1}||^2_{L^2(Q)} \leq C\big(||v_0||^2_{L^2(\Omega)} + \epsilon||u_{e,0}||^2_{L^2(\Omega)}\big), \tag{1.36}$$

*para una constante $C = C(\Omega, \omega_1, ||a||_{L^\infty}, T) > 0$.*

Como de costumbre cuando se estudia la controlabilidad de sistemas no lineales, la clave para probar el Teorema 1.7 será un resultado de controlabilidad uniforme a cero para el sistema linenalizado:

$$\left|\begin{array}{ll} c_m v_t^\epsilon - \frac{\mu}{\mu+1} Div\big(\mathbf{M}_e(x)\nabla v^\epsilon\big) + a(t,x)v^\epsilon = f^\epsilon 1_{\omega_1} & \text{en } Q, \\ \epsilon u_{e,t}^\epsilon - Div\big(\mathbf{M}(x)\nabla u_e^\epsilon\big) = Div\big(\mathbf{M}_i(x)\nabla v^\epsilon\big) & \text{en } Q, \\ v^\epsilon = u_e^\epsilon = 0 & \text{sobre } \Sigma, \\ v^\epsilon(0) = v_0;\ u_e^\epsilon(0) = u_{e,0} & \text{en } \Omega, \end{array}\right. \tag{1.37}$$

donde $a$ es una función acotada.

El resultado de controlabilidad uniforme a cero que probamos para (1.37) es el siguiente.

**Teorema 1.8.** *Dados $v_0$ y $u_{e,0}$ en $L^2(\Omega)$, para cada $\epsilon > 0$ existe un control $f^\epsilon \in L^2(\omega_1 \times (0,T))$ tal que la solución asociada de (1.37) es cero en el tiempo $T > 0$. Es decir, la solución asociada satisface:*

$$v^\epsilon(T) = 0,\ u_e^\epsilon(T) = 0.$$

*Además, el control $f^\epsilon$ tiene la siguiente estimación:*

$$||f^\epsilon 1_{\omega_1}||^2_{L^2(Q)} \leq C\big(||v_0||^2_{L^2(\Omega)} + \epsilon||u_{e,0}||^2_{L^2(\Omega)}\big), \tag{1.38}$$

*para una constante $C = C(\Omega, \omega_1, ||a||_{L^\infty}, T) > 0$.*

Una vez probado el Teorema 1.8, probamos el Teorema 1.7 mediante argumentos de punto fijo.

La idea de la prueba del Teorema 1.8 es considerar el sistema adjunto de (1.37) (una vez más, no escribimos el índice $\epsilon$):

$$\left|\begin{array}{ll} -c_m\varphi_t - \frac{\mu}{\mu+1}Div(\mathbf{M}_e(x)\nabla\varphi) + a(t,x)\varphi = Div(\mathbf{M}_i(x)\nabla\varphi_e) & \text{en } Q, \\ -\epsilon\varphi_{e,t} - Div(\mathbf{M}(x)\nabla\varphi_e) = 0 & \text{en } Q, \\ \varphi = \varphi_e = 0 & \text{sobre } \Sigma, \\ \varphi(T) = \varphi_T;\ \varphi_e(T) = \varphi_{e,T} & \text{en } \Omega \end{array}\right. \tag{1.39}$$

y probar la desigualdad de Carleman uniforme:

**Teorema 1.9.** *Existe una constante positiva $C = C(\Omega, \omega_1, ||a||_{L^\infty}, T)$ tal que, para todo $\varphi_T, \varphi_{e,T} \in L^2(\Omega)$ y $a \in L^\infty(Q)$, la solución $(\varphi, \varphi_e)$ de (1.39) verifica:*

$$\iint_Q \beta_1^2|\rho|^2 dxdt + \iint_Q \beta_2^2|\varphi|^2 dxdt \leq C\iint_{\omega_1\times(0,T)} \beta_3^2|\varphi|^2 dxdt, \tag{1.40}$$

*donde $\rho = Div(M(x)\nabla\varphi_e(x,t))$ y las funciones peso $\beta_i$, $i = 1,2,3$ son de la forma (1.20).*

Obsérvese que como el control está actuando en la primera ecuación de (1.37), necesitamos una desigualdad de Carleman que nos de estimaciones globales de $\varphi$ y $\varphi_e$ en función de una integral local de $\varphi$, uniformemente con respecto a $\varepsilon$. Una vez que el término de acoplamiento en la primera ecuación del sistema adjunto (1.39) es en $Div(\mathbf{M}_i(x)\nabla\varphi_e)$ y no en $\varphi_e$, no podemos aplicar directamente los resultados del capítulo 2. Entre tanto, podemos trabajar con soluciones regulares de (1.39) y considerar $\rho(x,t) = Div(\mathbf{M}(x)\nabla\varphi_e(x,t))$ como una nueva variable y trabajar con el sistema:

$$\left|\begin{array}{ll} -\varphi_t - Div(\mathbf{M}_e(x)\nabla\varphi) + a(x,t)\varphi = \rho & \text{en } Q, \\ -\epsilon\rho_t - Div(\mathbf{M}(x)\nabla\rho) = 0 & \text{en } Q, \\ \varphi = \rho = 0 & \text{sobre } \Sigma, \\ \varphi(T) = \varphi_T; \rho(T) = \rho_T & \text{en } \Omega. \end{array}\right. \tag{1.41}$$

Para el sistema (1.41) podemos aplicar los resultados del capítulo 2 y obtener la desigualdad (1.40). Entre tanto, es importante decir que en el capítulo 3 damos una prueba directa de este resultado, combinando desigualdades de Carleman y desigualdades de energía con peso que, en este caso en particular, es mucho más sencilla que la prueba obtenida a través de los argumentos del capítulo 2.

Por último, utilizando el Teorema 1.9 y estimaciones de energía, deducimos la desigualdad de observabilidad

$$||\varphi(0)||^2_{L^2(\Omega)} + \varepsilon||\varphi_e(0)||^2_{L^2(\Omega)} \leq e^{C(1+1/T+||a||^{2/3}_{L^\infty}+||a||_{L^\infty}T)} \iint_{\omega_1\times(0,T)} |\varphi|^2 dxdt, \quad (1.42)$$

para las soluciones de (1.39), donde $C = C(\Omega, \omega_1)$.

El Teorema 1.8 es entonces consecuencia de la desigualdad de observabilidad (1.42).

### Capítulo 4: Un resultado de controlabilidad uniforme para el sistema de Keller-Segel

El capítulo 4 está dedicado a la controlabilidad uniforme a cero del sistema de Keller-Segel:

$$\left| \begin{array}{ll} u_t - \Delta u = -\nabla\cdot(u\nabla v) & \text{en } Q, \\ \epsilon v_t - \Delta v = au - bv + g\chi & \text{en } Q, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{sobre } \Sigma, \\ u(x,0) = u_0;\ v(x,0) = v_0 & \text{en } \Omega, \end{array} \right. \quad (1.43)$$

donde $a$ y $b$ son constantes positivas, $g$ es un control interno y $\epsilon$ es un parámetro positivo, que convergerá a cero. En (1.43), $\chi : \mathbb{R}^N \to \mathbb{R}$ es una función $C^\infty$ tal que *supp* $\chi \subset\subset \omega$, $0 \leq \chi \leq 1$ y $\chi \equiv 1$ in $\omega'$.

En este capítulo sunpondremos que $\Omega \subset \mathbb{R}^N$ (N = 2, 3) es un abierto conexo cuya frontera $\partial\Omega$ es suficientemente regular y $\omega' \subset\subset \omega$.

El modelo de Keller-Segel (1.43) es una ecuación de la biología matemática que tiene como objetivo explicar los fenómenos de agregación de organismos debido a quimiotaxis, es decir, el cambio de movimiento cuando una población formada por individuos (tales como amebas, bacterias, células endoteliales, etc) reacciona en respuesta (taxis) a un estímulo químico externo propagado en el entorno en el que residen. La quimiotaxis es un proceso celular fundamental en el desarrollo de los organismos multicelulares y, en particular, desempeña un papel esencial en el desarrollo embrionario, la homeostasis del tejido, la cicatrización de heridas, respuesta inmune, la progresión de enfermedades, así como la búsqueda de alimentos, acción repelente y la formación de cuerpos multicelulares de protozoarios (ver [61, 62] y sus referencias).

En varias aplicaciones (ver [82, 105] y sus referencias), el sistema (1.43) es aproximado por el siguiente sistema parabólico-elíptico:

$$\left| \begin{array}{ll} u_t - \Delta u = -\nabla \cdot (u \nabla v) & \text{en } Q, \\ -\Delta v = au - bv + g\chi & \text{en } Q, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{sobre } \Sigma, \\ u(x,0) = u_0 & \text{en } \Omega. \end{array} \right. \tag{1.44}$$

Así como en los capítulos anteriores, es natural estudiar la controlabilidad uniforme del sistema de Keller-Segel (1.43).

En el capítulo 4 mostramos la controlabilidad uniforme (1.43) entorno a una solución particular de (1.44). Más prescisamente, probamos el siguiente resultado.

**Teorema 1.10.** *Sea $0 < \epsilon \leq 1$ y $(M_1, M_2) \in \mathbb{R}^2_+$ tal que $aM_1 - bM_2 = 0$. Entonces, existe $\delta > 0$ tal que, para cada $(u_0, v_0) \in H^1(\Omega) \times H^2(\Omega)$ satisfaciendo $\int_\Omega u_0 dx = M_1$, $\frac{\partial v_0}{\partial \nu} = 0$ sobre $\partial\Omega$ y $||(u_0 - M_1, v_0 - M_2)||_{H^1(\Omega)\times H^2(\Omega)} \leq \delta$, podemos encontrar $g = g(\epsilon) \in L^2(0,T; H^1(\Omega))$, uniformemente acotado con respecto a $\epsilon$, tal que la solución $(u, v)$ de (1.43) satisface:*

$$(u(T), v(T)) = (M_1, M_2) \text{ in } \Omega.$$

En (1.43) (y (1.44)), $u = u(x,t) \geq 0$ y $v = v(x,t) \geq 0$ son, respectivamente, la concentración de la especie (i.e., la densidad de la población) y la del químico (i.e., la concentración de la sustancia química). Por tanto, estamos controlando el sistema de Keller-Segel a través de la sustancia química, que es razonable desde el punto de vista biológico. La condición $aM_1 - bM_2 = 0$ simplesmente significa que $(M_1, M_2)$ es una solución estacionaria del sistema parabólico-elíptico (1.44).

Para poder estudiar la controlabilidad de (1.43) entorno a $(M_1, M_2)$, primeramente analizamos la controlabilidad uniforme a cero de su linealización entorno a esa trayectoria, es decir

$$\left| \begin{array}{ll} u_t - \Delta u = -M_1 \Delta v + h_1 & \text{en } Q, \\ \epsilon v_t - \Delta v = au - bv + g\chi_\omega + h_2 & \text{en } Q, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{sobre } \Sigma, \\ u(x,0) = u_0; \ v(x,0) = v_0 & \text{en } \Omega, \\ \int_\Omega u_0(x) dx = 0, & \end{array} \right. \tag{1.45}$$

donde $h_1$ y $h_2$ son funciones pertenecientes a ciertos espacios de Banach $X_1$ y $X_2$, respectivamente, teniendo decaimiento exponencial en $t = T$ y $\int_\Omega h_1(x,t)dx = 0$ para todo $t \in [0,T]$. La idea es entonces probar que podemos encontrar $g$ de forma que la solución de (1.45) satisface $(u(T), v(T)) = (0,0)$ y que la cantidad $\nabla \cdot (u\nabla v)$ pertenece a $X_1$. Entonces, aplicando un argumento de aplicación inversa introducido en [129] obtenemos la controlabilidad de (1.43) entorno a $(M_1, M_2)$.

Así como en los capítulos 2 y 3, mostramos la controlabilidad a cero del sistema lineal (1.45) por medio de *estimaciones globales de Carleman* para las soluciones de su sistema adjunto, es decir

$$\left|\begin{array}{ll} -\varphi_t - \Delta\varphi = a\xi + f_1 & \text{en } Q, \\ -\epsilon\xi_t - \Delta\xi = -b\xi - M_1\Delta\varphi + f_2 & \text{en } Q, \\ \frac{\partial\varphi}{\partial\nu} = \frac{\partial\xi}{\partial\nu} = 0 & \text{sobre } \Sigma, \\ \varphi(x,T) = \varphi_T;\ \xi(x,T) = \xi_T & \text{en } \Omega, \\ \int_\Omega \varphi_T(x)dx = 0, & \end{array}\right. \tag{1.46}$$

donde $f_1$ y $f_2$ son funciones $L^2(Q)$ arbitrarias.

Obsérvese que en el sistema adjunto (1.46) ponemos $\int_\Omega \varphi_T dx = 0$. Esta condición es reminiscente de la condición $\int_\Omega u(x,t)dx = 0$ para todo $t \in [0,T]$, donde $(u,v)$ es la solución de (1.45).

Obsérvese también que, como el control está aplicado a la segunda ecuación de (1.45), necesitamos estimar integrales globales de $\varphi$ y $\xi$ en términos de estimaciones locales de $\xi$ y estimaciones globales de $f_1$ y $f_2$. La principal dificultad cuando uno intenta probar una desiguald de Carleman de este tipo para (1.46) se debe a que el acoplamiento en la segunda ecuación es en $\Delta\varphi$ y no en $\varphi$, como en el capítulo 3. De hecho, la desigualdad que obtenemos contiene términos globales de normas $L^2$ con peso de $\Delta\varphi$ y $\xi$ en el lado izquierdo de la desigualdad, sin términos locales en $\varphi$, mientras que una integral local de $\xi$ e integrales globales de $f_1$ y $f_2$ aparecerán en el lado derecho.

La desigualdad de Carleman que probamos para (1.46) es la siguiente.

**Teorema 1.11.** *Existe $C = C(\Omega,\omega,T) > 0$ tal que, para todo $0 < \epsilon \leq 1$ y todas $f_1, f_2 \in L^2(Q)$, la solución $(\varphi,\xi)$ del sistema (1.46) satisface:*

$$\iint_Q \beta_1^2|\xi|^2 dxdt \iint_Q \beta_2^2|\Delta\varphi|^2 dxdt \leq C\Big(\iint_Q \beta_3^2|\chi|^2|\xi|^2 dxdt + \iint_Q \beta_4^2|f_1|^2 dxdt + \iint_Q \beta_5^2|f_2|^2 dxdt\Big), \tag{1.47}$$

*donde los pesos $\beta_i$, $i = 1, 2 \ldots, 5$ son de la forma (1.20).*

La prueba del Teorema 1.11 está divida en 3 partes.

*Etapa 1. Desigualdad de Carleman para $\Delta\varphi$.*

Escribimos $\beta_1\varphi = \eta + \psi$, donde $\eta$ es la solución de una ecuación de calor con $f_1$ como lado derecho y $\psi$ es solución de una ecuación del calor con lado derecho en $H^1(0,T;H^2(\Omega))$. Consideramos la ecuación de $\psi$ y aplicamos desigualdades de

Carleman clásicas para la ecuación del calor con condiciones de Neumann y combinanos esta desigualdad con estimaciones de energía para la ecuación de $\eta$. Esto nos dará estimaciones globales de $\Delta\varphi$ en términos de una integral local de $\Delta\psi$ e integrales globales de $\Delta\xi$ y $f_1$.

*Etapa 2. Desigualdad de Carleman para $\xi$.*

En la segunda parte de la prueba, obtenemos una desigualdad de Carleman para $\xi$, con una dependencia precisa con respecto al parámetro de degeneración multiplicando la derivada en tiempo y la juntamos con la desigualdad de Carleman de la etapa 1. Eso nos da una estimación global de $\xi$ y $\Delta\varphi$ en términos de una integral local de $\xi$ y otra en $\Delta\psi$ e integrales globales de $f_1$ y $f_2$.

*Etapa 3. Estimación de la integral local de $\Delta\psi$.*

En la última parte, estimamos la integral local de $\Delta\psi$ en términos de una integral local de $\xi$ e integrales globales $f_1$ y $f_2$. Combinando los resultados de las etapas 2 y 3 terminamos la prueba del Teorema 1.11.

Una vez probado el Teorema 1.11, nos concentramos en el problema de la controlabilidad a cero del sistema lineal (1.45) con un lado derecho que decae exponencialmente cuando $t \to T^-$.

Este resultado será crucial en la prueba de la controlabilidad local de (1.43). De hecho, queremos encontrar $g \in L^2(0,T;H^1(\Omega))$ tal que la solución de

$$\begin{cases} L(u,v) = (h_1, h_2 + g\chi) & \text{en } Q, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{sobre } \Sigma, \\ u(x,0) = u_0;\ v(x,0) = v_0 & \text{en } \Omega, \end{cases} \tag{1.48}$$

donde

$$L(u,v) = (u_t - \Delta u + M_1 \Delta v, \epsilon v_t - \Delta v + bv - au), \tag{1.49}$$

verifica

$$u(x,T) = v(x,T) = 0 \text{ en } \Omega. \tag{1.50}$$

Además, queremos probar la existencia de una solución para este problema en un espacio con pesos apropiado. Para eso consideramos pesos similares a (1.20), pero que no degerenan en $t = 0$ y probamos la siguiente desigualdad de Carleman.

**Lema 1.1.** *Existe $C = C(\Omega,\omega,T) > 0$ tal que , para todo $0 < \epsilon \leq 1$, toda solucón de* (1.46) *verifica:*

$$\iint_Q \gamma_1^2|\xi|^2 dxdt + \iint_Q \gamma_2^2|\varphi - (\varphi)_\Omega|^2 dxdt + ||\varphi(0) - (\varphi(0))_\Omega||^2 + \epsilon||\xi(0)||^2$$
$$\leq C\Big(\iint_Q \gamma_3^2|\chi|^2|\xi|^2 dxdt + \iint_Q \gamma_4^2|f_1|^2 dxdt + \iint_Q \gamma_5^2|f_2|^2 dxdt\Big), \quad (1.51)$$

*donde*

$$(\varphi)_\Omega(t) = \frac{1}{|\Omega|}\int_\Omega \varphi(x,t)dx$$

*y*

$$\gamma(x,t) = e^{\frac{\eta(x)}{(T-t)^k}}, \quad (1.52)$$

*para alguna constante* $k \geq 1$.

La prueba de este resultado es clásica, siendo simplemente la combinación de (1.47) y estimaciones de energía para el sistema adjunto (1.46).

Una vez obtenido (1.51) resolvemos (1.48)-(1.50). En realidad, probamos dos resultados de controlabilidad: primero, obtenemos un resultado de controlabilidad a cero para (1.48) sin hipóteses adicionales de regularidad para el control y el estado; segundo, probamos (1.48)-(1.50) por medio de controles y estados más regulares.

El primer resultado de controlabilidad a cero que probamos para (1.48) es el siguiente.

**Proposición 1.3.** *Sea* $0 < \epsilon \leq 1$ *y* $(M_1, M_2) \in \mathbb{R}^2_+$ *con* $aM_1 - bM_2$ *=0. Supongamos que* $(u_0, v_0) \in L^2_0(\Omega) \times L^2(\Omega)$, $\gamma_2^{-1}h_1 \in L^2_0(Q)$ *y* $\gamma_1^{-1}h_2 \in L^2(Q)$. *Entonces, podemos encontrar* $g = g(\epsilon) \in L^2(\omega \times (0,T))$, *acotado independientemente de* $\epsilon$, *tal que* (1.48)-(1.50) *se verifica.*

La demostración de la prososición 1.3 es consecuencia de que (1.51) implica la existencia de un único mínimo para el funcional

$$J_\delta(\varphi_T, \xi_T) = \frac{1}{2}\iint_{\omega\times(0,T)} |\xi|^2 dxdt + (u_0, (\varphi_0 - (\varphi(0))_\Omega)) + \epsilon(v_0, \xi(0))$$
$$+ \iint_Q h_1(\varphi - (\varphi)_\Omega)dxdt + \iint_Q h_2\xi dxdt.$$

El segundo resultado de controlabilidad a cero que probamos para (1.48) es la clave para obtener resultados de controlabilidad para el sistema no lineal (1.43).

El espacio de Banach donde resolvemos (1.48)-(1.50) es

$$E = \{(u,v,g) \in E_0 : e^{-2s\hat{\beta}}\hat{\gamma}^3\Pi_1 L(u,v) \in L^2(Q), e^{-s\beta - s\hat{\beta}}\hat{\gamma}^2\big(\Pi_2 L(u,v) - g\chi\big) \in L^2(0,T;H^1(\Omega)),$$
$$\int_\Omega \Pi_1 L(u,v)dx = 0 \text{ y } \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ sobre } \Sigma\},$$

con

$$E_0 = \Big\{ (u,v,g) : ||\gamma_4^{-1} u||_{L^2(Q)} + ||\gamma_5^{-1} v||_{L^2(Q)} + ||\gamma_3^{-1} g||_{L^2(Q)} < \infty,$$
$$\gamma_6^{-1} u \in L^2(0,T;H^2(\Omega)), \gamma_6^{-1} u \in L^\infty(0,T;H^1(\Omega)),$$
$$\gamma_7^{-1}\Delta v \in L^2(0,T;H^1(\Omega)),\ \gamma_7^{-1}\nabla v \in L^2(0,T;H^2(\Omega)) \Big\},$$

para pesos apropiados $\gamma_6$ y $\gamma_7$ de la forma (1.52).

El resultado de controlabilidad a cero, con estado regular, para (1.48) es el siguiente.

**Proposición 1.4.** *Sea $0 < \epsilon \le 1$ y sea $(M_1, M_2) \in \mathbb{R}_+^2$ tal que $aM_1 - bM_2 = 0$ y supongamos las siguientes hipóteses:*

$$(u_0, v_0) \in H^1(\Omega) \times H^2(\Omega),\ \int_\Omega u_0 dx = 0,\ \frac{\partial v_0}{\partial \nu} = 0 \text{ sobre } \partial\Omega \tag{1.53}$$

*y*

$$(\gamma_2^{-1} h_1, \gamma_1^{-1} h_2) \in L^2(Q) \times L^2(0,T;H^1(\Omega)). \tag{1.54}$$

*Entonces, existe un control $g = g(\epsilon) \in L^2((0,T);H^1(\Omega))$, acotado independientemente de $\epsilon$, tal que, si $(u,v)$ es la solución de (1.48), se tiene $(u,v) \in E$. En particular, (1.50) se verifica.*

Para probar la proposición 1.4, mostramos que existe una única $(\hat{u}, \hat{v}, \hat{g})$ solución del problema extremal:

$$\begin{cases} \inf \frac{1}{2}\left( \iint_Q |\chi(x)|^2 \gamma_3^{-2} |g|^2 + \iint_Q \gamma_4^{-2}|u|^2 + \iint_Q \gamma_5^{-2}|v|^2 \right) \\ \text{sujeto a } g \in L^2(0,T;H^1(\Omega)) \text{ y} \\ \begin{cases} u_t - \Delta u = -M_1 \Delta v + h_1 & \text{en } Q, \\ \epsilon v_t - \Delta v + bv = au + g\chi + h_2 & \text{en } Q, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \Sigma, \\ u(x,0) = u_0;\ v(x,0) = v_0 & \text{en } \Omega, \\ u(x,T) = 0;\ v(x,T) = 0 & \text{en } \Omega. \end{cases} \end{cases} \tag{1.55}$$

Utilizando el principio de Lagrange, mostramos que existen variables duales $(\hat{z}, \hat{w})$ tal que

$$\begin{cases} (\hat{u}, \hat{v}) = (\gamma_4^2 \Pi_1 L^*(\hat{z}, \hat{w}), \gamma_5^2 \Pi_2 L^*(\hat{z}, \hat{w})) & \text{en } Q, \\ \hat{g} = -\gamma_3^2 \hat{w} \chi & \text{en } Q, \end{cases} \tag{1.56}$$

donde $L^*$ es el operador adjunto de $L$, i.e.,

$$L^*(z,w) = (-z_t - \Delta z - aw, -\epsilon w_t - \Delta w + bw + M_1 \Delta z).$$

Finalmente, usando el efecto regularizante de las ecuaciones parabólicas, se muestra que $(\hat{u}, \hat{v}, \hat{g})$ tiene las propiedades de regularidad que necesitamos.

Para probar el Teorema 1.10 primero reducimos el problema a un resultado de controlabilidad local a cero. Escribimos $(z,w) = (u - M_1, v - M_2)$, donde $(u,v)$ es la solución de (1.43). El par $(z,w)$ será entonces solución del problema no lineal:

$$\left| \begin{array}{ll} L(z,w) = (-\nabla \cdot (z \nabla w), g\chi_\omega) & \text{en } Q, \\ \frac{\partial z}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 & \text{sobre } \Sigma, \\ z(x,0) = u_0 - M_1;\ w(x,0) = v_0 - M_2 & \text{en } \Omega. \end{array} \right. \tag{1.57}$$

Tenemos que $(z(T), w(T)) = (0,0)$ si y solamente si $(u(T), v(T)) = (M_1, M_2)$.

El final de la demostración es la aplicación del siguiente teorema de aplicación inversa (véase [27]):

**Teorema 1.12.** *Sean $E$ y $G$ dos espacios de Banach y sea $\mathcal{A} : E \to G$ una función continua de $E$ a $G$ definida en n $B_\eta(0)$ para algun $\eta > 0$ con $\mathcal{A}(0) = 0$. Sea $\Lambda$ un operador linear y continuo de $E$ sobre $Y$ y supóngase que existe $C_0 > 0$ tal que*

$$||e||_E \leq C_0 ||\Lambda(e)||_G \tag{1.58}$$

*y que existe $\delta < C_0^{-1}$ tal que*

$$||\mathcal{A}(e_1) - \mathcal{A}(e_2) - \Lambda(e_1 - e_2)|| \leq \delta ||e_1 - e_2|| \tag{1.59}$$

*siempre que $e_1, e_2 \in B_\eta(0)$. Entonces la ecuación $\mathcal{A}(e) = h$ tiene una solución $e \in B_\eta(0)$ siempre que $||h||_G \leq c\eta$, donde $c = C_0^{-1} - \delta$.*

En efecto, aplicamos este teorema con el espacio $E$ definido anteriormente y con el espacio

$$G = X \times L^2_0(\Omega) \times L^2(\Omega),$$

donde

$$X = \{(h_1, h_2) : \gamma_2^{-1} h_1 \in L^2(Q), \gamma_1^{-1} h_2 \in L^2(0,T; H^1(\Omega)), \int_\Omega h_1(x,t)dx = 0\}. \tag{1.60}$$

El operador $\mathcal{A}$ viene dado por

$$\mathcal{A}(z,w,g) = (L(u,v) + ((\nabla\cdot(z\nabla w), -g\chi)), z(0), w(0))\ \forall (z,w,g) \in E.$$

Utilizando la regularidad de las funciones de $E$ mostramos que $\mathcal{A} \in C^1(E;G)$ y por lo que

$$\mathcal{A}'(0,0,0) = (L(u,v) + (0,-g\chi)), z(0), w(0))\ \forall (z,w,g) \in E.$$

De la proposición 1.4, se termina la demostración del Teorema 1.10.

### Capítulo 5: Controlabilidad a cero para un sistema de viscoelasticidad por medio de controles móviles

El capítulo 5 de esta tesis estará dedicado a extender al caso mutlidimensional los resultados de P. Martin *et al.* en [95] en el caso $1-d$ del siguiente modelo de viscoelasticidad:

$$\left|\begin{array}{ll} y_{tt} - \Delta y - \Delta y_t + b(x) y_t = 1_{\omega(t)} h, & \text{en } Q, \\ y = 0 & \text{sobre } \Sigma, \\ y(x,0) = y_0(x),\ y_t(x,0) = y_1(x) & \text{en } \Omega. \end{array}\right. \tag{1.61}$$

Aquí, $\Omega$ es un abierto acotado y regular de $\mathbb{R}^N$ $(N \geq 1)$ y $b \in L^\infty(\Omega)$.

En [95], en el caso de $1-d$, los autores muestran que (1.61) es controlable a cero si la región de control se mueve de forma que su soporte cubre todo el dominio donde la ecuación evoluciona en el tiempo.

Para que la controlabilidad a cero sea posible, supondremos que la región de control se mueve de manera que cubra todo el dominio. En realidad, si $\omega(t) \equiv \omega$ para $0 < t < T$, i.e., si el soporte del control no se mueve en el tiempo (como se considera usualmente), el sistema (1.61) no es controlable, excepto para el caso trivial $\omega = \Omega$.

Las técnicas utilizadas en [95] están basadas en el análisis de Fourier y no pueden ser aplicadas al caso multidimensional. Así que usamos una estrategia distinta, basada en el hecho de que se puede escribir el sistema (1.61) como el siguiente sistema:

$$\left\{\begin{array}{l} y_t - \Delta y + (b-1) y = z, \\ z_t + z = 1_{\omega(t)} h + (b-1) y. \end{array}\right. \tag{1.62}$$

Nuestro análisis de la controlabilidad de (1.62) será a través de las desigualdades de Carleman para su sistema adjunto, que también es un sistema del tipo parabólico-EDO. La clave es el uso de la misma función peso para ambas desigualdades de Carleman, tanto la ecuación parabólica como la EDO. Por lo que sabemos, todas las desigualdades de Carleman que uno encuentra en la literatura son para el caso de regiones de control fijas. En el caso con el que lidiamos la región de control se mueve en el tiempo, de forma que un efecto de transporte es añadido a la EDO (véase el sistema (5.97)). Por lo tanto, la demostración de las desigualdades de Carleman para la ecuación del calor y EDO's cuando la región de control se mueve es también una de las principales novedades que presentamos en esta tesis.

Nuestro resultado de controlabilidad para (1.61) es el siguiente.

**Teorema 1.13.** *Con hipóteses apropiadas sobre la trayectoria de la región de control $\omega(t)$ (véase condiciones (5.9)-(5.13)), para todo $T > 0$ y todo $(y_0, y_1) \in L^2(\Omega)^2$ con $y_1 - \Delta y_0 \in L^2(\Omega)$, existe una función $h \in L^2(0,T;L^2(\Omega))$ para la cual la solución de*

$$\left| \begin{array}{ll} y_{tt} - \Delta y - \Delta y_t + b(x) y_t = 1_{\omega(t)} h, & \text{en } Q, \\ y(x,t) = 0, & \text{sobre } \Sigma, \\ y(.,0) = y_0; y_t(.,0) = y_1 & \text{en } \Omega, \end{array} \right. \tag{1.63}$$

*verifica $y(.,T) = y_t(.,T) = 0$.*

Las condiciones (5.9)-(5.13) en el Teorema 1.13 básicamente significan que al final de la evolución de $\omega(t)$ hemos cubierto todo el dominio $\Omega$.

La demostración del Teorema 1.13 se hace primeramente probando la controlabilidad a cero del sistema

$$\left| \begin{array}{ll} y_t - \Delta y + (b(x) - 1) y = z & \text{en } Q, \\ z_t + z = 1_{\omega(t)} h + (b(x) - 1) y & \text{en } Q, \\ y(x,t) = 0 & \text{sobre } \Sigma, \\ z(x,0) = z_0(x); y(x,0) = y_0(x) & \text{en } \Omega. \end{array} \right. \tag{1.64}$$

Más precisamente, mostramos el siguiente resultado.

**Teorema 1.14.** *Con hipóteses apropiadas sobre la trayectoria de la región de control $\omega(t)$ (véase condiciones (5.9)-(5.13)), para todo $T > 0$ y todo $(y_0, y_1) \in L^2(\Omega)^2$, existe un control $h \in L^2(0,T;L^2(\Omega))$ tal que la solución $(y,z)$ de (1.64) verifica $y(.,T) = z(.,T) = 0$.*

Del Teorema 1.14 vemos la necesidad de las condiciones (5.9)-(5.13). En efecto, el hecho de que al final de la evolución de $\omega(t)$ hayamos cubierto todo el dominio $\Omega$ es una condición necesaria para controlar la EDO $(1.64)_2$.

En lo que sigue daremos las principales ideas de la prueba del Teorema 1.14.

Primero consideramos el sistema adjunto de (1.64):

$$\left|\begin{array}{ll} -p_t - \Delta p + (b(x)-1)p = (b(x)-1)q & \text{en } Q, \\ -q_t + q = p & \text{en } Q, \\ p(x,t) = 0 & \text{sobre } \Sigma, \\ p(x,T) = p_0(x); q(x,T) = q_0(x) & \text{en } \Omega, \end{array}\right. \tag{1.65}$$

y probamos los siguientes resultados.

**Lema 1.2.** *Existe una constante $C_0 > 0$ tal que, para todo $p \in C([0,T]; L^2(\Omega))$ con $p_t + \Delta p \in L^2(0,T; L^2(\Omega))$, se tiene:*

$$\int_0^T\int_\Omega \theta^2|p|^2]e^{2s\alpha}dxdt \leq C_0\left(\int_0^T\int_\Omega \theta_1^2|p_t+\Delta p|^2dxdt + \int_0^T\int_{\omega(t)}\theta^2|p|^2dxdt\right). \tag{1.66}$$

**Lema 1.3.** *Existe $C_1 > 0$ tal que, para todo $q \in H^1(0,T; L^2(\Omega))$, se tiene:*

$$\int_0^T\int_\Omega \theta_1^2|q|^2dxdt \leq C_1\Big(\int_0^T\int_\Omega \theta_2^2|q_t|^2dxdt + \int_0^T\int_{\omega(t)}\theta_1^2|q|^2dxdt\Big). \tag{1.67}$$

En los lemas 1.2 y 1.3, los pesos $\theta_i = \theta_i(x,t)$, $i = 1,2$, son similares a (1.20), la principal diferencia es que aquí $\eta = \eta(x,t)$, que está relacionado al hecho de que nuestra región de control se mueve en el tiempo.

Para terminar, resumimos las etapas de la demostración del Teorema 1.14.

*Etapa 1. Desigualdades de Carleman con el mismo peso.*

Aplicamos las desigualdades de Carleman de los lemas 1.2 y 1.3 para $p$ y $q$, respectivamente. Combinando estas desigualdades obtenemos estimaciones globales de $p$ y $q$ en términos de integrales locales de $p$ y $q$.

*Etapa 2. Estimación de la integral local de $p$.*

Análogamente a lo que hemos hecho en el capítulo 2, estimamos la integral local de $p$ en términos de una integral local de $q$.

*Step 3. Desigualdad de Observabilidad*

Por último, utilizando la teoría de semigrupos, probamos la siguiente desigualdad de observabilidad:

$$\int_\Omega [|p(0)|^2 + |q(0)|^2]dx \leq C\int_0^T\int_{\omega(t)} |q(x,t)|^2\, dxdt, \tag{1.68}$$

para todo $(p_0, q_0) \in L^2(\Omega)^2$.

De la desigualdad de observabilidad (1.68) completamos la demostración del Teorema 1.14 y, consecuentemente, la demostración del Teorema 1.13.

### Capítulo 6: Un sistema hiperbólico y el coste de la controlabilidad para el sistema de Stokes

El último capítulo de esta tesis estará dedicado a analizar el coste de la controlabilidad a cero, cuando $T \longrightarrow 0^+$, del sistema de Stokes:

$$\left| \begin{array}{ll} y_t - \Delta y + \nabla p = g 1_\omega & \text{en } Q, \\ div\; y = 0 & \text{en } Q, \\ y = 0 & \text{sobre } \Sigma, \\ y(0) = y_0 & \text{en } \Omega. \end{array} \right. \tag{1.69}$$

Nuestra motivación para este capítulo se debe al conocido hecho de que el coste de la controlabilidad a cero de la ecuación del calor

$$\left| \begin{array}{ll} v_t - \Delta v = f 1_\omega & \text{en } Q, \\ v = 0 & \text{sobre } \Sigma, \\ v(0) = v_0 & \text{en } \Omega \end{array} \right. \tag{1.70}$$

es de la forma $e^{C(\Omega,\omega)(1+1/T)}$ cuando $T \longrightarrow 0^+$ (véase, por ejemplo, [98, 99]). Concretamente:

$$||f 1_\omega||_{L^2(Q)} \leq e^{C(\Omega,\omega)(1+1/T)} ||v_0||_{L^2(\Omega)}. \tag{1.71}$$

Entre tanto, al contrario del caso de la ecuación del calor, los resultados conocidos de controlabilidad a cero para la ecuación de Stokes (véase, por ejemplo, [38]) nos dan

$$||g 1_\omega||_{L^2(Q)} \leq e^{C(\Omega,\omega)(1+1/T^4)} ||y_0||_{L^2(\Omega)}. \tag{1.72}$$

Como se observa en [32], la principal razón para la forma del coste (1.71) se debe a que la solución fundamental de la ecuación del calor en $\mathbb{R}^N$ viene dada por

$$\Phi(x,t) = \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x|^2}{4t}}. \tag{1.73}$$

Una vez que las soluciones fundamentales de la ecuación del calor y del sistema de Stokes tienen, al menos para $N = 2, 3$, el mismo comportamiento en el tiempo (ver [56, 57, 121]), la siguiente cuestión surge naturalmente:

**Pregunta 1.1.** ¿ Tienen los costes de la controlabilidad para la ecucaión del calor y el sistema de Stokes el mismo orden en tiempo cuando $T \longrightarrow 0^+$?

El objetivo del capítulo 6 es dar una respuesta positiva a la pregunta 1.1. Para ello asumiremos que $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ es un abierto conexo, estrellado con respecto al origen y con frontera $\partial\Omega$ suficientemente regular. Nuestra región de control sera un conjunto no vacío $\omega$ de $\Omega$ satisfaciendo:

$$\exists \mathcal{O} \subset \mathbb{R}^N, \mathcal{O} \text{ es un entorno de } \partial\Omega \text{ y } \omega = \Omega \cap \mathcal{O}. \tag{1.74}$$

Introducimos los siquientes espacios usuales de la mecánica de fluidos

$$V = \{u \in H_0^1(\Omega)^N;\ div\ u = 0\},$$

$$H = \{u \in L^2(\Omega)^N;\ div\ u = 0, u \cdot \nu = 0 \text{ sobre } \partial\Omega\}.$$

El principal resultado del capítulo 6 será:

**Teorema 1.15.** *Sunpógase que $\omega$ satisface (1.74), $y_0 \in H$ y sea $0 < T \leq 1$. Entonces existe un control $g \in L^2(\omega \times (0,T))$ tal que la solución $y$ de (1.69) verifica:*

$$y(T) = 0.$$

*Además, existen constantes positivas $C_1$ y $C_2$, dependiendo solamente de $\Omega$ y $\omega$, tal que*

$$\iint_{\omega\times(0,T)} |g|^2 dxdt \leq C_1 e^{C_2/T} |y_0|_H^2, \tag{1.75}$$

*para todo $y_0 \in H$ y $0 < T \leq 1$.*

El primer intento con el fin de demostrar el teorema 1.15 es analizar las diferentes formas en que uno puede probar (1.71) y (1.72). En efecto, existen al menos dos métodos diferentes para demostrar (1.71). El primero de ellos se basa en la descomposición espectral de la ecuación, también conocida como estrategia de Lebeau-Robbiano (ver [82]). El segundo método se basa en las desigualdades Carleman (ver [44, 46]). Para el sistema de Stokes, una estrategia del tipo Lebeau-Robbiano es difícil de probar, ya que uno tiene que lidiar con la presión y, por lo que sabemos, esto todavía es un problema abierto, por lo cual, el método más conocido usado para probar (1.72) se basa en las desigualdades de Carleman (ver [38]).

La principal diferencia al probar (1.71) y (1.72) por medio de las desigualdades de Carleman son los pesos que uno debe utilizar. En efecto, para la ecuación del calor los pesos son de la forma

$$\rho(t) = \frac{e^{C/(t(T-t))}}{t(T-t)}, \tag{1.76}$$

mientras que para el sistema de Stokes los pesos son de la forma

$$\rho(t) = \frac{e^{C/(t^4(T-t)^4)}}{t^4(T-t)^4}. \tag{1.77}$$

La razón por la cual tenemos distintos pesos para el sistema de Stokes y la ecuación del calor se debe a que uno tiene que lidiar con la presión en el sistema de Stokes. Si fuéramos capaces de utilizar pesos de la forma (1.76) para el sistema de Stokes, entonces las dos ecuaciones tendrían costes de controlabilidad del mismo orden. Entre tanto, un análisis cuidadoso en ambas demostraciones indica que la obstrución es la presión y que, en principio, es apenas una cuestión técnica.

Consecuentemente, nuestra estrategia no estará basada en estimaciones de Carleman, pero sí en el Método de Transmutación de Controles (CTM). En efecto, consideramos el siguiente sistema hiperbólico con un término de presión:

$$\left|\begin{array}{ll} u_{tt} - \Delta u + \nabla p = h1_\omega, & \text{en } Q, \\ div\ u = 0 & \text{en } Q, \\ u = 0 & \text{sobre } \Sigma, \\ u(0) = u^0; u_t(0) = u^1 & \text{en } \ \Omega. \end{array}\right. \tag{1.78}$$

La idea para demostrar el Teorema 1.15 es la siguiente. Probamos la controlabilidad a cero de (1.78) y aplicamos el CTM para obtener la controlabilidad a cero para el sistema de Stokes (1.69). Además, una vez conocido el coste de la controlabilidad de (1.78), utilizamos la fórmula de transmutación para el control y obtenemos la estimación (1.75).

Probamos el siguiente resultado de controlabilidad para (1.78).

**Teorema 1.16.** *Supóngase que $\omega$ satisface* (1.74) *y sea $T/2 > R_0 = \text{máx}\{|x|, x \in \overline{\Omega}\}$. Dado $(u_0, u_1) \in V \times H$, existe un control $h \in L^2(\omega \times (0,T))$ tal que la solución $u$ de* (1.78) *verifica:*

$$u(T) = u_t(T) = 0.$$

*Además, existe una constante positiva $C$ tal que*

$$\iint_{\omega\times(0,T)} |h|^2 dxdt \leq C\big(||u_0||_V^2 + |u^1|_H^2\big). \tag{1.79}$$

Remarcamos que el sistema (1.78) no es del tipo Cauchy-Kowalewski, lo que hace imposible la aplicación del Teorema de Holgrem como en el caso de la ecuación de ondas. Por lo que sabemos, el Teorema 1.16 es un resultado nuevo en la literatura. La controlabdidad exacta de (1.78) cuando el control actúa en el borde fue demostrado en [107].

Daremos ahora las principales ideas de la prueba del Teorema 1.16.

Consideramos el sistema adjunto de (1.78):

$$\left|\begin{array}{ll} \phi_{tt} - \Delta\phi + \nabla q = 0 & \text{en } Q, \\ div\ \phi = 0 & \text{en } Q, \\ \phi = 0 & \text{sobre } \Sigma, \\ \phi(0) = \phi^0; \phi_t(0) = \phi^1 & \text{en } \Omega, \end{array}\right. \tag{1.80}$$

donde $\phi^0 \in H$ y $\phi^1 \in V'$.

Necesitamos mostrar que existe una constante $C > 0$ tal que la siguiente desigualdad de observabilidad se verifica

$$|\phi^0|_H^2 + ||\phi^1||_{V'}^2 \leq C \iint_{\omega\times(0,T)} |\phi|^2 dxdt, \tag{1.81}$$

para toda solución de (1.80).

Primeramente probamos los siguientes resultados.

**Teorema 1.17.** *Si tomamos $T/2 > R_0 = \text{máx}\{|x|, x \in \overline{\Omega}\}$ entonces, para toda solución de (1.80) con dato inicial $(\phi^0, \phi^1) \in V \times H$, se tiene:*

$$|\phi^1|_H^2 + ||\phi^0||_V^2 \leq \frac{R_0}{2(T-2R_0)} \iint_\Sigma \left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Sigma. \tag{1.82}$$

La demostración del Teorema 1.17 está basada en la aplicación del método de los multiplicadores, exactamente como en el caso de la ecuación de ondas, para ello primero mostramos que, para soluciones regulares,

$$\iint_Q \frac{\partial p}{\partial x_i} x_k \frac{\partial\phi^i}{\partial x_k} dxdt = 0.$$

**Teorema 1.18.** *Supóngase que $\omega$ satisface (1.74) y que existe una constante $C > 0$ tal que, para todo $(\phi^0, \phi^1) \in V \times H$, la solución $\phi$ de (1.80) verifica*

$$||\phi^0||_V^2 + |\phi^1|_H^2 \leq C \iint_{\omega\times(0,T)} |\phi_t|^2 dxdt, \tag{1.83}$$

*entonces la desigualdad (1.81) es cierta para toda solución de (1.80) con dato inicial $(\phi^0, \phi^1)$ en $H \times V'$.*

Probamos el Teorema 1.18 de manera análoga al caso de la ecuación de ondas.

Seguidamente, utilizando argumentos de contradicción similares a los de [136], pero adaptados a nuestro problema, probamos (1.83).

Una vez obtenida la controlabilidad a cero/exacta de (1.78), aplicamos el CTM como sigue.

Introdcucimos dos intervalos de tiempo $(0,T)$ y $(0,L)$ distintos y consideramos el sistema de Stokes

$$\left| \begin{array}{ll} y_t - \Delta y + \nabla p = g 1_\omega & \text{en } Q_t := \Omega \times (0,T), \\ div\ y = 0 & \text{en } Q_t, \\ y = 0 & \text{sobre } \Sigma_t := \partial\Omega \times (0,T), \\ y(0) = y_0 & \text{en } \Omega \end{array} \right. \tag{1.84}$$

y el sistema hiperbólico con un término de presión

$$\left| \begin{array}{ll} u_{ll} - \Delta u + \nabla q = h 1_\omega & \text{en } Q_l := \Omega \times (0,L), \\ div\ u = 0 & \text{en } Q_l, \\ u = 0 & \text{sobre } \Sigma_l := \partial\Omega \times (0,L), \\ u(0) = y_0; u_l(0) = 0 & \text{en } \Omega \end{array} \right. \tag{1.85}$$

en $\Omega \times (0,T)$ y $\Omega \times (0,L)$, respectivamente.

Tomamos $y_0 \in V$ y $L > 2R_0$. Utilizando el Teorema 1.16 vemos que el sistema (1.85) es controlable a cero con un control $h \in L^2(\omega \times (0,L))$ verificando (1.79).

Definimos
$$y(t) = \int k(t,s)u(s)ds$$
y
$$g(t) = \int k(t,s)h(s)ds,$$
donde $k$ es la solución de

$$\left| \begin{array}{l} k_t = \partial_s^2 k \ \text{ en } \mathcal{D}'((0,T) \times (-L,L)), \\ k(0,x) = \delta(0), \\ k(T,x) = 0, \\ ||k||^2_{L^2((0,T)\times(-L,L))} \leq \gamma e^{\alpha L^2/T}, \end{array} \right. \tag{1.86}$$

para constantes apropiadas $\alpha$ y $\gamma$.

Una vez que $u$ es solución de (1.85) y $k$ es solución de (1.86), se muestra que $(y,g)$ resuelve, junto con algún $p$, el sistema de Stokes (1.84). También se tiene que $y(T) = 0$, una vez que $k$ es una solución controlada de la ecuación del calor. De la definición del control $g$, tenemos

$$\iint_{\omega\times(0,T)} |g|^2 dxdt \leq ||k||^2_{L^2((0,T)\times(-L,L))} ||h||^2_{L^2(\omega\times(0,T))}.$$

Finalmente, de las propiedades de $k$ y el Teorema 1.16, tenemos que la desigualdad (1.75) se verifica cuando $y_0 \in V$.

Utilizando el efecto regularizante del sistema de Stokes, se termina la demostración del Teorema 1.15 .

# Chapter 2

# Controllability of fast diffusion coupled parabolic systems

## 2.1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded connected open set whose boundary $\partial\Omega$ is regular enough ($N \geq 1$). Let $T > 0$ and let $\omega_1$ and $\omega_2$ be two nonempty subsets of $\Omega$, which will be referred to as *control domains*. We will use the notation $Q := \Omega \times (0,T)$ and $\Sigma := \partial\Omega \times (0,T)$.

The aim of this chapter is to analyze the controllability of linear coupled parabolic systems in which one of the equations is degenerating into an elliptic equation.

In order to state the problem, we introduce the system:

$$\left| \begin{array}{ll} u_t - \Delta u = au + bv + f1_{\omega_1} & \text{in } Q, \\ \epsilon v_t - \Delta v = cu + dv + g1_{\omega_2} & \text{in } Q, \\ u = v = 0 & \text{on } \Sigma, \\ u(0) = u_0;\ v(0) = v_0 & \text{in } \Omega. \end{array} \right. \tag{2.1}$$

For any $\epsilon > 0$, $a = a(x,t)$, $b = b(x,t)$, $c = c(x,t)$ and $d = d(x,t)$ in $L^\infty(Q)$, $f$ and $g$ in $L^2(Q)$ and $u_0, v_0$ in $L^2(\Omega)$, it is standard to show, from [91] for example, that (2.1) has a unique solution $(u,v) \in \left(L^2(0,T;H_0^1(\Omega)\right)^2 \cap \left(C([0,T];L^2(\Omega)\right)^2$. For the purpose of the present chapter we will consider $a,b,c,d$ functions in $C^3(\overline{Q})$ and assume that $0 < \epsilon < 1/2$. In particular we want to study this problem when only one control is active, namely when $g \equiv 0$ or $f \equiv 0$ and analyze the dependence of the cost of the null controllability of system (2.1) with respect to the parameter $\epsilon$.

Our interest in this problem comes from the fact that in many physical situations system (2.1) is formally approximated by the following parabolic-elliptic system:

$$\left|\begin{array}{ll} u_t - \Delta u = au + bv + f1_{\omega_1} & \text{in } Q, \\ -\Delta v = cu + dv + g1_{\omega_2} & \text{in } Q, \\ u = v = 0 & \text{on } \Sigma, \\ u(0) = u_0 & \text{in } \Omega. \end{array}\right. \tag{2.2}$$

This is the case for instance for biological systems modeling aggregation phenomena or chemical systems having two different concentrations, see [61, 62] and the references therein. However, even if this approximation is consistent with the existence and uniqueness point of view, it is not clear at all what can be done from a control theory point of view. The main reason for that arises from the fact that we are considering systems having different physical properties and therefore, at least a priori, different control properties.

If $a, b, c, d$ are in $L^\infty(Q)$ and $d < \mu_1$, where $\mu_1$ is the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions, as in the parabolic-parabolic case, it is standard to show that for every $f$ and $g$ in $L^2(Q)$ and $u_0$ in $L^2(\Omega)$, that (2.2) possesses a unique solution $u \in L^2(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega))$ and $v \in L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega))$.

It is important to mention that this question of approximating an equation by another one having different physical properties was already studied in the case of a hyperbolic equation degenerating into a parabolic one and vice-versa. In fact, it was proved in [83] that system

$$\left|\begin{array}{ll} \epsilon u_{tt} - \Delta u + u_t = 1_\omega & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(0) = u_0;\ u_t(0) = u_1 & \text{in } \Omega, \end{array}\right. \tag{2.3}$$

is null controllable, for each $\epsilon$ fixed, and that the controls remains bounded when $\epsilon \to 0^+$ if we impose some geometric condition on $\Omega$. Furthermore, the control sequence converges, when $\epsilon \to 0^+$, to a control for the heat equation

$$\left|\begin{array}{ll} u_t - \Delta u = f1_\omega & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(0) = u_0 & \text{in } \Omega. \end{array}\right. \tag{2.4}$$

Another relevant work in this subject is [53], in which the authors consider the linear transport diffusion equation

$$\left|\begin{array}{ll} y_t - \epsilon \Delta y + M(x,t) \cdot \nabla y = f1_\omega & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega \end{array}\right. \tag{2.5}$$

and investigate the cost of the controllability in the vanishing viscosity limit $\epsilon \to 0^+$ and, in particular, they try to determine in which situation it is possible to obtain a control which remains bounded as $\epsilon \to 0^+$. In that paper the authors are able to prove boundedness of controls by assuming some conditions on the vector field $M$ and the time $T$. See also [26, 55] for the analysis of (2.5) in the 1-d case when $M$ is constant.

Concerning the case of parabolic systems converging to parabolic-elliptic systems, as far as we know, the first time this problem was addressed was in [5] and [6] (see also chapter 3 of this thesis), where the authors considered the case of a nonlinear parabolic-elliptic system appearing in electrocardiology as a simplification of a coupled parabolic system modeling electrical activities in the heart and, combining Carleman estimates and weighted energy inequalities, the authors are able to prove that the control properties of the parabolic-elliptic system can be viewed as a limit process of the control properties of a family of parabolic systems.

Let us denote by $\big(u(t;\epsilon,(u_0,v_0),f,g),v(t;\epsilon,(u_0,v_0),f,g)\big)$ the solution of (2.1) at time $t\in[0,T]$ associated to $(u_0,v_0)\in L^2(\Omega)^2$ and $(f,g)\in L^2(Q)^2$.

The first main result in this chapter is given by the following Theorem.

**Theorem 2.1.** *Let $(u_0,v_0)\in L^2(\Omega)^2$ and $a,b,c$ and $d$ be $C^3(\overline{Q})$ functions. Then:*

*(i) If $c\neq 0$ in $\overline{\omega}$, for some $\omega\subset\subset\omega_1$, and $d<\mu_1$ ($\mu_1$ being the first eigenvalue of $-\Delta$ with Dirichlet boundary condition), then system (2.1) is uniformly null controllable, with respect to $\epsilon$, with control only on the first equation. More precisely, for each $0<\epsilon<1/2$, there exists $f=f(\epsilon)\in L^2(Q)$ such that*

$$\big(u(T;\epsilon,(u_0,v_0),f(\epsilon),0),v(T;\epsilon,(u_0,v_0),f(\epsilon),0)\big)=(0,0). \tag{2.6}$$

*Moreover, we have the following estimate on the control*

$$||f(\epsilon)1_{\omega_1}||^2_{L^2(Q)}\leq C(||u_0||^2_{L^2(\Omega)}+\epsilon||v_0||^2_{L^2(\Omega)}), \tag{2.7}$$

*where $C$ is a constant that does not depend on $\epsilon$, $u_0$ and $v_0$.*

*(ii) If $b\neq 0$ in $\overline{\omega}$, for some $\omega\subset\subset\omega_2$, and $d<\mu_1$, then system (2.1) is uniformly null controllable, with respect to $\epsilon$, with control acting only on the second equation. More precisely, for each $0<\epsilon<1/2$, there exists $g=g(\epsilon)\in L^2(Q)$ such that*

$$\big(u(T;\epsilon,(u_0,v_0),0,g(\epsilon)),v(T;\epsilon,(u_0,v_0),0,g(\epsilon))\big)=(0,0). \tag{2.8}$$

*Moreover, we have the following estimate on the control*

$$||g(\epsilon)1_{\omega_2}||^2_{L^2(Q)} \leq C(||u_0||^2_{L^2(\Omega)} + \epsilon||v_0||^2_{L^2(\Omega)}), \tag{2.9}$$

*where $C$ does not depend on $\epsilon$, $u_0$ and $v_0$.*

In order to prove Theorem 2.1 we are led to consider the adjoint system of (2.1),

$$\left| \begin{array}{ll} -\varphi_t - \Delta\varphi = a\varphi + c\xi & \text{in } Q, \\ -\epsilon\xi_t - \Delta\xi = b\varphi + d\xi & \text{in } Q, \\ \varphi = \xi = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi_T;\ \xi(T) = \xi_T & \text{in } \Omega, \end{array} \right. \tag{2.10}$$

with $(\varphi_T, \xi_T) \in L^2(\Omega)^2$.

It is well known (see, for instance, [86]) that case $1$ of Theorem 2.1 is equivalent to prove the existence of a universal constant $C$, which does not depend on $\epsilon$, such that the observability inequality

$$||\varphi(0)||^2_{L^2(\Omega)} + \epsilon||\xi(0)||^2_{L^2(\Omega)} \leq C \iint_{\omega_1\times(0,T)} |\varphi|^2 dxdt, \tag{2.11}$$

holds for all solutions $(\varphi, \xi)$ of (2.10). Analogously, one can prove that case $2$ of Theorem 2.1 is equivalent to show that

$$||\varphi(0)||^2_{L^2(\Omega)} + \epsilon||\xi(0)||^2_{L^2(\Omega)} \leq C \iint_{\omega_2\times(0,T)} |\xi|^2 dxdt, \tag{2.12}$$

for all solutions $(\varphi, \xi)$ of (2.10).

The study of the controllability of systems of parabolic equations has obtained a lot of attention in the recent years. For instance, in [69] the authors analyze the controllability of a reaction diffusion system consisting of two parabolic equations coupled by zero-order terms, obtaining the null controllability for the linear system and the local null controllability of the semilinear system. In [51] the controllability of a quite general system of two coupled linear parabolic equations is studied and, combining Carleman inequalities and some energy inequalities, null controllability is proved.

Another relevant work concerning to the controllability of coupled systems is [47], in which the authors analyze the null controllability of a cascade system of $m$ coupled parabolic equations, obtaining the null controllability for the cascade system whenever they have a good coupling structure. For a general discussion about the controllability of parabolic systems, see the survey paper [70].

It is important to mention that, unlike the analysis done in this thesis, the aforementioned works are devoted to systems that do not degenerate. Actually, following [69] or [51], one can prove that, under the assumptions of case 2 in Theorem 2.1, the uniform null controllability, with respect to $\epsilon$, can be obtained in the case where one control is acting on the second equation of (2.1). On the other hand, following [69] or [51], if the control is acting only on the first equation of (2.1) one obtains a cost of null controllability of order $\epsilon^{-1}$.

Therefore, some analysis is required in order to guarantee the uniform controllability with respect to the degenerating parameter $\epsilon$. Thus, in this chapter we analyze this problem and obtain a uniform estimate on the cost of controllability of (2.1) in the case of a control acting only on the first equation (case 1 of Theorem 2.1). Our proof can also be applied in order to obtain the boundedness of the cost of the null controllability of (2.1) when the control is acting on the second equation (see Theorem 2.4).

## 2.2 A model problem

In this section we consider a system consisting of two ODE's in which one of them is converging to an algebraic equation. This example is useful to better understand the problems we have and techniques we shall use in the context of systems of PDE's like (2.1), where things are necessarily technically more involved and complex due to the much richer structure associated to the continuous character of the media under consideration and the needed analytical tools. It also shows how difficult it can be when analyzing the uniform controllability for degenerating systems, even for simple equations.

Given $a \in \mathbb{R}$, consider the controlled ODE system

$$\left| \begin{array}{l} x_t^\epsilon + x^\epsilon + a y^\epsilon = g^\epsilon, \\ \epsilon y_t^\epsilon + x^\epsilon + y^\epsilon = 0, \\ x^\epsilon(0) = x_0, \; y^\epsilon(0) = y_0 \end{array} \right. \tag{2.13}$$

and the respective adjoint system

$$\left| \begin{array}{l} \varphi_t + \varphi + \xi = 0, \\ \epsilon \xi_t + \xi + a\varphi = 0, \\ \varphi(0) = \varphi_0, \; \xi(0) = \xi_0. \end{array} \right. \tag{2.14}$$

The observability inequality which gives the uniform null (exact) controllability of (2.13) is the following

$$|\varphi(T)|^2 + \epsilon|\xi(T)|^2 \leq C \int_0^T |\varphi|^2 dt, \tag{2.15}$$

where $C$ is independent of $\epsilon$.

The following Theorem holds.

**Theorem 2.2.** *There exists a constant $C > 0$, independent of $0 < \epsilon < 1/2$, such that, for every $(u_0, v_0) \in \mathbb{R}^2$, the associated solution $(u, v)$ of (2.14) satisfies (2.15).*

*Proof.* We consider a function $\theta \in C^3([0,T])$, $0 \leq \theta \leq 1$, such that

$$\begin{cases} \theta \equiv 0 \text{ in } [0, T/4], \\ \theta \equiv 1 \text{ in } [3T/4, T]. \end{cases}$$

Multiplying $(2.14)_1$ by $\theta\varphi$ and $(2.14)_1$ by $\theta\xi$, respectively, we get

$$\frac{\theta}{2}\frac{d}{dt}|\varphi|^2 + \theta|\varphi|^2 = -\theta\varphi\xi$$

and

$$\frac{\epsilon\theta}{2}\frac{d}{dt}|\xi|^2 + \theta|\xi|^2 = -a\theta\varphi\xi.$$

Adding this two identities, we obtain

$$\frac{\theta}{2}\frac{d}{dt}(|\varphi|^2 + \epsilon|\xi|^2) + \theta(|\varphi|^2 + |\xi|^2) = -(1+a)\theta\varphi\xi.$$

Integrating this last expression from $0$ to $T$, we get

$$\frac{1}{2}(|\varphi(T)|^2 + \epsilon|\xi(T)|^2) + \int_0^T \theta(|\varphi|^2 + |\xi|^2)dt = -\int_0^T (1+a)\theta\varphi\xi dt + \int_0^T \frac{\theta'}{2}(|\varphi|^2 + \epsilon|\xi|^2)dt. \tag{2.16}$$

Now we analyze the terms involving $\xi$ in the right-hand side of (2.16). First, we have that

$$|\int_0^T (1+a)\theta\varphi\xi dt| \leq C_\delta \int_0^T \theta|\varphi|^2 dt + \delta\int_0^T \theta|\xi|^2 dt, \tag{2.17}$$

for any $\delta > 0$.

Next,

$$\int_0^T \frac{\theta'}{2}\epsilon|\xi|^2 dt = -\int_0^T \frac{\theta'}{2}\epsilon\xi(\varphi_t + \varphi)dt := A_1 + A_2. \tag{2.18}$$

We have

$$A_1 = \int_0^T \frac{\theta''}{2}\epsilon\varphi\xi dt + \int_0^T \frac{\theta'}{2}\epsilon\xi_t\varphi dt \tag{2.19}$$
$$= -\int_0^T \frac{\theta''}{2}\epsilon\varphi(\varphi_t+\varphi)dt - \int_0^T \frac{\theta'}{2}\varphi(\xi + a\varphi)dt$$
$$= \epsilon\int_0^T (\frac{\theta'''(t)}{4} - \frac{\theta''}{2})|\varphi|^2 dt - \int_0^T \frac{\theta'}{2}(\varphi\xi + a|\varphi|^2)dt$$

and

$$A_2 = \int_0^T \frac{\theta'}{2}\varphi(\varphi_t+\varphi)dt = \int_0^T (\frac{\theta'}{2} - \frac{\theta''}{4})|\varphi|^2 dt. \tag{2.20}$$

From (2.17), (2.18), (2.19) and (2.20), we see that

$$\frac{1}{2}(|\varphi(T)|^2 + \epsilon|\xi(T)|^2) + \int_0^T \theta(|\varphi|^2 + |\xi|^2)dt \le C_\delta \int_0^T |\varphi|^2 dt + \delta\int_0^T \theta|\xi|^2 dt. \tag{2.21}$$

Finally, taking $\delta$ small enough, the result follows.

□

Notice that the key point in the above proof is to use several times the adjoint system in order to estimate the integral on $\xi$.

The following uniform null controllability result for (2.13) is a consequence of Theorem 2.2.

**Corollary 2.1.** *Given $T > 0$ and $a \in \mathbb{R}$. For every $0 < \epsilon < 1/2$ and every $(x_0, y_0) \in \mathbb{R}^2$, there exists a control $g \in L^2(0,T)$ such the solution $(x^\epsilon, y^\epsilon)$ of (2.13) satisfies:*

$$x^\epsilon(T) = y^\epsilon(T) = 0$$

*and*

$$\int_0^T |g^\epsilon|^2 dt \le C(|x_0|^2 + \epsilon|y_0|^2), \tag{2.22}$$

*for a constant $C > 0$ does not depending on $\epsilon$.*

*Moreover, the solution $(x^\epsilon, y^\epsilon)$ converges to $(x^0, -x^0)$ in $L^\infty(0,T)\times L^2(0,T)$, where $x^0$ is the solution of*

$$\left| \begin{array}{l} x^0_t + (1+a)x^0 = g^0, \\ x^0(0) = x_0, \ x^0(T) = 0, \end{array} \right. \tag{2.23}$$

*with $g^0$ being the weak limit of $g^\epsilon$ in $L^2(0,T)$.*

## 2.3 Carleman estimates and an extended adjoint system

In this section we deduce Carleman type estimates that will be used to prove observability inequalities (2.11) and (2.12). To this end, we first define several weight functions which will be useful in the sequel.

The basic weight we need is given by the following Lemma.

**Lemma 2.1.** *Given a nonempty open set $\omega_0 \subset\subset \omega_1$ ($\omega_0 \subset\subset \omega_2$ for the case 2 of Theorem 2.1), there exists a function $\psi \in C^2(\overline{\Omega})$ verifying*

$$\psi(x) > 0 \text{ in } \Omega, \quad \psi = 0 \text{ on } \partial\Omega, \quad |\nabla\psi(x)| > 0 \; \forall x \in \overline{\Omega\backslash\omega_0}.$$

*Proof.* See [46]. □

Using this weight, we introduce:

$$\begin{aligned} &\phi(x,t) = \frac{e^{\lambda\psi(x)}}{t(T-t)}; \; \alpha(x,t) = \frac{e^{\lambda\psi(x)} - e^{2\lambda\|\psi\|_\infty}}{t(T-t)}, \\ &\hat{\phi}(t) = \min_{x\in\overline{\Omega}} \phi(x,t); \; \phi^*(t) = \max_{x\in\overline{\Omega}} \phi(x,t), \\ &\hat{\alpha}(t) = \min_{x\in\overline{\Omega}} \alpha(x,t), \; \alpha^*(t) = \max_{x\in\overline{\Omega}} \alpha(x,t), \end{aligned} \tag{2.24}$$

for some positive real number $\lambda$.

With this notation, now we state a Carleman inequality which will be very important to our purposes.

**Lemma 2.2.** *Let $\beta \in \mathbb{R}$, $0 < \sigma \leq 1$ and let $\omega$, $\omega_0$ be two nonempty subsets of $\Omega$ such that $\omega_0 \subset\subset \omega$. There exist $\lambda_0 = \lambda_0(\Omega, \omega) \geq 1$ and $s_0 = s_0(\Omega, \omega, \lambda_0)$ such that, for every $\lambda \geq \lambda_0$ and every $s \geq s_0(T + T^2)$, the following inequality holds:*

$$\begin{aligned} &s^{\beta-1} \iint_Q e^{2s\alpha}\phi^{\beta-1}(\sigma^2|q_t|^2 + \sum_{i,j=1}^N |\frac{\partial^2 q}{\partial x_i \partial x_j}|^2) dxdt \\ &+ s^{\beta+1} \iint_Q e^{2s\alpha}\phi^{\beta+1}|\nabla q|^2 dxdt + s^{\beta+3} \iint_Q e^{2s\alpha}\phi^{\beta+3}|q|^2 dxdt \\ &\leq C\left(s^\beta \iint_Q e^{2s\alpha}\phi^\beta|\sigma\partial_t q + \Delta q|^2 dxdt + s^{\beta+3} \iint_{\omega\times(0,T)} e^{2s\alpha}\phi^{\beta+3}|q|^2 dxdt\right), \end{aligned}$$

*for all $q \in C^2(\overline{Q})$, with $q = 0$ on $\Sigma$, where $C = C(\Omega, \omega, \lambda)$.*

*Proof.* This result is basically the same as the one proved in [46]. However, here the derivative in time is degenerating and a careful proof must be performed in this case. For the sake of completeness, we give a prove of this result in appendix A. □

The second main result of this chapter is a Carleman type estimate for the adjoint system (2.10).

**Theorem 2.3.** *Under the assumptions of case* 1 *of Theorem 2.1, there exist* $\lambda_0 = \lambda_0(\Omega, \omega_1) \geq 1$ *and* $s_0 = s_0(\Omega, \omega_1, \lambda_0) > 0$ *such that, for every* $\lambda \geq \lambda_0$ *and every* $s \geq s_0(T + T^2)$*, the solution* $(\varphi, \xi)$ *of* (2.10) *satisfies:*

$$s^4 \iint_Q e^{2s\alpha}\phi^4|\varphi|^2 dxdt + \iint_Q e^{2s\alpha}\phi^4|\xi|^2 dxdt \\ \leq Cs^{14} \iint_{\omega_1\times(0,T)} e^{4s\alpha^*-2s\hat{\alpha}}(\phi^*)^{14}|\varphi|^2 dxdt, \tag{2.25}$$

*with* $C$ *depending only on* $\Omega$*,* $\omega_1$ *and* $\lambda$*.*

**Remark 2.1.** Notice that $4s\alpha^* - 2s\hat{\alpha} < 0$ for $\lambda$ large enough and, therefore, that $e^{4s\alpha^*-2s\hat{\alpha}}(\phi^*)^{14}$ is bounded.

With the purpose of proving Theorem 2.3, we extend our adjoint system to a system of 4 equations. We set the notation:

$$\mathcal{L}_{\gamma,\theta} := \gamma\partial_t - \Delta - \theta, \text{ for } \gamma \in \mathbb{R} \text{ and } \theta \in L^\infty(Q). \tag{2.26}$$

Then we consider a new function

$$w = \mathcal{L}_{-\epsilon,d}\varphi$$

and, if $\varphi_T$ and $\xi_T$ are smooth and $(\varphi, \xi)$ is the solution of (2.10) associated to this initial data, a simple calculation gives

$$-w_t - \Delta w - aw = \varphi(cb + \mathcal{L}_{-\epsilon,0}a - \mathcal{L}_{-1,0}d) + \xi\mathcal{L}_{-\epsilon,0}c - 2\nabla\xi\nabla c + 2\nabla\varphi(\nabla d - \nabla a).$$

Therefore, we can add two more equations to our adjoint system, going from a system of 2 equations to a system of 4 equations, namely

$$\left|\begin{array}{ll} \mathcal{L}_{-1,a}w = \varphi(cb + \mathcal{L}_{-\epsilon,0}a - \mathcal{L}_{-1,0}d) + \xi\mathcal{L}_{-\epsilon,0}c - 2\nabla\xi\nabla c + 2\nabla\varphi(\nabla d - \nabla a) & \text{in } Q, \\ \mathcal{L}_{-\epsilon,d}\varphi = w & \text{in } Q, \\ \mathcal{L}_{-1,a}\varphi = c\xi & \text{in } Q, \\ \mathcal{L}_{-\epsilon,d}\xi = b\varphi & \text{in } Q, \\ \varphi = \xi = w = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi_T;\ \xi(T) = \xi_T;\ w(T) = -\epsilon\varphi_T - \Delta\varphi_T - d\varphi_T & \text{in } \Omega. \end{array}\right. \tag{2.27}$$

The plan of the proof of Theorem 2.3 contains five parts:

*First part*: We see equations of (2.27) as heat equations and apply the Carleman estimate for the heat equation given in Lemma 2.2. This yields a global estimate for $\varphi$, $w$ and $\xi$ in terms of local terms of $\varphi$, $w$ and $\xi$.

*Second part*: Using the second equation in (2.27), we eliminate the local integral of $w$ appearing in the Carleman estimate obtained in *step 1*.

*Third part*: We estimate a local integral of $\xi$ in terms of a local integral of $\varphi$, a local integral of $\varphi_t$ and some small order terms.

*Fourth part*: Using the extend adjoint system, we show that we can estimate $\varphi_t$ locally in terms of a local integral of $\varphi$ and some small order terms.

*Fifth part*: We gather the estimates from the previous steps and absorb the small order terms, obtaining our desired Carleman estimate.

Along this thesis, we will use the notation:

$$\begin{aligned} I_\beta(s,\sigma;\rho) = s^{\beta+3}\iint_Q e^{2s\alpha}\phi^{\beta+3}|\rho|^2dxdt + s^{\beta+1}\iint_Q e^{2s\alpha}\phi^{\beta+1}|\nabla\rho|^2dxdt \\ + s^{\beta-1}\iint_Q e^{2s\alpha}\phi^{\beta-1}(\sigma^2|\rho_t|^2 + \sum_{i,j=1}^N |\frac{\partial^2\rho}{\partial x_i\partial x_j}|^2)dxdt, \end{aligned} \tag{2.28}$$

where $s, \beta$ and $\sigma$ are real numbers and $\rho = \rho(x,t)$.

**Proof of Theorem 2.3**. For an easier comprehension, we divide the proof into several steps:

**Step 1.** *First Carleman inequalities.*
Let $\omega'$ be a nonempty set such that $\omega_0 \subset\subset \omega' \subset\subset \omega$, where $\omega$ is a subset of $\omega_1$ in which $c \neq 0$. We apply Lemma 2.2 to $(2.27)_1$, with $\beta = 2$, and to $(2.27)_3$ and $(2.27)_4$, with $\beta = 1$. Then

$$\begin{aligned} I_2(s,1;w) \leq & C\big(s^5\iint_{\omega'\times(0,T)} e^{2s\alpha}\phi^5|w|^2dxdt \\ & + s^2\iint_Q \phi^2 e^{2s\alpha}(|\varphi|^2 + |\nabla\varphi|^2 + |\xi|^2 + |\nabla\xi|^2)dxdt\big), \end{aligned} \tag{2.29}$$

$$I_1(s,1;\varphi) \leq C\big(s^4\iint_{\omega'\times(0,T)} e^{2s\alpha}\phi^4|\varphi|^2dxdt + s\iint_Q e^{2s\alpha}\phi|\xi|^2dxdt\big) \tag{2.30}$$

and

$$I_1(s,\epsilon;\xi) \leq C\big(s^4\iint_{\omega'\times(0,T)} e^{2s\alpha}\phi^4|\xi|^2dxdt + s\iint_Q e^{2s\alpha}\phi|\varphi|^2dxdt\big). \tag{2.31}$$

Adding (2.29), (2.30), (2.31), and absorbing the lower order terms, we get

$$\begin{aligned} &I_2(s,1;w) + I_1(s,\epsilon;\xi) + I_1(s,1;\varphi) \\ &\qquad \leq C\big(s^4 \iint_{\omega'\times(0,T)} e^{2s\alpha}\phi^4|\varphi|^2 dxdt + s^4 \iint_{\omega'\times(0,T)} e^{2s\alpha}\phi^4|\xi|^2 dxdt \\ &\qquad + s^5 \iint_{\omega'\times(0,T)} e^{2s\alpha}\phi^5|w|^2 dxdt\big). \end{aligned} \tag{2.32}$$

**Step 2.** *Estimate of the local integral of $w$.*
In this step we estimate the local integral of $w$ in the right-hand side of (2.32) in terms of a local integral of $\varphi$ and a small order term involving $w$. In order to do that, we introduce a cut-off function $\theta$ with

$$\theta \in C_0^\infty(\omega''), \text{with } 0 \leq \theta \leq 1 \text{ and } \theta \equiv 1 \text{ on } \omega',$$

where $\omega' \subset\subset \omega'' \subset\subset \omega_1$.
We use $(2.27)_2$ to write

$$\begin{aligned} s^5 \iint_{\omega''\times(0,T)} e^{2s\alpha}\phi^5|w|^2\theta dxdt &= s^5 \iint_{\omega''\times(0,T)} e^{2s\alpha}\phi^5\theta w(-\epsilon\varphi_t - \Delta\varphi - d\varphi) dxdt \\ &:= M_1 + M_2 + M_3. \end{aligned} \tag{2.33}$$

We estimate each term in the expression above. For the first term, we integrate by parts to see that

$$\begin{aligned} M_1 = 5\epsilon s^5 &\iint_{\omega''\times(0,T)} \phi^4\phi_t e^{2s\alpha}\theta w\varphi dxdt + 2\epsilon s^6 \iint_{\omega''\times(0,T)} \alpha_t\phi^5 e^{2s\alpha}\theta w\varphi dxdt \\ &+ \epsilon s^5 \iint_{\omega''\times(0,T)} \phi^5 e^{2s\alpha}\theta w_t\varphi dxdt \end{aligned} \tag{2.34}$$

and use Young's inequality to obtain

$$\begin{aligned} M_1 \leq \epsilon^2\big(&s^5 \iint_{\omega''\times(0,T)} \phi^5 e^{2s\alpha}|w|^2 dxdt + s\iint_{\omega''\times(0,T)} \phi e^{2s\alpha}|w_t|^2 dxdt\big) \\ &+ Cs^9 \iint_{\omega''\times(0,T)} \phi^9 e^{2s\alpha}|\varphi|^2 dxdt. \end{aligned} \tag{2.35}$$

Here we have used that $|\alpha_t| \leq C\phi^2$.

Next, since

$$
\begin{aligned}
M_2 = 5s^5 \iint_{\omega''\times(0,T)} \phi^4 e^{2s\alpha} w\theta(\nabla\phi\cdot\nabla\varphi)dxdt + 2s^6 \iint_{\omega''\times(0,T)} \phi^5 e^{2s\alpha} w\theta(\nabla\alpha\cdot\nabla\varphi)dxdt \\
+ s^5 \iint_{\omega''\times(0,T)} \phi^5 e^{2s\alpha} w(\nabla\theta\cdot\nabla\varphi)dxdt + s^5 \iint_{\omega''\times(0,T)} \phi^5 e^{2s\alpha}\theta(\nabla w\cdot\nabla\varphi)dxdt,
\end{aligned}
$$

it is not difficult to see that, for any $\delta > 0$ there exists $C = C(\delta)$ such that

$$
\begin{aligned}
M_2 + M_3 \le \delta(s^5 \iint_{\omega''\times(0,T)} e^{2s\alpha}\phi^5|w|^2 dxdt + s^3 \iint_{\omega''\times(0,T)} e^{2s\alpha}\phi^3|\nabla w|^2 dxdt) \\
+ C(s^7 \iint_{\omega''\times(0,T)} e^{2s\alpha}\phi^7|\nabla\varphi|^2 dxdt + s^5 \iint_{\omega''\times(0,T)} \phi^5 e^{2s\alpha}|\varphi|^2 dxdt). \quad (2.36)
\end{aligned}
$$

Hence

$$
\begin{aligned}
s^5 \iint_{\omega'\times(0,T)} e^{2s\alpha}\phi^5|w|^2 dxdt \le C(s^9 \iint_{\omega''\times(0,T)} \phi^9 e^{2s\alpha}|\varphi|^2 dxdt \quad (2.37) \\
+ s^7 \iint_{\omega''\times(0,T)} e^{2s\alpha}\phi^7|\nabla\varphi|^2 dxdt) + (\delta+\epsilon^2) I_2(s,1,w).
\end{aligned}
$$

Now we eliminate the local integral of $\nabla\varphi$. For this, we consider a set $\omega'''$ with $\omega'' \subset\subset \omega''' \subset\subset \omega_1$ and a cut-off function $\theta_1 \in C_0^\infty(\omega''')$ satisfying

$$
0 \le \theta_1 \le 1,\ \theta_1 \equiv 1 \text{ on } \omega''.
$$

Integration by parts gives

$$
\begin{aligned}
s^7 \iint_{\omega'''\times(0,T)} e^{2s\alpha}\theta_1\phi^7|\nabla\varphi|^2 dxdt = & - s^7 \iint_{\omega'''\times(0,T)} e^{2s\alpha}\theta_1\phi^7 \Delta\varphi\varphi dxdt \\
& - \frac{1}{2}s^7 \iint_{\omega'''\times(0,T)} \Delta(\theta_1 e^{2s\alpha}\phi^7)|\varphi|^2 dxdt. \quad (2.38)
\end{aligned}
$$

Using the fact that

$$
|\Delta(\theta_1 e^{2s\alpha}\phi^7)| \le Cs^2\phi^9 e^{2s\alpha} \ \text{ in } \omega'''\times(0,T),
$$

together with Young's inequality, we see that, for any $\delta > 0$ there exists $C = C(\delta)$ such that

$$s^7 \iint_{\omega'''\times(0,T)} e^{2s\alpha}\phi^7|\nabla\varphi|^2 dxdt \le \delta \iint_{\omega'''\times(0,T)} e^{2s\alpha}|\Delta\varphi|^2 dxdt + Cs^{14}\iint_{\omega'''\times(0,T)} e^{2s\alpha}\phi^{14}|\varphi|^2 dxdt. \tag{2.39}$$

Therefore

$$s^5 \iint_{\omega'\times(0,T)} e^{2s\alpha}\phi^5|w|^2 dxdt \le Cs^{14}\iint_{\omega'''\times(0,T)} e^{2s\alpha}\phi^{14}|\varphi|^2 dxdt + (\delta+\epsilon)I_2(s,1;w) + \delta I_1(s,1;\varphi), \tag{2.40}$$

for any $\delta > 0$.

Combining (2.32) and (2.40), we get

$$I_2(s,1;w) + I_1(s,\epsilon;\xi) + I_1(s,1;\varphi) \le C\big(s^{14}\iint_{\omega'''\times(0,T)} e^{2s\alpha}\phi^{14}|\varphi|^2 dxdt + s^4\iint_{\omega'\times(0,T)} e^{2s\alpha}\phi^4|\xi|^2 dxdt\big). \tag{2.41}$$

**Step 3.** *Estimate of the local integral of $\xi$.*
In this step we estimate the local integral of $\xi$ in the right-hand side of (2.41) in terms of a local integral of $\varphi$, a local integral of $\varphi_t$ and some small order terms.

Using equation $(2.27)_3$ and the fact that $c \neq 0$ in $\overline{\omega}$, we see that

$$s^4\iint_{\omega''\times(0,T)} e^{2s\alpha^*}\theta(\phi^*)^4|\xi|^2 dxdt = s^4\iint_{\omega''\times(0,T)} e^{2s\alpha^*}\frac{\theta}{c}(\phi^*)^4\xi(-\varphi_t - \Delta\varphi - a\varphi)dxdt := M_4 + M_5 + M_6, \tag{2.42}$$

where $\theta$ is the cut-off function introduced in *step 2*.

As in the previous steps, we estimate each term in the expression above. We have

$$M_4 \le \frac{s^4}{2}\iint_{\omega''\times(0,T)} e^{2s\alpha^*}\theta(\phi^*)^4|\xi|^2 dxdt + \frac{s^4}{2}\iint_{\omega''\times(0,T)} e^{2s\alpha^*}\frac{\theta}{c^2}(\phi^*)^4|\varphi_t|^2 dxdt. \tag{2.43}$$

Integration by parts gives

$$M_5 = -\, s^4\iint_{\omega''\times(0,T)} e^{2s\alpha^*}(\phi^*)^4(\Delta(\frac{\theta}{c})\xi + 2\nabla(\frac{\theta}{c})\nabla\xi + (\frac{\theta}{c})\Delta\xi)\varphi dxdt.$$

Using this last equality we can show that, for any $\delta > 0$, there exists $C = C(\delta)$ such that

$$M_5 + M_6 \leq \delta\big(s^4 \iint_{\omega''\times(0,T)} e^{2s\alpha}\phi^4|\xi|^2 dxdt + s^2 \iint_{\omega''\times(0,T)} e^{2s\alpha}\phi^2|\nabla\xi|^2 dxdt + \iint_{\omega''\times(0,T)} e^{2s\alpha}|\Delta\xi|^2 dxdt\big) + Cs^8 \iint_{\omega''\times(0,T)} e^{4s\alpha^*-2s\alpha}(\phi^*)^8|\varphi|^2 dxdt. \tag{2.44}$$

Hence

$$s^4 \iint_{\omega'\times(0,T)} e^{2s\alpha^*}\theta(\phi^*)^4|\xi|^2 dxdt \leq C\Big(s^4 \iint_{\omega''\times(0,T)} e^{2s\alpha^*}(\phi^*)^4|\varphi_t|^2 dxdt + s^8 \iint_{\omega''\times(0,T)} e^{4s\alpha^*-2s\alpha}(\phi^*)^8|\varphi|^2 dxdt\Big) + \delta I_1(s,\epsilon;\xi). \tag{2.45}$$

From (2.45), our objective is now reduced to estimate a local integral of $\varphi_t$ in terms of a local integral of $\varphi$ and small order terms. This will be done in the next steps.

**Step 4.** *Estimate of the local integral of $\varphi_t$.*

In this step we deal with the first term appearing in the right-hand side of (2.45). First, we integrate by parts to get

$$s^4 \iint_{\omega''\times(0,T)} e^{2s\alpha^*}(\phi^*)^4|\varphi_t|^2 dxdt = -\, s^4 \iint_{\omega''\times(0,T)} e^{2s\alpha^*}(\phi^*)^4\varphi_{tt}\varphi dxdt + \frac{s^4}{2} \iint_{\omega''\times(0,T)} (e^{2s\alpha^*}(\phi^*)^4)_{tt}|\varphi|^2 dxdt. \tag{2.46}$$

Since

$$s^4 \iint_{\omega''\times(0,T)} e^{2s\alpha^*}(\phi^*)^4\varphi_{tt}\varphi dxdt \leq \frac{s^{-6}}{2} \iint_{\omega''\times(0,T)} e^{2s\hat{\alpha}}\hat{\phi}^{-5}|\varphi_{tt}|^2 dxdt + \frac{s^{14}}{2} \iint_{\omega''\times(0,T)} e^{4s\alpha^*-2s\hat{\alpha}}(\phi^*)^8\hat{\phi}^5|\varphi|^2 dxdt, \tag{2.47}$$

we just have to estimate the local integral of $\varphi_{tt}$ in the right-hand side of (2.47). In order to do that, we use $(2.27)_2$ to see that

$$-\,\epsilon(e^{s\hat{\alpha}}\hat{\phi}^{-5/2}\varphi_{tt})_t - \Delta(e^{s\hat{\alpha}}\hat{\phi}^{-5/2}\varphi_{tt}) = e^{s\hat{\alpha}}\hat{\phi}^{-5/2}w_{tt} - \epsilon(e^{s\hat{\alpha}}\hat{\phi}^{-5/2})_t\varphi_{tt} + e^{s\hat{\alpha}}\hat{\phi}^{-5/2}(d_{tt}\varphi + 2d_t\varphi_t + d\varphi_{tt}), \tag{2.48}$$

with $e^{s\hat{\alpha}}\hat{\phi}^{-5/2}\varphi_{tt} = 0$ in $\partial\Omega$ and $e^{s\hat{\alpha}}\hat{\phi}^{-5/2}\varphi_{tt}(T) = e^{s\hat{\alpha}}\hat{\phi}^{-5/2}\varphi_{tt}(0) = 0$.

Next, multiplying both sides of (2.48) by $e^{s\hat{\alpha}}\hat{\phi}^{-5/2}\varphi_{tt}$, integrating over $Q$ and using

Young's inequality, we get

$$\begin{aligned}\iint_Q |\nabla(e^{s\hat{\alpha}}\hat{\phi}^{-5/2}\varphi_{tt})|^2 dxdt \leq C\Big(&\iint_Q |e^{s\hat{\alpha}}\hat{\phi}^{-5/2}w_{tt}|^2 dxdt + \epsilon^2 \iint_Q |(e^{s\hat{\alpha}}\hat{\phi}^{-5/2})_t\varphi_{tt}|^2 dxdt \\ &+ \iint_Q |e^{s\hat{\alpha}}\hat{\phi}^{-5/2}\varphi_t|^2 dxdt + \iint_Q |e^{s\hat{\alpha}}\hat{\phi}^{-5/2}\varphi|^2 dxdt\Big) \\ &+ \iint_Q d|e^{s\hat{\alpha}}\hat{\phi}^{-5/2}\varphi_{tt}|^2 dxdt \end{aligned} \tag{2.49}$$

and, since $d < \mu_1$, we have

$$\begin{aligned}s^{-6}\iint_Q |\nabla(e^{s\hat{\alpha}}\hat{\phi}^{-5/2}\varphi_{tt})|^2 dxdt \leq Cs^{-6}\Big(&\iint_Q |e^{s\hat{\alpha}}\hat{\phi}^{-5/2}w_{tt}|^2 dxdt + \iint_Q |e^{s\hat{\alpha}}\hat{\phi}^{-5/2}\varphi_t|^2 dxdt \\ &+ \iint_Q |e^{s\hat{\alpha}}\hat{\phi}^{-5/2}\varphi|^2 dxdt + \epsilon^2 \iint_Q |(e^{s\hat{\alpha}}\hat{\phi}^{-5/2})_t\varphi_{tt}|^2 dxdt\Big).\end{aligned} \tag{2.50}$$

**Step 4.1.** *Estimate of the term in* $\varphi_{tt}$.
Here, we estimate the last term in the right-hand side of (2.50). Using $(2.27)_2$ and $(2.27)_3$, we can show that

$$-\epsilon\varphi_{tt} = -\epsilon^2\varphi_{tt} - \epsilon w_t - \epsilon d_t\varphi - \epsilon d\varphi_t + \epsilon a_t\varphi + \epsilon a\varphi_t + \epsilon c_t\xi + \epsilon c\xi_t, \tag{2.51}$$

from where we see that

$$\begin{aligned}|(e^{s\hat{\alpha}}\hat{\phi}^{-5/2})_t\varphi_{tt}|^2 \leq \epsilon^2|(e^{s\hat{\alpha}}\hat{\phi}^{-5/2})_t|^2|\varphi_{tt}|^2 + |(e^{s\hat{\alpha}}\hat{\phi}^{-5/2})_t|^2|w_t|^2 \\ + C|(e^{s\hat{\alpha}}\hat{\phi}^{-5/2})_t|^2\big(|\varphi|^2 + |\varphi_t|^2 + |\xi|^2 + |\xi_t|^2\big).\end{aligned} \tag{2.52}$$

Since

$$|(e^{s\hat{\alpha}}\hat{\phi}^{-5/2})_t| \leq Cs^2\hat{\phi}^{-1/2}e^{s\hat{\alpha}},$$

and $0 < \epsilon < 1/2$, inequality (2.52) implies

$$\begin{aligned}&\epsilon^2 s^{-6}\iint_Q |(e^{s\hat{\alpha}}\hat{\phi}^{-5/2})_t\varphi_{tt}|^2 dxdt \\ &\quad\leq C\epsilon^2 s^{-2}\iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-1}(|\xi|^2 + |\xi_t|^2 + |\varphi_t|^2 + |\varphi|^2 + |w_t|^2)dxdt.\end{aligned} \tag{2.53}$$

**Step 4.2.** *Estimate of the term in* $w_{tt}$.

We estimate the first term in the right-hand side of (2.50). From $(2.27)_1$ we have

$$\begin{aligned}-w_{tt}-\Delta w_t-a_tw-aw_t&=(cb)_t\varphi+cb\varphi_t-\epsilon c_{tt}\xi-\epsilon c_t\xi_t-\epsilon a_{tt}\varphi-\epsilon a_t\varphi_t-\xi_t\Delta c-\xi\Delta c_t\\&-2\nabla\xi_t\nabla c-2\nabla\xi\nabla c_t+\varphi_t\Delta(d-a)+\varphi\Delta(d-a)_t+d_t\varphi_t\\&+d_{tt}\varphi+2\nabla\varphi\nabla(d-a)_t+2\nabla\varphi_t\nabla(d-a).\end{aligned}\tag{2.54}$$

We multiply both sides of (2.54) by $e^{2s\hat{\alpha}}\hat{\phi}^{-5}w_{tt}$ and integrate over $Q$, we obtain this way

$$\begin{aligned}&\iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-5}|w_{tt}|^2dxdt\\&=\iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-5}\Delta w_tw_{tt}dxdt+\iint_Q a_te^{2s\hat{\alpha}}\hat{\phi}^{-5}ww_{tt}dxdt\\&+\iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-5}w_{tt}\varphi\big((cb)_t-\epsilon a_{tt}+\Delta(d-a)_t+d_{tt}\big)dxdt\\&+\iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-5}w_{tt}\varphi_t\big(cb-\epsilon a_t+\Delta(d-a)+d_t\big)dxdt+\iint_Q ae^{2s\hat{\alpha}}\hat{\phi}^{-5}w_tw_{tt}dxdt\\&+2\iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-5}w_{tt}\nabla\varphi\nabla(d-a)_tdxdt+2\iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-5}w_{tt}\nabla\varphi_t\nabla(d-a)dxdt\\&+\iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-5}w_{tt}\xi\big(-\epsilon c_{tt}-\Delta c_t\big)dxdt+\iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-5}w_{tt}\xi_t\big(-\epsilon c_t-\Delta c\big)dxdt\\&+2\iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-5}w_{tt}\nabla\xi\nabla c_tdxdt+2\iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-5}w_{tt}\nabla\xi_t\nabla cdxdt.\end{aligned}\tag{2.55}$$

After a long, but straightforward calculation, we can show that

$$\begin{aligned}&s^{-6}\iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-5}|w_{tt}|^2dxdt\\&\quad\le Cs^{-6}\Big(\iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-5}\Delta w_tw_{tt}dxdt+\iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-5}(|w|^2+|w_t|^2)dxdt\\&\quad+\iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-5}(|\nabla\varphi|^2+|\nabla\varphi_t|^2+|\nabla\xi|^2+|\nabla\xi_t|^2)dxdt\Big).\end{aligned}\tag{2.56}$$

The rest of the proof of this step is devoted to estimate the integrals $\iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-5}\Delta w_tw_{tt}dxdt$, $\iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-5}|\nabla\varphi_t|^2dxdt$ and $\iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-5}|\nabla\xi_t|^2dxdt$ appearing in the right-hand side of (2.56).The other terms in the righ-hand side of (2.56) can be absorbed by the left had side of (2.41).

**Step 4.2.1.** *Estimate of the term in* $\varphi_t$.

We use $(2.27)_2$ to see that $-\Delta\varphi_t=w_t+\epsilon\varphi_{tt}+d_t\varphi+d\varphi_t$. From (2.51), and the fact that $-\Delta$ gives a norm in $H^2(\Omega)\cap H_0^1(\Omega)$, we get

$$s^{-6}\iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-5}|\nabla\varphi_t|^2dxdt \le s^{-6}\iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-5}|\Delta\varphi_t|^2dxdt$$
$$\le Cs^{-6}\iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-5}(\epsilon^2|\xi|^2+\epsilon^2|\xi_t|^2+|\varphi_t|^2+|\varphi|^2+|w_t|^2)dxdt. \tag{2.57}$$

All the terms in the right-hand side of (2.57) can be absorbed by the left had side of (2.41).

**Step 4.2.2.** *Estimate of the term in $\Delta w_t w_{tt}$.*

We have

$$s^{-6}\iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-5}\Delta w_t w_{tt}dxdt = \frac{s^{-6}}{2}\iint_Q (e^{2s\hat{\alpha}}\hat{\phi}^{-5})_t|\nabla w_t|^2dxdt$$
$$\le Cs^{-4}\iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-3}|\nabla w_t|^2dxdt, \tag{2.58}$$

since

$$|(e^{2s\hat{\alpha}}\hat{\phi}^{-5})_t| \le Cs^2e^{2s\hat{\alpha}}\hat{\phi}^{-3}.$$

Next, we use $(2.27)_1$ to see that

$$-w_{tt}-\Delta w_t - a_t w - a w_t = (cb-\epsilon a_t+\Delta(d-a)+d_t)\varphi_t - (\epsilon c_{tt}+\Delta c_t)\xi - (\epsilon c_t+\Delta c)\xi_t$$
$$+((cb)_t+\Delta(d-a)_t-\epsilon a_{tt}+d_{tt})\varphi - 2\nabla\xi_t\nabla c - 2\nabla\xi\nabla c_t$$
$$+2\nabla\varphi_t\nabla(d-a)+2\nabla\varphi\nabla(d-a)_t. \tag{2.59}$$

Multipyling both sides of (2.59) by $s^{-4}e^{2s\hat{\alpha}}\hat{\phi}^{-3}w_t$, integrating by parts and using Young's inequality, we get

$$s^{-4}\iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-3}|\nabla w_t|^2dxdt \le Cs^{-1}\Big(\iint_Q e^{2s\hat{\alpha}}(|\nabla\varphi|^2+|\nabla\xi|^2+|\nabla\xi_t|^2)dxdt$$
$$+\iint_Q e^{2s\hat{\alpha}}(|w|^2+|w_t|^2)dxdt\Big), \tag{2.60}$$

since

$$|(e^{2s\hat{\alpha}}\hat{\phi}^{-3})_t| \le Cs^2e^{2s\hat{\alpha}}\hat{\phi}^{-1}$$

and

$$s^{-1}\varphi^{-1}\le C. \tag{2.61}$$

**Step 4.2.3.** *Estimate of the term in $\nabla\xi_t$.*

We use $(2.27)_3$ to see that $-\epsilon\xi_{tt}-\Delta\xi_t = b_t\varphi+b\varphi_t+d_t\xi+d\xi_t$. Multiplying both sides by

$e^{2s\hat{\alpha}}\xi_t$ and integrating over $Q$, we obtain

$$\iint_Q e^{2s\hat{\alpha}}|\nabla\xi_t|^2dxdt \le \iint_Q de^{2s\hat{\alpha}}|\xi_t|^2dxdt + C\Big(\iint_Q e^{2s\hat{\alpha}}|\xi|^2dxdt + \iint_Q e^{2s\hat{\alpha}}(|\varphi|^2+|\varphi_t|^2)dxdt\Big), \tag{2.62}$$

which gives, from the assumption on $d$,

$$\iint_Q e^{2s\hat{\alpha}}|\nabla\xi_t|^2dxdt \le C\iint_Q e^{2s\hat{\alpha}}(|\xi|^2+|\varphi|^2+|\varphi_t|^2)dxdt. \tag{2.63}$$

**Step 5.** *Last arrangements and conclusion.*

From (2.46), (2.47), (2.50), (2.53), (2.56), (2.57), (2.60) and (2.63), we get

$$s^4\iint_{\omega''\times(0,T)} e^{2s\alpha^*}(\phi^*)^4|\varphi_t|^2dxdt \le Cs^4\iint_{\omega''\times(0,T)}(e^{2s\alpha^*}(\phi^*)^4)_{tt}|\varphi|^2dxdt + Cs^{-1}\iint_Q e^{2s\hat{\alpha}}(|\nabla\varphi|^2+|w_t|^2+|w|^2)dxdt + \delta I_1(s,\epsilon;\xi), \tag{2.64}$$

for any $\delta>0$. Putting (2.64) in (2.45), we obtain

$$s^4\iint_{\omega'\times(0,T)} e^{2s\alpha^*}\theta(\phi^*)^4|\xi|^2dxdt \le C\big(s^8\iint_{\omega''\times(0,T)}(e^{2s\alpha^*}+e^{4s\alpha^*-2s\alpha})(\phi^*)^8|\varphi|^2dxdt + s^{-1}\iint_Q e^{2s\hat{\alpha}}(|\nabla\varphi|^2+|w_t|^2+|w|^2)dxdt\big) + \delta I_1(s,\epsilon;\xi), \tag{2.65}$$

since

$$|(e^{2s\alpha^*}(\phi^*)^4)_{tt}| \le Cs^4e^{2s\alpha^*}(\phi^*)^8 \quad \text{and} \quad e^{2s\alpha^*} \le Ce^{4s\alpha^*-2s\hat{\alpha}}. \tag{2.66}$$

Finally, choosing $s$ large enough and $\delta$ small, putting (2.65) in (2.41) and absorb the small order terms, we obtain

$$I_2(s,1;w)+I_1(s,\epsilon;\xi)+I_1(s,1;\varphi) \le Cs^{14}\iint_{\omega_1\times(0,T)}(e^{2s\alpha^*}+e^{4s\alpha^*-2s\alpha})(\phi^*)^{14}|\varphi|^2dxdt. \tag{2.67}$$

This finishes the proof of Theorem 2.1. □

Noticing that the system formed by the first two equations in (2.27) has the same structure as the system formed by the third and fourth equation in (2.27), we can argue as in steps 1 and 2 above in order to prove the following result, which is the third main result of this chapter.

**Theorem 2.4.** *Under the assumptions of case 2 of Theorem 2.1, let $\psi$, $\phi$, $\alpha$ be the functions defined above. There exist $\lambda_0 = \lambda_0(\Omega, \omega_2) \geq 1$ and $s_0 = s_0(\Omega, \omega_2, \lambda_0) > 0$ such that, for any $\lambda \geq \lambda_0$ and any $s \geq s_0(T + T^2)$, the solution $(\varphi, \xi)$ of (2.10) satisfies:*

$$\begin{aligned} & s^3 \iint_Q e^{2s\alpha}\phi^3|\xi|^2 dxdt + s \iint_Q e^{2s\alpha}\phi|\nabla\xi|^2 dxdt + s^{-1} \iint_Q e^{2s\alpha}\phi^{-1}(\epsilon^2|\xi_t|^2 + |\Delta\xi|^2)dxdt \\ & + s^3 \iint_Q e^{2s\alpha}\phi^3|\varphi|^2 dxdt + s \iint_Q e^{2s\alpha}\phi|\nabla\varphi|^2 dxdt + s^{-1} \iint_Q e^{2s\alpha}\phi^{-1}(|\varphi_t|^2 + |\Delta\varphi|^2)dxdt \\ & \leq Cs^7 \iint_{\omega_2\times(0,T)} e^{2s\alpha}\phi^7|\xi|^2 dxdt, \end{aligned} \tag{2.68}$$

*with $C$ depending only on $\Omega$, $\omega_2$ and $\lambda_0$.*

## 2.4 Observability inequality

In this section we prove Theorem 2.1. As we said before, it is equivalent to prove an observability inequality for the adjoint system (i.e., inequality (2.11) in case 1 or inequality (2.12) in case 2). Since the proofs of (2.11) and (2.12) are similar, we just prove the first one. To do this, we first change the orientation of the adjoint system (2.10), i.e., instead of going from $T$ to 0 the system will evolve from 0 to $T$. Changing $t$ by $T - t$, we obtain the system

$$\left| \begin{array}{ll} \varphi_t - \Delta\varphi = a\varphi + c\xi & \text{in } Q, \\ \epsilon\xi_t - \Delta\xi = b\varphi + d\xi & \text{in } Q, \\ \varphi = \xi = 0 & \text{on } \Sigma, \\ \varphi(0) = \varphi_T;\ \xi(0) = \xi_T & \text{in } \Omega. \end{array} \right. \tag{2.69}$$

Our desired observability inequality becomes

$$||\varphi(T)||^2 + \epsilon||\xi(T)||^2 \leq C \iint_{\omega_1\times(0,T)} |\varphi|^2 dxdt, \tag{2.70}$$

where $C$ is a constant which does not depend on $\epsilon$.
In fact, multiplying $(2.69)_1$ by $\varphi$ and $(2.69)_2$ by $\xi$ and integrating over $\Omega$ we obtain

$$\frac{1}{2}\frac{d}{dt}||\varphi(t)||^2 + \frac{\epsilon}{2}\frac{d}{dt}||\xi(t)||^2 + ||\nabla\varphi(t)||^2 + ||\nabla\xi(t)||^2 \\ = \int_\Omega \big(a\varphi(t) + d\xi(t) + (b+c)\varphi(t)\xi(t)\big)dx. \tag{2.71}$$

Using the assumption on $d$ and Poincaré's inequality, we get

$$\frac{1}{2}\frac{d}{dt}||\varphi(t)||^2 + \frac{\epsilon}{2}\frac{d}{dt}||\xi(t)||^2 \leq C||\varphi(t)||^2. \tag{2.72}$$

Then, by Gronwall's inequality, we obtain

$$\frac{1}{2}||\varphi(T)||^2 + \frac{\epsilon}{2}||\xi(T)||^2 \leq C(||\varphi(t)||^2 + ||\xi(t)||^2), \tag{2.73}$$

where $C$ does not depend on $\epsilon$. Integrating from $T/4$ to $3T/4$, we get

$$\frac{1}{2}||\varphi(T)||^2 + \frac{\epsilon}{2}||\xi(T)||^2 \leq C\int_{T/4}^{3T/4}\int_\Omega (|\varphi(t)|^2 + |\xi(t)|^2)dxdt. \tag{2.74}$$

From the Carleman inequality given by Theorem 2.3 and Remark 2.1, we obtain the desired observability inequality:

$$\frac{1}{2}||\varphi(T)||^2 + \frac{\epsilon}{2}||\xi(T)||^2 \leq C\iint_{\omega_1\times(0,T)} |\varphi|^2 dxdt. \tag{2.75}$$

where $C$ does not depend on $\epsilon$.

Inequality (2.75) proves case 1 of Theorem 2.1. Using Theorem 2.4 we prove case 2 of Theorem 2.1 . In this way, the proof of Theorem 2.1 is established. $\square$

## 2.5 The heat equation with an inverse Laplacian

As explained in section 2.1, we can use Theorem 2.1 in order to obtain the null controllability of the parabolic-elliptic system (2.2). In this section we give a direct proof of this result in the simple case that $b = c = 1$ and $a = d = 0$, namely

$$\left| \begin{array}{ll} u_t - \Delta u + v = f1_{\omega_1} & \text{in } Q, \\ -\Delta v = u & \text{in } Q, \\ u = v = 0 & \text{on } \Sigma, \\ u(0) = u_0 & \text{in } \Omega. \end{array} \right. \tag{2.76}$$

The proof obtained through the passage to the limit in (2.1) is based on the use of Carleman inequalities for the adjoint system (2.10). However, a proof based on Carleman inequalities for the adjoint system of (2.76) is very hard to obtain. This can be easily seen by noticing that system (2.76) is equivalent to

$$\left| \begin{array}{ll} u_t - \Delta u + (-\Delta)^{-1} u = f 1_{\omega_1} & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(0) = u_0 & \text{in } \Omega. \end{array} \right. \tag{2.77}$$

Indeed, the adjoint system of (2.77) reads:

$$\left| \begin{array}{ll} -\varphi_t - \Delta \varphi + (-\Delta)^{-1} \varphi = 0 & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi_T. & \text{in } \Omega. \end{array} \right. \tag{2.78}$$

Applying the Carleman inequality given by Lemma 2.2 to (2.80), we obtain an extra term in the right-hand side of the inequality involving $(-\Delta)^{-1}\varphi$, this term must be absorbed, which is not straightforward. For this reason, we give an alternative proof of the null controllability of (2.77). Our proof is based on the Lebeau-Robiano strategy for the controllability of the heat equation (see [77]).

More precisely, we prove the following result.

**Theorem 2.5.** *Given $T > 0$ and $u_0 \in L^2(\Omega)$, there exists $f \in L^2(\omega_1 \times (0, T))$ such that the solution $(u, v)$ of (2.76) satisfies:*

$$u(T) = v(T) = 0.$$

*Moreover,*

$$||f||_{L^2(\omega_1 \times (0,T))} \leq C||u_0||_{L^2(\Omega)},$$

*for some $C > 0$.*

*Proof.* We consider $(e_n)_1^\infty$ the orthonormal basis of $L^2(\Omega)$ consisting of eigenvectors of the Laplacian, i.e., there exists a sequence $0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots$, such that

$$\left| \begin{array}{ll} -\Delta e_n = \lambda_n^2 e_n & \text{in } \Omega, \\ e_n = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{2.79}$$

Let $u_0 = \sum a^n e_n$ be in $L^2(\Omega)$. The solution of (2.77) can be written as $u(t, x) = \sum u^n(t) e_n(x)$, where $u^n$ solves

$$\left| \begin{array}{l} u_t^n + (\lambda_n^2 + \frac{1}{\lambda_n^2}) u_n = 0, \\ u_n(0) = a_n. \end{array} \right.$$

We have

$$u = \sum a_n e^{-(\lambda_n^2 + \frac{1}{\lambda_n^2})t} e_n.$$

We divide the proof into two steps.

**Step 1.** *Control of low frequencies.*

We want to find a control $f$ such that $u(T)$ is orthogonal to all eigenvalues $e_i$ for all $i$ such that $\lambda_i \leq \lambda$ or, equivalently, $< u(T), g >_{L^2(\Omega)} = 0$ for all $g = \sum_{\lambda_i \leq \lambda} g_i e_i$. We consider $\varphi$, the solution of

$$\left| \begin{array}{ll} -\varphi_t - \Delta\varphi + (-\Delta)^{-1}\varphi = 0 & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(T) = g & \text{in } \Omega, \end{array} \right. \tag{2.80}$$

and we define $F = \{\varphi|_{[0,T]\times\omega_1} \in L^2(\omega_1 \times (0,T)),\ \text{for } g = \sum_{\lambda_i \leq \lambda} g_i e_i\}$ and $K : F \longrightarrow \mathbb{R}$ given by $K\varphi = -\int_\Omega \varphi(0) u_0 dx$.

Let us assume that there exists a constant $C > 0$ such that

$$|K\varphi| \leq C||\varphi||_{L^2(\omega_1\times(0,T))}||u_0||_{L^2(\Omega)}. \tag{2.81}$$

Then, by Riesz Theorem, there exists $f \in L^2(\omega_1 \times (0,T))$ such that

$$K\varphi = -\int_\Omega \varphi(0) u_0 dx = \iint_{\omega_1\times(0,T)} f\varphi dx dt. \tag{2.82}$$

Using the duality between (2.77) and (2.80), we can show that

$$< u(T), g >_{L^2(\Omega)} = \iint_{\omega_1\times(0,T)} f\varphi dx + < u_0, \varphi(0) >_{L^2(\Omega)} . \tag{2.83}$$

From (2.82) and (2.83), we conclude that $u(T)$ is orthogonal to all $e_i$ such that $\lambda_i \leq \lambda$. Let us now prove (2.81). We need the following result.

**Proposition 2.1.** *There exists $C > 0$ such that for all $\lambda \geq 1$ and every function*

$$h = \sum_{\lambda_n \leq \lambda} b_n e_n$$

*the following inequality holds*

$$||h||_{L^2(\Omega)} \leq Ce^{C\lambda}||h|_{\omega_1}||_{L^2(\omega_1)}. \tag{2.84}$$

*Proof.* See [77]. □

Using Proposition 2.1, we now prove:

**Proposition 2.2.** *There exists $C > 0$ such that, for every $T > 0$, $\lambda \geq 1$ and $g = \sum_{\lambda_n \leq \lambda} g_n e_n$, we have*

$$||\varphi(0)||^2_{L^2(\Omega)} \leq \frac{Ce^{C\lambda}}{T} \iint_{\omega_1 \times (0,T)} |\varphi|^2 dxdt. \tag{2.85}$$

*Proof of Proposition 2.2.* We start observing that (2.85) is equivalent to

$$\begin{aligned} \sum_{\lambda_n \leq \lambda} |g_n|^2 e^{-2(\lambda_n^2 + 1/\lambda_n^2)T} &= || \sum_{\lambda_n \leq \lambda} g_n^{-(\lambda_n^2+1/\lambda_n^2)T} e_n||^2_{L^2(\Omega)} \\ &\leq \frac{Ce^{C\lambda}}{T} \iint_{\omega_1 \times (0,T)} | \sum_{\lambda_n \leq \lambda} e^{-(\lambda_n^2+1/\lambda_n^2)(T-s)} e_n| dxds. \end{aligned} \tag{2.86}$$

We apply (2.84) to $g_n^{-(\lambda_n^2+1/\lambda_n^2)(T-s)} e_n$, where $s \in [0, T]$ is a parameter. We get

$$|| \sum_{\lambda_n \leq \lambda} g_n^{-(\lambda_n^2+1/\lambda_n^2)(T-s)} e_n||^2_{L^2(\Omega)} \leq C^2 e^{2C\lambda} || \sum_{\lambda_n \leq \lambda} g_n^{-(\lambda_n^2+1/\lambda_n^2)(T-s)} e_n||^2_{L^2(\omega_1 \times (0,T))}. \tag{2.87}$$

Since $\sum_{\lambda_n \leq \lambda} |g_n|^2 e^{-2(\lambda_n^2+1/\lambda_n^2)(T-s)} \geq \sum_{\lambda_n \leq \lambda} |g_n|^2 e^{-2(\lambda_n^2+1/\lambda_n^2)T}$ and from (2.87), integration in time gives

$$T|| \sum_{\lambda_n \leq \lambda} g_n^{-(\lambda_n^2+1/\lambda_n^2)T} e_n||^2_{L^2(\Omega)} \leq C^2 e^{2C\lambda} \int_0^T || \sum_{\lambda_n \leq \lambda} g_n^{-(\lambda_n^2+1/\lambda_n^2)(T-s)} e_n||^2_{L^2(\omega_1 \times (0,T))},$$

which is exactly (2.85).

From Proposition 2.2, inequality (2.81) follows. We also have

$$||f||_{L^2(\omega_1 \times (0,T))} \leq \frac{Ce^{C\lambda}}{T} ||u_0||_{L^2(\Omega)} \tag{2.88}$$

and

$$||u(t,.)||_{L^2(\Omega)} \leq Ce^{C\lambda} ||u_0||_{L^2(\Omega)}, \tag{2.89}$$

for all $t \in [0, T]$.

□

**Step 2.** *Control of high frequencies.*

We start observing that, if $u_0 = \sum_{\lambda_n > \lambda} a^n e_n$, then

$$||u(t)||_{L^2(\Omega)} = || \sum_{\lambda_n>\lambda} a^n e^{-(\lambda_n^2+1/\lambda_n^2)t} e_n ||_{L^2(\Omega)} = \left( \sum_{\lambda_n>\lambda} |a^n|^2 e^{-2(\lambda_n^2+1/\lambda_n^2)t} \right)^{1/2}$$
$$\leq e^{-(\lambda_n^2+1/\lambda_n^2)t} \left( \sum_{\lambda_n>\lambda} |a^n|^2 \right)^{1/2} \leq e^{-(\lambda_n^2+1/\lambda_n^2)t} ||u_0||_{L^2(\Omega)}. \tag{2.90}$$

Let $A = \sum_{n=1}^{\infty} 2^{-n\alpha}$, where $\alpha > 0$ will be chosen later. We consider $T_m = \frac{T}{A} \sum_{n=1}^{m} 2^{-n\alpha}$ so that the sequence $(T_m)_1^\infty$ is crescent and converges to $T$. We define $T'_m = (T_m + T_{m+1})/2$. We have $T_{m+1} - T_m = \frac{T}{A} 2^{-(m+1)\alpha}$ and $T'_m - T_m = T_{m+1} - T'_m = \frac{T}{A 2^{1+\alpha}} 2^{-m\alpha}$.

As in the case of the single heat equation, we construct a control by recurrence, controlling the frequencies lower than $2^m$ in the time interval $[T_m, T'_m]$ then letting system evolve freely between $[T'_m, T_{m+1}]$.

In the rest of the proof we will use $u_m(.) = u(T_m, .)$ and, for $t \in [T_m, T_{m+1}]$, we write $u(t, .) = u_m(t - T_m, .)$.

From step 1, there exists $f_m \in L^2(\omega_1 \times (0, T'_m - T_m))$ such that

$$||f_m||_{L^2(\omega_1 \times (0, T'_m - T_m))} \leq C'_1 2^{m\alpha/2} e^{C'_1 2^m} ||u_m||_{L^2(\Omega)} \leq C_1 e^{C_1 2^m} ||u_m||_{L^2(\Omega)} \tag{2.91}$$

and

$$||u(T'_m, .)||_{L^2(\Omega)} \leq C_2 e^{C_2 2^m} ||u_m||_{L^2(\Omega)}, \tag{2.92}$$

for sufficiently large $C_1$ and $C_2$.

We consider the system without control in $[T'_m, T_{m+1}]$, i.e., $f \equiv 0$, and using the dissipation of the system, we get

$$||u_{m+1}||_{L^2(\Omega)} \leq e^{(T_{m+1} - T'_m) 2^{2m}} ||u(T'_m, .)||_{L^2(\Omega)} \leq C_3 e^{-C_4 2^{-m\alpha + 2m}} e^{C_2 2^m} ||u_m||_{L^2(\Omega)}$$
$$C_5 e^{-C_6 2^{m(2-\alpha)}} ||u_m||_{L^2(\Omega)}. \tag{2.93}$$

In fact, if $0 < \alpha < 1$, then there exists $C_6 > 0$ such that for all $m \in \mathbb{N}$, $-C_4 2^{-m\alpha+2m} + C_2 2^m \leq -C_6 2^{m(2-\alpha)}$.

By recurrence, there exists $C_7, C_8 > 0$ such that for all $m \in \mathbb{N}$

$$||u_m||_{L^2(\Omega)} \leq C_7 e^{-C_8 2^{m(2-\alpha)}} ||u_0||_{L^2(\Omega)}. \tag{2.94}$$

From (2.89), there exists $C_9, C_{10} > 0$ such that for all $t \in [T_m, T_{m+1}]$ we have

$$||u(t, .)||_{L^2(\Omega)} \leq C_9 e^{C_9 2^m} ||u_m||_{L^2(\Omega)} \leq C_{10} e^{-C_6 2^{m(2-\alpha)} + C_9 2^m} ||u_0||_{L^2(\Omega)}. \tag{2.95}$$

In particular, $u(t,.) \longrightarrow 0$ in $L^2(\Omega)$, when $t \longrightarrow T$, since, for $t \in [T_m, T_{m+1}]$, $T - t \sim 2^{-m\alpha}$.

From (2.91) and (2.94), there exists $C_{11}, C_{12}, C_{13} > 0$ such that

$$\begin{aligned} ||f_m||_{L^2(\omega_1 \times (0,T'_m - T_m))} &\leq C_1 e^{C_1 2^m} ||u_m||_{L^2(\Omega)} \\ &\leq C_{11} e^{C_1 2^m} e^{C_8 2^{m(2-\alpha)}} ||u_0||_{L^2(\Omega)} \\ &\leq C_{12} e^{-C_{13} 2^{m(2-\alpha)}} ||u_0||_{L^2(\Omega)}. \end{aligned} \tag{2.96}$$

The control over the whole time interval $[0,T]$ will be then define by $f(t,x) = \sum_{m=0}^{\infty} 1_{[T_m, T'_m]} f_m(t - T_m, x)$. It is not difficult to see that there exists $C_{14} > 0$ such that

$$\begin{aligned} ||f||_{L^2(\omega_1 \times (0,T))} &\leq \sum_{m=0}^{\infty} ||f_m(t - T_m, x)||_{L^2(\omega_1 \times (0,T'_m - T_m))} \\ &\leq C_{12} \sum_{m=0}^{\infty} e^{-C_{13} 2^{m(2-\alpha)}} ||u_0||_{L^2(\Omega)} \\ &\leq C_{14} ||u_0||_{L^2(\Omega)}. \end{aligned} \tag{2.97}$$

Therefore, proof of Theorem 2.5 is finished.

□

## 2.6 Regular controls for the heat equation

In this section we use the results obtained in previous sections in order to obtain more regular controls for the heat equation.

We consider

$$\left| \begin{array}{ll} u_t(x,t) - \Delta u(x,t) = f 1_{\omega_1} & \text{in } Q, \\ u(x,t) = 0 & \text{on } \Sigma, \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{array} \right. \tag{2.98}$$

It is well-known that null controllability for (2.98) is equivalent to an observability inequality for its adjoint equation

$$\left| \begin{array}{ll} -\varphi_t(x,t) - \Delta \varphi(x,t) = 0 & \text{in } Q, \\ \varphi(x,t) = 0 & \text{on } \Sigma, \\ \varphi(x,T) = \varphi_T(x) & \text{in } \Omega. \end{array} \right. \tag{2.99}$$

More precisely, the following inequality is required

$$||\varphi(0)||^2_{L^2(\Omega)} \leq C \iint_{\omega_1\times(0,T)} |\varphi|^2 dxdt. \tag{2.100}$$

Indeed, if inequality (2.100) holds, we define the functional

$$J_\epsilon(\varphi_T) = \frac{1}{2}\iint_{\omega_1\times(0,T)} |\varphi|^2 dxdt + \epsilon||\varphi(T)||_{L^2(\Omega)} + < \varphi(0), u(0) > \tag{2.101}$$

and the duality between (2.98) and (2.99) allow us to show that, for each $\epsilon > 0$, there exists a minimizer $(\varphi^\epsilon_T)$ for (2.101). It is an easy matter to show that $\varphi^\epsilon$, the solution of (2.99) associated to this minimizer, is a control for the heat equation (2.98) such that

$$||u^\epsilon(T)||_{L^2(\Omega)} \leq \epsilon. \tag{2.102}$$

It is also not difficult to show that $\varphi^\epsilon$ is bounded, with respect to $\epsilon$, in $L^2(\omega_1 \times (0,T))$. Therefore, it must converges weakly to some function $\varphi$ in $L^2(\omega_1 \times (0,T))$. Considering this limit function $\varphi$ as a control for (2.98), we conclude that

$$u(T) = 0,$$

where $u$ is the solution to (2.98) associated to a control $\varphi$ belonging to $L^2(\omega_1 \times (0,T))$. Hence, in some sense, the control for the heat equation is a solution to the backward heat equation, i.e., the solution of a parabolic equation.

We now give a way to obtain a control to the heat equation that minimizes some functional like (2.101) in such a way that the control obtained solves an elliptic equation, hence, being more regular in space.

Consider the system

$$\left|\begin{array}{ll} u_t - \Delta u = v1_{\omega_2} & \text{in } Q, \\ \epsilon v_t - \Delta v = f1_{\omega_1} & \text{in } Q, \\ u = v = 0 & \text{on } \Sigma, \\ u(0) = u_0;\ v(0) = v_0 & \text{in } \Omega. \end{array}\right. \tag{2.103}$$

with $\epsilon > 0$ some $v_0 \in L^2(\Omega)$ and a nonempty subset $\omega_2$ of $\Omega$ such that $\omega_2 \cap \omega_1 \neq \emptyset$.

The adjoint system of (2.103) is given by

$$\left|\begin{array}{ll} -\varphi_t - \Delta\varphi = 0 & \text{in } Q, \\ -\epsilon\xi_t - \Delta\xi = \varphi 1_{\omega_2} & \text{in } Q, \\ \varphi = \xi = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi_0;\ \xi(T) = \xi_0 & \text{in } \Omega. \end{array}\right. \tag{2.104}$$

The approximate controllability for (2.103) is equivalent to minimize the functional

$$J_\delta(\varphi_T, \xi_T) = \frac{1}{2}\iint_{\omega_1\times(0,T)} |\xi|^2 dxdt + \delta(||\varphi_T||_{L^2(\Omega)} + ||\xi_T||_{L^2(\Omega)}) \\ + < \varphi(0), u(0) > +\epsilon < \xi(0), v(0) > . \tag{2.105}$$

We can take the limit when $\delta \to 0$ and obtain the null controllability of (2.103). Using Theorem 2.1, we can take the limit when $\epsilon \to 0^+$ and obtain the following null controllable system

$$\left|\begin{array}{ll} u_t - \Delta u = v 1_{\omega_2} & \text{in } Q, \\ -\Delta v = f 1_{\omega_1} & \text{in } Q, \\ u = v = 0 & \text{on } \Sigma, \\ u(0) = u_0 & \text{in } \Omega. \end{array}\right. \tag{2.106}$$

Hence, the control of the heat equation can be taken of the form $(-\Delta)^{-1} f 1_{\omega_1}$, where $f 1_{\omega_1} \in L^2(Q)$.

In the same way, if we consider the system

$$\left|\begin{array}{ll} \partial_t y_1 - \Delta y_1 = y_2 1_{\omega_2} & \text{in } Q, \\ \epsilon\partial_t y_2 - \Delta y_2 = y_3 1_{\omega_3} & \text{in } Q, \\ \epsilon\partial_t y_3 - \Delta y_3 = y_4 1_{\omega_1} & \text{in } Q, \\ \vdots & \\ \epsilon\partial_t y_{m-1} - \Delta y_{m-1} = y_m 1_{\omega_{m-1}} & \text{in } Q, \\ \epsilon\partial_t y_m - \Delta y_m = f 1_{\omega_1} & \text{in } Q, \\ y_j(0) = y_{0,j} & \text{in } \Omega, \\ y_j = 0 & \text{on } \Sigma,\ 1 \le j \le m, \end{array}\right. \tag{2.107}$$

for some $y_{0,j} \in L^2(\Omega)$ and $\big(\cap_j \omega_j\big) \cap \omega_1 \neq \emptyset$, where $\omega_j \subset \Omega$ for $1 \le j \le m$. As before, we can show that there exists a control that drives the solution of the heat equation (2.98) to rest and is of the form $(-\Delta)^{-(m-1)} f 1_{\omega_1}$, for all $m \in \mathbb{N}$, where $f 1_{\omega_1} \in L^2(Q)$.

# Chapter 3

# Uniform null controllability for a degenerating reaction-diffusion system approximating a simplified cardiac model

## 3.1 Introduction

Let $T > 0$, $\Omega \subset \mathbb{R}^N (N = 1, 2, 3)$ be a bounded connected open set whose boundary $\partial\Omega$ is regular enough and $\omega_1$ and $\omega_2$ be two nonempty subsets of $\Omega$.

In this chapter we study the properties of controllability and observability for a family of reaction-diffusion systems which degenerates into a parabolic-elliptic system describing the cardiac electric activity in $\Omega$ ($\Omega \subset \mathbb{R}^3$ being the natural domain of the heart).

To state the model, we let $u_i = u_i(t, x)$ and $u_e = u_e(t, x)$ represent the *intracellular* and *extracellular* electric potentials, respectively. Their difference, $v = u_i - u_e$ is called the *transmembrane* potential. The anisotropic properties of the media are modeled by intracellular and extracellular conductivity tensors $\mathbf{M}_i(x)$ and $\mathbf{M}_e(x)$.

The system governing the electrical activity in the cardiac tissue reads as follows:

$$\begin{cases} c_m v_t - Div(\mathbf{M}_i(x)\nabla u_i) + h(v) = f 1_{\omega_1} & \text{in } Q, \\ c_m v_t + Div(\mathbf{M}_e(x)\nabla u_e) + h(v) = g 1_{\omega_2} & \text{in } Q, \end{cases} \tag{3.1}$$

where $c_m > 0$ is the surface capacitance of the membrane, the nonlinear function $h : \mathbb{R} \to \mathbb{R}$ is the transmembrane ionic current (the most interesting case being when $h$ is cubic polynomial) and $f$ and $g$ are stimulation currents applied to $\omega_1$ and $\omega_2$, respectively.

System (3.1) is known as the *bidomain model* and is completed with Dirichlet boundary conditions for the intra- and extracellular electric potentials

$$u_i = u_e = 0 \qquad \text{on } \Sigma \tag{3.2}$$

and with initial data for the transmembrane potential

$$v(0,x) = v_0(x), \quad x \in \Omega. \tag{3.3}$$

We point out that realistic models describing electrical activities in the heart include a system of ODE's for computing the ionic current as a function of the transmembrane potential and a serie of additional "gating variables" aiming to model the ionic transfer across the cell membrane (see [63, 65, 93, 103]).

In the case where $f1_{\omega_1} = g1_{\omega_2}$ and $\mathbf{M}_i = \mu\mathbf{M}_e$ for some constant $\mu \in \mathbb{R}$, the bidomain model is simplified into the following parabolic-elliptic system:

$$\left| \begin{array}{ll} c_m v_t - \frac{\mu}{\mu+1} Div\big(\mathbf{M}_e(x)\nabla v\big) + h(v) = f1_{\omega_1} & \text{in } Q, \\ -Div\big(\mathbf{M}(x)\nabla u_e\big) = Div\big(\mathbf{M}_i(x)\nabla v\big) & \text{in } Q, \\ v = u_e = 0 & \text{on } \Sigma, \\ v(0) = v_0 & \text{in } \Omega, \end{array} \right. \tag{3.4}$$

where $M = M_i + M_e$.

System (3.4) is known as *monodomain model* and is a very interesting model from the implementation point of view, since it conserves some of the essential features of the bidomain model as excitability phenomena (see [23, 74, 126]).

The main difference between the bidomain model (3.1) and the monodomain model (3.4) is the fact that the first model is a system of two coupled parabolic equations while the second one is a system of parabolic-elliptic type. Therefore, from the control point of view, one could expect these two systems to have, at least a priori, different control properties. In this work we show that, actually, the properties of controllability and observability for the monodomain model can be seen as a limit process of the controllability properties of a family of coupled parabolic systems.

Given any $\epsilon$ such that $0 < \epsilon < 1/2$, we approximate the monodomain model by the following family of parabolic systems:

$$\left| \begin{array}{ll} c_m v^\epsilon_t - \frac{\mu}{\mu+1} Div\big(\mathbf{M}_e(x)\nabla v^\epsilon\big) + h(v^\epsilon) = f^\epsilon 1_{\omega_1} & \text{in } Q, \\ \epsilon u^\epsilon_{e,t} - Div\big(\mathbf{M}(x)\nabla u^\epsilon_e\big) = Div\big(\mathbf{M}_i(x)\nabla v^\epsilon\big) & \text{in } Q, \\ v^\epsilon = u^\epsilon_e = 0 & \text{on } \Sigma, \\ v^\epsilon(0) = v_0;\ u^\epsilon_e(0) = u_{e,0} & \text{in } \Omega. \end{array} \right. \tag{3.5}$$

The aim of this chapter is to give a positive answer to the following question:

**Question 3.1.** If, for each $\epsilon > 0$, there exists a control $f^\epsilon$ that drives the solution $(v^\epsilon, u_e^\epsilon)$ of (3.5) to zero at time $t = T$, i.e.

$$v^\epsilon(T) = u_e^\epsilon(T) = 0,$$

is it true that when $\epsilon \to 0^+$ the control sequence $f^\epsilon$ converges to a function $f$, that drives the solution $(v, u_e)$ of (3.4) to zero at time $t = T$?

In order to answer the previous question, we consider the following linearized version of (3.5):

$$\left| \begin{array}{ll} c_m v_t^\epsilon - \frac{\mu}{\mu+1} Div\big(\mathbf{M}_e(x)\nabla v^\epsilon\big) + a(t,x)v^\epsilon = f^\epsilon 1_{\omega_1} & \text{in } Q, \\ \epsilon u_{e,t}^\epsilon - Div\big(\mathbf{M}(x)\nabla u_e^\epsilon\big) = Div\big(\mathbf{M}_i(x)\nabla v^\epsilon\big) & \text{in } Q, \\ v^\epsilon = u_e^\epsilon = 0 & \text{on } \Sigma, \\ v^\epsilon(0) = v_0;\ u_e^\epsilon(0) = u_{e,0} & \text{in } \Omega, \end{array} \right. \tag{3.6}$$

where $a$ is a bounded function.

Given $\varepsilon > 0$, the first obstacle to answering, positively, Question 3.1, will be to drive $(v^\varepsilon, u_e^\varepsilon)$, solution of (3.6), to zero at time $T$ by means of a control $f^\varepsilon$ in such a way that the sequence of controls $\{f^\varepsilon\}_{\varepsilon>0}$ converges when $\varepsilon \to 0^+$. Once showed that such a convergent sequence of control $\{f^\varepsilon\}_{\varepsilon>0}$ for the linear system (3.6) exists, we use a fixed point argument to conclude that the same is true for the nonlinear system (3.5).

Thus, we introduce the adjoint system of (3.6):

$$\left| \begin{array}{ll} -c_m \varphi_t^\epsilon - \frac{\mu}{\mu+1} Div(M_e(x)\nabla\varphi^\epsilon) + a(t,x)\varphi^\epsilon = Div(M_i(x)\nabla\varphi_e^\epsilon) & \text{in } Q, \\ -\epsilon\varphi_{e,t}^\epsilon - Div(M(x)\nabla\varphi_e^\epsilon) = 0 & \text{in } Q, \\ \varphi^\epsilon = \varphi_e^\epsilon = 0 & \text{on } \Sigma, \\ \varphi^\epsilon(T) = \varphi_T;\ \varphi_e^\epsilon(T) = \varphi_{e,T} & \text{in } \Omega. \end{array} \right. \tag{3.7}$$

Using duality arguments, it is very easy to prove that the task of building such a convergent sequence of controls $\{f^\varepsilon\}_{\varepsilon>0}$ for (3.6) is equivalent to prove the following (uniform) observability inequality for the solutions of (3.7):

$$\epsilon||\varphi_e^\epsilon(0)||^2_{L^2(\Omega)} + ||\varphi^\epsilon(0)||^2_{L^2(\Omega)} \le C \iint_{\omega_1\times(0,T)} |\varphi^\epsilon|^2 dxdt, \tag{3.8}$$

where $(\varphi_T, \varphi_{e,T}) \in L^2(\Omega)^2$ and a constant $C = C(\epsilon, \Omega, \omega_1, T)$ remains bounded when $\epsilon \to 0^+$.

We prove inequality (3.8) as a consequence of an appropriate Carleman inequality for the solution $(\varphi^\epsilon, \varphi_e^\epsilon)$ of (3.7) (see section 3.3). We notice that, due to the fact that the control is acting on the first equation of (3.6), in our Carleman inequality, we need to bound global integrals of $\varphi^\epsilon$ and $\varphi_e^\epsilon$ in terms of a local integral of $\varphi^\epsilon$, uniformly with respect to $\varepsilon$. Two main difficulties appear: first, the coupling in the first equation is in $Div(M_i(x)\nabla\varphi_e^\epsilon)$ and not in $\varphi_e^\epsilon$; second, we must to show that the constant we get in our Carleman inequality does not blow up when $\varepsilon \to 0^+$.

In chapter 2 we have discussed about some works devoted to the controllability of linear and semilinear parabolic equations. However, in what concerns to the controllability of the bidomain model, since in both equations the couplings are given by the time derivatives of the electrical potentials, it seems very difficult to analyze the controllability properties of such model. To our best knowledge, the problems of null and approximate controllability for the bidomain model (3.1) are still open (even with two controls). Regarding the null controllability of the monodomain model (3.4), since the solution of the equation enters as a source term in the elliptic one, the following controllability result holds:

**Theorem 3.1.** *Let $q_N \in \mathbb{R}$ be such that $\frac{5}{2} < q_N < 10$ if $N = 3$ and $q_N \in (2, \infty)$ if $N = 1, 2$.*

*(i) Assume $h$ is $C^1(\mathbb{R})$, globally Lipschitz and $h(0) = 0$. Given $v_0 \in L^2(\Omega)$, there exists a control $f \in L^2(\omega_1 \times (0,T))$ such that the solution $(v, u_e)$ of (3.4) satisfies:*

$$v(T) = u_e(T) = 0.$$

*Moreover, the control $f$ satisfies the estimate:*

$$||f 1_{\omega_1}||^2_{L^2(Q)} \leq C||v_0||^2_{L^2(\Omega)}, \tag{3.9}$$

*for a constant $C = C(\Omega, \omega_1, T) > 0$.*

*(ii) Assume $h$ is $C^1(\mathbb{R})$, $h(0) = 0$ and*

$$\frac{h(v_1) - h(v_2)}{v_1 - v_2} \geq -C, \quad \forall v_1 \neq v_2, \tag{3.10}$$

$$0 < \liminf_{|v|\to\infty} \frac{h(v)}{v^3} \leq \limsup_{|v|\to\infty} \frac{h(v)}{v^3} < \infty. \tag{3.11}$$

*If $v_0 \in H_0^1(\Omega) \cap W^{2(1-\frac{1}{q_N}),q_N}(\Omega)$, with $||v_0||_{L^\infty} \leq \gamma$, for sufficiently small $\gamma$, there exists a control $f \in L^{q_N}(\omega_1 \times (0,T))$ such that the solution $(v, u_e)$ of (3.4), with $(v, u_e) \in W^{2,1}_{q_N}(Q)^2$, satisfies:*

$$v(T) = u_e(T) = 0.$$

*Moreover, the control $f$ satisfies the estimate:*

$$||f1_{\omega_1}||^2_{L^{q_N}(Q)} \leq C||v_0||^2_{L^2(\Omega)}, \tag{3.12}$$

*for a constant $C = C(\Omega, \omega_1, T) > 0$.*

Case 1 of Theorem 3.1 follows from [46, Theorem 3.1] and case 2 follows from [48, Theorem 3.5] (see also [46, Theorem 4.2]).

This chapter is organized as follows. In section 3.2, we state our main results. In section 3.3, we prove a uniform Carleman inequality for the adjoint system (3.7). In section 3.4, we show the uniform null controllability for (3.6). In section 3.5 we deal with the uniform null controllability of the nonlinear system (3.5).

## 3.2 Main results

Throughout this chapter, we will assume that the matrices $M_j, j = i, e$ are $C^\infty$, bounded, symmetric and positive semidefinite.

The following existence theorem holds.

**Theorem 3.2.** *Suppose $h$ satisfies (3.10)-(3.11). If $(v_0, u_{0,e}) \in L^2(\Omega)^2$ and $f \in L^2(Q)$, then system (3.5) has a unique weak solution $(v^\epsilon, u_e^\epsilon)$ and $(v^\epsilon, u_e^\epsilon) \in L^2(0,T;H_0^1(\Omega)) \times L^2(0,T;H_0^1(\Omega))$ such that $v_t^\epsilon$ and $\epsilon u_{e,t}^\epsilon$ belong to $L^2(0,T,H^{-1}(\Omega)) + L^{4/3}(Q)$ and $L^2(0,T,H^{-1}(\Omega))$.*

The proof of this Theorem is performed exactly as in [7] and, being far from the aim of this thesis, it will be omitted.

The first main result of this chapter is a uniform Carleman estimate for the adjoint system (3.7).

**Theorem 3.3.** *Given $0 < \epsilon < 1/2$, there exist positive constants $C = C(\Omega, \omega_1)$, $\lambda_0 = \lambda_0(\Omega, \omega_1)$ and $s_0 = s_0(\Omega, \omega_1)$ such that, for every $(\varphi_T, \varphi_{e,T}) \in L^2(\Omega)^2$ and every $a \in L^\infty(Q)$, the solution $(\varphi^\epsilon, \varphi_e^\epsilon)$ of (3.7) satisfies:*

$$\iint_Q e^{3s\alpha}|\rho^\epsilon|^2 dxdt + s^3\lambda^4 \iint_Q \phi^3 e^{3s\alpha}|\varphi^\epsilon|^2 dxdt \leq Ce^{6\lambda||\psi||_\infty} s^8\lambda^4 \iint_{\omega_1\times(0,T)} \phi^8 e^{2s\alpha}|\varphi^\epsilon|^2 dxdt, \tag{3.13}$$

*where $\rho^\epsilon(x,t) = Div(M(x)\nabla\varphi_e^\epsilon(x,t))$, for every $s \geq (T + (1 + ||a||_\infty^{2/3})T^2)s_0$, $\lambda \geq \lambda_0$ and appropriate weight functions $\phi$ and $\alpha$ defined in (3.19) and (3.20), respectively.*

The proof of Theorem 3.3 is given in section 3.3 .

**Remark 3.1.** As a direct consequence of the Carleman inequality (3.13) we have, for any $\varepsilon > 0$, the unique continuation property

"if $(\varphi^\epsilon, \varphi_e^\epsilon)$, solution of (3.7) satisfies $\varphi^\epsilon = 0$ in $\omega_1 \times (0,T)$ then $(\varphi^\epsilon, \varphi_e^\epsilon) \equiv (0,0)$ in $Q$".

This unique continuation property for the adjoint problem (3.7) implies the approximate controllability, at time $T$, of system (3.6), for any $\epsilon > 0$, with a control acting only in the first equation.

Our second main result gives the null controllability of the linear system (3.6).

**Theorem 3.4.** *Given any $(v_0, u_{e,0}) \in L^2(\Omega)^2$ and any $0 < \epsilon < 1/2$, there exists a control $f^\epsilon \in L^2(\omega_1 \times (0,T))$ such that the associated solution of (3.6) is driven to zero at time $T$. That is to say, the associated solution satisfies:*

$$v^\epsilon(T) = 0, \; u_e^\epsilon(T) = 0.$$

*Moreover, the control $f^\epsilon$ satisfies the estimate:*

$$||f^\epsilon 1_{\omega_1}||^2_{L^2(Q)} \le C\big(||v_0||^2_{L^2(\Omega)} + \epsilon||u_{e,0}||^2_{L^2(\Omega)}\big), \tag{3.14}$$

*for a constant $C = C(\Omega, \omega_1, ||a||_{L^\infty}, T) > 0$.*

From Theorem 3.2, the proof of Theorem 3.4 is standard. However, for the sake of completeness, we prove Theorem 3.4 in section 3.4.

The third main result of this chapter is concerned with the uniform null controllability of the nonlinear parabolic system (3.5).

**Theorem 3.5.** *Let $q_N \in \mathbb{R}$ be as in Theorem 3.1 and $0 < \epsilon < 1/2$.*

*(i) Assume $h$ is $C^1(\mathbb{R})$, globally Lipschitz and $h(0) = 0$. Given $(v_0, u_{e,0}) \in L^2(\Omega)^2$, there exists a control $f^\epsilon \in L^2(\omega_1 \times (0,T))$ such that the solution $(v^\epsilon, u_e^\epsilon)$ of (3.5) satisfies:*

$$v^\epsilon(T) = u_e^\epsilon(T) = 0.$$

*Moreover, the control $f^\epsilon$ satisfies the estimate:*

$$||f^\epsilon 1_{\omega_1}||^2_{L^2(Q)} \le C\big(||v_0||^2_{L^2(\Omega)} + \epsilon||u_{e,0}||^2_{L^2(\Omega)}\big), \tag{3.15}$$

*for a constant $C = C(\Omega, \omega_1, ||a||_{L^\infty}, T) > 0$.*

*(ii) Assume $h$ is $C^1(\mathbb{R})$ and satisfies (3.10)-(3.11). If $(v_0, u_{e,0}) \in \big(H_0^1(\Omega) \cap W^{2(1-\frac{1}{q_N}),q_N}(\Omega)\big)^2$, with $||(v_0, u_{e,0})||_{L^\infty} \le \gamma$, for sufficiently small $\gamma$ does not depending on $\epsilon$, there exists a*

*control* $f^\epsilon \in L^{q_N}(\omega_1 \times (0,T))$ *such that the solution* $(v^\epsilon, u_e^\epsilon)$ *of (3.5), with* $(v^\epsilon, u_e^\epsilon) \in W^{2,1}_{q_N}(Q)^2$*, satisfies*

$$v^\epsilon(T) = u_e^\epsilon(T) = 0.$$

*Moreover, the control* $f^\varepsilon$ *satisfies the estimate:*

$$||f^\varepsilon 1_{\omega_1}||^2_{L^{q_N}(Q)} \leq C\big(||v_0||^2_{L^2(\Omega)} + \epsilon||u_{e,0}||^2_{L^2(\Omega)}\big), \tag{3.16}$$

*for a constant* $C = C(\Omega, \omega_1, ||a||_{L^\infty}, T) > 0$.

The proof of Theorem 3.5 is achieved applying fixed point arguments, and it will be done in section 3.5.

**Remark 3.2.** In our analysis, we restrict the dimension to $N = 1, 2, 3$, because the bidomain model makes sense only in such dimensions. Nevertheless, from the mathematical point of view, systems (3.4), (3.5) and (3.6) also make sense for any $N \in \mathbb{N}$ (the 1-d case corresponding to the cable equation) and, with the appropriate adaptations (when necessary), all the results of this chapter still hold for higher dimensions.

## 3.3 Carleman inequality

In this section we prove Theorem 3.3.

To simplify the notation, we neglect the index $\epsilon$ and, since the only constant which matters in our analysis is $\epsilon$, we assume that all other constants are normalized to be the unity. In this case the adjoint system (3.7) reads:

$$\left|\begin{array}{ll} -\varphi_t - Div(M_e(x)\nabla\varphi) + a(x,t)\varphi = Div(M_i(x)\nabla\varphi_e) & \text{in } Q, \\ -\epsilon\varphi_{e,t} - Div(M(x)\nabla\varphi_e) = 0 & \text{in } Q, \\ \varphi = \varphi_e = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi_T; \varphi_e(T) = \varphi_{e,T} & \text{in } \Omega. \end{array}\right. \tag{3.17}$$

We notice that, if $\varphi_T$ and $\varphi_{e,T}$ are regular enough, taking $\rho(x,t) = Div(M_i(x)\nabla\varphi_e(x,t))$, the pair $(\varphi, \rho)$ satisfies:

$$\left|\begin{array}{ll} -\varphi_t - Div(M_e(x)\nabla\varphi) + a(x,t)\varphi = \rho & \text{in } Q, \\ -\epsilon\rho_t - Div(M(x)\nabla\rho) = 0 & \text{in } Q, \\ \varphi = \rho = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi_T; \rho(T) = \rho_T & \text{in } \Omega. \end{array}\right. \tag{3.18}$$

We prove the Carleman inequality (3.13) using system (3.18).

Before starting the proof of the Carleman inequality, we introduce the following weights (which are slightly different from those used in chapter 2),

$$\phi(x,t)=\frac{e^{\lambda(\psi(x)+m||\psi||_\infty)}}{t(T-t)};\quad \phi^*(t)=\min_{x\in\overline{\Omega}}\phi(x,t)=\frac{e^{\lambda m||\psi||_\infty}}{t(T-t)}; \tag{3.19}$$

$$\alpha(x,t)=\frac{e^{\lambda(\psi(x)+m||\psi||_\infty)}-e^{2\lambda m||\psi||_\infty}}{t(T-t)};\quad \alpha^*(t)=\max_{x\in\overline{\Omega}}\alpha(x,t)=\frac{e^{\lambda(m+1)||\psi||_\infty}-e^{2\lambda m||\psi||_\infty}}{t(T-t)}, \tag{3.20}$$

for a parameter $\lambda>0$ and a constant $m>1$.

**Remark 3.3.** From the definition of $\alpha$ and $\alpha^*$ it follows that $3\alpha^*\le 2\alpha$ (for $\lambda$ large enough!). Moreover

$$\phi^*(t)\le\phi(x,t)\le e^{\lambda||\psi||}\phi^*(x,t)$$

and

$$|\alpha_t^*|\le e^{2\lambda||\psi||}T\phi^2.$$

The following Carleman inequality holds.

**Lemma 3.1.** *Let $\beta\in\{0,1\}$, $0<\sigma\le 1$, $\omega$ be a nonempty subset of $\Omega$ and $\omega_0\subset\subset\omega$. There exists a constant $\lambda_0=\lambda_0(\Omega,\omega)\ge 1$ such that for every $\lambda\ge\lambda_0$, there exists $s_0=s_0(\Omega,\omega,\lambda)$ and $C=C(\Omega,\omega)$ such that, for every $s\ge s_0(T+T^2)$, the following inequality holds:*

$$\begin{aligned}
&s^{\beta-1}\iint_Q e^{2s\alpha}\phi^{\beta-1}(\sigma^2|q_t|^2+\sum_{i,j=1}^N|\frac{\partial^2 q}{\partial x_i\partial x_j}|^2)dxdt\\
&+s^{\beta+1}\lambda^2\iint_Q e^{2s\alpha}\phi^{\beta+1}|\nabla q|^2dxdt+s^{\beta+3}\lambda^4\iint_Q e^{2s\alpha}\phi^{\beta+3}|q|^2dxdt\\
&\le Ce^{\beta\lambda||\psi||}\left(s^\beta\iint_Q e^{2s\alpha}\phi^\beta|\sigma\partial_t q+\sum_{i,j=1}^N\frac{\partial^2 q}{\partial x_i\partial x_j}|^2dxdt+s^{\beta+3}\lambda^4\iint_{\omega\times(0,T)}e^{2s\alpha}\phi^{\beta+3}|q|^2dxdt\right),
\end{aligned}$$

*for all $q\in C^2(\overline{Q})$, with $q=0$ on $\Sigma$.*

*Proof.* See appendix A. □

*Proof of Theorem 3.3 .* For an easier comprehension, we divide the proof into several steps:

***Step 1.*** *First estimate for the parabolic system.*

In this step we obtain a first Carleman estimate for the adjoint system. We will use some sharp Carleman inequalities, with respect to $\epsilon$, for the equation and get a global estimation of $\varphi$ and $\rho$ in terms of a local integral of $\varphi$ and another of $\rho$.

First, we consider a set $\omega'$ such that $\omega_0 \subset\subset \omega' \subset\subset \omega_1$ and apply the Carleman inequality given by Lemma 3.1, with $\sigma = 1$, $\beta = 0$ and $\sigma = \epsilon$, $\beta = 1$ to $\varphi$ and $\rho$, respectively, to obtain

$$\begin{aligned}
&\iint_Q s^{-1}\phi^{-1}e^{2s\alpha}|\varphi_t|^2 dxdt + s^{-1}\iint_Q \phi^{-1}e^{2s\alpha}\sum_{i,j=1}^{N}|\frac{\partial^2}{\partial x_i x_j}\varphi|^2 dxdt \\
&+ s^3\lambda^4\iint_Q \phi^3 e^{2s\alpha}|\varphi|^2 dxdt + s\lambda^2\iint_Q \phi e^{2s\alpha}|\nabla\varphi|^2 dxdt \\
&\quad \leq C\Big(\iint_Q e^{2s\alpha}(|\rho|^2+|\varphi|^2)dxdt + s^3\lambda^4\iint_{\omega'\times(0,T)}\phi^3 e^{2s\alpha}|\varphi|^2 dxdt\Big)
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
&\iint_Q e^{2s\alpha}|\rho_t|^2 dxdt + \epsilon^{-2}\iint_Q e^{2s\alpha}\sum_{i,j=1}^{N}|\frac{\partial^2}{\partial x_i x_j}\rho|^2 dxdt \\
&\quad + s^4\lambda^4\epsilon^{-2}\iint_Q \phi^4 e^{2s\alpha}|\rho|^2 dxdt + s^2\lambda^2\epsilon^{-2}\iint_Q \phi^2 e^{2s\alpha}|\nabla\rho|^2 dxdt \\
&\qquad \leq Ce^{\lambda||\psi||_\infty}s^4\lambda^4\epsilon^{-2}\iint_{\omega'\times(0,T)}\phi^4 e^{2s\alpha}|\rho|^2 dxdt.
\end{aligned} \tag{3.22}$$

Next, we add (3.21) and (3.22) and absorb the lower order terms in the right-hand side, we get

$$\begin{aligned}
&\iint_Q \phi^{-1}e^{2s\alpha}|\varphi_t|^2 dxdt + \iint_Q \phi^{-1}e^{2s\alpha}\sum_{i,j=1}^{N}|\frac{\partial^2}{\partial x_i x_j}\varphi|^2 dxdt \\
&+ s^4\lambda^4\iint_Q \phi^3 e^{2s\alpha}|\varphi|^2 dxdt + s^2\lambda^2\iint_Q \phi e^{2s\alpha}|\nabla\varphi|^2 dxdt \\
&+ \epsilon^2\iint_Q e^{2s\alpha}|\rho_t|^2 dxdt + \iint_Q e^{2s\alpha}\sum_{i,j=1}^{N}|\frac{\partial^2}{\partial x_i x_j}\rho|^2 dxdt \\
&+ s^4\lambda^4\iint_Q \phi^4 e^{2s\alpha}|\rho|^2 dxdt + s^2\lambda^2\iint_Q \phi^2 e^{2s\alpha}|\nabla\rho|^2 dxdt \\
&\leq C\Big(e^{\lambda||\psi||_\infty}s^4\lambda^4\iint_{\omega'\times(0,T)}\phi^4 e^{2s\alpha}|\rho|^2 dxdt + s^4\lambda^4\iint_{\omega'\times(0,T)}\phi^3 e^{2s\alpha}|\varphi|^2 dxdt\Big),
\end{aligned} \tag{3.23}$$

for $s \geq (T + (1+||a||_\infty^{2/3})T^2)s_0$.
At this point a remark has to be done. If we were trying to control (3.6) with controls on both equations, inequality (3.23) would be sufficient for such purposes.

***Step 2.*** *Estimate of the local integral of $\rho$.*
In this step we estimate the local integral involving $\rho$ in the right-hand side of (3.23). It will be done using equation $(3.18)_1$. Indeed, we consider a function $\xi$ satisfying

$$\xi \in C_0^\infty(\omega_1),\ 0 \le \xi \le 1,\ \xi(x) = 1\ \forall x \in \omega'.$$

Then

$$\begin{aligned} & Ce^{\lambda||\psi||_\infty} s^4\lambda^4 \iint_{\omega'\times(0,T)} e^{2s\alpha}\phi^4|\rho|^2 dxdt \\ & \le Ce^{\lambda||\psi||_\infty} s^4\lambda^4 \iint_{\omega_1\times(0,T)} e^{2s\alpha}\phi^4|\rho|^2 \xi dxdt \\ & \quad = Ce^{\lambda||\psi||_\infty} s^4\lambda^4 \iint_{\omega_1\times(0,T)} e^{2s\alpha}\phi^4\rho(-\varphi_t - Div(M_e\nabla\varphi) + a\varphi)\xi dxdt \\ & \quad := E + F + G. \end{aligned}$$

In the sequel we estimate each parcel in the expression above.

$$\begin{aligned} E = & Ce^{\lambda||\psi||_\infty} s^4\lambda^4 (\iint_{\omega_1\times(0,T)} s\alpha_t e^{2s\alpha}\phi^4\rho\varphi\xi dxdt \\ & + \iint_{\omega_1\times(0,T)} e^{2s\alpha}\phi^3\phi_t\rho\varphi\xi dxdt + \iint_{\omega_1\times(0,T)} e^{2s\alpha}\phi^4\rho_t\varphi\xi dxdt) \\ & := E_1 + E_2 + E_3. \end{aligned}$$

It is immediate to see that

$$E_1 + E_2 \le \frac{1}{10} s^4\lambda^4 \iint_{\omega_1\times(0,T)} e^{2s\alpha}\phi^4|\rho|^2 dxdt + Ce^{2\lambda||\psi||_\infty} s^8\lambda^4 \iint_{\omega_1\times(0,T)} e^{2s\alpha}\phi^8|\varphi|^2 dxdt \tag{3.24}$$

and

$$E_3 \le \frac{\epsilon^2}{2} \iint_{\omega_1\times(0,T)} e^{2s\alpha}|\rho_t|^2 dxdt + Ce^{2\lambda||\psi||_\infty} \varepsilon^{-2} s^8\lambda^8 \iint_{\omega_1\times(0,T)} e^{2s\alpha}\phi^8|\varphi|^2 dxdt.$$

Next,

$$\begin{aligned}
e^{-\lambda||\psi||_\infty} s^{-4}\lambda^{-4} F = & \sum_{i,j=1}^{N} \iint_{\omega_1\times(0,T)} s\partial_{x_i}\alpha\, e^{2s\alpha}\phi^4\rho(M_e^{ij}\partial_{x_j}\varphi)\xi dxdt \\
& + \sum_{i,j=1}^{N} \iint_{\omega_1\times(0,T)} e^{2s\alpha}\phi^3\partial_{x_i}\phi\, \rho(M_e^{ij}\partial_{x_j}\varphi)\xi dxdt \\
& + \sum_{i,j=1}^{N} \iint_{\omega_1\times(0,T)} e^{2s\alpha}\phi^4\partial_{x_i}\rho\, (M_e^{ij}\partial_{x_j}\varphi)\xi dxdt \\
& + \sum_{i,j=1}^{N} \iint_{\omega_1\times(0,T)} e^{2s\alpha}\phi^4\rho(M_e^{ij}\partial_{x_j}\varphi)\partial_{x_i}\xi dxdt
\end{aligned}$$

and we can show that

$$\begin{aligned}
F \leq & \frac{1}{10}s^4\lambda^4 \iint_{\omega_1\times(0,T)} e^{2s\alpha}\phi^4|\rho|^2 dxdt + \frac{1}{6}s^2\lambda^2 v e^{2s\alpha}\phi^2|\nabla\rho|^2 dxdt \\
& + Ce^{2\lambda||\psi||_\infty} s^8\lambda^8 \iint_{\omega_1\times(0,T)} e^{2s\alpha}\phi^8|\varphi|^2 dxdt + \frac{1}{2}\iint_{\omega_1\times(0,T)} e^{2s\alpha}|\frac{\partial^2}{\partial x_i x_j}\rho|^2 dxdt.
\end{aligned}$$

Finally,

$$\begin{aligned}
G \leq & \frac{1}{10}s^4\lambda^4 \iint_{\omega_1\times(0,T)} e^{2s\alpha}\phi^4|\rho|^2 dxdt \\
& + Ce^{2\lambda||\psi||_\infty} s^4\lambda^4||a||^2_{L^\infty} \iint_{\omega_1\times(0,T)} e^{2s\alpha}\phi^4|\varphi|^2 dxdt.
\end{aligned}$$

Putting $E$, $F$ and $G$ together in (3.23), we get

$$\begin{aligned}
& \iint_Q e^{2s\alpha}|\varphi_t|^2 dxdt + \iint_Q e^{2s\alpha}\sum_{i,j=1}^{N}|\frac{\partial^2}{\partial x_i x_j}\varphi|^2 dxdt \\
& + s^4\lambda^4 \iint_Q \phi^4 e^{2s\alpha}|\varphi|^2 dxdt + s^2\lambda^2 \iint_Q \phi^2 e^{2s\alpha}|\nabla\varphi|^2 dxdt\epsilon^2 \iint_Q e^{2s\alpha}|\rho_t|^2 dxdt \\
& + \iint_Q e^{2s\alpha}\sum_{i,j=1}^{N}|\frac{\partial^2}{\partial x_i x_j}\rho|^2 dxdt + s^4\lambda^4 \iint_Q \phi^4 e^{2s\alpha}|\rho|^2 dxdt \\
& + s^2\lambda^2 \iint_Q \phi^2 e^{2s\alpha}|\nabla\rho|^2 dxdt \\
& \leq Ce^{2\lambda||\psi||_\infty}\epsilon^{-2}s^8\lambda^8 \iint_{\omega_1\times(0,T)} e^{2s\alpha}\phi^8|\varphi|^2 dxdt.
\end{aligned} \tag{3.25}$$

Using (3.25) we can prove that, for every $\epsilon > 0$, system (3.6) is null controllable. However, the sequence of controls obtained this way will not be bounded when $\epsilon \to 0^+$. Therefore, we need to go further and improve estimate (3.25). This will be done in the next step.

***Step 3.*** *An energy Inequality.*

The reason why we do not obtain a bounded sequence of controls out of step 2 is because of the term $\epsilon^{-2}$ in the right-hand side of (3.25). In this step we prove a weighted energy inequality for equation $(3.18)_2$. This inequality will be used to compensate the $\epsilon^{-2}$ term in (3.25).

Let us introduce the function

$$y = e^{\frac{3}{2}s\alpha^*}\rho.$$

This new function satisfies:

$$\left| \begin{array}{ll} \epsilon\partial_t y - Div(M(x)\nabla y) = \epsilon\frac{3}{2}s\alpha_t^* e^{\frac{3}{2}s\alpha^*}\rho & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y(T) = 0 & \text{in } \Omega. \end{array} \right. \tag{3.26}$$

We multiply (3.26) by $y$ and integrate over $\Omega$, we get

$$\frac{\epsilon}{2}\frac{d}{dt}||y(t)||^2_{L^2(\Omega)} + C||\nabla y(t)||^2_{L^2(\Omega)} \leq \epsilon\frac{3}{2}\int_\Omega s\alpha_t^*(t)e^{\frac{3}{2}s\alpha^*(t)}\rho(t)y(t)dx.$$

Integrating from $0$ to $T$ and using Poincaré's and Young's inequalities, it is not difficult to see that

$$\iint_Q e^{3s\alpha^*}|\rho|^2 dxdt \leq C\varepsilon^2 e^{4\lambda||\psi||_\infty}\iint_Q s^4\phi^4 e^{2s\alpha}|\rho|^2 dxdt. \tag{3.27}$$

From (3.25) and (3.27), we obtain

$$\iint_Q e^{3s\alpha^*}|\rho|^2 dxdt \leq Ce^{6\lambda||\psi||_\infty}s^8\lambda^4\iint_{\omega_1\times(0,T)}\phi^8 e^{2s\alpha}|\varphi|^2 dxdt. \tag{3.28}$$

This inequality gives a global estimate of $\rho$ in terms of a local integral of $\phi$, with a bounded constant.

***Step 4.*** *Last estimates and conclusion.*

In order to finish the prove of Theorem 3.3, we combine inequality (3.28) and another Carleman inequality to equation $(3.18)_1$. Indeed, we have

$$\begin{aligned}\iint_Q s^{-1}\phi^{-1}e^{3s\alpha}|\varphi_t|^2dxdt + s^{-1}\iint_Q \phi^{-1}e^{3s\alpha}\sum_{i,j=1}^{N}|\frac{\partial^2}{\partial x_i x_j}\varphi|^2dxdt \\ + s^3\lambda^4\iint_Q \phi^3 e^{3s\alpha}|\varphi|^2dxdt + s\lambda^2\iint_Q \phi e^{3s\alpha}|\nabla\varphi|^2dxdt \\ \leq C\Big(\iint_Q e^{3s\alpha}|\rho|^2dxdt + s^3\lambda^4\iint_{\omega_1\times(0,T)}\phi^3e^{3s\alpha}|\varphi|^2dxdt\Big),\end{aligned} \tag{3.29}$$

where $\varphi$ is, together with $\rho$, solution of (3.18).

Notice that here we just changed the weight $e^{2s\alpha}$ by $e^{3s\alpha}$. The proof of (3.29) is the same as the one given by Lemma 3.1, just taking a slightly different change of variable (see appendix A).

Next, since $e^{3s\alpha} \leq e^{3s\alpha^*}$, we have

$$\iint_Q e^{3s\alpha}|\rho|^2dxdt \leq \iint_Q e^{3s\alpha^*}|\rho|^2dxdt$$

and, by (3.28),

$$\iint_Q e^{3s\alpha}|\rho|^2dxdt \leq Ce^{6\lambda||\psi||_\infty}s^8\lambda^4\iint_{\omega_1\times(0,T)}\phi^8e^{2s\alpha}|\varphi|^2dxdt.$$

From (3.28) and (3.29), it follows that

$$\iint_Q e^{3s\alpha}|\rho|^2dxdt + s^3\lambda^4\iint_Q \phi^3e^{3s\alpha}|\varphi|^2dxdt \leq Ce^{6\lambda||\psi||_\infty}s^8\lambda^4\iint_{\omega_1\times(0,T)}\phi^8e^{2s\alpha}|\varphi|^2dxdt, \tag{3.30}$$

which is exactly (3.13).

By density, we can show that (3.30) remains true if we take initial data in $L^2(\Omega)$. Therefore, the Carleman inequality (3.13) holds for all initial data in $L^2(\Omega)$.

This finishes the proof of Theorem 3.3.

□

## 3.4 Null controllability for the linearized system

This section is devoted to prove the null controllability of linearized equation (3.6). It will be done by showing the observability inequality (3.8) for the adjoint system (3.7) and solving a minimization problem. The arguments used here are classical in control theory for linear PDE's. Hence, we just give a sketch of the proof.

*Sketch of the proof of Theorem 3.4.* Combining the standard energy inequality for system (3.18) and the Carleman inequality given by Theorem 3.3, we can show the following observability inequality for the solutions of (3.18):

$$||\varphi(0)||^2_{L^2(\Omega)} + \varepsilon||\rho(0)||^2_{L^2(\Omega)} \leq e^{C(1+1/T+||a||^{2/3}_{L^\infty}+||a||_{L^\infty}T)} \iint_{\omega_1\times(0,T)} |\varphi|^2 dxdt, \qquad (3.31)$$

where $C = C(\Omega, \omega_1)$ is a positive constant.

Next, since $\rho(x,t) = Div(M(x)\nabla\varphi_e(x,t))$ and $\varphi_e = 0$ on $\partial\Omega$, we have

$$||\varphi_e(t)||_{H^2(\Omega)} \leq C||\rho(t)||_{L^2(\Omega)},$$

for all $t \in [0,T]$. Therefore, it follows from (3.31) that

$$||\varphi(0)||^2_{L^2(\Omega)} + \varepsilon||\varphi_e(0)||^2_{L^2(\Omega)} \leq e^{C(1+1/T+||a||^{2/3}_{L^\infty}+||a||_{L^\infty}T)} \iint_{\omega_1\times(0,T)} |\varphi|^2 dxdt, \qquad (3.32)$$

which is the observability inequality (3.8).

From (3.32) and the density of smooth solutions in the space of solutions of (3.17) with initial data in $L^2(\Omega)$, we see that the above observability inequality is satisfied by all solutions of (3.7) with initial data in $L^2(\Omega)$.

Now, in order to obtain the null controllability for linear system (3.6), we solve, for any $\delta > 0$, the following minimization problem:

Given $\varphi_T$ and $\varphi_{e,T}$ in $L^2(\Omega)$,

$$\text{Minimize } \mathcal{J}_\delta(\varphi_T, \varphi_{e,T}), \text{ with}$$
$$\mathcal{J}_\delta(\varphi_T, \varphi_{e,T}) = \left\{ \frac{1}{2} \iint_{\omega_1\times(0,T)} |\varphi^\epsilon|^2 dxdt + \epsilon(u_{e,0}, \varphi^\epsilon_e(0)) \right. \qquad (3.33)$$
$$\left. + (v_0, \varphi^\epsilon(0)) + \delta(||\varphi_T||_{L^2(\Omega)} + \epsilon^{1/2}||\varphi_{e,T}||_{L^2(\Omega)}) \right\},$$

where $(\varphi, \varphi_e)$ is the solution of the adjoint problem (3.7) with initital data $(\varphi_T, \varphi_{e,T})$.
It is an easy matter to check that $\mathcal{J}$ is strictly convex and continuous. So, in order to guarantee the existence of a minimizer, the only thing remaining to prove is the coercivity of $\mathcal{J}$.
Using the observability inequality (3.8) for the adjoint system (3.7), the coercivity of $\mathcal{J}$ is straightfoward. Therefore, for each $\delta > 0$, there exists a unique minimizer $(\varphi^\delta_{e,T}, \varphi^\delta_T)$ of $\mathcal{J}$. Let us denote by $\varphi^{\epsilon,\delta}$ the corresponding solution to (3.7) associated to this minimizer.

Taking $f^{\epsilon,\delta} = \varphi^{\epsilon,\delta} 1_{\omega_1}$ as a control for (3.6), the duality between (3.6) and (3.7) gives

$$||v^{\epsilon,\delta}(T)||_{L^2(\Omega)} + ||u_e^{\epsilon,\delta}(T)||_{L^2(\Omega)} \leq \delta, \tag{3.34}$$

where $(v^{\epsilon,\delta}, u_e^{\epsilon,\delta})$ is the solution associated to the control $f^{\epsilon,\delta}$. It also gives

$$||f^{\epsilon,\delta} 1_{\omega_1}||^2_{L^2(Q)} \leq C\big(||v_0||^2_{L^2(\Omega)} + \epsilon ||u_{e,0}||^2_{L^2(\Omega)}\big). \tag{3.35}$$

From (3.34) and (3.35), we get a control $f^\epsilon$ (the weak limit of a subsequence of $f^{\epsilon,\delta} 1_{\omega_1}$ in $L^2(\omega_1 \times (0,T))$) that drives the solution of (3.6) to zero at time $T$. From (3.35), we have the following estimate on the control $f^\epsilon$,

$$||f^\epsilon 1_{\omega_1}||^2_{L^2(Q)} \leq C\big(||v_0^\epsilon||^2_{L^2(\Omega)} + \epsilon ||u_{e,0}^\epsilon||^2_{L^2(\Omega)}\big). \tag{3.36}$$

This finishes the proof of Theorem 3.4. □

## 3.5 The nonlinear system

In this section we prove Theorem 3.5. The proof is achieved through fixed point arguments.

*Proof of Theorem 3.5 (case 1):* We consider the following linearization of system (3.5):

$$\left|\begin{array}{ll} c_m v_t^\epsilon - \frac{\mu}{\mu+1} Div\big(\mathbf{M}_e(x)\nabla v^\epsilon\big) + g(z)v^\epsilon = f^\epsilon 1_{\omega_1} & \text{in } Q, \\ \epsilon u_{e,t}^\epsilon - Div\big(\mathbf{M}(x)\nabla u_e^\epsilon\big) = Div\big(\mathbf{M}_i(x)\nabla v^\epsilon\big) & \text{in } Q, \\ v^\epsilon = u_e^\epsilon = 0 & \text{on } \Sigma, \\ v^\epsilon(0) = v_0;\ u_e^\epsilon(0) = u_{e,0} & \text{in } \Omega, \end{array}\right. \tag{3.37}$$

where

$$g(s) = \begin{cases} \dfrac{h(s)}{s} \text{ if } |s| > 0, \\ h'(0) \text{ if } s = 0. \end{cases} \tag{3.38}$$

It follows from Theorem 3.4 that, for each $(v_0, u_{e,0}) \in L^2(\Omega)^2$ and $z \in L^2(Q)$, there exists a control function $f^\epsilon \in L^2(Q)$ such that the solution of (3.37) satisfies:

$$v^\epsilon(T) = u_e^\epsilon(T) = 0.$$

As we said before, the idea is to use fixed point arguments. For that, we will use the following generalized version of Kakutani's fixed point theorem, due to Glicksberg [54].

**Theorem 3.6.** *Let $B$ be a non-empty convex, compact subset of a locally convex topological* *vector space $X$. If $\Lambda : B \longrightarrow B$ is a convex set-valued mapping with closed graph and $\Lambda(B)$ is closed, then $\Lambda$ has a fixed point.*

In order to apply Glicksberg's Theorem, we define a mapping $\Lambda : B \longrightarrow X$ as follows

$$\Lambda(z) = \{v^\epsilon;\ (v^\epsilon, u_e^\epsilon) \text{ is a solution of (3.37), such that } v^\epsilon(T) = u_e^\epsilon(T) = 0,$$
$$\text{for a control } \ f^\epsilon 1_{\omega_1} \text{ satisfying (3.14)}\}.$$

Here, $X = L^2(Q)$ and $B$ is the ball

$$B = \{z \in L^2(0, T, H_0^1(\Omega), z_t \in L^2(0, T, H^{-1}(\Omega));$$
$$||z||^2_{L^2(0,T;H_0^1(\Omega))} + ||z_t||^2_{L^2(0,T;H^{-1}(\Omega))} \le M\}.$$

It is easy to see that $\Lambda$ is well defined and that $B$ is a convex and compact subset of $L^2(Q)$.

Let us now prove that $\Lambda$ is convex, compact and has closed graphic. It will be done into the next few steps.

• $\Lambda(B) \subset B$.

Let $z \in B$ and $v^\epsilon \in \Lambda(z)$. Since $v^\epsilon$ satisfies $(3.37)_1$, the following inequality holds

$$||v^\epsilon||^2_{L^2(0,T;H_0^1(\Omega))} + ||v_t^\epsilon||^2_{L^2(0,T;H^{-1}(\Omega))} \le K_1. \tag{3.39}$$

In this way, if $z \in B$ then $\Lambda(z) \subset B$, if we take $M = K_1$.

• $\Lambda(z)$ is closed in $L^2(Q)$.

Let $z \in B$ fixed, and $v_n^\epsilon \in \Lambda(z)$, such that $v_n^\epsilon \to v^\epsilon$. Let us prove that $v^\epsilon \in \Lambda(z)$.

In fact, by definition we have that $v_n^\epsilon$ is, together with a function $u_{e,n}^\epsilon$ and a control $f_n^\epsilon$, the solution of (3.37), with $||f_n^\epsilon 1_{\omega_1}||^2_{L^2(Q)} \le C\big(||v_0||^2_{L^2(\Omega)} + \epsilon||u_{e,0}||^2_{L^2(\Omega)}\big)$. Therefore, we can extract a subsequence of $f_n^\epsilon$, denoted by the same index, such that

$$f_n^\epsilon 1_{\omega_1} \to f^\epsilon 1_{\omega_1} \text{ weakly in } \ L^2(Q).$$

Since $f_n^\epsilon$ is bounded, we can argue as in the previous section in order to obtain the inequality

$$||v_n^\epsilon||^2_{L^2(0,T;H_0^1(\Omega))} + ||v_{t,n}^\epsilon||^2_{L^2(0,T;H^{-1}(\Omega))} \le M. \tag{3.40}$$

Therefore,

$$\left|\begin{array}{l} v_n^\epsilon \to v^\epsilon \text{ weakly in } \ L^2(0,T;H_0^1(\Omega)), \\ v_n^\epsilon \to v^\epsilon \text{ strongly in } \ L^2(Q), \\ v_{t,n}^\epsilon \to v_t^\epsilon \text{ weakly in } \ L^2(0,T;H^{-1}(\Omega)). \end{array}\right.$$

Using the converges above and $(3.37)_2$, we see that there exists a function $u_e$ such that

$$\left|\begin{array}{l} u^\epsilon_{e,n} \to u^\epsilon_e \text{ weakly in } L^2(0,T;H^1_0(\Omega)),\\ u^\epsilon_{e,n} \to u^\epsilon_e \text{ strongly in } L^2(Q),\\ u^\epsilon_{t,e,n} \to u^\epsilon_{t,e} \text{ weakly in } L^2(0,T;H^{-1}(\Omega)).\end{array}\right.$$

It follows that $(v^\epsilon, u^\epsilon_e)$ is a controlled solution of (3.37) associated to the control $f$. Hence, $v^\epsilon \in \Lambda(z)$ and $\Lambda(z)$ is closed and compact of $L^2(Q)$.

• $\Lambda$ has closed graph in $L^2(Q) \times L^2(Q)$.

We need to prove that if $z_n \to z$, $v^\epsilon_n \to v^\epsilon$ strongly in $L^2(Q)$ and $v^\epsilon_n \in \Lambda(z_n)$, then $v^\epsilon \in \Lambda(z)$. Using the two steps before, it is easy to show that $v^\epsilon \in \Lambda(z)$

Therefore, we can apply Glicksberg Theorem to conclude that $\Lambda$ has a fixed point. This proves Theorem 3.5 in the case where the nonlinearity is a $C^1$ global Lipschitz function.

*Proof of Theorem 3.5 (case 2):* The proof of the local null controllability in the case 2 of Theorem 3.5 is done as in [69].

We consider the linear system

$$\left|\begin{array}{ll} c_m v^\epsilon_t - \frac{\mu}{\mu+1} Div\big(\mathbf{M}_e(x)\nabla v^\epsilon\big) + a(z)v^\epsilon = f^\epsilon 1_{\omega_1} & \text{in } Q,\\ \epsilon u^\epsilon_{e,t} - Div\big(\mathbf{M}(x)\nabla u^\epsilon_e\big) = Div\big(\mathbf{M}_i(x)\nabla v^\epsilon\big) & \text{in } Q,\\ v^\epsilon = u^\epsilon_e = 0 & \text{on } \Sigma,\\ v^\epsilon(0) = v_0; u^\epsilon_e(0) = u_{e,0} & \text{in } \Omega,\end{array}\right. \tag{3.41}$$

with $(v_0, u_{e,0}) \in \big(H^1_0(\Omega) \cap W^{2(1-\frac{1}{q_N}),q_N}(\Omega)\big)^2$, $z \in L^\infty(Q)$ and

$$a(z) = \int_0^1 \frac{dh}{dz}(sz)ds.$$

It is not difficult to show the null controllability of system (3.5) with a control in $L^2(\omega_1 \times (0,T))$. However, these $L^2$ controls are not sufficient to apply fixed point arguments and obtain the null controllability of the nonlinear system (3.4). For that reason, we modify a little the functional (3.33), obtaining controls which allows to use Schauder's fixed point Theorem. In fact, for any $\delta > 0$, we consider the minimization problem:

$$\begin{array}{l}\text{Minimize } \mathcal{J}_\delta(\varphi_T, \varphi_{eT}), \text{ with}\\ \mathcal{J}_\delta(\varphi_T, \varphi_{eT}) = \Bigg\{ \frac{1}{2}\int_0^T \int_{\omega_1} e^{2s\alpha}\phi^8 |\varphi^\epsilon|^2 dxdt + \epsilon(u_{e,0}, \varphi^\epsilon_e(0)) \\ \qquad + (v_0, \varphi^\epsilon(0)) + \delta\Big(||\varphi_T||_{L^2(\Omega)} + \epsilon^{1/2}||\varphi_{e,T}||_{L^2(\Omega)}\Big)\Bigg\},\end{array} \tag{3.42}$$

where $(\varphi^\epsilon, \varphi_e^\epsilon)$ is the solution of the adjoint system (3.7) with initial data $(\varphi_T, \varphi_{e,T})$.

As before, it can be proved that (3.42) has an unique minimizer $(\varphi^{\epsilon,\delta}, \varphi_e^{\epsilon,\delta})$. Defining $f^{\epsilon,\delta} = e^{2s\alpha}\phi^8\varphi^{\epsilon,\delta}$ and using the fact that $\varphi^{\epsilon,\delta}$ is, together with a $\varphi_e^{\epsilon,\delta}$, the solution of (3.7), we see that $f^{\epsilon,\delta}$ is the solution of a heat equation with null initial data, right-hand side in $L^2(Q)$ and Dirichlet boundary conditions. Using the regularizing effect of the heat equation, we can show the estimate

$$||f^{\epsilon,\delta}1_{\omega_1}||^2_{L^{q_N}(Q)} \leq C\big(||v_0||^2_{L^2(\Omega)} + \epsilon||u_{e,0}||^2_{L^2(\Omega)}\big). \tag{3.43}$$

Taking the limit when $\delta \to 0^+$, we get a control $f^\epsilon 1_{\omega_1} \in L^{q_N}(Q)$ such that the associated solution $(v^\epsilon, u_e^\epsilon)$ of (3.41) satisfies:

$$v^\epsilon(T) = u_e^\epsilon(T) = 0.$$

The proof is finished applying Schauder's fixed point Theorem for system (3.41).

# Chapter 4

# A uniform controllability result for the Keller-Segel system

## 4.1 Introduction

Let $\Omega \subset \mathbb{R}^N$ (N = 2, 3) be a bounded connected open set whose boundary $\partial\Omega$ is regular enough. Let $T > 0$ and $\omega'$ and $\omega$ be two (small) nonempty subsets of $\Omega$ with $\omega' \subset\subset \omega$ and we denote by $\nu(x)$ the outward normal to $\Omega$ at the point $x \in \partial\Omega$.

In this chapter we will be concerned with the following controlled Keller-Segel system:

$$\left|\begin{array}{ll} u_t - \Delta u = -\nabla \cdot (u \nabla v) & \text{in } Q, \\ \epsilon v_t - \Delta v = au - bv + g\chi & \text{in } Q, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \Sigma, \\ u(x,0) = u_0;\ v(x,0) = v_0 & \text{in } \Omega, \end{array}\right. \tag{4.1}$$

where, $a$ and $b$ are positive real constants, $g$ is an internal control and $\epsilon$ is a small positive parameter, which is intended to tend to zero. In (4.1), $\chi : \mathbb{R}^N \to \mathbb{R}$ is a $C^\infty$ function such that $supp\ \chi \subset\subset \omega$, $0 \le \chi \le 1$ and $\chi \equiv 1$ in $\omega'$.

System (4.1) is a classical equation in chemotaxis, describing the change of motion when a population reacts in response to an external chemical stimulus spread in the environment where they reside. In many applications (see [82, 105] and the references therein), system (4.1) is approximated by the following parabolic-elliptic system:

$$\left|\begin{array}{ll} u_t - \Delta u = -\nabla \cdot (u \nabla v) & \text{in } Q, \\ -\Delta v = au - bv + g\chi & \text{in } Q, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \Sigma, \\ u(x,0) = u_0 & \text{in } \Omega. \end{array}\right. \tag{4.2}$$

In (4.1) and (4.2), $u = u(x,t) \geq 0$ and $v = v(x,t) \geq 0$ represent, respectively, the concentrations of species (i.e, the population density) and that of the chemical (i.e., concentration of the chemical substance).

The goal of this chapter is to analyze the controllability of (4.1) around a fixed trajectory of (4.2), uniformly with respect to $\epsilon$. More precisely, we consider a constant solution $(M_1, M_2) \in \mathbb{R}^2$ of (4.2), with $g \equiv 0$ (which is equivalent to require that $aM_1 - bM_2 = 0$), and we seek for a control $g = g(\epsilon)$ such that $(u(T), v(T)) = (M_1, M_2)$ and $g$ is bounded with respect to $\epsilon$.

**Remark 4.1.** Each one of the models (4.1) and (4.2) can be viewed as a single nonlinear parabolic equation for $u$ with a *nonlocal* (either in $x$ or $(x,t)$) nonlinearity, since the term $\nabla v$ can be expressed as a linear integral operator acting on $u$. In the first model, the variations of the concentration $v$ are governed by the linear nonhomogeneous heat equation, and therefore are slower than in the latter system, where the response of $v$ to the variations of $u$ are instantaneous, and described by the integral operator $(-\Delta)^{-1}$ whose kernel has a singularity. Thus, one may expect the evolution described by (4.2) to be faster than in (4.1), especially for large values of $\epsilon$ when the diffusion of $v$ is rather slow compared to that of $u$. Moreover, the nonlinear effects for (4.2) should manifest themselves faster than for (4.1) (see [11]).

As usual in control theory, we study the controllability of (4.1) around $(M_1, M_2)$ by first analyzing the controllability of its linearization around this trajectory, namely

$$\left| \begin{array}{ll} u_t - \Delta u = -M_1 \Delta v + h_1 & \text{in } Q, \\ \epsilon v_t - \Delta v = au - bv + g\chi + h_2 & \text{in } Q, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \Sigma, \\ u(x,0) = u_0;\ v(x,0) = v_0 & \text{in } \Omega, \end{array} \right. \tag{4.3}$$

where $h_1$ and $h_2$ are given forces belonging to appropriate Banach spaces ( $X := X_1 \times X_2$, see (4.79)) and having exponential decay at $t = T$. Our objective is to prove that we can find $g$ so that the solution of (4.3) satisfies $(u(T), v(T)) = (0,0)$ and moreover we want that the quantity $\nabla \cdot (u \nabla v)$ belongs to $X_1$. Then we employ an inverse mapping argument introduced in [129] in order to obtain the controllability of (4.1) around $(M_1, M_2)$.

The most important tool to prove the null controllability of the linear system (4.3) is a *global Carleman inequality* for the solutions of its adjoint system, that is to say,

$$\left|\begin{array}{ll} -\varphi_t - \Delta\varphi = a\xi + f_1 & \text{in } Q, \\ -\epsilon\xi_t - \Delta\xi = -b\xi - M_1\Delta\varphi + f_2 & \text{in } Q, \\ \frac{\partial\varphi}{\partial\nu} = \frac{\partial\xi}{\partial\nu} = 0 & \text{on } \Sigma, \\ \varphi(x,T) = \varphi_T;\ \xi(x,T) = \xi_T & \text{in } \Omega, \\ \int_\Omega \varphi_T dx = 0. & \end{array}\right. \tag{4.4}$$

Here, $f_1$ and $f_2$ are arbitrary $L^2(Q)$ functions.

Actually, due to the fact that the control is acting on the second equation of (4.3), we need to bound global integrals of $\varphi$ and $\xi$ in terms of a local integral of $\xi$ and global integrals of $f_1$ and $f_2$. The main difficulty when proving a Carleman inequality of this type for (4.4) arises from the fact that the coupling in the second equation is in $\Delta\varphi$ and not in $\varphi$. In fact, the inequality we prove will contain global terms with the $L^2$-weighted norms of $\Delta\varphi$ and $\xi$ in the left hand side, no global terms in $\varphi$, while a local integral of $\xi$ and global integrals of $f_1$ and $f_2$ will appear in its right-hand side.

With the help of the Carleman inequality and an inverse function theorem, we prove the following result, which is the main result of this chapter.

**Theorem 4.1.** *Let $0 < \epsilon \leq 1$ and let $(M_1, M_2) \in \mathbb{R}^2_+$ be such that $aM_1 - bM_2 = 0$. Then there exists $\delta > 0$ such that, for any $(u_0, v_0) \in H^1(\Omega) \times H^2(\Omega)$ satisfying $\int_\Omega u_0 dx = M_1$, $\frac{\partial v_0}{\partial\nu} = 0$ on $\partial\Omega$ and $||(u_0 - M_1, v_0 - M_2)||_{H^1(\Omega)\times H^2(\Omega)} \leq \delta$, we can find $g \in L^2(0,T;H^1(\Omega))$, uniformly bounded with respect to $\epsilon$, such that the associated solution $(u,v)$ of (4.1) satisfies:*

$$(u(T), v(T)) = (M_1, M_2) \text{ in } \Omega.$$

**Remark 4.2.** Notice that all constant trajectories $(M_1, M_2) \in \mathbb{R}^2_+$ of (2.1) satisfy $aM_1 - bM_2 = 0$. On the other hand, condition $\int_\Omega u_0 dx = M_1$ in Theorem 4.1 is a necessary condition, since the mass of $u$ is preserved, i.e.,

$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx, \quad \forall t > 0. \tag{4.5}$$

Concerning the controllabity of the Keller-Segel system, the only result we know is the one obtained in [58], where the authors analyze the controllability of the Keller system (4.1), with $\epsilon = 1$, around a fixed trajectory of (4.1) (i.e., a solution of (4.1) with $g \equiv 0$), when a control is acting on the first equation, which is not natural from a physical point of view. The authors are able to show that the Keller-Segel system is controllable around this trajectory if the trajectory has good regularity properties. However, in their case, since the control is acting on the first equation, the problem is much easier from a mathematical point of view because the adjoint system of the linearization of the Keller-Segel system around the trajectory has a zero-order coupling term. Another interesting work in this subject is [37], in which the authors show that, in dimension 2, any global in time bounded solution of system (4.1) converges to a single equilibrium (a stationary solution of (4.1)) as time tends to infinity.

## 4.2 Carleman inequality

Let us consider the adjoint system (4.4). In this section, we obtain a suitable Carleman inequality for this system. This will provide a null controllability result for the linear system (4.3) for suitable $h_1$ and $h_2$ (section 4.3).

As in chapters 2 and 3, we consider the basic weight function given by Lemma 2.1 and introduce:

$$\phi(x,t) = \frac{e^{\lambda(m||\eta_0||_\infty + \eta_0(x))}}{t^4(T-t)^4}, \ \alpha(x,t) = \frac{e^{\lambda(m||\eta_0||_\infty + \eta^0(x))} - e^{5/4m\lambda||\eta_0||_\infty}}{t^4(T-t)^4},$$
$$\hat{\phi}(t) = \min_{x\in\overline{\Omega}} \phi(x,t), \ \phi^*(t) = \max_{x\in\overline{\Omega}} \phi(x,t), \ \alpha^*(t) = \max_{x\in\overline{\Omega}} \alpha(x,t), \ \hat{\alpha} = \min_{x\in\overline{\Omega}} \alpha(x,t), \tag{4.6}$$

where $m > 4$ is a fixed real number and $\lambda > 0$ is a real parameter.

We remark that these weights functions were already used in [38] in order to obtain a Carleman inequality for the Stokes system.

Throughout this chapter we will also use the notation:

$$I_\beta(s,\sigma;\rho) = s^{\beta+3}\iint_Q e^{2s\alpha}\phi^{\beta+3}|\rho|^2 dxdt + s^{\beta+1}\iint_Q e^{2s\alpha}\phi^{\beta+1}|\nabla\rho|^2 dxdt$$
$$+ s^{\beta-1}\iint_Q e^{2s\alpha}\phi^{\beta-1}(\sigma^2|\rho_t|^2 + \sum_{i,j=1}^{N} |\frac{\partial^2\rho}{\partial x_i \partial x_j}|^2) dxdt, \tag{4.7}$$

where $s, \beta$ and $\sigma$ are real numbers and $\rho = \rho(x,t)$.
The following Carleman inequality holds:

**Lemma 4.1.** *Let $f \in L^2(\Omega)$ be given. There exist $\lambda_0 = \lambda_0(\Omega,\omega')$, $s_0 = s_0(\Omega,\omega',\lambda)$ and $C = C(\Omega,\omega',\lambda,T)$ such that, for every $\lambda \geq \lambda_0$, any $s \geq s_0(T^4 + T^8)$ and any $q_0 \in L^2(\Omega)$, the weak solution of*

$$\left| \begin{array}{ll} \sigma q_t - \Delta u = f & in \ \ Q, \\ \frac{\partial q}{\partial \nu} = 0 & on \ \ \Sigma, \\ q(x,0) = q_0 & in \ \ \Omega \end{array} \right. \tag{4.8}$$

*satisfies:*

$$I_\beta(s,\sigma;\rho) \leq C\bigg(s^\beta \iint_Q e^{2s\alpha}\phi^\beta |f|^2 dxdt + s^{\beta+3}\iint_{\omega'\times(0,T)} e^{2s\alpha}\phi^{\beta+3}|q|^2 dxdt\bigg),$$

*for all* $\beta \in \mathbb{R}$ *and any* $0 < \sigma \leq 1$ .

A proof of Lemma 4.1 can be deduced from the Carleman inequality for the heat equation with homogeneous Neumann boundary condition (see [40]) by keeping track of the degenerating parameter $\sigma$, as in the case of the heat equation with Dirichlet boundary condition (see [46]). For the sake of completeness, we prove this result in appendix A.

The main result of this section is stated as follows.

**Theorem 4.2.** *Given* $0 < \epsilon \leq 1$, *there exist* $C_0 = C_0(\Omega,\omega) > 0$ *and* $s_0 = s_0(\Omega,\omega,\lambda,T)$ *such that for every* $\lambda \geq C_0$, *every* $s \geq s_0$ *and any* $f_1, f_2 \in L^2(Q)$, *the solution* $(\varphi,\xi)$ *of system* (4.4) *satisfies:*

$$\begin{aligned} &I_2(s,\epsilon,e^{s\hat\alpha}\hat\phi^{-9/2}\xi) + \iint_Q e^{2s\alpha+2s\hat\alpha}\hat\phi^{-6}|\Delta\varphi|^2 dxdt + \iint_Q e^{2s\hat\alpha+2s\hat\alpha}\hat\phi^{-6}|\nabla\varphi|^2 dxdt \\ &\leq C\bigg(s^7\iint_Q \phi^7 e^{2s\alpha}e^{2s\hat\alpha}|\chi|^2|\xi|^2 dxdt + s^3\iint_Q e^{2s\hat\alpha}\hat\phi^{-9}|f_1|^2 dxdt \\ &+ s^3\iint_Q \phi^3 e^{2s\alpha+2s\hat\alpha}|f_2|^2 dxdt\bigg), \end{aligned} \tag{4.9}$$

*with* $C = C(\Omega,\omega)$.

*Proof.* Since the proof is a bit technical, for a better understanding, we will divide it into several steps.

*Step 1. Carleman Inequality for* $\Delta\varphi$.

We write $e^{s\hat\alpha}\hat\phi^{-9/2}\varphi = \eta + \psi$, where $\eta$ solves a heat equation with a $L^2$ right-hand side and $\psi$ solves a heat equation with right-hand side in $H^1(0,T;H^2(\Omega))$. Applying a Carleman inequality for $\psi$ and energy estimates for $\eta$ we obtain a global estimate of $\Delta\varphi$ in terms of a local integral of $\Delta\psi$ and global integrals of $\Delta\xi$ and $f_1$.

The functions $\eta$ and $\psi$ stand to solve

$$\left|\begin{array}{ll} -\eta_t - \Delta\eta = e^{s\hat\alpha}\hat\phi^{-9/2}f_1 & \text{in } Q, \\ \frac{\partial\eta}{\partial\nu} = 0 & \text{on } \Sigma, \\ \eta(T) = 0 & \text{in } \Omega \end{array}\right. \tag{4.10}$$

and

$$\left|\begin{array}{ll} -\psi_t - \Delta\psi = ae^{s\hat{\alpha}}\hat{\phi}^{-9/2}\xi - (e^{s\hat{\alpha}}\hat{\phi}^{-9/2})_t\varphi & \text{in } Q, \\ \frac{\partial\psi}{\partial\nu} = 0 & \text{on } \Sigma, \\ \psi(T) = 0 & \text{in } \Omega, \end{array}\right. \tag{4.11}$$

respectively.

Using standard energy estimates for the heat equation with Neumann boundary condition, we have

$$||\eta||^2_{H^1(Q)} + ||\Delta\eta||^2_{L^2(Q)} \leq C||e^{s\hat{\alpha}}\hat{\phi}^{-9/2}f_1||^2_{L^2(Q)}. \tag{4.12}$$

Next, from (4.11) we see that

$$\left|\begin{array}{ll} -(\Delta\psi)_t - \Delta(\Delta\psi) = ae^{s\hat{\alpha}}\hat{\phi}^{-9/2}\Delta\xi + (e^{s\hat{\alpha}}\hat{\phi}^{-9/2})_t\Delta\varphi & \text{in } Q, \\ \frac{\partial(\Delta\psi)}{\partial\nu} = 0 & \text{on } \Sigma, \\ \Delta\psi(T) = 0 & \text{in } \Omega. \end{array}\right. \tag{4.13}$$

Let $\omega''$ be a nonempty set such that $\omega'' \subset\subset \omega'$, applying Lemma 4.1 to equation (4.11), with $\beta = 0$ and $\sigma = 1$, we get

$$\begin{aligned} I_0(s,1,\Delta\psi) \leq & C\bigg(s^3\iint_{\omega''\times(0,T)} e^{2s\alpha}\phi^3|\Delta\psi|^2dxdt + \iint_Q e^{2s\alpha+2s\hat{\alpha}}\hat{\phi}^{-9}|\Delta\xi|^2dxdt \\ & + s^{5/2}\iint_Q e^{2s\alpha+2s\hat{\alpha}}\hat{\phi}^{-13/2}|\Delta\varphi|^2dxdt\bigg), \end{aligned} \tag{4.14}$$

for any $s \geq s_0(\Omega, \omega'', T, \lambda)$ (a proof of (4.14) is achieved taking into account that

$$|(e^{s\hat{\alpha}}\hat{\phi}^{-9/2})_t| \leq Cs_0(T)\hat{\phi}^{-13/4}e^{s\hat{\alpha}},$$

since

$$|\hat{\alpha}_t| + |\hat{\phi}_t| \leq CT\hat{\phi}^{5/4} \text{ and } |\hat{\phi}^{-1}| \leq CT^8,$$

for some $C = C(\Omega, \omega)$ and any $s \geq s_0(\Omega, \omega, \lambda, T)$).
Since $e^{s\hat{\alpha}}\hat{\phi}^{-9/2}\Delta\varphi = \Delta\psi + \Delta\eta$, estimate (4.14) gives

$$\begin{aligned} I_0(s,1,\Delta\psi) \leq & C\bigg(s^3\iint_{\omega''\times(0,T)} e^{2s\alpha}\phi^3|\Delta\psi|^2dxdt + \iint_Q e^{2s\alpha+2s\hat{\alpha}}\hat{\phi}^{-9}|\Delta\xi|^2dxdt \\ & s^{5/2}\iint_Q e^{2s\alpha}\phi^{5/2}(|\Delta\psi|^2 + |\Delta\eta|^2)dxdt\bigg). \end{aligned} \tag{4.15}$$

The last term in the right-hand side of (4.15) can be estimated as follows

$$s^{5/2}\iint_Q e^{2s\alpha}\phi^{5/2}(|\Delta\psi|^2+|\Delta\eta|^2)dxdt \le Cs^3\iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-9}|f_1|^2dxdt + \delta I_0(s,1,\Delta\psi), \tag{4.16}$$

for any $\delta > 0$ and any $s \geq s_0(\Omega,\omega'',T,\lambda)$. Here we have used estimate (4.12).

Therefore, combining (4.12), (4.15) and (4.16), we obtain

$$\begin{aligned} I_0(s,1,\Delta\psi)+||\eta||^2_{H^1(Q)}+||\Delta\eta||^2_{L^2(Q)} \le C\Big( & s^3\iint_{\omega''\times(0,T)} e^{2s\alpha}\phi^3|\Delta\psi|^2dxdt \\ & +\iint_Q e^{2s\alpha+2s\hat{\alpha}}\hat{\phi}^{-9}|\Delta\xi|^2dxdt \\ & +s^3\iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-9}|f_1|^2dxdt\Big). \end{aligned} \tag{4.17}$$

Hence, we have the following Carleman estimate for $\Delta\varphi$:

$$\begin{aligned} \iint_Q e^{2s\alpha+2s\hat{\alpha}}\hat{\phi}^{-6}|\Delta\varphi|^2dxdt \le C\Big( & s^3\iint_{\omega''\times(0,T)} e^{2s\alpha}\phi^3|\Delta\psi|^2dxdt \\ & +\iint_Q e^{2s\alpha+2s\hat{\alpha}}\hat{\phi}^{-9}|\Delta\xi|^2dxdt \\ & +s^3\iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-9}|f_1|^2dxdt\Big). \end{aligned} \tag{4.18}$$

for any $s \geq s_0(\Omega,\omega'',T,\lambda)$.

*Step 2. Carleman inequality for $\xi$.*

In this step, a Carleman inequality for $\xi$ is obtained. Combining this inequality with the Carleman inequality from the previous step, global estimates of $\xi$ and $\Delta\varphi$ in terms of local integrals of $\xi$ another in $\Delta\psi$ and global integrals of $f_1$ and $f_2$ are obtained.

We consider the function $e^{s\hat{\alpha}}\hat{\phi}^{-9/2}\xi$, which fulfill the system:

$$\left|\begin{array}{ll} -\epsilon(e^{s\hat{\alpha}}\hat{\phi}^{-9/2}\xi)_t - e^{s\hat{\alpha}}\hat{\phi}^{-9/2}\Delta\xi = F & \text{in } Q, \\ \frac{\partial(e^{s\hat{\alpha}}\hat{\phi}^{-9/2}\xi)}{\partial\nu} = 0 & \text{on } \Sigma, \\ (e^{s\hat{\alpha}}\hat{\phi}^{-9/2}\xi)(T) = 0 & \text{in } \Omega, \end{array}\right. \tag{4.19}$$

where $F = -be^{s\hat{\alpha}}\hat{\phi}^{-9/2}\xi - M_1e^{s\hat{\alpha}}\hat{\phi}^{-9/2}\Delta\varphi - \epsilon(e^{s\hat{\alpha}}\hat{\phi}^{-9/2})_t\xi + e^{s\hat{\alpha}}\hat{\phi}^{-9/2}f_2$.
Applying Lemma 4.1 to the equation (4.19), with $\beta = 2$ and $\sigma = \epsilon$, absorbing the lower

order terms, we get

$$I_2(s,\epsilon,e^{s\hat{\alpha}}\hat{\phi}^{-9/2}\xi) \leq C\Big(s^5\iint_{\omega'\times(0,T)} e^{2s\alpha+2s\hat{\alpha}}\hat{\phi}^{-4}|\xi|^2dxdt \\ + s^2\iint_Q \phi^2e^{2s\alpha}(|\Delta\psi|^2+|\Delta\eta|^2)dxdt + s^2\iint_Q e^{2s\alpha+2s\hat{\alpha}}\hat{\phi}^{-7}|f_2|^2dxdt\Big). \tag{4.20}$$

Using estimate (4.16), we see that

$$I_2(s,\epsilon,e^{s\hat{\alpha}}\hat{\phi}^{-9/2}\xi) \leq C\Big(s^5\iint_{\omega'\times(0,T)} e^{2s\alpha+2s\hat{\alpha}}\phi^{-4}|\xi|^2dxdt + \delta I(s,\Delta\psi) \\ + s^3\iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-9}|f_1|^2dxdt + s^2\iint_Q e^{2s\alpha+2s\hat{\alpha}}\hat{\phi}^{-7}|f_2|^2dxdt, \tag{4.21}$$

for any $\delta > 0$ and any $s \geq s_0(\Omega,\omega',T,\lambda)$.

Adding (4.17), (4.18) and (4.21), absorbing the lower order terms, we obtain

$$I_2(s,\epsilon,e^{s\hat{\alpha}}\hat{\phi}^{-9/2}\xi) + I_0(s,1,\Delta\psi) + \iint_Q e^{2s\alpha+2s\hat{\alpha}}\hat{\phi}^{-6}|\Delta\varphi|^2dxdt \\ \leq C\Big(s^3\iint_{\omega''\times(0,T)} e^{2s\alpha}\phi^3|\Delta\psi|^2dxdt + s^5\iint_{\omega'\times(0,T)} e^{2s\alpha+2s\hat{\alpha}}\phi^{-4}|\xi|^2dxdt \\ + s^3\iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-9}|f_1|^2dxdt + s^2\iint_Q e^{2s\alpha+2s\hat{\alpha}}\hat{\phi}^{-7}|f_2|^2dxdt\Big), \tag{4.22}$$

for any $s \geq s_0(\Omega,\omega',T,\lambda)$.

*Step 3. Estimate of the local integral of $\Delta\psi$ and conclusion.*
In this step we estimate the local integral $\Delta\psi$ in the right hand side of (4.22) in terms of a local integral of $\xi$ and global integrals of $f_1$ and $f_2$. In order to do that, we introduce a cut-off function $\theta$ with

$$\theta \in C_0^\infty(\omega'), \text{ with } 0 \leq \theta \leq 1 \text{ and } \theta \equiv 1 \text{ on } \omega''.$$

Next, we use (4.4) to write

$$\Delta\psi = -\frac{e^{s\hat{\alpha}}\hat{\phi}^{-9/2}}{M_1}(-\epsilon\xi_t + \Delta\xi + be^{s\hat{\alpha}}\xi - f_2) + \Delta\eta$$

and then it follows that

$$
\begin{aligned}
& s^3 \iint_{\omega'' \times (0,T)} e^{2s\alpha} \phi^3 |\Delta\psi|^2 dxdt && (4.23)\\
& \leq -\frac{1}{M_1} s^3 \iint_{\omega' \times (0,T)} \theta e^{2s\alpha} \phi^3 \hat{\phi}^{-9/2} \Delta\psi (-\epsilon e^{s\hat{\alpha}} \xi_t + e^{s\hat{\alpha}} \Delta\xi + b e^{s\hat{\alpha}} \xi - e^{s\hat{\alpha}} f_2 + M_1 \Delta\eta) dxdt.
\end{aligned}
$$

The rest of this step is devoted to estimate each one of the terms in the right-hand side of the above integral. For the first one, integration by parts gives

$$
\begin{aligned}
\epsilon s^3 & \iint_{\omega' \times (0,T)} \theta e^{2s\alpha} \phi^3 \hat{\phi}^{-9/2} \Delta\psi e^{s\hat{\alpha}} \xi_t dxdt \\
& = \epsilon s^3 \iint_{\omega' \times (0,T)} \theta e^{2s\alpha} e^{s\hat{\alpha}} \hat{\phi}^{-9/2} \phi^3 \Delta\psi_t \xi dxdt \\
& + \epsilon s^4 \iint_{\omega' \times (0,T)} \theta e^{2s\alpha} e^{s\hat{\alpha}} \phi^3 \hat{\phi}^{-9/2} (2\alpha_t + \hat{\alpha}_t) \Delta\psi \xi dxdt \\
& + 3\epsilon s^3 \iint_{\omega' \times (0,T)} \theta e^{2s\alpha} e^{s\hat{\alpha}} \hat{\phi}^{-9/2} \phi^2 \phi_t \Delta\psi \xi dxdt \\
& - 9/2 \epsilon s^3 \iint_{\omega' \times (0,T)} \theta e^{2s\alpha} e^{s\hat{\alpha}} \phi^3 \hat{\phi}^{-11/2} \hat{\phi}_t \Delta\psi \xi dxdt \qquad (4.24)
\end{aligned}
$$

and it is not difficult to see that

$$
\begin{aligned}
|\epsilon s^3 \iint_{\omega' \times (0,T)} \theta e^{2s\alpha} \phi^3 \hat{\phi}^{-9/2} \Delta\psi e^{s\hat{\alpha}} \xi_t dxdt| & \leq C s^7 \iint_{\omega' \times (0,T)} e^{2s\alpha + 2s\hat{\alpha}} \hat{\phi}^7 |\xi|^2 dxdt \\
& + \epsilon^2 I_0(s, 1, \Delta\psi). \qquad (4.25)
\end{aligned}
$$

where $C$ does not depend on $\epsilon$.

Next, we integrate by parts the second term in (4.23) to obtain

$$
\begin{aligned}
s^3 & \iint_{\omega' \times (0,T)} \theta e^{2s\alpha} e^{s\hat{\alpha}} \phi^3 \hat{\phi}^{-9/2} \Delta\psi \Delta\xi dxdt \qquad (4.26)\\
& = s^3 \iint_{\omega' \times (0,T)} e^{2s\alpha} e^{s\hat{\alpha}} \phi^3 \hat{\phi}^{-9/2} \Delta\psi \nabla\theta \cdot \nabla\xi dxdt \\
& + s^3 \iint_{\omega' \times (0,T)} \theta \Delta\psi \nabla (e^{2s\alpha} e^{s\hat{\alpha}} \phi^3 \hat{\phi}^{-9/2}) \cdot \nabla\xi dxdt \\
& + s^3 \iint_{\omega' \times (0,T)} \theta e^{2s\alpha} e^{s\hat{\alpha}} \phi^3 \hat{\phi}^{-9/2} \nabla(\Delta\psi) \cdot \nabla\xi dxdt.
\end{aligned}
$$

Integrating by parts the terms in the right-hand side of (4.26), and using Young's inequality, we show that

$$|s^3 \iint_{\omega' \times (0,T)} \theta e^{2s\alpha} e^{s\hat{\alpha}} \phi^3 \hat{\phi}^{-9/2} \Delta\psi \Delta\xi dxdt| \leq Cs^7 \iint_{\omega' \times (0,T)} \phi^7 e^{2s\alpha + 2s\hat{\alpha}} |\xi|^2 dxdt + \delta I_0(s, 1, \Delta\psi). \quad (4.27)$$

Here, we need to used the fact that

$$|\nabla(e^{2s\alpha} e^{s\hat{\alpha}} \phi^3)| \leq Cs\phi^4 e^{2s\alpha + s\hat{\alpha}} \text{ and } |\Delta(e^{2s\alpha} e^{s\hat{\alpha}} \phi^3)| \leq Cs^2 \phi^5 e^{2s\alpha + s\hat{\alpha}},$$

for some $C > 0$. Finally, for the last two terms, we have

$$\begin{aligned} &|s^3 \iint_{\omega' \times (0,T)} \theta e^{2s\alpha} e^{s\hat{\alpha}} \phi^3 \hat{\phi}^{-9/2} \Delta\psi (b\xi + f_2) dxdt| \quad (4.28) \\ &\qquad \leq Cs^3 \iint_{\omega' \times (0,T)} \phi^3 e^{2s\alpha + 2s\hat{\alpha}} (|\xi|^2 + |f_2|^2) dxdt + \delta I_0(s, 1, \Delta\psi) \end{aligned}$$

and

$$|s^3 \iint_{\omega' \times (0,T)} \theta e^{2s\alpha} e^{s\hat{\alpha}} \phi^3 \hat{\phi}^{-9/2} \Delta\psi \Delta\eta dxdt| \leq Cs^3 \iint_Q e^{2s\hat{\alpha}} \hat{\phi}^{-9} |f_1|^2 dxdt + \delta I_0(s, 1, \Delta\psi). \quad (4.29)$$

Gathering (4.25), (4.27), (4.28) and (4.29) in (4.23), we obtain from (4.22), after absorbing the lower order terms, the Carleman inequality:

$$\begin{aligned} &I_2(s, \epsilon, e^{s\hat{\alpha}} \hat{\phi}^{-9/2} \xi) + I_0(s, 1, \Delta\psi) + \iint_Q e^{2s\alpha + 2s\hat{\alpha}} \hat{\phi}^{-6} |\Delta\varphi|^2 dxdt \\ &+ \iint_Q e^{2s\hat{\alpha} + 2s\hat{\alpha}} \hat{\phi}^{-6} |\nabla\varphi|^2 dxdt \\ &\leq C\Big( s^7 \iint_{\omega' \times (0,T)} \phi^7 e^{2s\alpha} e^{2s\hat{\alpha}} |\xi|^2 dxdt + s^3 \iint_Q e^{2s\hat{\alpha}} \hat{\phi}^{-9} |f_1|^2 dxdt \\ &\quad + s^3 \iint_Q \phi^3 e^{2s\alpha + 2s\hat{\alpha}} |f_2|^2 dxdt \Big), \quad (4.30) \end{aligned}$$

for $C = C(\Omega, \omega)$ and every $s \geq s_0(\Omega, \omega, T, \lambda)$. Observe that we can add the last term in the lef-hand side of (4.30) because $\frac{\partial\varphi}{\partial\nu} = 0$. From (4.30) and the definition of the function $\chi$, we obtain our desired Carleman inequality (4.9).

□

## 4.3 Null controllability of the linear system with a right-hand side

In this section we want to solve the null controllability problem for the system (4.3), with a right-hand side which decays exponentially as $t \to T^-$.

This result will be crucial when proving the local controllability of (4.1) in the next section.

Indeed, for any $0 < \epsilon \leq 1$, we would like to find $g = g(\epsilon) \in L^2(0,T;H^1(\Omega))$, bounded independently from $\epsilon$, such that the solution to

$$\begin{cases} L(u,v) = (h_1, h_2 + g\chi) & \text{in } Q, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \Sigma, \\ u(x,0) = u_0;\ v(x,0) = v_0 & \text{in } \Omega, \end{cases} \tag{4.31}$$

where

$$L(u,v) = (u_t - \Delta u + M_1 \Delta v, \epsilon v_t - \Delta v + bv - au), \tag{4.32}$$

verifies

$$u(x,T) = 0;\ v(x,T) = 0 \text{ in } \Omega. \tag{4.33}$$

Furthermore, it will be convenient to prove the existence of a solution of the previous problem in an appropriate weighted space. Before introducing such spaces, we improve the Carleman inequality obtained in the previous section. This Carleman inequality will contain only weight functions that do not vanish at $t = 0$. In order to introduce these new weights, let us consider the function

$$l(t) = \begin{cases} (T^2/4) & \text{if } 0 \leq t \leq T/2 \\ t(T-t) & \text{if } T/2 \leq t \leq T, \end{cases} \tag{4.34}$$

and we define our new weight functions as

$$\beta(x,t) = \frac{e^{\lambda(m||\eta_0||_\infty + \eta^0(x))} - e^{5/4m\lambda||\eta_0||_\infty}}{l^4(t)},\ \gamma(x,t) = \frac{e^{\lambda(m||\eta_0||_\infty + \eta_0(x))}}{l^4(t)},$$

$$\hat{\gamma}(t) = \min_{x\in\overline{\Omega}} \gamma(x,t),\ \gamma^*(t) = \max_{x\in\overline{\Omega}} \phi(x,t),\ \beta^*(t) = \max_{x\in\overline{\Omega}} \beta(x,t),\ \hat{\beta} = \min_{x\in\overline{\Omega}} \beta(x,t). \tag{4.35}$$

With this new weights, we state our refined Carleman estimate as follows.

**Lemma 4.2.** *There exists $C = C(\Omega, \omega, \lambda, T) > 0$ such that, for any $0 < \epsilon \leq 1$, every solution* *of* (4.4) *satisfies:*

$$\begin{aligned}
&\iint_Q e^{2s\beta+2s\hat{\beta}}\hat{\gamma}^{-4}|\xi|^2 dxdt + \int_0^T\!\!\int_\Omega e^{2s\beta+2s\hat{\beta}}\hat{\gamma}^{-6}|\nabla\xi|^2 dxdt \\
&+ \int_0^T \iint_Q e^{2s\hat{\beta}+2s\hat{\beta}}\hat{\gamma}^{-6}|\nabla\varphi|^2 dxdt + ||\varphi(0) - \big(\varphi(0)\big)_\Omega||^2 + \epsilon||\xi(0)||^2 \\
&+ \iint_Q e^{2s\hat{\beta}+2s\hat{\beta}}\hat{\gamma}^{-6}|\varphi - (\varphi)_\Omega|^2 dxdt \\
&\leq C\Big(\iint_Q e^{2s\hat{\beta}}\hat{\gamma}^{-9}|f_1|^2 dxdt + \iint_Q (\gamma^*)^3 e^{2s\beta^*+2s\hat{\beta}}|f_2|^2 dxdt \\
&\quad + \iint_Q e^{2s\beta^*+2s\hat{\beta}}(\gamma^*)^7|\chi|^2|\xi|^2 dxdt\Big),
\end{aligned} \tag{4.36}$$

*where*

$$(\varphi)_\Omega(t) = \frac{1}{|\Omega|}\int_\Omega \varphi(x,t)dx.$$

*Proof.* We start the proof by showing a simple a priori estimate for the linear system (4.3), with a precise dependence with respect to $\epsilon$. For this, let us introduce a function $\theta_0 \in C^2([0,T])$ with

$$\theta_0 = 1 \text{ in } [0,T/2], \;\; \theta_0 = 0 \text{ in } [3T/4,T], \;\; |\theta_0'| \leq C/T$$

and define $\overline{\varphi} = \theta_0\varphi$ and $\overline{\xi} = \theta_0\xi$, where $(\varphi,\xi)$ is the solution of (4.4). After a change of variable in the time scale from $t$ to $T-t$, the pair $(\overline{\varphi},\overline{\xi})$ solves:

$$\left|\begin{array}{ll}
\overline{\varphi}_t - \Delta\overline{\varphi} = a\overline{\xi} + \theta_0 f_1 + \theta_0'\varphi & \text{in } Q, \\
\epsilon\overline{\xi}_t - \Delta\overline{\xi} = -b\overline{\xi} - M_1\Delta\overline{\varphi} + \theta_0 f_2 + \theta_0'\xi & \text{in } Q, \\
\frac{\partial\overline{\varphi}}{\partial\nu} = \frac{\partial\overline{\xi}}{\partial\nu} = 0 & \text{on } \Sigma, \\
\overline{\varphi}(x,0) = 0; \; \overline{\xi}(x,0) = 0 & \text{in } \Omega.
\end{array}\right. \tag{4.37}$$

We multiply the first equation of (4.37) by $-\Delta\overline{\varphi}$ and the second by $\overline{\xi}$ and integrate over $\Omega$, we get this way

$$\begin{aligned}
&\frac{d}{dt}\big(||\nabla\overline{\varphi}(t)||^2 + \epsilon||\overline{\xi}(t)||^2\big) + ||\Delta\overline{\varphi}(t)||^2 + ||\nabla\overline{\xi}(t)||^2 + ||\overline{\xi}(t)||^2 \\
&\leq C(||\nabla\overline{\varphi}||^2 + ||\theta_0 f_1||^2 + ||\theta_0 f_2||^2 + ||\theta_0'\nabla\varphi||^2 + ||\theta_0'\xi||^2),
\end{aligned} \tag{4.38}$$

which, by Gronwall's inequality, gives

$$\begin{aligned}&||\nabla\overline{\varphi}(t)||^2+\epsilon||\overline{\xi}(t)||^2\\&\leq Ce^{CT}\int_0^T(||\theta_0 f_1||^2+||\theta_0 f_2||^2+||\theta_0'\nabla\varphi||^2+||\theta_0'\xi||^2)dt,\end{aligned}\tag{4.39}$$

for all $0\leq t\leq T$.

From inequality (4.39), we see that

$$\begin{aligned}&\int_0^T(||\nabla\overline{\varphi}(t)||^2+\epsilon||\overline{\xi}(t)||^2)dt\\&\leq CTe^{CT}\int_0^T(||\theta_0 f_1||^2+||\theta_0 f_2||^2+||\theta_0'\nabla\varphi||^2+||\theta_0'\xi||^2)dt.\end{aligned}\tag{4.40}$$

Integrating (4.38) from 0 to $T$, we get

$$\begin{aligned}&||\nabla\overline{\varphi}(T)||^2+\epsilon||\overline{\xi}(T)||^2+\int_0^T||\Delta\overline{\varphi}(t)||^2dt+\int_0^T||\nabla\overline{\xi}(t)||^2dt+\int_0^T||\overline{\xi}(t)||^2dt\\&\leq Ce^{CT}\int_0^T(||\theta_0 f_1||^2+||\theta_0 f_2||^2+||\theta_0'\nabla\varphi||^2+||\theta_0'\xi||^2)dt.\end{aligned}\tag{4.41}$$

Going back to the original variables, inequalities (4.40) and (4.41) then gives

$$\begin{aligned}&||\nabla\varphi||^2_{L^2(0,T/2;L^2(\Omega))}+||\Delta\varphi||^2_{L^2(0,T/2;L^2(\Omega))}+||\xi||^2_{L^2(0,T/2;H^1(\Omega))}\\&+||\nabla\varphi||^2_{L^\infty(0,T/2;L^2(\Omega))}+\epsilon||\xi||^2_{L^\infty(0,T/2;L^2(\Omega))}\\&\leq Ce^{CT}\big(||f_1||^2_{L^2(0,3T/4;L^2(\Omega))}+||f_2||^2_{L^2(0,3T/4;L^2(\Omega))}+\frac{1}{T}(||\nabla\varphi||^2_{L^2(T/2,3T/4;L^2(\Omega))}\\&+||\xi||^2_{L^2(T/2,3T/4;L^2(\Omega))})\big).\end{aligned}\tag{4.42}$$

As a consequence, we can obtain the following estimate in $\Omega\times(0,T/2)$:

$$\begin{aligned}&\int_0^{T/2}\int_\Omega e^{2s\beta+2s\hat{\beta}}\hat{\gamma}^{-4}|\xi|^2dxdt+\int_0^{T/2}\int_\Omega e^{2s\beta+2s\hat{\beta}}\hat{\gamma}^{-6}|\Delta\varphi|^2dxdt\\&+\int_0^{T/2}\int_\Omega e^{2s\beta+2s\hat{\beta}}\hat{\gamma}^{-6}|\nabla\xi|^2dxdt+\int_0^{T/2}\int_\Omega e^{2s\beta+2s\hat{\beta}}\hat{\gamma}^{-6}|\nabla\varphi|^2dxdt\\&+||\nabla\varphi(0)||^2+\epsilon||\xi(0)||^2\\&\leq C\Big(\int_0^{3T/4}\int_\Omega(e^{2s\hat{\beta}}\hat{\gamma}^{-9}|f_1|^2dxdt+\gamma^3e^{2s\beta+2s\hat{\beta}}|f_2|^2)dxdt\\&+\int_{T/2}^{3T/4}\int_\Omega e^{2s\beta+2s\hat{\beta}}\gamma^{-4}|\xi|^2+e^{2s\hat{\beta}+2s\hat{\beta}}\hat{\gamma}^{-6}|\nabla\varphi|^2dxdt\Big).\end{aligned}\tag{4.43}$$

On the other hand, since $\alpha = \beta$ in $\Omega \times (T/2, T)$, we have:

$$\begin{aligned}
&\int_{T/2}^{T}\int_{\Omega} e^{2s\beta+2s\hat{\beta}}\hat{\gamma}^{-4}|\xi|^2dxdt + \int_{T/2}^{T}\int_{\Omega} e^{2s\beta+2s\hat{\beta}}\hat{\gamma}^{-6}|\Delta\varphi|^2dxdt\\
&+ \int_{T/2}^{T}\int_{\Omega} e^{2s\beta+2s\hat{\beta}}\hat{\gamma}^{-6}|\nabla\xi|^2dxdt + \int_{T/2}^{T}\int_{\Omega} e^{2s\hat{\beta}+2s\hat{\beta}}\hat{\gamma}^{-6}|\nabla\varphi|^2dxdt\\
&= \int_{T/2}^{T}\int_{\Omega} e^{2s\alpha+2s\hat{\alpha}}\hat{\phi}^{-4}|\xi|^2dxdt + \int_{T/2}^{T}\int_{\Omega} e^{2s\alpha+2s\hat{\alpha}}\hat{\phi}^{-6}|\Delta\varphi|^2dxdt\\
&+ \int_{T/2}^{T}\int_{\Omega} e^{2s\alpha+2s\hat{\alpha}}\hat{\phi}^{-6}|\nabla\xi|^2dxdt + \int_{T/2}^{T}\int_{\Omega} e^{2s\hat{\alpha}+2s\hat{\alpha}}\hat{\phi}^{-6}|\nabla\varphi|^2dxdt. \qquad (4.44)
\end{aligned}$$

So, in virtue of (4.9), we have

$$\begin{aligned}
&\int_{T/2}^{T}\int_{\Omega} e^{2s\beta+2s\hat{\beta}}\hat{\gamma}^{-4}|\xi|^2dxdt + \int_{T/2}^{T}\int_{\Omega} e^{2s\beta+2s\hat{\beta}}\hat{\gamma}^{-6}|\Delta\varphi|^2dxdt\\
&+ \int_{T/2}^{T}\int_{\Omega} e^{2s\beta+2s\hat{\beta}}\hat{\gamma}^{-6}|\nabla\xi|^2dxdt + \int_{T/2}^{T}\int_{\Omega} e^{2s\hat{\beta}+2s\hat{\beta}}\hat{\gamma}^{-6}|\nabla\varphi|^2dxdt\\
&\leq C\Big(\iint_{Q}(e^{2s\hat{\alpha}}\hat{\gamma}^{-9}|f_1|^2 + \phi^3 e^{2s\alpha+2s\hat{\alpha}}|f_2|^2)dxdt + \iint_{Q} e^{2s\alpha+2s\hat{\alpha}}\phi^7|\chi|^2|\xi|^2dxdt\Big) \qquad (4.45)
\end{aligned}$$

and, from the definition of $\beta$, $\beta^*$, $\gamma$ and $\hat{\gamma}$, it follows that

$$\begin{aligned}
&\int_{T/2}^{T}\int_{\Omega} e^{2s\beta+2s\hat{\beta}}\hat{\gamma}^{-4}|\xi|^2dxdt + \int_{T/2}^{T}\int_{\Omega} e^{2s\beta+2s\hat{\beta}}\hat{\gamma}^{-6}|\Delta\varphi|^2dxdt\\
&+ \int_{T/2}^{T}\int_{\Omega} e^{2s\beta+2s\hat{\beta}}\hat{\gamma}^{-6}|\nabla\xi|^2dxdt + \int_{T/2}^{T}\int_{\Omega} e^{2s\hat{\beta}+2s\hat{\beta}}\hat{\gamma}^{-6}|\nabla\varphi|^2dxdt \qquad (4.46)\\
&\leq C\Big(\iint_{Q} e^{2s\hat{\beta}}\hat{\gamma}^{-9}|f_1|^2dxdt + \iint_{Q}\gamma^3 e^{2s\beta+2s\hat{\beta}}|f_2|^2dxdt + \iint_{Q} e^{2s\beta+2s\hat{\beta}}\gamma^7|\chi|^2|\xi|^2dxdt\Big).
\end{aligned}$$

Therefore, gathering (4.43) and (4.46), we obtain the following refined Carleman inequality

$$\begin{aligned}
&\iint_{Q} e^{2s\beta+2s\hat{\beta}}\hat{\gamma}^{-4}|\xi|^2dxdt + \iint_{Q} e^{2s\beta+2s\hat{\beta}}\hat{\gamma}^{-6}|\Delta\varphi|^2dxdt + \iint_{Q} e^{2s\beta+2s\hat{\beta}}\hat{\gamma}^{-6}|\nabla\xi|^2dxdt\\
&+ \iint_{Q} e^{2s\hat{\beta}+2s\hat{\beta}}\hat{\gamma}^{-6}|\nabla\varphi|^2dxdt + ||\nabla\varphi(0)||^2 + \epsilon||\xi(0)||^2 \qquad (4.47)\\
&\leq C\Big(\iint_{Q} e^{2s\hat{\beta}}\hat{\gamma}^{-9}|f_1|^2dxdt + \iint_{Q}\gamma^3 e^{2s\beta+2s\hat{\beta}}|f_2|^2dxdt + \iint_{Q} e^{2s\beta+2s\hat{\beta}}\gamma^7\chi|\xi|^2dxdt\Big).
\end{aligned}$$

Using Poincaré-Wirtinger's inequality, estimate (4.47) gives

$$\begin{aligned}
&\iint_Q e^{2s\beta+2s\hat{\beta}}\hat{\gamma}^{-4}|\xi|^2 dxdt + \iint_Q e^{2s\beta+2s\hat{\beta}}\hat{\gamma}^{-6}|\nabla\xi|^2 dxdt \\
&+ \iint_Q e^{2s\hat{\beta}+2s\hat{\beta}}\hat{\gamma}^{-6}|\nabla\varphi|^2 dxdt + ||\varphi(0) - \big(\varphi(0)\big)_\Omega||^2 + \epsilon||\xi(0)||^2 \\
&+ \iint_Q e^{2s\hat{\beta}+2s\hat{\beta}}\hat{\gamma}^{-6}|\varphi - \big(\varphi\big)_\Omega|^2 dxdt \\
&\leq C\Big(\iint_Q e^{2s\beta^*+2s\hat{\beta}}(\gamma^*)^7\chi|\xi|^2 dxdt + \iint_Q e^{2s\hat{\beta}}\hat{\gamma}^{-9}|f_1|^2 dxdt \\
&+ \iint_Q (\gamma^*)^3 e^{2s\beta^*+2s\hat{\beta}}|f_2|^2 dxdt\Big),
\end{aligned} \tag{4.48}$$

which is exactly (4.36). □

Once we have got (4.36), we are ready to solve (4.31)-(4.33). In fact, we will prove two controllability results: first, we will obtain a null controllability result for (4.31) with no supplementary regularity for the control and the state (see Proposition 4.1); secondly, we prove (4.31)-(4.33) with a more regular state (see Proposition 4.2).

Now we present our first null controllability result for (4.31).

**Proposition 4.1.** *Let $0 < \epsilon \leq 1$ and let $(M_1, M_2) \in \mathbb{R}^2$ be such that $aM_1 - bM_2 = 0$. Assume that $(u_0, v_0) \in L^2_0(\Omega) \times L^2(\Omega)$, $e^{-2s\hat{\beta}}\hat{\gamma}^3 h_1 \in L^2_0(Q)$ and that $e^{-s\beta-s\hat{\beta}}\hat{\gamma}^2 h_2 \in L^2(Q)$. Then we can find $g = g(\epsilon) \in L^2(\omega \times (0,T))$ such that (4.31)-(4.33) is satisfied. Moreover, $g$ is bounded independently from $\epsilon$.*

*Proof.* Assume, without loss of generality, that $\chi(x) = 1_\omega(x)$. Then, for each $(\varphi_T, \xi_T) \in L^2_0(\Omega) \times L^2(\Omega)$, we consider the solution of (4.4) with zero right-hand side, namely

$$\left|\begin{array}{ll}
-\varphi_t - \Delta\varphi = a\xi & \text{in } Q, \\
-\epsilon\xi_t - \Delta\xi = -b\xi - M_1\Delta\varphi & \text{in } Q, \\
\frac{\partial\varphi}{\partial\nu} = \frac{\partial\xi}{\partial\nu} = 0 & \text{on } \Sigma, \\
\varphi(x,T) = \varphi_T;\ \xi(x,T) = \xi_T & \text{in } \Omega
\end{array}\right. \tag{4.49}$$

and introduce, each $\delta > 0$, the following functional:

$$\begin{aligned}
J_\delta(\varphi_T, \xi_T) = \frac{1}{2}\iint_{\omega\times(0,T)} |\xi|^2 dxdt + \delta(||\varphi_T|| + \epsilon^{1/2}||\xi_T||) + (u_0, (\varphi_0 - \big(\varphi(0)\big)_\Omega)) + \epsilon(v_0, \xi(0)) \\
+ \iint_Q h_1(\varphi - \big(\varphi\big)_\Omega) dxdt + \iint_Q h_2\xi dxdt.
\end{aligned}$$

Using the Carleman inequality (4.36), it is not difficult to see that, for every $\delta > 0$, $J_\delta$ is coercive and possesses a unique minimum $(\varphi_{\delta,T}, \xi_{\delta,T}) \in L^2_0(\Omega) \times L^2(\Omega)$. Then, setting

$g_\delta = \xi_\delta 1_\omega$ and denoting by $(u_\delta, v_\delta)$ the associated solution of (4.31), from the fact that $J_\delta \le 0$ and using (4.36), we find

$$||g_\delta||_{L^2(Q)} \le C(||(u_0, \epsilon^{1/2} v_0)||_{L^2(\Omega)^2} + ||e^{-2s\hat{\beta}}\hat{\gamma}^3 h_1||_{L^2(Q)} + ||e^{-s\beta - s\hat{\beta}}\hat{\gamma}^2 h_2||_{L^2(Q)}). \quad (4.50)$$

In particular, we see that $g_\delta$ is uniformly bounded in $L^2(\omega \times (0,T))$.

On the other hand, writing explicitly the necessary condition satisfied by $J_\delta$ at this minimum $(\varphi_{\delta,T}, \xi_{\delta,T})$ and the duality between $(u_\delta, v_\delta)$ and $(\varphi_\delta, \xi_\delta)$, we deduce that

$$||(u(T), \epsilon^{1/2} v(T))|| \le \delta. \quad (4.51)$$

Combining (4.50) and (4.51), we get the existence of a control $g$ (the weak limit of a subsequence $g_\delta$ in $L^2(\omega \times (0,T))$ ) such that the associated solution to (4.31) verifies (4.33).

The fact that $g$ is bounded independently from $\epsilon$ follows from the fact that the constant $C$ in (4.50) is bounded independently from $\epsilon$. □

**Remark 4.3.** In the proof of Proposition 4.1 we have used implicitly that the solution $u$ of (2.1) satisfies $\int_\Omega u(x,t)dx = 0$ for any $t > 0$, which follows from the fact that $\int_\Omega u_0 dx = 0$ and that $\int_\Omega h_1(x,t)dx = 0$.

Now we present our second main null controllability result for (4.31), where we seek for a more regular solution $(u, v)$. This will be crucial to deduce controllability properties for the nonlinear system (4.1) in the last section.

To this end, we proceed to the definition of the spaces where (4.31) verifying (4.33) will be solved. Let $\Pi_i$ be the projection over the $i$-th component. The main space is:

$$\begin{aligned} E = \{(u,v,g) \in E_0 : \; & e^{-2s\hat{\beta}}\hat{\gamma}^3 \Pi_1 L(u,v) \in L^2(Q), \\ & e^{-s\beta - s\hat{\beta}}\hat{\gamma}^2 \big(\Pi_2 L(u,v) - g\chi\big) \in L^2(0,T;H^1(\Omega)), \\ & \int_\Omega \Pi_1 L(u,v)dx = 0 \text{ and } \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \Sigma\}, \end{aligned}$$

where

$$\begin{aligned} E_0 = \Big\{ & (u,v,g) : ||e^{-s\hat{\beta}}\hat{\gamma}^{9/2} u||_{L^2(Q)} + ||e^{-s\beta^* - s\hat{\beta}}(\gamma^*)^{-3/2} v||_{L^2(Q)} \\ & + ||\chi(x) e^{-s\beta^* - s\hat{\beta}}(\gamma^*)^{-7/2} g||_{L^2(Q)} < \infty, \\ & e^{-s\hat{\beta}}\hat{\gamma}^{13/4} u \in L^2(0,T;H^2(\Omega)), e^{-s\hat{\beta}}\hat{\gamma}^{13/4} u \in L^\infty(0,T;H^1(\Omega)), \\ & e^{-s\hat{\beta}}\hat{\gamma}^{-1/4} \Delta v \in L^2(0,T;H^1(\Omega)), \; e^{-s\hat{\beta}}\hat{\gamma}^{-1/4} \nabla v \in L^2(0,T;H^2(\Omega)) \Big\}. \end{aligned}$$

Notice that $E$ is a Banach space for the norm:

$$\begin{aligned}||(u,v,g)||_E =&||e^{-s\hat{\beta}}\hat{\gamma}^{9/2}u||_{L^2(Q)} + ||e^{-s\beta^*-s\hat{\beta}}(\gamma^*)^{-3/2}v||_{L^2(Q)} \\ &+ ||\chi(x)e^{-s\beta^*-s\hat{\beta}}(\gamma^*)^{-7/2}g||_{L^2(Q)} \\ &+ ||e^{-2s\hat{\beta}}\hat{\gamma}^3\Pi_1 L(u,v)||_{L^2(Q)} + ||e^{-s\beta-s\hat{\beta}}\hat{\gamma}^2\big(\Pi_2 L(u,v) - g\chi\big)||_{L^2(0,T;H^1(\Omega))} \\ &+ ||e^{-s\hat{\beta}}\hat{\gamma}^{13/4}u||_{L^2(0,T;H^2(\Omega))} + ||e^{-s\hat{\beta}}\hat{\gamma}^{13/4}u||_{L^\infty(0,T;H^1(\Omega))} \\ &+ ||e^{-s\hat{\beta}}\hat{\gamma}^{-1/4}\Delta v||_{L^2(0,T;H^1(\Omega))} + ||e^{-s\hat{\beta}}\hat{\gamma}^{-1/4}\nabla v||_{L^2(0,T;H^2(\Omega))}. \end{aligned} \tag{4.52}$$

**Remark 4.4.** For every $(u,v,g) \in E_0$, we have that $\nabla \cdot (u\nabla v) \in L^2(e^{-4s\hat{\beta}}\hat{\gamma}^6; Q)$. In fact,

$$\begin{aligned}&\iint_Q e^{-4s\hat{\beta}}\hat{\gamma}^6|\nabla\cdot(u\nabla v)|^2dxdt \le \iint_Q e^{-4s\hat{\beta}}\hat{\gamma}^6(|\nabla u|^2|\nabla v|^2 + |u|^2|\Delta v|^2)dxdt \\ &\le \iint_Q (|e^{-s\hat{\beta}}\hat{\gamma}^{13/4}\nabla u|^2|e^{-s\hat{\beta}}\hat{\gamma}^{-1/4}\nabla v|^2 + |e^{-s\hat{\beta}}\hat{\gamma}^{13/4}u|^2|e^{-s\hat{\beta}}\hat{\gamma}^{-1/4}\Delta v|^2)dxdt < \infty.\end{aligned}$$

**Remark 4.5.** If $(u,v,g) \in E$ then $u(T) = v(T) = 0$, so that $(u,v,g)$ solve a null controllability problem for system (4.3) with an appropriate right-hand side $(h_1,h_2)$.

We have the following result.

**Proposition 4.2.** *Let* $0 < \epsilon \le 1$ *and let* $(M_1,M_2) \in \mathbb{R}^2$ *be such that* $aM_1 - bM_2 = 0$. *Moreover, assume that*

$$(u_0,v_0) \in H^1(\Omega)\times H^2(\Omega),\ \int_\Omega u_0 dx = 0,\ \frac{\partial v_0}{\partial\nu} = 0 \text{ on } \partial\Omega \tag{4.53}$$

*and that*

$$(e^{-2s\hat{\beta}}\hat{\gamma}^3h_1, e^{-s\beta-s\hat{\beta}}\hat{\gamma}^2h_2) \in L^2(Q)\times L^2(0,T;H^1(\Omega)). \tag{4.54}$$

*Then there exists a control* $g \in L^2((0,T);H^1(\Omega))$, *bounded independently from* $\epsilon$, *such that, if* $(u,v)$ *is the associated solution of* (4.31), *one has* $(u,v) \in E$. *In particular,* (4.33) *holds.*

*Proof.* Following the ideas of [38], we introduce the extremal problem:

$$\begin{cases} \inf \frac{1}{2}\big(\iint_Q e^{-2s\hat\beta}\hat\gamma^9|u|^2 + \iint_Q e^{-2s\beta^*-2s\hat\beta}(\gamma^*)^{-3}|v|^2 + \iint_Q |\chi|^2(x)e^{-2s\beta^*-2s\hat\beta}(\gamma^*)^{-7}|g|^2\big) \\ \text{subject to } g\in L^2(0,T;H^1(\Omega)) \text{ and} \\ \begin{cases} u_t-\Delta u = -M_1\Delta v + h_1 & \text{in } Q, \\ \epsilon v_t - \Delta v + bv = au + g\chi + h_2 & \text{in } Q, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \Sigma, \\ u(x,0)=u_0;\ v(x,0)=v_0 & \text{in } \Omega, \\ u(x,T)=0;\ v(x,T)=0 & \text{in } \Omega. \end{cases} \end{cases} \tag{4.55}$$

We must show that (4.55) possesses a unique solution $(\hat u, \hat v, \hat g)$ and then, in view of Lagrange's principle, there will exist dual variables $(\hat z, \hat w)$ such that

$$\begin{cases} (\hat u,\hat v) = (e^{2s\hat\beta}\hat\gamma^{-9}\Pi_1 L^*(\hat z,\hat w), e^{2s\beta^*+2s\hat\beta}(\gamma^*)^3\Pi_2 L^*(\hat z,\hat w)) & \text{in } Q, \\ \hat g = -e^{2s\beta^*+2s\hat\beta}(\gamma^*)^7\hat w\chi & \text{in } Q, \end{cases} \tag{4.56}$$

where $L^*$ is the adjoint operator of $L$, i.e.,

$$L^*(z,w) = (-z_t - \Delta z - aw, -\epsilon w_t - \Delta w + bw + M_1\Delta z).$$

In order to prove the existence and uniqueness of solution for (4.55), let us intoduce:

$$P_0 = \{(z,w)\in C^\infty(\overline{Q});\ \frac{\partial z}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \text{ on } \Sigma, \int_\Omega z(x,T)dx = 0\ \forall t\in[0,T]\},$$

and the bilinear form

$$\begin{aligned} a\big((\hat z,\hat w),(z,w)\big) = & \iint_Q e^{2s\hat\beta}\hat\gamma^{-9}\Pi_1 L^*(\hat z,\hat w)\Pi_1 L^*(z,w)dxdt \\ & + \iint_Q e^{2s\beta^*+2s\hat\beta}(\gamma^*)^3\Pi_2 L^*(\hat z,\hat w)\Pi_2 L^*(z,w)dxdt \\ & + \iint_Q |\chi(x)|^2 e^{2s\beta^*+2s\hat\beta}(\gamma^*)^7\hat w w dxdt, \quad \forall (z,w)\in P_0. \end{aligned}$$

Then, if $(\hat u, \hat v, \hat g)$, given by (4.56), satisfy the control problem (4.31)-(4.33), we must have:

$$a\big((\hat z,\hat w),(z,w) = < l,(z,w) >, \ \ \forall (z,w)\in P_0, \tag{4.57}$$

where we have used the notation

$$< l,(z,w) >= \iint_Q h_1 z dxdt + \iint_Q h_2 w dxdt + \int_\Omega (u_0 z(0) + \epsilon v_0 w(0))dx. \tag{4.58}$$

We need to prove that there exists exactly one $(\hat{z}, \hat{w})$ satisfying (4.57) in a appropriate class. We will define then $(\hat{u}, \hat{v}, \hat{g})$ using (4.56) and check that $(\hat{u}, \hat{v}, \hat{g})$ fulfills the desired properties.

Thus, consider the linear space $P_0$ and the bilinear form $a(.,.)$ on $P_0$:

$$\begin{aligned} a\big((\zeta,\rho),(z,w)\big) &= \iint_Q e^{2s\hat{\beta}}\hat{\gamma}^{-9}\Pi_1 L^*(\zeta,\rho)\Pi_1 L^*(z,w)dxdt \\ &+ \iint_Q e^{2s\beta^*+2s\hat{\beta}}(\gamma^*)^3\Pi_2 L^*(\zeta,\rho)\Pi_2 L^*(z,w)dxdt \\ &+ \iint_Q e^{2s\beta^*+2s\hat{\beta}}(\gamma^*)^7|\chi(x)|^2\rho w dxdt, \quad \forall(\zeta,\rho),(z,w) \in P_0. \end{aligned} \tag{4.59}$$

The Carleman inequality (4.47) holds for all $(z,w) \in P_0$, i.e.,

$$\begin{aligned} &\iint_Q e^{2s\beta+2s\hat{\beta}}\hat{\gamma}^{-4}|w|^2dxdt + \iint_Q e^{2s\beta+2s\hat{\beta}}\hat{\gamma}^{-6}|\nabla w|^2dxdt + \iint_Q e^{2s\hat{\beta}+2s\hat{\beta}}\hat{\gamma}^{-6}|\nabla z|^2dxdt \\ &+ \iint_Q e^{2s\beta+2s\hat{\beta}}\hat{\gamma}^{-6}|\Delta z|^2dxdt + \iint_Q e^{2s\hat{\beta}+2s\hat{\beta}}\hat{\gamma}^{-6}|z-\big(z\big)_\Omega|^2dxdt \\ &+ ||z(0) - \big(z(0)\big)_\Omega||^2 + \epsilon||w(0)||^2 \\ &\le Ca((z,w),(z,w)), \quad \forall(w,z) \in P_0. \end{aligned} \tag{4.60}$$

From (4.60), it follows that we have a unique continuation property for the system

$$\begin{cases} L^*(z,w) = (0,0) & \text{in } Q, \\ \frac{\partial z}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 & \text{on } \Sigma, \end{cases} \tag{4.61}$$

which implies that $a(.,.)$ is a scalar product on $P_0$.

Therefore, we can consider the space $P$, the completion of $P_0$ with respect to the norm associated to $a(.,.)$ (which we denote by $||.||_P$). This is a Hilbert space and $a(.,.)$ is a continuous and coercive bilinear form on $P$.

Let us also introduce $l$, given by (4.58) for all $(z,w) \in P$. After a simple computation, we see that

$$
\begin{aligned}
|<l,(z,w)>| \leq{} & ||e^{-2s\hat{\beta}}\hat{\gamma}^3 h_1||_{L^2(Q)}||e^{2s\hat{\beta}}\hat{\gamma}^{-3}(z-(z)_\Omega)||_{L^2(Q)} \\
& + ||e^{-s\beta-s\hat{\beta}}\hat{\gamma}^2 h_2||_{L^2(Q)}||e^{s\beta+s\hat{\beta}}\hat{\gamma}^{-2}w||_{L^2(Q)} \\
& + ||(u_0,\epsilon^{1/2}v_0)||_{L^2(\Omega)^2}||(z(0))-\big(z(0)\big)_\Omega,\epsilon^{1/2}w(0))||_{L^2(\Omega)^2}, \quad \forall (z,w)\in P
\end{aligned} \tag{4.62}
$$

and, in particular, using (4.60), the density of $P_0$ in $P$, we get

$$
\begin{aligned}
|<l,(z,w)>| \leq & C\big(||e^{-2s\hat{\beta}}\hat{\gamma}^3 h_1||_{L^2(Q)} + ||e^{-s\beta-s\hat{\beta}}\hat{\gamma}^2 h_2||_{L^2(Q)} \\
& + ||(u_0,\epsilon^{1/2}v_0)||_{L^2(\Omega)^2}\big)||(z,w)||_P, \quad \forall (z,w)\in P.
\end{aligned} \tag{4.63}
$$

In other words, $l$ is a bounded linear form on $P$ and the constant $C$ in (4.63) does not depend on $\epsilon$. Consequently, in view of Lax-Milgran's lemma, there exists a unique $(\hat{z},\hat{w})$ satisfying:

$$
a((\hat{z},\hat{w}),(z,w)) = \big\langle l,(z,w)\big\rangle, \quad \forall (z,w)\in P \text{ and } (\hat{z},\hat{w})\in P. \tag{4.64}
$$

We set

$$
(\hat{u},\hat{v}) = \big(e^{2s\hat{\beta}}\hat{\gamma}^{-9}\Pi_1 L^*(\hat{z},\hat{w}), e^{2s\beta^*+2s\hat{\beta}}(\gamma^*)^3\Pi_2 L^*(\hat{z},\hat{w})\big) \text{ and } \hat{g} = -e^{2s\beta^*+2s\hat{\beta}}(\gamma^*)^7\hat{w}\chi \tag{4.65}
$$

and we must see that $(\hat{u},\hat{v})$ verifies

$$
\iint_Q e^{-2s\hat{\beta}}\hat{\gamma}^9|\hat{u}|^2 + \iint_Q e^{-2s\beta^*-2s\hat{\beta}}(\gamma^*)^{-3}|\hat{v}|^2 + \iint_Q e^{-2s\beta^*-2s\hat{\beta}}(\gamma^*)^{-7}|\chi(x)|^2|\hat{g}|^2 < \infty
$$

and is a solution of the reaction diffusion system (4.55).

The first property follows from the fact that $(\hat{z},\hat{w})\in P$ and

$$
\begin{aligned}
& \iint_Q e^{-2s\hat{\beta}}\hat{\gamma}^9|\hat{u}|^2 + \iint_Q e^{-2s\beta^*-2s\hat{\beta}}(\gamma^*)^{-3}|\hat{v}|^2 + \iint_Q e^{-2s\beta^*-2s\hat{\beta}}(\gamma^*)^{-7}|\chi(x)|^2|\hat{g}|^2 \\
& = a((\hat{z},\hat{w}),(\hat{z},\hat{w})).
\end{aligned}
$$

In particular, $(\hat{u},\hat{v})\in L^2(Q)^2$ and $\hat{g}\in L^2(Q)$ and, from (4.63) and (4.64), it follows that $\hat{g}$ is bounded independently from $\epsilon$.

Let us now consider $(\tilde{u}, \tilde{v})$, the weak solution of

$$\left| \begin{array}{ll} \tilde{u}_t - \Delta \tilde{u} = -M_1 \Delta \tilde{v} + h_1 & \text{in } Q, \\ \epsilon \tilde{v}_t - \Delta \tilde{v} + bv = au + \hat{g}\chi + h_2 & \text{in } Q, \\ \frac{\partial \tilde{u}}{\partial \nu} = \frac{\partial \tilde{v}}{\partial \nu} = 0 & \text{on } \Sigma, \\ \tilde{u}(x,0) = u_0;\ \tilde{v}(x,0) = v_0 & \text{in } \Omega. \end{array} \right. \tag{4.66}$$

We have that $(\tilde{u}, \tilde{v})$ is also the unique solution of (4.66) defined by transposition. Of course, this means that $(\tilde{u}, \tilde{v})$ is the unique function such that

$$\iint_Q (\tilde{u}, \tilde{v}) \cdot (F_1, F_2) dxdt = \iint_Q h_1 \varphi dxdt + \iint_Q h_2 \xi dxdt + \iint_Q g\chi\xi dxdt \\ + (u_0, \varphi(0)) + \epsilon(v_0, w(0)), \tag{4.67}$$

where $(\varphi, \xi)$ is the solution of

$$\left| \begin{array}{ll} -\varphi_t - \Delta \varphi = a\xi + F_1 & \text{in } Q, \\ -\epsilon \xi_t - \Delta \xi = -b\xi - M_1 \Delta \varphi + F_2 & \text{in } Q, \\ \frac{\partial \varphi}{\partial \nu} = \frac{\partial \xi}{\partial \nu} = 0 & \text{on } \Sigma, \\ \varphi(x,T) = 0;\ \xi(x,T) = 0 & \text{in } \Omega. \end{array} \right. \tag{4.68}$$

From (4.64) and (4.65), we see that $(\hat{u}, \hat{v})$ also satisfies (4.67). Consequently, $(\hat{u}, \hat{v}) = (\tilde{u}, \tilde{v})$ and $(\hat{u}, \hat{v})$ is the solution of (4.55).

Finally, we must see that $(\hat{u}, \hat{v}, \hat{g})$ belongs to $E$.

We already know that

$$||e^{-s\hat{\beta}} \hat{\gamma}^{9/2} \hat{u}||_{L^2(Q)} + ||e^{-s\beta^* - s\hat{\beta}} (\gamma^*)^{-3/2} \hat{v}||_{L^2(Q)} + ||\chi(x) e^{-s\beta^* - s\hat{\beta}} (\gamma^*)^{-7/2} \hat{g}||_{L^2(Q)} < \infty,$$

$$e^{-2s\hat{\beta}} \hat{\gamma}^3 \Pi_1 L(\hat{u}, \hat{v}) \in L^2(Q),$$

$$e^{-s\beta - s\hat{\beta}} \hat{\gamma}^2 \left(\Pi_2 L(\hat{u}, \hat{v}) - \hat{g}\chi\right) \in L^2(0,T; H^1(\Omega))$$

and that

$$\int_\Omega \Pi_1 L(u,v) dx = 0.$$

The only thing remaining to check is that

$$e^{-s\hat{\beta}} \hat{\gamma}^{13/4} u \in L^2(0,T; H^2(\Omega)), e^{-s\hat{\beta}} \hat{\gamma}^{13/4} u \in L^\infty(0,T; H^1(\Omega))$$

and that

$$e^{-s\hat{\beta}}\hat{\gamma}^{-1/4}\Delta v \in L^2(0,T;H^1(\Omega)),\ e^{-s\hat{\beta}}\hat{\gamma}^{-1/4}\nabla v \in L^2(0,T;H^2(\Omega)).$$

To this end, let us introduce the pair $(u^*, v^*) = \rho(t)(\hat{u}, \hat{v})$, which satisfies:

$$\begin{cases} u^*_t - \Delta u^* = -M_1 \Delta v^* + \rho h_1 + \rho_t \hat{u} & \text{in } Q, \\ \epsilon v^*_t - \Delta v^* + b v^* = a u^* + \rho \hat{g} \chi + \epsilon \rho_t \hat{v} + \rho h_2 & \text{in } Q, \\ \frac{\partial u^*}{\partial \nu} = \frac{\partial v^*}{\partial \nu} = 0 & \text{on } \Sigma, \\ u^*(x,0) = \rho(0) u_0;\ v^*(x,0) = \rho(0) v_0 & \text{in } \Omega \end{cases} \tag{4.69}$$

We will consider two cases:

*Case 1.* $\rho = e^{-s\hat{\beta}}(\hat{\gamma})^{13/4}$.

We have

$$|\rho_t| = |s\hat{\beta}_t(\hat{\beta})^{13/4} e^{-s\hat{\beta}} + 13/4 e^{-s\hat{\beta}}(\hat{\gamma})^{9/4}\hat{\gamma}_t| \leq C\hat{\gamma}^{9/2} e^{-s\hat{\beta}}. \tag{4.70}$$

Therefore, the products $\rho_t \hat{u}$ and $\rho_t \hat{v}$ both belong to $L^2(0,T;L^2(\Omega))$. From well-known regularity properties for parabolic systems (see, for instance, [75]), we deduce that

$$\begin{cases} e^{-s\hat{\beta}}(\hat{\gamma})^{13/4}\hat{u} \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)), \\ e^{-s\hat{\beta}}(\hat{\gamma})^{13/4}\hat{v},\ e^{-s\hat{\beta}}(\hat{\gamma})^{13/4}\nabla \hat{v},\ e^{-s\hat{\beta}}(\hat{\gamma})^{13/4}\Delta \hat{v} \in L^2(0,T;L^2(\Omega)). \end{cases} \tag{4.71}$$

*Case 2.* $\rho = e^{-s\hat{\beta}}(\hat{\gamma})^{-1/4}$.

In this case, a simple calculation gives

$$|\rho_t| = |s\hat{\beta}_t(\hat{\beta})^{-1/4} e^{-s\hat{\beta}} + 1/4 e^{-s\hat{\beta}}(\hat{\gamma})^{-5/4}\hat{\gamma}_t| \leq C\hat{\gamma} e^{-s\hat{\beta}}. \tag{4.72}$$

From the regularity obtained in case 1 we have that $\rho\hat{u}$ and $\rho_t\hat{v}$ belong to $L^2(0,T;H^1(\Omega))$.

Using definition of $\hat{g}$ and (4.60), we can easily see that

$$\iint_Q |\nabla(e^{-s\hat{\beta}}(\hat{\gamma})^{-1/4}\hat{g})|^2 \leq C a((\hat{z},\hat{w}),(\hat{z},\hat{w})) \tag{4.73}$$

where $C$ does not depend on $\epsilon$ and hence it follows that $e^{-s/2\beta^*}\hat{\gamma}^{-25/8}\hat{g} \in L^2(0,T;H^1(\Omega))$ and is bounded independently from $\epsilon$.

Therefore, from the regularity theory for parabolic systems we deduce that

$$\begin{cases} e^{-s\hat{\beta}}(\hat{\gamma})^{-1/4}\hat{u} \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)), \\ e^{-s\hat{\beta}}(\hat{\gamma})^{-1/4}\nabla\hat{v} \in L^2(0,T;H^2(\Omega)),\ e^{-s\hat{\beta}}(\hat{\gamma})^{-1/4}\Delta\hat{v} \in L^2(0,T;H^1(\Omega)). \end{cases} \tag{4.74}$$

This finishes the proof of Proposition 4.2.

**Remark 4.6.** Given any $\epsilon > 0$, any $f \in L^2(0,T;H^1(\Omega))$ and any $z_0 \in H^2(\Omega)$, with $\frac{\partial z_0}{\partial \nu} = 0$, the solution of

$$\left| \begin{array}{ll} \epsilon z_t - \Delta z + z = f & \text{in } Q, \\ \frac{\partial z}{\partial \nu} = 0 & \text{on } \Sigma, \\ z(x,0) = z_0 & \text{in } \Omega \end{array} \right. \tag{4.75}$$

satisfies

$$\|z\|_{L^2(0,T;H^3(\Omega))} \leq C\big(\|f\|_{L^2(0,T;H^1(\Omega))} + \|z_0\|_{H^2(\Omega)}\big),$$

where $C > 0$ is independent from $\epsilon$.

In fact, multiplying (4.75) by $\epsilon\Delta z_t$ and integrating over $\Omega$, we get

$$\epsilon^2 \int_\Omega |\nabla v_t|^2 dxdt + \frac{\epsilon}{2}\frac{d}{dt}\int_\Omega |\Delta v|^2 dx \leq \int_\Omega |\nabla f|^2 dx.$$

This last inequality gives $\epsilon v_t \in L^2(0,T;H^1(\Omega))$. Using elliptic regularity for (4.75), the result follows.

□

## 4.4 Uniform exact controllability to the trajectory

In this section we give the proof of Theorem 4.1 using similar arguments to those employed in [38]. We will see that the results obtained in the previous section allow us to locally invert the nonlinear system (4.1). In fact, the regularity deduced for the solution of the linearized system (4.31) will be sufficient to apply a suitable inverse function theorem (see Theorem 4.3 below).

Thus, let us set $u = M_1 + z$ and $v = M_2 + w$ and let us use these equalities in (4.1). We find:

$$\left| \begin{array}{ll} L(z,w) = (-\nabla\cdot(z\nabla w), g\chi_\omega) & \text{in } Q, \\ \frac{\partial z}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 & \text{on } \Sigma, \\ z(x,0) = u_0 - M_1;\ w(x,0) = v_0 - M_2 & \text{in } \Omega. \end{array} \right. \tag{4.76}$$

$$L(u,v) = (u_t - \Delta u + M_1\Delta v, \epsilon v_t - \Delta v + bv - au).$$

This way, we have reduced our problem to a local null controllability result for the solution $(z, w)$ of the nonlinear problem (4.76). For the sequel, the following inverse mapping theorem is needed (see [27]):

**Theorem 4.3.** *Let $E$ and $G$ be two Banach spaces and let $\mathcal{A} : E \to G$ be a continuous function from $E$ to $G$ defined in $B_\eta(0)$ for some $\eta > 0$ with $\mathcal{A}(0) = 0$. Let $\Lambda$ be a continuous and linear operator from $E$ onto $Y$ and suppose there exists $C_0 > 0$ such that*

$$||e||_E \leq C_0 ||\Lambda(e)||_G \tag{4.77}$$

*and that there exists $\delta < C_0^{-1}$ such that*

$$||\mathcal{A}(e_1) - \mathcal{A}(e_2) - \Lambda(e_1 - e_2)|| \leq \delta ||e_1 - e_2|| \tag{4.78}$$

*whenever $e_1, e_2 \in B_\eta(0)$. Then the equation $\mathcal{A}(e) = h$ has a solution $e \in B_\eta(0)$ whenever $||h||_G \leq c\eta$, where $c = C_0^{-1} - \delta$.*

In our setting, we use this theorem with the space $E$ defined before and

$$G = X \times Y,$$

where

$$X = \{(h_1, h_2) : e^{-2s\hat{\beta}}\hat{\gamma}^3 h_1 \in L^2(Q), e^{-s\beta - s\hat{\beta}}\hat{\gamma}^2 h_2 \in L^2(0, T; H^1(\Omega)) \tag{4.79}$$

$$\text{and } \int_\Omega h_1(x, t)dx = 0 \text{ a. e. } t \in (0, T)\}, \tag{4.80}$$

$$Y = \{(z_0, w_0) \in H^1(\Omega) \times H^2(\Omega); \int_\Omega z_0 dx = 0 \text{ and } \frac{\partial w_0}{\partial \nu} = 0 \text{ on } \partial\Omega\} \tag{4.81}$$

and, for each $0 < \epsilon \leq 1$, the operator

$$\mathcal{A}(z, w, g) = (L(u, v) + ((\nabla \cdot (z\nabla w), -g\chi)), z(0), w(0)) \; \forall (z, w, g) \in E.$$

We have

$$\mathcal{A}'(0, 0, 0) = (L(u, v) + (0, -g\chi)), z(0), w(0)) \; \forall (z, w, g) \in E.$$

In order to apply Theorem 4.3 to our problem, we must check that the previous framework fits the regularity required. This is done using the following proposition.

**Proposition 4.3.** $\mathcal{A} \in C^1(E; G)$.

*Proof.* All the terms appearing in $\mathcal{A}$ are linear (and consequently $C^1$), except for the term $\nabla \cdot (z\nabla w)$. However, the operator

$$\big((z_1, w_1, g_1), (z_2, w_2, g_2)\big) \mapsto (\nabla \cdot (z_1 \nabla w_2), 0) \tag{4.82}$$

is bilinear, so it suffices to prove its continuity from $E \times E$ to $X$.

In fact, we have

$$\begin{aligned}
||(\nabla \cdot (z_1 \nabla w_2), 0)||_X =& ||z_1 \Delta w_2 + \nabla z_1 \nabla w_2||_{L^2(e^{-4s\hat{\beta}}\hat{\gamma}^6(0,T);\Omega)} \\
&\leq \overline{C}||e^{-2s\hat{\beta}}\hat{\gamma}^3 z_1 \Delta w_2||_{L^2(Q)} + ||e^{-2s\hat{\beta}}\hat{\gamma}^3 \nabla z_1 \nabla w_2||_{L^2(Q)} \\
&\leq \overline{C}\Big( ||e^{-s\hat{\beta}}\hat{\gamma}^{13/4} z_1 e^{-s\hat{\beta}}\hat{\gamma}^{-1/4} \Delta w_2||_{L^2(Q)} \\
&\quad + ||e^{-s\hat{\beta}}\hat{\gamma}^{13/4} \nabla z_1 e^{-s\hat{\beta}}\hat{\gamma}^{-1/4} \nabla w_2||_{L^2(Q)} \Big) \\
&\leq \overline{C}\Big( ||e^{-s\hat{\beta}}\hat{\gamma}^{13/4} z_1||_{L^\infty(0,T;H^1(\Omega))} ||e^{-s\hat{\beta}}\hat{\gamma}^{-1/4} \Delta w_2||_{L^2(0,T;H^1(\Omega))} \\
&\quad + ||e^{-s\hat{\beta}}\hat{\gamma}^{13/4} \nabla z_1||_{L^\infty(0,T;L^2(\Omega))} ||e^{-s\hat{\beta}}\hat{\gamma}^{-1/4} \nabla w_2||_{L^2(0,T;H^2(\Omega))} \Big),
\end{aligned}$$

for a positive constant $\overline{C}$ which does not depend on $\epsilon$.

Therefore, continuity of (4.82) is established and the proof Proposition 4.3 is finished. □

An application of Theorem 4.3 gives the existence of $\delta, \eta > 0$, which a priori depend on $\epsilon$, such that if $||(u_0 - M_1, v_0 - M_2)|| \leq \eta/(C_0^{-1} - \delta)$, then there exists a control $g = g(\epsilon)$ such that the associated solution $(z, w)$ to (4.76) verifies $z(T) = w(T) = 0$ and $||(z, w, g)||_E \leq \eta$. To finish the proof of Theorem 4.1, we must show that $C_0$, $\eta$ and $\delta$ does not depend on $\epsilon$. This is a direct consequence from the fact that the constant $C_0$ in (4.77) does not depend on $\epsilon$ (see Theorem 4.2), that we can take any $\delta < C_0^{-1}$ and that $\eta$ can be chosen to be $\delta/\overline{C}$.

# Chapter 5

# Null controllability of a system of viscoelasticity with a moving control

## 5.1 Introduction

In this chapter we are concerned with the controllability of the following model of viscoelasticity consisting of a wave equation with both viscous Kelvin-Voigt and frictional damping:

$$\left| \begin{array}{ll} y_{tt} - \Delta y - \Delta y_t + b(x) y_t = 1_{\omega(t)} h, & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = y_0(x); y_t(x,0) = y_1(x) & \text{in } \Omega. \end{array} \right. \tag{5.1}$$

Here $\Omega$ is a smooth, bounded open set in $\mathbb{R}^N$ $(N \geq 1)$, $b \in L^\infty(\Omega)$ is a given function determining the frictional damping and $h = h(x,t)$ denotes the control. To simplify the presentation and notation, and without loss of generality, the viscous constant has been taken to be the unit one $\nu = 1$. The same system could be considered with an arbitrary viscosity constant $\nu > 0$ leading to the more general system

$$y_{tt} - \Delta y - \nu \Delta y_t + b(x) y_t = 1_{\omega(t)} h, \tag{5.2}$$

but the analysis would be the same.

The control $h$ acting on the right hand side term as an external force is, for all $0 < t < T$, localized in a subset of $\Omega$. This fact is modeled by the multiplicative factor $1_{\omega(t)}$ which stands for the characteristic function of the set $\omega(t)$ that, for any $0 < t < T$, constitutes the support of the control, localized in a moving subset $\omega(t)$ of $\Omega$.

Typically we shall consider control sets $\omega(t)$ determined by the evolution of a given reference subset $\omega$ of $\Omega$ through a smooth flow $X(x,t,0)$.

We consider the problem of null controllability. In other words, given a final time $T$ and initial data for the system $(y_0, y_1)$ in a suitable functional setting, we analyze the existence of a control $h = h(x,t)$ such that the corresponding solution satisfies the rest condition at the final time $t = T$:

$$y(x,T) \equiv y_t(x,T) \equiv 0, \quad \text{in } \Omega.$$

One of the distinguished features of the system under consideration is that, for this null controllability condition to be fulfilled, the control needs to move in time. Indeed, if $\omega(t) \equiv \omega$ for all $0 < t < T$, i.e. if the support of the control does not move in time as it is often considered, the system under consideration is not controllable. This can be easily seen at the level of the dual observability problem. In fact, the structure of the underlying PDE operator and, in particular, the existence of time-like characteristic hyperplanes, makes impossible the propagation of information in the space-like directions, thus making the observability inequality also impossible. This fact was observed in the work by P. Martin *et al.* in [95] in the $1-d$ setting. There, for the $1-d$ model, it was shown that this obstruction could be removed by making the control move so that its support covers the whole domain where the equation evolves.

More precisely, in [95], the $1-d$ version of the problem above was considered in the torus, with periodic boundary conditions, $b \equiv 0$ and $\omega(t) = \{x - t;\ x \in \omega\}$, i.e.

$$y_{tt} - y_{xx} - y_{xxt} = 1_{\omega(t)} h(x,t), \quad x \in \mathbb{T}. \tag{5.3}$$

Recall that this system with boundary control, i.e. $h \equiv 0$ and the boundary conditions

$$y(0,t) = 0, \quad y(1,t) = g(t),$$

$g = g(t)$ being the boundary control, fails to be spectrally controllable, because of the existence of a limit point in the spectrum of the adjoint system (see [109]). In the moving frame $x' = x + t$, (5.3) may be written as

$$z_{tt} - 2z_{xt} - z_{txx} + z_{xxx} = a(x)h(x+t,t) \tag{5.4}$$

where $z(x,t) = y(x+t,t)$. In [95] the spectrum of the adjoint system to (5.4) was shown to be split into a hyperbolic part and a parabolic one. As a consequence, equation (5.4) was proved to be null controllable in large time. A similar result was proved in [112] for the Benjamin-Bona-Mahony equation

$$y_t - y_{txx} + y_x + yy_x = a(x - ct)h(x,t), \qquad x \in \mathbb{T}.$$

Once again this system turns out to be globally controllable and exponentially stabilizable in $H^1(\mathbb{T})$ for any $c \neq 0$. But, as noticed in [96], the linearized equation fails to be spectrally controllable with a control supported in a fixed domain.

As mentioned above, in both cases, the lack of controllability of these systems with immobile controls is due to the fact that the underlying PDE operators exhibit the presence of time-like characteristic lines thus making propagation in the space-like directions impossible. By the contrary, when analyzing the problem in a moving frame, the characteristic lines are oblique ones in $(x, t)$, thus facilitating propagation properties.

In this chapter, we extend the $1-d$ analysis in [95] to the multi-dimensional case. This can not be done with the techniques in [95] based on Fourier analysis. Our approach is rather inspired on the fact that system (5.1) can be rewritten as a system coupling a parabolic equation with an ODE. The presence of this ODE, in the case of a fixed support of the control, independent of $t$, is responsible for the lack of controllability of the system, due to the absence of propagation in the space-like direction. Letting the control move introduces an effect similar to adding a transport term in the ODE but keeping the control immobile, thus changing the structure of the system into a parabolic-transport coupled one. This new system turns out to be controllable under the condition that all characteristics of the transport equation enter within the control set in the given control time, a condition that is reminiscent of the so-called Geometric Control Condition in the context of the wave equation (see [4]).

The approach in [95] would suggest to do the following splitting of (5.1):

$$\begin{cases} v_t - \Delta v = 1_{\omega(t)} h + (1-b)(v-y), \\ y_t + y = v. \end{cases} \tag{5.5}$$

However, the splitting can be performed in an alternative manner as follows:

$$\begin{cases} y_t - \Delta y + (b-1)y = z, \\ z_t + z = 1_{\omega(t)} h + (b-1)y. \end{cases} \tag{5.6}$$

It can easily be seen that $y$ solves equation $(5.1)_1$ if, and only if, it is the first component of the solution of system (5.6).

Our analysis of the Carleman inequalities for these systems is analog to that in [1] for a system of thermoelasticity coupling the heat and the wave equation. The key in [1] and in our own analysis is to use the same weight function both for the Carleman inequality of the heat and the hyperbolic model. In [1], since dealing with the wave equation, rather strong geometric conditions were needed on the subset where the control or the observation mechanism acts. In our case, since we are considering the simpler transport equation, the geometric assumptions will be milder, consisting mainly on a characteristic condition ensuring that all characteristic lines of the transport equation intersect the control/observation set. This suffices for the Carleman inequality to hold for the transport equation and is also sufficient for the heat equation that it is well-known to be controllable/observable from any open non-empty subset of the space-time cylinder where the equation is formulated.

It is important to mention that, as far as we know, all the Carleman inequalities for the heat equation available in the literature are done for the case where the control region is fixed. In the case we are dealing, the control region is moving in time. Therefore, a new Carleman inequality must be proved in this framework. The proof of a Carleman inequality for the heat equation when the control region is moving is one of the novelties we present in this chapter.

In order to state the main result of this chapter, we first describe precisely the class of moving trajectories for the control for which our null controllability result will hold.

**Admissible trajectories**: In practice the trajectory of the control can be taken to be determined by the flow $X(x,t,t_0)$ generated by some vector field

$$f \in C([0,T]; W^{2,\infty}(\mathbb{R}^N;\mathbb{R}^N)),$$

i.e., $X$ solves

$$\begin{cases} \dfrac{\partial X}{\partial t}(x,t,t_0) = f(X(x,t,t_0),t), \\ X(x,t_0,t_0) = x. \end{cases} \tag{5.7}$$

For instance, any translation of the form:

$$X(x,t,t_0) = x + \gamma(t) - \gamma(t_0), \tag{5.8}$$

where $\gamma \in C^1([0,T];\mathbb{R}^N)$, is admissible. (Pick $f(x,t) = \dot\gamma(t)$).

We assume that there exist a bounded, smooth, open set $\omega_0 \subset \mathbb{R}^N$, a curve $\Gamma \in C^\infty([0,T];\mathbb{R}^N)$, and two times $t_1, t_2$ with $0 \leq t_1 < t_2 \leq T$ such that:

$$\Gamma(t) \in X(\omega_0, t, 0) \cap \Omega, \quad \forall t \in [0, T]; \tag{5.9}$$

$$\overline{\Omega} \subset \cup_{t\in[0,T]} X(\omega_0, t, 0) = \{X(x, t, 0);\ \ x \in \omega_0,\ t \in [0, T]\}; \tag{5.10}$$

$$\Omega \setminus \overline{X(\omega_0, t, 0)} \text{ is nonempty and connected for } t \in [0, t_1] \cup [t_2, T]; \tag{5.11}$$

$$\Omega \setminus \overline{X(\omega_0, t, 0)} \text{ has two connected components for } t \in (t_1, t_2); \tag{5.12}$$

$$\forall \gamma \in C([0, T]; \Omega),\ \exists t \in [0, T], \quad \gamma(t) \in X(\omega_0, t, 0). \tag{5.13}$$

The main result in this chapter is as follows.

**Theorem 5.1.** *Let $T > 0$, $X(x, t, t_0)$ and $\omega_0$ be as in (5.9)-(5.13), and let $\omega$ be any open set in $\Omega$ such that $\overline{\omega}_0 \subset \omega$. Then for all $(y_0, y_1) \in L^2(\Omega)^2$ with $y_1 - \Delta y_0 \in L^2(\Omega)$, there exists a function $h \in L^2(0, T; L^2(\Omega))$ for which the solution of*

$$\left| \begin{array}{lll} y_{tt} - \Delta y - \Delta y_t + b(x) y_t = \mathbf{1}_{X(\omega,t,0)}(x) h, & \textit{in } Q, \\ y(x, t) = 0, & \textit{on } \Sigma, \\ y(., 0) = y_0; y_t(., 0) = y_1 & \textit{in } \Omega, \end{array} \right. \tag{5.14}$$

*fulfills $y(., T) = y_t(., T) = 0$.*

A few remarks are in order in what concerns the functional setting of this model:

- Viewing the system of viscoelasticity under consideration as a damped wave equation, a natural functional setting would be the following: For data in $H_0^1(\Omega) \times L^2(\Omega)$ and, say, right hand side term of $(5.1)_1$ in $L^2(0, T; H^{-1}(\Omega))$, there exists an unique solution $y \in C([0, T]; H_0^1(\Omega) \cap C^1([0, T]; L^2(\Omega))$. Furthermore $y_t \in L^2(0, T; H_0^1(\Omega))$. The latter is an added integrability/regularity property of the solution that is due to the strong damping effect of the system. This can be seen naturally by considering the energy of the system

$$E(t) = \frac{1}{2} \int_\Omega [|y_t|^2 + |\nabla y|^2] dx,$$

that fulfills the energy dissipation law

$$\frac{d}{dt} E(t) = -\int_\Omega [|\nabla y_t|^2 + b(x)|y_t|^2] dx + \int_{\omega(t)} h y_t dx.$$

- We can solve (5.6) so that $y$, solution of the heat equation, lies in the space $C([0,T];H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ and $z$, solution of the ODE, in $C([0,T];L^2(\Omega))$. This can be done provided $(y_0, y_1 - \Delta y_0) \in H_0^1(\Omega) \times L^2(\Omega)$. The functional setting is not exactly the same but this is due to the fact that, in some sense, in one case we see the system as a perturbation of the wave equation, while, in the other one, as a variant of the heat equation.

- From a control theoretical point of view it is much more efficient to analyze the system in the second setting, as a perturbation of the heat equation, through the coupling with the ODE or, after changing variables, with a transport equation. If we view the system of viscoelasticity as a perturbation of the wave equation, standard hyperbolic techniques such as multiplier, Carleman inequalities or microlocal tools do not apply since, actually, the viscoelastic term determines the principal part of the underlying PDE operator and cannot be viewed as a perturbation of the wave dynamics.

The analysis is particularly simple in the special case where $b \equiv 1$. Indeed, in that case, system (5.5) (with a second control incorporated in the ODE) takes the following cascade form

$$\begin{cases} v_t - \Delta v = 1_{\omega(t)}\tilde{h}, \\ y_t + y = 1_{\omega(t)}\tilde{k} + v, \end{cases} \tag{5.15}$$

where the parabolic equation in (5.15) is *uncoupled*. This system will be investigated in a separate section (section 5.2) since some of the basic ideas allowing to handle the general case emerge already in its analysis. Note that, in this particular case, roughly, one can first control the heat equation by a suitable control $\tilde{h}$ and then, once this is done, viewing $v$ as a given source term, control the transport equation by a convenient $\tilde{k}$. This case is also important because the only assumption needed to prove Theorem 5.1 in this case is (5.10) (i.e. we don't assume that (5.9) and (5.11)-(5.13) are satisfied). In this particular case $b \equiv 1$ a similar argument can be used with the second decomposition.

## 5.2 Analysis of the decoupled cascade system

In this section, we give a proof of Theorem 5.1 in the special situation when $b \equiv 1$, and $\omega(t) = X(\omega_0, t, 0)$, where $X$ is given by (5.7) for some $f \in C(\mathbb{R}^+; W^{2,\infty}(\mathbb{R}^N; \mathbb{R}^N))$.
As we said before, we will prove Theorem 5.1 in the case $b \equiv 1$ by proving a null controllability result for the decomposition (5.15). The idea of the proof is as follows. We take some appropriate $0 < \epsilon < T$ and then drive the solution of the heat equation $(5.15)_1$ to zero in time $\epsilon$ by means of a control $\tilde{h}$. Next, we let equation $(5.15)_2$ evolves

freely in $[0,\epsilon]$, i.e., $\tilde{k} \equiv 0$, and then we control this equation by means of a smooth control $\tilde{k}$ in the time interval $[\epsilon, T]$. This gives the null controllability of the system (5.15) in the whole time interval $[0,T]$.

**Proof of Theorem 5.1 in the case** $b \equiv 1$.

Suppose (5.10) is satisfied and let

$$T_0 = \inf\{T > 0;\ \overline{\Omega} \subset \cup_{0\le t\le T} X(\omega_0, t, 0)\}. \tag{5.16}$$

Pick any $T > T_0$, and pick some $\varepsilon \in (0, T - T_0)$ and some nonempty open set $\omega_{-1} \subset \omega_0$ such that

$$\overline{\Omega} \quad \subset \quad \cup_{\varepsilon \le t\le T} X(\omega_0, t, 0), \tag{5.17}$$

$$\omega_{-1} \quad \subset \quad X(\omega_0, t, 0) \quad \forall t \in (0,\varepsilon). \tag{5.18}$$

Let $T' = T - \varepsilon$, and pick any $(v_0, y_0) \in L^2(\Omega)^2$. It is well known (see [130]) that there exists some control input $h \in L^2(0,\varepsilon; L^2(\Omega))$ such that the solution $v = v(x,t)$ of

$$\left|\begin{array}{ll} v_t - \Delta v = \mathbf{1}_{\omega_{-1}} h & \text{in } \Omega \times (0,\varepsilon), \\ v(x,0) = v_0(x) & \text{in } \Omega, \end{array}\right. \tag{5.19}$$

satisfies

$$v(x,\varepsilon) = 0, \qquad x \in \Omega.$$

Set

$$\begin{aligned} \tilde{h}(x,t) = \mathbf{1}_{\omega_{-1}}(x) h(x,t), &\qquad x \in \Omega,\ t \in (0,\varepsilon), \\ \tilde{h}(x,t) = 0, &\qquad x \in \Omega,\ t \in (\varepsilon, T), \\ \tilde{k}(x,t) = 0, &\qquad x \in \Omega,\ t \in (0,\varepsilon). \end{aligned}$$

Then the solution $v$ of

$$\left|\begin{array}{ll} v_t - \Delta v = \mathbf{1}_{X(\omega_0,t,0)}(x)\tilde{h} & \text{in } Q, \\ v(x,0) = v_0(x) & \text{in } \Omega, \end{array}\right. \tag{5.20}$$

satisfies $v(x,t) = 0$ for $t \in [\varepsilon, T]$. We claim that the system

$$\left|\begin{array}{ll} y_t + y = \mathbf{1}_{X(\omega_0,t,0)}(x)k(x,t) & \text{in } \Omega \times (\varepsilon, T), \\ y(x,\varepsilon) = y_0(x) & \text{in } \Omega, \end{array}\right. \tag{5.21}$$

is exactly controllable in $L^2(\Omega)$ on the time interval $(\varepsilon, T)$. By duality, this is equivalent to proving that the corresponding observability inequality

$$\int_\Omega |q_0(x)|^2 dx \le C \int_\varepsilon^T \int_\Omega \mathbf{1}_{X(\omega_0,t,0)}(x)\ |q(x,t)|^2 dx dt \tag{5.22}$$

is fulfilled with a uniform constant $C > 0$ for all solution $q$ of the adjoint system

$$\left| \begin{array}{ll} -q_t + q = 0 & \text{in } \Omega \times (\varepsilon, T), \\ q(x,T) = q_0(x) & \text{in } \Omega. \end{array} \right. \tag{5.23}$$

Since the solution of (5.23) is given by $q(x,t) = e^{t-T} q_0(x)$, we have that

$$\int_\varepsilon^T \int_\Omega \mathbf{1}_{X(\omega_0,t,0)}(x)\ |q(x,t)|^2 dx dt \ge e^{2(\varepsilon-T)} \int_\Omega |q_0(x)|^2 (\int_\varepsilon^T \mathbf{1}_{X(\omega_0,t,0)}(x)\ dt) dx. \tag{5.24}$$

From (5.16) and the smoothness of $X$, we see that for all $x \in \overline{\Omega}$, there is some $t_0 \in (\varepsilon, T)$, and some $\delta > 0$ such that for any $y \in B(x,\delta)$ and any $t \in (\varepsilon,T) \cap (t_0 - \delta, t_0 + \delta)$ we have $y \in X(\omega_0, t, 0)$. From the compactness of $\overline{\Omega}$, we see that there exists a number $\delta_0 > 0$ such that

$$\int_\varepsilon^T \mathbf{1}_{X(\omega_0,t,0)}(x) dt > \delta_0, \qquad \forall x \in \overline{\Omega}.$$

Combined with (5.24), this yields (5.22). Thus, (5.21) is exactly controllable in $L^2(\Omega)$ on $(\varepsilon, T)$ with some controls $k \in C([\varepsilon, T]; L^2(\Omega))$.
Let $y_1(x) = e^{-\varepsilon} y_0(x) + \int_0^\varepsilon e^{s-\varepsilon} v(x,s) ds$. Extend $\tilde{k}$ to $(0,T)$ so that $\tilde{k} \in L^2(0,T; L^2(\Omega))$ and the solution of

$$\left| \begin{array}{ll} y_t + y = \mathbf{1}_{X(\omega_0,t,0)}(x) \tilde{k} & \text{in } \Omega \times (\varepsilon, T), \\ y(x,\varepsilon) = y_1(x) & \text{in } \Omega, \end{array} \right. \tag{5.25}$$

satisfies $y(.,T) = 0$. Thus the control $(\tilde{h}, \tilde{k})$ steers the solution of (5.15) from $(v_0, y_0)$ at $t = 0$ to $(0,0)$ at $t = T$. Applying the operator $\partial_t - \Delta$ in each side of $(5.15)_2$ results in

$$y_{tt} - \Delta y - \Delta y_t + y_t = \mathbf{1}_{X(\omega_0,t,0)}(x) \tilde{h} + (\partial_t - \Delta)[\mathbf{1}_{X(\omega_0,t,0)}(x) \tilde{k}], \tag{5.26}$$

This proves Theorem 5.1, except for the fact that the control does not live in $L^2(0,T; L^2(\Omega))$.
Assume now that $(v_0, y_0) \in L^2(\Omega) \times [H^2(\Omega) \cap H_0^1(\Omega)]$. To get a control $\tilde{k} \in L^2(0,T; H^2(\Omega))$, it is sufficient to replace $\mathbf{1}_{\omega(t)}$ by $a(X(x,0,t))$ in $(5.15)_2$, where $a$ is a function satisfying

$$a \in C_0^\infty(\omega),$$
$$a(x) = 1 \quad \forall x \in \omega_0.$$

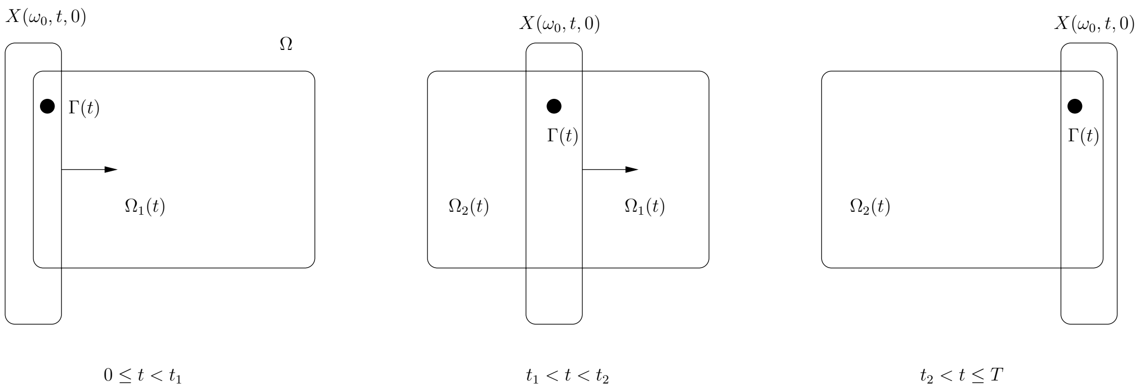


Figure 5.1: Example for which conditions (5.9)-(5.13) are satisfied.

The proof is completed by showing the observability inequality

$$||q_0||^2_{X'} \le C \int_{\varepsilon}^{T} ||(t-\varepsilon)a(X(.,0,t))q(.,t)||^2_{X'} dt$$

for the solution $q$ of system (5.23), where $X = H^2(\Omega) \cap H^1_0(\Omega)$ and $X'$ stands for its dual space. This can be done as in [111, Proposition 2.1]. Next, using the HUM operator, we notice that $\tilde{k} \in C^1([\varepsilon, T]; X)$ with $\tilde{k}(., \varepsilon) = 0$, since $q \in C^1([\varepsilon, T]; X')$ for any $q_0 \in X'$. Thus, with this small change, the right hand side term in (5.26) can be written $\mathbf{1}_{X(\omega,t,0)} u(x,t)$, where $u \in L^2(0,T; L^2(\Omega))$.

**Remark 5.1.** Observe that the situation when $X(\omega_0, t, 0)$ moves as in Figure 5.2, Figure 5.4 or in Figure 5.5 (see below) is admissible in the case when $b \equiv 1$.

## 5.3 Examples

In this section, we provide some geometric examples to illustrate the assumptions (5.9)-(5.13). We use simple shapes (like rectangles) just for convenience.

- Figure 5.1 shows how a control region should move in order to satisfy conditions (5.9)-(5.13).
- Figure 5.2 depicts a situation for which Theorem 5.1 cannot be applied, except in the case when $b \equiv 1$, as condition (5.12) fails.
- In Figure 5.3, we modify the example given in Figure 5.1 by shifting the time. Theorem 5.1 cannot be applied as it is, since (5.11) fails. However, the conclusion

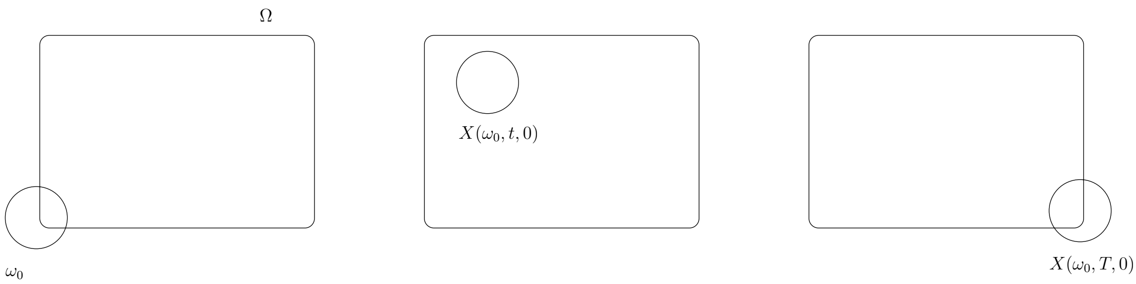


Figure 5.2: Example for which condition (5.12) fails.

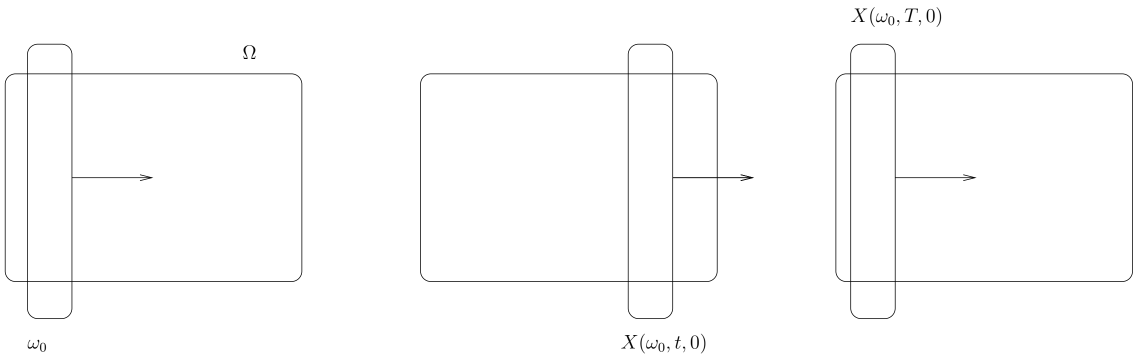


Figure 5.3: Example for which condition (5.11) fails.

of Theorem 5.1 remains valid. Indeed, assume that $\Omega\setminus\overline{\omega(t)}$ has two connected components (resp. one) for $t\in[0,\tau_1)\cup(\tau_2,T]$ (resp. for $t\in[\tau_1,\tau_2]$). Assume that the "jump" of $\omega(t)$ occurs at $t=\tau_3$, with $\tau_1<\tau_3<\tau_2$. Let

$$\mathcal{O}_1 := \cup_{0\le t\le \tau_3}\,\omega(t), \tag{5.27}$$

$$\mathcal{O}_2 := \cup_{\tau_3\le t\le T}\,\omega(t) \tag{5.28}$$

and let $\eta\in C^\infty(\Omega;[0,1])$ be such that

$$\text{supp}(\eta)\subset\mathcal{O}_1, \tag{5.29}$$

$$\text{supp}(1-\eta)\subset\mathcal{O}_2, \tag{5.30}$$

$$\text{supp}(\nabla\eta)\subset\omega_0. \tag{5.31}$$

Then, applying the Carleman estimate in Lemma 5.4 to $(p_1,q_1)=\eta\big(X(x,0,t)\big)(p,q)$ in $\Omega\cap\{\eta>0\}$ on the time interval $[0,\tau_3]$, and to $(p_2,q_2)=\big(1-\eta(X(x,0,t))\big)(p,q)$ in $\Omega\cap\{\eta<1\}$ on the time interval $[\tau_3,T]$, we can easily prove the observability inequality (5.34).

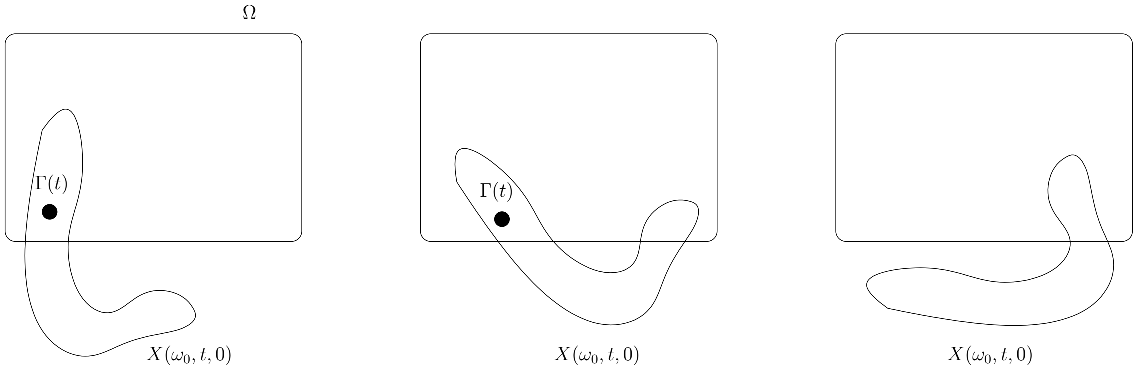


Figure 5.4: Example showing that $X(\omega_0, t, 0) \cap \Omega \neq \emptyset \; \forall t \in [0, T]$ does not imply (5.9).

- Figure 5.4 shows that the assumption (5.9), which is needed to construct the weight function $\psi$ in Lemma 5.1 cannot be replaced by the simpler condition

$$X(\omega_0, t, 0) \cap \Omega \neq \emptyset, \quad \forall t \in [0, T].$$

- Figure 5.5 shows that the assumption (5.13), which is also needed to construct the weight function $\psi$ in Lemma 5.1, does not result from the other assumptions (5.9)-(5.12).

## 5.4 Null controllability of system (5.15).

In this section we proof Theorem 5.1. Using decomposition (5.6), it is easy to see that the null controllability of (5.1) turns out to be equivalent to the null controllability of the system

$$\left| \begin{array}{ll} y_t - \Delta y + (b(x) - 1)y = z & \text{in } Q, \\ z_t + z = \mathbf{1}_{X(\omega,t,0)}(x)h + (b(x) - 1)y & \text{in } Q, \\ y(x,t) = 0 & \text{on } \Sigma, \\ z(x,0) = z_0(x); y(x,0) = y_0(x) & \text{in } \Omega. \end{array} \right. \tag{5.32}$$

More precisely, Theorem 5.1 is a direct consequence of the following result.

**Theorem 5.2.** *Let $T$, $X(x,t,t_0)$ and $\omega_0$ be as in (5.9)-(5.13), and let $\omega$ be as in Theorem 5.1. Then for all $(y_0, z_0) \in L^2(\Omega)^2$, there exists a control function $h \in L^2(0,T; L^2(\Omega))$ for which the solution $(y,z)$ of (5.32) satisfies $y(.,T) = z(.,T) = 0$.*

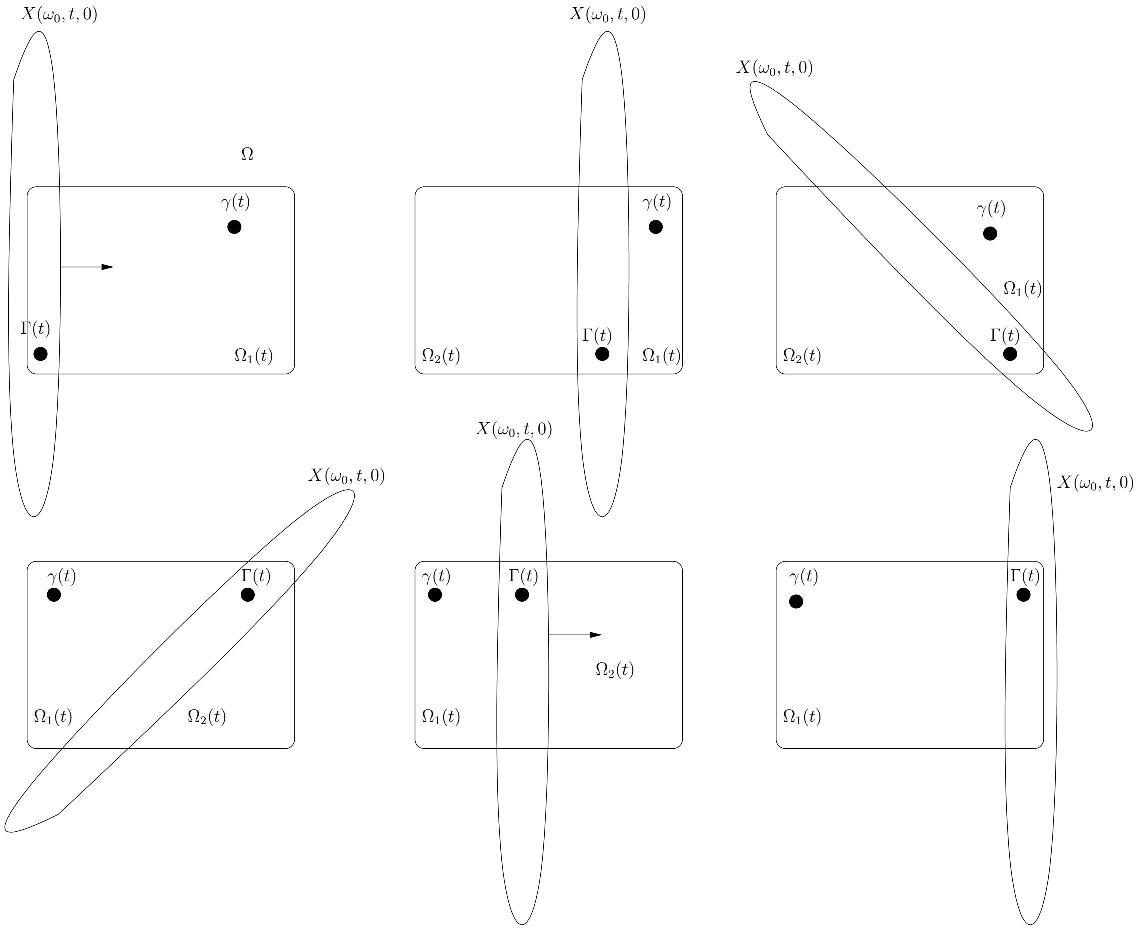


Figure 5.5: Example showing that (5.9)-(5.12) does not imply (5.13).

From now on we concentrate in the proof of Theorem 5.2.

Theorem 5.2 is equivalent to prove an observability inequality for the adjoint system of (5.32), namely

$$\left|\begin{array}{ll} -p_t - \Delta p + (b(x)-1)p = (b(x)-1)q & \text{in } Q, \\ -q_t + q = p & \text{in } Q, \\ p(x,t) = 0 & \text{in } \Sigma, \\ p(x,T) = p_0(x); q(x,T) = q_0(x) & \text{in } \Omega. \end{array}\right. \tag{5.33}$$

In fact, one can show that Theorem 5.2 is equivalent to the following:

**Proposition 5.1.** *Let $T$, $X$, $\omega_0$ and $\omega$ be as in Theorem 5.1. Then there exists a constant $C > 0$*

*such that for all* $(p_0, q_0) \in L^2(\Omega)^2$*, the solution* $(p, q)$ *of* (5.33) *satisfies:*

$$\int_\Omega [|p(x,0)|^2 + |q(x,0)|^2]dx \le C \int_0^T \int_{X(\omega,t,0)} |q(x,t)|^2 \, dxdt. \tag{5.34}$$

*Proof of Proposition 5.1.* Inspired in part by [1] (which was concerned with a heat-wave system[1]), we shall establish some Carleman estimates for the (backward) parabolic equation $(5.33)_1$ and the ODE $(5.33)_2$ with the *same singular weight*.

For a better comprehension, the proof will be divided into two steps as follows:

*Step 1.* We apply suitable Carleman estimates for the parabolic equation $(5.33)_1$ and the ODE $(5.33)_2$, with the same weights and with a moving control region.

*Step 2.* We estimate a local integral of $p$ in terms of a local integral of $q$ and some small order terms. Finally, we combine all the estimates obtained in the first step and derive the desired Carleman inequality.

The basic weight function we need in order to prove such inequalities is given by the following Lemma.

**Lemma 5.1.** *Given* $T > 0$*. Let* $X$*,* $\omega_0$ *and* $\omega$ *be as in Theorem 5.1, and let* $\omega_1$ *be a nonempty open set in* $\mathbb{R}^N$ *such that*

$$\overline{\omega_0} \subset \omega_1, \quad \overline{\omega_1} \subset \omega. \tag{5.35}$$

*Then there exist a number* $\delta \in (0, T/2)$ *and a function* $\psi \in C^\infty(\overline{\Omega} \times [0,T])$ *such that*

$$\nabla\psi(x,t) \neq 0, \qquad t \in [0,T],\ x \in \overline{\Omega} \setminus X(\omega_1, t, 0), \tag{5.36}$$

$$\psi_t(x,t) \neq 0, \qquad t \in [0,T],\ x \in \overline{\Omega} \setminus X(\omega_1, t, 0), \tag{5.37}$$

$$\psi_t(x,t) > 0, \qquad t \in [0,\delta],\ x \in \overline{\Omega} \setminus X(\omega_1, t, 0), \tag{5.38}$$

$$\psi_t(x,t) < 0, \qquad t \in [T-\delta,T],\ x \in \overline{\Omega} \setminus X(\omega_1, t, 0), \tag{5.39}$$

$$\frac{\partial\psi}{\partial n}(x,t) \le 0, \qquad t \in [0,T],\ x \in \partial\Omega, \tag{5.40}$$

$$\psi(x,t) > \frac{3}{4}||\psi||_{L^\infty(\Omega\times(0,T))}, \qquad t \in [0,T],\ x \in \overline{\Omega}. \tag{5.41}$$

The proof of Lemma 5.1 will be given in section 5.5.

[1]See also [29] for some Carleman estimates for a coupled system of parabolic-hyperbolic equations.

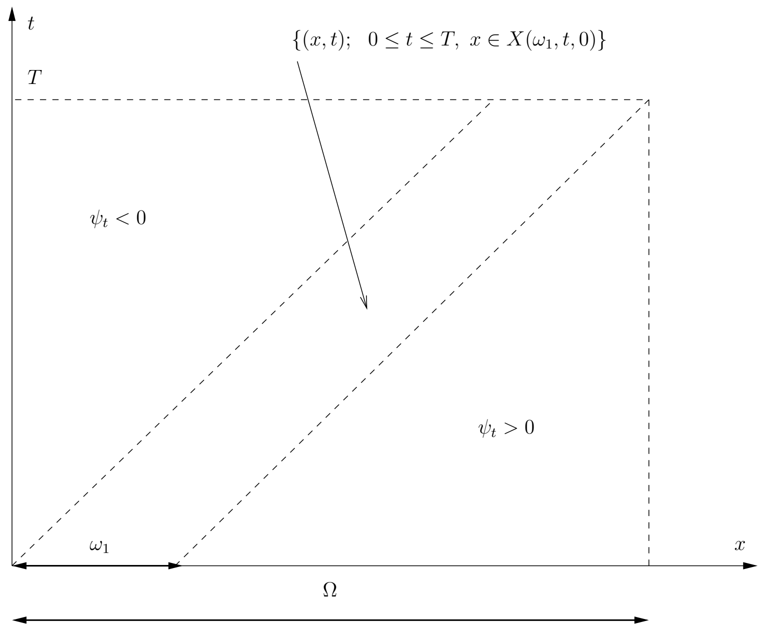


Figure 5.6: Sign of the time derivative of $\psi$.

Next, we pick a function $g \in C^{\infty}(0,T)$ such that

$$g(t) = \begin{cases} \frac{1}{t} & \text{for } 0 < t < \delta/2, \\ \text{strictly decreasing} & \text{for } 0 < t \leq \delta, \\ 1 & \text{for } \delta \leq t \leq \frac{T}{2}, \\ g(T-t) & \text{for } \frac{T}{2} \leq t < T \end{cases}$$

and define the weights

$$\begin{aligned} \varphi(x,t) &= g(t)(e^{\frac{3}{2}\lambda||\psi||_{L^\infty}} - e^{\lambda\psi(x,t)}), & (x,t) \in \Omega \times (0,T), \\ \theta(x,t) &= g(t)e^{\lambda\psi(x,t)}, & (x,t) \in \Omega \times (0,T), \end{aligned}$$

where $||\psi||_{L^\infty} = ||\psi||_{L^\infty(\Omega\times(0,T))}$ and $\lambda > 0$ is a parameter.

**Remark 5.2.** Basically, $\psi$ drags the critical points of $\psi(x,0)$ inside the control region during the evolution of the flow. This is the analogous of the function given by Lemma 2.1 for the case of static control region.

*Step 1. Carleman estimates with the same weight.*

In this step we apply a Carleman inequality for the heat-like equation $(5.33)_1$ and a Carleman inequality for the ODE $(5.33)_2$, both with the same weight. We combine such inequalities and obtain a global estimation of $p$ and $q$ in terms of local integrals of $p$ and $q$.

For the purpose of the proof, we assume that the following two lemmas are true (their proof are given in sections 5.6 and 5.7, respectively).

**Lemma 5.2.** *There exist some constants $\lambda_0 > 0$, $s_0 > 0$ and $C_0 > 0$ such that for all $\lambda \geq \lambda_0$, all $s \geq s_0$ and all $p \in C([0,T]; L^2(\Omega))$ with $p_t + \Delta p \in L^2(0,T; L^2(\Omega))$, the following holds*

$$\int_0^T\int_\Omega [(s\theta)^{-1}(|\Delta p|^2 + |p_t|^2) + \lambda^2(s\theta)|\nabla p|^2 + \lambda^4(s\theta)^3|p|^2]e^{-2s\varphi}dxdt \\ \leq C_0\left(\int_0^T\int_\Omega |p_t + \Delta p|^2 e^{-2s\varphi}dxdt + \int_0^T\int_{X(\omega_1,t,0)} \lambda^4(s\theta)^3|p|^2e^{-2s\varphi}dxdt\right). \quad (5.42)$$

**Lemma 5.3.** *There exist some numbers $\lambda_1 \geq \lambda_0$, $s_1 \geq s_0$ and $C_1 > 0$ such that for all $\lambda \geq \lambda_1$, all $s \geq s_1$ and all $q \in H^1(0,T; L^2(\Omega))$, the following holds*

$$\int_0^T\int_\Omega (\lambda^2 s\theta)|q|^2e^{-2s\varphi}dxdt \leq C_1\left(\int_0^T\int_\Omega |q_t|^2e^{-2s\varphi}dxdt \right. \\ \left. + \int_0^T\int_{X(\omega,t,0)} \lambda^2(s\theta)^2|q|^2e^{-2s\varphi}dxdt\right). \quad (5.43)$$

Applying the Carleman inequality given in Lemma 5.2 to the heat-like equation $(5.33)_1$, we obtain

$$\int_0^T\int_\Omega [(s\theta)^{-1}(|\Delta p|^2 + |p_t|^2) + \lambda^2(s\theta)|\nabla p|^2 + \lambda^4(s\theta)^3|p|^2]e^{-2s\varphi}dxdt \\ \leq C_0\left(\int_0^T\int_\Omega |(b(x-1)(p-q)|^2e^{-2s\varphi}dxdt + \int_0^T\int_{X(\omega_1,t,0)} \lambda^4(s\theta)^3|p|^2e^{-2s\varphi}dxdt\right). \quad (5.44)$$

Next, we apply the Carleman inequality given by Lemma 5.3 to the ODE $(5.33)_2$ and obtain

$$\int_0^T\int_\Omega (\lambda^2 s\theta)|q|^2e^{-2s\varphi}dxdt \\ \leq C_1\left(\int_0^T\int_\Omega |q-p|^2e^{-2s\varphi}dxdt + \int_0^T\int_{X(\omega,t,0)} \lambda^2(s\theta)^2|q|^2e^{-2s\varphi}dxdt\right). \quad (5.45)$$

Adding (5.44) and (5.45), it is not difficult to see that

$$\int_0^T\int_\Omega [(s\theta)^{-1}(|\Delta p|^2+|p_t|^2)+\lambda^2(s\theta)|\nabla p|^2+\lambda^4(s\theta)^3|p|^2]e^{-2s\varphi}dxdt$$
$$+\int_0^T\int_\Omega (\lambda^2 s\theta)|q|^2e^{-2s\varphi}dxdt$$
$$\leq C\left(\int_0^T\int_{X(\omega,t,0)}\lambda^2(s\theta)^2|q|^2e^{-2s\varphi}dxdt+\int_0^T\int_{X(\omega_1,t,0)}\lambda^4(s\theta)^3|p|^2e^{-2s\varphi}dxdt\right), \quad (5.46)$$

for appropriate $s\geq s_2\geq s_1$ and $\lambda\geq\lambda_2\geq\lambda_1$.

*Step 2. Arrangements and conclusion.*

In this step we estimate the local integral of $p$ appearing in (5.46) by a local integral of $q$ and some small order terms. Finally, using semigroup theory, we finish the proof of Proposition 5.1.

The main result of this step is the following.

**Lemma 5.4.** *There exist some numbers $\lambda_2\geq\lambda_1$, $s_2\geq s_1$ and $C_2>0$ such that for all $\lambda\geq\lambda_2$, all $s\geq s_2$ and all $(p_0,q_0)\in L^2(\Omega)^2$, the corresponding solution $(p,q)$ of system (5.33) fulfills*

$$\int_0^T\int_\Omega [(s\theta)^{-1}(|\Delta p|^2+|p_t|^2)+\lambda^2(s\theta)|\nabla p|^2+\lambda^4(s\theta)^3|p|^2]e^{-2s\varphi}dxdt$$
$$+\int_0^T\int_\Omega (\lambda^2 s\theta)|q|^2e^{-2s\varphi}dxdt\leq C_2\int_0^T\int_{X(\omega,t,0)}\lambda^8(s\theta)^7e^{-2s\varphi}|q|^2dxdt. \quad (5.47)$$

*Proof of Lemma 5.4.* In order to prove Lemma 5.4, we just need to estimate the $p$ appearing in the right-hand side of (5.46 ). For that, we introduce the function

$$\zeta(x,t):=\xi(X(x,0,t)), \quad (5.48)$$

where $\xi$ is a cut-off function satisfying

$$\xi\in C_0^\infty(\omega), \quad (5.49)$$
$$0\leq\xi(x)\leq 1, \quad x\in\mathbb{R}^N, \quad (5.50)$$
$$\xi(x)=1, \quad x\in\omega_1. \quad (5.51)$$

We have that

$$\int_0^T\int_{X(\omega_1,t,0)}\lambda^4(s\theta)^3|p|^2e^{-2s\varphi}dxdt\leq\int_0^T\int_\Omega \zeta\lambda^4(s\theta)^3|p|^2e^{-2s\varphi}dxdt \quad (5.52)$$

and we use $(5.33)_2$ to write

$$\begin{aligned}\int_0^T\int_\Omega \zeta\lambda^4(s\theta)^3|p|^2e^{-2s\varphi}dxdt &= \int_0^T\int_\Omega \zeta\lambda^4(s\theta)^3pqe^{-2s\varphi}dxdt \\ &+\int_0^T\int_\Omega \zeta\lambda^4(s\theta)^3p(-q_t)e^{-2s\varphi}dxdt \\ &=: M_1+M_2. \end{aligned} \tag{5.53}$$

It remains to estimate $M_1$ and $M_2$. Using Cauchy-Schwarz inequality and (5.49)-(5.50), we have, for every $\varepsilon>0$,

$$|M_1|\le \varepsilon\int_0^T\int_\Omega \lambda^4(s\theta)^3|p|^2e^{-2s\varphi}dxdt+\frac{1}{4\varepsilon}\int_0^T\int_{X(\omega,t,0)}\lambda^4(s\theta)^3|q|^2e^{-2s\varphi}dxdt. \tag{5.54}$$

On the other hand, integrating by parts with respect to $t$ in $M_2$ yields

$$\begin{aligned} M_2 &= \int_0^T\int_\Omega \zeta\lambda^4(s\theta)^3p_tqe^{-2s\varphi}dxdt+\int_0^T\int_\Omega \zeta\lambda^4(3s^3\theta^2\theta_t-2s^4\varphi_t\theta^3)pqe^{-2s\varphi}dxdt \\ &\quad -\int_0^T\int_\Omega \nabla\xi(X(x,0,t))\cdot\left(\frac{\partial X}{\partial x}\right)^{-1}(X(x,0,t),t,0)f(x,t)\lambda^4(s\theta)^3pqe^{-2s\varphi}dxdt \\ &=: M_2^1+M_2^2-M_2^3. \end{aligned}$$

For $M_2^1$, we notice that for every $\varepsilon>0$,

$$|M_2^1|\le \varepsilon\int_0^T\int_\Omega (s\theta)^{-1}|p_t|^2e^{-2s\varphi}dxdt+\frac{1}{4\varepsilon}\int_0^T\int_{X(\omega,t,0)}\lambda^8(s\theta)^7|q|^2e^{-2s\varphi}dxdt. \tag{5.55}$$

Since $|\theta_t|+|\varphi_t|\le C\lambda\theta^2$, we infer that

$$\begin{aligned} |M_2^2| &\le C\int_0^T\int_\Omega \zeta s^4(\lambda\theta)^5|pq|e^{-2s\varphi}dxdt \qquad (5.56)\\ &\le \varepsilon\int_0^T\int_\Omega \lambda^4(s\theta)^3|p|^2e^{-2s\varphi}dxdt+\frac{C}{\varepsilon s^2}\int_0^T\int_{X(\omega,t,0)}\lambda^6(s\theta)^7|q|^2e^{-2s\varphi}dxdt. \end{aligned}$$

Finally, $M_2^3$ is estimated as $M_1$:

$$|M_2^3|\le \epsilon\int_0^T\int_\Omega \lambda^4(s\theta)^3|p|^2e^{-2s\varphi}dxdt+\frac{C}{\varepsilon}\int_0^T\int_{X(\omega,t,0)}\lambda^4(s\theta)^3|q|^2e^{-2s\varphi}dxdt. \tag{5.57}$$

Gathering together (5.46) and (5.52)-(5.57) and taking $\epsilon$ small enough, we obtain (5.47).

Now we finish the proof of the observability inequality (5.34).

Pick any $(p_0, q_0) \in L^2(\Omega)^2$, and denote by $(p, q)$ the solution of (5.33). Note that $p \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$ and that $q \in H^1(0,T; L^2(\Omega))$. Using classical semigroup estimates, one derives at once (5.34) from (5.47).

□

## 5.5 Proof of Lemma 5.1

*Proof.* Pick any $\delta < \min(t_1, T - t_2, T/2)$. We search $\psi$ (see Figure 5.6) in the form

$$\psi(x,t) = \psi_1(x,t) + C_2\psi_2(x,t) + C_3 \tag{5.58}$$

where, roughly, $\psi_1$ fulfills (5.36), $\psi_2$ fulfills (5.37)-(5.39) together with $\nabla\psi_2 \equiv 0$ outside $X(\omega_1, t, 0)$, and $C_2, C_3$ are (large enough) positive constants.

*Step 1. Construction of $\psi_1$.*

Let $\Gamma \in C^\infty([0,T]; \mathbb{R}^N)$ be as in (5.9), and let $\varepsilon > 0$ be such that

$$B(\Gamma(t), 3\varepsilon) \subset X(\omega_0, t, 0) \cap \Omega, \qquad t \in [0,T].$$

We choose a vector field $\tilde{f} \in C^\infty(\mathbb{R}^N \times [0,T]; \mathbb{R}^N)$ such that

$$\tilde{f}(x,t) = \begin{cases} \dot{\Gamma}(t) & \text{if } t \in [0,T],\ x \in B(\Gamma(t), \varepsilon), \\ 0 & \text{if } t \in [0,T],\ x \in \mathbb{R}^N \setminus B(\Gamma(t), 2\varepsilon). \end{cases}$$

Let $\tilde{X}$ denote the flow associated with $\tilde{f}$; that is, $\tilde{X}$ solves

$$\begin{aligned} \frac{\partial \tilde{X}}{\partial t}(x,t,t_0) &= \tilde{f}(\tilde{X}(x,t,t_0),t), && (x,t,t_0) \in \mathbb{R}^N \times [0,T]^2, \\ \tilde{X}(x,t_0,t_0) &= x, && (x,t_0) \in \mathbb{R}^N \times [0,T]. \end{aligned}$$

Note that

$$\begin{aligned} \tilde{X}(y + \Gamma(0), t, 0) = y + \Gamma(t) \quad &\text{if} \quad (y,t) \in B(0,\varepsilon) \times [0,T], \\ \tilde{X}(x,t,t_0) = x \quad &\text{if} \quad \text{dist}\,(x, \partial\Omega) < \varepsilon,\ (t,t_0) \in [0,T]^2. \end{aligned}$$

By a well-known result (see [130, Lemma 1.2]), there exists a function $\tilde{\psi} \in C^\infty(\overline{\Omega})$ such that

$$\begin{aligned} \tilde{\psi}(x) > 0 &\quad \text{if} \quad x \in \Omega; \\ \tilde{\psi}(x) = 0 &\quad \text{if} \quad x \in \partial\Omega; \\ \nabla\tilde{\psi}(x) \neq 0 &\quad \text{if} \quad x \in \overline{\Omega} \setminus B(\Gamma(0), \varepsilon). \end{aligned}$$

Actually, the function $\tilde{\psi}$ given in [130] is only of class $C^2$, but the regularity $C^\infty$ can be obtained by mollification with a partition of unity (see e.g. [110, Lemma 4.2]). Let us set

$$\psi_1(x,t) = \tilde{\psi}(\tilde{X}(x,0,t)).$$

Then $\psi_1 \in C^\infty(\overline{\Omega} \times [0,T])$ and it fulfills

$$\psi_1(x,t) > 0 \quad \text{if} \quad (x,t) \in \Omega \times [0,T], \tag{5.59}$$

$$\psi_1(x,t) = 0 \quad \text{if} \quad (x,t) \in \partial\Omega \times [0,T], \tag{5.60}$$

$$\nabla\psi_1(x,t) = \nabla\tilde{\psi}(\tilde{X}(x,0,t))\frac{\partial \tilde{X}}{\partial x}(x,0,t) \neq 0 \quad \text{if} \quad x \in \overline{\Omega} \setminus X(\omega_0, t, 0). \tag{5.61}$$

For (5.61), we notice that if we write $x = \tilde{X}(\tilde{x}, t, 0)$, then $\tilde{x} = \tilde{X}(x,0,t)$ hence

$$\nabla\tilde{\psi}(\tilde{X}(x,0,t)) = \nabla\tilde{\psi}(\tilde{x}) \neq 0$$

if $\tilde{x} \notin B(\Gamma(0), \varepsilon)$, which is equivalent to $x \notin B(\Gamma(t), \varepsilon)$. The last condition is satisfied when $x \notin X(\omega_0, t, 0)$.

*Step 2. Construction of $\psi_2$.*

From (5.11), (5.12), and (5.13), we can pick two curves $\gamma_1 \in C^0([0, t_2); \Omega)$ and $\gamma_2 \in C^0((t_1, T]; \Omega)$ such that

$$\begin{aligned} \gamma_1(t) \notin \overline{X(\omega_0, t, 0)}, &\qquad 0 \le t < t_2, \\ \gamma_2(t) \notin \overline{X(\omega_0, t, 0)}, &\qquad t_1 < t \le T. \end{aligned}$$

We infer from (5.13) that for any $t \in (t_1, t_2)$, $\gamma_1(t)$ and $\gamma_2(t)$ do not belong to the same connected component of $\Omega \setminus \overline{X(\omega_0, t, 0)}$. Let $\Omega_1(t)$ (resp. $\Omega_2(t)$) denote the connected component of $\gamma_1(t)$ (resp. $\gamma_2(t)$) for $0 \le t < t_2$ (resp. for $t_1 < t \le T$). Clearly

$$\Omega \setminus \overline{X(\omega_0, t, 0)} = \begin{cases} \Omega_1(t), & \text{if } 0 \le t \le t_1, \\ \Omega_1(t) \cup \Omega_2(t), & \text{if } t_1 < t < t_2, \\ \Omega_2(t), & \text{if } t_2 \le t \le T. \end{cases}$$

Set $\Omega_1(t) = \emptyset$ for $t \in [t_2, T]$, and $\Omega_2(t) = \emptyset$ for $t \in [0, t_1]$. Let $\psi_2 \in C^\infty(\overline{\Omega} \times [0, T])$ with

$$\begin{aligned} \psi_2(x,t) &= t\left(\mathbf{1}_{\Omega_1(t)}(x) - \mathbf{1}_{\Omega_2(t)}(x)\right) \quad \text{for } t \in [0,T],\ x \in \Omega \setminus X(\omega_1, t, 0), \\ \frac{\partial \psi_2}{\partial n} &= 0 \quad \text{for } (x,t) \in \partial\Omega \times [0,T]. \end{aligned}$$

Such a function $\psi_2$ exists, since by (5.35)

$$\inf_{t_1<t<t_2} \text{dist}\left(\Omega_1(t) \setminus X(\omega_1, t, 0), \Omega_2(t) \setminus X(\omega_1, t, 0)\right) > 0.$$

Then

$$\frac{\partial \psi_2}{\partial t} = \begin{cases} 1 & \text{if } 0 \le t < t_2 \text{ and } x \in \Omega_1(t) \setminus X(\omega_1, t, 0), \\ -1 & \text{if } t_1 < t \le T \text{ and } x \in \Omega_2(t) \setminus X(\omega_1, t, 0). \end{cases}$$

and

$$\nabla \psi_2(x,t) = 0 \quad \text{if } x \in \Omega \setminus X(\omega_1, t, 0).$$

Note that (5.37)-(5.39) are satisfied for $\psi_2$. Note also that for any pair $(\tau_1, \tau_2)$ with $0 \le \tau_1 < \tau_2 \le T$, the set

$$K_{\tau_1,\tau_2} := \{(x,t) \in \mathbb{R}^{N+1};\ \tau_1 \le t \le \tau_2,\ x \in \overline{\Omega} \setminus X(\omega_1, t, 0)\}$$

is *compact*.

**Remark 5.3.** Let us now explain in details the construction of the function $\psi_2$.

First, we pick $\omega_2$ and $\omega_3$ such that $\omega_0 \subset\subset \omega_2 \subset\subset \omega_3 \subset\subset \omega_1$. We fix $t \in [0, T]$. Assume $t_1 < t < t_2$. Note that $dist(\Omega_1(t) \setminus X(\omega_2, t, 0), \Omega_2(t) \setminus X(\omega_2, t, 0) > 0$, since $X(\omega_2, t, 0)$ is an open neighborhood of $X(\overline{\omega}_0, t, 0)$. Thus, we can construct $\eta^t \in C^\infty(\overline{\Omega})$ such that

$$\eta^t(x) = \begin{cases} 1 & \text{if } x \in \Omega_1(t) \setminus X(\omega_2, t, 0), \\ -1 & \text{if } x \in \Omega_2(t) \setminus X(\omega_2, t, 0). \end{cases}$$

For $t \le t_1$ (resp. $t \ge t_2$), we set $\eta^t(x) = 1$ (resp. $\eta^t(x) = -1$). Note that we do not impose $\frac{\partial \eta^t}{\partial \nu} = 0$ on $\partial\Omega$.

Next, we consider a vector field $g \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$ such that $g(x) = \nu(x)$, for all $x \in \partial\Omega$ and $||g(x)||_\infty \le 2$ for all $x \in \mathbb{R}^N$ (actually, we can construct this vector field).

Let us denote by $\psi(x,t)$ the flow associated to $g$. Then, for $\epsilon > 0$ small enough, the set $G_1 = \{\psi(x,t); -\epsilon < t < \epsilon, x \in \partial\Omega\}$ is an open neighborhood of $\partial\Omega$. Let us also consider an open set $G_2 \subset \Omega$ such that $dist(G_2, \mathbb{R}^N \setminus \Omega) > 0$ and $\Omega \subset G_1 \cup G_2$.

The map $\left(t \in (-\epsilon, 0), x \in \partial\Omega\right) \mapsto \psi(t,x) \in \Omega$ is one to one for $\epsilon$ small enough. We denote by $\varphi^{-1}$ the $\partial\Omega$ component of the inverse mapping.

Denoting by $\zeta_1, \zeta_2$ the partition of unity associated to $G_1, G_2$. Then, $\tilde{\eta}^t(x) = \zeta_1(x)\eta^t(\varphi^{-1}(x)) +$

$\zeta_2(x)\eta^t(x)$ fullfil $\frac{\partial \tilde{\eta}^t}{\partial \nu} = 0$ in $\partial\Omega$.

Furthermore, for $\epsilon$ small enough,

$$\tilde{\eta}^t(x) = \begin{cases} 1 & \text{if } x \in \Omega_1(t) \setminus X(\omega_3, t, 0), \\ -1 & \text{if } x \in \Omega_2(t) \setminus X(\omega_3, t, 0) \end{cases}$$

and $\frac{\partial \tilde{\eta}^t}{\partial \nu} = 0$ on $\partial\Omega$.

We pick now a partition of $[0,T]$ denoted by $0 = t_0 < t_1 < \ldots < t_k = t$ and a partition of unity (in time) $(\mathcal{T}_i), 1 \leq i \leq k$, associated to $(-\delta, t_1) = \mathcal{T}_0, (t_0, t_2) = \mathcal{T}_1, (t_1, t_3) = \mathcal{T}_2, \ldots (t_{k-2}, t_k) = \mathcal{T}_{k-1}, (t_{k-1}, T+\delta) = \mathcal{T}_k$ and set

$$\eta(t,x) = \sum_{j=0}^{k} \tilde{\eta}^{t_j}(x)\mathcal{T}_j(t)$$

Then, $\eta \in C^\infty(\overline{\Omega} \times [0,T])$ and, if $\sup |t_{j+1} - t_j|$ is small enough, we have

$$\eta(t,x) = \begin{cases} 1 & \text{if } x \in \Omega_1(t) \setminus X(\omega_1, t, 0), \\ -1 & \text{if } x \in \Omega_2(t) \setminus X(\omega_1, t, 0). \end{cases}$$

Clearly, $\frac{\partial \eta}{\partial \nu} = 0$ on $\partial\Omega \times [0,T]$. Finally, set $\psi_2(x,t) = t\eta(t,x)$.

*Step 3. Construction of $\psi$.*

Let $\psi$ be defined as in (5.58), with $C_2 > 0$ and $C_3$ to be determined. Then (5.36) and (5.40) are satisfied. We pick $C_2$ large enough for (5.37)-(5.39) to be satisfied. Finally, (5.41) is satisfied for $C_3$ large enough. □

## 5.6 Proof of Lemma 5.2

*Proof of Lemma 5.2.* The method of the proof is widely inspired from [46], and the computations are presented as in [110, Proof of Proposition 4.3].

Let $v = e^{-s\varphi}p$ and $P = \partial_t + \Delta$. Then

$$e^{-s\varphi}Pp = e^{-s\varphi}P(e^{s\varphi}v) = P_s v + P_a v$$

where

$$P_s v = \Delta v + (s\varphi_t + s^2|\nabla\varphi|^2)v, \tag{5.62}$$

$$P_a v = v_t + 2s\nabla\varphi \cdot \nabla v + s(\Delta\varphi)v \tag{5.63}$$

denote the (formal) self-adjoint and skew-adjoint parts of $e^{-s\varphi}P(e^{s\varphi}\cdot)$, respectively. It follows that

$$||e^{-s\varphi}Pp||^2 = ||P_s v||^2 + ||P_a v||^2 + 2(P_s v, P_a v) \tag{5.64}$$

where $(f,g) = \int_0^T\!\int_\Omega fg\,dxdt$, $||f||^2 = (f,f)$. In the sequel, $\int_0^T\!\int_\Omega f(x,t)dxdt$ is denoted $\iint f$ for the sake of shortness. We have

$$\begin{aligned}(P_s v, P_a v) &= \big(\Delta v, v_t\big) + \big(\Delta v, 2s\nabla\varphi\cdot\nabla v\big) + (\Delta v, s(\Delta\varphi)v) + \big(s\varphi_t v + s^2|\nabla\varphi|^2 v, v_t\big)\\ &+ (s\varphi_t v + s^2|\nabla\varphi|^2 v, 2s\nabla\varphi\cdot\nabla v) + (s\varphi_t v + s^2|\nabla\varphi|^2 v, s(\Delta\varphi)v) =: I_1+I_2+I_3+I_4+I_5+I_6.\end{aligned} \tag{5.65}$$

First, observe that

$$I_1 = -\iint \nabla v\cdot\nabla v_t = 0. \tag{5.66}$$

Using the convention of repeated indices and denoting $\partial_i = \partial/\partial x_i$, we obtain that

$$\begin{aligned} I_2 &= 2s\iint \partial_j^2 v\,\partial_i\varphi\,\partial_i v\\ &= -2s\iint \partial_j v(\partial_j\partial_i\varphi\partial_i v + \partial_i\varphi\partial_j\partial_i v) + 2s\int_0^T\!\!\int_{\partial\Omega}(\partial_j v)n_j\partial_i\varphi\partial_i v d\sigma.\end{aligned}$$

Since $v=0$ for $(x,t)\in\partial\Omega\times(0,T)$, $\nabla v = (\partial v/\partial n)n$, so that $\nabla\varphi\cdot\nabla v = (\partial\varphi/\partial n)(\partial v/\partial n)$ and

$$\int_0^T\!\!\int_{\partial\Omega}(\partial_j v)n_j\partial_i\varphi\partial_i v\,d\sigma = \int_0^T\!\!\int_{\partial\Omega}(\partial\varphi/\partial n)|\partial v/\partial n|^2 d\sigma.$$

It follows that

$$\begin{aligned} I_2 &= -2s\iint \partial_j\partial_i\varphi\partial_j v\partial_i v - s\iint \partial_i\varphi\partial_i(|\partial_j v|^2) + 2s\int_0^T\!\!\int_{\partial\Omega}(\partial\varphi/\partial n)|\partial v/\partial n|^2 d\sigma\\ &= -2s\iint \partial_j\partial_i\varphi\partial_j v\partial_i v + s\iint \Delta\varphi|\nabla v|^2 + s\int_0^T\!\!\int_{\partial\Omega}(\partial\varphi/\partial n)|\partial v/\partial n|^2 d\sigma. \end{aligned} \tag{5.67}$$

On the other hand, integrations by parts in $x$ yields

$$I_3 = -s\iint \nabla v\cdot\big(v\nabla(\Delta\varphi) + (\Delta\varphi)\nabla v\big) = s\iint \Delta^2\varphi\frac{|v|^2}{2} - s\iint \Delta\varphi|\nabla v|^2 \tag{5.68}$$

and integration by parts with respect to $t$ gives

$$I_4 = -\iint (s\varphi_{tt} + s^2\partial_t|\nabla\varphi|^2)\frac{|v|^2}{2}\cdot$$

Integrating by parts with respect to $x$ in $I_5$ yields

$$I_5 = -\iint s^2 \nabla \cdot (\varphi_t \nabla \varphi)|v|^2 - \iint s^3 \nabla \cdot (|\nabla \varphi|^2 \nabla \varphi)|v|^2. \tag{5.69}$$

Gathering (5.65)-(5.69), we infer that

$$2(P_s v, P_a v) = -4s \iint \partial_j \partial_i \varphi \partial_j v \partial_i v + 2s \int_0^T \int_{\partial\Omega} (\partial\varphi/\partial n)|\partial v/\partial n|^2 d\sigma + \iint |v|^2 [s(\Delta^2 \varphi - \varphi_{tt}) - 2s^2 \partial_t |\nabla \varphi|^2 - 2s^3 \nabla \varphi \cdot \nabla |\nabla \varphi|^2].$$

Consequently, (5.64) may be rewritten

$$||e^{-s\varphi} P p||^2 = ||P_s v||^2 + ||P_a v||^2 - 4s \iint \partial_j \partial_i \varphi \partial_j v \partial_i v + 2s \int_0^T \int_{\partial\Omega} (\partial\varphi/\partial n)|\partial v/\partial n|^2 d\sigma + \iint |v|^2 [s(\Delta^2 \varphi - \varphi_{tt}) - 2s^2 \partial_t |\nabla \varphi|^2 - 2s^3 \nabla \varphi \cdot \nabla |\nabla \varphi|^2].$$

CLAIM 1. There exist some numbers $\lambda_1 > 0$, $s_1 > 0$ and $A \in (0,1)$ such that for all $\lambda \geq \lambda_1$ and all $s \geq s_1$,

$$\iint |v|^2 [s(\Delta^2 \varphi - \varphi_{tt}) - 2s^2 \partial_t |\nabla \varphi|^2 - 2s^3 \nabla \varphi \cdot \nabla |\nabla \varphi|^2] + A^{-1} \lambda s^3 \int_0^T \int_{X(\omega_1,t,0)} (\lambda\theta)^3 |v|^2 \geq A \lambda s^3 \iint (\lambda\theta)^3 |v|^2. \tag{5.70}$$

*Proof of Claim 1.* Easy computations show that

$$\partial_i \varphi = -\lambda g(t) e^{\lambda\psi} \partial_i \psi, \qquad \partial_j \partial_i \varphi = -g(t) e^{\lambda\psi} (\lambda^2 \partial_i \psi \partial_j \psi + \lambda \partial_j \partial_i \psi) \tag{5.71}$$

and

$$-\nabla |\nabla \varphi|^2 \cdot \nabla \varphi = -2(\partial_j \partial_i \varphi) \partial_i \varphi \partial_j \varphi = 2(\lambda g e^{\lambda\psi})^3 (\lambda |\nabla \psi|^4 + \partial_j \partial_i \psi \partial_i \psi \partial_j \psi).$$

It follows from (5.36) that for $\lambda$ large enough, say $\lambda \geq \lambda_1$, we have that

$$-\nabla |\nabla \varphi|^2 \cdot \nabla \varphi \geq A\lambda(\lambda\theta)^3, \qquad t \in [0,T],\ x \in \overline{\Omega} \setminus X(\omega_1,t,0), \tag{5.72}$$

$$|\nabla |\nabla \varphi|^2 \cdot \nabla \varphi| \leq A^{-1}\lambda(\lambda\theta)^3, \qquad t \in [0,T],\ x \in X(\omega_1,t,0), \tag{5.73}$$

for some constant $A \in (0,1)$. According to (5.41), we have for some constant $C > 0$

$$|\Delta^2 \varphi| + |\varphi_{tt}| + |\partial_t |\nabla \varphi|^2| \leq C\lambda(\lambda\theta)^3, \qquad t \in [0,T],\ x \in \overline{\Omega}.$$

Therefore, we infer that for $s$ large enough, say $s \geq s_1$, and for all $\lambda \geq \lambda_1$ we have that

$$s(\Delta^2\varphi - \varphi_{tt}) - 2s^2\partial_t|\nabla\varphi|^2 - 2s^3\nabla\varphi\cdot\nabla|\nabla\varphi|^2 \geq A\lambda s^3(\lambda\theta)^3,$$

for $t \in [0,T],\ x \in \overline{\Omega}\setminus X(\omega_1,t,0)$ and

$$|s(\Delta^2\varphi - \varphi_{tt}) - 2s^2\partial_t|\nabla\varphi|^2 - 2s^3\nabla\varphi\cdot\nabla|\nabla\varphi|^2| \leq 3A^{-1}\lambda s^3(\lambda\theta)^3,$$

for $t \in [0,T],\ x \in X(\omega_1,t,0)$.
This gives (5.70) with a possibly decreased value of $A$. □

Thus, using the fact that $\partial\varphi/\partial n \geq 0$ on $\partial\Omega$ by (5.40), we conclude that

$$||P_s v||^2 + ||P_a v||^2 + A\lambda s^3 \iint (\lambda\theta)^3|v|^2 \leq ||e^{-s\varphi}Pp||^2 + 4s\iint \partial_j\partial_i\varphi\partial_j v\partial_i v + A^{-1}\lambda s^3\int_0^T\int_{X(\omega_1,t,0)}(\lambda\theta)^3|v|^2. \quad (5.74)$$

CLAIM 2. There exist some numbers $\lambda_2 \geq \lambda_1$ and $s_2 \geq s_1$ such that for all $\lambda \geq \lambda_2$ and all $s \geq s_2$,

$$\lambda s\iint(\lambda\theta)|\nabla v|^2 + \lambda s^{-1}\iint(\lambda\theta)^{-1}|\Delta v|^2 \leq C\left(s^{-1}||P_s v||^2 + \lambda s^3\iint(\lambda\theta)^3|v|^2\right). \quad (5.75)$$

*Proof of Claim 2.* By (5.62), we have

$$\begin{aligned} s^{-1}\iint(\lambda\theta)^{-1}|\Delta v|^2 &= s^{-1}\iint(\lambda\theta)^{-1}|P_s v - s\varphi_t v - s^2|\nabla\varphi|^2 v|^2 \\ &\leq Cs^{-1}\iint(\lambda\theta)^{-1}\big(|P_s v|^2 + s^2|\varphi_t|^2|v|^2 + s^4(\lambda\theta)^4|v|^2\big) \\ &\leq C\left(\frac{||P_s v||^2}{\lambda s} + s^3\iint(\lambda\theta)^3|v|^2\right) \end{aligned} \quad (5.76)$$

provided that $s$ and $\lambda$ are large enough, where we used (5.41) in the last line to bound $\varphi_t$. On the other hand,

$$\begin{aligned} \lambda s\iint(\lambda\theta)|\nabla v|^2 &= \lambda s\{\iint(\lambda\theta)(-\Delta v)v - \iint(\nabla(\lambda\theta)\cdot\nabla v)v\} \\ &\leq \frac{\lambda}{2s}\iint(\lambda\theta)^{-1}|\Delta v|^2 + \frac{\lambda s^3}{2}\iint(\lambda\theta)^3|v|^2 + \frac{\lambda s}{2}\iint\Delta(\lambda\theta)|v|^2 \\ &\leq C\left(s^{-1}||P_s v||^2 + \lambda s^3\iint(\lambda\theta)^3|v|^2\right) \end{aligned} \quad (5.77)$$

by (5.76), provided that $s \geq s_2 \geq s_1$ and $\lambda \geq \lambda_2 \geq \lambda_1$. Then (5.75) follows from (5.76)-

(5.77). □

We infer from (5.74)-(5.75) that

$$||P_a v||^2 + \lambda s \iint (\lambda\theta)|\nabla v|^2 + \lambda s^{-1} \iint (\lambda\theta)^{-1}|\Delta v|^2 + \lambda s^3 \iint (\lambda\theta)^3|v|^2$$
$$\le C\left( ||e^{-s\varphi}Pp||^2 + 4s \iint \partial_j\partial_i\varphi\partial_j v\partial_i v + A^{-1}\lambda s^3 \int_0^T\int_{X(\omega_1,t,0)} (\lambda\theta)^3|v|^2 \right). \quad (5.78)$$

By (5.71),

$$s\iint \partial_j\partial_i\varphi\partial_j v\partial_i v \le -s\lambda \iint g(t)e^{\lambda\psi}\partial_j\partial_i\psi\partial_j v\partial_i v \le Cs\iint (\lambda\theta)|\nabla v|^2.$$

Therefore, for $\lambda$ large enough and $s \ge s_2$,

$$||P_a v||^2 + \lambda s^3 \iint (\lambda\theta)^3|v|^2 + \lambda s \iint (\lambda\theta)|\nabla v|^2 + \lambda s^{-1} \iint (\lambda\theta)^{-1}|\Delta v|^2$$
$$\le C\left( ||e^{-s\varphi}Pp||^2 + \lambda s^3 \int_0^T\int_{X(\omega_1,t,0)} (\lambda\theta)^3|v|^2 \right). \quad (5.79)$$

Using (5.63) and (5.79), we see that for $\lambda$ large enough and $s \ge s_2$

$$\begin{aligned}\lambda s^{-1}\iint (\lambda\theta)^{-1}|v_t|^2 &\le C\lambda s^{-1}\iint (\lambda\theta)^{-1}\big(|P_a v|^2 + s^2|\nabla\varphi|^2|\nabla v|^2 + s^2|\Delta\varphi|^2|v|^2\big)\\ &\le C\left( ||e^{-s\varphi}Pp||^2 + \lambda s^3 \int_0^T\int_{X(\omega_1,t,0)} (\lambda\theta)^3|v|^2 \right).\end{aligned}$$

Hence, there exists some number $\lambda_3 \ge \lambda_2$ such that for all $\lambda \ge \lambda_3$ and all $s \ge s_2$, we have

$$\lambda s^3 \iint (\lambda\theta)^3|v|^2 + \lambda s \iint (\lambda\theta)|\nabla v|^2 + \lambda s^{-1} \iint (\lambda\theta)^{-1}(|\Delta v|^2 + |v_t|^2)$$
$$\le C\left( ||e^{-s\varphi}Pp||^2 + \lambda s^3 \int_0^T\int_{X(\omega_1,t,0)} (\lambda\theta)^3|v|^2 \right). \quad (5.80)$$

Replacing $v$ by $e^{-s\varphi}p$ in (5.80) gives at once (5.42). The proof of Lemma 5.2 is complete. □

## 5.7 Proof of Lemma 5.3

*Proof of Lemma 5.3.* The proof is divided into three parts corresponding to the estimates for $t \in [0, \delta]$, for $t \in [\delta, T - \delta]$ and for $t \in [T - \delta, T]$. The estimates for $t \in [0, \delta]$ and for $t \in [T - \delta, T]$ being similar, we shall prove only the first ones.

Let $v = e^{-s\varphi} q$. Then

$$e^{-s\varphi} q_t = e^{-s\varphi}(e^{s\varphi} v)_t = s\varphi_t v + v_t =: P_s v + P_a v. \tag{5.81}$$

CLAIM 3.

$$\int_0^\delta\int_\Omega \lambda(s\theta)^2|v|^2 dxdt \leq C\left(\int_0^\delta\int_\Omega \lambda^{-1}|e^{-s\varphi}q_t|^2 dxdt \right. \\ \left. + \int_\Omega [(1-\zeta)^2(s\theta)|v|^2]_{|t=\delta} dx + \int_0^\delta\int_{X(\omega,t,0)} \lambda(s\theta)^2|v|^2 dxdt\right), \tag{5.82}$$

where $\zeta$ is the function introduced in (5.48).

To prove the claim, we compute in several ways

$$I := \int_0^\delta\int_\Omega (e^{-s\varphi}q_t)(1-\zeta)^2 s\theta v\, dxdt.$$

We split $I$ into

$$I = \int_0^\delta\int_\Omega (P_s v)(1-\zeta)^2 s\theta v\, dxdt + \int_0^\delta\int_\Omega (P_a v)(1-\zeta)^2 s\theta v\, dxdt =: I_1 + I_2.$$

Then

$$\begin{aligned} I_1 &= \int_0^\delta\int_\Omega (1-\zeta)^2 s^2\varphi_t\theta v^2\, dxdt \\ &= \int_0^\delta\int_\Omega [g'(e^{\frac{3}{2}\lambda||\psi||_{L^\infty}} - e^{\lambda\psi}) - g\lambda\psi_t e^{\lambda\psi}](1-\zeta)^2 s^2 g e^{\lambda\psi} v^2\, dxdt. \end{aligned}$$

On the other hand

$$\begin{aligned} I_2 &= \int_0^\delta\int_\Omega (1-\zeta)^2(sge^{\lambda\psi} v v_t)\, dxdt \\ &= \frac{1}{2}\int_\Omega [(1-\zeta)^2 s\theta|v|^2]_{|t=\delta} dx - \int_0^\delta\int_\Omega s[g'e^{\lambda\psi} + g\lambda\psi_t e^{\lambda\psi}](1-\zeta)^2\frac{v^2}{2} dxdt \\ &\quad - \int_0^\delta\int_\Omega (1-\zeta)\nabla\xi(X(x,0,t))\cdot\left(\frac{\partial X}{\partial x}\right)^{-1}(X(x,0,t),t,0)f(x,t)s\theta v^2 dxdt, \end{aligned}$$

where we used the fact that $[\theta|v|^2]_{|t=0} = 0$. Clearly, since $\theta \geq 1$, for $s \geq 1$

$$|\int_0^\delta\int_\Omega (1-\zeta)\nabla\xi(X(x,0,t))\cdot(\frac{\partial X}{\partial x})^{-1}(X(x,0,t),t,0)f(x,t)s\theta v^2 dxdt| \\ \leq C\int_0^\delta\int_{X(\omega,t,0)}(s\theta)^2|v|^2dxdt.$$

On the other hand, using (5.38), we see that there exist some constants $C > 0$ and $s_1 \geq s_0$ such that for all $s \geq s_1$ and all $\lambda \geq \lambda_0 > 0$, it holds

$$g\lambda\psi_t e^{\lambda\psi}(s^2 g e^{\lambda\psi} + \frac{s}{2}) \geq C\lambda(s\theta)^2, \qquad t\in(0,\delta),\ x\in\overline{\Omega}\setminus X(\omega_1,t,0)$$
$$-g'(t)\left((e^{\frac{3}{2}\lambda\|\psi\|_{L^\infty}} - e^{\lambda\psi})s^2 g e^{\lambda\psi} - \frac{s}{2}e^{\lambda\psi}\right) > 0 \qquad t\in(0,\delta),\ x\in\overline{\Omega}\setminus X(\omega_1,t,0).$$

It follows that for some positive constant $C' > C$

$$C\int_0^\delta\int_\Omega \lambda(s\theta)^2|v|^2dxdt \leq -I + \frac{1}{2}\int_\Omega[(1-\zeta)^2(s\theta)|v|^2]_{|t=\delta}dx + C'\int_0^\delta\int_{X(\omega,t,0)}\lambda(s\theta)^2|v|^2dxdt. \tag{5.83}$$

Finally, by Cauchy-Schwarz inequality, we have for any $\kappa > 0$

$$|I| \leq (4\kappa)^{-1}\int_0^\delta\int_\Omega |e^{-s\varphi}q_t|^2dxdt + \kappa\int_0^\delta\int_\Omega(s\theta)^2|v|^2dxdt. \tag{5.84}$$

Combining (5.83) with (5.84) gives (5.82) for $\kappa/\lambda > 0$ small enough. Therefore, Claim 3 is proved. □

We can prove in the same way the following estimate for $t \in [T-\delta, T]$:

$$\int_{T-\delta}^T\int_\Omega \lambda(s\theta)^2|v|^2dxdt \leq C\left(\int_{T-\delta}^T\int_\Omega \lambda^{-1}|e^{-s\varphi}q_t|^2dxdt \right.\\ \left. + \int_\Omega[(1-\zeta)^2(s\theta)|v|^2]_{|t=T-\delta}dx + \int_{T-\delta}^T\int_{X(\omega,t,0)}\lambda(s\theta)^2|v|^2dxdt\right). \tag{5.85}$$

Let us now consider the estimate for $t \in [\delta, T-\delta]$.

CLAIM 4.

$$\int_\delta^{T-\delta}\int_\Omega \lambda^2(s\theta)|v|^2dxdt + \int_\Omega[(1-\zeta)^2(\lambda s\theta)|v|^2]_{|t=\delta}dx + \int_\Omega[(1-\zeta)^2(\lambda s\theta)|v|^2]_{|t=T-\delta}dx \\ \leq C\left(\int_\delta^{T-\delta}\int_\Omega |e^{-s\varphi}q_t|^2dxdt + \int_\delta^{T-\delta}\int_{X(\omega,t,0)}\lambda^2(s\theta)|v|^2dxdt\right). \tag{5.86}$$

$\|\cdot\|$ and $(.,.)$ denoting here the Euclidean norm and scalar product in $L^2(\Omega\times(\delta,T-\delta))$, we have that

$$||e^{-s\varphi}q_t||^2 \geq ||(1-\zeta)(s\varphi_t v + v_t)||^2 \geq 2((1-\zeta)s\varphi_t v, (1-\zeta)v_t). \tag{5.87}$$

Next, we compute

$$((1-\zeta)s\varphi_t v, (1-\zeta)v_t) = \int_\Omega (1-\zeta)^2 s\varphi_t \frac{v^2}{2}dx\Big|_{t=\delta}^{T-\delta} - \frac{s}{2}\int_\delta^{T-\delta}\int_\Omega (1-\zeta)^2\varphi_{tt}|v|^2 dxdt$$
$$-\int_\delta^{T-\delta}\int_\Omega (1-\zeta)\nabla\xi(X(x,0,t))\cdot(\frac{\partial X}{\partial x})^{-1}(X(x,0,t),t,0)f(x,t)s\varphi_t v^2 dxdt. \tag{5.88}$$

Since $g(t)=1$ for $t\in[\delta, T-\delta]$, we have that $\varphi_t = -\lambda\psi_t e^{\lambda\psi}$. From (5.38)-(5.39), we infer that

$$\begin{array}{rclr} s\varphi_t(x,T-\delta) & \geq & C\lambda s e^{\lambda\psi} & x\in\overline{\Omega}\setminus X(\omega_1, T-\delta, 0), \\ -s\varphi_t(x,\delta) & \geq & C\lambda s e^{\lambda\psi} & x\in\overline{\Omega}\setminus X(\omega_1,\delta,0). \end{array}$$

Therefore, using (5.51),

$$\int_\Omega (1-\zeta)^2 s\varphi_t\frac{v^2}{2}dx\Big|_\delta^{T-\delta} \geq C\left(\int_\Omega [(1-\zeta)^2(\lambda s\theta)|v|^2]_{|t=\delta}dx + \int_\Omega [(1-\zeta)^2(\lambda s\theta)|v|^2]_{|t=T-\delta}dx\right). \tag{5.89}$$

Next, with $\varphi_{tt} = -\{(\lambda\psi_t)^2 + \lambda\psi_{tt}\}e^{\lambda\psi}$ and (5.37), we obtain for $\lambda\geq\lambda_1>\lambda_0$

$$-\frac{s}{2}\int_\delta^{T-\delta}\int_\Omega (1-\zeta)^2\varphi_{tt}|v|^2dxdt \geq C\int_\delta^{T-\delta}\int_\Omega (1-\zeta)^2\lambda^2 s\theta|v|^2 dxdt. \tag{5.90}$$

Finally

$$\left|\int_\delta^{T-\delta}\int_\Omega (1-\zeta)\nabla\xi(X(x,0,t))\cdot(\frac{\partial X}{\partial x})^{-1}(X(x,0,t),t,0)f(x,t)s\varphi_t v^2 dxdt\right|$$
$$\leq C\int_\delta^{T-\delta}\int_{X(\omega,t,0)}\lambda s\theta|v|^2. \tag{5.91}$$

Claim 4 follows from (5.87)-(5.91).

We infer from (5.82), (5.85) and (5.86) that for some constants $\lambda_1\geq\lambda_0$, $s_1\geq s_0$ and $C_1>0$ we have, for all $\lambda\geq\lambda_1$ and all $s\geq s_1$,

$$\int_0^T\int_\Omega \lambda^2(s\theta)|v|^2dxdt \leq C_1\left(\int_0^T\int_\Omega |e^{-s\varphi}q_t|^2dxdt + \int_0^T\int_{X(\omega,t,0)}\lambda^2(s\theta)^2|v|^2dxdt\right). \tag{5.92}$$

Replacing $v$ by $e^{-s\varphi}q$ in (5.92) gives at once (5.43). The proof of Lemma 5.3 is complete. □

## 5.8 Additional comments

- *Another decomposition*

  As commented in section 5.1, there is another splitting of the operator $\mathcal{L} = \partial_t^2 - \Delta - \Delta\partial_t + \partial_t$, given by

$$\mathcal{L} = (\partial_t - \Delta)(\partial_t + Id).$$

  Thus, letting

$$v(x,t) = y(x,t) + y_t(x,t),$$

  we see that $(5.1)_1$ may be written as

$$\begin{cases} v_t - \Delta v = 1_{\omega(t)}h + (1 - b(x))(v - y), \\ y_t + y = v, \end{cases} \tag{5.93}$$

  which is a coupled system of a parabolic equation and an ODE .

  This splitting was used to prove Theorem 5.1 with less assumptions on the trajectories (see section 5.2).

  The control term $h$ acts directly in the heat equation and indirectly in the ODE through the coupling term $v$. The problem can be treated directy as such, with requires further work at the level of the dual observability problem since both components of the adjoint system will be needed to be observed by partial measurements only on one of its components. The problem can also be addressed incorporating in $(5.93)_2$ an additional auxiliary control acting directly in the ODE. This leads to the system

$$\begin{cases} v_t - \Delta v = 1_{\omega(t)}h + (1 - b(x))(v - y), \\ y_t + y = 1_{\omega(t)}k + v, \end{cases} \tag{5.94}$$

  where $(v, y) \in L^2(\Omega)^2$ is the state function to be controlled and $(h, k) \in L^2(0, T; L^2(\Omega)^2)$ is the control input.

  Once the controllability of this system is proved, when going back to the original viscoelasticity equation, one gets

$$y_{tt} - \Delta y - \Delta y_t + b(x)y_t = 1_{\omega(t)}\left[h - (1 - b(x))k\right] + (\partial_t - \Delta)[1_{\omega(t)}k]. \tag{5.95}$$

But, then, the second control $1_{\omega(t)}k$ enters under the action of the heat operator. It is then necessary to ensure that the control $k$ is smooth enough and, furthermore, to replace in (5.94) the cut-off function $1_{\omega(t)}$ by a regularized version. These are technicalities that can be overcame with further work. To be more precise, the control in (5.95) takes the form $\mathbf{1}_{X(\omega,t,0)}(x)\tilde{h}$ with $\tilde{h} \in L^2(0,T;L^2(\Omega))$, provided that both $h,\ k \in L^2(0,T;L^2(\Omega))$ and

$$k \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)).$$

Therefore special attention has to be paid to obtain smooth controls for the transport equation (see section 5.2).

- *Manifolds without boundary*

  The lack of propagation properties of the ODE $(5.6)_2$ in the space variable requires the control to move in time. As we mentioned in the introduction, through a suitable change of variables, this is equivalent to keeping the support of the control fixed but replacing the ODE by a transport equation. Obviously, attention has to be paid to the Dirichlet boundary conditions when performing this change of variables. Of course, this is no longer an issue when the model is considered in a smooth manifold without boundary. As an example of such a situation we consider the periodic case in the torus

$$x \in \mathbb{T}^N := \mathbb{R}^N/\mathbb{Z}^N. \tag{5.96}$$

For a moving control with a constant velocity $\omega(t) = \{x - ct;\ x \in \omega\}$, $c \in \mathbb{R}^N \setminus \{0\}$, system (5.6) can be put in the form of a coupled system of parabolic-hyperbolic equations

$$\begin{cases} v_t - \Delta v - c \cdot \nabla v + (b(x+ct) - 1)v = w, \\ w_t - c \cdot \nabla w + w = 1_\omega(x)\tilde{h} + (b(x+ct) - 1)v, \end{cases} \tag{5.97}$$

by letting

$$v(x,t) = y(x+ct,t), \tag{5.98}$$
$$w(x,t) = z(x+ct,t), \tag{5.99}$$
$$\tilde{h}(x,t) = h(x+ct,t). \tag{5.100}$$

The system is now constituted by the coupling between a heat and a transport equation with control $\tilde{h}$ with fixed support. Once more, the problem now can be treated by means of the classical duality principle between the controllability problem and the observability property of the adjoint system. The later was solved in [95] in $1-d$ using Fourier analysis techniques and in this paper we do it using Carleman inequalities.

Note that the Carleman approach developed in this chapter cannot be applied as it is to the periodic case. Consider for instance the case of the torus $\mathbb{T}$. A weight $\psi \in C^\infty(\mathbb{T} \times (0,T))$ as in Lemma 5.1 does not exist, because of the periodicity in $x$ (see figure 5.6.) However, it is well known that the periodic case can be deduced from both the Dirichlet case and the Neumann case (using classical extensions by reflection, see e.g. [111]). Even if the Neumann case was not considered in this chapter, it is likely that it could be treated in much the same way as we did for the Dirichlet case.

# Chapter 6

# A hyperbolic system and the cost of null controllability for the Stokes system

## 6.1 Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded connected open set, whose boundary $\partial\Omega$ is regular enough (for instance of class $C^4$). Let $T > 0$ and let $\omega$ be a nonempty open subset of $\Omega$ which will usually be referred to as a *control domain*. We will use the notation $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$ and we will denote by $\nu(x)$ the outward normal to $\Omega$ at the point $x \in \partial\Omega$.

Given $u_0 \in L^2(\Omega)$, it is well-known (see, for instance, [44, 46]) that there exists $f \in L^2(\omega \times (0,T))$ such that the associated solution $v$ to the heat equation

$$\left| \begin{array}{ll} v_t - \Delta v = f 1_\omega & \text{in } Q, \\ v = 0 & \text{on } \Sigma, \\ v(0) = v_0 & \text{in } \Omega \end{array} \right. \tag{6.1}$$

satisfies:

$$v(T) = 0. \tag{6.2}$$

In other words, the heat equation is *null controllable* for any open control domain and any initial data $v_0 \in L^2(\Omega)$. Moreover, one also has the following estimate:

$$||f 1_\omega||_{L^2(Q)} \leq C_h(T) ||v_0||_{L^2(\Omega)}, \tag{6.3}$$

for a constant $C_h = C_h(T)$, the *cost of controllability for the heat equation*, of the form

$e^{C(\Omega,\omega)(1+1/T)}$, i.e., the heat equation has a cost of controllability of order $e^{C/T}$ as $T \longrightarrow 0^+$.

As pointed out in [32] (see also [31, 98, 99, 124]), the main reason for the form of the constant $C_h$ in (6.3) is due to the fact that the fundamental solution of the heat equation in $\mathbb{R}^N$ is given by

$$\Phi(x,t) = \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x|^2}{4t}}. \tag{6.4}$$

If one now considers the Stokes system

$$\left| \begin{array}{ll} y_t - \Delta y + \nabla p = g 1_\omega & \text{in } Q, \\ div\ y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega, \end{array} \right. \tag{6.5}$$

it is also well-known (see, for instance, [38]) that, given $y_0 \in L^2(\Omega)$ with $div\ y_0 = 0$, there exists $g \in L^2(\omega \times (0,T))$ such that the associated solution $y_0$ of (6.5) satisfies:

$$y(T) = 0.$$

Nevertheless, unlike the case of the heat equation, for the Stokes system, the known results in the literature (see, for instance, [38]) give

$$||g 1_\omega||_{L^2(Q)} \leq C_S(T) ||y_0||_{L^2(\Omega)}, \tag{6.6}$$

for a constant $C_S = C_S(T)$, the *cost of controllability for the Stokes system*, of the form $e^{C(\Omega,\omega)(1+1/T^4)}$, i.e., the Stokes system has a cost of controllability at most of order $e^{C/T^4}$ as $T \longrightarrow 0^+$.

Since the fundamental solutions of the heat and the Stokes system have, at least for $N = 2, 3$, the same behavior in time (see [56, 57, 121]), looking to (6.3) and (6.6), the following natural question arises:

**Question 6.1.** Do the costs of the controllability for the heat equation and the Stokes system have the same order in time as $T \longrightarrow 0^+$?

When trying to answer Question 6.1, the first attempt is to analyze the many different ways one can prove (6.3) and (6.6). In fact, there are at least two different ways to prove (6.3). The first one is based on spectral decompositions, the so-called Lebeau-Robbiano strategy (see [77]) and the second one based on the use of Carleman inequalities (see [44, 46]). For the Stokes system, it seems that a Lebeau-Robbiano strategy is very difficult to prove, since one must deal with the pressure and, as far as we know, it has not been proved yet to hold. Consequently, the most known method used to prove (6.6) is based on Carleman inequalities (see [38]).

The main difference when proving (6.3) and (6.6) by means of Carleman inequalities are the weights one must use. Indeed, for the heat equation the weights used are of the form

$$\rho(t) = \frac{e^{C/(t(T-t))}}{t(T-t)}, \tag{6.7}$$

while for the Stokes system the weights take the form

$$\rho(t) = \frac{e^{C/(t^4(T-t)^4)}}{t^4(T-t)^4}. \tag{6.8}$$

If we were able to use weights as (6.7) for the Stokes system then these two equations would have costs of controllability of the same order. However, a careful analysis in both proofs indicates that the obstruction to have weights of the form (6.7) for the Stokes system is due to the pressure term and that, probably, it is of purely technical nature.

The main objective of this part of the thesis is to show that, at least for good geometries, the heat equation and the Stokes system have costs of null controllability of the same order, as the time goes to zero. Our strategy will not be based on the use of Carleman inequalities but rather on the application of the Control Transmutation Method (CTM) (see [98]).

In order to use the CTM, we are led to study the null controllability of the following hyperbolic system with a pressure term:

$$\left| \begin{array}{ll} u_{tt} - \Delta u + \nabla p = h 1_\omega & \text{in } Q, \\ div\ u = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(0) = u^0, u_t(0) = u^1 & \text{in } \Omega. \end{array} \right. \tag{6.9}$$

The idea is as follows. If one can show that system (6.9) is null controllable, then the CTM can be applied to guarantee the null controllability of the Stokes system (6.5). Moreover, if the cost of controlling (6.9) is known, then the cost of the controllability for (6.5) can be obtained explicitly (see [98]).

It is important to mention that systems like (6.9) are simple models of dynamical elasticity for incompressible materials. They also appear in coupled elasto-thermicity problems where one of the coupling parameters (related to compressibility properties) tends to infinity (see [89]).

Concerning the controllability of (6.9), as far as we know, the only result available in the literature is [107]. There, the author shows the exact controllability of (6.9) when the control is acting on a part of the boundary. However, it seems that no controllability results are known when the control is acting internally, i.e., acting on a part of the domain. The main reason for that seems to be the fact that system (6.9) is not of Cauchy-Kowalewski type, which makes impossible to apply directly Holgrem's Theorem as in the case of the wave equation.

## 6.2 The Stokes system with regular initial data

In this section we prove that for regular initial data the Stokes system (6.5) is null controllable with a cost of order $e^{C/T}$ as $T \longrightarrow 0^+$. Our proof is based on the Control Transmutation Method in the spirit of [98] and a null controllability result for the system (6.9).

Throughout this chapter, we assume that $\Omega$ is star-shaped with respect to the origin, i.e., there exists $\gamma > 0$ such that

$$x \cdot \nu(x) \geq \gamma > 0, \quad \forall x \in \partial\Omega$$

and we define

$$R_0 = \max\{|x|,\ x \in \bar{\Omega}\}. \tag{6.10}$$

Our control region $\omega$ will be a nonempty subset of $\Omega$ satisfying

$$\exists \mathcal{O} \subset \mathbb{R}^N, \mathcal{O} \text{ being a neighborhood of } \partial\Omega \text{ and } \omega = \Omega \cap \mathcal{O}. \tag{6.11}$$

We also define the following usual spaces in the context of fluid mechanics:

$$\mathcal{V} = \{v \in C_0^\infty(\Omega);\ div\ v = 0\},$$
$$V = \overline{\mathcal{V}}^{H_0^1(\Omega)^N} = \{u \in H_0^1(\Omega)^N;\ div\ u = 0\},$$
$$H = \overline{\mathcal{V}}^{L^2(\Omega)^N} = \{u \in L^2(\Omega)^N;\ div\ u = 0, u \cdot \nu = 0 \text{ on } \partial\Omega\}.$$

The main result of this section is stated as follows.

**Theorem 6.1.** *Assume $\omega$ satisfies (6.11), $y_0 \in V$ and let $0 < T \leq 1$. Then there exists a control $g \in L^2(\omega \times (0,T))$ such that the solution $y$ of (6.5) satisfies:*

$$y(T) = 0.$$

*Moreover, there exist positive constants $C_1$ and $C_2$, depending only on $\Omega$ and $\omega$, such that*

$$\iint_{\omega\times(0,T)} |g|^2 dxdt \leq C_1 e^{C_2/T} ||y_0||_V^2, \tag{6.12}$$

*for all $y_0 \in V$ and $0 < T \leq 1$.*

*Proof of Theorem 6.1.* For the proof of Theorem 6.1, we need the following results.

**Theorem 6.2.** *Assume $\omega$ satisfies (6.11). Then there exists $T_0 > 0$ such that, for any $T > T_0$ and any $(u_0, u_1) \in V \times H$, we can find a control $h \in L^2(0,T;H)$ such that the associated solution $u$ of (6.9) satisfies:*

$$u(T) = u_t(T) = 0.$$

*Moreover, there exists $C > 0$ such that*

$$\iint_{\omega\times(0,T)} |h|^2 dxdt \leq C\big(||u^0||_V^2 + |u^1|_H^2\big). \tag{6.13}$$

**Lemma 6.1** (Miller, [98])**.** *There exists a positive constant $\alpha^*$ such that, for all $\alpha > \alpha^*$, there exists $\gamma > 0$ having the property that, for all $L > 0$ and $T \in (0, \inf(\pi/2, L)^2]$, there exists a distribution $k \in C([0,T]; \mathcal{M}(-L,L))$ satisfying*

$$\left|\begin{array}{l} k_t = \partial_s^2 k \ \ in \ \ \mathcal{D}'((0,T)\times(-L,L)), \\ k(0,x) = \delta(0), \\ k(T,x) = 0, \\ ||k||^2_{L^2((0,T)\times(-L,L))} \leq \gamma e^{\alpha L^2/T}. \end{array}\right. \tag{6.14}$$

We prove Theorem 6.2 in section 6.4.

Let us now introduce two different time intervals $(0,T)$ and $(0,L)$ and consider the two systems

$$\left|\begin{array}{ll} y_t - \Delta y + \nabla p = g 1_\omega & \text{in } Q_t := \Omega\times(0,T), \\ div\ y = 0 & \text{in } \ Q_t, \\ y = 0 & \text{on } \Sigma_t := \partial\Omega\times(0,T), \\ y(0) = y_0 & \text{in } \Omega \end{array}\right. \tag{6.15}$$

and

$$\left|\begin{array}{ll} u_{ll} - \Delta u + \nabla q = h 1_\omega & \text{in } Q_l := \Omega\times(0,L), \\ div\ u = 0 & \text{in } Q_l, \\ u = 0 & \text{on } \Sigma_l := \partial\Omega\times(0,L), \\ u(0) = y_0, \ u_l(0) = 0 & \text{in } \ \Omega \end{array}\right. \tag{6.16}$$

in $\Omega\times(0,T)$ and $\Omega\times(0,L)$, respectively. Here, $l \in (0,L)$ plays the role of a pseudo-time variable.

Taking $L > T_0$, where $T_0$ is the minimal time of Theorem 6.2, it follows from Theorem 6.2 that the system (6.16) is null controllable, with a control $h \in L^2(\omega \times (0, L))$ satisfying (6.13).

Next, we extend $k$ by zero outside $[0, T] \times (-L, L)$, $u$ and $h$ by reflection to $[-L, 0]$ and by zero outside $[-L, L]$ and set

$$y(t) = \int k(t, s)u(s)ds \tag{6.17}$$

and

$$g(t) = \int k(t, s)h(s)ds. \tag{6.18}$$

From (6.14), we see that

$$y(0) = y_0 \text{ and } y(T) = 0$$

and from (6.13) and $(6.14)_4$, we have that

$$\iint_{\omega\times(0,T)} |g|^2 dxdt \leq C\gamma e^{\alpha L^2/T} ||y_0||_V^2.$$

We finish the proof showing that the pair $(y, g)$ solves, together with some $p$, the Stokes system (6.15).

First, it is not difficult to see that

$$div\ y = 0 \text{ in } Q_t \text{ and } y = 0 \text{ on } \Sigma_t.$$

Now, for any $\varphi \in V$, we have

$$< y(t), \varphi >_H = < \int k(t, s)u(s)ds, \varphi >_H,$$

which implies

$$< y_t(t), \varphi >_H = < \int k_t(t, s)u(s)ds, \varphi >_H .$$

Using the properties of $k$, we see that

$$< y_t(t), \varphi >_H = < \int k_{ss}(t, s)u(s)ds, \varphi >_H .$$

Integrating by parts, and using the fact that $u(-L) = u(L) = u_l(-L) = u_l(L) = 0$, we obtain

$$< y_t(t), \varphi >_H = < \int k(t,s) u_{ss}(s) ds, \varphi >_H ds,$$

i.e.,

$$< y_t(t), \varphi >_H = \int k(t,s) < u_{ss}(s), \varphi >_H ds.$$

Since $u$ is, together with some $q$, solution of (6.16), we have

$$< y_t(t), \varphi >_H = \int k(t,s) < \Delta u(s) + h 1_\omega, \varphi >_H ds.$$

Therefore,

$$< y_t(t), \varphi >_H = < \int k(t,s) \Delta u(s) ds, \varphi >_H + < \int k(t,s) h 1_\omega ds, \varphi >_H .$$

This last identity gives

$$< y_t(t) - \Delta y(t), \varphi >_H = < g(t) 1_\omega, \varphi >_H, \tag{6.19}$$

and the proof is finished. □

## 6.3 The Stokes system with less regular data

In this section we improve the result obtained in section 6.2. Indeed, we prove that we can take less regular initial data and still have null controllability for the Stokes system with a cost of order $e^{C/T}$ as $T \longrightarrow 0^+$. In order to show the result, we combine Theorem 6.1, energy inequalities and the smoothing effect of the Stokes system.

The result is as follows.

**Theorem 6.3.** *Assume $\omega$ satisfies (6.11), $y_0 \in H$ and let $0 < T \leq 1$. Then there exists a control $g \in L^2(\omega \times (0,T))$ such that the solution $y$ of (6.5) satisfies:*

$$y(T) = 0.$$

*Moreover, there exist positive constants $C_1$ and $C_2$, depending only on $\Omega$ and $\omega$, such that*

$$\iint_{\omega \times (0,T)} |g|^2 dx dt \leq C_1 e^{C_2/T} |y_0|_H^2, \tag{6.20}$$

*for all $y_0 \in H$ and $0 < T \leq 1$.*

*Proof.* We begin choosing $\epsilon > 0$ small enough and letting system (6.15) evolve freely in the interval $(0, \epsilon)$. From the smoothing effect of the Stokes system, we have that

$y(\epsilon) = y_\epsilon$ belongs to $V$. We also have, thanks to Theorem 6.1, that there exists $g \in L^2(\omega \times (0, T-\epsilon))$ such that the associated solution $y$ to the problem

$$\left| \begin{array}{ll} y_t - \Delta y + \nabla p = g\chi_\omega & \text{in } (0, T-\epsilon) \times \Omega, \\ div\ y = 0 & \text{in } (0, T-\epsilon) \times \Omega, \\ y = 0 & \text{on } (0, T-\epsilon) \times \partial\Omega, \\ y(0) = y_\epsilon & \text{in } \ \Omega, \end{array} \right. \tag{6.21}$$

satisfies

$$y(T-\epsilon) = 0.$$

Moreover,

$$\int_0^{T-\epsilon}\int_\omega |g|^2 dx dt \leq C\gamma e^{\alpha L^2/T} ||y_\epsilon||_V^2. \tag{6.22}$$

Let us now define the functions $\overline{y}$ and $\overline{g}$ by $\overline{y}(t+\epsilon) = y(t)$, $\overline{g}(t+\epsilon) = g(t)$ for $0 < t < T - \epsilon$. The functions $\overline{y}$ and $\overline{g}$ are defined in $(\epsilon, T)$ and satisfy

$$\left| \begin{array}{ll} \overline{y}_t - \Delta \overline{y} + \nabla \overline{q} = \overline{g}\chi_\omega & \text{in } (\epsilon, T) \times \Omega, \\ div\ \overline{y} = 0 & \text{in } (\epsilon, T) \times \Omega, \\ \overline{y} = 0 & \text{on } (\epsilon, T) \times \partial\Omega, \\ \overline{y}(\epsilon) = y_\epsilon & \text{in } \Omega. \end{array} \right. \tag{6.23}$$

Inequality (6.22) then becomes

$$\int_\epsilon^T\int_\omega |\overline{g}|^2 dx dt \leq C\gamma e^{\alpha L^2/T} ||y_\epsilon||_V^2. \tag{6.24}$$

Next, we set

$$g(t) = \begin{cases} 0; & \text{if } \quad 0 < t < \epsilon, \\ \overline{g}(t); & \text{if } \quad \epsilon \leq t < T. \end{cases}$$

It is not difficult to see that the solution $y$ of (6.15), with $g$ as a control, fulfils $y(T) = 0$. From (6.24), and the definition of $g$, we have the following estimate

$$\int_0^T\int_\omega |y|^2 dx dt \leq C\gamma e^{\alpha L^2/T} ||y_\epsilon||_V^2. \tag{6.25}$$

Let us now consider system (6.15) in the interval $[0, \epsilon]$, i.e., we consider the system

$$\left| \begin{array}{ll} y_t - \Delta y + \nabla p = 0 & \text{in } (0,\epsilon)\times\Omega, \\ div\ y = 0 & \text{in } (0,\epsilon)\times\Omega, \\ y = 0 & \text{on } (0,\epsilon)\times\partial\Omega, \\ y(0) = y_0 & \text{in } \Omega, \end{array} \right. \tag{6.26}$$

with $y_0 \in H$.

We make the change of variable $z(t) = e^{-\frac{1}{t}}y(t)$. This new function $z$ solves

$$\left| \begin{array}{ll} z_t - \Delta z + \nabla p = \frac{1}{t^2}e^{-\frac{1}{t}}y & \text{in } (0,\epsilon)\times\Omega, \\ div\ z = 0 & \text{in } (0,\epsilon)\times\Omega, \\ z = 0 & \text{on } (0,\epsilon)\times\partial\Omega, \\ z(0) = 0 & \text{in } \Omega. \end{array} \right. \tag{6.27}$$

Using the fact that $\frac{1}{t^2}e^{-\frac{1}{t}}y \in L^2(0,\epsilon;H)$, and the regularity of the Stokes system, we conclude that $z \in L^2(0,\epsilon;H^2(\Omega))$ and that $z_t \in L^2(0,\epsilon;H)$.

Multiplying (6.27) by $z_t$ and integrating by parts, we get

$$2|z_t(t)|_H^2 + \frac{d}{dt}||z(t)||_V^2 = 2(\frac{1}{t^2}e^{-\frac{1}{t}}y(t), z_t)_H. \tag{6.28}$$

Integrating (6.28) from $0$ to $\epsilon$ and using Young's inequality, we obtain

$$2\int_0^\epsilon |z_t(t)|_H^2 dt + ||z(\epsilon)||_V^2 \le C_\delta \int_0^\epsilon |\frac{1}{t^2}e^{-\frac{1}{t}}y(t)|_H^2 dt + \delta\int_0^\epsilon |z_t|_H^2 dt,$$

for any $\delta > 0$.

Taking $\delta$ small enough, we have that

$$||z(\epsilon)||_V^2 \le C\int_0^\epsilon |\frac{1}{t^2}e^{-\frac{1}{t}}y(t)|_H^2 dt \tag{6.29}$$

and since, for $\epsilon$ sufficiently small, $\frac{1}{t^4}e^{-\frac{2}{t}} \le e^{\frac{1}{\epsilon}}$, it follows that

$$||z(\epsilon)||_V^2 \le e^{\frac{1}{\epsilon}}\int_0^\epsilon |y(t)|_H^2 dt.$$

Finally, using the fact that $||y||^2_{L^2(0,\epsilon;H)} \le \epsilon|y_0|_H^2$, we get from (6.29) that

$$||z(\epsilon)||_V^2 \le \epsilon e^{\frac{1}{\epsilon}}|y_0|_H^2,$$

and, in particular, using the fact that $z(t) = e^{-\frac{1}{t}} y(t)$, we conclude that

$$||y(\epsilon)||_V^2 \leq \epsilon e^{\frac{2}{\epsilon}} |y_0|_H^2. \tag{6.30}$$

From (6.25) and (6.30), the result follows. □

**Remark 6.1.** Since $y_\epsilon \longrightarrow y_0$ in $H$, the norm of $y_\epsilon$ is not bounded in $V$. Hence, the right-hand side of (6.25) is unbounded when $\epsilon \longrightarrow 0^+$.

## 6.4 Null controllability for the hyperbolic system

This section is devoted to prove Theorem 6.2 used in the proof of Theorem 6.1. In order to prove the result, it is convenient to write system (6.9) in an abstract way. For that, we introduce the Stokes operator $A : H^2(\Omega)^N \cap V \longrightarrow H$ given by

$$Au := P(\Delta u), \tag{6.31}$$

where $P : L^2(\Omega)^N \longrightarrow H$ is the orthogonal projection onto $H$ and $\Delta : H^2(\Omega)^N \cap H_0^1(\Omega)^N \longrightarrow L^2(\Omega)^N$ is the Laplace operator with Dirichlet boundary conditions. Thus, system (6.9) is equivalent to

$$\left| \begin{array}{l} u_{tt} = Au + h1_\omega, \\ u(0) = u^0, u_t(0) = u^1. \end{array} \right. \tag{6.32}$$

The following theorem holds.

**Theorem 6.4.** *Let $(u^0, u^1, h) \in V \times H \times L^2(0, T; H)$. Then there exists a unique (weak) solution $u$ of the problem (6.32) such that*

$$u \in C([0, T]; V) \cap C^1([0, T]; H)$$

*and $u$ satisfies:*

$$\frac{1}{2}|u_t(t)|_H^2 + \frac{1}{2}||u(t)||_V^2 = \frac{1}{2}|u^1|_H^2 + \frac{1}{2}||u^0||_V^2 + \int_0^t (h(s)1_\omega, u_t(s))_H ds, \ \ \forall t \in [0, T].$$

*Moreover, the linear mapping*

$$V \times H \times L^2(0, T; H) \longrightarrow C([0, T]; V) \cap C^1([0, T]; H)$$

$$(u^0, u^1, f) \mapsto u$$

*is continuous.*

The proof of Theorem 6.4 is standard and, being far from the aim of this thesis, it will not be reproduced here (for a proof see, for instance, [125]).

**Remark 6.2.** Arguing as in chapter 2 of [123], it is possible to show the existence of a function $p \in H^{-1}(0,T;L_0^2(\Omega))$ such that (6.9) is satisfied in $\mathcal{D}'(Q)$. Moreover, there exists $C > 0$ such that

$$||p||^2_{H^{-1}(0,T;L_0^2(\Omega))} \leq C(|u^1|^2_H + ||u^0||^2_V + ||h1_\omega||^2_{L^2(0,T;H)}).$$

By a classical duality argument (see, for instance, [46, 86, 97] ), it is not difficult to see that proving Theorem 6.2 is equivalent to show the existence of a positive constant $C$ such that

$$|\phi^0|^2_H + ||\phi^1||^2_{V'} \leq C \iint_{\omega\times(0,T)} |\phi|^2 dxdt, \tag{6.33}$$

for all solutions of

$$\left| \begin{array}{l} \phi_{tt} = A\phi, \\ \phi(0) = \phi^0, \phi_t(0) = \phi^1, \end{array} \right. \tag{6.34}$$

where $\phi^0 \in H$ and $\phi^1 \in V'$.

**Remark 6.3.** Since the Stokes operator $A$ is an isomorphism from $V$ to $V'$, given $(\phi^0, \phi^1) \in H \times V'$, we define the solution $\phi$ of (6.34) as

$$\phi = \psi_t,$$

where $\psi$ is the unique solution of

$$\left| \begin{array}{l} \psi_{tt} = A\psi, \\ \psi(0) = A^{-1}\phi^1, \psi_t(0) = \phi^0. \end{array} \right. \tag{6.35}$$

Following the ideas in [120], we can show that for regular initial data the abstract problem (6.34) is equivalent to

$$\left| \begin{array}{ll} \phi_{tt} - \Delta\phi + \nabla p = 0 & \text{in } Q, \\ div\ \phi = 0 & \text{in } Q, \\ \phi = 0 & \text{on } \Sigma, \\ \phi(0) = \phi^0, \phi_t(0) = \phi^1 & \text{in } \Omega. \end{array} \right. \tag{6.36}$$

Let us now concentrate on proving (6.33). The proof relies on some results that we prove below.

**Lemma 6.2.** *If, for every* $(\phi^0, \phi^1) \in V \times H$*, the solution* $\phi$ *of (6.34) satisfies*

$$||\phi^0||_V^2 + |\phi^1|_H^2 \leq C \iint_{\omega\times(0,T)} |\phi_t|^2 dxdt, \tag{6.37}$$

*for some constant* $C > 0$*, then inequality (6.33) holds for all solutions of (6.34) with initial data* $(\phi^0, \phi^1)$ *in* $H \times V'$*.*

*Proof of Lemma 6.2.* Given $(\phi^0, \phi^1) \in H \times V'$, we consider $\psi$ solution of (6.35), i.e.,

$$\left| \begin{array}{l} \psi_{tt} = A\psi, \\ \psi(0) = A^{-1}\phi^1, \psi_t(0) = \phi^0. \end{array} \right. \tag{6.38}$$

Next, using the fact that $\phi = \psi_t$, and inequality (6.37), we see that

$$||A^{-1}\phi^1||_V^2 + |\phi^0|_H^2 \leq C \iint_{\omega\times(0,T)} |\phi|^2 dxdt. \tag{6.39}$$

From (6.39) and the fact that $A : V \longrightarrow V'$ is an isomorphism, we finish the proof. □

**Lemma 6.3.** *Let* $m \in C^1(\overline{\Omega})^N$*. Then, for all regular solutions of (6.34), the following identity holds:*

$$\langle \nabla p, m \cdot \nabla\phi \rangle_{L^2(Q)^N} = -\langle \nabla p, \phi \cdot \nabla m \rangle_{L^2(Q)^N} + \langle \nabla p, \phi(div\ m) \rangle_{L^2(Q)^N}. \tag{6.40}$$

*Proof.* Let us set

$$X = -\iint_Q \frac{\partial p}{\partial x_i} m_k \frac{\partial \phi^i}{\partial x_k} dxdt.$$

Integrating by parts with respect to $x_k$, and using the fact that $\phi = 0$ on $\Sigma$, we get

$$X = \iint_Q \frac{\partial}{\partial x_k}\left(\frac{\partial p}{\partial x_i} m_k\right)\phi^i dxdt = \iint_Q \frac{\partial^2 p}{\partial x_k \partial x_i} m_k \phi^i dxdt + \iint_Q \frac{\partial p}{\partial x_i}\frac{\partial m_k}{\partial x_k}\phi^i dxdt.$$

Next, we integrate by parts again the first integral, this time with respect to $x_i$ and we obtain

$$\iint_Q \frac{\partial p}{\partial x_k}\frac{\partial}{\partial x_i}\left(m_k\phi^i\right) dxdt = -\iint_Q \nabla p\phi \cdot \nabla m dxdt.$$

Hence, we conclude that

$$X = -\iint_Q \nabla p\phi \cdot \nabla m dxdt + \iint_Q \nabla p\phi(div\ m) dxdt,$$

and the proof of Lemma 6.3 is finished. □

**Lemma 6.4.** *Assume $\omega$ satisfies (6.11) and let $T > 2R_0$. There exists $C > 0$ such that, for every $(\phi^0, \phi^1) \in V \times H$, the weak solution $\phi$ of (6.34) satisfies:*

$$||\phi^0||_V^2 + |\phi^1|_H^2 \le C \iint_{\omega\times(0,T)} \left(|\phi_t|^2 + |\phi|^2\right) dxdt. \tag{6.41}$$

*Proof.* We set the notation:

$$E(t) = |\phi_t(t)|_H^2 + ||\phi(t)||_V^2, \quad \forall t \in [0,T].$$

Without loss of generality, we assume that $\phi$ is regular and work with the equivalent problem (6.36), this is the case if we take, for instance, $\phi^0 \in V \cap H^4(\Omega)$ and $\phi^1 \in V \cap H^2(\Omega)$.

Using the change of variables $T\tau = (T-2\epsilon)t + T\epsilon$, which implies $\epsilon \le \tau \le T-\epsilon$, from the boundary observability inequality given in Theorem 6.6 in section 6.5, we have

$$E(0) \le C \int_\epsilon^{T-\epsilon} \int_{\partial\Omega} \left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Sigma.$$

Next, we consider a vector field $h \in C^2(\overline{\Omega})^N$ such that $h = \nu$ on $\partial\Omega$ and $h = 0$ on $\Omega \setminus \omega$ and let $\eta \in C^2([0,T])$ be such that $\eta(0) = \eta(T) = 0$ and $\eta(t) = 1$ in $(\epsilon, T-\epsilon)$. We define $\theta(x,t) = \eta(t)h(x)$, which belongs to $W^{2,\infty}(Q)$ and satisfies

$$\left|\begin{array}{l} \theta(x,t) = \nu(x) \ \text{ for all } \ (x,t) \in \partial\Omega \times (\epsilon, T-\epsilon), \\ \theta(x,0) = \theta(x,T) = 0, \ \text{ for all } \ x \in \Omega, \\ \theta(x,t) = 0 \ \text{ in } \ (\Omega \setminus \omega)\times(0,T). \end{array}\right.$$

Then we consider the multiplier $\theta \cdot \nabla\phi$ and, from Lemma 6.5 in the appendix, we obtain the following identity for all weak solution $\phi$ of (6.34):

$$\frac{1}{2}\iint_\Sigma \theta_k(x,t)\nu_k(x)\left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Sigma = (\phi_t(.), \theta(x,.)\cdot\nabla\phi(.))\big|_0^T + \iint_Q \frac{\partial\theta_k}{\partial x_j}\frac{\partial\phi^i}{\partial x_k}\frac{\partial\phi^i}{\partial x_j} dxdt \tag{6.42}$$
$$+\frac{1}{2}\iint_Q \frac{\partial\theta_k}{\partial x_k}\left(|\phi_t|^2 - |\nabla\phi|^2\right) dxdt + \iint_Q \frac{\partial p}{\partial x_i}\theta_k \frac{\partial\phi^i}{\partial x_k} dxdt.$$

Using the definition of $\theta$, we obtain

$$\frac{1}{2}\int_\epsilon^{T-\epsilon}\int_{\partial\Omega}\left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Sigma \le \frac{1}{2}\iint_\Sigma \theta_k(x,t)\nu_k(x)\left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Sigma,$$

because $\theta(x,t)=\nu(x)$ on $\partial\Omega\times(\epsilon,T-\epsilon)$ and

$$(\phi_t(.),\theta(x,.)\cdot\nabla\phi(.))\big|_0^T=0.$$

We also have

$$\left|\iint_Q \frac{\partial\theta_k}{\partial x_j}\frac{\partial\phi^i}{\partial x_k}\frac{\partial\phi^i}{\partial x_j}dxdt\right|\leq C\iint_{\omega\times(0,T)}|\nabla\phi|^2dxdt,$$

since $\theta\in C^1(\overline{\Omega}\times(0,T))$.

For the pressure, we use Lemma 6.3 to see that

$$\begin{aligned}\iint_Q \frac{\partial p}{\partial x_i}\theta_k\frac{\partial\phi^i}{\partial x_k}dxdt &= -\iint_Q \nabla p\phi\cdot\nabla\theta dxdt+\iint_Q\nabla p\phi(div\ \theta)dxdt\\ &=\langle\nabla p,-\phi\cdot\nabla\theta+\phi(div\ \theta)\rangle_{H^{-1}(Q)^N,H_0^1(Q)^N}.\end{aligned}$$

Consequently

$$\begin{aligned}\left|\iint_Q \frac{\partial p}{\partial x_i}\theta_k\frac{\partial\phi^i}{\partial x_k}dxdt\right| &\leq C_\delta\iint_{\omega\times(0,T)}(|\phi|^2+|\phi_t|^2+|\nabla\phi|^2)dxdt,\\ &+\delta\left\|\nabla p\right\|^2_{H^{-1}(Q)^N}\end{aligned}\tag{6.43}$$

for any $\delta>0$. Thus,

$$\frac{1}{2}\iint_\Sigma \theta_k(x,t)\nu_k(x)\left(\frac{\partial\phi}{\partial\nu}\right)^2d\Sigma\leq C\iint_{\omega\times(0,T)}\left(|\phi|^2+|\phi_t|^2+|\nabla\phi|^2\right)dxdt+\delta\left\|\nabla p\right\|^2_{H^{-1}(Q)^N}.$$

Using the fact that

$$\|\nabla p\|^2_{H^{-1}(Q)^N}\leq CE(0),$$

and choosing $\delta$ small enough, we conclude that

$$E(0)\leq C\int_\epsilon^{T-\epsilon}\int_{\partial\Omega}\left(\frac{\partial\phi}{\partial\nu}\right)^2d\Sigma\leq C\iint_{\omega\times(0,T)}\left(|\phi_t|^2+|\phi|^2+|\nabla\phi|^2\right)dxdt.\tag{6.44}$$

Hence, by a change of variables in time, we can show that

$$E(0)\leq C\int_\epsilon^{T-\epsilon}\int_\omega\left(|\phi|^2+|\phi_t|^2+|\nabla\phi|^2\right)dxdt.\tag{6.45}$$

Now, let $\omega_0$ be a neighborhood of $\partial\Omega$ such that $\Omega\cap\omega_0\subset\omega$. We observe that inequality

(6.45) is true for each neighborhood of $\partial\Omega$, and in particular for $\omega_0$, that is to say

$$E(0) \leq C \int_{\epsilon}^{T-\epsilon}\int_{\omega_0} \big(|\phi|^2 + |\phi_t|^2 + |\nabla\phi|^2\big)dxdt.$$

Now, we consider $\rho \in W^{1,\infty}(\Omega)$, $\rho \geq 0$, such that

$$\rho = 1 \text{ in } \omega_0, \text{ and } \rho = 0 \text{ in } \Omega \setminus \omega.$$

Defining $h = h(x,t)$ by $h(x,t) = \eta(t)\rho^2(x)$, where $\eta$ is defined above, it follows that

$$\left|\begin{array}{l} h(x,t) = 1 \text{ for all } (x,t) \in \omega_0 \times (\epsilon, T-\epsilon), \\ h(x,t) = 0, \text{ for all } (x,t) \in \big(\Omega \setminus \omega\big)\times(0,T), \\ h(x,0) = h(x,T) = 0, \text{ for all } x \in \Omega, \\ \frac{|\nabla h|}{h} \in L^\infty(Q). \end{array}\right.$$

Multiplying both sides of $(6.36)_1$ by $h\phi$ and integrating by parts in $Q$, we obtain

$$\iint_Q h\phi \cdot \phi_{tt} dxdt - \iint_Q h\phi \cdot \Delta\phi dxdt + \iint_Q h\nabla p \cdot \phi dxdt = 0.$$

We have

$$\iint_Q h\phi_{tt} \cdot \phi dxdt = -\iint_Q h|\phi_t|^2 dxdt - \iint_Q h_t\phi \cdot \phi_t dxdt. \tag{6.46}$$

For the second term in the right hand side of (6.46), since $\phi = 0$ on $\Sigma$, we have

$$-\iint_Q h\Delta\phi \cdot \phi dxdt = \iint_Q h|\nabla\phi|^2 dxdt + \iint_Q \phi \cdot \big(\nabla\phi \cdot \nabla h\big) dxdt.$$

Consequently,

$$\iint_Q h|\nabla\phi|^2 dxdt = \iint_Q h|\phi_t|^2 dxdt + \iint_Q h_t\phi \cdot \phi_t dxdt - \iint_Q \phi \cdot \big(\nabla\phi \cdot \nabla h\big) dxdt \\ - \iint_Q h\nabla p \cdot \phi dxdt.$$

It is immediate that

$$\left|\iint_Q \phi \cdot \big(\nabla\phi \cdot \nabla h\big) dxdt\right| \leq \frac{1}{2}\iint_Q h|\nabla\phi|^2 dxdt + \frac{1}{2}\iint_Q \frac{|\nabla h|^2}{h}|\phi|^2 dxdt.$$

Hence

$$\iint_Q h|\nabla\phi|^2 dxdt \le C\iint_{\omega\times(0,T)} (|\phi_t|^2+|\phi|^2)dxdt + 2\left|\iint_Q h\nabla p\cdot\phi dxdt\right|.$$

Next, observing that

$$\begin{aligned}\iint_Q h\nabla p\cdot\phi dxdt &= \langle p, \phi\cdot\nabla h\rangle_{H^{-1}(0,T;L^2(\Omega)^N),H_0^1(0,T;L^2(\Omega)^N)}\\ &\le \delta||p||^2_{H^{-1}(0,T;L^2(\Omega)^N)} + C_\delta||h\phi||^2_{H_0^1(0,T;L^2(\Omega)^N)},\end{aligned}$$

for any $\delta>0$, we conclude that

$$\int_\epsilon^{T-\epsilon}\int_{\omega_0}|\nabla\phi|^2dxdt \le C\iint_{\omega\times(0,T)}\left(|\phi_t|^2+|\phi|^2\right)dxdt + \delta||p||^2_{H^{-1}(0,T;L^2(\Omega)^N)}.$$

From this last estimate we infer that

$$E(0)\le C\iint_{\omega\times(0,T)}\left(|\phi_t|^2+|\phi|^2\right)dxdt + \delta||p||^2_{H^{-1}(0,T;L^2(\Omega)^N)}.$$

Finally, taking $\delta$ small enough, we obtain

$$E(0)\le C\iint_{\omega\times(0,T)}\left(|\phi_t|^2+|\phi|^2\right)dxdt, \tag{6.47}$$

which is exactly (6.41). □

**Proposition 6.1.** *Assume $\omega$ satisfies (6.11). Then there exist $T_0>0$ and a constant $C>0$ such that, for any $T>T_0$ and any $(\phi^0,\phi^1)\in V\times H$, the solution $\phi$ of (6.34) satisfies (6.37).*

*Proof of Proposition 6.1* . Let us suppose that (6.37) is not true. Then, given a natural number $n$, there exists an initial data $(\tilde{\phi}_n^0,\tilde{\phi}_n^1)$ such that $\tilde{\phi}_n$, the solution of (6.34) corresponding to this initial data, satisfies

$$||\tilde{\phi}_n^0||_V^2 + |\tilde{\phi}_n^1|_H^2 \ge n||\tilde{\phi}_{n,t}||_{L^2(\omega\times(0,T))}.$$

Without loss of generality, we assume that $(\tilde{\phi}_n^0,\tilde{\phi}_n^1)$ is smooth and set

$$K_n = \left(||\tilde{\phi}_n^0||_V^2 + |\tilde{\phi}_n^1|_H^2\right)^{1/2},$$

and

$$\phi_n^0 = \frac{\tilde{\phi}_n^0}{K_n},\quad \phi_n^1 = \frac{\tilde{\phi}_n^1}{K_n},\quad \phi_n = \frac{\tilde{\phi}_n}{K_n}.$$

We have

$$||\phi_{n,t}||^2_{L^2(\omega\times(0,T))} \leq \frac{1}{n} \tag{6.48}$$

and

$$||\phi_n^0||_V^2 + |\phi_n^1|_H^2 = 1. \tag{6.49}$$

From (6.48), there exist subsequences, denoted by the same index, such that

$$\liminf_{n\longrightarrow\infty} \iint_{\omega\times(0,T)} |\phi_{n,t}|^2 dxdt = 0, \tag{6.50}$$

$$\phi_n^0 \rightharpoonup \phi^0 \text{ in } V \tag{6.51}$$

and

$$\phi_n^1 \rightharpoonup \phi^1 \text{ in } H. \tag{6.52}$$

Since $\phi_n$ is the solution of (6.34) associated to the initial data $(\phi_n^0, \phi_n^1)$, we have:

$$\left| \begin{array}{l} \phi_n \text{ is bounded in } L^\infty(0,T;V), \\ \phi_{n,t} \text{ is bounded in } L^\infty(0,T;H). \end{array} \right. \tag{6.53}$$

Therefore, there exists a subsequence $\{\phi_n\}$ such that

$$\left| \begin{array}{l} \phi_n \longrightarrow \phi \text{ weak star in } L^\infty(0,T;V), \\ \phi_{n,t} \longrightarrow \phi_t \text{ weak star in } L^\infty(0,T;H). \end{array} \right. \tag{6.54}$$

From (6.54), it is not difficult to show that $\phi$ is the weak solution of (6.34) corresponding to the initial data $(\phi^0, \phi^1)$.

Next, since $V \hookrightarrow H$ compactly, estimate (6.54) and the Aubin-Lions compactness theorem give

$$\phi_n \longrightarrow \phi \text{ in } L^2(0,T;H). \tag{6.55}$$

Hence, it follows from (6.50) and (6.54) that

$$\phi_t \equiv 0 \text{ in } \omega \times (0,T) \tag{6.56}$$

and $\phi$ is independent of $t$ in $\omega$.

Let us now consider the system

$$\left| \begin{array}{l} \xi_{tt} = A\xi, \\ \xi(0) = \phi^1, \xi_t(0) = A\phi^0. \end{array} \right. \tag{6.57}$$

Taking $\psi(x,t) = \phi^0(x) + \int_0^t \xi(x,s)ds$, it is not difficult to see that $\psi$ solves (6.34), with $(\phi^0, \phi^1)$ as initial data. Therefore, from the uniqueness of solutions of (6.34), we have that $\psi \equiv \phi$ and thanks to (6.56) we have that $\xi \equiv 0$ in $\omega \times (0,T)$.

Let us now show that $\xi \equiv 0$. Indeed, applying the *curl* operator in (6.57), we see that $v = curl\ \xi$ satisfies

$$\left| \begin{array}{ll} v_{tt} - \Delta v = 0 & \text{in } Q, \\ v \equiv 0 & \text{in } \omega \times (0,T). \end{array} \right. \tag{6.58}$$

Then, by Holmgren's Uniqueness Theorem (see, for instance, [86]), there exists $T_0 > 0$ such that if $T > T_0$ then $v \equiv 0$. Therefore, there exists a scalar function $\Phi = \Phi(x,t)$ such that

$$\xi = \nabla \Phi \ \text{ in } \ Q.$$

In view of $(6.57)_2$, we have

$$\Delta \Phi = 0 \ \text{ in } \ Q.$$

Since $\xi = 0$ in $\omega \times (0,T)$, we also have that

$$\Phi = f(t) \ \text{ in } \ \omega \times (0,T).$$

From the unique continuation property of the Laplace equation, we deduce that

$$\Phi = f(t) \ \text{ in } \ Q,$$

which implies

$$\xi = \nabla \Phi = 0 \ \text{ in } \ Q. \tag{6.59}$$

Hence,

$$\phi^1 = \phi^0 = 0. \tag{6.60}$$

From (6.41), (6.55) and (6.60), we get a contradiction, and the proof is finished. □

As a consequence of Lemmas 6.2 and 6.4, and Proposition 6.1, we have the following result.

**Theorem 6.5.** *Assume $\omega$ satisfies (6.11). Then there exist $T_0 > 0$ and a constant $C > 0$ such that, for any $T > T_0$ and any $(\phi^0, \phi^1) \in H \times V'$, the solution $\phi$ of (6.34) satisfies (6.33).*

We end this section proving Theorem 6.2.

*Proof of Theorem 6.2.* We consider the functional

$$\mathcal{J} : H \times V' \longrightarrow \mathbb{R} \tag{6.61}$$

given by

$$\mathcal{J}(\phi^0,\phi^1)=\frac{1}{2}\iint_{\omega\times(0,T)}|\phi|^2dxdt+<\varphi^1,u^0>_{V,V'}-(\phi^0,u^1)_H,$$

where $\varphi$ is the solution of (6.34) corresponding to the initial data $(\phi^0,\phi^1)$.

Using the observability inequality (6.33) and energy estimates, we can show that the functional $\mathcal{J}$ is continuous, strictly convex and coercive. Therefore, $\mathcal{J}$ has a unique minimizer $(\hat{\phi}^0,\hat{\phi}^1)$. Using the Euler-Lagrange equation of $\mathcal{J}$, we conclude that $\hat{\phi}$, solution of (6.34) associated to $(\hat{\phi}^0,\hat{\phi}^1)$, is a control which drives $u$ to zero at time $T$. Inequality (6.13) then follows from the observability inequality (6.33) and the fact that $\mathcal{J}(\hat{\phi}^0,\hat{\phi}^1)\leq 0$. This finishes the proof of Theorem 6.2.

**Remark 6.4.** The minimal time $T_0$ in Proposition 6.1 and Theorems 6.2 and 6.5 must satisfy $T_0>2R_0$ and be such that Holgrem's Theorem can be applied to conclude that the solution of (6.58) vanishes (see [86]).

□

## 6.5 Boundary observability for the hyperbolic system

The main objective of this section is to prove the following result.

**Theorem 6.6.** *If we take $T>2R_0$ then, for every solution of (6.34) with initial data $(\phi^0,\phi^1)\in V\times H$, the following estimate holds:*

$$|\phi^1|_H^2+||\phi^0||_V^2\leq\frac{R_0}{2(T-2R_0)}\iint_\Sigma\left(\frac{\partial\phi}{\partial\nu}\right)^2d\Sigma. \tag{6.62}$$

For the proof of Theorem 6.6, we need the following two lemmas.

**Lemma 6.5.** *Let $\overline{q}=\overline{q}(x)$ be in $C^1(\bar{\Omega})^N$, then, for every regular solution $u$ of (6.32), the following identity holds:*

$$\begin{aligned}\frac{1}{2}\iint_\Sigma\overline{q}_k(x)\nu_k(x)\left(\frac{\partial u}{\partial\nu}\right)^2d\Sigma \;=\;& (u_t(t),\overline{q}(x)\nabla u(t))\Big|_0^T+\iint_Q\frac{\partial\overline{q}_k}{\partial x_j}\frac{\partial u^i}{\partial x_k}\frac{\partial u^i}{\partial x_j}dxdt\\ &+\frac{1}{2}\iint_Q\frac{\partial\overline{q}_k}{\partial x_k}\left(|u_t|^2-|\nabla u|^2\right)dxdt\\ &+\iint_Q\frac{\partial p}{\partial x_i}\overline{q}_k\frac{\partial u^i}{\partial x_k}dxdt+\iint_Q h^i\overline{q}_k\frac{\partial u^i}{\partial x_k}dxdt.\end{aligned} \tag{6.63}$$

The proof of Lemma 6.5 is the same as in the case of a single wave equation, the difference being that here we see the pressure as a force term in the right-hand side.

**Lemma 6.6.** *Let $(u^0, u^1, h) \in V \times H \times L^2(Q)^N$. Then the weak solution of (6.32) satisfies:*

$$\iint_\Sigma \left(\frac{\partial u}{\partial \nu}\right)^2 d\Sigma \le C\big(|u^1|_H^2 + ||u^0||_V^2 + ||h||^2_{L^2(Q)^N}\big).$$

*Proof.* The proof of Lemma 6.6 is obtained exactly as in the case of the wave equation, first showing the result for regular solutions. Indeed, here we take the vector field $\overline{q}$ in Lemma 6.5 to be the vector field $\overline{q}(x) = x$ and use the fact that

$$\iint_Q \frac{\partial p}{\partial x_i} \overline{q}_k \frac{\partial u^i}{\partial x_k} dxdt = 0.$$

□

*Proof of Theorem 6.6.* Without loss of generality, we assume that $\phi$ is regular and then work with the equivalent problem (6.36). Using Lemma 6.5, with $\overline{q}$ being the vector field $m(x) = x$, we have:

$$\frac{1}{2}\iint_\Sigma m{\cdot}\nu\left(\frac{\partial \phi}{\partial \nu}\right)^2 d\Sigma = (\phi_t(.), m(x)\nabla\phi(.))\big|_0^T + \iint_Q |\nabla\phi|^2 dxdt + \frac{N}{2}\iint_Q \Big(|\phi_t|^2 - |\nabla\phi|^2\Big) dxdt.$$

Next, multiplying $(6.36)_1$ by $\phi$ and integrating by parts, we easily see that

$$(\phi_t(.), \phi(.))\big|_0^T = \iint_Q |\phi_t|^2 dxdt - \iint_Q |\nabla\phi|^2 dxdt.$$

Then, using this last identity and the fact that

$$|\phi_t(t)|_H^2 + ||\phi(t)||_V^2 = |\phi^1|_H^2 + ||\phi^0||_V^2 \quad \forall t \in [0, T],$$

we obtain

$$(\phi_t(.), m\nabla u(.) + \frac{N-1}{2} u(.))\big|_0^T + T\big(|\phi^1|_H^2 + ||\phi^0||_V^2\big) = \frac{1}{2}\iint_\Sigma m \cdot \nu \left(\frac{\partial \phi}{\partial \nu}\right)^2 d\Sigma.$$

We also have

$$\big|m\nabla u(t) + \frac{N-1}{2} u(t)\big|^2 \le R_0 |\nabla\phi(t)|^2 \quad \forall t \in [0, T],$$

which implies, by Gronwall inequality, that

$$\left|(\phi_t(.), m\nabla\phi(.) + \frac{N-1}{2}\phi(.))\big|_0^T\right| \le 2R_0\big(|\phi^1|_H^2 + ||\phi^0||_V^2\big).$$

Finally, combining all the above estimates, we conclude that

$$\big(T-2R_0\big)\big(|\phi^1|_H^2+||\phi^0||_V^2\big)\leq \frac{R_0}{2}\iint_\Sigma \left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Sigma,$$

which is exactly (6.62).

□

# Chapter 7

# Perspectives and open problems

Let us now give some perspective about the many different chapters of this thesis.

- In chapter 2 we have analyzed the uniform null controllability of systems of the form

$$\left|\begin{array}{ll} u_t - \Delta u = au + bv + f1_{\omega_1} & \text{in } Q, \\ \epsilon v_t - \Delta v = cu + dv + g1_{\omega_2} & \text{in } Q, \\ u = v = 0 & \text{on } \Sigma, \\ u(0) = u_0;\ v(0) = v_0 & \text{in } \Omega, \end{array}\right. \tag{7.1}$$

  Inspired by the results in [51], it would be interesting, and natural, to consider the uniform null controllability, with only one control, of more general systems of the form

$$\left|\begin{array}{ll} u_t - \Delta u + cu + E\cdot\nabla u = P_1(x,t;D)(v\theta_1) + f1_{\omega_1} & \text{in } Q, \\ \epsilon v_t - \Delta v + hv + K\cdot\nabla v = P_2(x,t,D)(u\theta_2) + g1_{\omega_2} & \text{in } Q, \\ u = v = 0 & \text{on } \Sigma, \\ u(0) = u_0;\ v(0) = v_0 & \text{in } \Omega, \end{array}\right. \tag{7.2}$$

  where $c, E, h$, and $K$ are constants and $P_i(x,t;D)$ $(i = 1, 2)$ is a partial differential operator in the space variables of order $i$ and $\theta_i$ is such that $|\theta_i| \geq C > 0$ in, say, $\omega_i$.

  The methods of chapter 2 should be useful to treat such problems. However, here a careful analysis is in order, since a decomposition like (2.26) leads to a much more complicated extended adjoint system than (2.27).

  We remark that the null controllability of (7.2) by means of one control in the case where $c, E, h$, and $K$ are not constants is also an interesting open problem, even for a given fixed $\epsilon > 0$ (see [51]).

- In chapter 3, we consider the monodomain model (3.4) as a simplification of the bidomain model

$$\left|\begin{array}{ll} c_m v_t - Div(\mathbf{M}_i(x)\nabla u_i) + h(v) = f 1_{\omega_1} & \text{in } Q, \\ c_m v_t + Div(\mathbf{M}_e(x)\nabla u_e) + h(v) = g 1_{\omega_2} & \text{in } Q, \\ u_i = u_e = 0 & \text{on } \Sigma, \\ v(0,x) = v_0(x), & \text{in } \Omega, \end{array}\right. \tag{7.3}$$

where $v = u_i - u_e$.

It would be interesting to prove controllability results for the bidomain model (7.3). Notice that the difficult here is due to the fact that we have higher order terms in both equations, i.e., $\partial_t u_i$ and $\partial_t u_e$. As far as we know, controllability results for (7.3) has not been proved yet, even in the case where the two controls, $f$ and $g$, are acting on the system.

- In chapter 4, we consider the Keller-Segel system

$$\left|\begin{array}{ll} u_t - \Delta u = -\nabla \cdot (u \nabla v) & \text{in } Q, \\ \epsilon v_t - \Delta v = au - bv + g\chi & \text{in } Q, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \Sigma, \\ u(x,0) = u_0 & \text{in } \Omega. \end{array}\right. \tag{7.4}$$

We prove the uniform local null controllability of (7.4) around a stationary solution of the parabolic-elliptic system (4.2), i.e., a pair $(M_1, M_2) \in \mathbb{R}^2_+$ such that $aM_1 - bM_2 = 0$.

Given any $\epsilon > 0$, it is an open problem whether controllability result for (7.4) holds in the case of more general trajectories than the ones considered in chapter 4. In fact, this question is closely related to the null controllability of system (7.2) since, given any $(\overline{u}, \overline{v})$ solution of (7.4), the linearization of (7.4) around $(\overline{u}, \overline{v})$ gives a system like (7.2). The uniform local null controllability of (7.4) around other trajectories is also another interesting open problem.

- The results of chapter 5 concerns the controllability of a coupled system consisting on a heat equation and an ODE when the controller/observer moves in time. As seen in chapters 2 and 3, coupled systems consisting of parabolic equations and ode's are important since they appear in biological models of chemotaxis or interactions between cellular process and diffusing growth factors (see [61, 94, 106] and references therein). Systems governing these kind of phenomena are, in general, non linear and have the form

$$\left| \begin{array}{l} u_t = f(u, v), \\ v_t = D\Delta v + g(u, v), \end{array} \right. \tag{7.5}$$

where $v$ and $u$ are vectors, $D$ is a diagonal matrix with positive coefficients and $f$ and $g$ are real functions.

Using the arguments of chapter 5, it would be interesting to study the controllability of (7.5) for general functions $f$ and $g$.

The analysis of performed in chapter 5 can be also related to some recent works on the controllability of parabolic equations with memory terms. Indeed, notice that the system (5.6) in the particular case $b \equiv 0$ and $z(0) \equiv 0$, in the absence of the control $h$ and the addition of a control of the form $1_\omega k$ in the first equation, can be written as an integro-differential equation

$$y_t - \Delta y - y - \int_0^t e^{s-t} y(x, s) ds = 1_\omega k. \tag{7.6}$$

This system is closely related to the one considered in [52]. There, it is shown that the system (7.6) lacks to be null controllable. This is in agreement with our results that, in the particular case under consideration, show also that a moving control could bypass this limitation. It would be interesting to analyze to which extent this idea of controlling by moving the support of the control can be of use for more general parabolic equations with memory terms.

Finally, in the context of chapters 3 and 5, it would be interesting to study the controllability of the general bidomain model

$$\left| \begin{array}{l} c_m v_t - \text{div}\,(D_i \nabla u_i) + I_{ion}(v, w) = I^i_{app} + f 1_{\omega_1}, \\ c_m v_t - \text{div}\,(D_e \nabla u_e) - I_{ion}(v, w) = -I^e_{app} + g 1_{\omega_2}, \\ w_t - R(v, w) = 0. \end{array} \right. \tag{7.7}$$

As in the case of the bidomain model (7.3), it is unknown whether any controllability result for system (7.7) holds.

- In chapter 6, we study the optimal cost of the null controllability for the Stokes system

$$\left| \begin{array}{ll} y_t - \Delta y + \nabla p = g 1_\omega & \text{in } Q, \\ div\ y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega. \end{array} \right. \tag{7.8}$$

We prove that if the control region $\omega$ satisfies the geometrical control condition for the multiplier method to hold, then the Stokes system (7.8) has the same cost of the null controllability as the heat equation. It is unknown if the same result holds for any given control region $\omega \subset \Omega$.

# Appendix A

# Some degenerate Carleman inequalities

This appendix is devoted to prove some degenerated Carlmeman inequalities used in this thesis.

## A.1 Heat equation with homogeneous Dirichlet boundary condition

In this section we prove the Carleman inequalities given by Lemmas 2.2 and 3.1. Since both proofs are similar, we just prove the second result.

Let $g \in L^2(Q)$ and $q_T \in L^2(\Omega)$. We consider the equation

$$\left| \begin{array}{ll} -q_t(x,t) - \displaystyle\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}(a_{ij}(x)\frac{\partial}{\partial x_i}q(t,x)) = g(x,t) & \text{in } Q, \\ q = 0 & \text{on } \Sigma, \\ q(T) = q_T & \text{in } \Omega, \end{array} \right. \tag{A.1}$$

and assume that the matrix $a_{ij}$ take the form

$$a_{ij} = \frac{M_{ij}}{\sigma},$$

where $(M_{ij})_{ij}$ is an elliptic matrix, i.e., there exists $\gamma > 0$ such that $\sum_{i,j}^{N} M_{ij}\xi_j\xi_i \geq \gamma|\xi|^2$ for all $\xi \in \mathbb{R}^N$.

### A.1.1 The case $\beta = 0$

*Proof of Lemma 3.1 when $\beta = 0$.* For $s > 0$ and $\lambda > 0$, we consider the change of variable

$$w(t, w) = e^{s\alpha} q(t, w), \tag{A.2}$$

which implies

$$w(T, x) = w(0, x) = 0.$$

We have

$$L_1 w + L_2 w = g_s, \tag{A.3}$$

where

$$L_1 w = -w_t + 2s\lambda \sum_{i,j=1}^{N} \phi a_{ij} \frac{\partial}{\partial x_j} \psi \frac{\partial}{\partial x_i} w + 2s\lambda^2 \sum_{i,j=1}^{N} \phi a_{ij} \frac{\partial}{\partial x_i} \psi \frac{\partial}{\partial x_j} \psi w, \tag{A.4}$$

$$L_2 w = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} (a_{ij} \frac{\partial}{\partial x_j} w) - s^2 \lambda^2 \sum_{i,j=1}^{N} \phi^2 a_{ij} \frac{\partial}{\partial x_i} \psi \frac{\partial}{\partial x_j} \psi w + s\alpha_t w \tag{A.5}$$

and

$$g_s = e^{s\alpha} g + s\lambda^2 \sum_{i,j=1}^{N} \phi a_{ij} \frac{\partial}{\partial x_i} \psi \frac{\partial}{\partial x_j} \psi w - s\lambda \sum_{i,j=1}^{N} \phi \frac{\partial}{\partial x_i} (a_{ij} \frac{\partial}{\partial x_j} \psi) w. \tag{A.6}$$

From (A.3),

$$||L_1 w||^2_{L^2(Q)} + ||L_2 w||^2_{L^2(Q)} + 2(L_1 w, L_2 w)_{L^2(Q)} = ||g_s||^2_{L^2(Q)}. \tag{A.7}$$

We must analyze the terms appearing in $(L_1 w, L_2 w)_{L^2(Q)}$. We write

$$(L_1 w, L_2 w)_{L^2(Q)} = \sum_{i,j=1}^{N} I_{ij},$$

where $I_{ij}$ is the inner product in $L^2(Q)$ of the $ith$ term in the expression of $L_1 w$ and the $jth$ term of $L_2 w$ and, after a long, but straightforward, calculation, we can show that

$$\begin{aligned} 2(L_1 w, L_2 w)_{L^2(Q)} \geq\ & 2s^3\lambda^4\gamma^2\sigma^{-2} \iint_Q \phi^3 |\nabla\psi|^4 |w|^2 dxdt + 2s\lambda^2\gamma^2\sigma^{-2} \iint_Q \phi |\nabla\psi|^2 |\nabla w|^2 dxdt \\ & - C\sigma^{-2}\Big(T^2 s^2\lambda^4 + Ts^2\lambda^2 + T^2 s + s^3\lambda^3 + Ts^2\lambda\Big) \iint_Q \phi^3 |w|^2 dxdt \\ & - C\sigma^{-2}(s\lambda + \lambda^2) \iint_Q \phi |\nabla w|^2 dxdt. \end{aligned} \tag{A.8}$$

We take $\lambda \geq \lambda_0$ and $s \geq s_0(T+T^2)$, it follows from Remark A.1 that

$$\begin{aligned}&2(L_1w, L_2w)_{L^2(Q)} + 2s^3\lambda^4\gamma^2\sigma^{-2}\iint_{\omega_0\times(0,T)}\phi^3|w|^2dxdt\\&+2s\lambda^2\gamma^2\sigma^{-2}\iint_{\omega_0\times(0,T)}\phi|\nabla w|^2dxdt\\&\geq 2s^3\lambda^4\gamma^2\sigma^{-2}\iint_Q\phi^3|w|^2dxdt + 2s\lambda^2\gamma^2\sigma^{-2}\iint_Q\phi|\nabla w|^2dxdt. \qquad (A.9)\end{aligned}$$

**Remark A.1.** Since $\overline{\Omega\backslash\omega_0}$ is compact and $|\nabla\psi| > 0$ on $\overline{\Omega\backslash\omega_0}$, there exists $\delta > 0$ such that

$$\gamma|\nabla\psi| \geq \delta \text{ on } \overline{\Omega\backslash\omega_0}.$$

Putting (A.9) in (A.7), we get

$$\begin{aligned}&||L_1w||^2_{L^2(Q)} + ||L_2w||^2_{L^2(Q)} + 2\gamma^{-2}s^3\lambda^4\delta^4\sigma^{-2}\iint_Q\phi^3|w|^2dxdt\\&\qquad+2s\lambda^2\delta^2\sigma^{-2}\iint_Q\phi|\nabla w|^2dxdt\\&\leq ||g_s||^2_{L^2(Q)} + 2\gamma^{-2}s^3\lambda^4\delta^4\sigma^{-2}\iint_{\omega_0\times(0,T)}\phi^3|w|^2dxdt\\&\qquad+2s\lambda^2\delta^2\sigma^{-2}\iint_{\omega_0\times(0,T)}\phi|\nabla w|^2dxdt.\end{aligned} \qquad (A.10)$$

Now we want to deal with the local integral involving $\nabla w$ in the right-hand side of (A.10). To this end we introduce a cutt-off function $\xi$ such that

$$\xi \in C_0^\infty(\omega),\ 0 \leq \xi \leq 1,\ \xi(x) = 1\ \forall x \in \omega_0.$$

Using the ellipticity condition on $a_{ij}$, we prove that

$$\begin{aligned}&\gamma\sigma^{-1}\iint_{\omega\times(0,T)}\phi\xi^2|\nabla w|^2dxdt\\&\leq C\Big(\iint_Q L_2w\phi\xi^2wdxdt + (sT+\sigma^{-1}s^2\lambda^2)\iint_{\omega\times(0,T)}\phi^3|w|^2dxdt\\&\qquad+\lambda\sigma^{-1}\iint_{\omega\times(0,T)}\phi^{1/2}|\nabla w|\xi\phi^{1/2}wdxdt\Big).\end{aligned}$$

Therefore, using Young's inequality, we get

$$s\lambda^2\delta^2\sigma^{-2}\iint_{\omega\times(0,T)}\phi\xi^2|\nabla w|^2dxdt$$
$$\leq \frac{1}{4}\iint_Q |L_2 w|^2dxdt + C\gamma^{-2}s^3\lambda^4(\delta^4+\delta^2)\sigma^{-2}\iint_{\omega\times(0,T)}\phi^3|w|^2dxdt.$$

Thus, inequality (A.10) gives

$$\begin{aligned} ||L_1 w||^2_{L^2(Q)} + ||L_2 w||^2_{L^2(Q_T)} + \gamma^{-2}s^3\lambda^4\sigma^{-2}\iint_Q \phi^3|w|^2dxdt \\ + s\lambda^2\sigma^{-2}\iint_Q \phi|\nabla w|^2dxdt \\ \leq C\Big(||e^{s\alpha}g||^2_{L^2(Q)} + \gamma^{-2}s^3\lambda^4\sigma^{-2}\iint_{\omega\times(0,T)}\phi^3|w|^2dxdt\Big). \end{aligned} \tag{A.11}$$

Now we use the first two terms in left-hand side of (A.11) in order to add the integrals of $|\Delta w|^2$ and $|w_t|^2$ to the left-hand side of (A.11). This can be done using the expressions of $L_1 w$ and $L_2 w$. Indeed, from (A.4) and (A.5), we have

$$\begin{aligned} \iint_Q s^{-1}\phi^{-1}|w_t|^2dxdt + \sigma^{-2}\iint_Q s^{-1}\phi^{-1}\sum_{i,j=1}^N |\frac{\partial}{\partial x_i}(M_{ij}\frac{\partial}{\partial x_j}w)|^2dxdt \\ + s^3\lambda^4\sigma^{-2}\iint_Q \phi^3|w|^2dxdt + s\lambda^2\sigma^{-2}\iint_Q \phi|\nabla w|^2dxdt \\ \leq C\Big(||e^{s\alpha}g||^2_{L^2(Q)} + s^3\lambda^4\sigma^{-2}\iint_{\omega\times(0,T)}\phi^3|w|^2dxdt\Big). \end{aligned} \tag{A.12}$$

Using the term in $|\frac{\partial}{\partial x_i}(M_{ij}\frac{\partial}{\partial x_j}w)|^2$ in the lef-hand side of (A.12) and elliptic regularity, it is easy to show that

$$s^{-1}\sigma^{-2}\iint_Q \phi^{-1}\sum_{i,j=1}^N |\frac{\partial^2}{\partial x_i x_j}w|^2dxdt \;\leq\; C\Big(||e^{s\alpha}g||^2_{L^2(Q)} + s^3\lambda^4\sigma^{-2}\iint_{\omega\times(0,T)}\phi^3|w|^2dxdt\Big).$$

Estimate (A.12) then gives

$$\begin{aligned} \iint_Q s^{-1}\phi^{-1}|w_t|^2dxdt + s^{-1}\sigma^{-2}\iint_Q \phi^{-1}\sum_{i,j=1}^N |\frac{\partial^2}{\partial x_i x_j}w|^2dxdt \\ +s^3\lambda^4\sigma^{-2}\iint_Q \phi^3|w|^2dxdt + s\lambda^2\sigma^{-2}\iint_Q \phi|\nabla w|^2dxdt \\ \leq C\Big(||e^{s\alpha}g||^2_{L^2(Q)} + s^3\lambda^4\sigma^{-2}\iint_{\omega\times(0,T)}\phi^3|w|^2dxdt\Big). \end{aligned} \tag{A.13}$$

From (A.13) and the fact that $w = e^{s\alpha}v$, proof of Lemma 3.1 in the case when $\beta = 0$ is finished. □

### A.1.2 The case $\beta = 1$

Since the proof of Lemma 2.2 for a general $\beta$ is similar to the proof of Lemma 3.1 when $\beta = 1$, to fix ideas, we consider only the second case.

*Proof of Lemma 3.1 when $\beta = 1$.* The starting point is to apply the Carleman inequality given in Lemma 3.1 for $q$, i.e.,

$$\begin{aligned}\sigma^2 \iint_Q s^{-1}\phi^{-1}e^{2s\alpha}|q_t|^2 dxdt + \iint_Q s^{-1}\phi^{-1}e^{2s\alpha}\sum_{i,j=1}^{N}|\frac{\partial^2}{\partial x_i x_j}q|^2 dxdt \\ + s^3\lambda^4 \iint_Q \phi^3 e^{2s\alpha}|q|^2 dxdt + s\lambda^2 \iint_Q \phi e^{2s\alpha}|\nabla q|^2 dxdt \\ \leq C(\iint_Q e^{2s\alpha}|g|^2 dxdt + s^3\lambda^4 \iint_{\omega\times(0,T)} \phi^3 e^{2s\alpha}|q|^2 dxdt).\end{aligned} \tag{A.14}$$

Next, we introduce the function $y(x,t) = q(x,t)(\phi^*(t))^{\frac{1}{2}}$, which solves the system

$$\left|\begin{array}{ll}\sigma y_t - Div(M(x)\nabla y) = -\sigma\frac{(T-2t)}{2}\phi^* y + \sigma(\phi^*(t))^{\frac{1}{2}}g & \text{in } Q, \\ y = 0 & \text{on } \Sigma.\end{array}\right. \tag{A.15}$$

Applying again the Carleman inequality given by Lemma 3.1, at this time for $y$, we obtain, for $s$ large enough, that

$$\begin{aligned}\iint_Q s^{-1}\phi^{-1}e^{2s\alpha}|y_t|^2 dxdt + \sigma^{-2}\iint_Q s^{-1}\phi^{-1}e^{2s\alpha}\sum_{i,j=1}^{N}|\frac{\partial^2}{\partial x_i x_j}y|^2 dxdt \\ + s^3\lambda^4\sigma^{-2}\iint_Q \phi^3 e^{2s\alpha}|y|^2 dxdt + s\lambda^2\sigma^{-2}\iint_Q \phi e^{2s\alpha}|\nabla y|^2 dxdt \\ \leq C(\iint_Q \phi^* e^{2s\alpha}|g|^2 dxdt + s^3\lambda^4\sigma^{-2}\iint_{\omega\times(0,T)} \phi^3 e^{2s\alpha}|y|^2 dxdt).\end{aligned} \tag{A.16}$$

From the definition of $y$ it is easy to show that

$$\iint_Q s^{-1}\phi^{-1}e^{2s\alpha}|q_t(\phi^*)^{\frac{1}{2}}|^2 dxdt \leq \iint_Q s^{-1}\phi^{-1}e^{2s\alpha}|y_t|^2 dxdt + \iint_Q e^{2s\alpha}\phi|y|^2 dxdt. \tag{A.17}$$

Using (A.17), inequality (A.16) becomes

$$\begin{aligned}
&\iint_Q s^{-1}\phi^{-1}\phi^* e^{2s\alpha}|q_t|^2 dxdt + \sigma^{-2}\iint_Q s^{-1}\phi^*\phi^{-1}e^{2s\alpha}\sum_{i,j=1}^N |\frac{\partial^2}{\partial x_i x_j} q|^2 dxdt \\
&\quad + s^3\lambda^4\sigma^{-2}\iint_Q \phi^3\phi^* e^{2s\alpha}|q|^2 dxdt + s\lambda^2\sigma^{-2}\iint_Q \phi\phi^* e^{2s\alpha}|\nabla q|^2 dxdt \\
&\leq C(\iint_Q \phi^* e^{2s\alpha}|g|^2 dxdt + s^3\lambda^4\sigma^{-2}\iint_{\omega\times(0,T)}\phi^3\phi^* e^{2s\alpha}|q|^2 dxdt).
\end{aligned} \tag{A.18}$$

From Remark 3.3 the result follows. □

## A.2 Heat equation with homogeneous Neumann boundary condition

In this section we prove Lemma 4.1 in the case $\beta = 0$ (the proof in the general case is analogous as the one for the heat equation with Dirichlet boundary condition, see Lemma 3.1). This lemma was used in chapter 4 in order to obtain a controllability result for the Keller-Segel system (4.1).

*Proof of Lemma 4.1 in the case $\beta = 0$.*

We introduce the following new weight functions

$$\tilde{\phi}(x,t) = \frac{e^{\lambda(m||\eta_0||_\infty - \eta_0(x))}}{t^4(T-t)^4}, \ \tilde{\alpha}(x,t) = \frac{e^{\lambda(m||\eta_0||_\infty - \eta^0(x))} - e^{5/4m\lambda||\eta_0||_\infty}}{t^4(T-t)^4}, \tag{A.19}$$

and consider $\omega_0 \subset\subset \omega' \subset\subset \omega$.

For an easier comprehension the proof will be divided into several steps.

**Step. 1.** For $s > 0$ and $\lambda > 0$, we make the change of variable

$$w(t,w) = e^{s\alpha}q(t,w), \tag{A.20}$$

which implies

$$w(T,x) = w(0,x) = 0.$$

We have

$$L_1 w + L_2 w = f_s, \tag{A.21}$$

where

$$L_1 w = -\sigma w_t + 2s\lambda\phi\nabla\psi\cdot\nabla w + 2s\lambda^2\phi|\nabla\psi|^2 w, \tag{A.22}$$

$$L_2 w = -\Delta w - s^2\lambda^2\phi^2|\nabla\psi|^2 w + \sigma s\alpha_t w \tag{A.23}$$

and

$$f_s = e^{s\alpha} g + s\lambda^2\phi|\nabla\psi|^2 w - s\lambda\phi\Delta\psi w. \tag{A.24}$$

From (A.21), we have that

$$||L_1 w||^2_{L^2(Q)} + ||L_2 w||^2_{L^2(Q)} + 2(L_1 w, L_2 w)_{L^2(Q)} = ||f_s||^2_{L^2(Q)}. \tag{A.25}$$

In this step we analyze the terms appearing in $(L_1 w, L_2 w)_{L^2(Q)}$. First, we write

$$(L_1 w, L_2 w)_{L^2(Q)} = \sum_{i,j=1}^{N} I_{ij}.$$

We have

$$I_{11} = \sigma \iint_Q w_t \Delta w dxdt = \sigma \iint_\Sigma \frac{\partial w}{\partial \nu} w_t d\Sigma. \tag{A.26}$$

Then,

$$\begin{aligned} I_{12} &= \frac{\sigma}{2} \iint_Q s^2\lambda^2\phi^2|\nabla\psi|^2 \frac{d}{dt}|w|^2 dxdt = -\sigma \iint_Q s^2\lambda^2\phi\phi_t|\nabla\psi|^2|w|^2 dxdt \\ &\le C\sigma s^2\lambda^2 T^4 \iint_Q \phi^3|w|^2 dxdt \end{aligned} \tag{A.27}$$

and

$$I_{13} = -\sigma^2 \iint_Q s\alpha_t w w_t dxdt \le C\sigma^2 e^{\lambda||\psi||_\infty} sT^8 \iint_Q \phi^3|w|^2 dxdt, \tag{A.28}$$

since

$$|\alpha_{tt}| \le Ce^{\lambda||\psi||_\infty}\phi^2(1 + T^8\phi) \le Ce^{\lambda||\psi||_\infty}\phi^3.$$

Next,

$$\begin{aligned} I_{21} &= -2\iint_Q s\lambda\phi\nabla\psi\nabla w\Delta w dxdt \\ &= -2s\lambda \iint_\Sigma \phi\frac{\partial\psi}{\partial\nu}\Big|\frac{\partial w}{\partial\nu}\Big|^2 d\Sigma + 2s\lambda \iint_Q \frac{\partial^2\psi}{\partial x_i x_j}\phi\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j} dxdt \\ &+ 2s\lambda^2 \iint_Q \phi|\nabla\psi\cdot\nabla w|^2 dxdt + s\lambda \iint_Q \phi\nabla\psi\cdot\nabla|\nabla w|^2 dxdt \\ &:= A_1 + A_2 + A_3 + A_4. \end{aligned}$$

We keep the boundary term $A_1$, the term $A_3 \geq 0$ and estimate the other two terms:

$$|A_2| \leq Cs\lambda \iint_Q \phi |\nabla w|^2 dxdt.$$

For $A_4$, we first observe that

$$\begin{aligned}
A_4 &= s\lambda \iint_\Sigma \phi \frac{\partial \psi}{\partial \nu} |\nabla w|^2 d\Sigma - s\lambda^2 \iint_Q \phi |\nabla \psi|^2 |\nabla w|^2 dxdt \\
&- s\lambda \iint_Q \phi \Delta\psi |\nabla w|^2 dxdt \\
&:= A_{41} + A_{42} + A_{43}.
\end{aligned}$$

We keep $A_{41}$ and $A_{42}$ and estimate the last term as

$$|A_{43}| \leq Cs\lambda \iint_Q \phi |\nabla w|^2 dxdt.$$

Consequently,

$$\begin{aligned}
I_{21} &\geq -2s\lambda \iint_\Sigma \phi \frac{\partial \psi}{\partial \nu} \Big|\frac{\partial w}{\partial \nu}\Big|^2 d\Sigma + 2s\lambda^2 \iint_Q \phi |\nabla \psi \cdot \nabla w|^2 dxdt \\
&+ s\lambda \iint_\Sigma \phi \frac{\partial \psi}{\partial \nu} |\nabla w|^2 d\Sigma - s\lambda^2 \iint_Q \phi |\nabla \psi|^2 |\nabla w|^2 dxdt \\
&- Cs\lambda \iint_Q \phi |\nabla w|^2 dxdt. \qquad (A.29)
\end{aligned}$$

Next,

$$\begin{aligned}
I_{22} &= -2s^3\lambda^3 \iint_Q \nabla\psi \cdot \nabla w \phi^3 |\nabla \psi|^2 w dxdt \\
&= 3s^3\lambda^4 \iint_Q \phi^3 |\nabla\psi|^4 |w|^2 dxdt + s^3\lambda^3 \iint_Q \Delta\psi |\nabla\psi|^2 \phi^3 |w|^2 dxdt \\
&+ 2s^3\lambda^3 \iint_Q \frac{\partial \psi}{\partial x_i} \frac{\partial \psi}{\partial x_i x_j} \frac{\partial \psi}{\partial x_j} \phi^3 |w|^2 dxdt \\
&- s^3\lambda^3 \iint_\Sigma \phi^3 |\nabla\psi|^2 \frac{\partial \psi}{\partial \nu} |w|^2 d\Sigma \\
&:= B_1 + B_2 + B_3 + B_4.
\end{aligned}$$

We keep $B_1$ and $B_4$. Fro the other two, we have

$$|B_2| + |B_3| \leq Cs^3\lambda^3 \iint_Q \phi^3 |w|^2 dxdt.$$

Consequently,

$$\begin{aligned} I_{22} \geq 3s^3\lambda^4 \iint_Q \phi^3|\nabla\psi|^4|w|^2 dxdt - s^3\lambda^3 \iint_\Sigma \phi^3|\nabla\psi|^2 \frac{\partial\psi}{\partial\nu}|w|^2 d\Sigma \\ - Cs^3\lambda^3 \iint_Q \phi^3|w|^2 dxdt. \end{aligned} \tag{A.30}$$

We also have

$$\begin{aligned} I_{23} &= 2\sigma s^2\lambda \iint_Q \nabla\psi\cdot\nabla w\phi\alpha_t w dxdt \\ &= \sigma s^2\lambda \iint_\Sigma \phi\alpha_t \frac{\partial\psi}{\partial\nu}|w|^2 d\Sigma - \sigma s^2\lambda^2 \iint_Q |\nabla\psi|^2\phi\alpha_t|w|^2 dxdt \\ &- \sigma s^2\lambda \iint_Q \phi\nabla\psi\cdot\nabla\alpha_t|w|^2 dxdt - \sigma s^2\lambda \iint_Q \phi\Delta\psi\alpha_t|w|^2 dxdt \\ &:= D_1 + D_2 + D_3 + D_4. \end{aligned}$$

Observe that

$$|D_2| + |D_3| + D_4| \leq C\sigma e^{2\lambda||\psi||_\infty} s^2\lambda^2 T^4 \iint_Q |\phi|^3|w|^2 dxdt,$$

which gives

$$I_{23} \geq \sigma s^2\lambda \iint_\Sigma \phi\alpha_t \frac{\partial\psi}{\partial\nu}|w|^2 d\Sigma - C\sigma e^{2\lambda||\psi||_\infty} s^2\lambda^2 T^4 \iint_Q |\phi|^3|w|^2 dxdt. \tag{A.31}$$

Next,

$$\begin{aligned} I_{31} &= -2s\lambda^2 \iint_Q \phi|\nabla\psi|^2\psi w\Delta w dxdt \\ &= -2s\lambda^2 \iint_\Sigma \phi|\nabla\psi|^2 \frac{\partial w}{\partial\nu} w d\Sigma + 2s\lambda^2 \iint_Q \phi|\nabla\psi|^2|\nabla w|^2 dxdt \\ &+ 4s\lambda^2 \iint_Q \phi\frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_i x_j}\frac{\partial w}{\partial x_j} w dxdt + s\lambda^3 \iint_Q \phi|\nabla\psi|^2\nabla\psi\cdot\nabla|w|^2 dxdt \\ &= E_1 + E_2 + E_3 + E_4. \end{aligned}$$

We keep $E_1$ and $E_2$. For $E_3$ and $E_4$, we have

$$E_3 \leq Cs\lambda^4 \iint_Q \phi|w|^2 dxdt + Cs \iint_Q \phi|\nabla w|^2 dxdt$$

and

$$E_3 \le Cs^2\lambda^4 \iint_Q \phi^2|w|^2dxdt + C\lambda^2 \iint_Q |\nabla w|^2dxdt.$$

Hence,

$$\begin{aligned} I_{31} \ge & -2s\lambda^2 \iint_\Sigma \phi|\nabla\psi|^2\frac{\partial w}{\partial\nu}wd\Sigma + 2s\lambda^2 \iint_Q \phi|\nabla\psi|^2|\nabla w|^2dxdt \\ & - Cs\lambda^4 \iint_Q \phi^2|w|^2dxdt - Cs\iint_Q |\nabla w|^2dxdt \\ & - Cs^2\lambda^4 \iint_Q \phi^2|w|^2dxdt - C\lambda^2 \iint_Q |\nabla w|^2dxdt \end{aligned} \tag{A.32}$$

and

$$I_{32} = -2s^3\lambda^4 \iint_Q |\nabla\psi|^4\phi^3|w|^2dxdt. \tag{A.33}$$

Finally,

$$I_{33} = -2\sigma s^2\lambda^2 \iint_Q \phi|\nabla\psi|^2\psi\alpha_t|w|^2dxdt \le Ce^{2\lambda||\psi||_\infty}s^2\lambda^2T^4 \iint_Q \phi^3|w|^2dxdt. \tag{A.34}$$

Therefore,

$$\begin{aligned} (L_1w, L_2w)_{L^2(Q)} \ge & \ \sigma \iint_\Sigma \frac{\partial w}{\partial\nu}w_td\Sigma + \sigma s^2\lambda \iint_\Sigma \phi\alpha_t\frac{\partial\psi}{\partial\nu}|w|^2d\Sigma \\ & - 2s\lambda \iint_\Sigma \phi\frac{\partial\psi}{\partial\nu}\Big|\frac{\partial w}{\partial\nu}\Big|^2d\Sigma + s\lambda \iint_\Sigma \phi\frac{\partial\psi}{\partial\nu}|\nabla w|^2d\Sigma \\ & - 2s\lambda^2 \iint_\Sigma \phi|\nabla\psi|^2\frac{\partial w}{\partial\nu}wd\Sigma - s^3\lambda^3 \iint_\Sigma \phi^3|\nabla\psi|^2\frac{\partial\psi}{\partial\nu}|w|^2d\Sigma \\ & + s^3\lambda^4 \iint_Q \phi^3|\nabla\psi|^4|w|^2dxdt + s\lambda^2 \iint_Q \phi|\nabla\psi|^2|\nabla w|^2dxdt \\ & - Cs^2\lambda^4 \iint_Q \phi^2|w|^2dxdt - C\lambda^2 \iint_Q |\nabla w|^2dxdt \\ & - Cs\lambda^4 \iint_Q \phi^2|w|^2dxdt - Cs\iint_Q |\nabla w|^2dxdt \\ & - Ce^{2\lambda||\psi||_\infty}s^2\lambda^2T^4 \iint_Q \phi^3|w|^2dxdt - C\sigma s^2\lambda^2T^4 \iint_Q \phi^3|w|^2dxdt \\ & - Cs\lambda \iint_Q \phi|\nabla w|^2dxdt - C\sigma^2e^{\lambda||\psi||_\infty}sT^8 \iint_Q \phi^3|w|^2dxdt \\ & - C\sigma e^{2\lambda||\psi||_\infty}s^2\lambda^2T^4 \iint_Q \phi^3|w|^2dxdt - Cs^3\lambda^3 \iint_Q \phi^3|w|^2dxdt. \end{aligned} \tag{A.35}$$

As a consequence of the properties of $\psi$ (see Lemma 2.1), we have

$$\begin{aligned}&s^3\lambda^4\iint_Q \phi^3|\nabla\psi|^4|w|^2dxdt + s\lambda^2\iint_Q \phi|\nabla\psi|^2|\nabla w|^2dxdt\\&\geq Cs^3\lambda^4\iint_Q \phi^3|w|^2dxdt + Cs\lambda^2\iint_Q \phi|\nabla w|^2dxdt\\&- Cs^3\lambda^4\iint_{\omega'\times(0,T)} \phi^3|w|^2dxdt - Cs\lambda^2\iint_{\omega'\times(0,T)} \phi|\nabla w|^2dxdt. \end{aligned}\tag{A.36}$$

Taking $\lambda \geq \lambda_0(\Omega,\omega)$ and $s \geq s_0(\Omega,\omega)(e^{2\lambda||\psi||_\infty}T^4 + T^8)$, we can absorb the lower order terms in (A.35) and $f_s$ and obtain from (A.25) that

$$\begin{aligned}&||L_1w||^2_{L^2(Q)} + ||L_2w||^2_{L^2(Q)} + s^3\lambda^4\iint_Q \phi^3|w|^2dxdt + s\lambda^2\iint_Q \phi|\nabla w|^2dxdt\\&+\sigma\iint_\Sigma \frac{\partial w}{\partial\nu}w_t d\Sigma + \sigma s^2\lambda\iint_\Sigma \phi\alpha_t\frac{\partial\psi}{\partial\nu}|w|^2d\Sigma\\&- 2s\lambda\iint_\Sigma \phi\frac{\partial\psi}{\partial\nu}|\frac{\partial w}{\partial\nu}|^2d\Sigma + s\lambda\iint_\Sigma \phi\frac{\partial\psi}{\partial\nu}|\nabla w|^2d\Sigma\\&- 2s\lambda^2\iint_\Sigma \phi|\nabla\psi|^2\frac{\partial w}{\partial\nu}wd\Sigma - s^3\lambda^3\iint_\Sigma \phi^3|\nabla\psi|^2\frac{\partial\psi}{\partial\nu}|w|^2d\Sigma\\&\leq C\big(||e^{s\alpha}f||^2_{L^2(Q)} + s^3\lambda^4\iint_{\omega'\times(0,T)} \phi^3|w|^2dxdt + s\lambda^2\iint_{\omega'\times(0,T)} \phi|\nabla w|^2dxdt\big).\end{aligned}\tag{A.37}$$

Let us now add integrals of $\Delta w$ and $w_t$ in the left-hand side of (A.37). This can be done since

$$\sigma^2s^{-1}\iint_Q \phi^{-1}|w_t|^2dxdt \leq C\big(s\lambda^4\iint_Q \phi|w|^2dxdt + s\lambda^2\iint_Q \phi|\nabla w|^2dxdt + ||L_1w||^2_{L^2(Q)}\big)$$

and

$$\begin{aligned}s^{-1}\iint_Q \phi^{-1}|\Delta w|^2dxdt \leq& C\big(s^3\lambda^4\iint_Q \phi|w|^2dxdt\\&+ s\sigma T^8e^{4\lambda||\psi||_\infty}\iint_Q \phi^3|w|^2dxdt + ||L_2w||^2_{L^2(Q)}\big),\end{aligned}$$

for $s \geq CT^8$. Accordingly, we deduce from (A.37) that

$$\begin{aligned}
&s^{-1}\iint_Q \phi^{-1}(\sigma^2|w_t|^2+|\Delta w|^2)dxdt + s^3\lambda^4\iint_Q \phi^3|w|^2dxdt + s\lambda^2\iint_Q \phi|\nabla w|^2dxdt \\
&+\sigma\iint_\Sigma \frac{\partial w}{\partial \nu}w_t d\Sigma + \sigma s^2\lambda\iint_\Sigma \phi\alpha_t\frac{\partial\psi}{\partial\nu}|w|^2d\Sigma \\
&-2s\lambda\iint_\Sigma \phi\frac{\partial\psi}{\partial\nu}|\frac{\partial w}{\partial\nu}|^2d\Sigma + s\lambda\iint_\Sigma \phi\frac{\partial\psi}{\partial\nu}|\nabla w|^2d\Sigma \\
&-2s\lambda^2\iint_\Sigma \phi|\nabla\psi|^2\frac{\partial w}{\partial\nu}wd\Sigma - s^3\lambda^3\iint_\Sigma \phi^3|\nabla\psi|^2\frac{\partial\psi}{\partial\nu}|w|^2d\Sigma \\
&\leq C\big(||e^{s\alpha}f||^2_{L^2(Q)} + s^3\lambda^4\iint_{\omega'\times(0,T)}\phi^3|w|^2dxdt + s\lambda^2\iint_{\omega'\times(0,T)}\phi|\nabla w|^2dxdt\big).
\end{aligned} \tag{A.38}$$

Now we estimate the local integral of $\nabla w$ in (A.38). To this end, let us introduce a function $\theta = \theta(x)$, with

$$\theta \in C_0^\infty(\omega),\ \theta \equiv 1 \text{ in } \omega',\ 0 \leq \theta \leq 0$$

and we make some computations:

$$\begin{aligned}
s\lambda^2\iint_{\omega'\times(0,T)}\phi|\nabla w|^2dxdt &\leq s\lambda^2\iint_{\omega\times(0,T)}\theta\phi|\nabla w|^2dxdt \\
&= -s\lambda^2\iint_{\omega\times(0,T)}\phi w\Delta dxdt - s\lambda^2\iint_{\omega\times(0,T)}\phi(\nabla w\cdot\nabla\theta)wdxdt \\
&- s\lambda^3\iint_{\omega\times(0,T)}\theta\phi(\nabla w\cdot\nabla\psi)wdxdt \\
&\leq \delta s^{-1}\iint_{\omega\times(0,T)}\phi^{-1}|\Delta w|^2dxdt + s^3\lambda^4\iint_{\omega\times(0,T)}\phi^3|w|^2dxdt.
\end{aligned} \tag{A.39}$$

In view of this last estimate, we deduce that the integral on $\nabla w$ in the right-hand side of (A.38) can be supressed if we enlarge slightly the control domain. We have the following estimate:

$$
\begin{aligned}
&s^{-1}\iint_Q \phi^{-1}(\sigma^2|w_t|^2+|\Delta w|^2)dxdt + s^3\lambda^4\iint_Q \phi^3|w|^2dxdt + s\lambda^2\iint_Q \phi|\nabla w|^2dxdt\\
&+\sigma\iint_\Sigma \frac{\partial w}{\partial\nu}w_t d\Sigma + \sigma s^2\lambda\iint_\Sigma \phi\alpha_t\frac{\partial\psi}{\partial\nu}|w|^2d\Sigma\\
&-2s\lambda\iint_\Sigma \phi\frac{\partial\psi}{\partial\nu}|\frac{\partial w}{\partial\nu}|^2d\Sigma + s\lambda\iint_\Sigma \phi\frac{\partial\psi}{\partial\nu}|\nabla w|^2d\Sigma\\
&-2s\lambda^2\iint_\Sigma \phi|\nabla\psi|^2\frac{\partial w}{\partial\nu}wd\Sigma - s^3\lambda^3\iint_\Sigma \phi^3|\nabla\psi|^2\frac{\partial\psi}{\partial\nu}|w|^2d\Sigma\\
&\le C\big(||e^{s\alpha}f||^2_{L^2(Q)} + s^3\lambda^4\iint_{\omega\times(0,T)}\phi^3|w|^2dxdt\big).
\end{aligned}
\tag{A.40}
$$

**Step. 2.** We perform the same analysis as in *step 1*, but now with the weights defined in (A.19). For $s>0$ and $\lambda>0$, we consider the change of variable

$$\tilde{w}(t,w) = e^{s\tilde{\alpha}}q(t,w), \tag{A.41}$$

which implies

$$\tilde{w}(T,x) = \tilde{w}(0,x) = 0.$$

We have

$$\tilde{L}_1 w + \tilde{L}_2 w = \tilde{f}_s, \tag{A.42}$$

where

$$\tilde{L}_1\tilde{w} = -\sigma\tilde{w}_t - 2s\lambda\tilde{\phi}\nabla\psi\cdot\nabla\tilde{w} + 2s\lambda^2\tilde{\phi}|\nabla\psi|^2\tilde{w}, \tag{A.43}$$

$$\tilde{L}_2\tilde{w} = -\Delta\tilde{w} - s^2\lambda^2\tilde{\phi}^2|\nabla\psi|^2\tilde{w} + \sigma s\tilde{\alpha}_t\tilde{w} \tag{A.44}$$

and

$$\tilde{f}_s = e^{s\tilde{\alpha}}g + s\lambda^2\tilde{\phi}|\nabla\psi|^2\tilde{w} + s\lambda\tilde{\phi}\Delta\psi\tilde{w}. \tag{A.45}$$

From (A.42), we have that

$$||\tilde{L}_1\tilde{w}||^2_{L^2(Q)} + ||\tilde{L}_2\tilde{w}||^2_{L^2(Q)} + 2(\tilde{L}_1\tilde{w},\tilde{L}_2\tilde{w})_{L^2(Q)} = ||\tilde{f}_s||^2_{L^2(Q)}. \tag{A.46}$$

In this step we analyze the terms appearing in $(L_1w, L_2w)_{L^2(Q)}$. First, we write

$$(\tilde{L}_1\tilde{w},\tilde{L}_2\tilde{w})_{L^2(Q)} = \sum_{i,j=1}^N \tilde{I}_{ij}.$$

As before, the following inequality can be proved

$$
\begin{aligned}
& s^{-1}\iint_Q \tilde\phi^{-1}(\sigma^2|\tilde w_t|^2+|\Delta\tilde w|^2)dxdt + s^3\lambda^4\iint_Q \tilde\phi^3|\tilde w|^2dxdt + s\lambda^2\iint_Q \tilde\phi|\nabla\tilde w|^2dxdt \\
& + \sigma\iint_\Sigma \frac{\partial\tilde w}{\partial\nu}\tilde w_t d\Sigma - \sigma s^2\lambda\iint_\Sigma \tilde\phi\tilde\alpha_t\frac{\partial\psi}{\partial\nu}|\tilde w|^2d\Sigma \\
& + 2s\lambda\iint_\Sigma \tilde\phi\frac{\partial\psi}{\partial\nu}|\frac{\partial\tilde w}{\partial\nu}|^2d\Sigma - s\lambda\iint_\Sigma \tilde\phi\frac{\partial\psi}{\partial\nu}|\nabla\tilde w|^2d\Sigma \\
& - 2s\lambda^2\iint_\Sigma \tilde\phi|\nabla\psi|^2\frac{\partial\tilde w}{\partial\nu}\tilde w d\Sigma + s^3\lambda^3\iint_\Sigma \tilde\phi^3|\nabla\psi|^2\frac{\partial\psi}{\partial\nu}|\tilde w|^2d\Sigma \\
& \le C\big(||e^{s\tilde\alpha}f||^2_{L^2(Q)} + s^3\lambda^4\iint_{\omega\times(0,T)}\tilde\phi^3|\tilde w|^2dxdt\big), \qquad \text{(A.47)}
\end{aligned}
$$

for $\lambda\ge\lambda_0(\Omega,\omega)$ and $s\ge s_0(\Omega,\omega)(e^{2\lambda||\psi||_\infty}T^4+T^8)$.

**Step. 3.** Let us now add the inequalities (A.40) and (A.47) and let us check that the integrals on $\Sigma$ can be simplified, so that there will only remain integrals in $Q$.

First, observe that, since $\psi=0$ on $\partial\Omega$, we have

$$\phi=\tilde\phi,\ \alpha=\tilde\alpha \text{ and } w=\tilde w \text{ on } \Sigma.$$

From the definitions of $w$ and $\tilde w$, we have

$$\frac{\partial w}{\partial x_i} = e^{s\alpha}(\frac{\partial q}{\partial x_i} - s\lambda\frac{\partial\psi}{\partial x_i}\phi q) = e^{s\tilde\alpha}(\frac{\partial q}{\partial x_i} + s\lambda\frac{\partial\psi}{\partial x_i}\tilde\phi q), \qquad \text{(A.48)}$$

whence

$$\frac{\partial w}{\partial\nu} = s\lambda\frac{\partial\psi}{\partial\nu}\phi e^{s\alpha}q,\ \frac{\partial\tilde w}{\partial\nu} = s\lambda\frac{\partial\psi}{\partial\nu}\tilde\phi e^{s\tilde\alpha}q \text{ on } \Sigma$$

and it follows that

$$\frac{\partial w}{\partial\nu} = -\frac{\partial\tilde w}{\partial\nu} \text{ on } \Sigma. \qquad \text{(A.49)}$$

Therefore, adding (A.40) and (A.47) and using (A.49), we conclude that

$$
\begin{aligned}
& s^{-1}\iint_Q \tilde\phi^{-1}(\sigma^2|\tilde w_t|^2+|\Delta\tilde w|^2)dxdt + s^3\lambda^4\iint_Q \tilde\phi^3|\tilde w|^2dxdt + s\lambda^2\iint_Q \tilde\phi|\nabla\tilde w|^2dxdt \\
& s^{-1}\iint_Q \phi^{-1}(\sigma^2|w_t|^2+|\Delta w|^2)dxdt + s^3\lambda^4\iint_Q \phi^3|w|^2dxdt + s\lambda^2\iint_Q \phi|\nabla w|^2dxdt \\
& \le C\big(||e^{s\alpha}f||^2_{L^2(Q)} + ||e^{s\tilde\alpha}f||^2_{L^2(Q)} + s^3\lambda^4\iint_{\omega\times(0,T)}\phi^3|w|^2+\tilde\phi^3|\tilde w|^2dxdt\big), \qquad \text{(A.50)}
\end{aligned}
$$

for $\lambda\ge\lambda_0(\Omega,\omega)$ and $s\ge s_0(\Omega,\omega)(e^{2\lambda||\psi||_\infty}T^4+T^8)$.

From the definitions os $\phi$, $\tilde{\phi}$, $\alpha$ and $\tilde{\alpha}$, we have

$$\tilde{\phi} \le \phi, \; e^{2s\tilde{\alpha}} \le e^{2s\alpha} \text{ in } Q.$$

Hence, (A.50) yelds

$$s^{-1} \iint_Q \phi^{-1}(\sigma^2|w_t|^2 + |\Delta w|^2)dxdt + s^3\lambda^4 \iint_Q \phi^3|w|^2dxdt + s\lambda^2 \iint_Q \phi|\nabla w|^2dxdt$$
$$\le C\big(||e^{s\alpha}f||^2_{L^2(Q)} + s^3\lambda^4 \iint_{\omega\times(0,T)} \phi^3|w|^2\big), \tag{A.51}$$

for $\lambda \ge \lambda_0(\Omega,\omega)$ and $s \ge s_0(\Omega,\omega)(e^{2\lambda||\psi||_\infty}T^4 + T^8)$.

We finally turn back to $q$. For the moment, we have

$$s^{-1} \iint_Q \phi^{-1}(\sigma^2|w_t|^2 + |\Delta w|^2)dxdt + s^3\lambda^4 \iint_Q \phi^3 e^{2s\alpha}|q|^2dxdt + s\lambda^2 \iint_Q \phi|\nabla w|^2dxdt$$
$$\le C\big(||e^{s\alpha}f||^2_{L^2(Q)} + s^3\lambda^4 \iint_{\omega\times(0,T)} \phi^3 e^{2s\alpha}|q|^2\big). \tag{A.52}$$

Using (A.48), we find that

$$s\lambda^2 \iint_Q \phi e^{2s\alpha}|\nabla q|^2dxdt \le C\big(s\lambda^2 \iint_Q \phi|\nabla w|^2dxdt + s^3\lambda^4 \iint_Q \phi^3 e^{2s\alpha}|q|^2dxdt\big).$$

Accordingly, the previous integral in $\nabla q$ can be added to the left-hand side of (A.52):

$$s^{-1} \iint_Q \phi^{-1}(\sigma^2|w_t|^2 + |\Delta w|^2)dxdt + s^3\lambda^4 \iint_Q \phi^3 e^{2s\alpha}|q|^2dxdt + s\lambda^2 \iint_Q \phi e^{2s\alpha}|\nabla q|^2dxdt$$
$$\le C\big(||e^{s\alpha}f||^2_{L^2(Q)} + s^3\lambda^4 \iint_{\omega\times(0,T)} \phi^3 e^{2s\alpha}|q|^2\big). \tag{A.53}$$

For $\Delta q$, we use the identity

$$\Delta w = e^{s\alpha}(\Delta q - s\lambda\Delta\psi\phi q + s\lambda^2|\nabla\psi|^2\phi q - 2s\lambda\nabla\psi\cdot\nabla q + s^2\lambda^2|\nabla\psi|^2\phi^2 q)$$

and we obtain

$$s^{-1} \iint_Q \phi^{-1}e^{s\alpha}|\Delta q|^2dxdt \le C\big(s^{-1} \iint_Q \phi^{-1}|\Delta w|^2dxdt$$
$$+ s^3\lambda^4 \iint_Q \phi^3 e^{2s\alpha}|q|^2dxdt + s\lambda^2 \iint_Q \phi e^{2s\alpha}|\nabla q|^2dxdt.$$

Finally, for $q_t$, we get

$$s^{-1} \iint_Q \phi^{-1} e^{2s\alpha} \sigma^2 |q_t|^2 dxdt \leq s^{-1} \iint_Q \phi^{-1} \sigma^2 |w_t|^2 dxdt$$
$$+ s\sigma^2 e^{4\lambda||\psi||_\infty} T^8 \iint_Q \phi^3 e^{2s\alpha} |q|^2 dxdt,$$

since

$$q_t = e^{-s\alpha}(w_t - s\alpha_t w).$$

Therefore, if $\lambda \geq \lambda_0(\Omega, \omega)$ and $s \geq s_0(\Omega, \omega)(e^{2\lambda||\psi||_\infty} T^4 + T^8)$ we have

$$s^{-1} \iint_Q \phi^{-1} e^{2s\alpha} (\sigma^2 |q_t|^2 + |\Delta q|^2 dxdt + s^3 \lambda^4 \iint_Q \phi^3 e^{2s\alpha} |q|^2 dxdt \tag{A.54}$$
$$+ s\lambda^2 \iint_Q \phi e^{2s\alpha} |\nabla w|^2 dxdt$$
$$\leq C\big(||e^{s\alpha} f||^2_{L^2(Q)} + s^3 \lambda^4 \iint_{\omega \times (0,T)} \phi^3 e^{2s\alpha} |q|^2\big) \tag{A.55}$$

and the proof of Lemma 4.1 is finished. □

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