Null controllability for a heat equation with a singular inverse-square potential involving the distance to the boundary function

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Abstract

This article is devoted to the analysis of control properties for a heat equation with singular potential μ/δ^2, defined on a bounded C^2 domain $\Omega \subset \mathbb{R}^N$, where δ is the distance to the boundary function. More precisely, we show that for any $\mu \leq 1/4$ the system is exactly null controllable using a distributed control located in any open subset of Ω, while for $\mu > 1/4$ there is no way of preventing the solutions of the equation from blowing-up. The result is obtained applying a new Carleman estimate.

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1. Introduction and main results

Let $T > 0$ and set $Q := \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded and C^2 domain, and let $\Gamma := \partial\Omega$. Moreover, let $\delta(x) := \text{dist}(x, \partial\Omega)$ be the distance to the boundary function. We are interested in proving the exact null controllability for a heat equation with singular inverse-square potential of the type $-\mu/\delta^2$, that is, given the operator

$$\mathcal{A} = \mathcal{A}(\mu) := -\Delta - \frac{\mu}{\delta^2}\mathfrak{I}, \quad \mu \in \mathbb{R}, \tag{1.1}$$

where $\mathfrak{I}$ indicates the identical operator, we are going to consider the following parabolic equation

$$\begin{cases} u_t - \Delta u - \dfrac{\mu}{\delta^2}u = f, & (x, t) \in Q \\ u = 0, & (x, t) \in \Gamma \times (0, T) \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

with the intent of proving that it is possible to choose the control function f in an appropriate functional space X such that the corresponding solution of (1.2) satisfies

$$u(x, T) = 0, \quad \text{for all } x \in \Omega. \tag{1.3}$$

In particular, the main result of this paper will be the following.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^2 domain and assume $\mu \leq 1/4$. Given any non-empty open set $\omega \subset \Omega$, for any time $T > 0$ and any initial datum $u_0 \in L^2(\Omega)$, there exists a control function $f \in L^2(\omega \times (0, T))$ such that the solution of (1.2) satisfies (1.3).*

The upper bound for the coefficient μ is related to a generalisation of the classical Hardy-Poincaré presented in [5] and plays a fundamental role in our analysis. Indeed, in [6] is shown that, for $\mu > 1/4$, (1.2) admits no positive weak solution for any u_0 positive and $f = 0$. Moreover, there is instantaneous and complete blow-up of approximate solutions.

As it is by now classical, for proving Theorem 1.1 we will apply the Hilbert Uniqueness Method (HUM, [18]); hence the controllability property will be equivalent to the observability of the adjoint system associated to (1.2), namely

$$\begin{cases} v_t + \Delta v + \dfrac{\mu}{\delta^2} v = 0, & (x, t) \in Q \\ v = 0, & (x, t) \in \Gamma \times (0, T) \\ v(x, T) = v_T(x), & x \in \Omega. \end{cases} \tag{1.4}$$

More in details, for any $\mu \leq 1/4$ we are going to prove that there exists a positive constant C_T such that, for all $v_T \in L^2(\Omega)$, the solution of (1.4) satisfies

$$\int_{\Omega} v(x, 0)^2 \, dx \leq C_T \int_{\omega \times (0, T)} v(x, t)^2 \, dxdt. \tag{1.5}$$

The inequality above, in turn, will be obtained as a consequence of a Carleman estimate for the solution of (1.4), which is derived taking inspiration from the works [8] and [11].

Furthermore, the bound $\mu \leq 1/4$ is sharp for our controllability result, as we are going to show later in this work.

Singular inverse-square potentials arise in quantum cosmology ([2]), in electron capture problems ([16]), but also in the linearisation of reaction-diffusion problems involving the heat equation with supercritical reaction term ([15]); also for these reasons, evolution problems involving this kind of potentials have been intensively studied in the last decades.

In the pioneering work of 1984 [1], Baras and Goldstein considered a heat equation in a bounded domain $\Omega \subset \mathbb{R}^N$, for $N \geq 3$, with potential $-\mu/|x|^2$ and positive initial data, and proved that the Cauchy problem is well posed in the case $\mu \leq \mu^* := (N-2)^2/4$, while it has no solution if $\mu > \mu^*$. We remind here that μ^* is the critical value for the constant in the Hardy inequality, guaranteeing that, for any $u \in H_0^1(\Omega)$, it holds

$$\int_{\Omega} |\nabla u|^2 \, dx \geq \mu^* \int_{\Omega} \frac{u^2}{|x|^2} \, dx. \tag{1.6}$$

The result by Baras and Goldstein was, in our knowledge, the first on the topic and it has later been improved by Vazquez and Zuazua in [23]. There the authors present a complete description of the functional framework in which it is possible to obtain well-posedness for the singular heat equation they analyse; in particular, they prove that when $\mu < \mu^*$ the corresponding operator generates a coercive quadratic form form in $H^1_0(\Omega)$ and this allows to show well-posedness in the classical variational setting. On the contrary, when $\mu = \mu^*$, the space $H^1_0(\Omega)$ has to be slightly enlarged, due to the logarithmic singularity of the solutions at $x = 0$.

Also the question of whether it is possible to control heat equations involving singular inverse-square potentials has already been addressed in the past, and there is nowadays an extended literature on this topic.

Among other works, we remind here the one by Ervedoza, [11], and the one by Vancostenoble and Zuazua, [21]. In both, the authors consider the case of an equation defined on a smooth domain containing the origin and prove exact null controllability choosing a control region inside of the domain, away from the singularity point $x = 0$.

In particular, in [21] the null controllability result is obtained choosing a control region containing an annular set around the singularity and using appropriate cut-off functions in order to split the problem in two:

- in a region of the domain away from the singularity, in which it is possible to employ classical Carleman estimates;

- in the remaining part of the domain, a ball centred in the singularity, in which the authors can apply polar coordinates and reduce themselves to a one-dimensional equation, which is easier to handle.

In [11], instead, the author generalises the result by Vancostenoble and Zuazua, proving controllability from any open subset of Ω that does not contains the singularity. This result is obtained deriving a new Carleman estimate, involving a weight that permits to avoid the splitting argument introduced in is [21].

Finally, it is worth to mention also the work [8], by Cazacu. In this paper, it is treated the case of a potential with singularity located on the boundary of the domain and it is proved again null controllability with an internal control. Also this result follows from a new Carleman estimate that is derived using the same kind of weight function proposed by Ervedoza, but with some suitable modifications that permit to deal with the case of boundary singularities. Moreover, the author shows that the presence of the singularity on the boundary of the domain allows to slightly enlarge the critical value for the constant μ, up to $\mu^* := N^2/4$.

In this article we analyse the case of a potential with singularity distributed all over the boundary. To the best of our knowledge, this is a problem that has never been treated in precedence, although it is a natural generalisation of the results of the works presented above.

This paper is organized as follows: in Section 2 we present the classical Hardy-Poincaré inequality introduced by Brezis and Marcus in [5], which will then be applied for obtaining well-posedness of the equation we consider; we also give some extensions of this inequality, needed for obtaining the Carleman estimate. These results are then employed for obtaining the well-posedness of our equation, applying classical semi-group theory. In Section 3 we present the Carleman estimate, showing what are the main differences between our result and previous ones obtained, for instance, in [11], [21] and, later, in [8]. In Section 4 we derive the observability in-

equality (1.5) and we apply it in the proof of Theorem 1.1. In Section 5 we prove that the bound $1/4$ for the Hardy constant μ is sharp for control, showing the impossibility of preventing the solutions of the equation from blowing-up in the case of supercritical potentials. The Carleman estimates is proved in Section 6. Section 7 is dedicated to some interesting open problems related to our results. Finally, we conclude our article with an appendix in which we prove several technical Lemmas that are fundamental in our analysis.

2. Hardy-Poincaré inequalities and well-posedness

When dealing with equations involving singular inverse-square potentials, it is by now classical that of great importance is an Hardy-type inequality. Inequalities of this kind have been proved to hold also in the more general case of for the potential μ/δ^2 (see, for instance [5],[19]); in particular, we have

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^2 domain; then, for any $u \in H_0^1(\Omega)$, and for any $\mu \leq 1/4$, the following inequality holds*

$$\int_\Omega |\nabla u|^2 \, dx \geq \mu \int_\Omega \frac{u^2}{\delta^2} \, dx. \tag{2.1}$$

Inequality (2.1) will be applied for obtaining the well-posedness of (1.2), as well as the observability inequality (1.5). For obtaining the Carleman estimate, instead, we are going to need the following Propositions

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^2 domain. For any $\mu \leq 1/4$ and any $\gamma \in (0, 2)$ there exist two positive constants A_1 and A_2, depending on γ and Ω such that, for any $u \in H_0^1(\Omega)$, the following inequality holds*

$$A_1 \int_\Omega \frac{u^2}{\delta^\gamma} \, dx + \mu \int_\Omega \frac{u^2}{\delta^2} \, dx \leq \int_\Omega |\nabla u|^2 \, dx + A_2 \int_\Omega u^2 \, dx. \tag{2.2}$$

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^2 domain. For any $\mu \leq 1/4$ and any $\gamma \in (0, 2)$ there exists a positive constant A_3 depending on γ, μ and Ω such that, for any $u \in H_0^1(\Omega)$, the following inequality holds*

$$\int_\Omega \delta^{2-\gamma} |\nabla u|^2 \, dx \leq R_\Omega^{2-\gamma} \int_\Omega \left(|\nabla u|^2 - \mu \frac{u^2}{\delta^2} \right) dx + A_3 \int_\Omega u^2 \, dx. \tag{2.3}$$

Proposition 2.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^2 domain. For any $\mu \leq 1/4$ and any $\gamma \in (0, 2)$ there exist two positive constants A_4 and A_5 depending on γ, μ and Ω such that, for any $u \in H_0^1(\Omega)$, the following inequality holds*

$$\int_\Omega \left(|\nabla u|^2 - \mu \frac{u^2}{\delta^2} \right) dx + A_4 \int_\Omega u^2 \, dx \geq A_5 \int_\Omega \left(\delta^{2-\gamma} |\nabla u|^2 + A_1 \frac{u^2}{\delta^\gamma} \right) dx, \tag{2.4}$$

where A_1 is the positive constant introduced in Proposition 2.2.

The proof of 2.2 follows immediately from the inequalities with weighted integral presented in [5, Section 4] and we are going to omit it here; moreover, 2.4 is a direct consequence of the

application of 2.2 and 2.3. Concerning the proof of Proposition 2.3, instead, we will presented it in appendix B.

We conclude this section analysing existence and uniqueness of solutions for equation (1.2), applying classical semi-group theory; at this purpose, we apply the same argument presented in [8]. Therefore, for any fixed $\gamma \in [0, 2)$ let us define the set

$$\mathcal{L}^\gamma := \left\{ A > 0 \ \text{ s.t. } \ \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega \left(|\nabla u|^2 - \mu^* u^2/\delta^2 + Au^2 \right) dx}{A_1 \int_\Omega u^2/\delta^\gamma \, dx} \geq 1 \right\}. \tag{2.5}$$

We remind here that μ^* is the critical Hardy constant and that in our case we have $\mu^* = 1/4$. Moreover, the set (2.5) is clearly non empty since it contains the constant A_2 in the inequality (2.2). Now, we define

$$A_0^\gamma := \inf_{A \in \mathcal{L}^\gamma} A \tag{2.6}$$

and, for any $\mu \leq \mu^*$, we introduce the functional

$$\Phi_\mu^\gamma(u) := \int_\Omega |\nabla u|^2 \, dx - \mu \int_\Omega \frac{u^2}{\delta^2} \, dx + A_0^\gamma \int_\Omega u^2 \, dx;$$

we remark that this functional is positive for any test function, due to (2.2) and to the particular choice of the constant A_0^γ.

Next, let us define the Hilbert space H_μ^γ as the closure of $C_0^\infty(\Omega)$ with respect to the norm induced by Φ_μ^γ; if $\mu \leq \mu^*$ we obtain

$$\left(1 - \frac{\mu^+}{\mu^*} \right) \int_\Omega \left(|\nabla u|^2 + A_0^\gamma u^2 \right) dx + \frac{\mu^+}{\mu^*} \int_\Omega \frac{u^2}{\delta^\gamma} \, dx \leq \|u\|_H^2 \leq \left(1 + \frac{\mu^-}{\mu^*} \right) \int_\Omega \left(|\nabla u|^2 + A_0^\gamma u^2 \right) dx, \tag{2.7}$$

where $\mu^+ := \max\{0, \mu\}$ and $\mu^- := \max\{0, -\mu\}$.

From the norm equivalence (2.7), in the sub-critical case $\mu < \mu^*$ it follows the identification $H_\mu^\gamma = H_0^1(\Omega)$; in the critical case $\mu = \mu^*$, instead, this identification does not hold anymore and the space H_μ^γ is slightly larger than $H_0^1(\Omega)$. For more details on the characterisation of these kind of spaces, we refer to [23].

Let us now consider the unbounded operator $\mathcal{B}_\mu^\gamma : \mathcal{D}(\mathcal{B}_\mu^\gamma) \subset L^2(\Omega) \to L^2(\Omega)$ defined as

$$\mathcal{D}(\mathcal{B}_\mu^\gamma) := \left\{ u \in H_\mu^\gamma \ \text{ s.t. } \ -\Delta u - \frac{\mu}{\delta^2} u + A_0^\gamma u \in L^2(\Omega) \right\},$$

$$\mathcal{B}_\mu^\gamma u := -\Delta u - \frac{\mu}{\delta^2} u + A_0^\gamma u, \tag{2.8}$$

whose norm is given by

$$\|u\|_{\mathcal{B}_\mu^\gamma} = \|u\|_{L^2(\Omega)} + \left\| \mathcal{B}_\mu^\gamma u \right\|_{L^2(\Omega)}.$$

With the definitions we just gave, by standard semi-group theory we have that for any $\mu \leq \mu^*$ the operator $(\mathcal{B}_\mu^\gamma, \mathcal{D}(\mathcal{B}_\mu^\gamma))$ generates an analytic semi-group in the pivot space $L^2(\Omega)$ for the equation (1.2). For more details we refer to the Hille-Yosida theory, presented in [4, Chapter 7], which can be adapted in the context of the space H_μ^γ introduced above.

Therefore, from the construction we just presented we immediately have the following well-posedness result

Theorem 2.1. *Given $u_0 \in L^2(\Omega)$ and $f \in C([0,T]; L^2(\Omega))$, for any $\mu \leq 1/4$ the problem* (1.2) *admits a unique weak solution*

$$u \in C^0([0,T]; L^2(\Omega)) \cap L^2((0,T); H_\mu^\gamma).$$

3. Carleman estimate

3.1. Choice of the weight σ

The observability inequality (1.5) will be proved, as it is classical in controllability problems for parabolic equations, applying a Carleman estimate.

The main problem when designing a Carleman estimate is the choice of a proper weight function $\sigma(x,t)$. In our case, this σ will be an adaptation of the one used in [8], that we conveniently modify in order to deal with the presence of the singularities distributed all over the boundary. In particular, the weight we propose is the following

$$\sigma(x,t) = \theta(t)\left(C_\lambda - \delta^2 \psi - \left(\frac{\delta}{r_0}\right)^\lambda \phi\right), \quad \phi = e^{\lambda\psi}, \tag{3.1}$$

where

$$\theta(t) = \left(\frac{1}{t(T-t)}\right)^3. \tag{3.2}$$

Here, C_λ is a positive constant large enough as to ensure the positivity of σ, and λ is a positive parameter aimed to be large; besides, r_0 satisfies

$$r_0 \leq \min\left\{1, \frac{2|\psi|_\infty}{4|D\psi|_\infty + |D^2\psi|_\infty}, \frac{1}{R_\Omega \sqrt{4|D\psi|_\infty^2 + 2|D^2\psi|_\infty}}, \frac{|\psi|_\infty}{2(2-\gamma)|D\psi|_\infty}, \left(\frac{M_2}{4|\mu||D\psi|_\infty}\right)^{1/(\gamma-1)},\right.$$

$$\frac{1}{\sqrt{8D_{\psi_1}|D\psi|_\infty/\varpi_0 + 3|D^2\psi|_\infty}}, \frac{2|\psi|_\infty}{|D\psi|_\infty^2 + (1+2|\psi|_\infty)|D\psi|_\infty}, \frac{1}{|D\psi|_\infty^2 + 2|D\psi|_\infty},$$

$$\left.\frac{3|\psi|_\infty^2}{4|D\psi|_\infty}, \frac{1}{|D\psi|_\infty \sqrt{D_3|\psi|_\infty^2 + D_4}}\right\}, \tag{3.3}$$

where γ is the parameter appearing in the Hardy inequalities presented above, with the particular choice $\gamma \in (1,2)$, while M_2 is a positive constant that will be introduced later. The choice of r_0 as in (3.3) is motivated by technical reasons that will be carefully justified throughout the paper. Finally, ψ is a bounded regular function (at least $C^4(\overline{\Omega})$) defined as

$$\psi = \varpi(\psi_1 + 1), \tag{3.4}$$

with $\psi_1 \in C^4(\overline{\Omega})$ and bounded, satisfying the conditions

$$\begin{cases} \psi_1(x) = \delta(x) & \forall x \in \Omega_{r_0}, \\ \psi_1(x) > r_0 & \forall x \in \Omega \setminus \overline{\Omega_{r_0}}, \\ \psi_1(x) = r_0 & \forall x \in \Sigma_{r_0}, \\ |\nabla\psi_1(x)| \geq \varpi_0 > 0 & \forall x \in \Omega \setminus \overline{\omega_0}, \end{cases} \tag{3.5}$$

for $\varpi\varpi_0 > 2C_\Omega$, where C_Ω is the constant introduced in [8, Section 2]. Such function exists but its construction is not trivial. See [8, Section 2] for more details. In particular, under these conditions ψ satisfies the following useful properties

$$
\begin{cases}
\psi(x) = 1 & \forall x \in \Gamma, \\
\psi(x) > 1 & \forall x \in \Omega, \\
|\nabla\psi(x)| \geq 2C_\Omega & \forall x \in \Omega \setminus \overline{\omega_0},
\end{cases}
\tag{3.6}
$$

In (3.5) and (3.6), $\omega_0 \subset\subset \omega$ is a non-empty subset of the control region ω; moreover, due to technical computations, we fix ϖ such that

$$
\varpi \geq \max\left\{ 1, \frac{1}{\varpi_0^2}\left(1 + \frac{2D_{\psi_1}}{r_0} + |D^2\psi|_\infty\right), \frac{2}{\varpi_0^2}\left(1 + \frac{2D_{\psi_1}}{r_0}\right), \frac{4D_{\psi_1}}{\varpi_0^2}, \frac{24D_{\psi_1}R_\Omega}{\varpi_0^2}, \frac{2}{\varpi_0} \right\},
\tag{3.7}
$$

where R_Ω is the diameter of the domain Ω, while D_{ψ_1} is a positive constant that will be introduced later. Furthermore, throughout the paper, formally, for a given function f we apply the notations

$$
|f|_\infty := \|f\|_{L^\infty(\Omega)}, \qquad\qquad\qquad |Df|_\infty := \|\nabla f\|_{L^\infty(\Omega)},
$$

$$
D^2 f(\xi,\xi) := \sum_{i,j=1}^{N} \partial^2_{x_i x_j} f \xi_i \xi_j, \quad \forall \xi \in \mathbb{R}^N, \qquad |D^2 f|_\infty := \sum_{i,j=1}^{N} \left\|\partial^2_{x_i x_j} f\right\|_{L^\infty(\Omega)}, \tag{3.8}
$$

and we denote

$$
\Omega_{r_0} := \{ x \in \Omega \mid \delta(x) < r_0 \}, \quad \mathcal{O} := \Omega \setminus \left(\overline{\omega_0} \cup \overline{\Omega_{r_0}} \right), \quad \tilde{\mathcal{O}} := \Omega \setminus \overline{\Omega_{r_0}}. \tag{3.9}
$$

3.2. Motivation for the choice of σ

The weigh σ that we propose for our Carleman estimates is not the standard one; we had to modify it in order to deal with some critical terms that emerge in our computations due to the presence of the singular potential. We justify here our choice, highlighting the reasons why the weights presented in previous works ([8],[11],[14]) are not suitable for the problem we consider.

In general, the weight used to obtain Carleman estimates for parabolic equations is assumed to be positive and to blow-up at the extrema of the time interval; besides, it has to be taken in separated variables. Therefore, we are looking for a function $\sigma(x,t)$ satisfying

$$
\begin{cases}
\sigma(x,t) = \theta(t)p(x), & (x,t) \in Q, & \text{(3.10a)} \\
\sigma(x,t) > 0, & (x,t) \in Q, & \text{(3.10b)} \\
\lim\limits_{t\to 0^+} \sigma(x,t) = \lim\limits_{t\to T^-} \sigma(x,t) = +\infty, & x \in \Omega. & \text{(3.10c)}
\end{cases}
$$

The function θ is usually chosen in the form

$$
\theta(t) = \left(\frac{1}{t(T-t)} \right)^k
$$

for $k \geq 1$, and this choice in particular ensures the validity of (3.10c); in our case we assume $k = 3$ which, as we will remark later, is the minimum value for obtaining some important estimates that

we need in the proof of the Carleman inequality.

While the choice of θ is standard, the main difficulty when building a proper σ is to identify a suitable $p(x)$ which is able to deal with the specificity of the equation we are analysing.

In [14], Fursikov and Imanuvilov obtained the controllability of the standard heat equation employing a positive weight in the form

$$\sigma_1 = \theta(t)\left(C_\lambda - e^{\lambda\psi}\right),$$

with a function $\psi \in C^2(\overline{\Omega})$ satisfying

$$\begin{cases} \psi(x) > 0, & x \in \Omega, \\ \psi(x) = 0, & x \in \partial\Omega, \\ |\nabla\psi(x)| > 0, & x \in \overline{\Omega} \setminus \omega_0. \end{cases}$$

An example of a ψ with this behaviour is shown in Figure 1 below; in particular, we notice that this function is required to be always strictly monotone outside of the control region.
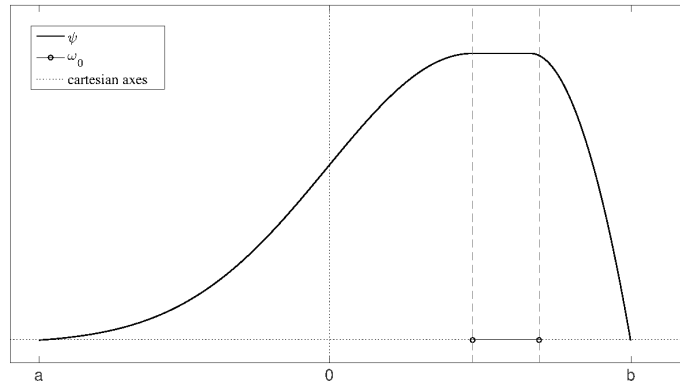

Figure 1: Function ψ of Fursikov and Imanuvilov in one space dimension on the interval (a, b)

This standard weight was later modified by Ervedoza in [11], for dealing with problems with interior quadratic singularities; in this case, the author applies the weight

$$\sigma_2 = \theta(t)\left(C_\lambda - \frac{1}{2}|x|^2 - e^{\lambda\psi(x)}\right),$$

with a function ψ such that

$$\begin{cases} \psi(x) = \ln(|x|), & x \in B(0, 1), \\ \psi(x) = 0, & x \in \partial\Omega, \\ \psi(x) > 0, & x \in \Omega \setminus \overline{B}(0, 1), \\ |\nabla\psi(x)| \geq \gamma > 0, & x \in \overline{\Omega} \setminus \omega_0. \end{cases}$$

This choice is motivated by some critical terms appearing due to the presence of the potential,
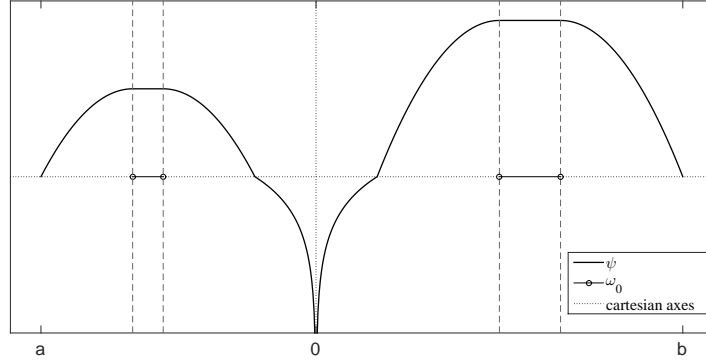
Figure 2: Function ψ of Ervedoza in one space dimension on the interval (a, b)

that must be absorbed outside ω in the Carleman estimate (see [11, Eq. 2.15]).

In particular, in order to take advantage of the Hardy inequality, the author needs to get rid of singular terms in the form $\Delta\sigma/|x|^2$ and $(x \cdot \nabla\sigma)/|x|^4$. The weight proposed allows to deal with this terms; indeed near the singularity, when λ is large enough σ_2 behaves like

$$\sigma_2 \sim \theta(t)\left(C_\lambda - \frac{1}{2}|x|^2\right),$$

which is the weight employed by Vancostenoble and Zuazua in [22] for their proof of the controllability of the heat equation with a singular potential and which satisfies $\nabla\sigma_2 \sim x$ and $\Delta\sigma_2 \sim C$ as $x \to 0$. On the other hand, away from the origin, where no correction is needed, σ_2 maintains the behaviour of the classical weight σ_1.

A further modification is proposed by Cazacu in [8], in the case of an equation with boundary singularity. In this case, indeed, the terms $\Delta\sigma/|x|^2$ and $(x \cdot \nabla\sigma)/|x|^4$ generates singularities that cannot be absorbed in a neighbourhood of the origin employing σ_2, since this weight involves a function ψ which is assumed to be zero on the boundary. Therefore, the author proposes a new weight

$$\sigma_3 = \theta(t)\left(C_\lambda - |x|^2\psi - \left(\frac{|x|}{r_0}\right)^\lambda e^{\lambda\psi}\right),$$

where ψ is now chosen as in (3.4), with the fundamental property of being constant and positive on the boundary.

Finally, when dealing as in our case with a singularity distributed all over the boundary the weights presented above do not allow anymore to manage properly the terms containing the singularities, since they now have a different nature. Therefore, we need to introduce further modifications in the weight we want to employ, designing it in a way that could compensate this kind of degeneracies. At this purpose, it is sufficient to modify σ_3 replacing the terms of the form $|x|$ with the distance function δ; being still in the case of boundary singularities the function ψ introduced in [8] (see (3.4) above) turns out to be a suitable one also in our case.

For concluding, we want to emphasise the fact that all the changes in the classical weight we introduced above are purely local, around the points where the singularity of the potential arises.
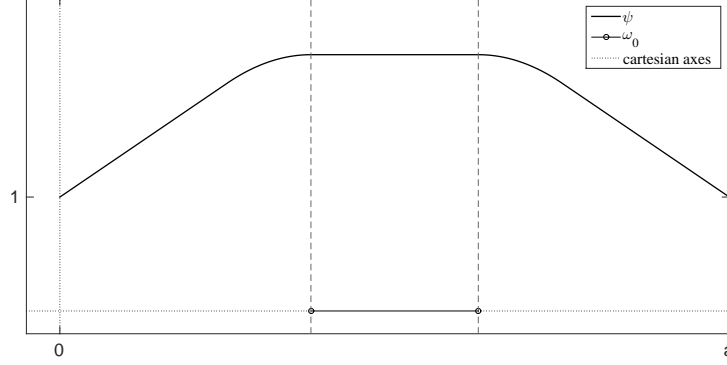
Figure 3: Function ψ of Cazacu in one space dimension on the interval $(0,a)$

This, of course, because as long as the potential remains bounded it can be handled with the same techniques as for the classical heat equation.

We now have all we need for introducing the Carleman estimate.

Theorem 3.1. *Let σ be the weight defined in* (3.1). *There exist two positive constants λ_0 and $\mathcal{M}$ such that for any $\lambda \geq \lambda_0$ there exists $R_0 = R_0(\lambda)$ such that for any $R \geq R_0$ and for any solution v of* (1.4) *it holds*

$$
R \int_Q \theta e^{-2R\sigma} \left(\delta^{2-\gamma} |\nabla v|^2 + A_1 \frac{v^2}{\delta^\gamma} \right) dxdt + \lambda R \int_{\Omega_{r_0} \times (0,T)} \theta \left(\frac{\delta}{r_0} \right)^{\lambda-2} e^{-2R\sigma} |\nabla v|^2 \, dxdt
$$

$$
+ \lambda^2 R \int_{\mathcal{O} \times (0,T)} \theta \left(\frac{\delta}{r_0} \right)^\lambda \phi e^{-2R\sigma} |\nabla v|^2 \, dxdt + R^3 \int_{\Omega_{r_0} \times (0,T)} \theta^3 \delta^2 e^{-2R\sigma} v^2 \, dxdt
$$

$$
+ \lambda^4 R^3 \int_{\mathcal{O} \times (0,T)} \theta^3 \left(\frac{\delta}{r_0} \right)^{3\lambda} \phi^3 e^{-2R\sigma} v^2 \, dxdt
$$

$$
\leq \mathcal{M} \left(\lambda^4 R^3 \int_{\omega_0 \times (0,T)} \theta^3 \left(\frac{\delta}{r_0} \right)^{3\lambda} \phi^3 e^{-2R\sigma} v^2 \, dxdt + \lambda^2 R \int_{\omega_0 \times (0,T)} \theta \left(\frac{\delta}{r_0} \right)^\lambda \phi e^{-2R\sigma} |\nabla v|^2 \, dxdt \right)
$$

$$
\tag{3.11}
$$

The proof of Theorem 3.1 is very technical and will be presented in Section 6. It relies on several technical Lemmas that we are going to prove in the appendix.

4. Proof of the observability inequality (1.5) and of the controllability Theorem 1.1

We now apply the Carleman estimate we just obtained for proving the observability inequality (1.5). This inequality will then be employed in the proof of our main result, Theorem 1.1.

Prooof of the observability inequality (1.5). Let us fix $\lambda \geq \lambda_0$ and $R \geq R_0(\lambda)$ such that (3.11) holds. These parameters now enter in the constant $\mathcal{M}$; in particular we have

$$\int_Q \theta e^{-2R\sigma} \frac{v^2}{\delta^\gamma} \, dxdt \leq \mathcal{M} \left(\int_{\omega_0 \times (0,T)} \theta^3 \phi^3 e^{-2R\sigma} v^2 \, dxdt + \int_{\omega_0 \times (0,T)} \theta \phi e^{-2R\sigma} |\nabla v|^2 \, dxdt \right).$$

Now, it is straightforward to check that there exists a positive constant $\mathcal{P}$ such that

$$\theta e^{-2R\sigma} \frac{1}{\delta^\gamma} \geq \mathcal{P}, \qquad (x,t) \in \Omega \times \left[\frac{T}{4}, \frac{3T}{4} \right],$$

$$\theta^3 \phi^3 e^{-2R\sigma} \leq \mathcal{P}, \qquad (x,t) \in \omega_0 \times (0,T),$$

$$\theta \phi e^{-2R\sigma} \leq \mathcal{P} e^{-R\sigma}, \quad (x,t) \in \omega_0 \times (0,T).$$

Thus the inequality above becomes

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_\Omega v^2 \, dxdt \leq \mathcal{N} \left(\int_{\omega_0 \times (0,T)} v^2 \, dxdt + \int_{\omega_0 \times (0,T)} e^{-R\sigma} |\nabla v|^2 \, dxdt \right).$$

Moreover, multiplying equation (1.4) by v and integrating over Ω we obtain

$$\frac{1}{2} \frac{d}{dt} \int_\Omega v^2 \, dx = \int_\Omega |\nabla v|^2 \, dx - \mu \int_\Omega \frac{v^2}{\delta^2} \, dx,$$

which, applying (2.1), implies

$$\frac{d}{dt} \int_\Omega v^2 \, dx \geq -C \int_\Omega v^2 \, dx.$$

Hence, the function $t \mapsto e^{2Ct} \|v(\cdot,t)\|_{L^2(\Omega)}$ is increasing, that is

$$e^{-2CT} \int_\Omega v(x,0)^2 \, dx \leq \int_\Omega v(x,t)^2 \, dx,$$

and, integrating in time between $T/4$ and $3T/4$ we have

$$\frac{T}{2} e^{-2CT} \int_\Omega v(x,0)^2 \, dx \leq \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_\Omega v(x,t)^2 \, dx.$$

Thus, we obtain the inequality

$$\int_\Omega v(x,0)^2 \, dxdt \leq \frac{2\mathcal{N} e^{2CT}}{T} \left(\int_{\omega_0 \times (0,T)} v^2 \, dxdt + \int_{\omega_0 \times (0,T)} e^{-R\sigma} |\nabla v|^2 \, dxdt \right).$$

Therefore to conclude the proof of (1.5), it is sufficient to apply the following lemma:

Lemma 4.1 (Cacciopoli's inequality). *Let* $\bar{\sigma} : (0,T) \times \omega_0 \to \mathbb{R}_+^*$ *be a smooth non-negative function such that*

$$\bar{\sigma}(x,t) \to +\infty, \quad as\ t \to 0^+ \quad and\ as\ t \to T^-,$$

and let $\mu \le \mu^*$. *Then, there exists a constant* Υ *independent of* μ *such that any solution* v *of* (1.4) *satisfies*

$$\int_{\omega_0 \times (0,T)} e^{-R\bar{\sigma}} |\nabla v|^2 \, dxdt \le \Upsilon \int_{\omega \times (0,T)} v^2 \, dxdt. \tag{4.1}$$

Lemma 4.1 is a trivial adaptation of an analogous result, [21, Lemma 3.3], and its proof is left to the reader. It is now straightforward that, applying (4.1) for σ as in (3.1) we finally get

$$\int_\Omega v(x,0)^2 \, dxdt \le C_T \int_{\omega_0 \times (0,T)} v^2 \, dxdt,$$

that clearly implies (1.5), due to the definition of ω_0. □

Proof of Theorem (1.1). Once the observability inequality (1.5) is known to hold, we can immediately obtain the controllability of our equation through a control $f \in L^2(\omega \times (0,T))$. To do that, we are going to introduce the functional

$$J(v_T) := \frac{1}{2} \int_{\omega \times (0,T)} v^2 \, dxdt + \int_\Omega v(x,0) u_0(x) \, dx, \tag{4.2}$$

defined over the Hilbert space

$$H := \left\{ v_T \in L^2(\Omega) \,\middle|\, \text{the solution } v \text{ of } (1.4) \text{ satisfies } \int_{\omega \times (0,T)} v^2 \, dxdt \le +\infty \right\}. \tag{4.3}$$

To be more precise, H is the completion of $L^2(\Omega)$ with respect to the norm $\left(\int_0^T \int_\omega v^2 \, dxdt \right)^{1/2}$.

Observe that J is convex and, according to (1.5), it is also continuous in H; on the other hand, again (1.5) gives us also the coercivity of J. Therefore, there exists $v^* \in H$ minimizing J. The corresponding Euler-Lagrange equation is

$$\int_{\omega \times (0,T)} v(x,t) F(x,t) \, dxdt + \int_\Omega u_0(x) v(x,0) \, dx = 0, \tag{4.4}$$

where $F(x,t) := v^*(x,t) \chi_\omega$. F will be our control function; we observe that, by definition $F \in L^2(\omega \times (0,T))$. Now, considering equation (1.2) with $f = F$, multiplying it by v and integrating by parts, we get

$$\int_0^1 u(x,T) v_T(x) \, dx = \int_{\omega \times (0,T)} v(x,t) F(x,t) \, dxdt + \int_\Omega u_0(x) v(x,0) \, dx,$$

for any $v_T \in L^2(\Omega)$. Hence, from (4.4) we immediately conclude $u(x,T) = 0$. □

5. Non existence of a control in the supercritical case

As we mentioned before, in [6] is proved that in the super-critical case, i.e. for $\mu > 1/4$, the Cauchy problem for our singular heat equation is severely ill-posed. However, a priori this fact does not exclude that, given $u_0 \in L^2(\Omega)$, it is possible to find a control $f \in L^2((0,T); L^2(\Omega))$ localised in ω such that there exists a solution of (1.2). If this fact occurs, it would mean that we can prevent blow-up phenomena by acting on a subset of the domain.

However, as we are going to show in this section, this control function f turns out to be impossible to find for $\mu > 1/4$ and, in this case, we cannot prevent the system from blowing up. Therefore, the upper bound $1/4$ for the Hardy constant μ shows up to be sharp for control.

The proof of this fact will rely on an analogous result presented in [11]; therefore, following the ideas of optimal control, for any $u_0 \in L^2(\Omega)$ we consider the functional

$$J_{u_0}(u, f) := \frac{1}{2} \int_Q |u(x,t)|^2 \, dx dt + \frac{1}{2} \int_0^T \|f(t)\|_{L^2(\Omega)}^2 \, dt,$$

defined on the set

$$\mathcal{C}(u_0) := \left\{ (u, f) \in L^2((0,T), H_0^1(\Omega)) \times L^2((0,T), L^2(\Omega)) \,\middle|\, u \text{ satisfies } (1.2) \right\}.$$

We say that it is possible to stabilise system (1.2) if we can find a constant A such that

$$\inf_{(u,f) \in \mathcal{C}(u_0)} J_{u_0}(u, f) \le A \|u_0\|_{L^2(\Omega)}^2.$$

Now, for $\varepsilon > 0$, we approximate (1.2) by the system

$$\begin{cases} u_t - \Delta u - \dfrac{\mu}{\delta^2 + \varepsilon^2} u = f, & (x,t) \in Q \\ u = 0, & (x,t) \in \Gamma \times (0,T) \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{5.1}$$

Due to the boundedness of the potential, (5.1) is well-posed; therefore, we can define the functional

$$J_{u_0}^\varepsilon(f) := \frac{1}{2} \int_Q |u(x,t)|^2 \, dx dt + \frac{1}{2} \int_0^T \|f(t)\|_{L^2(\Omega)}^2 \, dt,$$

where $f \in L^2((0,T); L^2(\Omega))$ is localised in ω and u is the corresponding solution of (5.1). We are going to prove the following

Theorem 5.1. *Assume that $\mu > 1/4$. There is no constant A such that, for all $\varepsilon > 0$ and all $u_0 \in L^2(\Omega)$,*

$$\inf_{f \in L^2((0,T); L^2(\Omega))} J_{u_0}^\varepsilon(f) \le A \|u_0\|_{L^2(\Omega)}^2.$$

We are going to prove Theorem 5.1 in two steps: firstly, we give some basic estimates on the spectrum of the operator

$$\mathcal{L}^\varepsilon := -\Delta - \frac{\mu}{\delta^2 + \varepsilon^2} \mathcal{I} \tag{5.2}$$

on Ω with Dirichlet boundary conditions; secondly, we will apply these estimates for proving the main result of this section, Theorem 5.1.

5.1. Spectral estimates

Since the function $1/(\delta^2 + \varepsilon^2)$ is smooth and bounded in Ω for any $\varepsilon > 0$, the spectrum of $\mathcal{L}^\varepsilon$ is given by a sequence of real eigenvalues $\lambda_0^\varepsilon \leq \lambda_1^\varepsilon \leq \ldots \leq \lambda_k^\varepsilon \leq \ldots$, with $\lambda_k^\varepsilon \to +\infty$ as $k \to +\infty$, to which corresponds a family of eigenfunctions ϕ_k^ε that forms an orthonormal basis of $L^2(\Omega)$.

Proposition 5.1. *Assume $\mu > 1/4$ and let Ω_β be as in (3.9). Then we have*

$$\lim_{\varepsilon \to 0^+} \lambda_0^\varepsilon = -\infty \tag{5.3}$$

and, for all $\beta > 0$,

$$\lim_{\varepsilon \to 0^+} \left\| \phi_0^\varepsilon \right\|_{H^1(\Omega \setminus \overline{\Omega}_\beta)} = 0. \tag{5.4}$$

Proof. We argue by contradiction and we assume that λ_0^ε is bounded from below by some constant M. From the Rayleigh formula we have

$$\mu \int_\Omega \frac{u^2}{\delta^2 + \varepsilon^2}\, dx \leq \int_\Omega |\nabla u|^2\, dx - M \int_\Omega u^2\, dx,$$

for all $\varepsilon > 0$ and any $u \in H_0^1(\Omega)$. Taking now $u \in \mathcal{D}(\Omega)$, we pass to the limit as $\varepsilon \to 0^+$ in the inequality above and we get

$$\mu \int_\Omega \frac{u^2}{\delta^2}\, dx \leq \int_\Omega |\nabla u|^2\, dx - M \int_\Omega u^2\, dx, \tag{5.5}$$

that holds for any $u \in H_0^1(\Omega)$ by a density argument.

Now, given $\beta_0 > 0$, let us choose $u \in H_0^1(\Omega_{\beta_0})$, that we extend by zero on $\mathbb{R}^N$, and let us define, for $a \geq 1$,

$$u_a(x) := a^N u(ax).$$

This function is clearly in $H_0^1(\Omega_{\beta_0})$, and consequently in $H_0^1(\Omega)$; therefore, we can apply (5.5) to it and find

$$a^2 \left(\mu \int_\Omega \frac{u^2}{\delta^2}\, dx - \int_\Omega |\nabla u|^2\, dx \right) \leq -M \int_\Omega u^2\, dx.$$

Passing to the limit as $a \to +\infty$, we obtain

$$\mu \int_\Omega \frac{u^2}{\delta^2}\, dx \leq \int_\Omega |\nabla u|^2\, dx,$$

for any $u \in H_0^1(\Omega_{\beta_0})$. Therefore, we should have $\mu \leq 1/4$, since this is the Hardy inequality in the set Ω_{β_0}; then, we have a contradiction.

Now, consider the first eigenfunction $\phi_0^\varepsilon \in H_0^1(\Omega)$ of $\mathcal{L}^\varepsilon$, that by definition satisfies

$$-\Delta \phi_0^\varepsilon - \mu \frac{\phi_0^\varepsilon}{\delta^2 + \varepsilon^2} = \lambda_0^\varepsilon \phi_0^\varepsilon, \tag{5.6}$$

in Ω. Observe that, since the potential is smooth in Ω, also the function ϕ_0^ε is smooth by classical elliptic regularity.

Set $\beta > 0$ and let ξ_β be a non-negative smooth function, vanishing in $\Omega_{\beta/2}$ and equals to 1 in $\mathbb{R}^N \setminus \Omega_\beta$, with $\|\xi_\beta\|_\infty \leq 1$. Multiplying 5.6 by $\xi_\beta \phi_0^\varepsilon$ and integrating by parts we obtain

$$\int_\Omega \xi_\beta \left|\nabla \phi_0^\varepsilon\right|^2 dx + \left|\lambda_0^\varepsilon\right| \int_\Omega \xi_\beta \left(\phi_0^\varepsilon\right)^2 dx = \mu \int_\Omega \xi_\beta \frac{\left(\phi_0^\varepsilon\right)^2}{\delta^2 + \varepsilon^2} dx + \frac{1}{2} \int_\Omega \Delta \xi_\beta \left(\phi_0^\varepsilon\right)^2 dx. \tag{5.7}$$

Therefore, since ϕ_0^ε is of unit L^2-norm, and due to the definition of ξ_β, we get

$$\left|\lambda_0^\varepsilon\right| \int_{\Omega\setminus\Omega_\beta} \left(\phi_0^\varepsilon\right)^2 dx \leq \frac{4\mu}{\beta^2} + \frac{1}{2} \|\Delta\xi_\beta\|_{L^\infty(\Omega)}.$$

Since $\left|\lambda_0^\varepsilon\right| \to \infty$ as $\varepsilon \to 0^+$, we obtain that for any $\beta > 0$

$$\lim_{\varepsilon \to 0^+} \int_{\Omega\setminus\Omega_\beta} \left(\phi_0^\varepsilon\right)^2 dx = 0. \tag{5.8}$$

Furthermore, using again (5.7) and the definition of ξ_β

$$\int_{\Omega\setminus\Omega_\beta} \left|\nabla \phi_0^\varepsilon\right|^2 dx \leq \left(\frac{4\mu}{\beta^2} + \frac{1}{2} \|\Delta\xi_\beta\|_{L^\infty(\Omega)}\right) \int_{\Omega\setminus\Omega_{\beta/2}} \left(\phi_0^\varepsilon\right)^2 dx.$$

Hence, the proof of (5.4) is completed by using (5.8) for $\beta/2$. □

Proof of Theorem 5.1. Fix $\varepsilon > 0$ and choose $u_0^\varepsilon = \phi_0^\varepsilon$, that by definition is of unit L^2-norm. We want to show that

$$\inf_{f \in L^2((0,T);L^2(\Omega))} J_{u_0^\varepsilon}^\varepsilon(f) \to +\infty$$

as $\varepsilon \to 0^+$.

Hence, let $f \in L^2((0,T);L^2(\Omega))$ and consider the corresponding solution u of (1.2) with initial data $u_0^\varepsilon = \phi_0^\varepsilon$. Set

$$\rho(t) = \int_\Omega u(x,t)\phi_0^\varepsilon(x)\,dx, \quad \text{and} \quad \zeta(t) = \langle f(t), \phi_0^\varepsilon\rangle_{L^2(\Omega)};$$

then, $\rho(t)$ satisfies the first order differential equation

$$\begin{cases} \rho'(t) + \lambda_0^\varepsilon \rho(t) = \zeta(t), \\ \rho(0) = 1. \end{cases}$$

By the Duhamel's formula we obtain

$$\rho(t) = e^{-\lambda_0^\varepsilon t} + \int_0^t e^{-\lambda_0^\varepsilon(t-s)}\zeta(s)\,ds.$$

Therefore,

$$\int_Q u^2\,dxdt \geq \int_0^T \rho(t)^2\,dt \geq \frac{1}{2}\int_0^T e^{-\lambda_0^\varepsilon t}\,dt - \int_0^T \left(\int_0^t e^{-\lambda_0^\varepsilon(t-s)}\zeta(s)\,ds\right)^2 dt. \tag{5.9}$$

Of course

$$\frac{1}{2} \int_0^T e^{-\lambda_0^\varepsilon t}\, dt = \frac{1}{4\lambda_0^\varepsilon}\left(e^{2\lambda_0^\varepsilon T} - 1\right);$$

on the other hand, by trivial computations we have

$$\int_0^T \left(\int_0^t e^{-\lambda_0^\varepsilon(t-s)}\zeta(s)\, ds\right)^2 dt \le \frac{1}{4\left(\lambda_0^\varepsilon\right)^2} e^{2\lambda_0^\varepsilon T} \int_0^T \zeta(s)^2\, ds.$$

Besides, from the definition of $\zeta(t)$, and since f is localized in ω, it immediately follows

$$|\zeta(t)|^2 \le \|f(t)\|_{L^2(\Omega)}^2 \|\phi_0^\varepsilon\|_{L^2(\omega)}^2.$$

Hence, we deduce from (5.9) that

$$\frac{1}{4\lambda_0^\varepsilon}\left(e^{2\lambda_0^\varepsilon T} - 1\right) \le \int_Q u^2\, dxdt + \frac{\|\phi_0^\varepsilon\|_{L^2(\omega)}^2}{4\left(\lambda_0^\varepsilon\right)^2} e^{2\lambda_0^\varepsilon T} \int_0^T \|f(t)\|_{L^2(\Omega)}^2\, dt,$$

that implies either

$$\frac{1}{8\lambda_0^\varepsilon}\left(e^{2\lambda_0^\varepsilon T} - 1\right) \le \int_Q u^2\, dxdt$$

or

$$\frac{1}{8\lambda_0^\varepsilon}\left(e^{2\lambda_0^\varepsilon T} - 1\right) \le \frac{\|\phi_0^\varepsilon\|_{L^2(\omega)}^2}{4\left(\lambda_0^\varepsilon\right)^2} e^{2\lambda_0^\varepsilon T} \int_0^T \|f(t)\|_{L^2(\Omega)}^2\, dt.$$

In any case, for any $f \in L^2((0,T); L^2(\Omega))$ with support in ω we get

$$J_{u_0^\varepsilon}^\varepsilon(f) \ge \inf\left\{\frac{e^{2\lambda_0^\varepsilon T} - 1}{16\lambda_0^\varepsilon}, \ \frac{\lambda_0^\varepsilon}{4\|\phi_0^\varepsilon\|_{L^2(\omega)}^2}\left(1 - e^{2\lambda_0^\varepsilon T}\right)\right\}.$$

This last bound blows up as $\varepsilon \to 0^+$, due to the estimates (5.3) and (5.4). Indeed, by definition of ω, we can find $\beta > 0$ such that $\omega \subset \Omega \setminus \Omega_\beta$ and therefore

$$\|\phi_0^\varepsilon\|_{L^2(\omega)} \le \|\phi_0^\varepsilon\|_{L^2(\Omega \setminus \Omega_\beta)} \le \|\phi_0^\varepsilon\|_{H^1(\Omega \setminus \Omega_\beta)} \to 0,$$

as $\varepsilon \to 0^+$. This concludes the proof. $\qquad\qquad\square$

6. Proof of the Carleman estimate

Before giving the proof of the Carleman estimate (3.11), it is important to remark that, in principle, the solutions of (1.4) do not have enough regularity to justify the computations; in particular, the H^2 regularity in the space variable that would be required for applying standard integration by parts may not be guaranteed. For this reason, we need to add some regularisation

argument.

In our case, this can be done by regularising the potential, i.e. by considering, instead of the operator $\mathcal{A}$ defined in (1.1), the following

$$\mathcal{A}_n v := \Delta v + \frac{\mu_1}{(\delta + 1/n)^2} \, v, \quad n > 0. \tag{6.1}$$

The domain of this new operator is $\mathcal{D}(\mathcal{A}_n) = \mathcal{D}(-\Delta) = H_0^1(\Omega) \cap H^2(\Omega)$, due to the fact that now our potential is bounded on Ω, and the solution v_n of the corresponding parabolic equation possess all the regularity needed to justify the computations. Passing to the limit as $n \to +\infty$, we can then recover our result for the solution v of (1.4).

In order to simplify our presentation, we will skip this regularisation process and we will write directly the formal computations for the solution of (1.4). Moreover, we are going to present here the main ideas of the proof of the inequality, using some some technical Lemmas, which will be proved in appendix A.

Step 1. Notation and rewriting of the problem

For any solution v of the adjoint problem (1.4), and for any $R > 0$, we define

$$z(x,t) := v(x,t)e^{-R\sigma(x,t)}, \tag{6.2}$$

which satisfies

$$z(x,0) = z(x,T) = 0 \tag{6.3}$$

in $H_0^1(\Omega)$, due to the definition of σ. The positive parameter R is meant to be large. Plugging $v(x,t) = z(x,t)e^{R\sigma(x,t)}$ in (1.4), we obtain that z satisfies

$$z_t + \Delta z + \frac{\mu}{\delta^2} z + 2R\nabla z \cdot \nabla \sigma + Rz\Delta \sigma + z\left(R\sigma_t + R^2|\nabla \sigma|^2\right) = 0, \quad (x,t) \in \Omega \times (0,T) \tag{6.4}$$

with boundary conditions

$$z(x,t) = 0, \quad (x,t) \in \Gamma \times (0,T). \tag{6.5}$$

Next, we define a smooth positive function $\alpha(x)$ such that

$$\alpha(x) = \begin{cases} 0 & x \in \Omega_{r_0/2} \\ 1 & x \in \Omega \setminus \Omega_{r_0} \end{cases} \tag{6.6}$$

where Ω_{r_0} has been introduced in (3.9). Setting

$$\mathrm{S}z := \Delta z + \frac{\mu}{\delta^2} z + z\left(R\sigma_t + R^2|\nabla \sigma|^2\right), \quad \mathrm{A}z := z_t + 2R\nabla z \cdot \nabla \sigma + Rz\Delta \sigma(1 + \alpha), \quad \mathrm{P}z := -R\alpha z\Delta \sigma,$$

one easily deduce from (6.4) that

$$\mathrm{S}z + \mathrm{A}z + \mathrm{P}z = 0, \quad \|\mathrm{S}z\|_{L^2(Q)}^2 + \|\mathrm{A}z\|_{L^2(Q)}^2 + 2\langle \mathrm{S}, \mathrm{A}\rangle_{L^2(Q)} = \|\mathrm{P}z\|_{L^2(Q)}^2.$$

In particular, we obtain that the quantity

$$I = \langle \mathrm{S}z, \mathrm{A}z\rangle_{L^2(Q)} - \frac{1}{2}\|R\alpha z\Delta \sigma\|_{L^2(Q)}^2 \tag{6.7}$$

is not positive.

Step 2. Computation of the scalar product

Lemma 6.1. *The following identity holds:*

$$
\begin{aligned}
I &= R \int_Q |\partial_n z|^2 \partial_n \sigma \, dxdt - 2R \int_Q D^2\sigma(\nabla z, \nabla z) \, dxdt - R \int_Q \alpha \Delta\sigma |\nabla z|^2 \, dxdt \\
&\quad + R \int_Q (\nabla(\Delta\sigma) \cdot \nabla\alpha) z^2 \, dxdt + \frac{R}{2} \int_Q \Delta\sigma \Delta\alpha \, z^2 \, dxdt + R\mu \int_Q \alpha \Delta\sigma \frac{z^2}{\delta^2} \, dxdt \\
&\quad + 2R\mu \int_Q (\nabla\delta \cdot \nabla\sigma) \frac{z^2}{\delta^3} \, dxdt + \frac{R}{2} \int_Q \Delta^2\sigma(1+\alpha) z^2 \, dxdt - 2R^3 \int_Q D^2\sigma(\nabla\sigma, \nabla\sigma) z^2 \, dxdt \\
&\quad + R^3 \int_Q \alpha \Delta\sigma |\nabla\sigma|^2 z^2 \, dxdt - \frac{R^2}{2} \int_Q \alpha^2 |\Delta\sigma|^2 z^2 \, dxdt - \frac{1}{2} \int_Q \left(R\sigma_{tt} + 2R^2(|\nabla\sigma|^2)_t \right) z^2 \, dxdt \\
&\quad + R^2 \int_Q \alpha \sigma_t \Delta\sigma \, z^2 \, dxdt.
\end{aligned}
\tag{6.8}
$$

The proof of Lemma 6.1 will be presented in the appendix. Moreover, in what follows we will split (6.8) in four parts; first of all, let us define the boundary term

$$
I_{bd} = R \int_\Sigma |\partial_n z|^2 \partial_n \sigma \, dsdt,
\tag{6.9}
$$

where $\Sigma := \partial\Omega \times (0, T)$.

Secondly, we define I_l as the sum of the integrals linear in σ which do not involve any time derivative

$$
\begin{aligned}
I_l &= -2R \int_Q D^2\sigma(\nabla z, \nabla z) \, dxdt - R \int_Q \alpha \Delta\sigma |\nabla z|^2 \, dxdt + R \int_Q (\nabla(\Delta\sigma) \cdot \nabla\alpha) z^2 \, dxdt \\
&\quad + \frac{R}{2} \int_Q \Delta\sigma \Delta\alpha \, z^2 \, dxdt + R\mu \int_Q \alpha \Delta\sigma \frac{z^2}{\delta^2} \, dxdt \\
&\quad + 2R\mu \int_Q (\nabla\delta \cdot \nabla\sigma) \frac{z^2}{\delta^3} \, dxdt + \frac{R}{2} \int_Q \Delta^2\sigma(1+\alpha) z^2 \, dxdt.
\end{aligned}
\tag{6.10}
$$

Then, we consider the sum of the integrals involving non-linear terms in σ and without any time derivative, that is

$$
I_{nl} = -2R^3 \int_Q D^2\sigma(\nabla\sigma, \nabla\sigma) z^2 \, dxdt + R^3 \int_Q \alpha \Delta\sigma |\nabla\sigma|^2 z^2 \, dxdt - \frac{R^2}{2} \int_Q \alpha^2 |\Delta\sigma|^2 z^2 \, dxdt. \tag{6.11}
$$

Finally, we define the terms involving the time derivative in σ as

$$
I_t = -\frac{1}{2} \int_Q \left(R\sigma_{tt} + 2R^2(|\nabla\sigma|^2)_t \right) z^2 \, dxdt + R^2 \int_Q \alpha \sigma_t \Delta\sigma \, z^2 \, dxdt.
\tag{6.12}
$$

Step 3. Bounds for the quantities I_b, I_l, I_{nl} and I_t

We now estimates the four quantities (6.9), (6.10), (6.11) and (6.12) separately.

Lemma 6.2. *It holds that $I_{bd} = 0$ for any $\lambda > 1$*

Lemma 6.3. *There exists λ_0 such that for any $\lambda \geq \lambda_0$ and any $R > 0$, and for any r_0 as in (3.3), it holds*

$$
I_l \geq B_1 R \int_Q \theta \left(\delta^{2-\gamma} |\nabla z|^2 + \frac{z^2}{\delta^\gamma} \right) dxdt + \frac{\lambda R}{2} \int_{\Omega_{r_0} \times (0,T)} \theta \left(\frac{\delta}{r_0} \right)^{\lambda-2} |\nabla z|^2 \, dxdt
$$

$$
- B_2 \lambda^2 R \int_{\omega_0 \times (0,T)} \theta \left(\frac{\delta}{r_0} \right)^\lambda \phi |\nabla z|^2 \, dxdt + B_3 \lambda^2 R \int_{\mathcal{O} \times (0,T)} \theta \left(\frac{\delta}{r_0} \right)^\lambda \phi |\nabla z|^2 \, dxdt - B_\lambda R \int_Q \theta z^2 \, dxdt,
$$

$$(6.13)$$

where B_1, B_2 and B_3 are positive constants independent on R and λ, and B_λ is a positive constant independent on R.

Lemma 6.4. *There exists λ_0 such that for any $\lambda \geq \lambda_0$ there exists $R_0 = R_0(\lambda)$ such that for any $R \geq R_0$ and for any r_0 as in (3.3) it holds*

$$
I_{nl} \geq \frac{R^3}{2} \int_{\Omega_{r_0} \times (0,T)} \theta^3 \delta^2 z^2 \, dxdt + B_5 \lambda^4 R^3 \int_{\mathcal{O} \times (0,T)} \theta^3 \left(\frac{\delta}{r_0} \right)^{3\lambda} \phi^3 z^2 \, dxdt
$$

$$
- B_6 \lambda^4 R^3 \int_{\omega_0 \times (0,T)} \theta^3 \left(\frac{\delta}{r_0} \right)^{3\lambda} \phi^3 z^2 \, dxdt,
$$

$$(6.14)$$

for some positive constants B_5 and B_6 uniform in R and λ.

Taking into account the negative terms in the expression of I_l that we want to get rid of, we define

$$
I_r = I_t - B_\lambda R \int_Q \theta z^2 \, dxdt. \tag{6.15}
$$

Lemma 6.5. *There exists λ_0 such that for any $\lambda \geq \lambda_0$ there exists $R_0 = R_0(\lambda)$ such that for any $R \geq R_0$ and for any r_0 as in (3.3) it holds*

$$
|I_r| \leq \frac{B_1}{2} R \int_Q \theta \frac{z^2}{\delta^\gamma} \, dxdt + \frac{B_5}{2} \lambda^4 R^3 \int_{\mathcal{O} \times (0,T)} \theta^3 \left(\frac{\delta}{r_0} \right)^{3\lambda} \phi^3 z^2 \, dxdt + \frac{R^3}{4} \int_{\Omega_{r_0} \times (0,T)} \theta^3 \delta^2 z^2 \, dxdt, \quad (6.16)
$$

where B_1 and B_5 are the positive constants introduced in Lemmas 6.3 and 6.4, respectively.

The proofs of Lemmas 6.2, 6.3, 6.4 and 6.5 will be presented again in the appendix.

Step 4. Conclusion
From the Lemmas above, we obtain the Carleman estimates in the variable z as follows

Theorem 6.1. *There exist two positive constants λ_0 and $\mathcal{L}$ such that for any $\lambda \geq \lambda_0$ there exists $R_0 = R_0(\lambda)$ such that for any $R \geq R_0$ it holds*

$$
R \int_Q \theta \left(\delta^{2-\gamma} |\nabla z|^2 + \frac{1}{2} \frac{z^2}{\delta^\gamma} \right) dxdt + \lambda R \int_{\Omega_{r_0} \times (0,T)} \theta \left(\frac{\delta}{r_0} \right)^{\lambda-2} |\nabla z|^2 \, dxdt + \lambda^2 R \int_{\mathcal{O} \times (0,T)} \theta \left(\frac{\delta}{r_0} \right)^\lambda \phi |\nabla z|^2 \, dxdt
$$

$$+ R^3 \int\limits_{\Omega_{r_0} \times (0,T)} \theta^3 \delta^2 z^2 \, dxdt + \lambda^4 R^3 \int\limits_{\mathcal{O} \times (0,T)} \theta^3 \left(\frac{\delta}{r_0}\right)^{3\lambda} \phi^3 z^2 \, dxdt$$

$$\leq \mathcal{L} \left(\lambda^4 R^3 \int\limits_{\omega_0 \times (0,T)} \theta^3 \left(\frac{\delta}{r_0}\right)^{3\lambda} \phi^3 z^2 \, dxdt + \lambda^2 R \int\limits_{\omega_0 \times (0,T)} \theta \left(\frac{\delta}{r_0}\right)^{\lambda} \phi |\nabla z|^2 \, dxdt \right)$$

$$(6.17)$$

Coming back from the variable z to the solution v of (1.4), we finally obtain Theorem 3.1.

7. Open problems and perspectives

We conclude this paper with some open problem and perspective related to our work.

- *Boundary controllability.* In this article it is treated the controllability problem for the equation

$$u_t - \Delta u - \frac{\mu}{\delta^2} u = 0, \quad (x,t) \in \Omega \times (0,T) \qquad (7.1)$$

with a distributed control located in an open set $\omega \subset \Omega$. An immediate and interesting extension of the result we obtained, would be the analysis of boundary controllability for equation (7.1). In this framework, a first approach to this problem in one space dimension is given in [3], where the author is able to obtain boundary controllability for a heat equation with an inverse-square potential presenting singularities all-over the boundary. The multi-dimensional case, instead, remains at the moment unaddressed. As it is explained in [3], the main difficulty of this problem is to understand the behaviour of the normal derivative of the solution when approaching the boundary. Indeed, due to the presence of the singularity this normal derivative degenerates and this degeneracy would need to be properly compensated, in order to build the control for our equation. More in details, always referring to [3], we believe that we need to introduce a weighted normal derivative in the form $\delta^\alpha \partial_\nu u$, with a coefficient α which has to be identified. Then, the weight σ we employ in our Carleman has to be modified accordingly; we propose $\tilde{\sigma}(x,t) = \theta(t)(C_\lambda + \delta^{1+2\alpha} \psi - (\delta/r_0)^\lambda \phi)$, with θ and ψ as in (3.1), since this function would allow to obtain the weighted normal derivative we mentioned above in the boundary term of the Carleman inequality. The main difficulty would then be to show that, with this choice of the weight, it is possible to obtain suitable bounds for the distributed terms that shall lead to the inequality we seek.

- *Wave equation.* It would be interesting to investigate controllability properties also for a wave equation with singular inverse-square potential of the type μ/δ^2. Even if there are already results in the literature on this topic (see, for instance [7] and [22]), in our knowledge nobody treated the case of a potential with singularities arising all over the boundary. This is a very challenging issue; indeed, already in the one dimensional case, the presence of the singularity all over the boundary makes the multiplier approach extremely tricky, in the sense that is very difficult to identify, if possible, the correct multiplier for obtaining a Pohozaev identity. On the other hand, this would be surely a problem which deserves a more deep analysis.

- *Optimality of our results.* In the definition of the weight σ we consider an exponent $k = 3$ for our function θ; the motivation of this choice is that for lower exponents we are not able to bound some terms in our Carleman inequality. However, this has consequences on the cost of the control as the time tends to zero (see, for instance, [12], [20]), which is not of the order of $\exp(C/T)$, as expected for the heat equation, but rather of $\exp(C/T^3)$. Therefore, it would be interesting to reduce the exponent in the definition of θ up to $k = 1$ and try to obtain a Carleman estimate with this new choice for the weight.

A. Proof of technical Lemmas

The computations for obtaining the Carleman estimate are very long; in order to simplify the presentation, in Section 6 we divided these computations in four step and we introduced several preliminary results, Lemmas 6.1 to 6.5. We present now the proof of these Lemmas.
At this purpose, we remind that the distance function δ satisfies the following properties

$$\delta \in C^{0,1}(\overline{\Omega}), \tag{A.1a}$$

$$|\nabla u| = 1, \text{ a.e. in } \Omega, \tag{A.1b}$$

$$\text{there exists a constant } P > 0 \text{ such that } |\Delta\delta| \leq P/\delta, \text{ a.e. in } \Omega. \tag{A.1c}$$

Furthermore, we are going to need the following result

Lemma A.1. *Assume that ψ is the function defined in (3.4) by means of ψ_1 and ϖ. Then, there exists a constant $D_{\psi_1} > 0$, which depends on ψ_1, such that*

$$|\nabla\delta \cdot \nabla\psi(x) - \varpi\psi_1(x)| \leq \varpi D_{\psi_1}. \tag{A.2}$$

Proof. By definition of ψ and Cauchy-Scwarz inequality, using (A.1b) and since ψ_1 is bounded, we immediately have

$$|\nabla\delta \cdot \nabla\psi(x) - \varpi\psi_1(x)| = \varpi|\nabla\delta \cdot \nabla\psi_1(x) - \psi_1(x)| \leq \varpi|\nabla\psi_1 - \psi_1| \leq \varpi D_{\psi_1}.$$

$\square$

Now, for σ as in (3.1) we introduce the notations

$$\sigma_\delta = -\theta\tau_\delta = -\theta\delta^2\psi, \qquad \sigma_\phi = -\theta\tau_\phi = -\theta\left(\frac{\delta}{r_0}\right)^\lambda\phi, \qquad \tau = \tau_\delta + \tau_\phi,$$

so that $\sigma(x, t) = C_\lambda\theta(t) + \sigma_\delta(x, t) + \sigma_\phi(x, t)$. Next, we deduce some formulas for τ_δ and τ_ϕ that we are going to use later in our computations. More precisely, for all $x \in \mathbb{R}^N$ and any $i, j \in \{1, \ldots, N\}$ we have

$$\partial_{x_i}\tau_\delta = 2\psi\delta\delta_{x_i} + \delta^2\psi_{x_i}, \tag{A.3}$$

$$\partial^2_{x_ix_j}\tau_\delta = 2\psi\delta_{x_i}\delta_{x_j} + 2\delta(\psi_{x_j}\delta_{x_i} + \psi\delta_{x_ix_j}) + 2\delta\psi_{x_i}\delta_{x_j} + \delta^2\psi_{x_ix_j} \tag{A.4}$$

and

$$\Delta\tau_\delta = 2\psi + 4\delta(\nabla\delta \cdot \nabla\psi) + 2\delta\psi\Delta\delta + \delta^2\Delta\psi, \tag{A.5}$$

$$D^2 \tau_\delta(\xi, \xi) = 2\psi(\xi \cdot \nabla \delta)^2 + 2\delta\psi D^2\delta(\xi, \xi) + 4\delta(\xi \cdot \nabla \delta)(\xi \cdot \nabla \psi) + \delta^2 D^2\psi(\xi, \xi), \quad \forall \xi \in \mathbb{R}^N. \quad \text{(A.6)}$$

On the other hand

$$\partial_{x_i} \tau_\phi = \frac{\phi}{r_0^\lambda}(\lambda\delta^{\lambda-1}\delta_{x_i} + \lambda\delta^\lambda\psi_{x_i}), \quad \text{(A.7)}$$

$$\partial_{x_i x_j}^2 \tau_\phi = \frac{\phi}{r_0^\lambda}\Big(\lambda(\lambda-1)\delta^{\lambda-2}\delta_{x_i}\delta_{x_j} + \lambda\delta^{\lambda-1}\delta_{x_i x_j} + \lambda^2\delta^{\lambda-1}(\psi_{x_j}\delta_{x_i} + \psi_{x_i}\delta_{x_j}) + \lambda\delta^\lambda\psi_{x_i x_j} + \lambda^2\delta^\lambda\psi_{x_i}\psi_{x_j}\Big)$$
$$\text{(A.8)}$$

and

$$\Delta\tau_\phi = \frac{\phi}{r_0^\lambda}\Big(\lambda(\lambda-1)\delta^{\lambda-2} + \lambda\delta^{\lambda-1}\Delta\delta + 2\lambda^2\delta^{\lambda-1}(\nabla\delta \cdot \nabla\psi) + \lambda\delta^\lambda\Delta\psi + \lambda^2\delta^\lambda|\nabla\psi|^2\Big), \quad \text{(A.9)}$$

$$D^2\tau_\phi(\xi, \xi) = \frac{\phi}{r_0^\lambda}\Big(\lambda(\lambda-1)\delta^{\lambda-2}(\xi \cdot \nabla\delta)^2 + \lambda\delta^{\lambda-1}D^2\delta(\xi, \xi) + 2\lambda^2\delta^{\lambda-1}(\xi \cdot \nabla\delta)(\xi \cdot \nabla\psi)$$
$$+ \lambda\delta^\lambda D^2\psi(\xi, \xi) + \lambda^2\delta^\lambda(\xi \cdot \nabla\psi)^2\Big), \quad \forall \xi \in \mathbb{R}^N. \quad \text{(A.10)}$$

Upper and lower bounds for $\Delta\tau_\delta$, $\Delta\tau_\phi$, $D^2\tau_\delta(\xi, \xi)$ and $D^2\tau_\phi(\xi, \xi)$

Proposition A.1. *For r_0 as in (3.3) we have*

$$\Delta\tau_\delta \geq 0, D^2\tau_\delta \geq 0, \qquad\qquad \forall x \in \Omega_{r_0}, \forall \xi \in \mathbb{R}^N, \quad \text{(A.11)}$$

$$|D^2\tau_\delta(\xi, \xi)| \leq C_1|\xi|^2, \qquad\qquad \forall x \in \Omega, \forall \xi \in \mathbb{R}^N, \quad \text{(A.12)}$$

$$|\Delta\tau_\delta| \leq C_2, \qquad\qquad \forall x \in \Omega_{r_0}. \quad \text{(A.13)}$$

where C_1 and C_2 are constants depending on Ω and ψ.

Proposition A.2. *For r_0 and ϖ as in (3.3) and (3.7) we have*

$$D^2\tau_\phi \geq \frac{\lambda}{2}\left(\frac{\delta}{r_0}\right)^{\lambda-2}\phi|\xi|^2, \qquad\qquad \forall x \in \Omega_{r_0}, \forall \xi \in \mathbb{R}^N, \quad \text{(A.14)}$$

$$\Delta\tau_\phi \geq \lambda^2\left(\frac{\delta}{r_0}\right)^\lambda\phi, \qquad\qquad \forall x \in \mathcal{O}, \quad \text{(A.15)}$$

$$D^2\tau_\phi \geq -\lambda C_3\left(\frac{\delta}{r_0}\right)^{\lambda-2}\phi|\xi|^2, \qquad\qquad \forall x \in \Omega, \forall \xi \in \mathbb{R}^N, \quad \text{(A.16)}$$

for λ large enough, where C_3 is a constant depending on Ω, r_0 and ψ.

Proof of Proposition A.1. Observe that the proofs of (A.12) and (A.13) are trivial. To prove (A.11), instead, it is enough to show that $D^2\tau_\delta(\xi, \xi) \geq 0$ in Ω_{r_0} since this also implies that $\Delta\tau_\delta \geq 0$ in Ω_{r_0}, simply choosing $\xi = e_i$ for all $i \in \{1, \ldots, N\}$. Now, we have that, for $x \in \Omega_{r_0}$

$$\delta(x) = |x - \text{pr}(x)| \quad \text{(A.17)}$$

where $\text{pr}(x)$ is the projection of x on Γ. Hence (A.6) becomes

$$D^2\tau_\delta(\xi, \xi) = 2\psi|\xi|^2 + 4\left(\xi \cdot \left(x - \text{pr}(x)\right)\right)(\xi \cdot \nabla\psi) + \delta^2 D^2\psi(\xi, \xi), \quad \forall \xi \in \mathbb{R}^N.$$

Now, using Cauchy-Scwarz inequality we obtain

$$D^2\tau_\delta(\xi,\xi) \geq (2\psi - 4\delta|D\psi|_\infty - \delta^2|D^2\psi|_\infty)|\xi|^2 \geq (2\psi - r_0(4|D\psi|_\infty + |D^2\psi|_\infty))|\xi|^2 \geq 0.$$

since r_0 satisfies (3.3). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Proposition A.2. First of all, we rewrite (A.10) as $D^2\tau_\phi(\xi,\xi) = \phi(1/r_0)^\lambda \mathcal{S}_\phi$, where

$$\begin{aligned}\mathcal{S}_\phi &= \lambda(\lambda-1)\delta^{\lambda-2}(\xi\cdot\nabla\delta)^2 + \lambda\delta^{\lambda-1}D^2\delta(\xi,\xi) + 2\lambda^2\delta^{\lambda-1}(\xi\cdot\nabla\delta)(\xi\cdot\nabla\psi) \\ &\quad + \lambda\delta^\lambda D^2\psi(\xi,\xi) + \lambda^2\delta^\lambda(\xi\cdot\nabla\psi)^2.\end{aligned} \tag{A.18}$$

Next, we have

$$|2\lambda^2\delta^{\lambda-1}(\xi\cdot\nabla\delta)(\xi\cdot\nabla\psi)| \leq a\lambda^2\delta^{\lambda-2}(\xi\cdot\nabla\delta)^2 + \frac{\lambda^2}{a}\delta^\lambda(\xi\cdot\nabla\psi)^2, \quad \forall a > 0,$$

which combined with (A.18) leads to

$$\mathcal{S}_\phi \geq (\lambda^2 - \lambda - a\lambda^2)\delta^{\lambda-2}(\xi\cdot\nabla\delta)^2 + \lambda\delta^{\lambda-1}D^2\delta(\xi,\xi) + \lambda\delta^\lambda D^2\psi(\xi,\xi) + \left(\lambda^2 - \frac{\lambda^2}{a}\right)\delta^\lambda(\xi\cdot\nabla\psi)^2.$$

Choosing now a such that $\lambda^2(1-a) - \lambda = 0$, i.e. $a = (\lambda-1)/\lambda$, we have

$$\mathcal{S}_\phi \geq \lambda\delta^{\lambda-1}D^2\delta(\xi,\xi) + \lambda\delta^\lambda D^2\psi(\xi,\xi) - \frac{\lambda^2}{\lambda-1}\delta^\lambda|\nabla\psi|^2|\xi|^2. \tag{A.19}$$

Applying (A.19) for $x \in \Omega_{r_0}$ we deduce

$$\begin{aligned}\mathcal{S}_\phi &\geq \frac{\lambda}{2}\delta^{\lambda-2}|\xi|^2 + \lambda\delta^{\lambda-2}|\xi|^2\left(\frac{1}{2} - \frac{\lambda}{\lambda-1}\delta^2|D\psi|_\infty^2 - \delta^2|D^2\psi|_\infty\right) \\ &\geq \frac{\lambda}{2}\delta^{\lambda-2}|\xi|^2 + \lambda\delta^{\lambda-2}|\xi|^2\left(\frac{1}{2} - r_0^2\left(2|D\psi|_\infty^2 + |D^2\psi|_\infty\right)\right) \geq \frac{\lambda}{2}\delta^{\lambda-2}|\xi|^2,\end{aligned}$$

for r_0 as in (3.3). This immediately yields the proof of (A.14).

Let us now prove (A.15). According to Lemma A.1, to the definition of ψ and to (A.1c) and (A.9) we get

$$\begin{aligned}\Delta\tau_\phi &\geq \frac{\phi}{r_0^\lambda}\left(\lambda(\lambda-1-P)\delta^{\lambda-2} + 2\lambda^2\delta^{\lambda-1}(\varpi\psi_1 - \varpi D_{\psi_1}) + \lambda\delta^\lambda\Delta\psi + \lambda^2\delta^\lambda|\nabla\psi|^2\right) \\ &\geq \lambda^2\left(\frac{\delta}{r_0}\right)^\lambda\phi\left(|\nabla\psi|^2 - \frac{2\varpi D_{\psi_1}}{r_0} - \varpi\frac{|\Delta\psi|}{\lambda}\right) \geq \lambda^2\left(\frac{\delta}{r_0}\right)^\lambda\phi\left(\varpi^2\varpi_0^2 - \frac{2\varpi D_{\psi_1}}{r_0} - \varpi\frac{|\Delta\psi|}{\lambda}\right) \\ &\geq \lambda^2\left(\frac{\delta}{r_0}\right)^\lambda\phi\end{aligned}$$

for all $x \in \mathcal{O}$, if we take ϖ as in (3.7) and λ large enough.

We conclude with the proof of (A.16). From (A.10) for any $x \in \Omega$ we have

$$D^2\tau_\phi(\xi,\xi) = \frac{\phi}{r_0^\lambda}\left(\lambda^2\left(\delta^{\frac{\lambda}{2}-1}(\xi\cdot\nabla\delta) + \delta^{\frac{\lambda}{2}}(\xi\cdot\nabla\psi)\right)^2 + \lambda\delta^{\lambda-1}D^2\delta(\xi,\xi) + \lambda\delta^\lambda D^2\psi(\xi,\xi) - \lambda\delta^{\lambda-2}(\xi\cdot\nabla\delta)^2\right)$$

$$\geq \lambda \left(\frac{\delta}{r_0}\right)^{\lambda-2} \phi\left(\frac{1}{r_0^2}\left(\delta D^2\delta(\xi,\xi) + \delta^2 D^2\psi(\xi,\xi) - (\xi\cdot\nabla\delta)^2\right)\right)$$

$$\geq -\lambda \left(\frac{\delta}{r_0}\right)^{\lambda-2} \phi\left(\frac{1}{r_0^2}\left(|D^2\delta|_\infty + R_\Omega^2|D^2\psi|_\infty + 1\right)\right)|\xi|^2,$$

which gives us the validity of (A.16) for $C_3 = \left(|D^2\delta|_\infty + R_\Omega^2|D^2\psi|_\infty + 1\right)/r_0^2$. □

Bounds for $2D^2\tau(\nabla\tau,\nabla\tau) - \alpha\Delta\tau|\nabla\tau|^2$

We provide here pointwise estimates for the quantity

$$2D^2\tau(\nabla\tau,\nabla\tau) - \alpha\Delta\tau|\nabla\tau|^2,$$

which appears in the identity in Lemma 6.1.
First of all, we have

$$\partial_{x_i}\tau = 2\psi\delta\delta_{x_i} + \delta^2\psi_{x_i} + \frac{\phi}{r_0^\lambda}(\lambda\delta^{\lambda-1}\delta_{x_i} + \lambda\delta^\lambda\psi_{x_i}),$$

$$\partial^2_{x_ix_j}\tau = 2\psi\delta_{x_i}\delta_{x_j} + 2\delta(\psi_{x_j}\delta_{x_i} + \psi\delta_{x_ix_j}) + 2\delta\psi_{x_i}\delta_{x_j} + \delta^2\psi_{x_ix_j}$$

$$+ \frac{\phi}{r_0^\lambda}\Big(\lambda(\lambda-1)\delta^{\lambda-2}\delta_{x_i}\delta_{x_j} + \lambda\delta^{\lambda-1}\delta_{x_ix_j} + \lambda^2\delta^{\lambda-1}(\psi_{x_j}\delta_{x_i} + \psi_{x_i}\delta_{x_j}) + \lambda\delta^\lambda\psi_{x_ix_j} + \lambda^2\delta^\lambda\psi_{x_i}\psi_{x_j}\Big),$$

and in consequence

$$\Delta\tau = 2\psi + 4\delta(\nabla\delta\cdot\nabla\psi) + 2\psi\Delta\delta + \delta^2\Delta\psi$$

$$+\frac{\phi}{r_0^\lambda}\Big(\lambda(\lambda-1)\delta^{\lambda-2} + \lambda\delta^{\lambda-1}\Delta\delta + 2\lambda^2\delta^{\lambda-1}(\nabla\delta\cdot\nabla\psi) + \lambda\delta^\lambda\Delta\psi + \lambda^2\delta^\lambda|\nabla\psi|^2\Big), \qquad (A.20)$$

$$D^2\tau(\nabla\tau,\nabla\tau) = 2\psi(\nabla\tau\cdot\nabla\delta)^2 + 2\delta\psi D^2\delta(\nabla\tau,\nabla\tau) + 4\delta(\nabla\tau\cdot\nabla\delta)(\nabla\tau\cdot\nabla\psi) + \delta^2 D^2\psi(\nabla\tau,\nabla\tau)$$

$$+\frac{\phi}{r_0^\lambda}\Big(\lambda(\lambda-1)\delta^{\lambda-2}(\nabla\tau\cdot\nabla\delta)^2 + \lambda\delta^{\lambda-1}D^2\delta(\nabla\tau,\nabla\tau) + 2\lambda^2\delta^{\lambda-1}(\nabla\tau\cdot\nabla\delta)(\nabla\tau\cdot\nabla\psi)$$

$$+ \lambda\delta^\lambda D^2\psi(\nabla\tau,\nabla\tau) + \lambda^2\delta^\lambda(\nabla\tau\cdot\nabla\psi)^2\Big). \qquad (A.21)$$

Using the expressions above we obtain the following useful formulas

$$(\nabla\delta\cdot\nabla\tau)^2 = |\nabla\tau|^2 + \left((\nabla\delta\cdot\nabla\psi)^2 - |\nabla\psi|^2\right)\left(\delta^2 + \lambda\frac{\phi}{r_0^\lambda}\delta^\lambda\right)^2,$$

$$(\nabla\delta\cdot\nabla\tau)(\nabla\psi\cdot\nabla\tau) = |\nabla\tau|^2(\nabla\delta\cdot\nabla\psi) + \left(|\nabla\psi|^2 - (\nabla\delta\cdot\nabla\psi)^2\right)\left(2\delta\psi + \lambda\frac{\phi}{r_0^\lambda}\delta^{\lambda-1}\right)\left(\delta^2 + \lambda\frac{\phi}{r_0^\lambda}\delta^\lambda\right),$$

$$(\nabla\psi\cdot\nabla\tau)^2 = |\nabla\psi|^2|\nabla\tau|^2 + \left((\nabla\delta\cdot\nabla\psi)^2 - |\nabla\psi|^2\right)\left(2\delta\psi + \lambda\frac{\phi}{r_0^\lambda}\delta^{\lambda-1}\right)^2,$$

and we finally conclude

$$2D^2\tau(\nabla\tau,\nabla\tau) - \alpha\Delta\tau|\nabla\tau|^2 = T_1 + T_2 + T_3,$$

where

$$T_1 = 2\psi(2-\alpha)|\nabla\tau|^2 + 4\delta\psi D^2\delta(\nabla\tau,\nabla\tau) + 2\delta^2 D^2\psi(\nabla\tau,\nabla\tau) + 4(2-\alpha)\delta(\nabla\delta\cdot\nabla\psi)|\nabla\tau|^2$$
$$- 2\delta\psi\alpha\Delta\delta|\nabla\tau|^2 - \delta^2\alpha\Delta\psi|\nabla\tau|^2, \tag{A.22}$$

$$T_2 = 4\left(|\nabla\psi|^2 - (\nabla\delta\cdot\nabla\psi)^2\right)\left(\delta^2 + \lambda\frac{\phi}{r_0^\lambda}\delta^\lambda\right)\left(5\delta^2\psi + \lambda(2-\psi)\frac{\phi}{r_0^\lambda}\delta^\lambda\right)$$
$$+ \frac{\phi}{r_0^\lambda}\left(|\nabla\psi|^2 - (\nabla\delta\cdot\nabla\psi)^2\right)\left(2\lambda^3\delta^{3\lambda-2}\left(\frac{\phi}{r_0^\lambda}\right)^2 + \lambda^2(8\psi(1-\psi)-2)\delta^{\lambda+2} + 4\lambda^2\frac{\phi}{r_0^\lambda}\delta^{2\lambda} + 2\lambda\delta^{\lambda+2}\right), \tag{A.23}$$

$$T_3 = \frac{\phi}{r_0^\lambda}\left\{\left[(\lambda^2(2-\alpha) - \lambda(2-\alpha+\alpha\delta\Delta\delta))\delta^{\lambda-2} + 2\lambda^2\delta^{\lambda-1}(2-\alpha)(\nabla\delta\cdot\nabla\psi)\right.\right.$$
$$\left.\left. + \lambda^2\delta^\lambda(2-\alpha)|\nabla\psi|^2 - \lambda\alpha\delta^\lambda\Delta\psi\right]|\nabla\tau|^2 + 2\lambda\delta^{\lambda-1}D^2\delta(\nabla\tau,\nabla\tau) + 2\lambda\delta^\lambda D^2\psi(\nabla\tau,\nabla\tau)\right\}. \tag{A.24}$$

Proposition A.3. *For r_0 as in (3.3), there exist two positive constants D_1 and D_2 depending on Ω and ψ such that the term T_1 in (A.22) satisfies*

$$T_1 \geq |\nabla\tau|^2, \qquad\qquad \forall x \in \Omega_{r_0}, \tag{A.25}$$

$$T_1 \geq -D_1|\nabla\tau|^2, \qquad\qquad \forall x \in \mathcal{O}, \tag{A.26}$$

$$|T_1| \leq D_2|\nabla\tau|^2, \qquad\qquad \forall x \in \omega_0. \tag{A.27}$$

Proposition A.4. *There exists λ_0 large enough such that, for any $\lambda \geq \lambda_0$ and r_0 as in (3.3), the term T_2 in (A.23) satisfies*

$$T_2 \geq -\frac{\phi}{r_0^\lambda}|D\psi|_\infty^2\left(D_3\lambda^2\psi^2 + D_4\lambda^2\right)\delta^{\lambda+2}, \qquad\qquad \forall x \in \Omega_{r_0}, \tag{A.28}$$

$$T_2 \geq 0, \qquad\qquad \forall x \in \tilde{\mathcal{O}}, \tag{A.29}$$

for some positive constants D_3 and D_4 depending on ...

Proposition A.5. *There exists λ_0 large enough such that, for any $\lambda \geq \lambda_0$ and r_0 and ϖ as in (3.3) and (3.7), the term T_3 in (A.24) satisfies*

$$T_3 \geq \lambda^2\left(\frac{\phi}{r_0^\lambda}\delta^{\lambda-2} + \left(\frac{\delta}{r_0}\right)^\lambda\phi\right)|\nabla\tau|^2, \qquad\qquad \forall x \in \Omega\setminus\overline{\omega_0}, \tag{A.30}$$

$$T_3 \leq \lambda^2 D_5\frac{\phi}{r_0^\lambda}\delta^{\lambda-2}|\nabla\tau|^2, \qquad\qquad \forall x \in \Omega, \tag{A.31}$$

for some positive constant D_5, not depending on λ.

Proposition A.6. *For any r_0 and ϖ as in (3.3) and (3.7) it holds*

$$|\nabla\tau|^2 \geq \delta^2, \qquad\qquad \forall x \in \Omega_{r_0}, \tag{A.32}$$

$$|\nabla\tau|^2 \geq \lambda^2\left(\frac{\delta}{r_0}\right)^{2\lambda}\phi^2, \qquad\qquad \forall x \in \mathcal{O}, \tag{A.33}$$

$$|\nabla\tau|^2 \leq \lambda^2 D_6\left(\frac{\delta}{r_0}\right)^{2\lambda}\phi^2, \qquad\qquad \forall x \in \omega_0, \tag{A.34}$$

where D_6 is a positive constant depending only on Ω and ψ.

Proof of Proposition A.3. The inequalities (A.26) and (A.27) are obvious. Hence, we only need to prove (A.25). Due to the definition of α, to the properties of ψ and to Lemma A.1, and using (A.17), we have (see also [8, Prop. 3.4])

$$T_1 \geq \left(2 - r_0^2(8\varpi D_{\psi_1} + 3|D^2\psi|_\infty)\right)|\nabla\tau|^2 \geq \left(2 - r_0^2\left(8\frac{D_{\psi_1}}{\varpi_0}|D\psi|_\infty + 3|D^2\psi|_\infty\right)\right)|\nabla\tau|^2 \geq |\nabla\tau|^2,$$

in Ω_{r_0}, for r_0 as in (3.3). $\square$

Proof of Proposition A.4. Due to Cuachy-Scwarz inequality, the term $|\nabla\psi|^2 - (\nabla\delta\cdot\nabla\psi)^2$ in (A.23) is positive; hence

$$4\left(|\nabla\psi|^2 - (\nabla\delta\cdot\nabla\psi)^2\right)\left(\delta^2 + \lambda\frac{\phi}{r_0^\lambda}\delta^\lambda\right)\left(5\delta^2\psi + \lambda(2-\psi)\frac{\phi}{r_0^\lambda}\delta^\lambda\right) \geq 4D_7\delta^2\left(5\delta^2\psi + \lambda(2-\psi)\frac{\phi}{r_0^\lambda}\delta^\lambda\right)$$

$$\geq -4D_7\lambda\psi\frac{\phi}{r_0^\lambda}\delta^{\lambda+2} \geq -D_8\lambda^2\frac{\phi}{r_0^\lambda}\delta^{\lambda+2}$$

for λ large enough. From this (A.28) follows trivially.

Concerning (A.29), it is straightforward to check that the inequality holds for λ large enough, since the term in λ^3 is positive and it dominates all the other terms far away from the boundary. $\square$

Proof of Proposition A.5. For $x \in \Omega_{r_0}$, due to (A.17), the proof is analogous to the one of [8, Prop. 3.6] and we omit it here. Therefore, let us assume now $x \in \tilde{O}$. Due to the definition of α, for λ large enough we have

$$\lambda^2(2-\alpha) - \lambda(2 - \alpha - \alpha\delta\Delta\delta) \geq \lambda^2.$$

Hence, from Lemma A.1 and from the properties of ψ, for $x \in \Omega \setminus \overline{\omega_0}$ we have

$$T_3 \geq \frac{\phi}{r_0^\lambda}\left(\lambda^2\delta^{\lambda-2} + 2\lambda^2\delta^{\lambda-1}(2-\alpha)(\varpi\psi_1 - \varpi D_{\psi_1}) + \lambda^2\delta^\lambda(2-\alpha)|\nabla\psi|^2 - \lambda\alpha\delta^\lambda|D^2\psi|_\infty\right.$$

$$\left. -2\lambda\delta^{\lambda-2}|D^2\delta|_\infty - 2\lambda\delta^\lambda|D^2\psi|_\infty\right)|\nabla\tau|^2$$

$$\geq \lambda^2\frac{\phi}{r_0^\lambda}\delta^{\lambda-2}|\nabla\tau|^2 + \lambda^2\frac{\phi}{r_0^\lambda}\delta^\lambda\left(\varpi^2|\nabla\psi_1|^2 - \frac{2\varpi D_{\psi_1}}{\delta} - \frac{2+\alpha}{\lambda}|D^2\psi|_\infty - 2\frac{|D^2\delta|_\infty}{\delta^2\lambda}\right)|\nabla\tau|^2$$

$$\geq \lambda^2\frac{\phi}{r_0^\lambda}\delta^{\lambda-2}|\nabla\tau|^2 + \lambda^2\frac{\phi}{r_0^\lambda}\delta^\lambda\left(\varpi^2|\nabla\psi_1|^2 - \frac{2\varpi D_{\psi_1}}{r_0} - \frac{2+\alpha}{\lambda}|D^2\psi|_\infty - 2\frac{|D^2\delta|_\infty}{r_0^2\lambda}\right)|\nabla\tau|^2$$

$$\geq \lambda^2\frac{\phi}{r_0^\lambda}\delta^{\lambda-2}|\nabla\tau|^2 + \lambda^2\frac{\phi}{r_0^\lambda}\delta^\lambda\left(\frac{\varpi^2\varpi_0^2}{2} - \frac{2\varpi D_{\psi_1}}{r_0}\right)|\nabla\tau|^2 \geq \lambda^2\frac{\phi}{r_0^\lambda}\delta^{\lambda-2}|\nabla\tau|^2 + \lambda^2\frac{\phi}{r_0^\lambda}\delta^\lambda|\nabla\tau|^2,$$

for λ large enough and ϖ as in (3.7). Concerning (A.31), once again the proof is trivial and we omit it here. $\square$

Proof of Proposition A.6. We have

$$|\nabla\tau|^2 = 4\delta^2\psi^2 + \delta^4|\nabla\psi|^2 + 4\delta^3(\nabla\delta\cdot\nabla\psi) + \lambda^2\left(\frac{\phi}{r_0^\lambda}\right)^2\left(\delta^{2\lambda-2} + \delta^{2\lambda}|\nabla\psi|^2 + 2\delta^{2\lambda-1}(\nabla\delta\cdot\nabla\psi)\right)$$

$$+ \lambda \frac{\phi}{r_0^\lambda} \left(2\delta^{2+\lambda} |\nabla \psi|^2 + 4\delta^\lambda \psi + 2(1 + 2\psi)\delta^{1+\lambda}(\nabla\delta \cdot \nabla\psi) \right) \tag{A.35}$$

Now we observe that, for r_0 as in (3.3), we have

$$3\delta^2\psi^2 + 4\delta^3(\nabla\delta \cdot \nabla\psi) \geq \delta^2(3\psi^2 - 4\delta|\nabla\psi|) \geq \delta^2(3\psi^2 - 4r_0|\nabla\psi|) \geq 0,$$

$$2\delta^{2+\lambda}|\nabla\psi|^2 + 4\delta^\lambda\psi + 2(1 + 2\psi)\delta^{1+\lambda}(\nabla\delta \cdot \nabla\psi) \geq 2\delta^\lambda \left(2\psi - \delta^2|\nabla\psi|^2 - (1 + 2\psi)\delta(\nabla\delta \cdot \nabla\psi) \right)$$
$$\geq 2\delta^\lambda \left(2\psi - r_0 \left(|\nabla\psi|^2 + (1 + 2\psi)|\nabla\psi| \right) \right) \geq 0$$

and

$$\delta^{2\lambda-2} + \delta^{2\lambda}|\nabla\psi|^2 + 2\delta^{2\lambda-1}(\nabla\delta \cdot \nabla\psi) = \delta^{2\lambda-2}\left(1 + \delta^2|\nabla\psi|^2 + 2\delta(\nabla\delta \cdot \nabla\psi) \right)$$
$$\geq \delta^{2\lambda-2}\left(1 - \delta^2|\nabla\psi|^2 - 2\delta|\nabla\psi| \right) \geq \delta^{2\lambda-2}\left(1 - r_0 \left(|\nabla\psi|^2 + 2|\nabla\psi| \right) \right) \geq 0.$$

Therefore, (A.32) immediately follows.

Let us now prove (A.33). Firstly, we observe that, thanks to Lemma A.1 and to the properties of ψ, we get

$$\delta^{2\lambda-2} + \delta^{2\lambda}|\nabla\psi|^2 + 2\delta^{2\lambda-1}(\nabla\delta \cdot \nabla\psi) \geq \delta^{2\lambda}\left(|\nabla\psi|^2 + \frac{2}{\delta}(\nabla\delta \cdot \nabla\psi) \right) \geq \delta^{2\lambda}\left(\varpi^2\varpi_0^2 - \frac{2\varpi D_{\psi_1}}{r_0} \right)$$
$$\geq \frac{\varpi^2\varpi_0^2}{2}\delta^{2\lambda},$$

for all $x \in \tilde{\mathcal{O}}$ and for ϖ as in (3.7). Moreover,

$$2\delta^{2+\lambda}|\nabla\psi|^2 + 4\delta^\lambda\psi + 2(1 + 2\psi)\delta^{1+\lambda}(\nabla\delta \cdot \nabla\psi) \geq -2(1 + 2\psi)\varpi D_{\psi_1}\delta^{\lambda+1};$$

hence

$$|\nabla\tau|^2 \geq \lambda^2 \frac{\varpi^2\varpi_0^2}{2}\left(\frac{\delta}{r_0} \right)^{2\lambda}\phi^2 - 2(1 + 2\psi)\varpi D_{\psi_1}R_\Omega\left(\frac{\delta}{r_0} \right)^\lambda\phi.$$

Now, since by definition $\lambda\psi \leq \phi$,

$$\lambda^2\frac{\varpi^2\varpi_0^2}{4}\left(\frac{\delta}{r_0} \right)^{2\lambda}\phi^2 - 2(1 + 2\psi)\varpi D_{\psi_1}R_\Omega\left(\frac{\delta}{r_0} \right)^\lambda\phi$$
$$= \frac{\varpi^2\varpi_0^2}{4}\left(\frac{\delta}{r_0} \right)^{2\lambda}\phi^2\left(\lambda^2 - \lambda\frac{8(1 + 2\psi)\varpi D_{\psi_1}R_\Omega}{\varpi^2\varpi_0^2}\left(\frac{r_0}{\delta} \right)^\lambda\frac{1}{\phi} \right)$$
$$\geq \frac{\varpi^2\varpi_0^2}{4}\left(\frac{\delta}{r_0} \right)^{2\lambda}\phi^2\left(\lambda^2 - \lambda\frac{24\psi\varpi D_{\psi_1}R_\Omega}{\varpi^2\varpi_0^2}\frac{1}{\phi} \right) \geq \frac{\varpi^2\varpi_0^2}{2}\left(\frac{\delta}{r_0} \right)^{2\lambda}\phi^2\left(\lambda^2 - \frac{\lambda\psi}{\phi} \right)$$
$$\geq \frac{\varpi^2\varpi_0^2}{2}\left(\frac{\delta}{r_0} \right)^{2\lambda}\phi^2\left(\lambda^2 - 1 \right)$$

for ϖ as in (3.7). Therefore we can conclude

$$|\nabla\tau|^2 \geq \lambda^2\frac{\varpi^2\varpi_0^2}{4}\left(\frac{\delta}{r_0} \right)^{2\lambda}\phi^2,$$

which implies (A.34), again for ϖ as in (3.7). $\qquad\qquad\square$

A.1. Proof of the lemmas from Section 6

Proof of Lemma 6.1. To simplify the presentation, we define

$$S_1 := \Delta z, \qquad S_2 := \frac{\mu}{\delta^2} z, \qquad S_3 := \left(R\sigma_t + R^2|\nabla\sigma|^2\right) z,$$

$$A_1 := z_t, \qquad A_2 := 2R\,\nabla\sigma \cdot \nabla z, \qquad A_3 := R\Delta\sigma(1+\alpha)z,$$

and we denote by $I_{i,j}$, $i, j = 1, 2, 3$, the scalar product $\langle S_i, A_j \rangle$. We compute each term separately. Moreover, the computations for $I_{1,j}$ and $I_{3,j}$, $j = 1, 2, 3$, are the same as in [11, Lemma 2.4] and we will omit them here.

Computations for $I_{2,1}$. Due to the boundary conditions (6.3), we immediately have

$$I_{2,1} = \frac{\mu}{2} \int_Q \frac{\partial_t(z^2)}{\delta^2}\,dxdt = \frac{\mu}{2} \int_\Omega \frac{z^2}{\delta^2}\Big|_0^T\,dx - \frac{\mu}{2} \int_Q z^2 \partial_t\left(\frac{1}{\delta^2}\right)dxdt = 0.$$

Computations for $I_{2,2}$. Applying integration by parts and (6.5) we have

$$I_{2,2} = R\mu \int_Q \frac{1}{\delta^2}\left(\nabla\sigma \cdot \nabla(z^2)\right)dxdt = -R\mu \int_Q \Delta\sigma \frac{z^2}{\delta^2}\,dxdt + 2R\mu \int_Q (\nabla\delta \cdot \nabla\sigma)\frac{z^2}{\delta^3}\,dxdt.$$

Computations for $I_{2,3}$.

$$I_{2,3} = R\mu \int_Q \Delta\sigma(1+\alpha)\frac{z^2}{\delta^2}\,dxdt.$$

Identity (6.8) follows immediately $\qquad\qquad\qquad\square$

Proof of Lemma 6.2. It is sufficient to prove that $\nabla\sigma \cdot n = 0$ for all $(x, t) \in \Gamma \times (0, T)$ and $\lambda > 1$. First of all, we have

$$\nabla\sigma = \theta\left(-2\delta\psi\nabla\delta - \delta^2\nabla\psi - \frac{\lambda}{r_0^\lambda}\left(\delta^{\lambda-1}\nabla\delta + \delta^\lambda\nabla\psi\right)\phi\right).$$

Moreover, because of the assumptions we made on the function ψ, for any $x \in \Gamma$ we have $\nabla\psi \cdot n = -|\nabla\psi|$; furthermore, it is a classical property of the distance function that $\nabla\delta \cdot n = -1$. Therefore,

$$\nabla\sigma \cdot n = \theta\left(-2\delta\psi(\nabla\delta \cdot n) + \delta^2|\nabla\psi| - \frac{\lambda}{r_0^\lambda}\left(\delta^{\lambda-1}\nabla\delta \cdot n - \delta^\lambda|\nabla\psi|\right)\phi\right)$$

$$= \theta\left(2\delta + \delta^2|\nabla\psi| + \frac{\lambda}{r_0^\lambda}\delta^{\lambda-1}\left(1 + \delta|\nabla\psi|\right)\phi\right).$$

It is thus evident that, for any $\lambda > 1$, $\nabla\sigma \cdot n = 0$ on $\Gamma \times (0, T)$. $\qquad\qquad\square$

Proof of Lemma 6.3. We split I_l in two parts, $I_l = I_l^1 + I_l^2$, where

$$I_l^1 = -2R \int_Q D^2\sigma(\nabla z, \nabla z)\,dxdt - R \int_Q \alpha\Delta\sigma|\nabla z|^2\,dxdt + 2R\mu \int_Q (\nabla\delta \cdot \nabla\sigma)\frac{z^2}{\delta^3}\,dxdt, \quad (\text{A.36})$$

$$I_l^2 = -\frac{R}{2}\int_Q \Delta^2\sigma(1+\alpha)z^2\,dxdt + R\int_Q (\nabla(\Delta\sigma)\cdot\nabla\alpha)\,z^2\,dxdt + \frac{R}{2}\int_Q \Delta\sigma\Delta\alpha z^2\,dxdt$$

$$+ R\mu\int_Q \alpha\Delta\sigma\frac{z^2}{\delta^2}\,dxdt. \tag{A.37}$$

Moreover, we also split $I_l^1 = I_{l,\delta}^1 + I_{l,\phi}^1$ where

$$I_{l,\delta}^1 = -2R\int_Q D^2\sigma_\delta(\nabla z,\nabla z)\,dxdt - R\int_Q \alpha\Delta\sigma_\delta|\nabla z|^2\,dxdt + 2R\mu\int_Q (\nabla\delta\cdot\nabla\sigma_\delta)\frac{z^2}{\delta^3}\,dxdt, \tag{A.38}$$

$$I_{l,\phi}^1 = -2R\int_Q D^2\sigma_\phi(\nabla z,\nabla z)\,dxdt - R\int_Q \alpha\Delta\sigma_\phi|\nabla z|^2\,dxdt + 2R\mu\int_Q (\nabla\delta\cdot\nabla\sigma_\phi)\frac{z^2}{\delta^3}\,dxdt. \tag{A.39}$$

Estimates for $I_{l,\delta}^1$. From (A.5) and (A.6) we have

$$I_{l,\delta}^1 = 4R\int_Q \theta\psi(\nabla\delta\cdot\nabla z)^2\,dxdt + 4R\int_Q \theta\psi\delta D^2\delta(\nabla z,\nabla z)\,dxdt$$

$$+ 8R\int_Q \theta\delta(\nabla\delta\cdot\nabla z)(\nabla\psi\cdot\nabla z)\,dxdt + R\int_Q \theta\delta^2 D^2\psi(\nabla z,\nabla z)\,dxdt - R\int_Q \alpha\Delta\sigma_\delta|\nabla z|^2\,dxdt$$

$$- 4R\mu\int_Q \theta\psi\frac{z^2}{\delta^2}\,dxdt - 2R\mu\int_Q \theta(\nabla\delta\cdot\nabla\psi)\frac{z^2}{\delta}\,dxdt.$$

Hence

$$I_{l,\delta}^1 \geq -4R\int_Q \theta\psi(\nabla\delta\cdot\nabla z)^2\,dxdt + 4R\int_Q \theta\psi\delta D^2\delta(\nabla z,\nabla z)\,dxdt$$

$$+ 8R\int_Q \theta\delta(\nabla\delta\cdot\nabla z)(\nabla\psi\cdot\nabla z)\,dxdt + R\int_Q \theta\delta^2 D^2\psi(\nabla z,\nabla z)\,dxdt - R\int_Q \alpha\Delta\sigma_\delta|\nabla z|^2\,dxdt$$

$$- 4R\mu\int_Q \theta\psi\frac{z^2}{\delta^2}\,dxdt - 2R\mu\int_Q \theta(\nabla\delta\cdot\nabla\psi)\frac{z^2}{\delta}\,dxdt$$

$$\geq 4R\int_Q \theta\psi\left(|\nabla z|^2 - \mu\frac{z^2}{\delta^2}\right)dxdt - 8R\int_Q \theta\psi|\nabla z|^2\,dxdt + 4R\int_Q \theta\psi\delta D^2\delta(\nabla z,\nabla z)\,dxdt$$

$$+ 8R\int_Q \theta\delta(\nabla\delta\cdot\nabla z)(\nabla\psi\cdot\nabla z)\,dxdt + R\int_Q \theta\delta^2 D^2\psi(\nabla z,\nabla z)\,dxdt - R\int_Q \alpha\Delta\sigma_\delta|\nabla z|^2\,dxdt$$

$$- 2R\mu\int_Q \theta(\nabla\delta\cdot\nabla\psi)\frac{z^2}{\delta}\,dxdt.$$

Therefore,

$$I_{l,\delta}^1 \geq 4R\int_Q \theta\psi\left(|\nabla z|^2 - \mu\frac{z^2}{\delta^2}\right)dxdt - 8R\int_Q \theta\psi|\nabla z|^2\,dxdt - 4R|D^2\delta|_\infty\int_Q \theta\psi|\nabla z|^2\,dxdt$$

$$- 8R|D\psi|_\infty R_\Omega\int_Q \theta|\nabla z|^2\,dxdt - R|D^2\psi|_\infty R_\Omega^2\int_Q \theta|\nabla z|^2\,dxdt - R\int_Q \alpha\Delta\sigma_\delta|\nabla z|^2\,dxdt$$

$$- 2R\mu\int_Q \theta(\nabla\delta\cdot\nabla\psi)\frac{z^2}{\delta}\,dxdt$$

$$\geq 4R \int_Q \theta\psi \left(|\nabla z|^2 - \mu\frac{z^2}{\delta^2} \right) dxdt - RM_1 \int_Q \theta|\nabla z|^2 \, dxdt - R \int_Q \alpha\Delta\sigma_\delta|\nabla z|^2 \, dxdt$$

$$- 2R\mu \int_Q \theta(\nabla\delta \cdot \nabla\psi)\frac{z^2}{\delta} \, dxdt.$$

where $M_1 = M_1(\mu, \psi, \Omega)$ is a positive constant.

Next, we estimate the first term in the expression above applying the Hardy-Poincaré inequality (2.4). First of all, by integration by parts we obtain the identities

$$\int_\Omega z(\nabla\psi \cdot \nabla z) \, dx = -\frac{1}{2} \int_\Omega z^2 \Delta\psi \, dx$$

$$\int_\Omega \delta^{2-\gamma} z(\nabla\psi \cdot \nabla z) \, dx = -\frac{1}{2} \int_\Omega \delta^{2-\gamma} \Delta\psi z^2 \, dx - \frac{2-\gamma}{2} \int_\Omega \delta^{1-\gamma}(\nabla\delta \cdot \nabla\psi) \, dx.$$

Secondly, we apply (2.4) for $u := z\sqrt{\psi}$ and, after integrating in time, we get

$$A_4 \int_Q \theta\psi z^2 \, dxdx + \int_Q \theta\psi \left(|\nabla z|^2 - \mu\frac{z^2}{\delta^2} \right) dxdt + \frac{1}{4} \int_Q \theta\frac{|\nabla\psi|^2}{\psi} z^2 \, dxdt - \frac{1}{2} \int_Q \theta z^2 \Delta\psi \, dxdt$$

$$\geq A_5 \int_Q \theta\psi \left(\delta^{2-\gamma}|\nabla z|^2 + A_1\frac{z^2}{\delta^\gamma} \right) dxdt + \frac{A_5}{4} \int_Q \theta\delta^{2-\gamma}\frac{|\nabla\psi|^2}{\psi} z^2 \, dxdt - \frac{A_5}{2} \int_Q \theta\delta^{2-\gamma} z^2 \Delta\psi \, dxdt$$

$$- A_5\frac{2-\gamma}{2} \int_Q \theta\delta^{1-\gamma}(\nabla\delta \cdot \nabla\psi)z^2 \, dxdt,$$

where A_5 and A_5 are the constants of Proposition 2.4. Now, for r_0 as in (3.3) we have

$$\frac{A_5\psi}{4\delta^\gamma} \geq \frac{A_5}{2}(2-\gamma)\delta^{1-\gamma}|D\psi|_\infty, \quad \forall x \in \Omega_{r_0};$$

therefore,

$$\frac{A_5}{2} \int_Q \theta\psi \left(\delta^{2-\gamma}|\nabla z|^2 + \frac{1}{2}\frac{z^2}{\delta^\gamma} \right) dxdt - A_5\frac{2-\gamma}{2} \int_Q \theta\delta^{1-\gamma}(\nabla\delta \cdot \nabla\psi)z^2 \, dxdt$$

$$\geq -\frac{A_5}{2}(2-\gamma)|D\psi|_\infty \left| \sup_{\delta > r_0} \delta^{1-\gamma} \right| \int_{\tilde{O}\times(0,T)} \theta z^2 \, dxdt;$$

combing the two expressions above, we finally obtain

$$\int_Q \theta\psi \left(|\nabla z|^2 - \mu^*\frac{z^2}{\delta^2} \right) dxdt \geq \frac{A_5}{2} \int_Q \theta\psi \left(\delta^{2-\gamma}|\nabla z|^2 + \frac{1}{2}\frac{z^2}{\delta^\gamma} \right) dxdt - A_6 \int_Q \theta z^2 \, dxdx,$$

where

$$A_6 := \frac{A_5}{4} \left(R_\Omega^{2-\gamma}|D\psi|_\infty^2 + 2R_\Omega^{2-\gamma} + 2(2-\gamma)|D\psi|_\infty \left| \sup_{\delta > r_0} \delta^{1-\gamma} \right| \right).$$

Therefore

$$I_{l,\delta}^1 \geq M_2 R \int_Q \theta\psi \left(\delta^{2-\gamma}|\nabla z|^2 + \frac{z^2}{\delta^\gamma} \right) dxdt - RM_1 \int_Q \theta|\nabla z|^2 \, dxdt - R \int_Q \alpha\Delta\sigma_\delta|\nabla z|^2 \, dxdt$$

$$- 2R\mu \int_Q \theta(\nabla\delta \cdot \nabla\psi)\frac{z^2}{\delta} \, dxdt - A_6 R \int_Q \theta z^2 \, dxdx.$$

Since $\gamma > 1$, for r_0 as in (3.3) we have

$$\frac{2|\mu||D\psi|_\infty}{\delta} \leq \frac{M_2}{2\delta^\gamma}, \quad \forall x \in \Omega_{r_0};$$

knowing this, we can finally conclude

$$I^1_{l,\delta} \geq B_1 R \int_Q \theta\psi \left(\delta^{2-\gamma}|\nabla z|^2 + \frac{z^2}{\delta^\gamma}\right) dxdt - RM_1 \int_Q \theta|\nabla z|^2 \, dxdt - R \int_Q \alpha\Delta\sigma_\delta|\nabla z|^2 \, dxdt$$

$$- A_6 R \int_Q \theta z^2 \, dxdx, \tag{A.40}$$

where $B_1 := M_2/2$.

Estimates for $I^1_{l,\phi}$. In order to get rid of the gradient terms with negative signs in (A.40), we introduce the quantity

$$\mathcal{T} = I^1_{l,\phi} - R \int_Q \alpha\Delta\sigma_\delta|\nabla z|^2 \, dxdt - RM_1 \int_Q \theta|\nabla z|^2 \, dxdt$$

$$= - 2R \int_Q D^2\sigma_\phi(\nabla z, \nabla z) \, dxdt - R \int_Q \alpha\Delta\sigma_\phi|\nabla z|^2 \, dxdt + 2R\mu \int_Q (\nabla\delta \cdot \nabla\sigma_\phi)\frac{z^2}{\delta^3} \, dxdt$$

$$- R \int_Q \alpha\Delta\sigma_\delta|\nabla z|^2 \, dxdt - RM_1 \int_Q \theta|\nabla z|^2 \, dxdt \tag{A.41}$$

and we need to estimate it from below. To do that, according to Propositions A.1 and A.2 we remark that

$$2D^2\tau_\phi(\nabla z, \nabla z) + \alpha\Delta\tau_\phi|\nabla z|^2 + \alpha\Delta\tau_\delta|\nabla z|^2 \geq \lambda\left(\frac{\delta}{r_0}\right)^{\lambda-2} \phi|\nabla z|^2, \qquad \forall x \in \Omega_{r_0},$$

$$\left|2D^2\tau_\phi(\nabla z, \nabla z) + \alpha\Delta\tau_\phi|\nabla z|^2 + (\alpha\Delta\tau_\delta - M_1)|\nabla z|^2\right| \leq M_2\lambda^2\left(\frac{\delta}{r_0}\right)^\lambda \phi|\nabla z|^2, \qquad \forall x \in \omega_0,$$

$$2D^2\tau_\phi(\nabla z, \nabla z) + \alpha\Delta\tau_\phi|\nabla z|^2 + (\alpha\Delta\tau_\delta - M_1)|\nabla z|^2 \geq M_3\lambda^2\left(\frac{\delta}{r_0}\right)^\lambda \phi|\nabla z|^2, \qquad \forall x \in \mathcal{O},$$

for λ large enough and for some positive constants M_2 and M_3 not depending on λ. On the other hand, there exists a positive constant M_4, again not depending on λ, such that it holds

$$\left|\frac{2|\mu||(\nabla\delta \cdot \nabla\tau_\phi)|}{\delta^3}\right| \leq M_4\lambda\left(\frac{\delta}{r_0}\right)^{\lambda-4} \phi, \quad \forall x \in \Omega.$$

Therefore it follows

$$\mathcal{T} \geq \frac{\lambda R}{2} \int\limits_{\Omega_{r_0}\times(0,T)} \theta\left(\frac{\delta}{r_0}\right)^{\lambda-2} |\nabla z|^2 \, dxdt - M_2\lambda^2 R \int\limits_{\omega_0\times(0,T)} \theta\left(\frac{\delta}{r_0}\right)^\lambda \phi|\nabla z|^2 \, dxdt$$

$$+ M_3\lambda^2 R \int_{\mathcal{O}\times(0,T)} \theta\left(\frac{\delta}{r_0}\right)^\lambda \phi|\nabla z|^2 \, dxdt - M_4\lambda R \int_Q \theta\left(\frac{\delta}{r_0}\right)^{\lambda-4} \phi z^2 \, dxdt,$$

for λ large enough. Joining the two expression obtained for $I_{l,\delta}^1$ and $\mathcal{T}$ we finally have

$$I_l^1 \geq B_1 R \int_Q \theta\psi\left(\delta^{2-\gamma}|\nabla z|^2 + \frac{z^2}{\delta^\gamma}\right) dxdt - A_6 R \int_Q \theta z^2 \, dxdx + \frac{\lambda R}{2} \int_{\Omega_{r_0}\times(0,T)} \theta\left(\frac{\delta}{r_0}\right)^{\lambda-2} |\nabla z|^2 \, dxdt$$

$$- B_2\lambda^2 R \int_{\omega_0\times(0,T)} \theta\left(\frac{\delta}{r_0}\right)^\lambda \phi|\nabla z|^2 \, dxdt + B_3\lambda^2 R \int_{\mathcal{O}\times(0,T)} \theta\left(\frac{\delta}{r_0}\right)^\lambda \phi|\nabla z|^2 \, dxdt$$

$$- M_5\lambda R \int_Q \theta\left(\frac{\delta}{r_0}\right)^{\lambda-4} \phi z^2 \, dxdt, \tag{A.42}$$

Estimates for I_l^2. Using the fact that the support of α is located away from the origin, we note that

$$\left|\alpha\frac{\Delta\tau_\delta}{\delta^2}\right|, \quad \left|\alpha\frac{\Delta\tau_\psi}{\delta^2}\right|, \quad |\Delta\alpha\Delta\tau_\delta|, \quad |\Delta\alpha\Delta\tau_\psi|, \quad |\nabla(\Delta\tau_\delta)\cdot\nabla\alpha|, \quad |\nabla(\Delta\tau_\psi)\cdot\nabla\alpha|, \quad |\Delta^2\tau_\delta| \leq A_\lambda, \quad \forall x \in \Omega.$$

Moreover, there exists a positive constant Υ such that

$$|\Delta^2\tau_\delta(1+\alpha)| \leq \frac{2\Upsilon}{\delta^2}, \quad \forall x \in \Omega.$$

Hence

$$I_l^2 \geq -A_\lambda R \int_Q \theta z^2 \, dxdt - \Upsilon R \int_Q \theta|\nabla z|^2 \, dxdt$$

and, for λ large enough, we finally have (6.13) with $B_\lambda := A_\lambda + A_6 + M_5\lambda \sup_{\in\Omega}\{(\delta/r_0)^{\lambda-4}\phi\}$. $\square$

Proof of Lemma 6.4. We split $I_{nl} = I_{nl,1} + I_{nl,2}$, where $I_{nl,1}$ indicates the integrals in I_{nl} restricted to Ω_{r_0}, while $I_{nl,2}$ are the terms in I_{nl} restricted to $\tilde{\mathcal{O}}$. Moreover, if we put $\sigma = -\theta\tau$, then I_{nl} can be rewritten as

$$I_{nl} = 2R^3 \int_Q \theta^3 D^2\tau(\nabla\tau, \nabla\tau)z^2 \, dxdt - R^3 \int_Q \theta^3\alpha\Delta\tau|\nabla\tau|^2 z^2 \, dxdt - \frac{R^2}{2} \int_Q \theta^2\alpha^2|\Delta\tau|^2 z^2 \, dxdt.$$

Computations for $I_{nl,1}$. From (A.28), (A.30) and (A.32), for any $x \in \Omega_{r_0}$ we have

$$T_2 + T_3 \geq \lambda^2\left(\frac{\phi}{r_0^\lambda}\delta^{\lambda-2} + \left(\frac{\delta}{r_0}\right)^\lambda \phi\right)|\nabla\tau|^2 - \lambda^2\frac{\phi}{r_0^\lambda}|D\psi|_\infty^2\left(D_3\psi^2 + D_4\right)\delta^{\lambda+2}$$

$$= \lambda^2\frac{\phi}{r_0^\lambda}\delta^{\lambda-2}\left(|\nabla\tau|^2 + \delta^2|\nabla\tau|^2 - |D\psi|_\infty^2\left(D_3\psi^2 + D_4\right)\delta^4\right)$$

$$\geq \lambda^2\frac{\phi}{r_0^\lambda}\delta^\lambda\left(1 - |D\psi|_\infty^2\left(D_3\psi^2 + D_4\right)\delta^2\right) \geq \lambda^2\frac{\phi}{r_0^\lambda}\delta^\lambda\left(1 - |D\psi|_\infty^2\left(D_3\psi^2 + D_4\right)r_0^2\right) \geq 0$$

for r_0 as in (3.3). Hence, using (A.25) and (A.32) we conclude

$$2D^2\tau(\nabla\tau,\nabla\tau) - \alpha\Delta\tau|\nabla\tau|^2 \geq \delta^2, \quad \forall x \in \Omega_{r_0};$$

as a consequence,

$$I_{nl,1} \geq R^3 \int\limits_{\Omega_{r_0}\times(0,T)} \theta^3\delta^2 z^2 \, dxdt - \frac{R^2}{2} \int\limits_{\Omega_{r_0}\times(0,T)} \theta^2\alpha^2|\Delta\tau|^2 z^2 \, dxdt.$$

Moreover, since α is supported away from the boundary we also have

$$\alpha^2|\Delta\tau|^2 \leq A'_\lambda\delta^2, \quad \forall x \in \Omega_{r_0};$$

hence, finally, there exists $R_0 = R_0(\lambda)$ large enough such that, for any $R \geq R_0$

$$I_{nl,1} \geq \frac{R^3}{2} \int\limits_{\Omega_{r_0}\times(0,T)} \theta^3\delta^2 z^2 \, dxdt.$$

Computations for $I_{nl,2}$. According to Propositions A.3, A.4 and A.5 and to (A.33), for all $x \in \mathcal{O}$ we have

$$2D^2\tau(\nabla\tau,\nabla\tau) - \alpha\Delta\tau|\nabla\tau|^2 \geq G_1\lambda^2 \left(\frac{\delta}{r_0}\right)^\lambda \phi|\nabla\tau|^2 \geq G_1\lambda^4 \left(\frac{\delta}{r_0}\right)^{3\lambda} \phi^3.$$

In addition, it holds

$$\alpha^2|\Delta\tau|^2 \leq G_2\lambda^4 \left(\frac{\delta}{r_0}\right)^{2\lambda} \phi^2, \qquad\qquad\qquad \forall x \in \tilde{\mathcal{O}},$$

$$\left|2D^2\tau(\nabla\tau,\nabla\tau) - \alpha\Delta\tau|\nabla\tau|^2\right| \leq G_3\lambda^2 \left(\frac{\delta}{r_0}\right)^\lambda \phi|\nabla\tau|^2 \leq G_4\lambda^4 \left(\frac{\delta}{r_0}\right)^{3\lambda} \phi^3, \qquad \forall x \in \omega_0.$$

The previous inequalities follows from (A.20), (A.21) and (A.34); the constants G_1, G_2, G_3 and G_4 are all positive and independent on λ. Therefore we obtain

$$I_{nl,2} \geq G_1\lambda^4 R^3 \int\limits_{\mathcal{O}\times(0,T)} \theta^3 \left(\frac{\delta}{r_0}\right)^{3\lambda} \phi^3 z^2 \, dxdt - G_4\lambda^4 R^3 \int\limits_{\omega_0\times(0,T)} \theta^3 \left(\frac{\delta}{r_0}\right)^{3\lambda} \phi^3 z^2 \, dxdt$$

$$- \frac{G_2}{2}\lambda^4 R^2 \int\limits_{\tilde{\mathcal{O}}\times(0,T)} \theta^2 \left(\frac{\delta}{r_0}\right)^{2\lambda} \phi^2 \, dxdt.$$

Joining now the two expressions we get for $I_{nl,1}$ and $I_{nl,2}$, we finally obtain that there exists $R_0 = R_0(\lambda)$ large enough such that for $R \geq R_0$

$$I_{nl} \geq \frac{R^3}{2} \int\limits_{\Omega_{r_0}\times(0,T)} \theta^3\delta^2 z^2 \, dxdt + G_5\lambda^4 R^3 \int\limits_{\mathcal{O}\times(0,T)} \theta^3 \left(\frac{\delta}{r_0}\right)^{3\lambda} \phi^3 z^2 \, dxdt$$

$$- G_6\lambda^4 R^3 \int\limits_{\omega_0\times(0,T)} \theta^3 \left(\frac{\delta}{r_0}\right)^{3\lambda} \phi^3 z^2 \, dxdt,$$

where $G_5 := G_1/2$ and $G_6 := G_2/2 + G_4$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Lemma 6.5. According to the expression of θ, there exists a constant $\varsigma > 0$ such that

$$|\theta\theta_t| \leq \varsigma\theta^3, \quad |\theta_{tt}| \leq \varsigma\theta^{5/3};$$

on the other hand, from the definition of σ we obtain

$$\begin{aligned}
|\Delta\sigma| &\leq E_\lambda\theta, \quad |\sigma_t| \leq E_\lambda\theta_t, & \forall x \in \Omega, \\
\partial_t\left(|\nabla\sigma|^2\right) &\leq E_\lambda\theta\theta_t\delta^2, & \forall x \in \Omega_{r_0}, \\
\partial_t\left(|\nabla\sigma|^2\right) &\leq E_\lambda\theta\theta_t\left(\frac{\delta}{r_0}\right)^{2\lambda}\phi^2 & \forall x \in \tilde{\mho}, & \qquad (A.43)
\end{aligned}$$

for some positive constant E_λ big enough.

Since α is supported away from the boundary, we can write

$$R^2 \int_Q \left|\alpha\sigma_t\Delta\sigma z^2\right| dxdt \leq \frac{\varsigma E_\lambda^2}{r_0^2}R^2 \int_{\Omega_{r_0}\times(0,T)} \theta^3\delta^2 z^2 \, dxdt + \varsigma E_\lambda^2 R^2 \int_{\tilde{\mho}\times(0,T)} \theta^3\delta^2 z^2 \, dxdt.$$

Furthermore, from (A.43) we obtain

$$R^2\left|\int_Q \partial_t\left(|\nabla\sigma|^2\right)z^2 \, dxdt\right| \leq \varsigma E_\lambda R^2 \int_{\Omega_{r_0}\times(0,T)} \theta^3\delta^2 z^2 \, dxdt + \varsigma E_\lambda R^2 \int_{\tilde{\mho}\times(0,T)} \theta^3\left(\frac{\delta}{r_0}\right)^{2\lambda}\phi^2 z^2 \, dxdt.$$

Now we define

$$\Theta := -\frac{R}{2}\int_Q \sigma_{tt}z^2 \, dxdt - B_\lambda R\int_Q \theta z^2 \, dxdt,$$

where B_λ is the same introduced in Lemma 6.3. It is straightforward that there exists a positive constant F_λ such that

$$|\Theta| \leq 2F_\lambda R\int_Q \theta^{5/3}z^2 \, dxdt.$$

Next, for $1 < q, q' < \infty$ such that $1/q + 1/q' = 1$ and $\ell > 0$ we can write

$$\int_Q \theta^{5/3}z^2 \, dxdt = \int_Q \left(\ell\theta^{5/3-1/q'}\delta^{1/q'}z^{2/q}\right)\left(\frac{1}{\ell}\theta^{1/q'}\delta^{-1/q'}z^{2/q'}\right) dxdt;$$

choosing $q = 3$ and $q' = 3/2$ in the previous expression, and using Young's inequality, we obtain

$$\int_Q \theta^{5/3}z^2 \, dxdt \leq \frac{\ell^3}{3}\int_Q \theta^3\delta^2 z^2 \, dxdt + \frac{2R_\Omega^{\gamma-1}}{3\ell^{3/2}}\int_Q \theta\frac{z^2}{\delta^\gamma} \, dxdt,$$

for some positive parameter $\gamma \in (1, 2)$. Therefore we have

$$|\Theta| \leq 2F_\lambda R\left(\frac{\ell^3}{3}\int_Q \theta^3\delta^2 z^2 \, dxdt + \frac{2R_\Omega^{\gamma-1}}{3\ell^{3/2}}\int_Q \theta\frac{z^2}{\delta^\gamma} \, dxdt\right).$$

Consequently, it follows that

$$|I_r| \le G_\lambda \left(R^2 \int_{\Omega_{r_0} \times (0,T)} \theta^3 \delta^2 z^2 \, dxdt \right.$$

$$\left. + \ell^3 R \int_Q \theta^3 \delta^2 z^2 \, dxdt + \frac{R}{\ell^{3/2}} \int_Q \theta \frac{z^2}{\delta^\gamma} \, dxdt + R^2 \int_{\tilde{\mathcal{O}}} \theta^3 \left(\frac{\delta}{r_0} \right)^{2\lambda} z^2 \, dxdt \right),$$

for some new constant $G_\lambda > 0$. Take now ℓ such that $G_\lambda / \ell^{3/2} = B_1/2$; then there exists $R_0 = R_0(\lambda)$ such that for any $R \ge R_0$ (6.16) holds.

We conclude pointing out that, if we choose an exponent $k < 3$ for the function θ in the definition of our weight σ (see Section 3), it is straightforward to check that some of the passages in the computations above are not true anymore and there are terms in the expression I_r that we are not able to handle. Therefore, the value $k = 3$ turns out to be sharp for obtaining our Carleman inequality. $\qquad \square$

B. Proof of the Propositions of Section 2

Proof of Proposition 2.3. We split the proof in two parts: firstly, we derive (2.3) in Ω_{r_0} and, in a second moment, we extend the result to the whole Ω.

Step 1. inequality on Ω_{r_0}. Let us consider a smooth function $\phi > 0$ which satisfies

$$-\Delta\phi \ge \mu \frac{\phi}{\delta^2} + \phi^p, \quad \forall p \in \left[1, \frac{N-k+2}{N-k-2} \right), \tag{B.1}$$

for $k \in (1, N-2)$. According to [13], for $\delta < 1$ the function

$$\delta^{-A_k^{1/2}(1-\delta^{1/2})} \left(1 + \frac{1}{\log \delta} \right), \quad A_k := \left(\frac{N-k-2}{2} \right)^2 \tag{B.2}$$

satisfies (B.1). Hence, for any $x \in \Omega_{r_0}$ with $r_0 \le 1$ we define $v := \phi z$ for $z \in C_0^\infty(\Omega_{r_0})$; in particular, $v \in C_0^\infty(\Omega_{r_0})$ and

$$|\nabla v|^2 = \phi^2 |\nabla z|^2 + z^2 |\nabla \phi|^2 + \frac{1}{2} \nabla(\phi^2) \cdot \nabla(z^2).$$

By applying integration by parts, it is simply a matter of computations to show

$$\int_{\Omega_{r_0}} |\nabla v|^2 \, dx = \int_{\Omega_{r_0}} \phi^2 |\nabla z|^2 \, dx - \int_{\Omega_{r_0}} \frac{\Delta\phi}{\phi} v^2 \, dx$$

and

$$\frac{1}{2} \int_{\Omega_{r_0}} \delta^{2-\gamma} \nabla(\phi^2) \cdot \nabla(z^2) \, dx = -(2-\gamma) \int_{\Omega_{r_0}} \delta^{1-\gamma} \frac{\nabla\phi \cdot \nabla\delta}{\phi} v^2 \, dx - \int_{\Omega_{r_0}} \delta^{2-\gamma} \frac{\Delta\phi}{\phi} v^2 \, dx$$

$$- \int_{\Omega_{r_0}} \delta^{2-\gamma} |\nabla\phi|^2 z^2 \, dx.$$

The two identities above implies

$$\int_{\Omega_{r_0}} \delta^{2-\gamma}\phi^2|\nabla z|^2\,dx \le R_\Omega^{2-\gamma}\int_{\Omega_{r_0}} \phi^2|\nabla z|^2\,dx = R_\Omega^{2-\gamma}\int_{\Omega_{r_0}} \left(|\nabla v|^2 + \frac{\Delta\phi}{\phi}v^2\right)dx$$

$$\le R_\Omega^{2-\gamma}\int_{\Omega_{r_0}} \left(|\nabla v|^2 - \mu\frac{v^2}{\delta^2}\right)dx - R_\Omega^{2-\gamma}\int_{\Omega_{r_0}} \phi^{p-1}v^2\,dx$$

and

$$\int_{\Omega_{r_0}} \delta^{2-\gamma}|\nabla v|^2\,dx = \int_{\Omega_{r_0}} \delta^{2-\gamma}\phi^2|\nabla z|^2\,dx - (2-\gamma)\int_{\Omega_{r_0}} \delta^{1-\gamma}\frac{\nabla\phi\cdot\nabla\delta}{\phi}v^2\,dx - \int_{\Omega_{r_0}} \delta^{2-\gamma}\frac{\Delta\phi}{\phi}v^2\,dx;$$

hence

$$\int_{\Omega_{r_0}} \delta^{2-\gamma}|\nabla v|^2\,dx \le R_\Omega^{2-\gamma}\int_{\Omega_{r_0}} \left(|\nabla v|^2 - \mu\frac{v^2}{\delta^2}\right)dx - R_\Omega^{2-\gamma}\int_{\Omega_{r_0}} \phi^{p-1}v^2\,dx + \mu\int_{\Omega_{r_0}} \delta^{2-\gamma}\frac{v^2}{\delta^2}\,dx$$

$$+ \int_{\Omega_{r_0}} \delta^{2-\gamma}\phi^{p-1}v^2\,dx - (2-\gamma)\int_{\Omega_{r_0}} \delta^{1-\gamma}\frac{\nabla\phi\cdot\nabla\delta}{\phi}v^2\,dx.$$

Now, again by integration by parts we have

$$-(2-\gamma)\int_{\Omega_{r_0}} \delta^{1-\gamma}\frac{\nabla\phi\cdot\nabla\delta}{\phi}v^2\,dx$$

$$= \int_{\Omega_{r_0}} \delta^{2-\gamma}\frac{\Delta\phi}{\phi}v^2\,dx - \int_{\Omega_{r_0}} \frac{\delta^{2-\gamma}}{\phi^2}|\nabla\phi|^2 v^2\,dx + 2\int_{\Omega_{r_0}} \delta^{2-\gamma}\frac{\nabla\phi\cdot\nabla v}{\phi}v\,dx$$

$$\le -\mu\int_{\Omega_{r_0}} \delta^{2-\gamma}\frac{v^2}{\delta^2}\,dx - \int_{\Omega_{r_0}} \delta^{2-\gamma}\phi^{p-1}v^2\,dx + 2\int_{\Omega_{r_0}} \delta^{2-\gamma}\frac{\nabla\phi\cdot\nabla v}{\phi}v\,dx;$$

therefore

$$\int_{\Omega_{r_0}} \delta^{2-\gamma}|\nabla v|^2\,dx$$

$$\le R_\Omega^{2-\gamma}\int_{\Omega_{r_0}} \left(|\nabla v|^2 - \mu\frac{v^2}{\delta^2}\right)dx - R_\Omega^{2-\gamma}\int_{\Omega_{r_0}} \phi^{p-1}v^2\,dx + 2\int_{\Omega_{r_0}} \delta^{2-\gamma}\frac{\nabla\phi\cdot\nabla v}{\phi}v\,dx$$

$$\le R_\Omega^{2-\gamma}\int_{\Omega_{r_0}} \left(|\nabla v|^2 - \mu\frac{v^2}{\delta^2}\right)dx + P_1\int_{\Omega_{r_0}} \phi^{p-1}v^2\,dx + 2\int_{\Omega_{r_0}} \delta^{2-\gamma}\frac{\nabla\phi\cdot\nabla v}{\phi}v\,dx$$

$$\le R_\Omega^{2-\gamma}\int_{\Omega_{r_0}} \left(|\nabla v|^2 - \mu\frac{v^2}{\delta^2}\right)dx + P_2\int_{\Omega_{r_0}} v^2\,dx + 2\int_{\Omega_{r_0}} \delta^{2-\gamma}\frac{\nabla\phi\cdot\nabla v}{\phi}v\,dx.$$

By definition of ϕ we have

$$\frac{\nabla\phi\cdot\nabla v}{\phi} = \left(1 + \frac{1}{\log\delta}\right)^{-1}\left(\frac{A_k^{1/2}}{2}\frac{\log\delta}{\delta^{1/2}} - A_k^{1/2}\frac{1-\delta^{1/2}}{\delta} - \frac{1}{\delta\log^2\delta}\right)(\nabla\delta\cdot\nabla v);$$

plugging this expression in the inequality above we immediately get

$$\int_{\Omega_{r_0}} \delta^{2-\gamma}|\nabla v|^2 dx \le R_\Omega^{2-\gamma}\int_{\Omega_{r_0}} \left(|\nabla v|^2 - \mu\frac{v^2}{\delta^2}\right)dx + P_2\int_{\Omega_{r_0}} v^2 dx + P_3\int_{\Omega_{r_0}} \delta^{2-\gamma}\frac{\log\delta}{\delta^{1/2}}(\nabla\delta\cdot\nabla v)v\,dx$$

with

$$P_3 := A_k^{1/2} \left| \sup_{x \in \Omega_{r_0}} \left(1 + \frac{1}{\log \delta} \right)^{-1} \right|.$$

Now, using another time integration by parts, and since $\log \delta < \delta^{3/2}$, we finally obtain

$$\int_{\Omega_{r_0}} \delta^{2-\gamma} |\nabla v|^2 \, dx \leq R_\Omega^{2-\gamma} \int_{\Omega_{r_0}} \left(|\nabla v|^2 - \mu \frac{v^2}{\delta^2} \right) dx + P_2 \int_{\Omega_{r_0}} v^2 \, dx + P_3 \int_{\Omega_{r_0}} \delta^{3-\gamma} (\nabla \delta \cdot \nabla(v^2)) \, dx$$

$$\leq R_\Omega^{2-\gamma} \int_{\Omega_{r_0}} \left(|\nabla v|^2 - \mu \frac{v^2}{\delta^2} \right) dx + A_2 \int_{\Omega_{r_0}} v^2 \, dx,$$

where

$$A_2 := P_2 + P_3 \left[R_\Omega^{2-\gamma}(3 - \gamma) + R_\Omega^{3-\gamma} |\Delta \delta| \right].$$

Step 2. inequality on Ω. We apply a cut-off argument to recover the validity of the inequality on the whole Ω. More in details, we consider a function $\psi \in C_0^\infty(\mathbb{R}^N)$ such that

$$\psi(x) = \begin{cases} 1, & \forall x \in \Omega_{r_0/2}, \\ 0, & \forall x \in \Omega \setminus \Omega_{r_0} \end{cases}$$

and we split $v \in C_0^\infty(\Omega)$ as $v = \psi v + (1 - \psi)v := v_1 + v_2$. Thus, we get

$$\int_\Omega \delta^{2-\gamma} |\nabla v|^2 \, dx = \int_{\Omega_{r_0}} \delta^{2-\gamma} |\nabla v_1|^2 \, dx + \int_{\Omega \setminus \Omega_{r_0/2}} \delta^{2-\gamma} |\nabla v_2|^2 \, dx + 2 \int_{\Omega_{r_0} \setminus \Omega_{r_0/2}} \delta^{2-\gamma} (\nabla v_1 \cdot \nabla v_2) \, dx.$$

Applying (2.3) to the previous identity we obtain

$$\int_\Omega \delta^{2-\gamma} |\nabla v|^2 \, dx \leq R_\Omega^{2-\gamma} \left(\int_\Omega |\nabla v|^2 \, dx - \mu \int_{\Omega_{r_0}} \frac{v^2}{\delta^2} \, dx \right) - \int_{\Omega_{r_0} \setminus \Omega_{r_0/2}} 2 \left(R_\Omega^{2-\gamma} - \delta^{2-\gamma} \right) (\nabla v_1 \cdot \nabla v_2) \, dx$$

$$+ J_1 \int_\Omega v^2 \, dx.$$

As shown in [7, Lemma 5.1], for a smooth function $q : C^\infty(\Omega) \to \mathbb{R}$ which is bounded and non-negative, there exists a constant $C > 0$ depending on Ω and q such that it holds

$$\int_\Omega q(x)(\nabla v_1 \cdot \nabla v_2) \, dx \geq -C \int_\Omega v^2 \, dx; \tag{B.3}$$

hence, considering (B.3) with

$$q = 2 \left(R_\Omega^{2-\gamma} - \delta^{2-\gamma} \right) \Big|_{\Omega_{r_0} \setminus \Omega_{r_0/2}}$$

we get

$$\int_\Omega \delta^{2-\gamma} |\nabla v|^2 \, dx \leq R_\Omega^{2-\gamma} \left(\int_\Omega |\nabla v|^2 \, dx - \mu \int_{\Omega_{r_0}} \frac{v^2}{\delta^2} \, dx \right) + J_2 \int_\Omega v^2 \, dx. \tag{B.4}$$

On the other hand we have

$$\int_{\Omega_{r_0}} \frac{v^2}{\delta^2} \, dx \geq \int_\Omega \frac{v^2}{\delta^2} \, dx - J_3 \int_\Omega v^2 \, dx.$$

Plugging this last inequality in (B.4), we finally obtain (2.3). $\qquad\square$

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