Self-similar stochastic models with stationary increments for symmetric space-time fractional diffusion

Gianni Pagnini
BCAM – Basque Center for Applied Mathematics, Alameda de Mazarredo 14, E–48009 Bilbao,
Ikerbasque, Alameda Urquijo 36–5, Plaza Bizkaia, E–48011 Bilbao, Basque Country – Spain
Email: gpagnini@bcamath.org

***Abstract*—An approach to develop stochastic models for studying anomalous diffusion is proposed. In particular, in this approach the stochastic particle trajectory is based on the fractional Brownian motion but, for any realization, it is multiplied by an independent random variable properly distributed. The resulting probability density function for particle displacement can be represented by an integral formula of subordination type and, in the single-point case, it emerges to be equal to the solution of the spatially symmetric space-time fractional diffusion equation. Due to the fractional Brownian motion, this class of stochastic processes is self-similar with stationary increments in nature and uniquely defined by the mean and the auto-covariance structure analogously to the Gaussian processes. Special cases are the time-fractional diffusion, the space-fractional diffusion and the classical Gaussian diffusion.**

I. Introduction

Space-time fractional diffusion was originally introduced in physics by Zaslavsky to study chaotic Hamiltonian dynamics in low dimensional systems [1], [2], [3], with the specific aim to model the so-called *anomalous diffusion*, see also [4], [5], [6]. The label *anomalous diffusion* is assigned to processes whose particle displacement variance does not grow linearly in time [7], in opposition to Gaussian *normal diffusion* that is mainly characterized by a linear law. In this respect, the terms *subdiffusion* and *superdiffusion* are used for those processes whose variance grows in time slower or faster than linear, respectively. *Anomalous diffusion* has been experimentally observed several times and definitively established in nature not only in chaotic dynamical systems, see e.g. [8], [9], [10], [11], [12]. Zaslavsky argued that, since chaotic dynamics is a physical phenomenon whose evolution bridges between a completely regular integrable system and a completely random process [5], kinetic equations and statistical tools arise as modelling methods. Because in these cases the classical diffusion paradigm based on a local flux-gradient relationship does not hold, a non-local relationship is necessary. Fractional Calculus [13], [14] is emerged to be a useful mathematical tool for modelling such non-local effects. In this framework, non-locality can be considered in time (*time-fractional diffusion*) [15], [16], [17], [18], [19] or in space (*space-fractional diffusion*) [16], [20], [21], [22], as well as both in space and time (*space-time fractional diffusion*) [6], [20], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36]. Moreover, when there is no separation of timescale between the microscopic and the macroscopic level of the process, the randomness of the microscopic level is transmitted to the macroscopic level and the correct description of the macroscopic dynamics has to be in terms of Fractional Calculus [37]. Furthermore, fractional integro/differential equations are related to the fractal properties of phenomena as showed in pioneering publications by Le Méhauté in 1990 [38] and Nigmatullin in 1992 [39], subsequently discussed by many authors [40], [41], [42], [43], and addressed also with a different approach by Rocco & West in 1999 [44]. However, fractional kinetics strongly differs from the usual kinetics because the fluctuations from the equilibrium state have a broad distribution of relaxation time [5], [45], [46] and because some moments of the probability density function (PDF) of particle displacement can be infinite. In fact, usual kinetics, as described by Einstein in 1905, relies on the assumption of the existence of a time interval such that the motion of any particle during different intervals is independent. The macroscopic observation of this motion leads to the known diffusion equation. This random walk picture is valid in processes lacking long-term memory and they are referred to as Markovian. However, many natural processes have a broad distribution of timescales so that the Markovian assumption is violated and they exhibit anomalous diffusion [47], [48]. What concerns particle displacement PDFs, they generally are not Gaussian [45] and mainly characterized by the decay of their tails. In particular, the PDF related to the time-fractional diffusion equation is a M-Wright/Mainardi [49], [50] density that is unimodal (diffusive) and bimodal (wave-like behaviour) in subdiffusion and superdiffusion cases, respectively. In subdiffusion cases, tails decay with a stretched exponential that is fatter than the Gaussian and in superdiffusion cases with a stretched exponential that is thinner than the Gaussian. The PDF obtained by the space-time fractional diffusion equation is a Lévy stable density, that is always unimodal [51] and shows tail decay with a power law.

The *space-time fractional diffusion* equation reads [25]

$$ {}_tD_*^{\beta}\, u(x;t) = {}_xD_\theta^{\alpha}\, u(x;t)\,, \quad x \in R\,, \quad t \in R_0^+\,, \qquad (1) $$

where ${}_tD_*^{\beta}$ is the *Caputo time-fractional derivative* of order β and ${}_xD_\theta^{\alpha}$ is the *Riesz-Feller space-fractional derivative* of order α and asymmetry parameter θ. The real parameters α, θ and β are restricted as follows: $0 < \alpha \le 2$, $|\theta| \le \min\{\alpha, 2-\alpha\}$, $0 < \beta \le 1$ or $1 < \beta \le \alpha \le 2$. Caputo time-fractional derivative ${}_tD_*^\beta$ is defined by its Laplace transform as $\int_0^{+\infty} \mathrm{e}^{-st} \left\{ {}_tD_*^\beta\, u(x;t) \right\} dt = s^\beta\, \widetilde{u}(x;s) - \sum_{n=0}^{m-1} s^{\beta-1-n}\, u^{(n)}(x;0^+)$, with $m-1 < \beta \le m$ and $m \in N$. Riesz–Feller space-fractional derivative ${}_xD_\theta^\theta$ is defined by its Fourier transform according to $\int_{-\infty}^{+\infty} \mathrm{e}^{+i\kappa x} \{ {}_xD_\theta^\alpha\, u(x;t)\}\, dx = -|\kappa|^\alpha\, \mathrm{e}^{i(\mathrm{sign}\,\kappa)\theta\pi/2}\, \widehat{u}(\kappa;t)$.

Solution of (1) can be represented as $u(x;t) = \int_{-\infty}^{+\infty} K_{\alpha,\beta}^\theta(x-\xi;t)\, u(\xi;0)\, d\xi$, where $K_{\alpha,\beta}^\theta(x;t)$ is the fundamental solution, or Green function, which is obtained by equipping (1) with the initial and boundary conditions $u(x;0) = \delta(x)$ and $u(\pm\infty;t) = 0$. Particular cases of (1) are the space-fractional diffusion ($\beta = 1$), the time-fractional diffusion ($\alpha = 2$) and the Gaussian diffusion ($\alpha = 2$ and $\beta = 1$). Furthermore when $\alpha = \beta = 2$ the D'Alembert wave equation is recovered. Space-time fractional diffusion equation (1) was analytically considered by many authors [8], [23], [25], [52], [53], [54]. The fundamental solution can be expressed by the Mellin–Barnes integral representation [54], [25], which highlights that Green functions of (1) belong to the family of the H-Fox functions [54], [55].

Solutions of (1) are emerged to show good data interpolation for anomalous diffusion processes, as for example, non-diffusive chaotic transport by Rossby waves in zonal flow [34], transport in pressure-gradient-driven plasma turbulence [34], [33], [32], transport with perturbative effects in magnetically confined fusion plasmas [56], non-diffusive tracer transport in a zonal flow under the effects of finite Larmor radius [57] or transport in point vortex flow [58]. On the physical ground, the time-fractional derivative is related to the non-Markovianity and the space-fractional derivative to non-Gaussian particle displacement PDF with heavy tails. In particular, in the Rossby waves problem, the trapping effect of the vortices gives rise to non-Markovian effects, and the zonal shear flows give rise to non-Gaussian particle displacement [34]. In plasma physics problems, the non-Markovian effects are due to the trapping in electrostatic eddies and the non-Gaussian particle displacements result from avalanche-like radial relaxation events [32], [33], [34]. Moreover, in the context of flows in porous media, fractional time derivatives describes particles that remain motionless for extended periods of time while fractional space derivatives model large motions through highly conductive layers or fractures [31], [59]. Castiglione and co-authors have highlighted that fractional diffusion equation (1) fails to model strong anomalous diffusion [60], i.e. diffusion processes where the power laws of statistical moments depend on the order of the moment under consideration. However the fractional diffusion approximation does not always breakdown [7], [28].

The *symmetric space-time fractional diffusion* equation follows from (1) when $\theta = 0$, i.e.

$$ {}_tD_*^\beta\, u(x;t) = {}_xD_0^\alpha\, u(x;t) = \frac{\partial^\alpha u}{\partial |x|^\alpha}\,, \qquad (2) $$

with $x \in R$, $t \in R_0^+$. Preliminary solutions of the symmetric case (2) were computed by Saichev & Zaslavsky [23] and Gorenflo, Iskenderov & Luchko [53].

Since (1) can be understood as a master equation and its solution as a PDF, the formulation of the underlying stochastic process is important to physically depict the anomalous diffusion at the microscopic scale. In this respect, it is well-known that the classical Gaussian diffusion is stochastically described by the *Brownian motion* (Bm). The stochastic solution of (2), i.e. a stochastic process whose single-point PDF is the solution of master equation (2), has been obtained by the continuous time random walk (CTRW) [61], [62], [63], or with proper CTRW scaling limits [64], [65], [66], by the parametric subordination [64], [65], [67], and by the subordinated Langevin equation [20], [19], [68]. However, all these methods, which are also interconnected, do not have stationary increments. Here an approach is proposed which permits to have a stochastic solution with stationary increments and fully characterized by the first and the second moments. First a new subordination-type formula for the fundamental solution of the asymmetric space-time fractional diffusion equation (1) is derived. Later, by using the identity between the PDFs resulting from the parent-directing subordination process and from the product of two independent random variables, the stochastic solution with stationary increments of the symmetric space-time fractional diffusion equation (2) is established. Solution is made up by the product of a *fractional Brownian motion* (fBm) and an independent non-negative random variable distributed according to the directing process. This representation, which is inspired by the *generalized grey Brownian motion* [69], [70], [71], [72], highlights the self-similar nature with stationary increments of the process and provides a simple approach for trajectory simulations. But, the stochastic process here proposed solves symmetric space-time fractional diffusion equation (2) while the *generalized grey Brownian motion* solves the so called Erdélyi–Kober fractional diffusion equation [73], [74], [75].

The rest of the paper is organized as follows. In Section II the parent-directing subordination process for space-time fractional diffusion equation (1) is considered and a new subordination formula is derived. In Section III the stochastic solution of (2) is obtained by using the identity between the resulting PDFs of a parent-directing subordination process and the product of two independent variables. In Section IV summary and conclusion are given with an applied perspective.

II. Subordination formulae for the space-time fractional diffusion

Let $Y(\tau)$, $\tau \in R_0^+$, be a stochastic process. If the parameter τ is randomized according to a second stochastic process with non-negative increments as $\tau = T(t)$, then the resulting process $X(t) = Y(T(t))$ is said to be *subordinated* to $Y(\tau)$, which is called the *parent process*, and to be directed by $T(t)$, which is the *directing process* [76]. In diffusive processes, the parameter τ is a time-like variable and it is referred to as the *operational time* [77]. In terms of PDFs, the subordination process is embodied by the following integral formula

$$p(x;t) = \int_0^\infty \psi(x;\tau)\,\varphi(\tau;t)\,d\tau\,, \tag{3}$$

where $\psi(x;\tau)$ is the PDF (of x evolving in τ) of the parent process $Y(\tau)$ and $\varphi(\tau,t)$ is the PDF (of τ evolving in t) of the directing process $T(t)$. Formula (3) can be studied in the framework of the Mellin transform theory and interpreted as a convolution integral [78], [79]. The following valuable subordination formula for $K^\theta_{\alpha,\beta}(x;t)$ was derived by Uchaikin & Zolotarev [80], [52],

$$K^\theta_{\alpha,\beta}(x;t) = \int_0^\infty L^\theta_\alpha(x;(t/y)^\beta)\,L^{-\beta}_\beta(y)\,dy\,, \tag{4}$$

that, by putting $t/y = \xi^{1/\beta}$, becomes [25] $K^\theta_{\alpha,\beta}(x;t) = \int_0^\infty L^\theta_\alpha(x;\xi)\,L^{-\beta}_\beta(t;\xi)\,\frac{t}{\beta\,\xi}\,d\xi$. Actually it is used in the elegant *parametric subordination* approach, developed by Gorenflo and co-authors in a number of papers [81], [64], [65], [67], and it is based on a systematic and consequent application of the CTRW integral equation to the various processes involved. This approach considers the parent process $Y(\tau)$ and the random walk $t = T^{-1}(\tau)$, which is the inverse of $\tau = T(t)$ and it is called the *leading process*. Hence, after the identification of the particle trajectory with the parent process, from the system composed by $X = Y(\tau)$ and $t = t(\tau)$ the dummy variable τ can be eliminated and the evolution of $X(t)$ obtained. This method is similar to the set of subordinated Langevin equations proposed by Fogedby [20], see also [82], [81], [19], [68], [83], [77], i.e. $dX/d\tau = \eta(\tau)$ and $dt/d\tau = \xi(\tau)$ where $\eta(\tau)$ and $\xi(\tau)$ are independent noises whose distributions are related to the parent and the leading process, respectively. Subordination formulae were used also in a theoretical statistical approach to study space-time fractional diffusion [84], [35], [85], [86], as well as to derive the stochastic solution of (1) [84], [31].

Consider the following two subordination formulae [25], [78]: $i)$ for $0 < \beta \le 1$

$$K^\theta_{\alpha,\beta}(x;t) = 2\int_0^\infty K^\theta_{\alpha,1}(x;\tau)\,K^0_{2,2\beta}(\tau;t)\,d\tau\,, \tag{5}$$

and $ii)$ for $0 < \beta/\alpha \le 1$: $K^\theta_{\alpha,\beta}(x;t) = 2\int_0^\infty K^\theta_{\alpha,\alpha}(x;\tau)\,K^0_{2,2\beta/\alpha}(\tau;t)\,d\tau$. Formula (5) shows that the solution of the space-time fractional diffusion equation (1) can be expressed in terms of the solution of the space-fractional diffusion equation of order α, i.e. $K^\theta_{\alpha,1}(x;t) = L^\theta_\alpha(x;t)$, where $L^\theta_\alpha(x;t)$ is the Lévy stable density, and of the solution of the time-fractional diffusion equation of order 2β, i.e. $K^0_{2,2\beta}(\tau;t) = M_\beta(\tau;t)/2$, $\tau \in R^+_0$, where $M_\nu(x)$, $0 < \nu < 1$, is the M-Wright/Mainardi density. [49], [50]. Moreover, since non-negative functions are involved, when $0 < \beta < 1$ and $1 < \beta \le \alpha \le 2$ $K^\theta_{\alpha,\beta}(x;t)$ can be interpreted as a PDF. By using (5) a new subordination formula for the space-time fractional diffusion can be derived. In fact, it is well known that the following subordination formula for Lévy stable density holds [76], [78], [79], $L^\theta_\alpha(x;t) = \int_0^\infty L^\gamma_\eta(x;\xi) L^{-\nu}_\nu(\xi;t)\,d\xi$, where $\alpha = \eta\nu$, $\theta = \gamma\nu$ and $0 < \alpha \le 2$, $|\theta| \le \min\{\alpha, 2-\alpha\}$, $0 < \eta \le 2$, $|\gamma| \le \min\{\eta, 2-\eta\}$, $0 < \nu \le 1$. Hence, inserting this formula into (5) gives

$$K^\theta_{\alpha,\beta}(x;t) = \int_0^\infty L^\gamma_\eta(x,\xi)\left\{\int_0^\infty L^{-\nu}_\nu(\xi;\tau) M_\beta(\tau;t)\,d\tau\right\} d\xi\,, \tag{6}$$

where the exchange of integration is allowed by the fact that the involved functions are normalized PDFs. Finally, using again (5) to compute the integral into braces in (6), when $0 < \beta \le 1$, the following new subordination formula is obtained

$$K^\theta_{\alpha,\beta}(x;t) = \int_0^\infty L^\gamma_\eta(x;\xi)\,K^{-\nu}_{\nu,\beta}(\xi,t)\,d\xi\,, \tag{7}$$

with $\alpha = \eta\nu$ and $\theta = \gamma\nu$. In the particular case $\eta = 2$ and $\gamma = 0$, so that $\nu = \alpha/2$ and $\theta = 0$, a Gaussian subordination follows. In fact $L^0_2(x;t) = \mathcal{G}(x;t) = \dfrac{e^{-x^2/(4\,t)}}{\sqrt{4\,\pi\,t}}$ so that formula (7) becomes

$$K^0_{\alpha,\beta}(x;t) = \int_0^\infty \mathcal{G}(x;\xi)\,K^{-\alpha/2}_{\alpha/2,\beta}(\xi;t)\,d\xi\,, \tag{8}$$

with $0 < \alpha \le 2$ and $0 < \beta \le 1$.

III. Stochastic solution with stationary increments of the symmetric space-time fractional diffusion equation

In this Section it is proposed a new method to obtain the stochastic solution $X_{\alpha,\beta}(t)$ of the symmetric space-time fractional diffusion equation (2) which has the valuable property to have stationary increments and to be fully characterized by the first and the second moments like Gaussian processes. This method is based on the identity between the PDFs resulting from the parent-directing subordination process and from the product of two independent random variables. In fact, it is well known that the PDF of the product of two independent random variables is represented by an integral formula that can be interpreted as a subordination formula for self-similar stochastic processes [76], [78], [79]. Recently, the correspondence between both approaches, the parent-directing subordination and the product of two independent random variables, for self-similar processes have been highlighted [72]. This relationship is useful to establish classes of *Hurst self-similar with stationary increments* (H-sssi) stochastic processes to model anomalous diffusion [72], similarly to the *generalized grey Brownian motion* [69], [70], [71], [72].

In fact, let Z_1 and Z_2 be two real independent random variables whose PDFs are $p_1(z_1)$ and $p_2(z_2)$, respectively, with $z_1 \in R$ and $z_2 \in R^+$. The joint PDF is $p(z_1, z_2) = p_1(z_1)p_2(z_2)$. Let Z be the random variable obtained by the product of Z_1 and Z_2^γ, i.e. $Z = Z_1\,Z_2^\gamma$, so that $z = z_1 z_2^\gamma$, then, carrying out the variable transformations $z_1 = z/\lambda^\gamma$ and $z_2 = \lambda$, it follows that $p(z,\lambda)\,dzd\lambda = p_1(z/\lambda^\gamma)p_2(\lambda)\,J\,dzd\lambda$, where $J = 1/\lambda^\gamma$ is the Jacobian of the transformation. Integrating in $d\lambda$, the PDF of Z emerges to be

$$p(z) = \int_0^\infty p_1\left(\frac{z}{\lambda^\gamma}\right) p_2(\lambda)\,\frac{d\lambda}{\lambda^\gamma}\,. \tag{9}$$

By applying $z = xt^{-\gamma\omega}$ and $\lambda = \tau t^{-\omega}$, subordination integral (3) is recovered from (9) by setting $\frac{1}{t^{\gamma\omega}} p\left(\frac{x}{t^{\gamma\omega}}\right) \equiv p(x;t)$, $\frac{1}{\tau^{\gamma}} p_1\left(\frac{x}{\tau^{\gamma}}\right) \equiv \psi(x;\tau)$, $\frac{1}{t^{\omega}} p_2\left(\frac{\tau}{t^{\omega}}\right) \equiv \varphi(\tau;t)$.

The correspondence between the two mechanisms can be understood as follows. Let $W(t)$, $t \in R_0^+$, be a self-similar process with self-similarity Hurst exponent H, then, for all $a > 0$, the processes $W(at)$ and $a^H W(t)$ have the same finite-dimensional distributions. Hence, if the parameter a is turned into a random variable, in the parent-directing subordination approach, the resulting process emerges to be $X(t) = Y(T(t)) = W(at)$ where it holds $Y(\tau) = W(\tau)$ and $\tau = T(t) = a\,t$, and, in the approach based on the product of two independent random variables, the resulting process is $Z(t) = Z_2^{\gamma} Z_1(t) = a^H W(t)$ where it holds $Z_2^{\gamma} = a^H$ and $Z_1(t) = W(t)$. Due to the self-similarity nature of $W(t)$, processes $X(t) = W(at)$ and $Z(t) = a^H W(t)$ have the same finite-dimensional distributions. This means that the process $Z(t)$ has the same single-point density of a subordinated stochastic process where the parent process $Y(\tau)$ is a self-similar process, i.e. $Y(\tau) = W(\tau)$, and the operational time τ is a line with stochastic slope, i.e. $\tau = T(t) = a\,t$. Finally, by setting $p(z) \equiv K^0_{\alpha,\beta}(z)$, $p_1(z_1) \equiv \mathcal{G}(z_1)$, $p_2(z_2) \equiv K^{-\alpha/2}_{\alpha/2,\beta}(z_2)$, and $\gamma = 1/2$, $\omega = 2\beta/\alpha$, $\gamma\omega = \beta/\alpha$, formula (9) becomes the new symmetric subordination formula (8). In terms of random variables it follows that $Z = X\,t^{-\beta/\alpha}$ and $Z = Z_1 Z_2^{1/2}$, hence it holds $X = Z\,t^{\beta/\alpha} = Z_1\,t^{\beta/\alpha}\,Z_2^{1/2}$. Since the random variable Z_1 is Gaussian, because $p_1(z_1) \equiv \mathcal{G}(z_1)$. the stochastic process $Z_1 t^{\beta/\alpha} = G_{2\beta/\alpha}(t)$ is a standard fBm with Hurst exponent $H = \beta/\alpha < 1$. The random variable $Z_2 = \Lambda_{\alpha/2,\beta}$ emerges to be distributed according to $p_2(z_2) \equiv K^{-\alpha/2}_{\alpha/2,\beta}(z_2)$. Following the same constructive approach adopted by Mura to built up the *generalized grey Brownian motion* [69], [70], [71], [72], the following class of H-sssi processes is established.

Let $X_{\alpha,\beta}(t)$, $t \in R_0^+$, be an H-sssi process defined by

$$X_{\alpha,\beta}(t) = \sqrt{\Lambda_{\alpha/2,\beta}}\; G_{2\beta/\alpha}(t)\,, \tag{10}$$

with $0 < \alpha \leq 2$, $0 < \beta \leq 1$, where the stochastic process $G_{2\beta/\alpha}(t)$ is a fBm with Hurst exponent $H = \beta/\alpha < 1$ and $\Lambda_{\alpha/2,\beta}$ is an independent non-negative random variable distributed according to the PDF $K^{-\alpha/2}_{\alpha/2,\beta}(\lambda)$, $\lambda \geq 0$, then the single-point PDF of $X_{\alpha,\beta}(t)$ is the solution of (2) namely $K^0_{\alpha,\beta}(x;t)$.

In other words, $X_{\alpha,\beta}(t)$ is the stochastic solution with stationary increments of equation (2). By using the Kolmogorov extension theorem, the finite-dimensional distribution of $X_{\alpha,\beta}(t)$ is obtained from (9) according to

$$f_{\alpha,\beta}(x_1, x_2, \ldots, x_n; \gamma_{\alpha,\beta}) = \frac{(2\pi)^{-(n-1)/2}}{\sqrt{\det \gamma_{\alpha,\beta}}} \int_0^{\infty} \frac{1}{\lambda^{n/2}} \mathcal{G}\left(\frac{z_n}{\lambda^{1/2}}\right) K^{-\alpha/2}_{\alpha/2,\beta}(\lambda)\, d\lambda\,, \tag{11}$$

where z_n is the n-dimensional particle position vector $z_n = \left(\sum_{i,j=1}^{n} x_i\, {\gamma_{\alpha,\beta}}^{-1}(t_i,t_j)\, x_j\right)^{1/2}$, and $\gamma_{\alpha,\beta}(t_i,t_j)$ is the covariance matrix $\gamma_{\alpha,\beta}(t_i,t_j) = \frac{1}{2}(t_i^{2\beta/\alpha} + t_j^{2\beta/\alpha} - |t_i - t_j|^{2\beta/\alpha})$. In the single-point case, i.e. $n = 1$, formula (11) reduces to

$$f_{\alpha,\beta} = \int_0^{\infty} \frac{1}{\lambda^{1/2}} \mathcal{G}\left(\frac{x\,t^{-\beta/\alpha}}{\lambda^{1/2}}\right) K^{-\alpha/2}_{\alpha/2,\beta}(\lambda)\, d\lambda = K^0_{\alpha,\beta}(x;t)\,, \tag{12}$$

or, after the change of variable $\lambda = \tau\, t^{-2\beta/\alpha}$, $\int_0^{\infty} \frac{1}{\tau^{1/2}} \mathcal{G}\left(\frac{x}{\tau^{1/2}}\right) K^{-\alpha/2}_{\alpha/2,\beta}\left(\frac{\tau}{t^{2\beta/\alpha}}\right) \frac{d\tau}{t^{2\beta/\alpha}} = \frac{1}{t^{\beta/\alpha}} K^0_{\alpha,\beta}\left(\frac{x}{t^{\beta/\alpha}}\right)$, which finally proves that the single-point PDF of $X_{\alpha,\beta}(t)$ is the solution of the symmetric space-time fractional diffusion equation (2). Formula (12) can be understood as the superposition of Gaussian processes depending on a random parameter λ (say, for example, related to the diffusion coefficient) and weighted according to its distribution, in analogy with ideas discussed elsewhere [75], [48]. Stochastic process (10) generalizes Gaussian processes, which are recovered when $\alpha = 2$ and $\beta = 1$, and it is uniquely determined by only the mean and the autocovariance structure that is a striking property of Gaussian processes that is met also by the *generalized grey Brownian motion* [69], [70], [71], [72]. This properties follow directly from the fact that $G_{2\beta/\alpha}(t)$ is a fBm, i.e. a Gaussian based stochastic processes, and $\Lambda_{\alpha/2,\beta}$ is a suitable chosen independent non-negative random variable. Furthermore, it is worth noting to highlight that the foundation of (10) on the fBm $G_{2/\beta/\alpha}(t)$ is a remarkable properties in general because it permits to reduce a large number of issues to the analysis of the fBm that has been largely studied [87]. Moreover, the stationarity of increments let stochastic solution (10) be suitable for the simulations of trajectories.

IV. Summary and Conclusion

In the present paper the problem to find a stochastic solution of the symmetric space-time fractional diffusion equation (2) in order to provide a microscopic physical insight to anomalous diffusion is addressed. The adopted method is based on the fact that the resulting PDFs from the parent-directing subordination process and from the product of two independent variables are equal. Hence, first the new subordination-type formula (7), which holds for the fundamental solution of the asymmetric space-time fractional diffusion (1), has been established. In the symmetric case, space-time fractional diffusion equation (1) reduces to (2) and integral formula (7) turns to (8). Later, this result is used to select two suitable independent variables to made up by their product the stochastic process (10) whose single-point PDF is the solution of (2). Actually, the stochastic process (10) emerges to be the product of a fBm and an independent random variable distributed according to the directing process in (7), or (8). Since the particle motion is driven by the fBm, the resulting stochastic process (10) is an H-sssi process unlike previous stochastic solutions derived in literature, see e.g. [20], [31], [67], [61]. Moreover, process (10) shares with Gaussian processes and the *generalized grey Brownian motion* [70], which is the stochastic solution of the Erdélyi–Kober fractional diffusion equation [74], [75], the valuable property to be fully characterized by only the first and second moments. Unphysical memory effects can arise in fractional systems, as discussed by Sabatier and collaborators [88], [89], [90]. The same problem may emerge for the present results because of the stated equality between the single-point PDF of the proposed stochastic process and the solution of the symmetric space-time fractional diffusion equation. However, it is here remarked that, no fractional calculus operators are used and no fractional differential equations are solved to derive all the present results. In particular, results in Section II follow from manipulations of integral formulae and results in Section III from stochastic analysis. Hence, the particle PDF obtained from the stochastic process is free from such unphysical effects. This means that, because this equality stated with a PDF obtained by using a method which is totally independent of these fractional operators' issues, even the considered solution of the symmetric space-time fractional diffusion equation is free from such unphysical effects. The possibility to built up a stochastic solution that is consistent with an evolution equation of particle PDF is important under the point of view of applications. In fact, since the stochastic solution can be understood as a generating method of particle random trajectories, the available information concerning the process increases and this allows for a stronger validation of the modelling approach and for a deeper understanding of the investigated physical phenomenon. In particular, by using the covariance function of particle trajectories, valuable spatial and temporal characteristics can be obtained about both the diffusive process and the medium. Among these computable characteristics there are, for example, the lengthscales and timescales associated to memory decay and to trapping effects, the diffusion features according to the Taylor–Green–Kubo formula and also the ergodicity property can be verified. These characteristics cannot be acquired when only the evolution equation of particle PDF is known.

Acknowledgment

The author would like to thank F. Mainardi. This research is supported by the Basque Government through the BERC 2014–2017 program and by the Spanish Ministry of Economy and Competitiveness MINECO: BCAM Severo Ochoa accreditation SEV-2013-0323.

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