# Radiation of water waves by a submerged nearly circular plate

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**Abstract**

A thin nearly circular plate is submerged below the free surface of deep water. The problem is reduced to a hypersingular integral equation over the surface of the plate which is conformally mapped onto the unit disc. The solution is computed by a spectral method proved to be efficient for the case of a circular disc. Numerical results are presented for the heave added mass and damping coefficients for two types of nearly circular plates.

**Keywords:** Water waves, Hypersingular integral equations, Expansion-collocation method, Heave added mass and damping, Orthogonal polynomials, Spectral method.

## 1 Introduction

A horizontally submerged thin plate is a component present in several structures used in offshore and coastal engineering. One of the major inovations used in the renewable energy sector are semi-submersible floating offshore wind turbines (FOWT). These structures often have column-stabilised platforms with submerged plates which provide extra heave added mass, wave damping far removed from wave excitation [2, 13].

A number of authors have considered the radiation or scattering by a submerged thin plate. Yu and Chwang [21] have used matched eigenfunctions expansions for studying the scattering by a horizontal disc in water of finite depth. Not long after, the wave scattering and radiation by a submerged elliptical disc was investigated by Zhang and Williams [23, 22] by similar techniques.

Martin and Farina [16] have considered the radiation of water waves by a submerged horizontal solid disc. They transformed the governing hypersingular integral equation for the jump in the velocity potential across the plate, $[\phi]$ into a one-dimensional Fredholm integral equation of the second kind for a new unknown function; the new equation is a generalisation of Love's integral equation, common in the theory of electrostatics of a circular-plate capacitor [14]. Numerical results of the heaving added mass and damping were presented. Farina [6] extended this work by considering the effects of taking the disc very close to the free surface and relating the hydrodynamic force to resonant frequencies. Both numerical and asymptotic methods have been used. The three-dimensional scattering by a thin disc, in deep water was investigated by Farina and Martin [7]. The authors solved the governing hypersingular integral equation numerically using a expansion-collocation spectral method. Similarly to the radiation problem, they found that the scattering problem presents a strong dependence on the frequency when the plate is close to the free surface. Relationships between the scattering cross-section and the peaks in the added mass have been presented.

Yu [20] uses analytical, numerical, and semi-empirical methods and summarises the functional performance of a submerged and essentially horizontal plate for offshore wave control. The authors focus on the hydrodynamics force and on the reflection and transmission coefficients.

The wave radiation by disc which are perturbed out of it original plane was considered by Ziebell and Farina [24]. Perturbing the plate surface, the authors formulated the problem in terms of hypersingular integral equations and the hydrodynamic force on circular caps and random rough discs were computed.

Recently, Porter [19] used a method based on Fourier/Hankel transforms to treat the scattering and the radiation by thin horizontal plates in two dimensions and also by a circular disc in three dimensions. For the latter, numerical results for the scattering cross section were presented and compared with [7].

In this work we consider the radiation of water waves by nearly circular plates. We formulate the problem by means of a hypersingular integral equation and as in the work by Martin on flat pressurised cracks [15], the new geometries are conformally mapped onto the unit disc. This preserves the kernel singularity and an efficient spectral method based on orthogonal polynomials can thus be applied to obtain the numerical solution. The convergence of the Galerkin version of this method has been recently proved by Farina et. al. [8].

The heave added mass and damping coefficients are computed and numerical results are shown for two different nearly circular plates.

## 2 Formulation

A Cartesian coordinate system is chosen, in which z is directed vertically downwards into the fluid. We take the mean free surface lying at $z = 0$. We assume the presence of a submerged body into the fluid with a smooth, closed and bounded surface $S$. We suppose that the motions of the fluid are of small-amplitude, time-harmonic, that the fluid is incompressible and inviscid, and that the motion is irrotational. We denote $\phi$ as the potential flow and $[\phi]$ as the discontinuity in $\phi$ across $S$. Thus, the time-dependent velocity potential is

$\mathrm{Re}\{\phi(x,z,t)\}\mathrm{e}^{-i\omega t}$, where $\omega$ is the angular frequency.

The conditions to be satisfied by $\phi$ are Laplace's equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi=0 \quad \text{in the fluid,}$$

along with the free-surface condition

$$K\phi+\frac{\partial\phi}{\partial z}=0 \ \text{ on } \ z=0,$$

where $K=\omega^2/g$; $g$ being the acceleration due to gravity.

On the surface of the body, the normal velocity is prescribed by

$$\frac{\partial\phi}{\partial n}=V \ \text{ on } \ S, \tag{1}$$

where $V$ is a given function and $\frac{\partial}{\partial n}$ denotes the normal derivative on $S$.

Additionally, $\phi$ must satisfy a radiation condition:

$$r^{1/2}\left(\frac{\partial\phi}{\partial r}-iK\phi\right)\to 0 \ \text{ when } \ r=(x^2+y^2)^{1/2}\to\infty.$$

In what follows, the points $P$, $Q$ denote points in the fluid and the points $p$, $q$ denote points on the submerged body.

The free surface Green function for this problem is given by

$$G(P,Q)\equiv G(x_0,y_0,z_0;x,y,z)=G_0(R,z-z_0)+G_1(R,z+z_0), \tag{2}$$

where $R=((x-x_0)^2+(y-y_0)^2)^{1/2}$, $G_0(R,z-z_0)=(R^2+(z-z_0)^2)^{-1/2}$ and

$$G_1(R,z+z_0)=\oint_0^\infty \mathrm{e}^{-k(z+z_0)}J_0(kR)\frac{k+K}{k-K}\,dk. \tag{3}$$

Here $J_0$ is the Bessel function of order zero. The integral above defining $G_1$ has its countour of integration passing below the the singularity $K$ to satisfy the radiation condition. $G$ also satisfies the free surface condition, the Laplace equation, and have a weak singularity at $P=Q$.

For any harmonic function $\phi$, satisfying $\phi=O(r^{-1})$ as $r\to\infty$, we have from Green's second identity, the following integral representation.

$$\phi(P)=\frac{1}{4\pi}\int_S\left(\phi(q)\frac{\partial}{\partial n_q}G(P,q)-G(P,q)\frac{\partial\phi}{\partial n_q}\right)dS_q, \tag{4}$$

where $\frac{\partial}{\partial n_q}$ denotes normal differentiation at $q$ on $S$.

Now, for a thin body with surface $\Omega$, denote the two sides of $\Omega$ by $\Omega^+$ and $\Omega^-$ and define the discontinuity in $\phi$ across $\Omega$ by

$$[\phi]=\lim_{Q\to q^+}\phi(Q)-\lim_{Q\to q^-}\phi(Q),$$

where $q \in \Omega$, $q^- \in \Omega^-$, $q^+ \in \Omega^+$ and $Q$ is a point in the fluid. The boundary condition now reads

$$\frac{\partial \phi}{\partial n^\pm} = V(q^\pm) \quad \text{for } q^\pm \text{ on } \Omega^\pm,$$

where $\frac{\partial}{\partial n^\pm}$ denote normal differentiation at a point on $\Omega^\pm$ in the direction from $\Omega^\pm$ into the fluid. Note that $\frac{\partial}{\partial n^+} = -\frac{\partial}{\partial n^-}$. We require that $\phi$ be bounded in the neighbourhood of $\partial\Omega$. As the plate is rigid, we have $V(q^+) = -V(q^-)$. Thus, equation (4) reduces to

$$\phi(P) = \frac{1}{4\pi} \int_\Omega [\phi(q)] \frac{\partial}{\partial n_q} G(P, q)\, dS, \tag{5}$$

where $n_q = n_q^+$ denotes now the normal unit vector at $q$ on $\Omega^+$. Applying boundary condition (1) to (5) gives

$$\frac{1}{4\pi} \neq\!\!\!\!\!\!\int_\Omega [\phi(q)] \frac{\partial^2}{\partial n_q \partial n_q} G(p, q)\, dS_q = V(p), \quad p \in \Omega, \tag{6}$$

where the integral must be interpreted in the Hadamard's finite-part sense. Equation (6) is the governing hypersingular integral equation for $[\phi]$; this is to be solved subject to the edge condition

$$[\phi] = 0 \quad \text{in} \quad \partial\Omega.$$

Suppose now that $\Omega$ is horizontal and submerged at a depth of $d$. Then, from (2) and (3), we have

$$\frac{\partial^2}{\partial n_q \partial n_q} G = \frac{1}{R^3} + M,$$

where

$$M = \oint_0^\infty \mathrm{e}^{-2kd} k^2 J_0(kR) \frac{k + K}{k - K}\, dk$$

and (6) bebomes

$$\frac{1}{4\pi} \neq\!\!\!\!\!\!\int_\Omega [\phi(q)] \frac{1}{R^3}\, dS_q + \frac{1}{4\pi} \int_\Omega [\phi(q)] M(p, q)\, dS_q = V(p), \quad p \in \Omega. \tag{7}$$

This is the same integral equation found by Martin and Farina [16, eq. (4.1)] in their treatment of horizontaly submerged circular disc and by Ziebell and Farina [24, eq. 23] as the first approximation of a perturbation method for a rough disc.

## 2.1 Plate geometry

Let us now define the plate geometry. The idea is to map the body surface $\Omega$, onto the unit disc. This procedure has at least two motivations. The first is that equation (7) can be efficiently solved by a spectral method for circular geometries ([7, 24, 8]). The other reason is justified by the Riemann mapping theorem, given below (see [1, p. 222]).

**Theorem 2.1** *Given any simply connected region $\Omega$ which is not the whole plane, and a point $\boldsymbol{x}_0 \in \Omega$, there exist a unique analytic function $f(\boldsymbol{x}) \in \Omega$, normalized by the conditions $f(\boldsymbol{x}_0) = 0, f'(\boldsymbol{x}_0) > 0$, such that $f(\boldsymbol{x})$ defines a one-to-one mapping of $\Omega$ onto the disc $|\boldsymbol{\xi}| < 1$.*

Assume that the plate is given by

$$\Omega = \{(r, \theta, d) : 0 \leq r < \rho(\theta), \quad -\pi \leq \theta < \pi\}. \tag{8}$$

Then the change of cylindrical coordinates from $(r, \theta, d)$ to $(s, \alpha, d)$ given by

$$r = r\rho(\alpha), \quad \theta = \alpha$$

maps $\Omega$ onto the horizontal unit disc

$$D = \{(s, \alpha, d) : 0 \leq s < 1, \quad -\pi \leq \alpha < \pi\}. \tag{9}$$

In general, such a change of variables made in the integral equation (7) can modify the kernel's singularity. Martin [15] proved that instead of the singularity $\frac{1}{R^3}$ the new kernel will behave like $\frac{\sigma(\Phi)}{S^3}$, where $(S, \Phi)$ are the polar coordinates at the singular point. This can, then, produce a different integral equation in its essence.

In order to preserve the singularity and keep the same type of equation, one needs only to use a *conformal* mapping between $\Omega$ and $D$.

In fact, let $\boldsymbol{x} = x + iy$ and $\boldsymbol{x}_0 = x_0 + iy_0$ be complex variables. Then, supported by Riemann mapping theorem 2.1 we can map the region $\Omega$ in the $\boldsymbol{x}$-plane onto the unit disc, $|\boldsymbol{\xi}| < 1$, in the $\boldsymbol{\xi}$-plane, using the conformal mapping

$$\boldsymbol{\xi} = F(\boldsymbol{x}), \tag{10}$$

where $F$ is an analytic function in $\Omega$. The choice of notation $\boldsymbol{x}$ and $\boldsymbol{\xi}$ is made to avoid confusion with the vertical cartesian coordinates $z$, already in use.

In our context, the property of a conformal mappinng $F$ that

$$|F'(\boldsymbol{x}_0)| = \lim_{\boldsymbol{x} \to \boldsymbol{x}_0} \frac{|F(\boldsymbol{x}) - F(\boldsymbol{x}_0)|}{|\boldsymbol{x} - \boldsymbol{x}_0|}$$

will be essential. This assures that in the limit, any small line segment near $\boldsymbol{x}_0$ is contracted or expanded by $F$, in the ratio $|F'(\boldsymbol{x}_0)|$. Following Martin [15], we then, impose $|F'(\boldsymbol{x}_0)| \neq 0$ for $\boldsymbol{x}_0 \in \Omega$. This condition is also sufficient to assure that we can invert (10). Thus, consider

$$\boldsymbol{x} = af(\boldsymbol{\xi}), \text{ for } |\boldsymbol{\xi}| < 1, \tag{11}$$

which conformally maps $D$ onto $\Omega$. This relationship between $\Omega$ and $D$ will be more convenient for our precedure. Here $a$ is a free parameter which could represent a length-scale of $\Omega$.

We assume now that $|f'(\boldsymbol{\xi})| \neq 0$ when $|\boldsymbol{\xi}| < 1$ and will next analyse the effect of the mapping (11) in the integral equation (7). Note that

$$R = |\boldsymbol{x} - \boldsymbol{x}_0| = a|f(\boldsymbol{\xi}) - f(\boldsymbol{\xi}_0)| \simeq a|f'(\boldsymbol{\xi}_0)|S$$

for small $S$, where

$$S = |\boldsymbol{\xi} - \boldsymbol{\xi}_0|.$$

By symmetry, we can conclude also that $R \simeq a|f'(\boldsymbol{\xi}_0)|S$ and then

$$a^3 R^{-3} = |f'(\boldsymbol{\xi})|^{-3/2} |f'(\boldsymbol{\xi}_0)|^{-3/2} \{S^{-3} + L(\boldsymbol{\xi}, \boldsymbol{\xi}_0)\},$$

where $L$ is defined by

$$L(\boldsymbol{\xi}, \boldsymbol{\xi}_0) = \frac{|f'(\boldsymbol{\xi})|^{3/2} |f'(\boldsymbol{\xi}_0)|^{3/2}}{|f(\boldsymbol{\xi}) - f(\boldsymbol{\xi}_0)|^3} - \frac{1}{S^3}. \tag{12}$$

The function $L$ above is the new additional kernel in our model. Although this can pose some numerical difficulties in its evaluation, it has only a weak singularity. In fact, Martin [15] proved that $L = O(S^{-1})$ as $S \to 0$.

Now, considering the Jacobian, $J(f) = a^2 |(f'(\boldsymbol{\xi}))|^2$ and writing

$$\boldsymbol{\xi} = \xi + i\eta \quad \text{and} \quad \boldsymbol{\xi}_0 = \xi_0 + i\eta_0,$$

we have

$$d\Omega = dxdy = J(f) d\xi d\eta = a^2 |f'(\boldsymbol{\xi})|^2 d\xi d\eta.$$

Then, defining the new functions $w, v, G_r$ by

$$[\phi](x(\boldsymbol{\xi}), y(\boldsymbol{\xi})) = a|f'(\boldsymbol{\xi})|^{-1/2} w(\xi, \eta), \tag{13}$$

$$V(x(\boldsymbol{\xi}_0), y(\boldsymbol{\xi}_0)) = |f'(\boldsymbol{\xi}_0)|^{-3/2} v(\xi_0, \eta_0), \tag{14}$$

$$M(x(\boldsymbol{\xi}), y(\boldsymbol{\xi})) = a^{-3} |f'(\boldsymbol{\xi})|^{-3/2} G_r(\boldsymbol{\xi}, \boldsymbol{\xi}_0), \tag{15}$$

and substituting (13), (14) and (15) in (7), we get

$$\frac{1}{4\pi} \int\!\!\!\!\!\!\times_D \frac{w(\xi, \eta)}{S^3} d\xi d\eta + \frac{1}{4\pi} \int_D w(\xi, \eta) \Big\{ L(\boldsymbol{\xi}, \boldsymbol{\xi}_0) + G_r(\boldsymbol{\xi}, \boldsymbol{\xi}_0) \Big\} d\xi d\eta = v(\xi_0, \eta_0), \tag{16}$$

for $(\xi_0, \eta_0) \in D$, where all quantities are nondimensional. This equation is to be solved constrained by $w = 0$ for $|\boldsymbol{\xi}| < 1$.

In operator notation, (16) can be written as

$$(H + \mathcal{L} + \mathcal{G}_r) w = v, \tag{17}$$

where $H$ is the hypersingular integral operator defined by the first term in (16). $\mathcal{L}$ and $\mathcal{G}_r$ are the integral operators defined by the kernels $L$ and $G_r$ above and are weakly singular, with $\mathcal{G}_r$ incorporating all the wave effects captured by the regular part (3) of the Green function.

## 2.2 Added mass and damping coefficients

We will consider the vertical oscillations performed by the plate and the resulting heave hydrodynamic force.

The heave added mass $\mathcal{A}$ and damping $\mathcal{B}$ coefficients ([5]) can be written as

$$\begin{aligned}\mathcal{A}+i\mathcal{B} &= -\int\limits_{\Omega}[\phi]d\Omega = -\int\limits_{\Omega}[\phi](x,y)dxdy \\ &= -\int\limits_{D} a|f'(\boldsymbol{\xi})^{-1/2}|w(\xi,\eta)\, a^2|f'(\boldsymbol{\xi})|^2 d\xi d\eta.\end{aligned}$$

Then,

$$\begin{aligned}\mathcal{A}+i\mathcal{B} &= -a^3\int\limits_{D} w(\xi,\eta)\ |f'(\boldsymbol{\xi})^{3/2}|d\xi d\eta \\ &= -a^3\int_0^{2\pi}\int_0^1 w(\xi(s,\alpha),\eta(s,\alpha))\ |f'(\boldsymbol{\xi}(s,\alpha))|^{3/2} s ds d\alpha.\end{aligned}$$

# 3 Method of solution

Equation (16), obtained in section 2.1, can efficiently be solved by the spectral method proposed in [7]. The convergence of the Galerkin version of it has recently been established by Farina et. al. [8].

The governing equation (17), can be written in the same general form as the models treated in [7, 24], which is

$$(H+\mathcal{Q})u = v, \tag{18}$$

where $v$ is a known function, $H$ is a hypersingular operator and $\mathcal{Q}$ is not strongly singular.

As a particular case, the horizontal heaving disc problem has an axisymmetric solution and for the case of a *circular* disc it can be solved by reducing it to a non singular one dimensional Fredholm integral equation of the second kind [16, eq. 7.6]. A simple numerical method can be used for this equation; for instance a Nyström method combined with the Gauss-Legendre quadrature rule, as employed by Martin and Farina [16].

However, in general, we need a more general method of solution. We employ the expansion-collocation method used by Farina and Martin [7] and by Ziebell and Farina [24] for solving equations of the form of (18). The forcing $v$ could be any function of two variables; for instance, this could represent an incident wave and in this way, the problem would be a scattering one. Now, going into the details, we see from (8) and (9) that (16) becomes

$$\frac{1}{4\pi}\, ⨎_D w(s,\alpha)\left\{\frac{1}{S^3}+Q(s,\alpha;t,\beta;b,K)\right\} s\, ds\, d\alpha = v(t,\beta), \quad (t,\beta)\in[0,1)\times[-\pi,\pi), \tag{19}$$

where $S^3 = [s^2 + t^2 - 2st \cos(\alpha - \beta)]^{3/2}$ and $Q = L + G_r$.

We shall expand $w$ using the basis functions $B_k^m$, defined by

$$B_k^m(s, \alpha) = P_{m+2k+1}^m(\sqrt{1 - s^2})\, e^{im\alpha}, \qquad k, m = 0, 1, \ldots,$$

where $P_n^m$ is an associated Legendre function. The radial part of these basis functions can also be expressed in terms of Gegenbauer polynomials [4].

The functions $\{B_k^m\}$ are orthogonal over the unit disc with respect to the weight $(1 - s^2)^{-1/2}$.

The next formula, due to Krenk [12] is essential in the construction of the method:

$$\frac{1}{4\pi} \int\!\!\!\!\!\!\times_S \frac{1}{S^3}\, B_k^m(s, \alpha)\, s\, ds\, d\alpha = C_k^m\, \frac{B_k^m(t, \beta)}{\sqrt{1 - t^2}}, \tag{20}$$

where

$$C_k^m = -\frac{\pi}{4} \frac{(2k + 1)!}{(2m + 2k + 1)!} [P_{m+2k+1}^{m+1}(0)]^2$$

Equation (20) allows us to evaluate the hypersingular integrals analytically[1]. To exploit (20), we expand $w$ in terms of the functions $B_k^m$. For brevity, we write

$$w \approx \sum_{k,m}^{N} a_k^m\, B_k^m := \sum_{k=0}^{N_1} \sum_{m=0}^{N_2} a_k^m\, B_k^m. \tag{21}$$

Substituting (21) in the integral equation (19) and then evaluating the hypersingular integrals analytically using (20), we obtain

$$\sum_{k,m}^{N} a_k^m \left\{ C_k^m\, \frac{B_k^m(t, \beta)}{\sqrt{1 - t^2}} + \frac{1}{4\pi} \int_S B_k^m(s, \alpha)\; Q(s, \alpha; t, \beta; b, K)\, s\, ds\, d\alpha \right\} = v(t, \beta), \tag{22}$$

for $(t, \beta) \in [0, 1) \times [-\pi, \pi)$.

It remains to determine the unknown coefficients $a_k^m$. We use a collocation method, in which evaluation of (22) at $(N_1 + 1)(N_2 + 1)$ points on the disc gives a linear system, $D\boldsymbol{a} = \boldsymbol{b}$ for the coefficients $\boldsymbol{a} = a_k^m$.

# 4 Computational aspects and numerical results

## 4.1 Implementation

In this subsection we describe some computational aspects of the proposed spectral method.

[1] Another consequence of formula (20) is that the functions $B_k^m(r, \theta)/\sqrt{1 - r^2}$ can bee seen as eigenfunctions of the integral operator $\bar{\mathcal{H}}$ defined by

$$\bar{\mathcal{H}} v(r, \theta) = \int\!\!\!\!\!\!\times_D \frac{1}{R^3} v(s, \alpha)\; \sqrt{1 - s^2}\, s\, ds\, d\alpha.$$

The computation of the free surface Green function and its derivatives is a major numerical aspect in our method. Alternative and efficient formulas for the computation of the regualar kernel $M$ and its mapped $G_r$ can be found in [7] or in [24, eq. 32].

In order to evaluate $P_l^m$ for $l > m$ we used a recurrence relation. Since most recurrences on $m$ are unstable, the following recurrence on $l$, which is stable, is adopted:

$$(l-m)\,P_l^m(x) = (2l-1)x\,P_{l-1}^m(x) - (l+m-1)\,P_{l-2}^m(x).$$

The functions $\mathbf{H}_0$ and $Y_0$ are computed by approximating Chebyshev polynomials [18]. This procedure produces an efficient way of evaluating these special functions since polynomials of sixth-degree are sufficient to give an accurate approximation.

We used $N_1 = N_2 = 6$ which gives 49 collocation points over the plate. These values provided numerical evidence of convergence of the expansions used to approximate $w$. We assume that the solution is symmetric about $\alpha = 0$. This means that we can assume that the collocation points lie in the semi-disc given by $\{(s,\alpha) : 0 \le s < 1, 0 \le \alpha \le \pi\}$; collocating at symmetric points on the other semi-disc would give no further information for obtaining the solution of the linear system. The collocation points are taken as the intersection of concentric circles (radius $r_l$) with equally separated rays emanating from the origin (angle $\theta_n$). Precisely, the chosen points are

$$(s_l, \alpha_n), \qquad l = 0, 1, \ldots, N_1, \quad n = 0, 1, \ldots, N_2,$$

where $\{s_l\}$ is a certain set of distinct points in $(0, 1)$ and

$$\alpha_n = (2n+1)\pi/(2N_2+2), \quad n = 0, 1, \ldots, N_2,$$

are the zeros of $\cos{(N_2+1)\theta}$ in $(0, \pi)$. However care must be taken when choosing the distribution of the numbers $s_l$. For instance, choosing equally spaced numbers in $(0, 1)$ gives a badly conditioned system. We adopt a *Chebyshev tensor-product* collocation, which means that the points in the radial variable are zeros of Chebyshev polynomials of the first kind, $T_{2N_1+2}(r)$ in $[0, 1]$; explicitly,

$$s_l = \cos{[(2l+1)\pi/(4N_1+4)]}, \quad l = 0, 1, \ldots, N_1.$$

The numerical method described above was implemented in Fortran and the LAPACK routine `cgerfs` was used for solving the linear system $D\boldsymbol{a} = \boldsymbol{b}$.

For further details on the computational issues above, see [7].

Two computational difficulties are intrinsic in the kind of problem we are treating. When the submergence depth $d$ decreases, the so-called resonant frequencies are associated with increasingly high values in the solution $w$ and in the hydrodynamic coefficients. This fact is associated with a singularity in the matrix $D$. To study the case where a plate is very close to the free surface, approximations such as the ones adopted in [16] and [6] are an alternative. The second numerical challenge is new and found in the development of the present work. As the plate $\Omega$ differs from a circular aspect, the function $f$ differs from the identity mapping. This causes the kernel $K$, given by (12) to become *nearly-hypersingular*. The numerical integration scheme we used to compute the integrals involving the kernels $K$ and $G_r$ is based on the Korobov-Conroy number theoretic method [10, 11, 3]. Although this method is very robust for dealing with weak singularities, a limitation has been still posed by the difficulty above.

## 4.2 Numerical results

The model for heaving motions of the plate corresponds $v = 1$. We, thus use this condition and take $a = 1$, without lost of generality. The method described in the previous section has been employed to compute the heave added mass and damping coefficients for nearly cicular plates. This is defined by conformal mappings of the type

$$f(\boldsymbol{\xi}) = \boldsymbol{\xi} + \epsilon g(\boldsymbol{\xi}),$$

where $\epsilon$ is dimensionless and $g(\boldsymbol{\xi})$ is analytic. For small $\epsilon$, $f$ defines a somewhat circular plate $\Omega$. These geometries have been considered as cracks by Gao and Rice [9] and Martin [15] in their works on elasticity. If $g(\boldsymbol{\xi}) = \boldsymbol{\xi}^{n+1}$, for example, by (11), a plate $\Omega$ is obtained with its boundary given by $r = \rho(\theta) = a(1 + \epsilon \cos n\theta)$ with an error of $O(\epsilon^2)$ ([15]).

In the next subsections, we will show the numerical results of two different types of nearly circular regions. For a heaving circular disc, the added mass and damping characteristics are well known [16] and some of their features are the occurence of negative added mass for sufficiently small $d$ and increasingly maxima and minima with the decrease of submergence. The height of the peak in the damping coefficient is almost exactly the same as the total height of the corresponding spikes in the added-mass coefficient. Furthermore, the peak of the damping is very close to a zero of the added mass. The spikes in the graphs of the hydrodynamic coefficients are associated with resonant frequencies which correspond to singularities in the system matrix $D = D(K, d)$. These spikes are, thus, also present in the total scattering cross section by the same object [7, 19].

**Case 1: Nearly circular plates of type A (elliptical plate)**

Let us consider mappings of the form

$$f(\boldsymbol{\xi}) = \boldsymbol{\xi} + \epsilon 0.5\boldsymbol{\xi} + \epsilon 0.5\boldsymbol{\xi}^2, \quad \boldsymbol{\xi} \in D,$$

and call the resulting $\Omega$ of plates of type A. This geometry is a simple approximation, for small eccentricity, of an ellipse given by [17, p. 265]

$$\Omega = \{(x, y) : x^2 + (1 + \epsilon)^2 y^2 < 1\}.$$

In figure 1 the added mass is plotted as function of $Ka$, for the elliptical plate with $\epsilon = 0.2$. We note main features observed for a circular plate plus additional ones. In particular, more pronounced secondary spikes occur in the band $0.5 < Ka < 1$, while the lower frequency ones are significantly smaller than those of the circular plate. This feature suggests an energy conservation in the hydrodynamic force. In the domain $1 < Ka < 2$, a third maxima is observed but in a much wider frequency band. It is also remarkable that the first peak is shifted to a higher frequency, as compared to the circular case. The same conclusions could be drawn for the damping coefficient, shown in figure 2. Simulations with other values of $\epsilon$ (not shown) presented the same characteristics.

**Case 2: Nearly circular plates of type B**

Consider now the mapping

$$f(\boldsymbol{\xi}) = \boldsymbol{\xi} + \epsilon(e^{\boldsymbol{\xi}^3} - 1)$$

which, for $\epsilon = 0.045$, produces the nearly circular plate ploted in figure 3. In figures 4 and 5 the added mass and damping coefficients, respectively, are shown. The hydrodynamic coefficients are very similar to the ones of the circular disc, even though deviations from the circular character are noticable. It is seen the absence of resonance as the submergence increases and this is clear for $d = 0.2$.

# 5 Discussion

We formulated the problem of water wave radiation by a nearly circular plate in terms of a hypersingular integral equation, by using a conformal mapping. The new equation allows the application of a spectral method which is known to be efficient and convergent for the numerical solution of integral equations over circular domains. We presented numerical results for two plate geometries. While there is little variation in comparison to the circular disc, in one case, for an approximate ellipse, we found a shift to higher frequency in the dominant maxima of the hydrodynamic coefficients. Although smaller in magnitude, these resonances are wider, compared to the circular plate. We remark the occurrence of secondary and terciary peaks in the graphs of the added mass and damping coefficientes which are significantly higher and wider than their circular counterpart.

The approach will also be applicable to the water wave and acoustic scattering problems, as the fundamental solutions present the same singularity.

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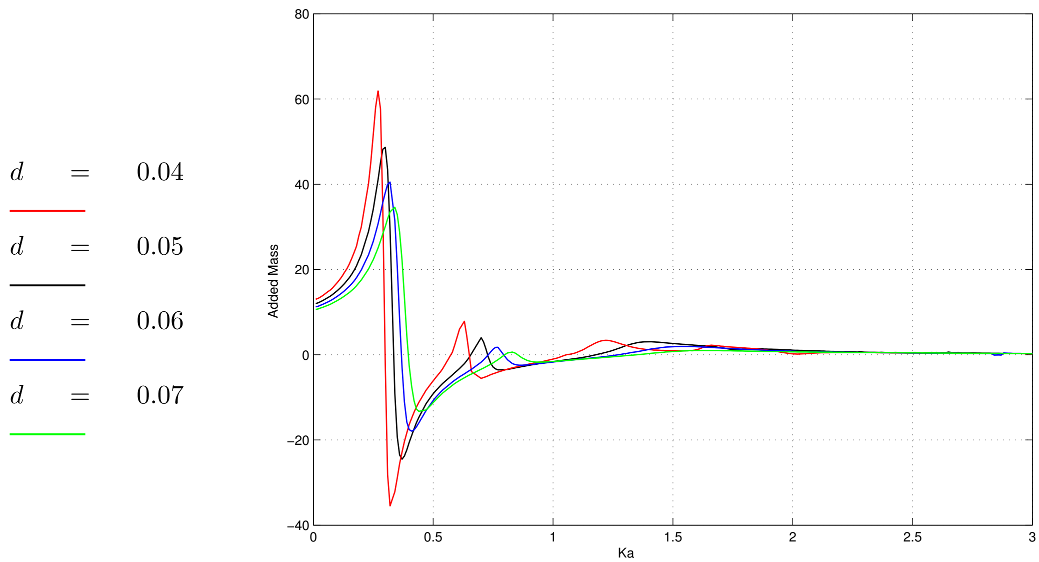


Figure 1: The heave added mass, as function of $Ka$, for a plate of type A, given by $f(\zeta) = \zeta + 0.1\zeta + 0.1\zeta^2$.

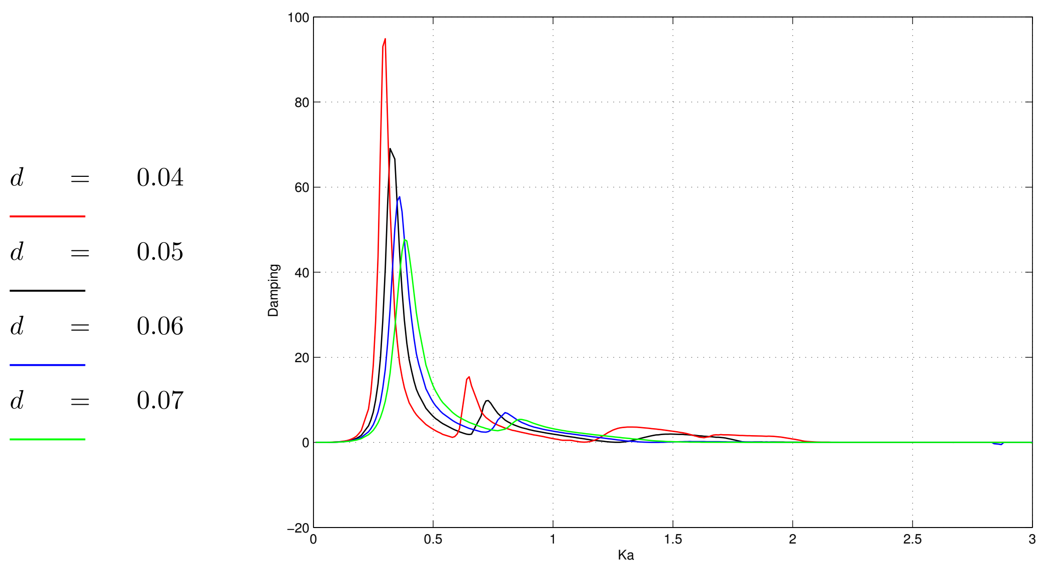


Figure 2: The damping coefficient as function of $Ka$, for a plate of type A, given by $f(\zeta) = \zeta + 0.1\zeta + 0.1\zeta^2$.

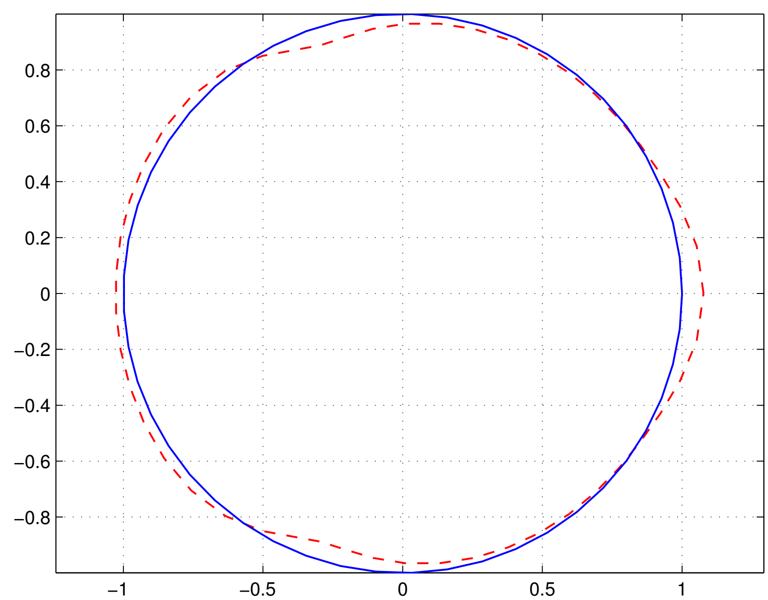


Figure 3: The nearly circular plate of type B, defined by $h(\zeta) = \zeta + 0.045(e^{\zeta^3} - 1)$ (dotted red line) and the unit disc (in blue).

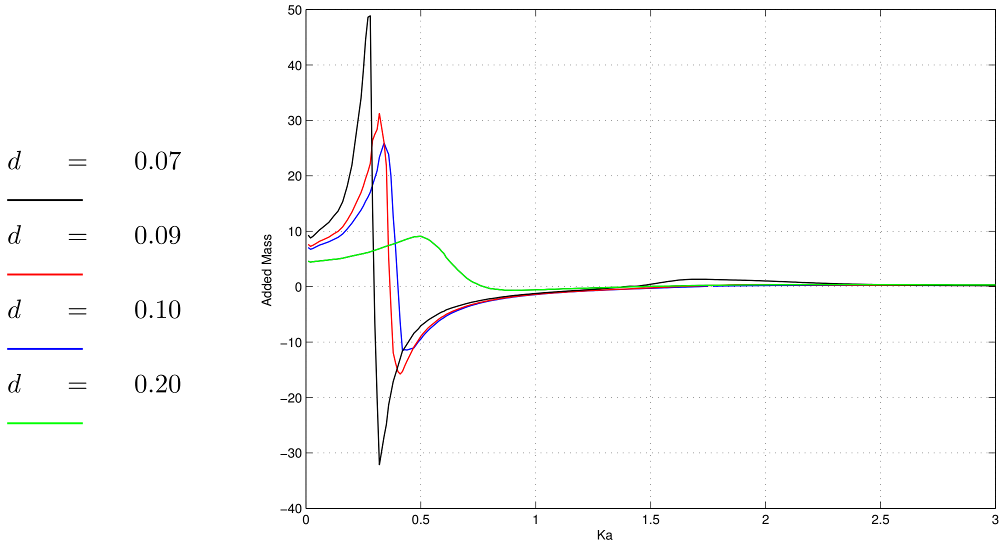


Figure 4: The heave added mass, as a function of $Ka$, for the nearly circular plate of the type B, given by $f(\boldsymbol{\xi}) = \boldsymbol{\xi} + 0.045(e^{\boldsymbol{\xi}^3} + 1)$.

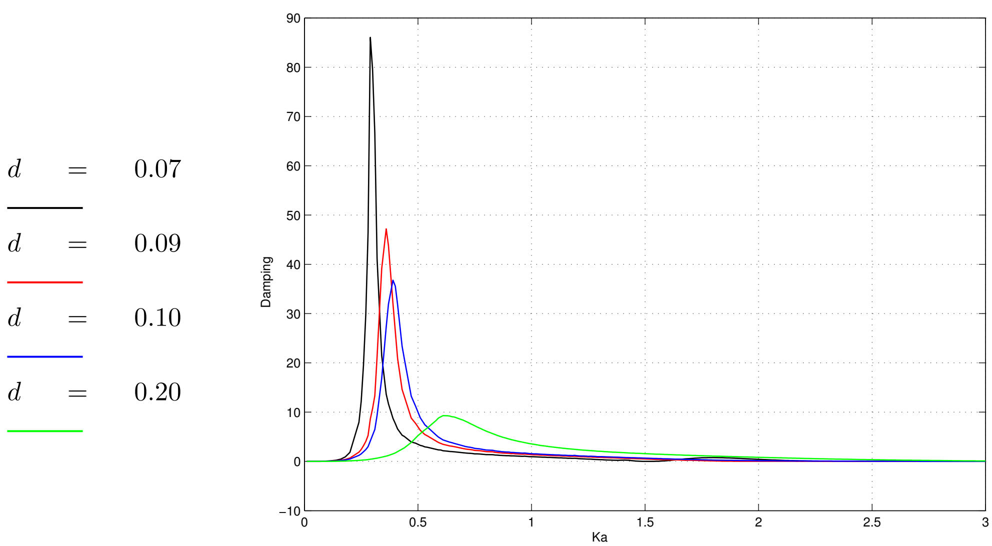


Figure 5: The damping coefficient, as a function of $Ka$, for the nearly circular plate of the type B, given by $f(\boldsymbol{\xi}) = \boldsymbol{\xi} + 0.045(e^{\boldsymbol{\xi}^3} + 1)$.