The Airfoil equation on near disjoint intervals: Approximate models and polynomial solutions

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Abstract

The airfoil equation is considered over two disjoint intervals. Assuming the distance between the intervals is small an approximate solution is found and relationships between this approximation and the solution of the classical airfoil equation are obtained. Numerical results show the convergence of the solution of the original problem to the approximation. Polynomial solutions for an approximate model are obtained and a spectral method for the generalized airfoil equation on near disjoint intervals is proposed.

Keywords: airfoil equation, singular integral equations, singular perturbation, orthogonal polynomials, spectral method

1. Introduction

Several works deal with the airfoil equation over an interval and its generalizations, see e.g. [1, 2, 3, 4]. In its original form, this equation can be written as

$$\frac{1}{\pi} \int_{-1}^{1} \frac{f(t)}{x-t} \mathrm{d}t = g(x) \ , \quad x \in (-1, 1) \ , \tag{1.1}$$

and it models the flow over an infinitely thin airfoil without viscosity and in the bidimensional case [5]. Here f is the local circulation density and it can be

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shown to be equal to the difference in induced velocity over the upper and lower side of the airfoil. The right-hand side g is given by the product of 2π, the onset velocity and the slope of airfoil which in case of a flat plate is nothing but the product of 2π, the onset velocity and the angle of incidence. The case when g is an odd function includes for instance an airfoil with zero angle of incidence and parabolic camber distribution. Applications in elasticity and surface wave scattering are also described by this equation [6].

In this work, we consider the airfoil equation with a Cauchy type of singularity and set in close disjoint intervals (2.1); that is, in two intervals with a small aperture of size $2\varepsilon > 0$. Tricomi [2] derived analytical solutions for the airfoil equation set in two disjoint intervals. Dutta and Banerjea [7] presented this solution in a slightly different form and have used it to deduce a solution of the associated hypersingular integral equation.

We investigate formally and numerically the asymptotic behavior for the solution f_ε of the airfoil equation when the small parameter ε tends to zero. We first identify the formal limit of f_ε as ε tends to zero : The background model f_0 is given by a Cauchy singular integral. We compare the formal limit with the solution of the airfoil equation set in an interval $(-1, 1)$, in Section 3.2. We exhibit the first two terms of an expansion in power series of ε for particular analytical solutions f_ε and we specify a functional framework in weighted Sobolev spaces, Section 3.3. We illustrate numerically convergence results of f_ε towards f_0 in ε^2 in this framework, Section 4. We then find polynomials as solutions for the approximate model and we exhibit a spectral method to solve a generalized airfoil equation set on close disjoint intervals, in Section 5.

2. The mathematical model

We consider the airfoil equation with Cauchy type singularity and set in the union of two disjoint intervals with a small hole of size $2\varepsilon > 0$, $G_\varepsilon = (-1, -\varepsilon) \cup (\varepsilon, 1)$

$$\frac{1}{\pi} \int_{-1}^{-\varepsilon} \frac{f_\varepsilon(t)}{x - t} \mathrm{d}t + \frac{1}{\pi} \int_{\varepsilon}^{1} \frac{f_\varepsilon(t)}{x - t} \mathrm{d}t = -\psi(x) , \quad x \in G_\varepsilon . \tag{2.1}$$

Here, f_ε is the unknown and ψ represents the data of the problem. We assume that ψ is Hölder continuous for $x \in G_\varepsilon$, $\psi \in \mathcal{C}^{0,\alpha}(G_\varepsilon)$.

Let $\varepsilon > 0$ be a fixed parameter. It is possible to solve the airfoil equation (2.1) by the method of Tricomi [2] (see also e.g. [7]). The solution writes

$$
f_\varepsilon(x) = \begin{cases} \dfrac{1}{\pi R_\varepsilon(x)} [C_{1,\varepsilon} + C_{2,\varepsilon} x + \Psi_\varepsilon(x)] , & x \in (-1, -\varepsilon) \\ -\dfrac{1}{\pi R_\varepsilon(x)} [C_{1,\varepsilon} + C_{2,\varepsilon} x + \Psi_\varepsilon(x)] , & x \in (\varepsilon, 1) \end{cases} \tag{2.2}
$$

where Ψ is given by the Cauchy integrals

$$
\Psi_\varepsilon(x) = \int_{-1}^{-\varepsilon} \frac{\psi(t) R_\varepsilon(t)}{x-t} \mathrm{d}t - \int_{\varepsilon}^{1} \frac{\psi(t) R_\varepsilon(t)}{x-t} \mathrm{d}t , \tag{2.3}
$$

$C_{1,\varepsilon}$ and $C_{2,\varepsilon}$ are two arbitrary constants, and $R_\varepsilon(x) = \sqrt{(1-x^2)(x^2 - \varepsilon^2)}$, $\quad x \in G_\varepsilon$.

We will take the constants $C_{1,\varepsilon}$ and $C_{2,\varepsilon}$ as zero and will call the associated solution as the *null-circulation solution*. The physical meaning and motivation for chosing homogeneous constants is due to the fact that this case relates to the solution of the Airfoil equation on an interval when the circulation around the airfoil is zero. We will see more details of this in section 3.2.

3. Convergence results

In this section, we present convergence results and approximate models for the solution f_ε of (2.1) when ε tends to 0 (Prop. 3.2). We derive the background model f_0 which is nothing but the pointwise limit of f_ε as ε tends to 0. We compare the background model f_0 with the solution of the airfoil equation over an interval (1.1) (Prop. 3.4). Finally, we exhibit the first two terms of an asymptotic expansion of f_ε in power series of ε when the right hand side is a Chebyshev polynomial.

3.1. Asymptotic behavior. Background model

We first investigate the behavior of the solution of the airfoil equation (2.1) when ε goes to 0.

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Lemma 3.1. *Let $x \in G_\varepsilon$ and $\psi \in \mathcal{C}^{0,\alpha}(-1,1)$. Define the function g_ε on $G_\varepsilon \setminus \{x\}$ as*

$$g_\varepsilon(t) = \frac{\psi(t)R_\varepsilon(t)}{x-t}\mathbb{1}_{(-1,-\varepsilon)}(t) - \frac{\psi(t)R_\varepsilon(t)}{x-t}\mathbb{1}_{(\varepsilon,1)}(t) . \tag{3.1}$$

Then, g_ε satisfies the pointwise convergence result : for a.e. $t \in (-1,1)$,

$$g_\varepsilon(t) \longrightarrow -\frac{\sqrt{1-t^2}}{x-t}t\psi(t)\mathbb{1}_{(-1,1)}(t) \quad as \quad \varepsilon \to 0 . \tag{3.2}$$

Proof.

For all $t \in G_\varepsilon \setminus \{x\}$, $g_\varepsilon(t) = \frac{\psi(t)R_\varepsilon(t)}{x-t}\left(\mathbb{1}_{(-1,-\varepsilon)} - \mathbb{1}_{(\varepsilon,1)}\right)(t)$. Observe that $R_\varepsilon(t) \longrightarrow |t|\sqrt{1-t^2}$ as $\varepsilon \to 0$ for all $t \in G_\varepsilon$ and $\left(\mathbb{1}_{(-1,-\varepsilon)} - \mathbb{1}_{(\varepsilon,1)}\right)(t) \longrightarrow -\operatorname{sgn}(t)\mathbb{1}_{(-1,1)}(t)$ as $\varepsilon \to 0$ for a.e. $t \in (-1,1)$. As a consequence, there holds for a.e. $t \in (-1,1)$

$$R_\varepsilon(t)\left(\mathbb{1}_{(-1,-\varepsilon)} - \mathbb{1}_{(\varepsilon,1)}\right)(t) \longrightarrow -\sqrt{1-t^2}t\mathbb{1}_{(-1,1)}(t) \quad as \quad \varepsilon \to 0 .$$

We infer the convergence result (3.2) ∎

In the framework of Lemma 3.1, one can not conclude in general that

$$\int_{-1}^{1} g_\varepsilon(t)\,\mathrm{d}t \longrightarrow -\int_{-1}^{1} \frac{\sqrt{1-t^2}}{x-t}t\psi(t)\,\mathrm{d}t \quad \text{when} \quad \varepsilon \to 0 \tag{3.3}$$

for such function $\psi \in \mathcal{C}^{0,\alpha}(-1,1)$.

Nevertheless, when we restrict our considerations to a class of functions $\psi \in \mathcal{C}^{0,\alpha}(-1,1)$ such that

$$|g_\varepsilon(t)| \leqslant g(t) \quad \text{for a.e.} \quad t \in (-1,1)$$

where $g \in \mathrm{L}^1(-1,1)$ is a given function independent of ε, then as a consequence of the Lebesgue's Dominated Convergence Theorem the convergence result (3.3) holds. We then infer the following result.

Proposition 3.2. *Let $x \in G_\varepsilon$ and $\psi \in \mathcal{C}^{0,\alpha}(-1,1)$. Assume that there exists a function $g \in \mathrm{L}^1(-1,1)$ such that*

$$|g_\varepsilon(t)| \leqslant g(t) \quad \text{for a.e.} \quad t \in (-1,1), \tag{3.4}$$

where g_ε is defined by (3.1). Then, in the framework of Sec. 2, the null-circulation solution f_ε of the airfoil equation (2.1) satisfies the following point-wise convergence result when $\varepsilon \to 0$:

$$f_\varepsilon(x) \longrightarrow f_0(x) \ . \tag{3.5}$$

Here the function f_0 is defined as

$$f_0(x) = -\frac{1}{\pi x \sqrt{1-x^2}} \Psi_0(x) \quad \text{for all} \quad x \in G_0 = (-1,1) \setminus \{0\} \ , \tag{3.6}$$

where Ψ_0 is defined by a Cauchy type singular integral

$$\Psi_0(x) = -\int_{-1}^{1} \frac{\sqrt{1-t^2}}{x-t} t\psi(t)\, \mathrm{d}t \ . \tag{3.7}$$

Proof.

The proof is a consequence of Lemma 3.1 together with Lebesgue's Dominated Convergence Theorem. As a consequence of Lemma 3.1, there holds for a.e. $t \in (-1,1)$:

$$g_\varepsilon(t) \longrightarrow -\frac{\sqrt{1-t^2}}{x-t} t\psi(t)\mathbb{1}_{(-1,1)}(t) \quad \text{as} \quad \varepsilon \to 0 \ .$$

Using the condition (3.4), we can apply Lebesgue's Dominated Convergence Theorem to obtain the following result

$$\int_{-1}^{1} g_\varepsilon(t)\, \mathrm{d}t \longrightarrow -\int_{-1}^{1} \frac{\sqrt{1-t^2}}{x-t} t\psi(t)\, \mathrm{d}t \quad \text{when} \quad \varepsilon \to 0$$

According to Section 2, there holds

$$f_\varepsilon(x) = \frac{1}{\pi R_\varepsilon(x)} \left(\mathbb{1}_{(-1,-\varepsilon)} - \mathbb{1}_{(\varepsilon,1)}\right)(x) \int_{-1}^{1} g_\varepsilon(t)\, \mathrm{d}t$$

Since $R_\varepsilon(x) \longrightarrow |x|\sqrt{1-x^2}$ as $\varepsilon \to 0$, we infer the convergence result (3.5) for all $x \in G_\varepsilon$. ∎

The following proposition provides a convergence result that will be relevant in section 5.2, in the discussion on spectral methods for the generalized Airfoil equation on disjoint intervals.

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Proposition 3.3. *Let $x \in G_\varepsilon$ and assume that there exists a function $v \in$ L$^1(-1, 1)$ such that the null-circulation solution f_ε of the airfoil equation (2.1) satisfies*

$$\left| \frac{f_\varepsilon(t)}{x - t} \mathbb{1}_{G_\varepsilon}(t) \right| \leqslant v(t) \quad \textit{for a.e.} \quad t \in (-1, 1) \ . \tag{3.8}$$

Then, for f_0 given by (3.6), there holds

$$\int_{G_\varepsilon} \frac{f_\varepsilon(t)}{x - t} \, dt \longrightarrow \int_{G_0} \frac{f_0(t)}{x - t} \, dt \quad \textit{as} \quad \varepsilon \to 0. \tag{3.9}$$

Proof.

As a consequence of Prop. 3.2 there holds for a.e. $t \in (-1, 1)$

$$\frac{f_\varepsilon(t)}{x - t} \mathbb{1}_{G_\varepsilon}(t) \longrightarrow \frac{f_0(t)}{x - t} \mathbb{1}_{(-1,1)}(t) \quad \text{as} \quad \varepsilon \to 0 \ .$$

Then using the condition (3.8) we can apply Lebesgue's Dominated Convergence Theorem to obtain the convergence result (3.9). ∎

3.2. Comparison with the solution of the airfoil equation set on an interval

When ψ is Hölder continuous in $(-1, 1)$, the general solution of (1.1) with a data $g = -\psi$ writes (see e.g. [2, pp. 173-180] or [8])

$$f(x) = \frac{1}{\pi} \int_{-1}^{1} \sqrt{\frac{1 - t^2}{1 - x^2}} \frac{\psi(t)}{x - t} \, dt + \frac{A}{\sqrt{1 - x^2}} \ . \tag{3.10}$$

Here A is an arbitrary constant. The constant A physically represents the circulation around the airfoil [5]. We consider that the circulation vanishes. Thus, $A = 0$, and the solution f is unique, given by

$$f(x) = \frac{1}{\pi} \int_{-1}^{1} \sqrt{\frac{1 - t^2}{1 - x^2}} \frac{\psi(t)}{x - t} \, dt \ . \tag{3.11}$$

Therefore, for all $x \in (-1, 1) \setminus \{0\}$ there holds

$$f(x) = f_0(x) + \frac{1}{x} \frac{1}{\pi} \int_{-1}^{1} \sqrt{\frac{1 - t^2}{1 - x^2}} \psi(t) \, dt \tag{3.12}$$

where f_0 is given by (3.6) as an approximation of f_ε.

3.2.1. Particular cases of the function $\psi(t)$

For some choices of the right hand side ψ, the solution f_ε of the airfoil equation (2.1) (set in G_ε) coincides with the solution f of the airfoil equation set in $(-1,1)$, modulo a residual term and when ε is small enough.

Proposition 3.4. *Let $\psi \in \{T_m, U_n, \quad m \in \mathbb{Z}^+\backslash\{0,2\}, n \in \mathbb{Z}^+\backslash\{0\}\}$ or let ψ be an odd function in $\mathcal{C}^{0,\alpha}(-1,1)$. Here T_n and U_n are the Chebyshev polynomials of the first and second types, respectively. Then, in the framework of section 2,*

$$f(x) = f_0(x) \quad \text{for all} \quad x \in (-1,1) \setminus \{0\}. \tag{3.13}$$

Moreover, setting $x \in G_\varepsilon$ and assuming (3.4), then the null-circulation solution f_ε of the airfoil equation (2.1) satisfies the following convergence result

$$f_\varepsilon(x) \to f(x) \quad \text{when} \quad \varepsilon \to 0 . \tag{3.14}$$

Proof.

From (3.12), we can write

$$f(x) = f_0(x) + \frac{1}{\pi x \sqrt{1 - x^2}} I \tag{3.15}$$

where $I = \int_{-1}^1 \sqrt{1 - t^2}\psi(t)dt$. Thus, it is obvious that (3.13) holds when ψ is odd. In order to show that (3.13) is also valid when ψ is equal to T_m or $U_n, \quad m \in \mathbb{Z}^+\backslash\{0,2\}, n \in \mathbb{Z}^+\backslash\{0\}$, we will look into the integral on the right hand side of (3.15). Note that for $\psi(t) = T_n(t)$ we have

$$I = \int_0^\pi \sin^2(\theta)T_n(\cos(\theta))d\theta$$

, using the change of variables $t = cos(\theta)$.

Now, using $\sin^2(\theta) = \dfrac{1 - \cos(2\theta)}{2}$ and $T_n(\cos(\theta)) = \cos(n\theta)$, we obtain

$$I = \underbrace{\frac{1}{2} \int_0^\pi \cos(n\theta) \, d\theta}_{I_1} \underbrace{-\frac{1}{2} \int_0^\pi \cos(2\theta) \cos(n\theta) \, d\theta}_{I_2}$$

Note that

$$I_1 = \begin{cases} \dfrac{\pi}{2} & , \quad \text{for} \;\; n = 0 \\ 0 & , \quad \text{elsewhere} \end{cases} \quad \text{and} \quad I_2 = \begin{cases} -\dfrac{\pi}{4} & , \quad \text{for} \;\; n = 2 \\ 0 & , \quad \text{elsewhere.} \end{cases}$$

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Thus, when $\psi = T_n$ and $n \in \mathbb{N} \setminus \{0, 2\}$, $I = 0$ and then $f(x) = f_0(x)$.

Analagously, for $\psi(t) = U_n(t)$ we have

$$J = \int_{-1}^{1} \sqrt{1 - t^2} \, U_n(t) \, dt = \int_{0}^{\pi} \sin^2(\theta) U_n(\cos(\theta)) d\theta.$$

As $U_n(\cos(\theta)) = \dfrac{\sin((n+1)\theta)}{\sin(\theta)}$, we get

$$J = \int_{0}^{\pi} \sin(\theta) \sin((n+1)\theta) d\theta = \begin{cases} \dfrac{\pi}{2} & , \quad \text{for} \quad n = 0 \\ 0 & , \quad \text{elsewhere.} \end{cases}$$

Thus, when $\psi = U_n$ and $n \in \mathbb{N} \setminus \{0\}$ there holds $J = 0$ and again $f(x) = f_0(x)$. Finally, as a consequence of the convergence result (3.5) we obtain (3.14). ∎

3.3. First two terms of an asymptotic expansion for particular analytical solutions f_ε

In this section, we derive the first two terms of an asymptotic expansion for the solution f_ε of the airfoil equation (2.1) (set on G_ε) when the right hand side ψ is a Chebyshev polynomial, Prop. 3.5. As a consequence, we exhibit weight functions together with weighted Sobolev spaces for the first two terms of this expansion.

Proposition 3.5. *Let $\psi \in \{T_m \, , \quad m \in \{0, 1, 2, 3\}\}$ (where T_n are the Chebyshev polynomials of the first types). Then, in the framework of section 2, the solution of the airfoil equation (2.1) set in G_ε admits the following asymptotic expansion*

$$f_\varepsilon \approx f_0 + \varepsilon^2 f_2 \quad \text{when} \quad \varepsilon \to 0 . \tag{3.16}$$

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Here, for all $x > 0$, $f_0(x)$ and $f_2(x)$ are given by :

$$f_0(x) = \frac{x^2 - \frac{1}{2}}{x\sqrt{1 - x^2}} \quad and \quad f_2(x) = \frac{-1}{x^3\sqrt{1 - x^2}} \quad when \quad \psi = T_0 , \tag{3.17a}$$

$$f_0(x) = \frac{x^2 - \frac{1}{2}}{\sqrt{1 - x^2}} \quad and \quad f_2(x) = \frac{-1}{4x^2\sqrt{1 - x^2}} \quad when \quad \psi = T_1 , \tag{3.17b}$$

$$f_0(x) = \frac{2x^4 - 2x^2 + \frac{1}{4}}{x\sqrt{1 - x^2}} \quad and \quad f_2(x) = \frac{-1}{8x^3\sqrt{1 - x^2}} \quad when \quad \psi = T_2 ,$$

$$\tag{3.17c}$$

$$f_0(x) = \frac{4x^4 - 5x^2 + 1}{\sqrt{1 - x^2}} \quad and \quad f_2(x) = \frac{1}{2x^2\sqrt{1 - x^2}} \quad when \quad \psi = T_3 . \tag{3.17d}$$

Proof.

The proof of this proposition is based on the derivation of the first two terms of an asymptotic expansion in power series of ε for the analytical solution f_ε of the airfoil equation (2.1) when $\psi \in \{T_m , \quad m \in \{0, 1, 2, 3\}\}$. This derivation is based on the following closed-form expressions, obtained in [9].

$$f_\varepsilon(x) = \frac{\mathrm{sgn}(x)}{R_\varepsilon(x)} \left(x^2 - \frac{1 + \varepsilon^2}{2} \right) \quad when \quad \psi = T_0 \tag{3.18}$$

$$f_\varepsilon(x) = \frac{x\,\mathrm{sgn}(x)}{R_\varepsilon(x)} \left(x^2 - \frac{1 + \varepsilon^2}{2} \right) \quad when \quad \psi = T_1 \tag{3.19}$$

$$f_\varepsilon(x) = \frac{\mathrm{sgn}(x)}{R_\varepsilon(x)} \left(-2x^4 + (2 + \varepsilon^2)x^2 + \frac{\varepsilon^4 - 4\varepsilon^2 - 1}{4} \right) \quad when \quad \psi = T_2 \tag{3.20}$$

$$f_\varepsilon(x) = \frac{x\,\mathrm{sgn}(x)}{R_\varepsilon(x)} \left(4x^4 - (5 + 2\varepsilon^2)x^2 + \frac{-\varepsilon^4 + 5\varepsilon^2 + 2}{2} \right) \quad when \quad \psi = T_3$$

$$\tag{3.21}$$

Observe that the function R_ε^{-1} has the following asymptotic expansion when $\varepsilon \to 0$: for all $x \in G_\varepsilon$,

$$R_\varepsilon^{-1}(x) = \frac{1}{|x|\sqrt{1 - x^2}} \left(1 + \frac{1}{2}\frac{\varepsilon^2}{x^2} + \mathcal{O}(\varepsilon^4) \right) \tag{3.22}$$

Substituting the expansion (3.22) into (3.18) (and (3.19), (3.20), (3.21), respectively) and performing the identification of terms with the same power of ε we infer the expressions of f_0 and f_2 given by (3.17a) (and (3.17b), (3.17c), (3.17d), respectively). ∎

As a consequence of Prop. 3.5, it is possible to specify a functional framework for f_2. We define two weight functions w_0 and w_1 as $w_0(x) := |x|^{11/2}\sqrt{1-x^2}$ and $w_1(x) := |x|^{7/2}\sqrt{1-x^2}$, and we denote by $\mathrm{L}^2_{w_0}(G_0)$ and $\mathrm{L}^2_{w_1}(G_0)$ the associated L^2- weighted Sobolev spaces.

Proposition 3.6. *In the framework of Prop. 3.5, the solution of the airfoil equation (2.1) satisfies at least formally the following expansion*

$$(f_\varepsilon - f_0) \approx \varepsilon^2 f_2, \quad when \quad \varepsilon \to 0 \tag{3.23}$$

with

$$f_2 \in \mathrm{L}^2_{w_0}(G_0) \ when \ \psi \in \{T_0, T_2\}$$

and

$$f_2 \in \mathrm{L}^2_{w_1}(G_0) \ when \ \psi \in \{T_1, T_3\}$$

Proof.

Let $\psi \in \{T_0, T_2\}$. As a consequence of (3.17a) and (3.17c), the function f_2 belongs to the space $\mathrm{L}^2_{w_0}(G_0)$. When $\psi \in \{T_1, T_3\}$, (3.17b) and (3.17d) imply that the function f_2 belongs to the space $\mathrm{L}^2_{w_1}(G_0)$. ∎

The validation of this asymptotic expansion (3.23) consists to prove uniform estimates when ε is small enough

$$\|f_\varepsilon - f_0\|_{\mathrm{L}^2_{w_0}} = \mathcal{O}(\varepsilon^2) \quad when \quad \psi \in \{T_0, T_2\}$$

and

$$\|f_\varepsilon - f_0\|_{\mathrm{L}^2_{w_1}} = \mathcal{O}(\varepsilon^2) \quad when \quad \psi \in \{T_1, T_3\} \ .$$

This part of the work is investigated numerically in Sec. 4 and illustrated with numerical convergence rates.

4. Numerical results

In this section, we show a number of numerical results in corroboration with our asymptotic analysis above. Thus, numerical values of $\|f_\varepsilon - f_0\|_{\mathrm{L}^2_{w_0}}$ are shown

in a table and figures presenting the graphs of the functions $f_0(t)$ and $f_\varepsilon(t)$ for different values of ε are also given.

In figures 1 and 2 the asymptotic behavior of f_ε and its convergence to f_0 is depicted for several cases of the function ψ.

The numerical convergence of $f_\varepsilon(t)$ to $f_0(t)$ is clearly seen in table 4 for $T_j, \quad j = 1, \ldots, 7$.

The plots of $\log(\|f_\varepsilon - f_0\|_{L^2_{w_0}})$ against $\log(\varepsilon)$ in figure 3, demonstrate numerically $\|f_\varepsilon - f_0\|_{L^2_{w_0}} = \mathcal{O}(\varepsilon^2)$ for $\psi \in \{T_0, T_1, T_6\}$ and indicate a convergence rate of order one for $\psi \in \{T_4, T_7\}$.
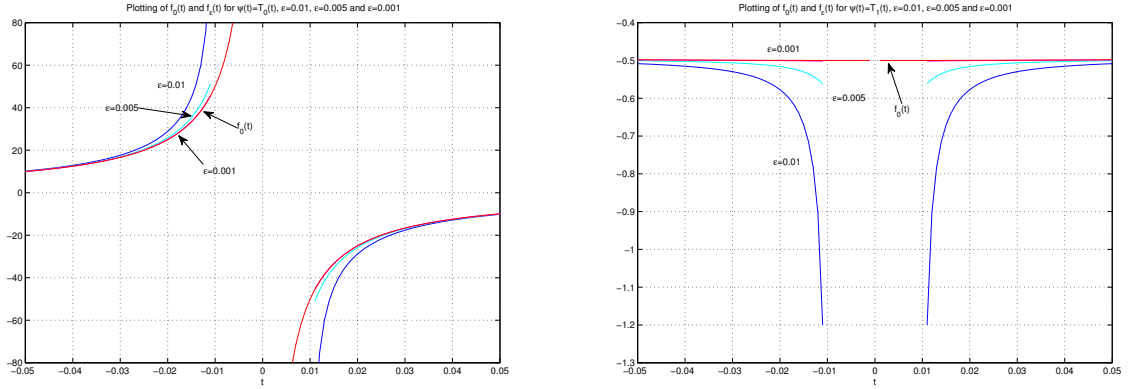
Figure 1: On the left, $f_\varepsilon(t)$ and $f_0(t)$(red) for $\psi(t) = 1$. On the right figure, $f_\varepsilon(t)$ and $f_0(t)$(red) for $\psi(t) = t$. In both figures, $\varepsilon = 0.01$ (blue), $\varepsilon = 0.005$ (cyan), $\varepsilon = 0.001$ (magenta).
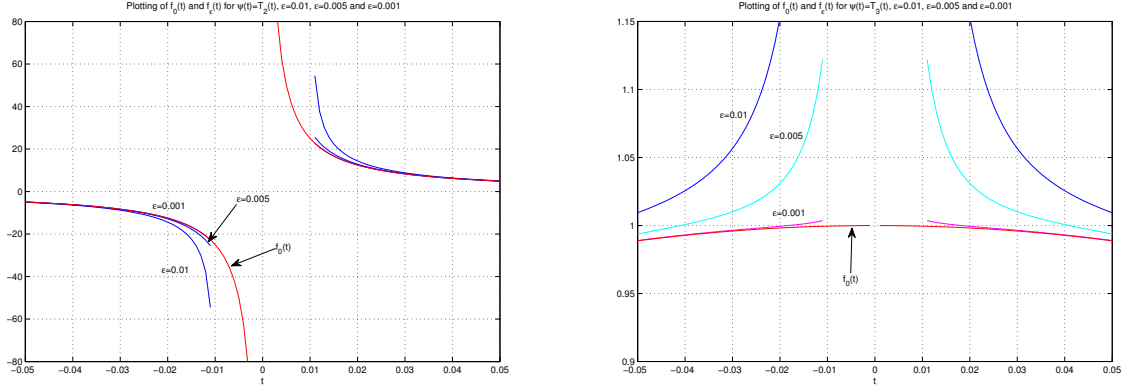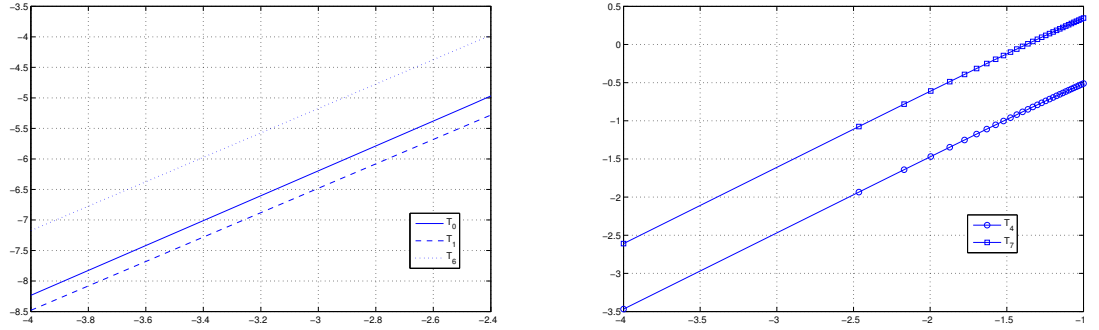
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Figure 2: On the left, $f_\varepsilon(t)$ and $f_0(t)$(red) for $\psi(t) = T_2(t)$. On the right, $f_\varepsilon(t)$ and $f_0(t)$(red) for $\psi(t) = T_3(t)$. In both figures, $\varepsilon = 0.01$ (blue), $\varepsilon = 0.005$ (cyan), $\varepsilon = 0.001$ (magenta)

Figure 3: $\log(\|f_\varepsilon - f_0\|_{\mathrm{L}^2_{w_0}})$ plotted against $\log(\varepsilon)$.

$\psi(t)$ ＼ ε	0.1	0.01	0.001	0.0001
T_0	0.016735069278020	0.000122221398303	0.000000619117636	0.000000005827215
T_1	0.003706513707562	0.000033062469290	0.000000330534162	0.000000003305335
T_2	0.008133702636602	0.000061094486497	0.000000309558293	0.000000002913607
T_3	0.007347436293331	0.000066119977949	0.000000661067827	0.000000006610670
T_4	0.307798166131477	0.033680203253810	0.003398455037006	0.000340151486471
T_5	0.007174486542688	0.000066106750706	0.000000661066506	0.000000006610670
T_6	0.067700651694066	0.000670502998051	0.000006704449030	0.000000067044432
T_7	2.220447951981798	0.243445284662592	0.024569234228085	0.002459182003860

Table 1: The norm $\|f_\varepsilon - f_0\|_{\mathrm{L}^2_{w_0}}$ for different values of ε and functions ψ.

5. Polynomial solutions

In this section we address the issue of existence of polynomial solutions for the airfoil equation (2.1). We propose to answer partially this question using the approximate solution f_0 given by (3.5).

5.1. Approximation of order 1

According to (3.5), we consider f_0 as an approximation of the solution f_ε for the integral equation (2.1). Then we look for polynomial solutions for the Cauchy type singular integral (3.6)

$$\frac{1}{\pi x\sqrt{1-x^2}} \int_{-1}^{1} \frac{\sqrt{1-t^2}}{x-t} t\psi(t)\, \mathrm{d}t = f_0(x) \quad \text{for all} \quad x \in (-1,1) \setminus \{0\} . \qquad (5.1)$$

It is known that for all $x \in (-1,1)$, and $n = 0, 1, 2, \cdots$,

$$\frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1-t^2}}{x-t} U_n(t)\, \mathrm{d}t = T_{n+1}(x) ,$$

where T_n and U_n are Chebyshev polynomials of the first and second kinds, respectively (see e.g. [8]). Hence, setting

$$\psi(x) = x^{-1} U_n(x) \qquad (5.2)$$

for all $x \in (-1,1)$ (and $x \neq 0$ when n is even) in (5.1), there holds

$$f_0(x) = \frac{1}{x\sqrt{1-x^2}} T_{n+1}(x). \qquad (5.3)$$

5.2. A spectral method of the generalized airfoil equation in close disjoint intervals

Given the results represented above we can now propose a spectral method for an approximate of the generalized equation [10]

$$\frac{1}{\pi} \int_{G_\epsilon} \left\{ \frac{1}{x-t} + K(x,t) \right\} f_\epsilon(t)\, w(t)\mathrm{d}t = h(x) , \quad x \in G_\varepsilon, \qquad (5.4)$$

where K is a regular (non strongly singular) function and $w(t) = \frac{1}{t\sqrt{1-t^2}}$.

Using (3.5) and Proposition 3.3, we get

$$\frac{1}{\pi} \int_{G_\epsilon} \frac{f_0(t)}{x-t} dt \approx -\psi(x), \quad x \in G_\epsilon, \quad \epsilon << 1 \qquad (5.5)$$

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and from (5.2),(5.3) and (5.5),

$$\frac{1}{\pi}\int_{G_\epsilon}\frac{1}{x-t}T_{n+1}(t)\frac{1}{t\sqrt{1-t^2}}dt \approx -x^{-1}U_n(x), \quad x \in G_\epsilon, \quad \epsilon << 1 \qquad (5.6)$$

Expand the solution as

$$f_\varepsilon(t) \simeq \sum_{n=0}^{N} a_n T_{n+1}(t). \qquad (5.7)$$

Substituting in (5.4) and using (5.6),

$$\sum_{n=0}^{N} a_n[-x^{-1}U_n(x) + \mathcal{K}T_{n+1}] = \psi(x), \qquad (5.8)$$

where $\mathcal{K}$ is the integral operator given by

$$\mathcal{K}(g) = \frac{1}{\pi}\int_{G_\epsilon} K(x,t)g(t)\ w(t)\mathrm{d}t.$$

Equation (5.8) can be solved by a Galerkin or collocation method.

The spectral method consists of (5.7) and (5.8).

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