A study of the apsidal angle and a proof of monotonicity in the logarithmic potential case

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Abstract

This paper concerns the behaviour of the apsidal angle for orbits of central force system with homogeneous potential of degree $-2 \leq \alpha \leq 1$ and logarithmic potential. We derive a formula for the apsidal angle as a fixed end-points integral and we study the derivative of the apsidal angle with respect to the angular momentum ℓ. The monotonicity of the apsidal angle as function of ℓ is discussed and it is proved in the logarithmic potential case.

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1. Introduction

In this paper we investigate the apsidal precession for the orbits of central force systems of the form

$$\ddot{u} = \nabla V_\alpha(|u|), \quad u \in \mathbb{R}^2 \tag{1}$$

where

$$\begin{aligned} V_\alpha(x) &= \frac{1}{\alpha}\frac{1}{x^\alpha}, \qquad \alpha \in [-2,1] \setminus \{0\} \\ V_0(x) &= -\log(x) \quad \alpha = 0. \end{aligned} \tag{2}$$

The solutions of system (1) preserve the mechanical energy $E = \frac{1}{2}|\dot{u}|^2 - V$ and the angular momentum $\vec{\ell} = u \wedge \dot{u}$ and, while rotating, all the bounded non-collision orbits oscillate between the apses, that is, the points of minimal (pericenter) and maximal (apocenter) distance from the origin. The apsidal angle is defined as the angle at the origin swept by the orbit between two consecutive apses.

For an admissible choice of E and ℓ (being ℓ the scalar angular momentum, positive for counterclockwise orbits and negative for clockwise orbits), the apsidal angle of an orbit $u(t)$ is given by

$$\Delta_\alpha\theta(u) = \int_{r_-}^{r_+} \frac{\ell}{r^2\sqrt{2\big(E+V(r)\big) - \frac{\ell^2}{r^2}}}\, dr \tag{3}$$

where $r_\pm$ are the points where the denominator vanishes.

The extremal cases $\alpha = -2$ and $\alpha = 1$ consist in the harmonic oscillator and the Kepler problem respectively. Systems with $-2 < \alpha < 1$ are of relevant interest in quantum mechanics and astrophysics, for instance in describing the galactic dynamics in presence of power law densities or massive black-holes and in modelling the gravitational lensing, see [1, 2]. Also, the logarithmic potential appears in particle physics [3, 4] and in a model for the dynamics of vortex filaments in ideal fluid, see e.g. [5].

Since Newton, the behaviour of the apsidal angle has been extensively studied, in particular for its implications to celestial mechanics. In the Book I of the *Principia* Newton derived a formula that relates the apsidal angle to the magnitude of the attracting force: for "close to circular" orbits, a force field of the form κr^{n-3} leads to an apsidal angle equal to $\pi/\sqrt{n}$. Hence, the experimental measurement of the precession of an orbit may give the exponent of the force law. Newton himself in Book III, looking at the orbit of the Earth and of the Moon, concludes that the attracting force of the Sun and of the Earth must be inverse square of the distance. "Close to circular" means that this formula has been proved for orbits with small eccentricity, or equivalently, with angular momentum close to ℓ_{max}.

In the singular case ($\alpha \geq 0$), the behaviour of the apsidal angle plays a fundamental role in dealing with the collision solutions of (1) and related systems: a possible regularisation of the singularities of the flow is possible only in case that twice the apsidal angle tends to a multiple of π as $\ell \to 0$, see for instance [6, 7, 8, 9]. Also a variational approach to system (1) may lead to collision avoidance provided $\Delta_\alpha\theta(u) > \pi$ as $|u| \to 0$, see [10]. Moreover, as pointed out in [11], the derivative of the apsidal angle with respect to the angular momentum has common feature with the derivative of the potential scattering phase shift and applications in a variety of fields, such as nuclear physics and astrophysics.

Two are the main technical hurdles that make both the analytical treatment and the numerical investigation of the integral (3) a difficult problem: the integrand is singular at the end-points and the end-points themselves are only implicitly defined as function of E and ℓ. Nevertheless, partial and important results are very well known. In 1873, Bertrand proved that there are only two central force laws for which all bounded orbits are closed, namely the linear and the inverse square [12]. It means that the apsidal angle of any bounded solution of the elastic and Kepler problem is rationally proportional to π for any value of E and ℓ. In the first case it values $\pi/2$ and π in the second. In addition, for $\alpha = -\frac{2}{3}, -1, \frac{1}{2}, \frac{2}{3}$, the apsidal angle can be expressed in terms of elliptic functions, see [13], but in general $\Delta_\alpha\theta(u)$ cannot be written in closed form.

Valluri et al. [11] propose an expansion of $\Delta_\alpha\theta(u)$ and $\partial_\ell\Delta_\alpha\theta(u)$ in terms of the eccentricity of the orbit for α close 1. A study for the logarithmic case using the $p - ellipse$ approximation and the Lambert W function is presented in [2]. Touma and Tremaine in [1], by means of the Mellin transform, provide an asymptotic series on ℓ for any $\alpha \in [-2, 1]$. Also, by applying a generalisation of the Bürmann-Lagrange series for multivalued inverse function, Santos et al. in [14] obtain a series expression for the apsidal angle valid for any central force field.

Several numerical simulations, like the ones proposed in the cited papers, show that for any fixed α and E the apsidal angle is a monotonic function of the angular momentum. However, at the best of the author's knowledge, no proof of this statement is available in the literature. The purpose of this paper is to review the notion of the apsidal angle and to derive a formula for its derivative with respect to the angular momentum; from where, the monotonicity will be evident for any $\alpha \in (-2, 1)$ and proved in the logarithmic potential case.

The plan of the paper is the following: in the next section we recall the orbital structure of the solutions of (1) and we write a formula for the apsidal angle as a fixed-ends integral (7), where the parameter q plays the role of the angular momentum. Then, section 3 is dedicated to the analytical study of the integrand, in particular of the function $\mathcal{E}_\alpha(s, q)$. In section 4 the results so far obtained are used to compute the limit values of the apsidal angle for radial and close to circular orbits. In section 5 we compute the derivative of the apsidal angle and we provide insight into the monotonicity in the α-homogeneous case. Finally, in section 6 the apsidal angle for $\alpha = 0$ is proved to be monotonically increasing as function of the angular momentum. In Appendix some technical lemmas are collected.

2. Orbital structure and apsidal angle

In polar coordinates $u = (r, \theta)$ the energy associated to a solution $u(t)$ of the problem (1) is

$$E(u) = \frac{1}{2}\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{r^2} - V_\alpha(r)$$

hence the motion is allowed only for those r where $E > V_\alpha^{eff} = \frac{1}{2}\frac{\ell^2}{r^2} - V_\alpha(r)$. From the analysis of the V_α^{eff}, see also Fig. 1, we realise that

- for $\alpha \in (0, 1]$: bounded orbits exist for negative energies and, for any value $E < 0$, the modulus of the angular momentum ℓ ranges between 0 and $\ell_{max} = \left(\frac{2\alpha}{\alpha-2}E\right)^{-\frac{2-\alpha}{2\alpha}}$. Values $E \geq 0$ give rise to unbounded orbits;
- for $\alpha = 0$: since $\log(r) \to \infty$ as $r \to \infty$, all the possible orbits are bounded. For any value of E, the motion is possible provided $0 \leq |\ell| \leq \ell_{max} = e^{E-\frac{1}{2}}$;
- for $\alpha \in [-2, 0)$: the potential function V_α is negative therefore the motion exists only for positive values of E. In particular all the orbits are bounded and the $|\ell|$ ranges within the interval $[0, \ell_{\max}]$ with $\ell_{max} = \left(\frac{2\alpha}{\alpha-2}E\right)^{-\frac{2-\alpha}{2\alpha}}$.

The radial coordinate $r(t)$ of the orbit $u(t)$ solves the differential equation

$$\ddot{r} - \frac{\ell^2}{r^3} + \frac{1}{r^{\alpha+1}} = 0$$

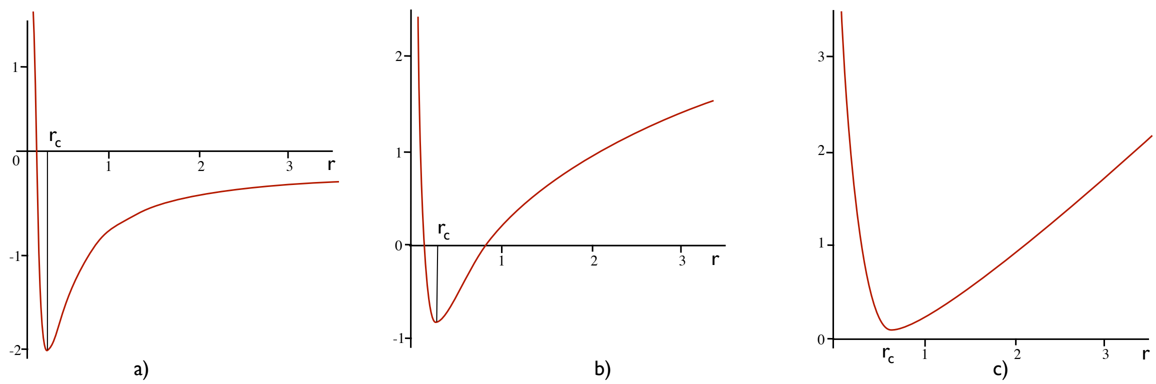

Figure 1: Plot of the function $V^{eff}_\alpha(r)$ for $a)$ $\alpha \in (0,1]$, $b)$ $\alpha = 0$, $c)$ $\alpha \in [-2,0)$. In each graphic r_c is the radius where the minimum of $V^{eff}_\alpha(r)$ is achieved and it corresponds to the radius of the circular orbit. It holds : in $a)$ and $c)$ $r_c = \ell^{\frac{2}{2-\alpha}}$, $V^{eff}_\alpha(r_c) = -\frac{2-\alpha}{2\alpha}\ell^{-\frac{2\alpha}{2-\alpha}}$, b) $r_c = \ell$, $V^{eff}_0(r_{c)} = \frac{1}{2} + \log(\ell)$.

and oscillates periodically between the extremal values r_+ and r_- given as positive solutions of the apses equation $\frac{1}{2}\frac{\ell^2}{r^2} - V_\alpha(r) - E = 0$. While oscillating, the solution rotates around the centre of attraction, in direction according to the sign of ℓ.

From the relation $\ell = r^2\dot\theta$ it follows that the apsidal angle, that is the rate of rotation between two consecutive apses, is given by $\int_{r_-}^{r_+} \frac{\ell}{r^2\dot r}\,dr$, from where formula (3) comes.

Remark 2.1. *The apsidal angle $\Delta_\alpha\theta(u)$ is odd as function of ℓ.*

Remark 2.2. *From* (3) *it descends that $\Delta_\beta\theta(u) = \frac{2-\alpha}{2}\Delta_\alpha\theta(u)$, where $\beta = -\frac{2\alpha}{2-\alpha}$, see also [15]. Thus apsidal angle for $\alpha \in [-2,0)$ is determined by the value for $\alpha \in (0,1]$.*

In light of the two remarks and since the case $\alpha = 1$ is completely solved, henceforth we restrict our analysis to

$$\alpha \in [0,1), \quad \ell \in (0,1).$$

A standard approach when studying the apsidal angle is to consider the differential equation

$$\frac{d^2}{d\theta^2}z + z = \phi\left(\frac{1}{z}\right)\frac{1}{z^2\ell^2}$$

where $z = \frac{1}{r}$ and $\phi(r)$ is the central force. From here, following Bertrand [12] and Griffin [16], it holds

$$\Delta_\alpha\theta(u) = \int_a^b \frac{\sqrt{w(b)-w(a)}}{\sqrt{a^2w(b) - b^2w(a) + (b^2-a^2)w(z) - z^2(w(b)-w(a))}}dz \tag{4}$$

where the function $w(z) = 2\int \psi(z)\, dz$ and $\psi(z) = r^2\phi(r)$. In practice, for the force field we are interest in, it holds

$$\phi(r) = -\frac{1}{r^{\alpha+1}}, \quad \psi(z) = z^{\alpha-1}, \quad w(z) = \frac{2}{\alpha} z^{\alpha}, \qquad \alpha \neq 0$$

$$\phi(r) = -\frac{1}{r}, \quad \psi(z) = z^{-1}, \quad w(z) = 2\log(z), \qquad \alpha = 0.$$

The dependence on ℓ and on the energy of the integral (4) is through the values of the $a = 1/r_+$ and $b = 1/r_-$. Indeed, in the logarithmic potential case, such values are given as solutions of the equation $\log(z^2) - \ell^2 z^2 = -2E$, whose roots can only be expressed as

$$z = \sqrt{\frac{-W_j(-\ell^2 e^{-2E})}{\ell}}$$

where W_j represent the two branches of the Lambert W function, [17]. Similarly, in the homogeneous case, a and b are the solutions of $-\ell^2 z^2 + \frac{2}{\alpha} z^{\alpha} = -2E$.

From (4), by easy manipulation it follows

$$\Delta_\alpha\theta(u) = \int_a^b \frac{dz}{\sqrt{b^2 - z^2 - (b^2 - a^2)\frac{w(b)-w(z)}{w(b)-w(a)}}} = \int_a^b \frac{dz}{\sqrt{(b-z)\left((b+z) - (b+a)\frac{\frac{w(b)-w(z)}{b-z}}{\frac{w(b)-w(a)}{b-a}}\right)}}$$

$$= \int_a^b \frac{1}{\sqrt{z-a}} \frac{1}{\sqrt{b-z}} \frac{1}{\sqrt{1+\varepsilon(z)}}\, dz$$

where

$$\varepsilon(z) = \frac{b+a}{z-a}\left[1 - \frac{\frac{w(b)-w(z)}{b-z}}{\frac{w(b)-w(a)}{b-a}}\right]. \tag{5}$$

Let $L := b - a$ and perform the change of variable $s = \frac{z-a}{L}$; it results

$$\Delta_\alpha\theta(u) = \int_0^1 \frac{1}{\sqrt{s(1-s)}} \frac{1}{\sqrt{1+\epsilon(s)}}\, ds \tag{6}$$

where

$$\epsilon(s) = \frac{(2b-L)}{Ls}\left[1 - \frac{w(b) - w\big(b - L(1-s)\big)}{(1-s)\big(w(b) - w(b-L)\big)}\right].$$

Then, taking $q = \frac{L}{b}$, we obtain

$$\Delta_\alpha\theta(u) = \int_0^1 \frac{1}{\sqrt{s(1-s)}} \frac{1}{\sqrt{1+\mathcal{E}_\alpha(s,q)}} ds \tag{7}$$

with

$$
\begin{aligned}
\mathcal{E}_\alpha(s,q) &= \frac{2-q}{qs}\left[1-\frac{1-\big(1-q(1-s)\big)^\alpha}{(1-s)(1-(1-q)^\alpha)}\right], \quad \alpha\in(0,1) \\
\mathcal{E}_0(s,q) &= \frac{2-q}{qs}\left[1-\frac{\log(1-q(1-s))}{(1-s)\log(1-q)}\right], \qquad \alpha=0
\end{aligned}
\tag{8}
$$

For any fixed value of the mechanical energy, the parameter $q = 1-\frac{a}{b}$ is a function of the angular momentum ℓ. In particular, as ℓ increases from zero to $\ell_{\max}$, the ratio a/b increases from 0 to 1. Therefore

$$
q = q(\ell)\in(0,1), \qquad \frac{dq}{d\ell}<0, \quad \forall \ell\in(0,1).
$$

The integral in (7) is the main object of investigation of the present paper. Two are the reasons why it is advantageous to analyse (7) instead of the classical one (3): in the first integral the end-points are fixed and the dependence on the angular momentum is confined and well separated from the singularities of the integrand function.

The behaviour of the apsidal angle as the angular momentum ℓ varies is intimately related to the behaviour of the function $\mathcal{E}_\alpha(q,s)$ when s, q vary in their domain. Hence, we first present a detailed analysis of the function $\mathcal{E}_\alpha(s,q)$ and its derivative $\partial_q\mathcal{E}_\alpha(s,q)$.

3. Analysis of the functions $\mathcal{E}_\alpha(s,q)$

It is convenient to introduce the weights

$$
\omega_n^\alpha := \begin{cases} -\binom{\alpha}{n}(-1)^n & \alpha\in(0,1) \\ \frac{1}{n} & \alpha=0 \end{cases} \qquad \forall n\geq 1 \tag{9}
$$

and two auxiliary functions: for any $n\geq 2$ let

$$
A_n(s) := \frac{1-(1-s)^{n-1}}{s}, \quad s\in(0,1)
$$

and, for any $n\geq 2$ and $\alpha\in[0,1)$ let

$$
K_n^\alpha(s) := 2\frac{\omega_{n+1}^\alpha}{\omega_n^\alpha}A_{n+1}(s)-A_n(s) = 2\frac{n-\alpha}{n+1}A_{n+1}(s)-A_n(s), \quad s\in(0,1). \tag{10}
$$

Note that $\omega_n^\alpha>0$ for any $n\geq1$ and $\alpha\in[0,1)$. We collect in the Appendix all the properties of the functions $A_n(s)$ and $K_n^\alpha(s)$ used throughout the paper.

Recalling the power series expansions:

$$
(1-x)^\alpha = 1+\sum_{n\geq1}\binom{\alpha}{n}(-1)^n x^n, \quad \log(1-x) = -\sum_{n\geq1}\frac{x^n}{n}, \quad |x|<1
$$

it holds, for $\alpha \in (0,1)$,

$$\mathcal{E}_\alpha(s,q) = \frac{2-q}{qs}\left[\frac{-(1-s)\sum_{n\geq 1}\binom{\alpha}{n}(-1)^n q^n + \sum_{n\geq 1}\binom{\alpha}{n}(-1)^n q^n (1-s)^n}{(1-s)\left(-\sum_{n\geq 1}\binom{\alpha}{n}(-1)^n q^n\right)}\right]$$

$$= \frac{2-q}{qs\left(-\sum_{n\geq 1}\binom{\alpha}{n}(-1)^n q^n\right)}\left[-\sum_{n\geq 2}\binom{\alpha}{n}(-1)^n q^n + \sum_{n\geq 2}\binom{\alpha}{n}(-1)^n q^n (1-s)^{n-1}\right]$$

$$= \frac{1}{\sum_{n\geq 1}\binom{\alpha}{n}(-1)^n q^{n-1}}\sum_{n\geq 2}\binom{\alpha}{n}(-1)^n q^{n-2}(2-q)A_n(s)$$

and similarly, for $\alpha = 0$,

$$\mathcal{E}_0(s,q) = \frac{1}{\sum_{n\geq 1}\frac{1}{n}q^{n-1}}\sum_{n\geq 2}\frac{1}{n}(2-q)q^{n-2}A_n(s).$$

By means of the weights ω_n^α, for any $\alpha \in [0,1)$ we can write in compact form

$$\mathcal{E}_\alpha(s,q) = \frac{1}{\sum_{n\geq 1}\omega_n^\alpha q^{n-1}}\sum_{n\geq 2}\omega_n^\alpha (2-q)q^{n-2}A_n(s). \tag{11}$$

Proposition 3.1. *For any $\alpha \in [0,1)$ and $q \in (0,1)$ the function $\mathcal{E}_\alpha(q,s)$ is monotonically decreasing and convex in s for any $s \in (0,1)$.*

Proof. Since $A_2'(s) = 0$, for any $\alpha \in [0,1)$

$$\frac{d}{ds}\mathcal{E}_\alpha(s,q) = \frac{1}{\sum_{n\geq 1}\omega_n^\alpha q^{n-1}}\sum_{n\geq 3}\omega_n^\alpha q^{n-2}(2-q)A_n'(s)$$

and

$$\frac{d^2}{ds^2}\mathcal{E}_\alpha(s,q) = \frac{1}{\sum_{n\geq 1}\omega_n^\alpha q^{n-1}}\sum_{n\geq 4}\omega_n^\alpha q^{n-2}(2-q)A_n''(s).$$

Since $\omega_n^\alpha > 0$ and, for point *iii*) of lemma 7.1, $A_n' < 0$ and $A_n'' > 0$, the thesis follows. ■

Corollary 3.2. *For any $\alpha \in [0,1)$ the function $\mathcal{E}_\alpha(s,q) > 0$ for any $s \in (0,1)$, $q \in (0,1)$.*

Proof. From the previous proposition

$$\mathcal{E}_\alpha(s,q) \geq \mathcal{E}_\alpha(1,q) = \begin{cases} \frac{2-q}{q}\left(\frac{1-(1-q)^\alpha - \alpha q}{1-(1-q)^\alpha}\right), & 0 < \alpha < 1 \\ \frac{2-q}{q}\left(1 + \frac{q}{\log(1-q)}\right), & \alpha = 0. \end{cases}$$

Since $(1-q)^\alpha < 1-\alpha q$ and $-\log(1-q) > q$, it follows $\mathcal{E}_\alpha(1,q) > 0$ for any $q \in (0,1)$. ■

Consider now the derivative of $\mathcal{E}_\alpha(s,q)$ with respect to the variable q. For $\alpha \in [0,1)$

$$\begin{aligned}\partial_q \mathcal{E}_\alpha(s,q) &= \frac{1}{\left(\sum_{n\geq 1}\omega_n^\alpha q^{n-1}\right)^2}\cdot\Bigg\{\left(\sum_{n\geq 2}\omega_n^\alpha\Big((n-2)(2-q)-q\Big)q^{n-3}A_n(s)\right)\left(\sum_{n\geq 1}\omega_n^\alpha q^{n-1}\right)\\ &\qquad -\left(\sum_{n\geq 2}\omega_n^\alpha (2-q)q^{n-2}A_n(s)\right)\left(\sum_{n\geq 1}\omega_n^\alpha (n-1)q^{n-2}\right)\Bigg\} \qquad (12)\\ &= \frac{1}{\left(\sum_{n\geq 1}\omega_n^\alpha q^{n-1}\right)^2}\cdot\sum_{n\geq 2, m\geq 1}\omega_n^\alpha\omega_m^\alpha\Big((2-q)(n-m)-2\Big)A_n(s)q^{(n-2)+(m-2)}.\end{aligned}$$

Henceforth, let us denote

$$C_\alpha(q) := \left(\sum_{n\geq 1}\omega_n^\alpha q^{n-1}\right)^{-2}.$$

In the sequel we prove that the function $\partial_q\mathcal{E}_\alpha(s,q)$ is monotonic and convex in s. For that let us introduce the quantity

$$H_\alpha(n,m) := \omega_{n+2}^\alpha\omega_m^\alpha(n-m)$$

and we prove the following.

Lemma 3.3. *Let $n,m\geq 1$ and $n>m$. Then, for any $\alpha\in[0,1)$,*

$$H_\alpha(n,m)+H_\alpha(m,n) > 0.$$

Proof. Case $\alpha = 0$.

$$H_0(n,m)+H_0(m,n) = \left(\frac{1}{n+2}\frac{1}{m}-\frac{1}{m+2}\frac{1}{n}\right)(n-m) = 2\frac{(n-m)^2}{nm(n+2)(m+2)} > 0.$$

Let be now $\alpha\in(0,1)$. For any integer t, we remind the relations

$$\binom{\alpha}{t+1} = \frac{\alpha-t}{t+1}\binom{\alpha}{t},\qquad \binom{\alpha}{t+2} = \frac{\alpha-(t+1)}{t+2}\frac{\alpha-t}{t+1}\binom{\alpha}{t}.$$

Then

$$\begin{aligned}H_\alpha(n,m)+H_\alpha(m,n) &= \left(\frac{\alpha-(n+1)}{n+2}\frac{\alpha-n}{n+1}-\frac{\alpha-(m+1)}{m+2}\frac{\alpha-m}{m+1}\right)\omega_n^\alpha\omega_m^\alpha(n-m)\\ &= \left(\frac{(n+1)-\alpha}{n+2}\frac{n-\alpha}{n+1}-\frac{(m+1)-\alpha}{m+2}\frac{m-\alpha}{m+1}\right)\omega_n^\alpha\omega_m^\alpha(n-m)\ .\end{aligned} \qquad (13)$$

Since $\frac{K-\alpha}{K+1} > \frac{P-\alpha}{P+1} > 0$ whenever $K > P \geq 1$, it follows $H_\alpha(n,m)+H_\alpha(m,n) > 0$ for any $n > m$. ■

Proposition 3.4. *For any $\alpha \in [0,1)$ and $q \in (0,1)$*

i)

$$\frac{d}{ds}\Big(\partial_q \mathcal{E}_\alpha(s,q)\Big) < 0, \quad \forall s \in (0,1)$$

ii)

$$\frac{d^2}{ds^2}\Big(\partial_q \mathcal{E}_\alpha(s,q)\Big) > 0, \quad \forall s \in (0,1).$$

Proof. *i*) From (12), since $A_2'(s) = 0$,

$$\begin{aligned}\frac{d}{ds}\left(\partial_q \mathcal{E}_\alpha(s,q)\right) &= C_\alpha(q) \sum_{n\geq 3, m\geq 1} \omega_n^\alpha \omega_m^\alpha \Big((2-q)(n-m)-2\Big) A_n'(s) q^{(n-2)+(m-2)} \\ &= C_\alpha(q) \sum_{p\geq 4} \left[\sum_{\substack{n+m=p \\ n\geq 3, m\geq 1}} \omega_n^\alpha \omega_m^\alpha \Big((2-q)(n-m)-2\Big) A_n'(s) \right] q^{p-4}.\end{aligned}$$

Denote

$$Coef(s,q,\alpha,p) := \sum_{\substack{n+m=p \\ n\geq 3, m\geq 1}} \omega_n^\alpha \omega_m^\alpha \Big((2-q)(n-m)-2\Big) A_n'(s).$$

We prove that $Coef(s,q,\alpha,p) < 0$ for any $s \in (0,1)$, $q \in (0,1)$ and $\alpha \in [0,1)$ and $p \geq 4$. Compute first the derivative in q:

$$\frac{d}{dq} Coef(s,q,\alpha,p) = -\sum_{\substack{n+m=p \\ n\geq 3, m\geq 1}} \omega_n^\alpha \omega_m^\alpha (n-m) A_n'(s).$$

For point *iii*) in lemma 7.1 $A_n'(s) < 0$, then if $p \leq 6$ the above sum is obviously positive, being $n \geq m$ in any contribution. For $p \geq 7$, we write

$$\frac{d}{dq} Coef(s,q,\alpha,p) = -\omega_{p-1}^\alpha \omega_1^\alpha (p-2) A_{p-1}'(s) - \omega_{p-2}^\alpha \omega_2^\alpha (p-4) A_{p-2}'(s) - \sum_{\substack{n+m=p \\ n\geq 3, m\geq 3}} \omega_n^\alpha \omega_m^\alpha (n-m) A_n'(s).$$

The first and the second terms in the right hand side are positive then, collecting the contributions to the sum due to the couples (n,m) and (m,n), and for point *iv*) in lemma 7.1, it holds

$$\frac{d}{dq} Coef(s,q,\alpha,p) > -\sum_{\substack{n+m=p \\ n > m\geq 3}} \omega_n^\alpha \omega_m^\alpha (n-m)(A_n'(s) - A_m'(s)) > 0.$$

Therefore, $\frac{d}{dq}Coef(s,q,\alpha,p) > 0$ for any $p \geq 4$ and $s \in (0,1)$, $q \in (0,1)$, $\alpha \in [0,1)$. It means that

$$Coef(s,q,\alpha,p) < Coef(s,1,\alpha,p).$$

Now,

$$\begin{aligned} Coef(s,1,\alpha,p) &= \sum_{\substack{n+m=p \\ n\geq 3, m\geq 1}} \omega_n^\alpha \omega_m^\alpha \Big(n-m-2\Big) A_n'(s) = \sum_{\substack{n+m=p-2 \\ n\geq 1, m\geq 1}} \omega_{n+2}^\alpha \omega_m^\alpha \Big(n-m\Big) A_{n+2}'(s) \\ &= \sum_{\substack{n+m=p-2 \\ n>m\geq 1}} \Big[\omega_{n+2}^\alpha \omega_m^\alpha \Big(n-m\Big) A_{n+2}'(s) + \omega_{m+2}^\alpha \omega_n^\alpha \Big(m-n\Big) A_{m+2}'(s)\Big]. \end{aligned}$$

Since $\omega_{n+2}^\alpha \omega_m^\alpha > 0$ and, again for property $iii)$ of lemma 7.1, $A_{n+2}'(s) < A_{m+2}'(s)$, it follows

$$\begin{aligned} Coef(s,1,\alpha,p) &< \sum_{\substack{n+m=p \\ n>m\geq 1}} \Big[\omega_{n+2}^\alpha \omega_m^\alpha \Big(n-m\Big) + \omega_{m+2}^\alpha \omega_n^\alpha \Big(m-n\Big)\Big] A_{m+2}'(s) \\ &= \sum_{\substack{n+m=p \\ n>m\geq 1}} (H_\alpha(n,m) + H_\alpha(m,n)) A_{m+2}'(s) < 0 \end{aligned}$$

in light of lemma 3.3. Therefore all the coefficients $Coef(s,q,\alpha,p) < 0$ and $\frac{d}{ds}\left(\partial_q \mathcal{E}_\alpha(s,q)\right) < 0$.

$ii)$ Consider now the second derivative

$$\frac{d^2}{ds^2}\Big(\partial_q \mathcal{E}(s,q)\Big) = C_\alpha(q) \sum_{p\geq 5} \sum_{\substack{n+m=p \\ n\geq 4, m\geq 1}} \omega_n^\alpha \omega_m^\alpha \Big((2-q)(n-m) - 2\Big) A_n''(s) q^{p-4} .$$

Arguing as before, we realise that $q = 1$ represents the worst case and, since $A_n''(s) > 0$, it follows

$$\sum_{\substack{n+m=p \\ n\geq 4, m\geq 1}} \omega_n^\alpha \omega_m^\alpha \Big((n-m) - 2\Big) A_n''(s) > 0, \quad \forall s \in (0,1), q \in (0,1), \alpha \in (0,1), p \geq 5.$$

$\blacksquare$

Up to now we investigated the monotonicity and the convexity of $\partial_q \mathcal{E}_\alpha(q,s)$ without mentioning the value of $\partial_q \mathcal{E}_\alpha(q,s)$. To the purpose of proving the monotonicity of the apsidal angle, it would be enough that $\partial_q \mathcal{E}_\alpha(q,s)$ is positive for any s. Unfortunately that is not the case, indeed for any α there is a region of q where $\partial_q \mathcal{E}_\alpha(q,s)$ is not everywhere positive. Nevertheless, we prove now that $\partial_q \mathcal{E}_\alpha(q,s) > 0$ at $s = 1/2$ for any α and q. For the previous result it follows that $\partial_q \mathcal{E}_\alpha(q,s) > 0$ for any $s \in (0,1/2)$, for any q and α.

Proposition 3.5. *For any $q \in (0,1)$ and $\alpha \in [0,1)$*

$$\partial_q \mathcal{E}_\alpha \left(\frac{1}{2}, q\right) > 0.$$

Proof.

$$\begin{aligned}
\partial_q \mathcal{E}_\alpha \left(\frac{1}{2}, q\right) &= C_\alpha(q) \sum_{p\geq 3} \sum_{\substack{n+m=p\\ n\geq 2, m\geq 1}} \omega_n^\alpha \omega_m^\alpha \Big((2-q)(n-m)-2\Big) A_n(1/2) q^{p-4} \\
&= C_\alpha(q) \sum_{p\geq 3} \left[\sum_{\substack{n+m=p\\ n\geq 2, m\geq 1}} \omega_n^\alpha \omega_m^\alpha \Big(2(n-m-1)\Big) A_n(1/2) q^{p-4} \right. \\
&\qquad \left. - \sum_{\substack{n+m=p\\ n\geq 2, m\geq 1}} \omega_n^\alpha \omega_m^\alpha \Big(n-m\Big) A_n(1/2) q^{p-3} \right].
\end{aligned}$$

When $p = 3$ the first sum does not contribute, then we can write

$$\partial_q \mathcal{E}_\alpha \left(\frac{1}{2}, q\right) = C_\alpha(q) \sum_{p\geq 4} \left[\sum_{\substack{n+m=p\\ n\geq 2, m\geq 1}} \omega_n^\alpha \omega_m^\alpha 2(n-m-1) A_n(1/2) - \sum_{\substack{n+m=p-1\\ n\geq 2, m\geq 1}} \omega_n^\alpha \omega_m^\alpha \Big(n-m\Big) A_n(1/2) \right] q^{p-4}.$$

Denote by

$$Q_p(q) := \sum_{\substack{n+m=p\\ n\geq 2, m\geq 1}} \omega_n^\alpha \omega_m^\alpha 2(n-m-1) A_n(1/2) - \sum_{\substack{n+m=p-1\\ n\geq 2, m\geq 1}} \omega_n^\alpha \omega_m^\alpha (n-m) A_n(1/2)$$

the coefficient of q^{p-4}. For $p \geq 4$

$$\begin{aligned}
Q_p(q) &= \sum_{\substack{n+m=p-1\\ n\geq 1, m\geq 1}} \omega_{n+1}^\alpha \omega_m^\alpha 2(n-m) A_{n+1}(1/2) - \sum_{\substack{n+m=p-1\\ n\geq 2, m\geq 1}} \omega_n^\alpha \omega_m^\alpha (n-m) A_n(1/2) \\
&= \omega_2^\alpha \omega_{p-2}^\alpha 2(3-p) A_2(1/2) + \sum_{\substack{n+m=p-1\\ n\geq 2, m\geq 1}} \omega_n^\alpha \omega_m^\alpha (n-m) \Big(2\frac{\omega_{n+1}^\alpha}{\omega_n^\alpha} A_{n+1}(1/2) - A_n(1/2) \Big) \\
&= \omega_2^\alpha \omega_{p-2}^\alpha 2(3-p) A_2(1/2) + \omega_{p-2}^\alpha \omega_1^\alpha (p-3) \left(2\frac{\omega_{p-1}^\alpha}{\omega_{p-2}^\alpha} A_{p-1}(1/2) - A_{p-2}(1/2) \right) \\
&\quad + \sum_{\substack{n+m=p-1\\ n\geq 2, m\geq 2}} \omega_n^\alpha \omega_m^\alpha (n-m) \left(\frac{2\omega_{n+1}^\alpha}{\omega_n^\alpha} A_{n+1}(1/2) - A_n(1/2) \right).
\end{aligned}$$

Reminding the definition (10) of $K_n^\alpha(s)$ and collecting the contributions due to the couple (n, m) and (m, n), it results

$$Q_p(q) = \omega_{p-2}^\alpha (p-3)\left[2\omega_1^\alpha \frac{\omega_{p-1}^\alpha}{\omega_{p-2}^\alpha} A_{p-1}(1/2) - \omega_1^\alpha A_{p-2}(1/2) - 2\omega_2^\alpha A_2(1/2)\right] + \sum_{\substack{n+m=p-1\\ n>m\geq 2}} \omega_n^\alpha \omega_m^\alpha (n-m)\Big(K_n^\alpha(1/2) - K_m^\alpha(1/2)\Big).$$

For point $ii)$ in lemma 7.2, the last sum is positive. Moreover, for $\alpha \in (0, 1)$, the function

$$\begin{aligned}\eta^\alpha(p) &:= 2\omega_1^\alpha \frac{\omega_{p-1}^\alpha}{\omega_{p-2}^\alpha} A_{p-1}(1/2) - \omega_1^\alpha A_{p-2}(1/2) - 2\omega_2^\alpha A_2(1/2)\\ &= \frac{2\alpha(p-2-\alpha)}{p-1} A_{p-1}(1/2) - \alpha A_{p-2}(1/2) - \alpha(1-\alpha)\end{aligned}$$

is such that

$$\eta^\alpha(4) = 0, \quad \dot\eta^\alpha(p) = \frac{2\alpha(2+2\alpha)}{(p-1)^2 2^{p-2}}\Big(2^{p-2} - 1 - (p-1)\log(2)\Big) > 0 \quad \forall p \geq 4.$$

A similar result holds for $\eta^0(p)$. Thus $Q_p(q) > 0$ for any $p \geq 4$ and $\alpha \in [0, 1)$ and the thesis follows. ■

4. Limits of $\Delta_\alpha \theta(u)$ as $\ell \to 0$ and $\ell \to \ell_{max}$

This section concerns the limits of the apsidal angle for near-circular and near-radial orbits, that is when $\ell \to \ell_{max}$ and $\ell \to 0$.

Although these limit values are well known in the literature, we present the result for the sake of completeness and because it descends directly from the above computations.

Theorem 4.1. *For any* $\alpha \in [0, 1)$

$$\lim_{\ell\to 0} \Delta_\alpha \theta(u) = \frac{\pi}{2-\alpha}, \qquad \lim_{\ell \to \ell_{\max}} \Delta_\alpha \theta(u) = \frac{\pi}{\sqrt{2-\alpha}}.$$

Proof. According to the discussion in section 2, the limit values for the apsidal angle are given as the limit for $q \to 0$ and $q \to 1$ of the integral (7). In particular, $q \to 0$ corresponds to $\ell \to \ell_{max}$ and $q \to 1$ to $\ell \to 0$. Since $\mathcal{E}_\alpha(s, q) > 0$ for any $\alpha \in [0, 1), q \in (0, 1), s \in (0, 1)$, see Corollary 3.2, for the Lebesgue's dominated convergence theorem,

$$\lim_{q\to 0} \Delta_\alpha \theta = \int_0^1 \frac{1}{\sqrt{s(1-s)}} \frac{1}{\sqrt{1 + \lim_{q\to 0} \mathcal{E}_\alpha(s, q)}} ds = \int_0^1 \frac{1}{\sqrt{s(1-s)}} \frac{1}{\sqrt{1 + (1-\alpha)}} ds = \frac{\pi}{\sqrt{2-\alpha}}$$

and

$$\lim_{q\to 1}\Delta_\alpha\theta = \int_0^1 \frac{1}{\sqrt{s(1-s)}}\frac{1}{\sqrt{1+\lim_{q\to 1}\mathcal{E}_\alpha(s,q)}}\,ds = \int_0^1 \frac{1}{\sqrt{s(1-s)}}\frac{1}{\sqrt{1+\frac{1}{s}\frac{s^\alpha - s}{1-s}}}\,ds = \frac{\pi}{2-\alpha}.$$

■

5. The derivative of $\Delta_\alpha\theta(u)$

We now examine the derivative of the apsidal angle as a function of the angular momentum ℓ. That is equivalent to compute the derivative of $\Delta_\alpha\theta(u)$ given in (7) with respect to q. From Proposition 3.4 and 3.5

$$|\partial_q\mathcal{E}_\alpha(s,q)| < \partial_q\mathcal{E}_\alpha(0,q) = M_\alpha(q)$$

where

$$M_\alpha(q) =$$

$$\begin{cases} \dfrac{\begin{gathered}\alpha^2\,(1-q)^\alpha\,q^3 + \Big((2-\alpha)\,(1-q)^{2\,\alpha} + \left(-2\,\alpha^2+\alpha-4\right)\,(1-q)^\alpha + 2\Big)\,q^2 \\ +\Big(-4\,(1-q)^{2\,\alpha} + 8\,(1-q)^\alpha - 4\Big)\,q + 2((1-q)^\alpha - 1)^2\end{gathered}}{\left((1-q)^\alpha - 1\right)^2\,(q-1)^2\,q^2,} & \alpha\in(0,1) \\ \\ \dfrac{q^3 - \log(1-q)\,q^2 - 2\,q^2 + 2\,\left(\log(1-q)\right)^2 q^2 - 4\,\left(\log(1-q)\right)^2 q + 2\,\left(\log(1-q)\right)^2}{\left(\log(1-q)\right)^2 q^2\,(1-2\,q+q^2)} & \alpha = 0 \end{cases}$$

is $\mathcal{C}^1((0,1))$ for any α. Therefore, for any $q\in(0,1)$ we can pass the derivative under the integral sign and write

$$\frac{d}{dq}\Delta_\alpha\theta(u) = \int_0^1 \frac{1}{\sqrt{s(1-s)}}\partial_q\left(\frac{1}{\sqrt{1+\mathcal{E}_\alpha(s,q)}}\right)ds = -\frac{1}{2}\int_0^1 \frac{1}{\sqrt{s(1-s)}}\frac{\partial_q\mathcal{E}_\alpha(s,q)}{(1+\mathcal{E}_\alpha(s,q))^{\frac{3}{2}}}\,ds. \quad (14)$$

The following analysis aims at proving that the integral in (14) is positive for any $q\in(0,1)$ and $\alpha\in[0,1)$. It would mean that the apsidal angle is monotonically increasing as a function of the angular momentum ℓ for any $\alpha\in[0,1)$.

Denote

$$I_\alpha(q) := \int_0^1 \frac{1}{\sqrt{s(1-s)}}\frac{\partial_q\mathcal{E}_\alpha(s,q)}{(1+\mathcal{E}_\alpha(s,q))^{\frac{3}{2}}}\,ds.$$

Taking advantage from the symmetry in s, it holds

$$I_\alpha(q) = \int_0^{1/2} \frac{1}{\sqrt{s(1-s)}}\left(\frac{\partial_q\mathcal{E}_\alpha(s,q)}{(1+\mathcal{E}_\alpha(s,q))^{\frac{3}{2}}} + \frac{\partial_q\mathcal{E}_\alpha(1-s,q)}{(1+\mathcal{E}_\alpha(1-s,q))^{\frac{3}{2}}}\right)ds.$$

By Propositions 3.4, 3.5, the first term of the integrand is positive, but, as said earlier, $\partial_q \mathcal{E}_\alpha(1-s,q)$ could be negative. A sufficient condition for $I_\alpha(q) > 0$ is that

$$\frac{\partial_q \mathcal{E}_\alpha(s,q)}{(1+\mathcal{E}_\alpha(s,q))^{\frac{3}{2}}} > -\frac{\partial_q \mathcal{E}_\alpha(1-s,q)}{(1+\mathcal{E}_\alpha(1-s,q))^{\frac{3}{2}}}, \quad \forall s \in (0,1/2). \tag{15}$$

The convexity of $\partial_q \mathcal{E}_\alpha(s,q)$ in the variable s implies that $\partial_q \mathcal{E}_\alpha(s,q) > |\partial_q \mathcal{E}_\alpha(1-s,q)|$ for any $s \in (0,1/2)$. On the other side, from Proposition 3.1, $\mathcal{E}_\alpha(s,q) > \mathcal{E}_\alpha(1-s,q)$ hence the validity of (15) cannot be achieved from the global behaviour of $\mathcal{E}_\alpha$ and $\partial_q \mathcal{E}_\alpha$, rather it deserves a carefully analysis.

Consider the function

$$\mathcal{I}_\alpha(s,q) := \partial_q \mathcal{E}_\alpha(s,q)\,(1+\mathcal{E}_\alpha(1-s,q))^{\frac{3}{2}} + \partial_q \mathcal{E}_\alpha(1-s,q)\,(1+\mathcal{E}_\alpha(s,q))^{\frac{3}{2}}.$$

Clearly, the statement $\mathcal{I}_\alpha(s,q) > 0$ for any $s \in (0,1/2)$ is equivalent to (15), then we aim at proving that $\mathcal{I}_\alpha(s,q)$ is positive. The next lemma allows to replace the power $\frac{3}{2}$ by 2.

Lemma 5.1. *Let $A \geq C > 0,\ B, D > 0$. If*

$$AB^2 - CD^2 > 0 \Rightarrow AB^{\frac{3}{2}} - CD^{\frac{3}{2}} > 0.$$

Proof.

$$AB^2 > CD^2 \Rightarrow A^{\frac{3}{4}} B^{\frac{3}{2}} > C^{\frac{3}{4}} D^{\frac{3}{2}} \Rightarrow AB^{\frac{3}{2}} > C^{\frac{3}{4}} A^{\frac{1}{4}} D^{\frac{3}{2}} > CD^{\frac{3}{2}}.$$

■

For those s where $\partial_q \mathcal{E}_\alpha(q,1-s) \geq 0$ it holds $\mathcal{I}_\alpha(s,q) > 0$ and (15) is true. While for those s where $\partial_q \mathcal{E}_\alpha(q,1-s) < 0$ it holds $\partial_q \mathcal{E}_\alpha(q,s) > -\partial_q \mathcal{E}_\alpha(q,1-s)$, then for the lemma, we can replace the above function $\mathcal{I}_\alpha(s,q)$ by

$$\mathcal{I}_\alpha(s,q) := \partial_q \mathcal{E}_\alpha(q,s)\,(1+\mathcal{E}_\alpha(q,1-s))^2 + \partial_q \mathcal{E}_\alpha(q,1-s)\,(1+\mathcal{E}_\alpha(q,s))^2 \tag{16}$$

and we investigate whether the new defined $\mathcal{I}_\alpha(s,q)$ is positive.

From (12) it descends, for $\alpha \in (0,1)$,

$$\mathcal{I}_\alpha(q,s) = C_\alpha(q) \sum_{n\geq 2, m\geq 1} \omega_n^\alpha \omega_m^\alpha \Big((2-q)(n-m)-2\Big) \left[\begin{array}{l} A_n(s)\,(1+\mathcal{E}_\alpha(1-s,q))^2 \\ +A_n(1-s)\,(1+\mathcal{E}_\alpha(s,q))^2 \end{array} \right] q^{(n-2)+(m-2)}. \tag{17}$$

Algebraic manipulations lead to the expansion

$$\mathcal{I}_\alpha(s,q) = C_\alpha(q) \sum_{p\geq 4} Q_\alpha(p,s) q^{p-4} \tag{18}$$

where

$$Q_\alpha(p,s) = \sum_{\substack{n\geq 2, m\geq 1\\ n+m=p}} \omega_n^\alpha \omega_m^\alpha 2(n-m-1)\left[A_n(s)\Big(1+\mathcal{E}_\alpha(1-s,q)\Big)^2 + A_n(1-s)\Big(1+\mathcal{E}_\alpha(s,q)\Big)^2\right]$$
$$- \sum_{\substack{n\geq 2, m\geq 1\\ n+m=p-1}} \omega_n^\alpha \omega_m^\alpha (n-m)\left[A_n(s)\Big(1+\mathcal{E}_\alpha(1-s,q)\Big)^2 + A_n(1-s)\Big(1+\mathcal{E}_\alpha(s,q)\Big)^2\right]. \tag{19}$$

Note that the quantities in square brackets are positive and increasing in n and it can be proved that both the sums appearing in $Q_\alpha(p,s)$ are positive for any $p \geq 4$. However, $Q_\alpha(p,s)$ has not a definite sign as q ranges in $(0,1)$ then we cannot conclude about the sign of $\mathcal{I}_\alpha(s,q)$ looking at the sign of each of the $Q_\alpha(p,s)$'s. Therefore we have to extrapolate the dependence on q from each of the $\mathcal{E}_\alpha(s,q)$ and $\mathcal{E}_\alpha(1-s,q)$. By substituting the expansion (11) in (17), straightforward computations give

$$\mathcal{I}_\alpha(q,s) = C_\alpha^2(q) \sum_{\substack{n\geq 2, m\geq 1\\ k_1,k_2\geq 2}} \omega_n^\alpha \omega_m^\alpha \omega_{k_1}^\alpha \omega_{k_2}^\alpha \Big((2-q)(n-m)-2\Big) q^{(n-2)+(m-2)+(k_1-2)+(k_2-2)}.$$
$$\cdot \left[\begin{array}{l} A_n(s)\left(\frac{\omega_{k_1-1}^\alpha}{\omega_{k_1}^\alpha} + (2-q)A_{k_1}(1-s)\right)\left(\frac{\omega_{k_2-1}^\alpha}{\omega_{k_2}^\alpha} + (2-q)A_{k_2}(1-s)\right) \\ +A_n(1-s)\left(\frac{\omega_{k_1-1}^\alpha}{\omega_{k_1}^\alpha} + (2-q)A_{k_1}(s)\right)\left(\frac{\omega_{k_2-1}^\alpha}{\omega_{k_2}^\alpha} + (2-q)A_{k_2}(s)\right)\end{array}\right]. \tag{20}$$

Note that in the series there is no the term q^{-1}. Indeed the only possibility for this term to appear is for the choice $(n,m,k_1,k_2) = (2,1,2,2)$ but the first bracket results $-q$. Also the coefficient of q^0 is equal to zero. Therefore, collecting on the powers of q, we can write

$$\mathcal{I}_\alpha(q,s) = C_\alpha^2(q) \sum_{p\geq 1} Coef_p(\alpha,s) q^p$$

where, for any p, only a finite number of quart-uple (n,m,k_1,k_2) contributes to $Coef_p(\alpha,s)$. Here we list the first few of these coefficients, for the case $\alpha \in (0,1)$.

$$Coef_1(\alpha,s) = \frac{1}{9}(2-\alpha)^2\big((5+\alpha)s^2\alpha - (5+\alpha)s + \alpha + 2\big)$$
$$Coef_2(\alpha,s) = \frac{1}{18}\alpha^4(1-\alpha)(7-4\alpha)(2-\alpha)^3\big((5+\alpha)s^2\alpha - (5+\alpha)s + \alpha + 2\big)$$

$$Coef_3(\alpha,s)=\frac{1}{90}\alpha^4(2-\alpha)^3(1-\alpha)\left(\begin{array}{l}\left(\alpha^3-5\,\alpha^2-17\,\alpha+69\right)s^4+\\ \left(10\,\alpha^2-138-2\,\alpha^3+34\,\alpha\right)s^3+\\ \left(482-346\,\alpha+15\,\alpha^2+23\,\alpha^3\right)s^2+\\ \left(-22\,\alpha^3-20\,\alpha^2-413+329\,\alpha\right)s+\\ 21\,\alpha^3-84\,\alpha-39\,\alpha^2+156\end{array}\right)$$

$$Coef_4(\alpha,s)=\frac{1}{540}\alpha^4(1-\alpha)(2-\alpha)^3(9-4\alpha)\left(\begin{array}{l}\left(3\,\alpha^3-15\,\alpha^2-51\,\alpha+207\right)s^4+\\ \left(-6\,\alpha^3+30\,\alpha^2+102\,\alpha-414\right)s^3+\\ \left(29\,\alpha^3-5\,\alpha^2-428\,\alpha+746\right)s^2+\\ \left(-26\,\alpha^3-10\,\alpha^2+377\,\alpha-539\right)s+\\ 23\,\alpha^3-47\,\alpha^2-92\,\alpha+188\end{array}\right).$$

The first two coefficients are clearly positive for any $s\in(0,1/2)$ and $\alpha\in(0,1)$ and the same holds for $Coef_3(\alpha,s)$, $Coef_4(\alpha,s)$. By symbolic computation and numerical visualisation it appears evident that even for larger p the coefficients $Coef_p(\alpha,s)$ are positive and, therefore, that $\mathcal{I}_\alpha(q,s)>0$. On the other side, although rigorous, the check of the positivity of any large but finite number of coefficients does not provide a proof for $\mathcal{I}_\alpha(q,s)>0$, unless a proof that $Coef_p(\alpha,s)>0$ for any $p>p^\star$ is provided.

That is exactly what we are presenting in the next section, where the monotonicity of the apsidal angle is proved in the case $\alpha=0$.

6. The monotonicity of the apsidal angle in the logarithmic potential case

We aim at proving that, for any fixed value of the energy, the apsidal angle $\Delta_0\theta(u)$ is monotonically increasing as a function of the angular momentum ℓ. According to the discussion of the previous sections, it is enough to prove that the function

$$\mathcal{I}_0(s,q)=\partial_q\mathcal{E}_0(s,q)\left(1+\mathcal{E}_0(1-s,q)\right)^2+\partial_q\mathcal{E}_0(1-s,q)\left(1+\mathcal{E}_0(s,q)\right)^2 \tag{21}$$

is positive for any $s\in(0,1/2)$ and $q\in(0,1)$.

The proof is split in different steps and analytical arguments are sometimes combined with rigorous numerical computations based on interval arithmetics, [18]. All the numerical computations have been performed in *Matlab* with the interval arithmetic package *Intlab*. [19]. That assures that the results we obtain are reliable in the strict mathematical sense.

First we show that $\partial_q\mathcal{E}_0(1-s,q)>0$ for any $s\in(0,1/2)$ for any $q\in[0.9,1)$, yielding $\mathcal{I}_0(s,q)>0$ for any $s\in(0,1/2)$, $q\in[0.9,1)$.

Proposition 6.1. *For any $q\in[0.9,1)$, $s\in(0,1/2)$, $\partial_q\mathcal{E}_0(1-s,q)>0$.*

Proof. From Proposition 3.4, it is enough to show that $\lim_{s\to1}\partial_q\mathcal{E}_0(s,q)>0$ for $q\in[0.9,1)$.

It holds

$$F(q):=\lim_{s\to1}\partial_q\mathcal{E}_0(s,q)=-\frac{2}{q^2}-\frac{1}{\log(1-q)}+\frac{2-q}{(1-q)\log^2(1-q)}.$$

Compute

$$F'(q) = \frac{4(1-q)^2\log^3(1-q) + q^4\log(1-q) + 2q^3(2-q)}{q^3(1-q)^2\log^3(1-q)} =: \frac{N(q)}{D(q)}.$$

The denominator $D(q)$ is negative for any $q \in (0,1)$, while the numerator

$$N(q) < -4(1-q)^2q^3 + q^4\log(1-q) + 2q^3(2-q) = q^4\Big(\log(1-q) - 4q + 6\Big) =: q^4B(q).$$

The function $B(q)$ is decreasing in q and, by rigorous computation it holds $B(0.91) < -0.04$. Moreover, for any $q \in [0.9, 0.91]$ it holds $N(q) < -0.2$. Therefore $N(q) < 0$ and $F(q)$ is increasing for any $q \in [0.9, 1)$. Since $F(0.9) \in [0.0394, 0.040]$ it follows that $F(q) > 0$ for any $q \in [0.9, 1)$.

■

From now on, we restrict our analysis to the case $q \in (0, 0.9]$. For these values of q the function $\partial_q\mathcal{E}_0(1-s,q)$ is not everywhere positive for $s \in (0, 1/2)$ then the argument above adopted is not anymore valid to prove that $\mathcal{I}_0(s,q) > 0$.

From (18), we can separate

$$\mathcal{I}_0(s,q) = C_0(q)\left(\mathcal{I}_0^{\boldsymbol{f}}(s,q) + \mathcal{I}_0^{\infty}(s,q)\right) = C_0(q)\left(\sum_{p=4}^{10} Q_0(s,p)q^{p-4} + \sum_{p\geq 11} Q_0(s,p)q^p\right) \tag{22}$$

and we prove separately, in Section 6.1 and 6.2, that the series $\mathcal{I}_0^{\infty}(s,q)$ and the finite sum $\mathcal{I}_0^{\boldsymbol{f}}(s,q)$ are positive for any $s \in (0, 1/2)$ and $q \in (0, 0.9]$. Reminding the definition (19) and (9), for the logarithmic potential case, we have

$$\begin{aligned} Q_0(p,s) = &\sum_{\substack{n\geq 2, m\geq 1\\ n+m=p}} \frac{1}{n}\frac{1}{m}2(n-m-1)\left(A_n(s)\Big(1+\mathcal{E}_0(1-s,q)\Big)^2 + A_n(1-s)\Big(1+\mathcal{E}_0(s,q)\Big)^2\right) \\ &- \sum_{\substack{n\geq 2, m\geq 1\\ n+m=p-1}} \frac{1}{n}\frac{1}{m}(n-m)\left(A_n(s)\Big(1+\mathcal{E}_0(1-s,q)\Big)^2 + A_n(1-s)\Big(1+\mathcal{E}_0(s,q)\Big)^2\right). \end{aligned} \tag{23}$$

It is convenient to express the coefficients $Q_0(p,s)$ in a different way: by a change of index $n \to n+1$

in the first sum, it follows

$$
\begin{aligned}
Q_0(p,s) =& \sum_{\substack{n\geq 1, m\geq 1\\ n+m=p-1}} \frac{1}{n+1}\frac{1}{m} 2(n-m)\left(A_{n+1}(s)\Big(1+\mathcal{E}_0(1-s,q)\Big)^2 + A_{n+1}(1-s)\Big(1+\mathcal{E}_0(s,q)\Big)^2\right)\\
&- \sum_{\substack{n\geq 2, m\geq 1\\ n+m=p-1}} \frac{1}{n}\frac{1}{m}(n-m)\left(A_n(s)\Big(1+\mathcal{E}_0(1-s,q)\Big)^2 + A_n(1-s)\Big(1+\mathcal{E}_0(s,q)\Big)^2\right)\\
=& \frac{1}{2}\frac{1}{p-2} 2(1-(p-2))\left(A_2(s)\Big(1+\mathcal{E}_0(1-s,q)\Big)^2 + A_2(1-s)\Big(1+\mathcal{E}_0(s,q)\Big)^2\right)\\
&+ \sum_{\substack{n\geq 2, m\geq 1\\ n+m=p-1}} \frac{1}{n+1}\frac{1}{m} 2(n-m)\left(A_{n+1}(s)\Big(1+\mathcal{E}_0(1-s,q)\Big)^2 + A_{n+1}(1-s)\Big(1+\mathcal{E}_0(s,q)\Big)^2\right)\\
&- \sum_{\substack{n\geq 2, m\geq 1\\ n+m=p-1}} \frac{1}{n}\frac{1}{m}(n-m)\left(A_n(s)\Big(1+\mathcal{E}_0(1-s,q)\Big)^2 + A_n(1-s)\Big(1+\mathcal{E}_0(s,q)\Big)^2\right).
\end{aligned}
$$

Collecting all the contributions and reminding the definition (10) it results

$$
\begin{aligned}
Q_0(p,s) =& \sum_{\substack{n\geq 2, m\geq 1\\ n+m=p-1}} \frac{1}{n}\frac{1}{m}(n-m)\left[K_n(s)\Big(1+\mathcal{E}_0(1-s,q)\Big)^2 + K_n(1-s)\Big(1+\mathcal{E}_0(s,q)\Big)^2\right]\\
&- \frac{p-3}{p-2}\left(\Big(1+\mathcal{E}_0(1-s,q)\Big)^2 + \Big(1+\mathcal{E}_0(s,q)\Big)^2\right).
\end{aligned}
\tag{24}
$$

6.1. Analytical estimate of the tail elements

The goal of this section is to prove that $Q_0(s,p) \geq 0$ for any $s \in (0,1/2)$, $q \in (0,1)$ and $p \geq 11$. That implies that $\mathcal{I}_0^{\infty}(s,q) \geq 0$ for any $s \in (0,1/2)$ and $q \in (0,1)$.

In practice, we are going to prove that

$$
\sum_{\substack{n\geq 2, m\geq 1\\ n+m=p-1}} \frac{1}{n}\frac{1}{m}(n-m)K_n^0(s) > \frac{p-3}{p-2}, \quad \forall p \geq 11, s \in (0,1).
$$

The first estimate is the following:

Lemma 6.2. *For any $p \geq 10$*

$$
\sum_{\substack{n\geq 2, m\geq 1\\ n+m=p-1}} \frac{1}{n}\frac{1}{m}(n-m)K_n^0(s) \geq \sum_{\substack{n\geq 2, m\geq 1\\ n+m=p-1}} \frac{1}{n}\frac{1}{m}(n-m)\frac{n-1}{n+1}, \quad \forall s \in (0,1).
$$

Proof. Set

$$g(s) := \sum_{\substack{n\geq 2, m\geq 1\\ n+m=p-1}} \frac{1}{n}\frac{1}{m}(n-m)K_n^0(s).$$

Assume p is odd, $p = 2l+1$. Then

$$\begin{aligned} g(s) &= \sum_{\substack{n=l+1\\ (m=2l-n)}}^{2l-2} \frac{1}{n}\frac{1}{m}(n-m)\Big(K_n^0(s) - K_m^0(s)\Big) + \frac{2l-2}{2l-1}K_{2l-1}^0(s) \\ &= \sum_{\substack{n=l+1\\ (m=2l-n)}}^{2l-3} \frac{1}{n}\frac{1}{m}(n-m)\Big(K_n^0(s) - K_m^0(s)\Big) + \frac{2l-2}{2l-1}K_{2l-1}^0(s) + \frac{l-2}{2l-2}\Big(K_{2l-2}^0(s) - K_2^0(s)\Big). \end{aligned}$$

Any term $(K_n^0(s) - K_m^0(s))$ inside the series is such that $m \geq 3$ and $n \geq m+2$. By adding and subtracting equal terms, we obtain

$$K_n^0(s) - K_m^0(s) = \Big(K_n^0(s) - K_{n-1}^0(s)\Big) + \Big(K_{n-1}^0(s) - K_{n-2}^0(s)\Big)\cdots + \Big(K_{m+1}^0(s) - K_m^0(s)\Big), \quad \text{if } m > 3$$

or

$$K_n^0(s) - K_m^0(s) = \Big(K_n^0(s) - K_{n-1}^0(s)\Big) + \Big(K_{n-1}^0(s) - K_{n-2}^0(s)\Big)\cdots + \Big(K_5^0(s) - K_3^0(s)\Big), \quad \text{if } m = 3.$$

In both the cases, for point *iii*) of Proposition 7.2 and for Remark 7.3, we infer

$$K_n^0(s) - K_m^0(s) \geq K_n^0(1) - K_m^0(1).$$

Again, for point *i*) of Proposition 7.2,

$$K_{2l-2}^0(s) \geq K_{2l-2}^0(1)$$

and, for Lemma 7.4,

$$\frac{2l-2}{2l-1}K_{2l-1}^0(s) - \frac{l-2}{2l-2}K_2^0(s) \geq \frac{2l-2}{2l-1}K_{2l-1}^0(1) - \frac{l-2}{2l-2}K_2^0(1)$$

for any $l \geq 1$.

Therefore, collecting all these bounds, we conclude

$$g(s) \geq g(1) = \sum_{\substack{n\geq 2, m\geq 1\\ n+m=p-1}} \frac{1}{n}\frac{1}{m}(n-m)K_n^0(1) = \sum_{\substack{n\geq 2, m\geq 1\\ n+m=p-1}} \frac{1}{n}\frac{1}{m}(n-m)\left[\frac{2n}{n+1} - 1\right].$$

The case p even is completely equivalent. Indeed setting $p = 2l + 2$ we obtain

$$g(s) = \sum_{\substack{n=l+1 \\ (m=2l+1-n)}}^{2l-2} \frac{1}{n}\frac{1}{m}(n-m)\Big(K_n^0(s) - K_m^0 s)\Big) + \frac{2l-3}{2(2l-1)} K_{2l-1}^0(s) + \frac{2l-1}{2l} K_{2l}^0(s) - \frac{2l-3}{2(2l-1)} K_2^0(s).$$

Note that when $m = 3$ then $n = 2l-2 = p-4 \geq 6$, therefore the term $K_4(s) - K_3(s)$ is not present. Arguing as before, we conclude that $g(s) \geq g(1)$.

■

Define

$$S(p) := \sum_{\substack{n \geq 2, m \geq 1 \\ n+m=p-1}} \frac{1}{n}\frac{1}{m}(n-m)\frac{1}{n+1}.$$

Lemma 6.3. *$S(p)$ is decreasing for any $p > 4$ and $S(p) < 0$ for any $p \geq 11$.*

Proof.

$$S(p) = \sum_{n=2}^{p-2} \left(\frac{1}{p-1-n} - \frac{1}{n}\right)\frac{1}{n+1}.$$

Then

$$\begin{aligned} S(p) - S(p+1) &= \sum_{n=2}^{p-2} \left(\frac{1}{p-1-n} - \frac{1}{n}\right)\frac{1}{n+1} - \sum_{n=2}^{p-1} \left(\frac{1}{p-n} - \frac{1}{n}\right)\frac{1}{n+1} \\ &= \sum_{n=2}^{p-2} \left(\frac{1}{p-1-n} - \frac{1}{p-n}\right)\frac{1}{n+1} - \left(1 - \frac{1}{p-1}\right)\frac{1}{p}. \end{aligned}$$

Since $\left(\frac{1}{p-1-n} - \frac{1}{p-n}\right) > 0$ for any $n = 2 \ldots, p-2$, it holds

$$\begin{aligned} S(p) - S(p+1) &\geq \frac{1}{p-1} \sum_{n=2}^{p-2} \left(\frac{1}{p-1-n} - \frac{1}{p-n}\right) - \frac{p-2}{p(p-1)} \\ &= \frac{1}{p-1}\left[\sum_{n=3}^{p-1} \frac{1}{p-n} - \sum_{n=2}^{p-2} \frac{1}{p-n}\right] - \frac{p-2}{p(p-1)} \\ &= \frac{1}{p-1}\left(1 - \frac{1}{p-2}\right) - \frac{p-2}{p(p-1)} = \frac{p-4}{(p-2)(p-1)p} > 0 \quad \forall p > 4. \end{aligned}$$

Then $S(p)$ is decreasing for any $p > 4$. In particular $S(11) = -\frac{29}{1260}$.

■

We are now in the position to prove

Theorem 6.4. *For any $s \in (0,1)$ and any $p \geq 11$*

$$\sum_{\substack{n\geq 2, m\geq 1\\ n+m=p-1}} \frac{1}{n}\frac{1}{m}(n-m)K_n^0(s) > \frac{p-3}{p-2}.$$

Proof. By Lemma 6.2, for any $p \geq 10$

$$\sum_{\substack{n\geq 2, m\geq 1\\ n+m=p-1}} \frac{1}{n}\frac{1}{m}(n-m)K_n^0(s) \geq \sum_{\substack{n\geq 2, m\geq 1\\ n+m=p-1}} \frac{1}{n}\frac{1}{m}(n-m)\frac{n-1}{n+1} = \sum_{\substack{n\geq 2, m\geq 1\\ n+m=p-1}} \frac{1}{n}\frac{1}{m}(n-m)\left(1-\frac{2}{n+1}\right)$$
$$= \sum_{\substack{n\geq 2, m\geq 1\\ n+m=p-1}} \left(\frac{1}{m}-\frac{1}{n}\right) - 2\sum_{\substack{n\geq 2, m\geq 1\\ n+m=p-1}} \frac{1}{n}\frac{1}{m}(n-m)\frac{1}{n+1}.$$

The first sum gives

$$\sum_{\substack{n\geq 2, m\geq 1\\ n+m=p-1}} \left(\frac{1}{m}-\frac{1}{n}\right) = \sum_{n=2}^{p-2}\left(\frac{1}{p-1-n}-\frac{1}{n}\right) = 1-\frac{1}{p-2}+\sum_{n=2}^{p-3}\frac{1}{p-1-n}-\sum_{n=2}^{p-3}\frac{1}{n} = \frac{p-3}{p-2}$$

then, by Lemma 6.3, the thesis follows. ∎

Corollary 6.5. *For any $s \in (0, \frac{1}{2}]$ and $p \geq 11$*

$$Q_0(p,s) \geq 0.$$

Proof. From (24)

$$\begin{aligned} Q_0(p,s) = &\left[\sum_{\substack{n\geq 2, m\geq 1\\ n+m=p-1}} \frac{1}{n}\frac{1}{m}(n-m)K_n^0(s) - \frac{p-3}{p-2}\right]\Big(1+\mathcal{E}_0(1-s,q)\Big)^2 \\ &+ \left[\sum_{\substack{n\geq 2, m\geq 1\\ n+m=p-1}} \frac{1}{n}\frac{1}{m}(n-m)K_n^0(1-s) - \frac{p-3}{p-2}\right]\Big(1+\mathcal{E}_0(s,q)\Big)^2. \end{aligned} \tag{25}$$

By Theorem 6.4 both the terms are non-negative whenever $p \geq 11$. ∎

6.2. Rigorous bound for the finite sum

We are now concerning the finite part in the splitting (22). We wish to prove that

$$\mathcal{I}_0^{\boldsymbol{f}}(s,q) = \sum_{p=4}^{10} Q_0(p,s) q^{p-4} \geq 0, \quad \forall s \in (0,1/2), q \in (0,0.9].$$

By inserting $Q_0(p,s)$ in the form (23), it results

$$\mathcal{I}_0^{\boldsymbol{f}}(s,q)$$

$$\begin{aligned}
&\mathcal{I}_0^{\boldsymbol{f}}(s,q) = \\
&\sum_{p=4}^{10} \left[\sum_{\substack{n\geq 2, m\geq 1\\ n+m=p}} \frac{1}{n}\frac{1}{m} 2(n-m-1) A_n(s) - \sum_{\substack{n\geq 2, m\geq 1\\ n+m=p-1}} \frac{1}{n}\frac{1}{m}(n-m)A_n(s) \right] q^{p-4} \Big(1+\mathcal{E}_0(1-s,q)\Big)^2 \\
&+ \sum_{p=4}^{10} \left[\sum_{\substack{n\geq 2, m\geq 1\\ n+m=p}} \frac{1}{n}\frac{1}{m} 2(n-m-1) A_n(1-s) - \sum_{\substack{n\geq 2, m\geq 1\\ n+m=p-1}} \frac{1}{n}\frac{1}{m}(n-m)A_n(1-s) \right] q^{p-4} \Big(1+\mathcal{E}_0(s,q)\Big)^2 \\
&= \mathcal{R}(s,q)\Big(1+\mathcal{E}_0(1-s,q)\Big)^2 + \mathcal{R}(1-s,q)\Big(1+\mathcal{E}_0(s,q)\Big)^2
\end{aligned} \tag{26}$$

where

$$\mathcal{R}(s,q) := \sum_{p=4}^{10} \left[\sum_{\substack{n\geq 2, m\geq 1\\ n+m=p}} \frac{1}{n}\frac{1}{m} 2(n-m-1) A_n(s) - \sum_{\substack{n\geq 2, m\geq 1\\ n+m=p-1}} \frac{1}{n}\frac{1}{m}(n-m)A_n(s) \right] q^{p-4}.$$

If we perform all the computations, we obtain

$$\begin{aligned}
\mathcal{R}(s,q) = &-\frac{2}{3}\,s + \frac{1}{3} + \left(1 - \frac{7}{3}\,s + s^2\right) q + \left(\frac{349}{180} - \frac{239}{45}\,s - \frac{6}{5}\,s^3 + \frac{43}{10}\,s^2\right) q^2 \\
&+ \left(-\frac{94}{15}\,s^3 + \frac{31}{10} + \frac{4}{3}\,s^4 + \frac{689}{60}\,s^2 - \frac{197}{20}\,s\right) q^3 \\
&+ \left(\frac{249}{56} - \frac{4523}{280}\,s - \frac{10}{7}\,s^5 + \frac{4093}{168}\,s^2 - \frac{823}{42}\,s^3 + \frac{173}{21}\,s^4\right) q^4 \\
&+ \left(\frac{3}{2}\,s^6 + \frac{3739}{126}\,s^4 - \frac{15367}{630}\,s + \frac{3749}{630} - \frac{29941}{630}\,s^3 - \frac{143}{14}\,s^5 + \frac{28313}{630}\,s^2\right) q^5 \\
&+ \left(\frac{95663}{12600} - \frac{218681}{6300}\,s + \frac{474889}{6300}\,s^2 - \frac{620681}{6300}\,s^3 + \frac{5125}{63}\,s^4 - \frac{10517}{252}\,s^5 + \frac{439}{36}\,s^6 - \frac{14}{9}\,s^7\right) q^6.
\end{aligned}$$

Note that the coefficient of q^0 is odd in $s = 1/2$.

Theorem 6.6. *For any $s \in (0, 1/2)$ and $q \in (0, 0.9]$*

$$\mathcal{I}_0^{\boldsymbol{f}}(s,q) > 0.$$

Proof. For any $q \in (0,1)$ the function $\mathcal{E}(s,q)$ is decreasing in s, that is $\mathcal{E}(1,q) \leq \mathcal{E}(s,q) \leq \mathcal{E}(0,q)$. We know that $\mathcal{R}(s,q) \geq 0$ for any $q \in [0,1]$ and $s \in [0, \frac{1}{2}]$ while it could be negative for larger values of s. Therefore, $\mathcal{I}_0^{\boldsymbol{f}}(s,q) > 0$ for those s where $\mathcal{R}(1-s)$ is positive, otherwise

$$\mathcal{I}_0^{\boldsymbol{f}}(s,q) \geq \mathcal{B}(s,q) := \mathcal{R}(s,q)\Big(1 + \mathcal{E}(1,q)\Big)^2 + \mathcal{R}(1-s,q)\Big(1 + \mathcal{E}(0,q)\Big)^2. \tag{27}$$

Hence, it is enough to prove that $\mathcal{B}(s,q) > 0$. Since $\mathcal{B}(s,q)$ is made up a finite number of terms and s, q are within bounded intervals, the problem $\mathcal{B}(s,q) > 0$ is, at least theoretically, well suited to be solved by rigorous numerics. Let us first show that $\mathcal{B}(s,q) > 0$ for any $s \in (0, 1/2)$ and $q \in (0, 0.1]$ then we address the case $q \in [0.1, 0.9]$. The reason for this further splitting is that $\mathcal{R}(s,q) + \mathcal{R}(1-s,q) = \mathcal{O}(q)$, meaning $\mathcal{B}(s,q) \to 0$ for any s when $q \to 0$. This behaviour underlies an obstacle for a rigorous computational scheme to be completely successful: indeed any numerical tool based on interval arithmetics will definitely result a non-positive output when computing $\mathcal{B}(s,q)$ at small values of q. Therefore, the case when q is small deserves a particular treatment.

First computation: $q \in [0, 1/10]$

Let us prove that $\mathcal{B}(s,q) \geq 0$ for any $s \in [0,1]$, $q \in [0, \frac{1}{10}]$. We compute

$$\begin{aligned}
\mathcal{E}(1,q) &:= \lim_{s \to 1} \mathcal{E}(s,q) = \frac{2}{q} - 1 + \frac{2-q}{\ln(1-q)} \\
\mathcal{E}(0,q) &:= \lim_{s \to 0} \mathcal{E}(s,q) = 1 - \frac{2}{q} - \frac{2-q}{(1-q)\ln(1-q)}.
\end{aligned}$$

Using the relation $\ln(1-x)^{-1} > -\Big(q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5}\Big)^{-1}$ it follows

$$\mathcal{E}(1,q) > 1 - \frac{1}{3}q + \frac{1}{90}q^3 - \frac{29}{90}q^4, \quad \forall q \in (0,1).$$

Similarly,

$$\begin{aligned}
\mathcal{E}(0,q) &< 1 + \frac{1}{3}q + \frac{1}{3}q^2 + \frac{1}{3}\,\frac{\big(58 + 87\,q + 8\,q^2 + 15\,q^3 + 12\,q^4\big)}{(1-q)\,(60 + 30\,q + 20\,q^2 + 15\,q^3 + 12\,q^4)}q^3 \\
&< 1 + \frac{1}{3}q + \frac{1}{3}q^2 + 0.4q^3, \quad \forall q \in [0, 0.1].
\end{aligned}$$

By inserting the above estimates into (27)

$$\mathcal{B}(s,q) \geq \mathcal{R}(s,q)\Big(2-\frac{1}{3}q+\frac{1}{90}q^3-\frac{29}{90}q^4\Big)^2+\mathcal{R}(1-s,q)\Big(2+\frac{1}{3}q+\frac{1}{3}q^2+0.4q^3\Big)^2, \quad \forall q\in[0,0.1].$$

As expected, the 0^{th} order term in q vanishes. Let be introduced $\tilde{\mathcal{R}}(s,q)$, $\tilde{\mathcal{E}}(1,q)$, $\tilde{\mathcal{E}}(0,q)$ so that

$$\mathcal{R}(s,q)=\left(\frac{1}{3}-\frac{2}{3}s\right)+q\tilde{\mathcal{R}}(s,q)$$

$$\Big(2-\frac{1}{3}q+\frac{1}{90}q^3-\frac{29}{90}q^4\Big)^2=4+q\tilde{\mathcal{E}}(1,q), \quad \Big(2+\frac{1}{3}q+\frac{1}{3}q^2+0.4q^3\Big)^2=4+q\tilde{\mathcal{E}}(0,q).$$

It results

$$\mathcal{B}(s,q)\geq q\left[\begin{array}{l} 4\Big(\tilde{\mathcal{R}}(s,q)+\tilde{\mathcal{R}}(1-s,q)\Big)+\left(\frac{1}{3}-\frac{2}{3}s\right)\Big(\tilde{\mathcal{E}}(1,q)-\tilde{\mathcal{E}}(0,q)\Big) \\ +q\Big(\tilde{\mathcal{R}}(s,q)\tilde{\mathcal{E}}(1,q)+\tilde{\mathcal{R}}(1-s,q)\tilde{\mathcal{E}}(0,q)\Big)\end{array}\right].$$

For the choice $\Delta(s)=0.02$, $\Delta(q)=0.01$, define the intervals

$$s_j:=[(j-1)\Delta(s),j\Delta_s], \quad j=1,\dots,25, \qquad q_k:=[(k-1)\Delta_q,k\Delta_q], \quad k=1,\dots,10$$

so that $\bigcup_j s_j\supset[0,1/2]$ and $\bigcup_k q_k\supset[0,1/10]$, Then, for any j,k in their range, we compute using interval arithmetics the quantities

$$\begin{aligned}\mathcal{M}(j,k)=&4\Big(\tilde{\mathcal{R}}(s_j,q_k)+\tilde{\mathcal{R}}(1-s_j,q_k)\Big)+\left(\frac{1}{3}-\frac{2}{3}s_j\right)\Big(\tilde{\mathcal{E}}(1,q_k)-\tilde{\mathcal{E}}(0,q_k)\Big)\\ &+q_k\Big(\tilde{\mathcal{R}}(s_j,q_k)\tilde{\mathcal{E}}(1,q_k)+\tilde{\mathcal{R}}(1-s_j,q_k)\tilde{\mathcal{E}}(0,q_k)\Big).\end{aligned} \tag{28}$$

It results $\min_{j,k}(\min(\mathcal{M}(j,k)))>0.2744$, proving that $\mathcal{B}(s,q)>0$ for any $s\in\Big(0,\frac{1}{2}\Big)$, $q\in\Big(0,\frac{1}{10}\Big]$.

Second computation, $q\in[0.1,0.9]$

We are now concerned with the remaining part, that is when $q\in[\frac{1}{10},\frac{9}{10}]$. We rigorously compute the lower bound for $\mathcal{B}(s,q)$, as given in (27), for $s\in(0,1)$, $q\in[0.1,0.9]$. With the choice of $\Delta_s=2\cdot10^{-3}$ and $\Delta_q=2\cdot10^{-4}$ it results that $\mathcal{B}(s,q)\geq0.0013$.

■

We can now state the theorem:

Theorem 6.7. *For any value of the energy E, the apsidal angle $\Delta_0\theta(u)$ is monotonically increasing as a function of the angular momentum.*

Proof. From proposition 6.1, corollary 6.5 and theorem 6.6 it follows that $\mathcal{I}_0(s,q)>0$ for any $s\in(0,1)$ and $q\in(0,1)$. Then, for lemma 5.1, the derivative in (14) is negative for any $q\in(0,1)$. Since $q=q(\ell)$ is decreasing, it follows that

$$\frac{d}{d\ell}\Delta_0\theta(u)>0, \quad \forall\ell\in(0,\ell_{max}).$$

■

7. Appendix

This appendix is intended to collect all the technical results concerning the functions $A_n(s)$ and $K_n^\alpha(s,q)$.

7.1. Properties of $A_n(s)$

$$A_n(s) = \frac{1-(1-s)^{n-1}}{s}, \quad n \geq 2, s \in (0,1)$$

Lemma 7.1.

i) $A_2(s) = 1$, $A_n(1) = 1$ *for any* $n \geq 2$.

ii) $A_{n+1}(s) - A_n(s) = (1-s)^{n-1}, \quad A_{n+1}(1-s) - A_n(1-s) = s^{(n-1)}$.

iii) For any $n \geq 3$ *and* $s \in (0,1)$, $A_n(s)$ *is decreasing and convex, i.e.* $A_n'(s) < 0$ *and* $A_n''(s) > 0$.

iv) For any $n > m$, $A_n'(s) < A_m'(s)$ *for any* $s \in (0,1)$.

Proof. $i), ii)$ are obvious.
$iii)$

$$\frac{d}{ds}A_n(s) = \frac{-1+(1-s)^{n-1}+(n-1)(1-s)^{(n-2)}s}{s^2}.$$

Looking at the numerator, say $N(s)$, we see that $N(0) = 0$, $N(1) = -1$ and

$$N'(s) = -(n-1)(1-s)^{n-2} - (n-1)(n-2)(1-s)^{n-3}s + (n-1)(1-s)^{(n-2)} < 0$$

hence $A_n(s)$ is decreasing.

$$\frac{d^2}{ds^2}A_n(s) = \frac{2s-2s(1-s)^{(n-1)}-2(n-1)(1-s)^{n-2}s^2-(n-2)(n-1)(1-s)^{n-3}s^3}{s^4}$$

$$= \frac{2-2(1-s)^{(n-1)}-2(n-1)(1-s)^{n-2}s-(n-2)(n-1)(1-s)^{n-3}s^2}{s^3}.$$

Looking at the numerator, $N(s)$, we note that $N(0) = 0, N(1) = 2$ and

$$N' = (n-3)(n-2)(n-1)(1-s)^{n-3} > 0, s \in (0,1)$$

hence $A_n(s)$ is convex.

iv)

$$\frac{d}{dn}A'_n(s) = \frac{(1-s)^{n-1}\log(1-s) + (1-s)^{n-2}s + (n-1)(1-s)^{n-2}s\log(1-s)}{s^2}$$
$$= \frac{(1-s)^{(n-2)}}{s^2}(\log(1-s)(1-s) + s + (n-1)s\log(1-s))$$
$$= \frac{(1-s)^{(n-2)}}{s^2}(\log(1-s) + s + (n-2)s\log(1-s)) < 0.$$

■

7.2. Properties of $K_n^\alpha(s)$

$$K_n^\alpha(s) = 2\frac{n-\alpha}{n+1}A_{n+1}(s) - A_n(s), \quad s \in (0,1), \alpha \in [0,1), n \geq 2.$$

Lemma 7.2. *For any $\alpha \in [0,1)$ and $n \geq 2$:*

i) $K_n^\alpha(s)$ is decreasing and convex for any $s \in (0,1)$.

ii) $K_{n+1}^\alpha(s) - K_n^\alpha(s) > 0$ for any $s \in (0,1)$.

For any $\alpha \in [0,1)$ and $n \geq 4$

iii) $K_{n+1}^\alpha(s) - K_n^\alpha(s)$ is decreasing for any $s \in (0,1)$.

Proof. *i*) By a direct computation the property is proved for $n = 2, 3$. Then,

$$\begin{aligned} K_n^\alpha(s) =& \frac{2(n-\alpha)}{n+1}A_{n+1}(s) - \frac{1-(1-s)^{n-1} + (1-s)^n - (1-s)^n}{s} \\ =& \left(\frac{2(n-\alpha)}{n+1} - 1\right) A_{n+1}(s) + (1-s)^{n-1}, \end{aligned} \tag{29}$$

hence

$$\frac{d}{ds}K_n^\alpha(s) = \left(\frac{n-2\alpha-1}{n+1}\right) A'_{n+1}(s) - (n-1)(1-s)^{n-2} < 0, \quad \forall s \in (0,1), n \geq 4.$$

Also,

$$\frac{d^2}{ds^2}K_n^\alpha(s) = \left(\frac{n-2\alpha-1}{n+1}\right) A''_{n+1}(s) + (n-2)(n-1)(1-s)^{n-3} > 0, \quad \forall s \in (0,1), n \geq 4.$$

ii) The case $n = 2$ gives $K_3^\alpha(s) - K_2^\alpha(s) = \frac{5}{6} - \frac{13}{6}\,s + \frac{3}{2}\,s^2 - \frac{1}{6}\,a + \frac{5}{6}\,\alpha s - \frac{1}{2}\alpha s^2$ that is positive for any $s \in (0,1)$ and $\alpha \in [0,1)$.

In the general case

$$K_{n+1}^{\alpha}(s) - K_n^{\alpha}(s) = \frac{2(n+1-\alpha)}{n+2} A_{n+2}(s) - \frac{3n+1-2\alpha}{n+1} A_{n+1}(s) + A_n(s)$$

$$= \frac{1}{(n+1)(n+2)} \left[2(n+1)(n+1-\alpha)A_{n+2}(s) - (3n+1-2\alpha)(n+2)A_{n+1}(s) + (n+1)(n+2)A_n(s)\right].$$

Using twice property *ii*) of Lemma 7.1 we obtain

$$K_{n+1}^{\alpha}(s) - K_n^{\alpha}(s) = \frac{1}{(n+1)(n+2)} \cdot$$

$$\left[2(\alpha+1)A_n(s) + 2(n+1)(n+1-\alpha)(1-s)^n - (n^2+3n-2\alpha)(1-s)^{n-1}\right].$$

Denote $\eta_n^{\alpha}(s) := 2(n+1)(n+1-\alpha)(1-s)^n - (n^2+3n-2\alpha)(1-s)^{n-1}$. Its derivative $\dot{\eta}_n^{\alpha}(s) = (1-s)^{n-2}((n^2+3n-2\alpha)(n-1) - 2n(n+1)(n+1-\alpha)(1-s))$ is zero for $(1-s) = \frac{(n^2+3n-2\alpha)(n-1)}{2n(n+1)(n+1-\alpha)}$ where it corresponds the minimum for $\eta_n^{\alpha}(s)$. We infer

$$\begin{aligned} \eta_n^{\alpha}(s) &\geq 2(n+1)(n+1-\alpha)\frac{(n^2+3n-2\alpha)^n(n-1)^n}{(2n)^n(n+1-\alpha)^n(n+1)^n} - \frac{(n^{2+3n-2\alpha})^n(n-1)^{n-1}}{(2n)^{n-1}(n+1-\alpha)^{n-1}(n+1)^{n-1}} \\ &= -\frac{(n^2+3n-2\alpha)^n(n-1)^{n-1}}{2^{n-1}n^n(n+1-\alpha)^{n-1}(n+1)^{n-1}}. \end{aligned} \tag{30}$$

For any $\alpha \in [0,1)$ we can uniformly bound

$$\eta_n^{\alpha}(s) \geq \eta(n) := -\frac{(n^2+3n)^n(n-1)^{n-1}}{2^{n-1}n^n(n)^{n-1}(n+1)^{n-1}} = -\frac{(n+3)^n(n-1)^{n-1}}{(2n)^{n-1}(n+1)^{n-1}}.$$

The last ratio is monotonically decreasing for $n \geq 1$, and it equals $\frac{3}{2}$ at $n = 3$. Therefore $\eta_n^{\alpha}(s) \geq -\frac{3}{2}$ for any $s \in (0,1)$ and $n \geq 3$, $\alpha \in [0,1)$. Since $A_n(s) \geq 1$ for any n and $s \in (0,1)$, it follows

$$K_{n+1}^{\alpha}(s) - K_n^{\alpha}(s) \geq \frac{1}{(n+1)(n+2)}\left(2 - \frac{3}{2}\right) > 0.$$

iii) The case $n = 4, 5$ is proved by a direct computation.

For $n \geq 6$

$$\frac{d}{ds}\Big(K_{n+1}^{\alpha}(s) - K_n^{\alpha}(s)\Big) = \frac{1}{(n+1)(n+2)} \cdot$$

$$\left[(2+\alpha)A_n'(s) - 2n(n+1)(n+1-\alpha)(1-s)^{n-1} + (n-1)(n^{2+}3n-2\alpha)(1-s)^{n-2}\right].$$

Denote $\nu_n^{\alpha}(s) := -2n(n+1)(n+1-\alpha)(1-s)^{n-1} + (n-1)(n^{2+}3n-2\alpha)(1-s)^{n-2}$. Then, arguing as before, we prove that

$$\max_{s\in(0,1)} (\nu_n^{\alpha}(s)) = \frac{(n-2)^{n-2}(n^2+3n-2\alpha)^{n-1}}{(2n)^{n-2}(n+1)^{n-2}(n+1-\alpha)^{n-2}}.$$

Therefore, for any $\alpha \in [0,1)$,

$$\nu_n^\alpha(s) \le \nu(n) := \frac{(n-2)^{n-2}(n^2+3n)^{n-1}}{(2n)^{n-2}(n+1)^{n-2}(n)^{n-2}} = \frac{(n-2)^{n-2}n(n+3)^{n-1}}{(2n)^{n-2}(n+1)^{n-2}}.$$

The function $\nu(n)$ is decreasing in n and $\nu(6) < 2$. Since $A_n'(s) \le -1$ for any s and n, it follows that $K_{n+1}^\alpha(s) - K_n^\alpha(s)$ is decreasing in s for any $n \ge 6$.

■

Remark 7.3. *$K_4^\alpha(s) - K_3^\alpha(s)$ is not decreasing in $s \in (0,1)$ and the minimum is not achieved at $s = 1$. However the function $K_5^0(s) - K_3^0(s)$, for $s \in [0,1]$ attains the minimum at $s = 1$. Indeed*

$$K_5^0(s) - K_3^0(s) = \frac{11}{6} - \frac{43}{6}\,s + \frac{67}{6}\,s^2 - \frac{22}{3}\,s^3 + \frac{5}{3}\,s^4$$

and the function has a local minimum and a local maximum at s_1, s_2 respectively, where $s_1 \in I_1 = [0.7230, 0.7233]$ and $s_2 \in I_2 = [0.871, 0.872]$. For any $s \in I_1$, $K_5^0(s) - K_3^0(s) \in [0.1672, 0.1785] > \frac{1}{6} = K_5^0(1) - K_3^0(1)$.

Lemma 7.4. *For any $n \ge 2$, the function*

$$g_n(s) := \frac{n-1}{n} K_n^0(s) - \frac{n-3}{2(n-1)} K_2^0(s)$$

is monotonically decreasing for any $s \in [0,1]$.

Proof.

$$g_n(s) = \frac{n-1}{n} K_n^0(s) - \frac{n-3}{2(n-1)} \left(\frac{4}{3} A_3(s) - 1\right)$$

then, since $A_3'(s) = -1$,

$$\dot{g}_n(s) = \frac{n-1}{n} \frac{d}{ds} K_n^0(s) + \frac{4}{3} \frac{n-3}{2(n-1)}.$$

For point $i)$ of lemma 7.2, for any $s \in (0,1)$,

$$\dot{g}_n(s) \le -\frac{n-1}{n}\left(\frac{n-1}{n+1}\right) + \frac{2(n-3)}{3(n-1)} = \frac{-3(n-1)^3 + 2n(n-3)(n+1)}{3n(n+1)(n-1)} < 0, \quad \forall n \ge 2.$$

■

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