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A POSTERIORI ERROR ANALYSIS OF A STABILIZED MIXED FEM FOR CONVECTION-DIFFUSION PROBLEMS

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Abstract. We present an augmented dual-mixed variational formulation for a linear convection-diffusion equation with homogeneous Dirichlet boundary conditions. The approach is based on the addition of suitable least squares type terms. We prove that for appropriate values of the stabilization parameters, that depend on the diffusion coefficient and the magnitude of the convective velocity, the new variational formulation and the corresponding Galerkin scheme are well-posed, and a Céa estimate holds. In particular, we derive the rate of convergence when the flux and the concentration are approximated, respectively, by Raviart-Thomas and continuous piecewise polynomials. In addition, we introduce a simple a posteriori error estimator which is reliable and locally efficient. Finally, we provide numerical experiments that illustrate the behavior of the method.

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1. **Introduction.** The numerical solution of convection-diffusion problems by a mixed method presents two main difficulties. On the one hand, stability can be ensured only when the discrete spaces satisfy the inf-sup condition. On the other hand, instabilities can also occur when convection is dominant. In this paper, we follow the ideas in [4, 1] and propose an augmented dual-mixed variational formulation for linear convection-diffusion equations with homogeneous Dirichlet boundary conditions. The approach is based on the addition of suitable least squares type terms. We prove that for appropriate values of the stabilization parameters, that depend on the diffusion coefficient and the magnitude of the convective velocity, the new variational formulation and the corresponding Galerkin scheme are well-posed, and a Céa estimate holds for any finite element subspaces. In particular, we derive the rate of convergence when the flux and the concentration are approximated, respectively, by Raviart-Thomas and continuous piecewise polynomials. In addition, we introduce a simple a posteriori error estimator which is reliable and locally efficient. Finally, we provide numerical experiments that illustrate the behavior of the method.

The paper is organized as follows. In Section 2 we recall the usual dual-mixed variational formulation of the convection-diffusion problem. Then, in Section 3 we introduce and analyze the stabilized dual-mixed variational formulation. In Section 4 we analyze the stabilized mixed finite element method. In Section 5, we derive a new a posteriori error estimator and prove its reliability and local efficiency. Finally, in Section 6 we provide some numerical results.

2. **The model problem.** Let Ω be a bounded connected subset of $\mathbb{R}^d$ $(d = 2, 3)$ with Lipschitz-continuous boundary Γ, and let ϵ be a positive parameter. Then, given $f \in L^2(\Omega)$ and $\mathbf{b} \in [L^\infty(\Omega)]^d$, we consider the following linear convection-diffusion problem: find $u : \Omega \to \mathbb{R}$ such that

$$\left\{ \begin{array}{rcll} -\epsilon\Delta u + \mathbf{b}\cdot\nabla u &=& f & \text{in } \Omega\,, \\ u &=& 0 & \text{on } \Gamma\,, \end{array} \right. \tag{1}$$

where we assume that $\mathbf{b}$ is solenoidal.

We consider the following dual-mixed variational formulation of problem (1): find $\sigma \in H(\mathrm{div};\Omega)$ and $u \in H_0^1(\Omega)$ such that

$$\left\{ \begin{array}{l} \displaystyle \frac{1}{\epsilon}\int_\Omega \sigma\cdot\tau + \int_\Omega u\,\mathrm{div}(\tau) \;=\; 0\,, \quad \forall\,\tau \in H(\mathrm{div};\Omega)\,, \\ \displaystyle \int_\Omega \mathrm{div}(\sigma)\,v - \int_\Omega \mathbf{b}\cdot\nabla u\, v \;=\; -\int_\Omega f v\,, \quad \forall\, v \in H_0^1(\Omega)\,, \end{array} \right. \tag{2}$$

where $\sigma := \epsilon\nabla u$.

Problem (2) was analyzed by Douglas and Roberts in [2]. It is well-known that the Galerkin scheme associated to the variational formulation (2) is not stable for any combination of finite element subspaces. In order to allow a greater set of stable interpolations, we must consider a stabilization technique.

3. **Augmented dual-mixed variational formulation.** We follow [4, 1] and subtract the second equation in (2) from the first one. Then, we add the following least-squares type terms:

$$\kappa_1 \int_\Omega (\mathrm{div}(\sigma) - \mathbf{b}\cdot\nabla u)(\mathrm{div}(\tau) + \mathbf{b}\cdot\nabla v) \;=\; -\kappa_1 \int_\Omega f\,(\mathrm{div}(\tau) + \mathbf{b}\cdot\nabla v) \tag{3}$$

and

$$\kappa_2 \int_\Omega (\nabla u - \epsilon^{-1}\sigma)\,\cdot\,(\nabla v + \epsilon^{-1}\tau) \;=\; 0\,, \tag{4}$$

where (σ, u) is the solution of (2) and $(\tau, v) \in H(\mathrm{div};\Omega)\times H_0^1(\Omega)$. The stabilization parameters, κ_1 and κ_2, are (in principle) any positive constants.

In what follows, we denote $\mathbf{H} := H(\mathrm{div};\Omega)\times H_0^1(\Omega)$, and endow this space with the product norm

$$\|(\tau, v)\|_{\mathbf{H}} := (\|\tau\|^2_{H(\mathrm{div};\Omega)} + \|v\|^2_{H^1(\Omega)})^{1/2}\,, \quad \forall\,(\tau, v)\in\mathbf{H}. \tag{5}$$

The augmented variational formulation reads as follows: *find* $(\sigma, u) \in \mathbf{H}$ *such that*

$$A_s((\sigma, u), (\tau, v)) = F_s(\tau, v)\,, \quad \forall\,(\tau, v)\in\mathbf{H}, \tag{6}$$

where the bilinear form $A_s : \mathbf{H}\times\mathbf{H}\to\mathbb{R}$ and the linear functional $F_s : \mathbf{H}\to\mathbb{R}$ are defined by

$$\begin{aligned} A_s((\sigma, u), (\tau, v)) := \;& \frac{1}{\epsilon}\int_\Omega \sigma\,\cdot\,\tau \;+\; \int_\Omega u\,\mathrm{div}(\tau) \;-\; \int_\Omega \mathrm{div}(\sigma)\,v \;+\; \int_\Omega \mathbf{b}\cdot\nabla u\,v \\ & + \kappa_1 \int_\Omega (\mathrm{div}(\sigma) - \mathbf{b}\cdot\nabla u)\,(\mathrm{div}(\tau) + \mathbf{b}\cdot\nabla v) \\ & + \kappa_2 \int_\Omega (\nabla u - \epsilon^{-1}\,\sigma)\,\cdot\,(\nabla v + \epsilon^{-1}\,\tau) \end{aligned} \tag{7}$$

and

$$F_s(\tau, v) := \int_\Omega f\,v \;-\; \kappa_1 \int_\Omega f\,(\mathrm{div}(\tau) + \mathbf{b}\cdot\nabla v) \tag{8}$$

for all $(\sigma, u), (\tau, v) \in \mathbf{H}$.

It is clear that the bilinear form $A_s(\cdot,\cdot)$ and the linear functional F_s are continuous in $\mathbf{H}$. Indeed, the continuity constant of the bilinear form $A_s(\cdot,\cdot)$ can be taken as $C_{\mathtt{cont}} = \alpha + \beta + \kappa_1\,\alpha^2 + \kappa_2\,\beta^2$, with $\alpha = 1 + \sqrt{d}\,\|\mathbf{b}\|_{[L^\infty(\Omega)]^d}$ and $\beta = 1 + \epsilon^{-1}$.

On the other hand, since $\mathbf{b}$ is solenoidal, for $v \in H_0^1(\Omega)$ there holds

$$\int_\Omega \mathbf{b}\cdot\nabla v\,v \;=\; 0\,. \tag{9}$$

Using this identity we deduce that

$$\begin{aligned} A_s((\tau, v), (\tau, v)) \;\geq\;& \frac{1}{\epsilon}(1 - \frac{\kappa_2}{\epsilon})||\tau||^2_{[L^2(\Omega)]^d} \;+\; \kappa_1\,||\mathrm{div}(\tau)||^2_{L^2(\Omega)} \\ & + (\kappa_2 - \kappa_1\,d\,||\mathbf{b}||^2_{[L^\infty(\Omega)]^d})\,||\nabla v||^2_{[L^2(\Omega)]^d}\,. \end{aligned} \tag{10}$$

Therefore, if we choose the stabilization parameters κ_1 and κ_2 such that

$$\kappa_1 > 0, \quad 1 - \frac{\kappa_2}{\epsilon} > 0 \quad \text{and} \quad \kappa_2 - \kappa_1\,d\,||\mathbf{b}||^2_{[L^\infty(\Omega)]^d} > 0\,, \tag{11}$$

then the bilinear form $A_s(\cdot,\cdot)$ will be coercive in $\mathbf{H}$, with a coercivity constant

$$C_{\texttt{ell}} := \min\{\frac{1}{\epsilon}(1-\frac{\kappa_2}{\epsilon}), \kappa_1, (\kappa_2 - \kappa_1\, d\,||\mathbf{b}||^2_{[L^\infty(\Omega)]^d})\, C_\Omega\}\,, \tag{12}$$

where C_Ω is the Poincaré constant. We have the following result.

Theorem 1. *Assume that*

$$0 < \kappa_2 < \epsilon \quad \textit{and} \quad 0 < \kappa_1 < \frac{\kappa_2}{d\,||\mathbf{b}||^2_{[L^\infty(\Omega)]^d}}\,. \tag{13}$$

Then, problem (6) *is well-posed.*

Proof. It follows from the previous considerations and the Lax-Milgram Lemma. □

4. **Augmented mixed finite element method.** Let $\{\mathcal{T}_h\}_{h>0}$ be a family of shape-regular meshes of $\bar{\Omega}$ made up of triangles if $d=2$ or tetrahedra if $d=3$. We denote by h_T the diameter of an element $T\in\mathcal{T}_h$ and define $h := \max_{T\in\mathcal{T}_h} h_T$. Let $H_h \subset H(\mathrm{div};\Omega)$ and $V_h \subset H^1_0(\Omega)$ be any finite element subspaces. Then, the Galerkin scheme associated to problem (6) reads: *find* $(\sigma_h, u_h) \in H_h \times V_h$ *such that*

$$A_s((\sigma_h,u_h),(\tau_h,v_h)) = F_s(\tau_h,v_h)\,, \qquad \forall\,(\tau_h,v_h)\in H_h\times V_h\,. \tag{14}$$

Under the assumptions of Theorem 1, problem (14) has a unique solution $(\sigma_h,u_h)\in H_h\times V_h$. Moreover,

$$||(\sigma-\sigma_h, u-u_h)||_{\mathbf{H}} \le \frac{C_{\texttt{cont}}}{C_{\texttt{ell}}} \inf_{(\tau_h,v_h)\in H_h\times V_h} ||(\sigma-\tau_h, u-v_h)||_{\mathbf{H}}\,. \tag{15}$$

In order to establish a rate of convergence result, we consider specific finite element subspaces H_h and V_h. Hereafter, given $T\in\mathcal{T}_h$ and an integer $l\ge 0$, we denote by $\mathcal{P}_l(T)$ the space of polynomials of total degree at most l defined on T and, given an integer $r\ge 0$, we denote by $\mathcal{RT}_r(T)$ the local Raviart-Thomas space of order r (cf. [5]). Given $r\ge 0$ and $m\ge 1$, we define

$$\begin{aligned} H_h &:= \Big\{\tau_h\in H(\mathrm{div};\Omega)\,:\,\tau_h\big|_T\in\mathcal{RT}_r(T), \quad \forall\,T\in\mathcal{T}_h\Big\},\\ V_h &:= \Big\{v_h\in\mathcal{C}(\overline{\Omega})\cap H^1_0(\Omega)\,:\,v_h\big|_T\in\mathcal{P}_m(T), \quad \forall\,T\in\mathcal{T}_h\Big\}\,. \end{aligned} \tag{16}$$

The corresponding rate of convergence is given in the next theorem.

Theorem 2. *Assume* (13). *Moreover, assume that* $\sigma\in[H^t(\Omega)]^d$, $\mathrm{div}(\sigma)\in H^t(\Omega)$ *and* $u\in H^{t+1}(\Omega)$. *Then, there exists* $C_{\texttt{err}}>0$*, independent of* h*, such that*

$$\begin{aligned} ||(\sigma-\sigma_h,u-u_h)||_{\mathbf{H}} \le\ & C_{\texttt{err}}\, h^{\min\{t,m,r+1\}}\\ & \Big(||\sigma||_{[H^t(\Omega)]^d} + ||\mathrm{div}(\sigma)||_{H^t(\Omega)} + ||u||_{H^{t+1}(\Omega)}\Big)\,. \end{aligned} \tag{17}$$

Proof. It follows straightforwardly from inequality (15) and the approximation properties of the corresponding finite element subspaces. □

5. **A posteriori error analysis.** In this section, we assume that the stabilization parameters satisfy the hypotheses of Theorems 1 and 2. We develop a residual-based a posteriori error analysis of the augmented mixed finite element method (14). We derive a simple a posteriori error estimator and prove that it is reliable and locally efficient.

Let $H_h \subset H(\mathrm{div};\Omega)$ and $V_h \subset H_0^1(\Omega)$ be any finite element subspaces, and let $(\sigma, u) \in \mathbf{H}$ and $(\sigma_h, u_h) \in H_h \times V_h$ be the unique solutions to problems (6) and (14), respectively. Then, we consider the residual

$$R_h(\tau, v) := F_s(\tau, v) - A_s((\sigma_h, u_h), (\tau, v)), \qquad \forall\,(\tau, v) \in \mathbf{H}. \tag{18}$$

Using the coercivity of the bilinear form $A_s(\cdot,\cdot)$ and the definition of the residual (18), we deduce

$$\|(\sigma - \sigma_h, u - u_h)\|_{\mathbf{H}} \;\leq\; C_{\mathtt{ell}}^{-1} \sup_{\substack{(\tau,v)\in\mathbf{H}\\ (\tau,v)\neq(0,0)}} \frac{R_h(\tau, v)}{\|(\tau, v)\|_{\mathbf{H}}}\,. \tag{19}$$

In the next lemma, we obtain an upper bound for the residual.

Lemma 1. *The right-hand side of inequality* (19) *can be bounded by*

$$C_{\mathtt{ell}}^{-1}\, C \left(||f + \mathrm{div}(\sigma_h) - \mathbf{b}\cdot\nabla u_h||_{L^2(\Omega)} \;+\; ||\nabla u_h - \epsilon^{-1}\,\sigma_h||_{[L^2(\Omega)]^d} \right), \tag{20}$$

with $C = \max(1 + \kappa_1(1 + \sqrt{d}\,\|\mathbf{b}\|_{[L^\infty(\Omega)]^d}), 1 + \kappa_2(1 + \epsilon^{-1}))$.

Proof. Using the definitions of the linear functional F_s and the bilinear form $A_s(\cdot,\cdot)$, we can write

$$R_h(\tau, v) = R_1(\tau) + R_2(v)\,, \qquad \forall\,\tau \in H(\mathrm{div};\Omega)\,, \quad \forall\,v \in H_0^1(\Omega)\,, \tag{21}$$

where $R_1 : H(\mathrm{div};\Omega) \to \mathbb{R}$ and $R_2 : H_0^1(\Omega) \to \mathbb{R}$ are defined by

$$\begin{aligned} R_1(\tau) \;:=\;\; & -\frac{1}{\epsilon}\int_\Omega \sigma_h\,\cdot\,\tau \;-\; \int_\Omega u_h\,\mathrm{div}(\tau) \\ & - \kappa_1 \int_\Omega (f + \mathrm{div}(\sigma_h) - \mathbf{b}\cdot\nabla u_h)\,\mathrm{div}(\tau) \\ & - \kappa_2 \int_\Omega (\nabla u_h - \frac{1}{\epsilon}\sigma_h)\,\cdot\,\frac{1}{\epsilon}\,\tau \end{aligned} \tag{22}$$

and

$$\begin{aligned} R_2(v) \;:=\;\; & \int_\Omega (f + \mathrm{div}(\sigma_h) - \mathbf{b}\cdot\nabla u_h)\,v \\ & - \kappa_1 \int_\Omega (f + \mathrm{div}(\sigma_h) - \mathbf{b}\cdot\nabla u_h)\,\mathbf{b}\cdot\nabla v \\ & - \kappa_2 \int_\Omega (\nabla u_h - \frac{1}{\epsilon}\sigma_h)\,\cdot\,\nabla v \end{aligned} \tag{23}$$

The proof follows integrating by parts the second term in R_1, using the boundary condition and applying the Cauchy-Schwarz inequality. □

Motivated by this result, we define the a posteriori error indicator θ

$$\theta^2 \;:=\; \sum_{T\in\mathcal{T}_h} \theta_T^2\,, \tag{24}$$

with

$$\theta_T^2 := \|f + \mathrm{div}(\sigma_h) - \mathbf{b}\cdot\nabla u_h\|_{L^2(T)}^2 \; + \; \|\nabla u_h - \frac{1}{\epsilon}\sigma_h\|_{[L^2(T)]^d}^2 \,. \tag{25}$$

In the next theorem, we establish the equivalence between the total error and the estimator θ. Inequality (26) means that the a posteriori error estimator θ is reliable, whereas inequality (27) means that θ is locally efficient.

Theorem 3. *Let $(\sigma, u) \in \mathbf{H}$ and $(\sigma_h, u_h) \in H_h \times V_h$ be the unique solutions to problems* (6) *and* (14)*, respectively. Then, there exists a positive constant $C_{\mathtt{rel}}$, independent of h, such that*

$$\|(\sigma - \sigma_h, u - u_h)\|_{\mathbf{H}} \;\le\; C_{\mathtt{rel}}\,\theta\,, \tag{26}$$

and there exists a positive constant $C_{\mathtt{eff}}$, independent of h and T, such that

$$C_{\mathtt{eff}}\,\theta_T \;\le\; \|(\sigma - \sigma_h, u - u_h)\|_{H(div,T)\times H^1(T)}\,, \qquad \forall\, T \in \mathcal{T}_h\,. \tag{27}$$

Proof. Using (19), Lemma 1 and the definition of θ, we deduce that the a posteriori error estimator θ is reliable, with $C_{\mathtt{rel}} = \sqrt{2}\, C_{\mathtt{ell}}^{-1}\, C$, where C is the constant of Lemma 1. On the other hand, to prove the local efficiency of θ, we recall that $f = -\mathrm{div}(\sigma) + \mathbf{b}\cdot\nabla u$ and $\frac{1}{\epsilon}\sigma - \nabla u = \mathbf{0}$ in Ω. Then, the proof follows using the triangle inequality with $C_{\mathtt{eff}}^{-2} := 2\,\max\{\frac{1}{\epsilon^2}, 1 + d\,\|\mathbf{b}\|_{[L^\infty(\Omega)]^d}^2\}$. □

6. **Computational results.** In order to study the behavior of this method in practice, we consider a test problem from [3]. We take $\Omega = (0,1)\times(0,1)$, $\mathbf{b} = (1,0)$ and define f such that the exact solution is

$$u(x_1, x_2) \;=\; 0.5\, x_1\,(1.0 - x_1)\, x_2\,(1.0 - x_2)\,(1.0 - tanh((0.5 - x_1)/\gamma)) \tag{28}$$

with $\gamma = 0.05$. We remark that u satisfies the homogeneous Dirichlet boundary condition on Γ and that it has a boundary layer along the line $x_1 = 0.5$.

We approximated the flux and the concentration by piecewise linear elements and start with an initial mesh of 20×20 elements. As stabilization parameters, we choose $\kappa_1 = \frac{\epsilon}{4\,d\,\|\mathbf{b}\|_{[L^\infty(\Omega)]^d}^2}$ and $\kappa_2 = \frac{\epsilon}{2}$. In particular, in the convection-dominated regime ($\epsilon << \|\mathbf{b}\|_{[L^\infty(\Omega)]^d}$), the efficiency index, which is defined as the ratio between the estimator and the total error, will stay in the range

$$\frac{\epsilon}{6\sqrt{2}d\|\mathbf{b}\|_{[L^\infty(\Omega)]^d}^2} \approx C_{\mathtt{rel}}^{-1} \le \frac{\theta}{\|(\sigma - \sigma_h, u - u_h)\|_{\mathbf{H}}} \le C_{\mathtt{eff}}^{-1} \approx \frac{\sqrt{2}}{\epsilon}\,. \tag{29}$$

In Figure 1, we provide, on the left, the decay of the error and error estimate for the uniform and adaptive refinements for the values $\epsilon = 1, 10^{-2}, 10^{-4}$. We observe that the adaptive algorithm performs better for moderately small values of ϵ. On the right we show the efficiency indices for the adaptive refinement algorithm. In all cases they approach one, which confirms the equivalence of the a posteriori error estimate and the total error in practice. This result is much better than that predicted by the theory.

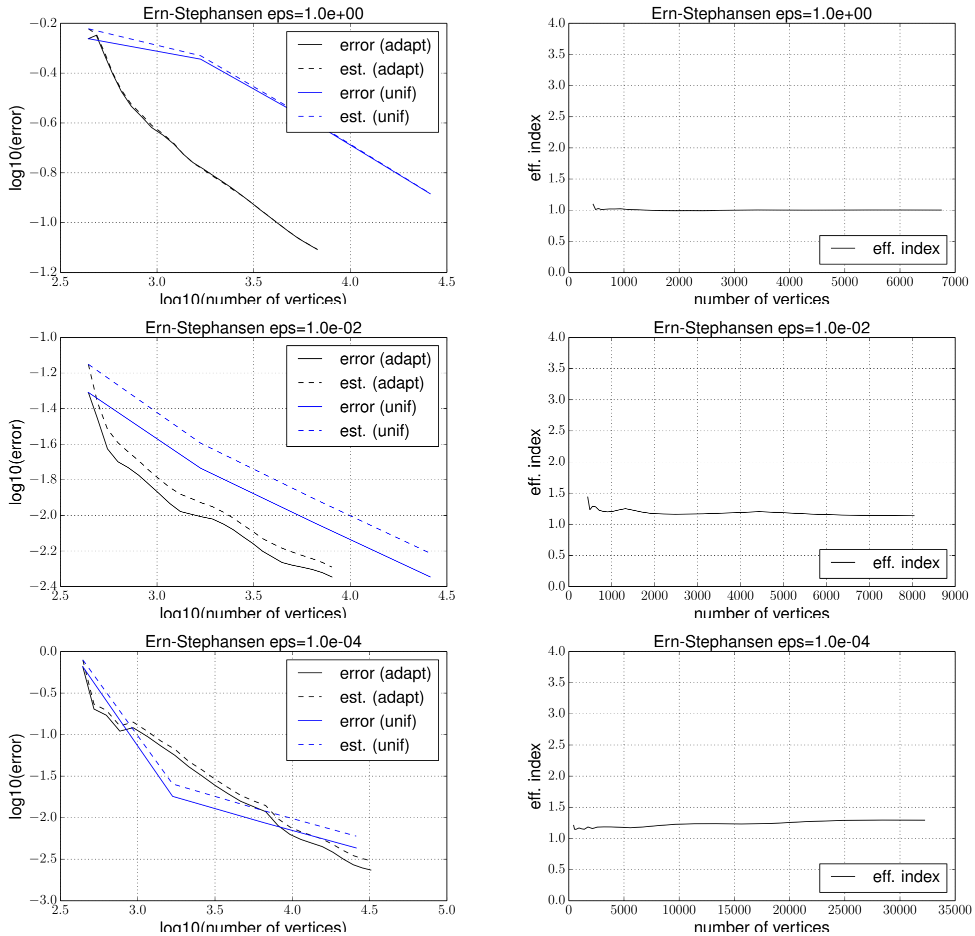

FIGURE 1. Graphs for the error and error estimate (left) and efficiency index (right) for $\epsilon = 1, 10^{-2}, 10^{-4}$.

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