AN ISOPERIMETRIC-TYPE INEQUALITY FOR ELECTROSTATIC SHELL INTERACTIONS FOR DIRAC OPERATORS

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Abstract. In this article we investigate spectral properties of the coupling $H+V_\lambda$, where $H=-i\alpha\cdot\nabla+m\beta$ is the free Dirac operator in $\mathbb{R}^3$, $m>0$ and V_λ is an electrostatic shell potential (which depends on a parameter $\lambda\in\mathbb{R}$) located on the boundary of a smooth domain in $\mathbb{R}^3$. Our main result is an isoperimetric-type inequality for the admissible range of λ's for which the coupling $H+V_\lambda$ generates pure point spectrum in $(-m,m)$. That the ball is the unique optimizer of this inequality is also shown. Regarding some ingredients of the proof, we make use of the Birman-Schwinger principle adapted to our setting in order to prove some monotonicity property of the admissible λ's, and we use this to relate the endpoints of the admissible range of λ's to the sharp constant of a quadratic form inequality, from which the isoperimetric-type inequality is derived.

1. Introduction

We investigate spectral properties of operators that are obtained as the coupling of the free Dirac operator in $\mathbb{R}^3$ with singular measure-valued potentials. Given $m\geq 0$, the free Dirac operator in $\mathbb{R}^3$ is defined by $H=-i\alpha\cdot\nabla+m\beta$, where $\alpha=(\alpha_1,\alpha_2,\alpha_3)$,

$$\alpha_j=\begin{pmatrix}0&\sigma_j\\ \sigma_j&0\end{pmatrix}\quad\text{for } j=1,2,3,\quad \beta=\begin{pmatrix}I_2&0\\0&-I_2\end{pmatrix},\quad I_2=\begin{pmatrix}1&0\\0&1\end{pmatrix},$$
$$\text{and}\quad \sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix} \tag{1}$$

compose the family of *Pauli matrices.* Although one can take $m=0$ in the definition of H, throughout this article we always assume $m>0$ to allow the existence of a nontrivial pure point spectrum in the interval $(-m,m)$ for the corresponding couplings.

Following [1, 2], we consider Hamiltonians of the form $H+V$, being V a singular potential located at the boundary of a bounded smooth domain. These type of couplings are usually referred as *shell interactions* for H. The particular case of the sphere was studied in [3], while in [1, 2] we considered boundaries of general bounded smooth domains. Due to the singularity of the potentials under study, a first issue to be treated is the self-adjoint character of the operator, something that we dealt with in [1]. Our approach fits within the abstract one developed in [9, 10], although we were interested in some concrete potentials that allowed us to obtain more specific results.

This article is addressed to the particular case of electrostatic shell potentials. Let $\Omega\subset\mathbb{R}^3$ be a bounded smooth domain, let σ and N be the surface measure and outward unit normal

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vector field on $\partial\Omega$, respectively. For convenience, we also set $\Omega_+ = \Omega$ and $\Omega_- = \mathbb{R}^3 \setminus \overline{\Omega}$, so $\partial\Omega = \partial\Omega_\pm$. Given $\lambda \in \mathbb{R}$ and $\varphi : \mathbb{R}^3 \to \mathbb{C}^4$, the electrostatic shell potential V_λ applied to φ is formally defined as

$$V_\lambda(\varphi) = \frac{\lambda}{2}(\varphi_+ + \varphi_-)\sigma,$$

where $\varphi_\pm$ denote the boundary values of φ (whenever they exist in a reasonable sense) when one approaches to $\partial\Omega$ from $\Omega_\pm$. Therefore, V_λ maps functions defined in $\mathbb{R}^3$ to vector measures of the form $f\sigma$ with $f : \partial\Omega \to \mathbb{C}^4$. In particular, one can interpret V_λ as the distribution $\lambda\delta_{\partial\Omega}$ when acting on functions which have a well-defined trace on $\partial\Omega$, where $\delta_{\partial\Omega}$ denotes the Dirac-delta distribution on $\partial\Omega$.

Our interest is focused on the study of the existence of stable energy states in $(-m, m)$ for $H + V_\lambda$, where $m > 0$ is interpreted as the mass of the particle whose evolution is modeled by the coupling $\partial_t + i(H + V_\lambda)$. More precisely, we look for a description of the set of λ's in $\mathbb{R}$ for which there exist $a \in (-m, m)$ and a nontrivial spinor φ in $L^2(\mathbb{R}^3)^4$ (actually, in the domain of definition of $H + V_\lambda$) such that

$$(H + V_\lambda)(\varphi) = a\varphi. \tag{2}$$

In [2] we found that this is not possible if $|\lambda|$ is either too big or too small. More precisely, we showed that there exist upper and lower thresholds $\lambda_u(\partial\Omega)$ and $\lambda_l(\partial\Omega)$, respectively, with $0 < \lambda_l(\partial\Omega) \le 2 \le \lambda_u(\partial\Omega)$ and such that if $|\lambda| \notin [\lambda_l(\partial\Omega), \lambda_u(\partial\Omega)]$ then there exists no nontrivial φ verifying (2) for any $a \in (-m, m)$.

The main purpose of this paper is to determine how small can $[\lambda_l(\partial\Omega), \lambda_u(\partial\Omega)]$ be under some constraint on the size of $\partial\Omega$ and/or Ω. In Section 5.2 we show that a natural condition is to consider

$$\frac{\text{Area}(\partial\Omega)}{\text{Cap}(\Omega)} = \text{constant},$$

where $\text{Cap}(\Omega)$ stands for the Newtonian capacity of Ω (see Section 5.2 for the details). In particular, our main result in this direction is the following theorem (see also Remark 5.5). The symbol "kr" in the statement of the theorem denotes the *kernel*, referring to (2).

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary and assume that*

$$m\,\frac{\text{Area}(\partial\Omega)}{\text{Cap}(\Omega)} > \frac{1}{4\sqrt{2}}. \tag{3}$$

Then

$$\begin{aligned}&\sup\{|\lambda| : \text{kr}(H + V_\lambda - a) \neq 0 \textit{ for some } a \in (-m, m)\}\\ &\qquad\ge 4\left(m\,\frac{\text{Area}(\partial\Omega)}{\text{Cap}(\Omega)} + \sqrt{m^2\left(\frac{\text{Area}(\partial\Omega)}{\text{Cap}(\Omega)}\right)^2 + \frac{1}{4}}\right),\\ &\inf\{|\lambda| : \text{kr}(H + V_\lambda - a) \neq 0 \textit{ for some } a \in (-m, m)\}\\ &\qquad\le 4\left(-m\,\frac{\text{Area}(\partial\Omega)}{\text{Cap}(\Omega)} + \sqrt{m^2\left(\frac{\text{Area}(\partial\Omega)}{\text{Cap}(\Omega)}\right)^2 + \frac{1}{4}}\right).\end{aligned}$$

In both cases, the equality holds if and only if Ω is a ball.

The first step to prove this result is to use the connection made in [2] between (2) and the existence of a nontrivial eigenvalue $c(a)$ of C_σ^a, a Cauchy type operator defined on $\partial\Omega$ in the principal value sense, and whose precise definition we postpone to Section 2.1. This connection corresponds to the so-called Birman-Schwinger principle (see [13]) adapted to our setting (see Proposition 3.1).

The second step is to show that $c(a)$ is a monotone function of $a \in (-m, m)$. This has important consequences because it reduces the problem to the study of the limiting cases $a = \pm m$. Using the well-known properties of the Cauchy operator stated in Lemma 2.2 below, it is sufficient to consider just the case $a = m$. This latter problem is equivalent to find $\lambda \in \mathbb{R}$ and $u, h \in L^2(\sigma)^2$ with $u, v \neq 0$ such that

$$\begin{cases} 2mK(u) + W(h) = -u/\lambda, \\ W(u) = -h/\lambda, \end{cases}$$

where K is an operator on $\partial\Omega$ defined by the convolution with the Newtonian kernel $k(x) = (4\pi|x|)^{-1}$ (a positive and compact operator), and W is a "Clifford algebra" version of the 2-dimensional Riesz transform on $\partial\Omega$ whose precise definition we postpone to Section 4.

At this point two results become crucial. On one hand, we use that $2W$ is an isometry when $\partial\Omega$ is a sphere. This is indeed something specific of the sphere, in [6] the authors prove that the spheres are the only boundaries of bounded domains for which $2W$ is an isometry (under some extra assumptions). On the other hand, to deal with K, we use the fact proved in [11, 12] which says that if the Newtonian capacity $\mathrm{Cap}(\Omega)$ is attained on the normalized surface measure of $\partial\Omega$ and Ω is regular enough, then $\partial\Omega$ is a sphere. By a simple argument, we relate K and $\mathrm{Cap}(\Omega)$. In order to use these two ingredients, we first prove that to solve our optimization problem is equivalent to minimize, in terms of Ω, the infimum over all $\lambda > 0$ such that

$$\left(\frac{4}{\lambda}\right)^2 \int_{\partial\Omega} |W(f)|^2 \, d\sigma + \frac{8m}{\lambda} \int_{\partial\Omega} K(f) \cdot \overline{f} \, d\sigma \leq \int_{\partial\Omega} |f|^2 \, d\sigma \tag{4}$$

for all $f \in L^2(\sigma)^2$. It is to this infimum λ to which we prove an isoperimetric-type inequality like the first one in Theorem 1.1 (see Lemma 5.3). The constraint (3) appears as a technical obstruction on the arguments that we use to connect the infimum λ of the quadratic form inequality to the admissible λ's that generate eigenvalues as in (2) (see Theorem 4.3(iv) and Corollary 4.6, see also Remark 4.5 for a related result). We should mention that the free Dirac operator H is not bounded neither above nor below, so that characterizing eigenvalues by minimizing some appropriately chosen quadratic form is not straightforward as can be seen in [4]. Since W is self-adjoint, (4) can be read as

$$\int_{\partial\Omega} \left(\left(\frac{4W}{\lambda} \right)^2 + \frac{8mK}{\lambda} \right)(f) \cdot \overline{f} \, d\sigma \leq \int_{\partial\Omega} |f|^2 \, d\sigma.$$

The paper is organized as follows. In Section 2 we state the preliminaries, where we introduce some notation and recall some properties of the resolvent of H, as well as the construction of $H + V_\lambda$. Section 3 is devoted to the Birman-Schwinger principle and the above-mentioned monotone character of the eigenvalues of C_σ^a. The relation with the limiting case $a = m$ for the optimization problem and the optimal constant of the quadratic form inequality (4) is explored in Section 4. Finally, Section 5 is about the isoperimetric-type result and contains the proof of Theorem 1.1 in Section 5.2. Previously, some other natural constraint conditions not including Newtonian capacity are discarded.

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2. Preliminaries

This article continues the study developed in [1, 2], so we assume that the reader is familiar with the notation, methods and results in there. However, in this section we recall some basic rudiments for the sake of completeness.

Given a positive Borel measure ν in $\mathbb{R}^3$, set

$$L^2(\nu)^4=\left\{f:\mathbb{R}^3\to\mathbb{C}^4\ \nu\text{-measurable}:\int|f|^2\,d\nu<\infty\right\},$$

and denote by $\langle\cdot,\cdot\rangle_\nu$ and $\|\cdot\|_\nu$ the standard scalar product and norm in $L^2(\nu)^4$, i.e., $\langle f,g\rangle_\nu=\int f\cdot\overline{g}\,d\nu$ and $\|f\|_\nu^2=\int|f|^2\,d\nu$ for $f,g\in L^2(\nu)^4$. We write I_4 or 1 interchangeably to denote the identity operator on $L^2(\nu)^4$. We say that ν is d-dimensional if there exists $C>0$ such that $\nu(B(x,r))\le Cr^d$ for all $x\in\mathbb{R}^3$, $r>0$.

We denote by μ the Lebesgue measure in $\mathbb{R}^3$. Concerning $\partial\Omega$, note that σ is 2-dimensional. Since we are not interested in optimal regularity assumptions, for the sequel we assume that $\partial\Omega$ is of class $\mathcal{C}^\infty$. Finally, we introduce the auxiliary space of locally finite measures

$$\mathcal{X}=\left\{G\mu+g\sigma:\ G\in L^2(\mu)^4,\ g\in L^2(\sigma)^4\right\}.$$

2.1. **A fundamental solution of $H-a$.** Observe that H, which is symmetric and initially defined in $\mathcal{C}_c^\infty(\mathbb{R}^3)^4$ ($\mathbb{C}^4$-valued functions in $\mathbb{R}^3$ which are $\mathcal{C}^\infty$ and with compact support), can be extended by duality to the space of distributions with respect to the test space $\mathcal{C}_c^\infty(\mathbb{R}^3)^4$ and, in particular, it can be defined on $\mathcal{X}$. The following lemma (see [2, Lemma 2.1]) is concerned with the resolvent of H, which will be very useful for the results below.

Lemma 2.1. *Given $a\in\mathbb{R}$, a fundamental solution of $H-a$ is given by*

$$\phi^a(x)=\frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|}\left(a+m\beta+\left(1+\sqrt{m^2-a^2}|x|\right)i\alpha\cdot\frac{x}{|x|^2}\right)\quad\text{for } x\in\mathbb{R}^3\setminus\{0\},$$

i.e., $(H-a)\phi^a=\delta_0I_4$ in the sense of distributions, where δ_0 denotes the Dirac measure centered at the origin. Furthermore, if $a\in(-m,m)$ then ϕ^a satisfies

- (i) *$\phi^a_{j,k}\in\mathcal{C}^\infty(\mathbb{R}^3\setminus\{0\})$ for all $1\le j,k\le4$,*
- (ii) *$\phi^a(x-y)=\overline{(\phi^a)^t}(y-x)$ for all $x,y\in\mathbb{R}^3$ such that $x\ne y$,*
- (iii) *there exist $\gamma,\delta>0$ such that*
  - (a) *$\sup_{1\le j,k\le4}|\phi^a_{j,k}(x)|\le C|x|^{-2}$ for all $|x|<\delta$,*
  - (b) *$\sup_{1\le j,k\le4}|\phi^a_{j,k}(x)|\le Ce^{-\gamma|x|}$ for all $|x|>1/\delta$,*
  - (c) *$\sup_{1\le j,k\le4}\sup_{\xi\in\mathbb{R}^3}(1+|\xi|^2)^{1/2}|\mathcal{F}(\phi^a_{j,k})(\xi)|<\infty$, where $\mathcal{F}$ denotes the Fourier transform in $\mathbb{R}^3$.*

In the lemma above we denoted the complex conjugate of the transpose of ϕ^a by $\overline{(\phi^a)^t}$, that is,

$$((\phi^a)^t)_{j,k}=\phi^a_{k,j}\quad\text{and}\quad(\overline{\phi^a})_{j,k}=\overline{\phi^a_{j,k}}\quad\text{for all } 1\le j,k\le4.$$

Note that the assumption $a\in(-m,m)$ in Lemma 2.1 is only relevant for the validity of properties (ii), $(iii)(b)$ and $(iii)(c)$.

Given a positive Borel measure ν in $\mathbb{R}^3$, $f\in L^2(\nu)^4$, and $x\in\mathbb{R}^3$, we set

$$(\phi^a*f\nu)(x)=\int\phi^a(x-y)f(y)\,d\nu(y),$$

whenever the integral makes sense. By Lemma 2.1 and [1, Lemma 2.1], if $a\in(-m,m)$ and ν is a d-dimensional measure in $\mathbb{R}^3$ with $1<d\le3$, then there exists $C>0$ such that

$$\|\phi^a*g\nu\|_\mu\le C\|g\|_\nu\quad\text{for all } g\in L^2(\nu)^4. \tag{5}$$

The next lemma (see [2, Lemma 2.2]), will be used in the sequel.

Lemma 2.2. *Given $g \in L^2(\sigma)^4$ and $x \in \partial\Omega$, set*

$$C_\sigma^a(g)(x) = \lim_{\epsilon \searrow 0} \int_{|x-z|>\epsilon} \phi^a(x-z)g(z)\,d\sigma(z) \quad \text{and} \quad C_\pm^a(g)(x) = \lim_{\Omega_\pm \ni y \xrightarrow{nt} x} (\phi^a * g\sigma)(y),$$

where $\Omega_\pm \ni y \xrightarrow{nt} x$ means that $y \in \Omega_\pm$ tends to $x \in \partial\Omega$ non-tangentially. Then C_σ^a and $C_\pm^a$ are bounded linear operators in $L^2(\sigma)^4$. Moreover, the following holds:

(i) $C_\pm^a = \mp\frac{i}{2}(\alpha \cdot N) + C_\sigma^a$ *(Plemelj–Sokhotski jump formulae),*
(ii) *for any $a \in [-m,m]$, C_σ^a is self-adjoint and $-4(C_\sigma^a(\alpha\cdot N))^2 = I_4$.*

2.2. **On the divergence theorem for $H-a$.** A simple computation involving the divergence theorem shows that

$$\int_{\Omega_\pm} \Big((\alpha\cdot\nabla)\varphi \cdot \overline{\psi} + \varphi \cdot \overline{(\alpha\cdot\nabla)\psi}\Big)\,d\mu = \pm \int_{\partial\Omega} (\alpha\cdot N)\varphi \cdot \overline{\psi}\,d\sigma$$

for all $\varphi, \psi \in W^{1,2}(\chi_{\Omega_\pm}\mu)^4$, where $W^{1,2}(\chi_{\Omega_\pm}\mu)^4$ denotes the Sobolev space of $\mathbb{C}^4$-valued functions defined on $\Omega_\pm$ such that all its components have all their derivatives up to first order in $L^2(\chi_{\Omega_\pm}\mu)$. As a consequence, we easily deduce that

$$\int_{\Omega_\pm} \Big((H-a)\varphi \cdot \overline{\psi} - \varphi \cdot \overline{(H-a)\psi}\Big)\,d\mu = \mp i \int_{\partial\Omega} (\alpha\cdot N)\varphi \cdot \overline{\psi}\,d\sigma. \tag{6}$$

2.3. **The construction of $H+V$ and its domain of definition.** In what follows we use a nonstandard notation, Φ^a, to define the convolution of measures in $\mathcal{X}$ with the fundamental solution of $H-a$, ϕ^a. Capital letters, as F or G, in the argument of Φ^a denote elements of $L^2(\mu)^4$, and the lowercase letters, as f or g, denote elements in $L^2(\sigma)^4$. Despite that this notation is nonstandard, it is very convenient in order to shorten the forthcoming computations.

Given $G\mu + g\sigma \in \mathcal{X}$, we define

$$\Phi^a(G+g) = \phi^a * G\mu + \phi^a * g\sigma.$$

Then (5) shows that $\|\Phi^a(G+g)\|_\mu \leq C(\|G\|_\mu + \|g\|_\sigma)$ for some constant $C>0$ and all $G\mu + g\sigma \in \mathcal{X}$, so $\Phi^a(G+g) \in L^2(\mu)^4$. Moreover, following [1, Section 2.3] one can show that $(H-a)(\Phi^a(G+g)) = G\mu + g\sigma$ in the sense of distributions. This allows us to define a "generic" potential V acting on functions $\varphi = \Phi^a(G+g)$ by

$$V(\varphi) = -g\sigma,$$

so that $(H-a+V)(\varphi) = G\mu$ in the sense of distributions. For simplicity of notation, we write $(H-a+V)(\varphi) = G \in L^2(\mu)^4$.

In order to construct a domain of definition where $H+V$ is self-adjoint, in [1] we used the trace operator on $\partial\Omega$. For $G \in \mathcal{C}_c^\infty(\mathbb{R}^3)^4$, one defines the trace operator on $\partial\Omega$ by $\mathrm{t}_{\partial\Omega}(G) = G\chi_{\partial\Omega}$. Then, $\mathrm{t}_{\partial\Omega}$ extends to a bounded linear operator

$$\mathrm{t}_\sigma : W^{1,2}(\mu)^4 \to L^2(\sigma)^4$$

(see [1, Proposition 2.6], for example). From Lemma 2.1$(iii)(c)$, we have

$$\|\Phi^a(G)\|_{W^{1,2}(\mu)^4} \leq C\|G\|_\mu$$

for some $C>0$ and all $G \in L^2(\mu)^4$ (see [1, Lemma 2.8]), thus we can define

$$\Phi_\sigma^a(G) = \mathrm{t}_\sigma(\Phi^a(G)) = \mathrm{t}_\sigma(\phi^a * G\mu)$$

and it satisfies $\|\Phi^a_\sigma(G)\|_\sigma \leq C\|G\|_\mu$ for all $G \in L^2(\mu)^4$. In accordance with the notation introduced in [1], for the case $a = 0$, we write Φ, Φ_σ, $C_\pm$ and C_σ instead of Φ^0, Φ^0_σ, $C^0_\pm$ and C^0_σ, respectively.

Finally, we recall our main tool to construct domains where $H + V$ is self-adjoint, namely [1, Theorem 2.11]. Actually, the following theorem, which corresponds to [2, Theorem 2.3], is a direct application of [1, Theorem 2.11] to $H + V$, and we state it here in order to make the exposition more self-contained. Given an operator between vector spaces $S : X \to Y$, denote

$$\mathrm{kr}(S) = \{x \in X : S(x) = 0\} \quad \text{and} \quad \mathrm{rn}(S) = \{S(x) \in Y : x \in X\}.$$

Theorem 2.3. *Let $\Lambda : L^2(\sigma)^4 \to L^2(\sigma)^4$ be a bounded operator. Set*

$$D(T) = \{\Phi(G + g) : G\mu + g\sigma \in \mathcal{X},\ \Phi_\sigma(G) = \Lambda(g)\} \subset L^2(\mu)^4 \text{ and } T = H + V \text{ on } D(T),$$

where $V(\varphi) = -g\sigma$ and $(H + V)(\varphi) = G$ for all $\varphi = \Phi(G + g) \in D(T)$. If Λ is self-adjoint and $\mathrm{rn}(\Lambda)$ is closed, then $T : D(T) \to L^2(\mu)^4$ is an essentially self-adjoint operator. In that case, if $\{\Phi(h) : h \in \mathrm{kr}(\Lambda)\}$ is closed in $L^2(\mu)^4$, then T is self-adjoint.

In particular, if Λ is self-adjoint and Fredholm, then the operator T given by Theorem 2.3 is self-adjoint.

2.4. **Electrostatic shell potentials.** In [1, Theorem 3.8] we proved that, if $\lambda \in \mathbb{R} \setminus \{0, \pm 2\}$ and T is the operator defined by

$$D(T) = \big\{\Phi(G + g) : G\mu + g\sigma \in \mathcal{X},\ \Phi_\sigma(G) = \Lambda(g)\big\} \quad \text{and} \quad T = H + V_\lambda \text{ on } D(T),$$

where

$$\Lambda = -(1/\lambda + C_\sigma), \qquad V_\lambda(\varphi) = \frac{\lambda}{2}(\varphi_+ + \varphi_-)\sigma \tag{7}$$

and $\varphi_\pm = \Phi_\sigma(G) + C_\pm(g)$ for $\varphi = \Phi(G+g) \in D(T)$, then $T : D(T) \subset L^2(\mu)^4 \to L^2(\mu)^4$ is self-adjoint. Moreover, we also showed that $V_\lambda = V$ on $D(T)$ for all $\lambda \neq 0$, so the self-adjointness was a consequence of Theorem 2.3. Let us mention that if one replaces $\Phi_\sigma(G) = \Lambda(g)$ by $\lambda\Phi_\sigma(G) = \lambda\Lambda(g)$ in the definition of $D(T)$ above, one recovers the well-known fact that $D(H + V_0) = D(H) = W^{1,2}(\mu)^4$ when $\lambda = 0$.

3. Birman-Schwinger principle and monotonicity

We will make use of the following proposition, which corresponds to [2, Proposition 3.1] and can be understood as the classical Birman-Schwinger principle adapted to our setting.

Proposition 3.1. *Let T be as in* Theorem 2.3. *Given $a \in (-m, m)$, there exists $\varphi = \Phi(G + g) \in D(T)$ such that $T(\varphi) = a\varphi$ if and only if $\Lambda(g) = (C^a_\sigma - C_\sigma)(g)$ and $G = a\Phi^a(g)$. Therefore, $\mathrm{kr}(T - a) \neq 0$ if and only if $\mathrm{kr}(\Lambda + C_\sigma - C^a_\sigma) \neq 0$.*

The following lemma contains the monotonicity property mentioned in the introduction.

Lemma 3.2. *Given $a \in [-m, m]$, the eigenvalues of C^a_σ form a finite or countable sequence $\emptyset \neq \{c_j(a)\}_j \subset \mathbb{R}$, being $1/4$ the only possible accumulation point of $\{c_j(a)^2\}_j$. Moreover, $\frac{d}{da}\, c_j(a) > 0$ for all $a \in (-m, m)$ and all j.*

As a consequence, given $a \in (-m, m)$, the set of real λ's such that $\mathrm{kr}(H+V_\lambda-a) \neq 0$ form a finite or countable sequence $\emptyset \neq \{\lambda_j(a)\}_j \subset \mathbb{R}$, being 4 the only possible accumulation point of $\{\lambda_j(a)^2\}_j$. Furthermore, $\lambda_j(a)$ is a strictly monotonous increasing function of $a \in (-m, m)$ for all j.

Proof. For any $a \in [-m, m]$, the existence of the sequence $\emptyset \neq \{c_j(a)\}_j \subset \mathbb{R}$ stated in the lemma and its possible accumulation point are guaranteed by [2, Remark 3.5] (which also holds for $a = \pm m$) and the self-adjointness of C_σ^a.

Given $a \in [-m, m]$ and $c_j(a)$, let $g_j(a) \in L^2(\sigma)^4$ be such that $\|g_j(a)\|_\sigma = 1$ and

$$C_\sigma^a(g_j(a)) = c_j(a) g_j(a). \tag{8}$$

To differentiate $c_j(a)$ with respect of a, we take the scalar product of (8) with $g_j(a)$, so

$$c_j(a) = \langle c_j(a) g_j(a), g_j(a)\rangle_\sigma = \langle C_\sigma^a(g_j(a)), g_j(a)\rangle_\sigma.$$

We abreviate $\partial_a \equiv \frac{d}{da}$. Then, at a formal level,

$$\begin{aligned}\partial_a c_j(a) &= \langle \partial_a\big(C_\sigma^a(g_j(a))\big), g_j(a)\rangle_\sigma + \langle C_\sigma^a(g_j(a)), \partial_a g_j(a)\rangle_\sigma\\ &= \langle (\partial_a C_\sigma^a)(g_j(a)), g_j(a)\rangle_\sigma + \langle C_\sigma^a(\partial_a g_j(a)), g_j(a)\rangle_\sigma + \langle C_\sigma^a(g_j(a)), \partial_a g_j(a)\rangle_\sigma\\ &= \langle (\partial_a C_\sigma^a)(g_j(a)), g_j(a)\rangle_\sigma + 2\mathrm{Re}\langle \partial_a g_j(a), C_\sigma^a(g_j(a))\rangle_\sigma,\end{aligned} \tag{9}$$

where we used in the last equality above that C_σ^a is self-adjoint. Recall that $\|g_j(a)\|_\sigma = 1$ for all $a \in (-m, m)$, thus (8) gives

$$\begin{aligned}0 &= c_j(a)\partial_a \langle g_j(a), g_j(a)\rangle_\sigma = \langle \partial_a g_j(a), c_j(a) g_j(a)\rangle_\sigma + \langle c_j(a) g_j(a), \partial_a g_j(a)\rangle_\sigma\\ &= 2\mathrm{Re}\langle \partial_a g_j(a), C_\sigma^a(g_j(a))\rangle_\sigma,\end{aligned}$$

which plugged into (9) yields

$$\partial_a c_j(a) = \langle (\partial_a C_\sigma^a)(g_j(a)), g_j(a)\rangle_\sigma. \tag{10}$$

To justify the above computations, in particular in what respects to the issue of the principal value in the definition of C_σ^a, one can decompose the kernel

$$\begin{aligned}\phi^a(x) &= \frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|}\left(a + m\beta + i\sqrt{m^2-a^2}\,\alpha\cdot\frac{x}{|x|}\right) + \frac{e^{-\sqrt{m^2-a^2}|x|}-1}{4\pi}\,i\left(\alpha\cdot\frac{x}{|x|^3}\right)\\ &\quad + \frac{i}{4\pi}\left(\alpha\cdot\frac{x}{|x|^3}\right) =: \omega_1(x) + \omega_2(x) + \omega_3(x)\end{aligned}$$

and note that the principal value only concerns ω_3, since the kernels ω_1 and ω_2 are absolutely integrable on $\partial\Omega$ and actually define compact operators, but ω_3 does not depend on a. At this point, standard arguments in perturbation theory (by compact operators which depend continuously on the perturbation parameter) allow us to justify the formal computations carried out above concerning ∂_a.

Our aim now is to understand the operator $\partial_a C_\sigma^a$. One may guess that, since C_σ^a is defined as the convolution operator on $\partial\Omega$ with the fundamental solution of $H - a$, and formally $\partial_a((H-a)^{-1}) = (H-a)^{-2}$, then $\partial_a C_\sigma^a$ should be defined as the convolution operator on $\partial\Omega$ with the fundamental solution of $(H-a)^2$. This is indeed the case. In the following lines, we are going to prove the details of this argument. We can easily compute

$$\partial_a(\phi^a(x)) = \frac{a e^{-\sqrt{m^2-a^2}|x|}}{4\pi\sqrt{m^2-a^2}}\left(a + m\beta + i\sqrt{m^2-a^2}\,\alpha\cdot\frac{x}{|x|}\right) + \frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|}. \tag{11}$$

Note that

$$-i\alpha\cdot\nabla(e^{-\sqrt{m^2-a^2}|x|}) = i\sqrt{m^2-a^2}e^{-\sqrt{m^2-a^2}|x|}\,\alpha\cdot\frac{x}{|x|},$$

so (11) gives

$$\partial_a(\phi^a(x)) = a\,(H+a)\,\frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi\sqrt{m^2-a^2}} + \frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|}. \tag{12}$$

A simple calculation shows that

$$(-\Delta+m^2-a^2)\frac{e^{-\sqrt{m^2-a^2}|x|}}{8\pi\sqrt{m^2-a^2}}=\frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|} \tag{13}$$

which, combined with (12) and using that $-\Delta+m^2-a^2=(H-a)(H+a)$, yields

$$\begin{aligned}\partial_a(\phi^a(x))&=\left(a(H+a)+\frac{1}{2}(-\Delta+m^2-a^2)\right)\frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi\sqrt{m^2-a^2}}\\&=\left(a+\frac{1}{2}(H-a)\right)(H+a)\,\frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi\sqrt{m^2-a^2}}=(H+a)^2\,\frac{e^{-\sqrt{m^2-a^2}|x|}}{8\pi\sqrt{m^2-a^2}}.\end{aligned} \tag{14}$$

Recall that $(4\pi|x|)^{-1}e^{-\sqrt{m^2-a^2}|x|}$ is a fundamental solution of $-\Delta+m^2-a^2$, that is

$$(-\Delta+m^2-a^2)\frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|}=\delta_0$$

in the sense of distributions. In particular, from (13) we get that

$$(-\Delta+m^2-a^2)^2\,\frac{e^{-\sqrt{m^2-a^2}|x|}}{8\pi\sqrt{m^2-a^2}}=\delta_0. \tag{15}$$

Since $-\Delta+m^2-a^2$ commutes with $H+a$, we easily see that

$$(H-a)^2(H+a)^2=(-\Delta+m^2-a^2)^2,$$

and then, from (14) and (15), we finally deduce that

$$(H-a)^2\,\partial_a(\phi^a(x))=\delta_0, \tag{16}$$

which means that $\partial_a(\phi^a(x))$ is a fundamental solution of $(H-a)^2$, and $\partial_a C^a_\sigma$ corresponds to the operator of convolution on $\partial\Omega$ with this kernel, as suggested before (11). Note that $\partial_a(\phi^a(x))=O(1/|x|)$ for $|x|\to 0$, so in particular $\partial_a C^a_\sigma$ is compact in $L^2(\sigma)^4$.

Given $g\in L^2(\sigma)^4$, set

$$u(x)=\int\partial_a(\phi^a(x-y))g(y)\,d\sigma(y)\qquad\text{for } x\in\mathbb{R}^3,$$

so $u=(\partial_a C^a_\sigma)(g)$ on $\partial\Omega$. Using (14), that $-\Delta+m^2-a^2$ and $H+a$ commute and (13), we see that for any $x\in\mathbb{R}^3\setminus\partial\Omega$,

$$\begin{aligned}(H-a)u(x)&=\int(H_x-a)\partial_a(\phi^a(x-y))g(y)\,d\sigma(y)\\&=\int(-\Delta_x+m^2-a^2)(H_x+a)\,\frac{e^{-\sqrt{m^2-a^2}|x-y|}}{8\pi\sqrt{m^2-a^2}}\,g(y)\,d\sigma(y)\\&=\int(H_x+a)\,\frac{e^{-\sqrt{m^2-a^2}|x-y|}}{4\pi|x-y|}\,g(y)\,d\sigma(y)=\Phi^a(g)(x),\end{aligned} \tag{17}$$

because $\phi^a(x)=(H+a)(4\pi|x|)^{-1}e^{-\sqrt{m^2-a^2}|x|}$ by construction. Concernig the notation employed, we mention that Δ_x and H_x denote the Laplace and Dirac operators acting as a derivative on the x variable. Since ϕ^a is a fundamental solution of $H-a$, we see from (17) that $(H-a)^2u=0$ outside $\partial\Omega$, a fact that we already knew in view of (16) and the definition of u.

From Lemma 2.2(i), we have $g = i(\alpha\cdot N)(C^a_+(g) - C^a_-(g))$. Therefore, using (6), that $(H-a)\Phi^a(g) = 0$ outside $\partial\Omega$ and (17), we finally get

$$\begin{aligned}\langle(\partial_a C^a_\sigma)(g), g\rangle_\sigma &= -i\int (\alpha\cdot N)u\cdot\overline{(C^a_+(g) - C^a_-(g))}\,d\sigma\\ &= \int_{\mathbb{R}^3\setminus\partial\Omega}\Big((H-a)u\cdot\overline{\Phi^a(g)} - u\cdot\overline{(H-a)\Phi^a(g)}\Big)\,d\mu\\ &= \int_{\mathbb{R}^3\setminus\partial\Omega}|\Phi^a(g)|^2\,d\mu.\end{aligned}\tag{18}$$

Thanks to the Plemelj–Sokhotski jump formulae from Lemma 2.2(i), we see that if $g\in L^2(\sigma)^4$ is such that $\Phi^a(g) = 0$ in $\mathbb{R}^3\setminus\partial\Omega$ then $C^a_\pm(g) = 0$, and thus $g = 0$. Therefore, applying (18) to $g_j(a)$ and plugging it into (10) yields

$$\partial_a c_j(a) = \langle(\partial_a C^a_\sigma)(g_j(a)), g_j(a)\rangle_\sigma = \int_{\mathbb{R}^3\setminus\partial\Omega}|\Phi^a(g_j(a))|^2\,d\mu > 0,$$

because $g_j(a)$ is not identically zero (since $\|g_j(a)\|_\sigma = 1$ by assumption).

To finish the proof of the lemma, it only remains to be shown the stated conclusions about $\{\lambda_j(a)\}_j$. By Proposition 3.1 and the definition of Λ in (7), if $a\in(-m,m)$ then

$$\operatorname{kr}(H+V_\lambda - a)\neq 0\quad\text{if and only if}\quad \operatorname{kr}(1/\lambda + C^a_\sigma)\neq 0,$$

thus the existence of a sequence $\emptyset\neq\{\lambda_j(a)\}_j\subset\mathbb{R}$ such that $\operatorname{kr}(H+V_{\lambda_j(a)} - a)\neq 0$ and the fact stated in the lemma concerning its unique possible accumulation point follow from the first part of the lemma. Moreover, by setting $c_j(a) = -1/\lambda_j(a)$ we see that $\lambda_j(a)$ is a strictly monotonous increasing function of $a\in(-m,m)$ for all j. □

Corollary 3.3. *Given $a\in(-m,m)$, we have*

$$\sup\{\lambda<0 :\, \operatorname{kr}(H+V_\lambda - a)\neq 0\} = -4/\sup\{\lambda>0 :\, \operatorname{kr}(H+V_\lambda - a)\neq 0\},\tag{19}$$

and the same holds replacing sup *by* inf *on both sides of* (19). *Set*

$$\lambda^s_{\pm m} = \sup\{\lambda\in\mathbb{R} :\, \operatorname{kr}(1/\lambda + C^{\pm m}_\sigma)\neq 0\},\qquad \lambda^i_{\pm m} = \inf\{\lambda\in\mathbb{R} :\, \operatorname{kr}(1/\lambda + C^{\pm m}_\sigma)\neq 0\}.$$

Then $\lambda^s_{\pm m} > 0 > \lambda^i_{\pm m}$ and the following hold:

(i) $\sup\{\lambda\in\mathbb{R} :\, \operatorname{kr}(H+V_\lambda - a)\neq 0$ *for some* $a\in(-m,m)\} = \lambda^s_m$,
(ii) $\inf\{\lambda\in\mathbb{R} :\, \operatorname{kr}(H+V_\lambda - a)\neq 0$ *for some* $a\in(-m,m)\} = \lambda^i_{-m}$,
(iii) $\sup\{|\lambda| :\, \operatorname{kr}(H+V_\lambda - a)\neq 0$ *for some* $a\in(-m,m)\} = \max\{\lambda^s_m, -\lambda^i_{-m}\}$,
(iv) $\inf\{|\lambda| :\, \operatorname{kr}(H+V_\lambda - a)\neq 0$ *for some* $a\in(-m,m)\} = 4/\max\{\lambda^s_m, -\lambda^i_{-m}\}$.

Proof. Given $a\in(-m,m)$, by [2, Remark 3.5] we see that $\{\lambda\in\mathbb{R}\setminus\{0\} :\, \operatorname{kr}(H+V_\lambda - a)\neq 0\}$ is a non empty set. Furthermore, in [2, Theorem 3.3] we also proved that

$$\operatorname{kr}(H+V_\lambda - a)\neq 0\quad\text{if and only if}\quad \operatorname{kr}(H+V_{-4/\lambda} - a)\neq 0.\tag{20}$$

In particular, we see that $\{\lambda<0 :\, \operatorname{kr}(H+V_\lambda - a)\neq 0\}$ and $\{\lambda>0 :\, \operatorname{kr}(H+V_\lambda - a)\neq 0\}$ are non empty sets. Then, a simple argument using (20) proves (19).

Note that [2, Remark 3.5] still aplies to the case $a = \pm m$, so $\{\lambda\in\mathbb{R} :\, \operatorname{kr}(1/\lambda + C^{\pm m}_\sigma)\neq 0\}$ is a non empty set, and thus $\lambda^s_{\pm m}$ and $\lambda^i_{\pm m}$ are well defined. An inspection of the proof of [2, Theorem 3.3] shows that, for any $a\in[-m,m]$,

$$\operatorname{kr}(1/\lambda + C^a_\sigma)\neq 0\quad\text{if and only if}\quad \operatorname{kr}(-\lambda/4 + C^a_\sigma)\neq 0,\tag{21}$$

which in fact is a consequence of Lemma 2.2(*ii*) (note that (20) follows by (21) and Proposition 3.1). A straightforward application of (21) proves that $\lambda^s_{\pm m} > 0 > \lambda^i_{\pm m}$. Furthermore, (*i*) and (*ii*) are a direct consequence of the monotonicity property proved in Lemma 3.2, and (*iii*) follows from (*i*), (*ii*) and the fact that $\lambda^s_m > 0 > \lambda^i_{-m}$. Regarding (*iv*), note that $\inf\{|\lambda| : \operatorname{kr}(H+V_\lambda - a) \neq 0 \text{ for some } a \in (-m,m)\}$ is the minimum between $-\sup\{\lambda < 0 : \operatorname{kr}(H+V_\lambda - a) \neq 0 \text{ for some } a \in (-m,m)\}$ and $\inf\{\lambda > 0 : \operatorname{kr}(H+V_\lambda - a) \neq 0 \text{ for some } a \in (-m,m)\}$, which by (19) and Lemma 3.2 correspond to $4/\lambda^s_m$ and $-4/\lambda^i_{-m}$, respectively. This yields (*iv*). □

4. Quadratic forms

For $a \in \mathbb{R}$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, where the σ_j's compose the family of Pauli matrices introduced in (1), define the kernels

$$k^a(x) = \frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|} I_2 \quad \text{and} \quad w^a(x) = \frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|^3}\left(1+\sqrt{m^2-a^2}|x|\right) i\,\sigma\cdot x$$

for $x \in \mathbb{R}^3 \setminus \{0\}$. Given $f \in L^2(\sigma)^2$ and $x \in \partial\Omega$, set

$$K^a(f)(x) = \int k^a(x-z) f(z)\, d\sigma(z) \quad \text{and} \quad W^a(f)(x) = \lim_{\epsilon \searrow 0} \int_{|x-z|>\epsilon} w^a(x-z) f(z)\, d\sigma(z).$$

That K^a and W^a are bounded operators in $L^2(\sigma)^2$ can be verified similarly to the case of C^a_σ in $L^2(\sigma)^4$, we omit the details. Moreover, note that

$$C^a_\sigma = \begin{pmatrix} (a+m)K^a & W^a \\ W^a & (a-m)K^a \end{pmatrix}. \tag{22}$$

The results in the following lemma are contained in [2, Section 4].

Lemma 4.1. *For any $a \in [-m,m]$, K^a is positive and self-adjoint, W^a is also self-adjoint and the following hold:*
- (*i*) *the anticommutator $\{(\sigma\cdot N)K^a, (\sigma\cdot N)W^a\}$ vanishes identically,*
- (*ii*) $((\sigma\cdot N)W^a)^2 + (a^2-m^2)((\sigma\cdot N)K^a)^2 = -1/4$.

For simplicity of notation, we write k, w, K and W instead of k^m, w^m, K^m and W^m, respectively. Observe that $k(x) = 1/(4\pi|x|)$ and $w(x) = i\sigma\cdot x/(4\pi|x|^3)$. If T denotes a bounded operator in $L^2(\sigma)^2$, we write $\|T\|_\sigma$ instead of $\|T\|_{L^2(\sigma)^2 \to L^2(\sigma)^2}$.

The following lemma is essentially contained in [6], but we give a simple proof for the sake of completeness.

Lemma 4.2. $\|W\|_\sigma \geq 1/2$. *Moreover, $\|W\|_\sigma = 1/2$ if and only if $\{\sigma\cdot N, W\} = 0$, and in this case $2W$ is an isometry in $L^2(\sigma)^2$.*

Proof. From Lemma 4.1(*ii*) we have

$$\frac{1}{16}\int |f|^2\, d\sigma = \int |((\sigma\cdot N)W)^2(f)|^2\, d\sigma \leq \|W\|^2_\sigma \int |W(f)|^2\, d\sigma \leq \|W\|^4_\sigma \int |f|^2\, d\sigma \tag{23}$$

for all $f \in L^2(\sigma)^2$. From this we see that $\|W\|_\sigma \geq 1/2$.

On one hand, if $\|W\|_\sigma = 1/2$, then (23) yields

$$\frac{1}{4}\int |f|^2\, d\sigma = \int |W(f)|^2\, d\sigma \qquad \text{for all } f \in L^2(\sigma)^2, \tag{24}$$

which shows that $2W$ is an isometry in $L^2(\sigma)^2$. By Lemma 4.1 and (24) we conclude that

$$\begin{aligned}\int |\{\sigma\cdot N, W\}(f)|^2\,d\sigma &= \int ((\sigma\cdot N)W + W(\sigma\cdot N))(f)\cdot \overline{((\sigma\cdot N)W + W(\sigma\cdot N))(f)}\,d\sigma\\ &= \int \Big(|W(f)|^2 + |W(\sigma\cdot N)(f)|^2 - \frac{1}{4}|f|^2 - \frac{1}{4}|(\sigma\cdot N)f|^2\Big)\,d\sigma = 0\end{aligned}$$

for all $f \in L^2(\sigma)^2$, which implies that $\{\sigma\cdot N, W\} = 0$.

On the other hand, if $\{\sigma\cdot N, W\} = 0$ then, once again by Lemma 4.1,

$$\begin{aligned}\int |W(f)|^2\,d\sigma &= \int W(\sigma\cdot N)(\sigma\cdot N)W(f)\cdot \overline{f}\,d\sigma\\ &= \int W(\sigma\cdot N)\{\sigma\cdot N, W\}(f)\cdot\overline{f}\,d\sigma + \frac{1}{4}\int |f|^2\,d\sigma = \frac{1}{4}\int |f|^2\,d\sigma\end{aligned}\tag{25}$$

for all $f \in L^2(\sigma)^2$. In particular, $\|W\|_\sigma = 1/2$. □

We must mention that in [6] the authors show that $\{\sigma\cdot N, W\} = 0$ (or, equivalently, $\|W\|_\sigma = 1/2$) if and only if $\partial\Omega$ is a plane or a sphere, as commented in the introduction in reference to the isometric character of $2W$.

The following theorem explores the connection between the quadratic form inequality (4) and the eigenvalues of $C_\sigma^{\pm m}$, and it is a key ingredient to derive the isoperimetric-type inequalities contained in Theorem 1.1.

Theorem 4.3. *Let λ_Ω be the infimum over all $\lambda > 0$ such that*

$$\Big(\frac{4}{\lambda}\Big)^2 \int |W(f)|^2\,d\sigma + \frac{8m}{\lambda}\int K(f)\cdot\overline{f}\,d\sigma \le \int |f|^2\,d\sigma \tag{26}$$

for all $f \in L^2(\sigma)^2$. Then, λ_Ω is also the infimum over all $\lambda > 0$ such that

$$\int |f|^2\,d\sigma + 2m\lambda \int K(f)\cdot\overline{f}\,d\sigma \le \lambda^2 \int |W(f)|^2\,d\sigma \tag{27}$$

for all $f \in L^2(\sigma)^2$, and the following hold:

- (*i*) $2 < 4\big(m\|K\|_\sigma + \sqrt{m^2\|K\|_\sigma^2 + 1/4}\big) \le \lambda_\Omega \le 4\big(m\|K\|_\sigma + \sqrt{m^2\|K\|_\sigma^2 + \|W\|_\sigma^2}\big)$,
- (*ii*) *if $\lambda > 0$ is such that* $\mathrm{kr}(1/\lambda + C_\sigma^m) \neq 0$ *then $\lambda \le \lambda_\Omega$,*
- (*iii*) *if $\lambda < 0$ is such that* $\mathrm{kr}(1/\lambda + C_\sigma^{-m}) \neq 0$ *then $\lambda \ge -\lambda_\Omega$,*
- (*iv*) (26) *holds for all $\lambda \ge \lambda_\Omega$ and it is sharp for $\lambda = \lambda_\Omega$. If $\lambda = \lambda_\Omega > 2\sqrt{2}$ then the equality in* (26) *is attained. In this case, the minimizers of* (26) *(that is, functions that attain the equality) give rise to functions in* $\mathrm{kr}(1/\lambda_\Omega + C_\sigma^m)$ *and vice versa; the same holds replacing* $\mathrm{kr}(1/\lambda_\Omega + C_\sigma^m)$ *by* $\mathrm{kr}(-1/\lambda_\Omega + C_\sigma^{-m})$,
- (*v*) (*iv*) *also holds replacing* (26) *by* (27).

Proof. Given $\lambda > 0$ and $f \in L^2(\sigma)^2$, set

$$A(\lambda, f) = \Big(\frac{4}{\lambda}\Big)^2\int |W(f)|^2\,d\sigma + \frac{8m}{\lambda}\int K(f)\cdot\overline{f}\,d\sigma.$$

Let us prove (*i*). Note that

$$A(\lambda, f) \le \bigg(\Big(\frac{4\|W\|_\sigma}{\lambda}\Big)^2 + \frac{8m\|K\|_\sigma}{\lambda}\bigg)\|f\|_\sigma^2. \tag{28}$$

Hence, if $\lambda \ge 4\big(m\|K\|_\sigma + \sqrt{m^2\|K\|_\sigma^2 + \|W\|_\sigma^2}\big)$ then (28) easily yields $A(\lambda, f) \le \|f\|_\sigma^2$ for all $f \in L^2(\sigma)^2$, which in turn implies that $\lambda_\Omega \le 4\big(m\|K\|_\sigma + \sqrt{m^2\|K\|_\sigma^2 + \|W\|_\sigma^2}\big)$.

The inequality from below is a bit more involved. Let $\lambda > 0$ be such that

$$A(\lambda, f) \leq \|f\|_\sigma^2 \qquad \text{for all } f \in L^2(\sigma)^2. \tag{29}$$

If we set $h = \frac{4}{\lambda}(\sigma \cdot N)W(f) \in L^2(\sigma)^2$, then $f = -\lambda(\sigma \cdot N)W(h)$ by Lemma 4.1(ii) taking $a = m$. Furthermore,

$$\int |W(f)|^2\, d\sigma = \left(\frac{\lambda}{4}\right)^2 \int |(\sigma \cdot N)h|^2\, d\sigma = \left(\frac{\lambda}{4}\right)^2 \int |h|^2\, d\sigma \tag{30}$$

and

$$\int |f|^2\, d\sigma = \lambda^2 \int |(\sigma \cdot N)W(h)|^2\, d\sigma = \lambda^2 \int |W(h)|^2\, d\sigma. \tag{31}$$

Moreover, using Lemma 4.1,

$$\begin{aligned} \int K(f) \cdot \overline{f}\, d\sigma &= \lambda^2 \int K(\sigma \cdot N)W(h) \cdot \overline{(\sigma \cdot N)W(h)}\, d\sigma \\ &= -\lambda^2 \int K(\sigma \cdot N)W(\sigma \cdot N)W(h) \cdot \overline{h}\, d\sigma = \frac{\lambda^2}{4} \int K(h) \cdot \overline{h}\, d\sigma. \end{aligned} \tag{32}$$

Gathering (29) with (30), (31) and (32) yields

$$\int |h|^2\, d\sigma + 2m\lambda \int K(h) \cdot \overline{h}\, d\sigma \leq \lambda^2 \int |W(h)|^2\, d\sigma \qquad \text{for all } h \in L^2(\sigma)^2. \tag{33}$$

Note that this argument is reversible, thus in particular we have proven that

$$\lambda_\Omega = \inf\left\{\lambda > 0 : \int |f|^2\, d\sigma + 2m\lambda \int K(f) \cdot \overline{f}\, d\sigma \leq \lambda^2 \int |W(f)|^2\, d\sigma \quad \forall\, f \in L^2(\sigma)^2\right\},$$

which yields (27). If we multiply (33) by $16/\lambda^4$ we get

$$\frac{16}{\lambda^4} \int |f|^2\, d\sigma + \frac{32m}{\lambda^3} \int K(f) \cdot \overline{f}\, d\sigma \leq \frac{16}{\lambda^2} \int |W(f)|^2\, d\sigma \qquad \text{for all } f \in L^2(\sigma)^2,$$

which added to (29) gives

$$2m \int K(f) \cdot \overline{f}\, d\sigma \leq \left(\frac{\lambda}{4} - \frac{1}{\lambda}\right) \int |f|^2\, d\sigma \qquad \text{for all } f \in L^2(\sigma)^2.$$

Since K is bounded, positive and self-adjoint, we see from the above inequality that

$$2m\|K\|_\sigma = 2m \sup_{\|f\|_\sigma = 1} \int K(f) \cdot \overline{f}\, d\sigma \leq \frac{\lambda}{4} - \frac{1}{\lambda},$$

which in turn is equivalent to

$$\lambda^2 - 8m\|K\|_\sigma \lambda - 4 \geq 0,$$

since $\lambda > 0$ by assumption. Therefore, we must have $\lambda \geq 4\big(m\|K\|_\sigma + \sqrt{m^2\|K\|_\sigma^2 + 1/4}\big)$ for all $\lambda > 0$ satisfying (29). This gives the desired inequality from below for λ_Ω, and finishes the proof of (i). Observe that this lower bound for λ_Ω is strictly greater than 2 because $\|K\|_\sigma > 0$.

We now prove (ii). Assume that $\lambda > 0$ is such that $\mathrm{kr}(1/\lambda + C_\sigma^m) \neq 0$. Let $0 \neq g \in L^2(\sigma)^4$ be such that $C_\sigma^m(g) = -g/\lambda$. In view of (22),

$$\text{if} \quad g = \begin{pmatrix} u \\ h \end{pmatrix} \quad \text{then} \quad \left\{ \begin{aligned} 2mK(u) + W(h) &= -u/\lambda, \\ W(u) &= -h/\lambda. \end{aligned} \right. \tag{34}$$

From Lemma 4.1(*ii*) and the last equality in (34) we deduce that

$$u = -4((\sigma \cdot N)W)^2(u) = \frac{4}{\lambda}(\sigma \cdot N)W(\sigma \cdot N)(h), \tag{35}$$

which plugged in the other equation in (34) yields

$$\Big(\frac{8m}{\lambda}K(\sigma \cdot N)W(\sigma \cdot N) + W + \frac{4}{\lambda^2}(\sigma \cdot N)W(\sigma \cdot N)\Big)(h) = 0. \tag{36}$$

Using Lemma 4.1, we may write

$$\begin{aligned}
&\frac{8m}{\lambda}K(\sigma \cdot N)W(\sigma \cdot N) + W + \frac{4}{\lambda^2}(\sigma \cdot N)W(\sigma \cdot N)\\
&\quad = -\frac{8m}{\lambda}W(\sigma \cdot N)K(\sigma \cdot N) + W(\sigma \cdot N)(\sigma \cdot N) - \frac{16}{\lambda^2}(W(\sigma \cdot N))^2(\sigma \cdot N)W(\sigma \cdot N)\\
&\quad = W(\sigma \cdot N)\Big(-\frac{8m}{\lambda}K + 1 - \frac{16}{\lambda^2}W^2\Big)(\sigma \cdot N).
\end{aligned} \tag{37}$$

Since $W(\sigma \cdot N)$ is invertible by Lemma 4.1(*ii*), from (36) and (37) we get that

$$\Big(-\frac{8m}{\lambda}K + 1 - \frac{16}{\lambda^2}W^2\Big)(f) = 0, \tag{38}$$

where we have set $f = (\sigma \cdot N)h$. Note that u is given in terms of h by (35) and $g \neq 0$ by assumption, thus we can also assume that $f \neq 0$. In conclusion, we have seen that if $\mathrm{kr}(1/\lambda + C^m_\sigma) \neq 0$ then there exists $0 \neq f \in L^2(\sigma)^2$ such that (38) holds. Actually, since all the involved arguments are reversible, we see that

$$\mathrm{kr}(1/\lambda + C^m_\sigma) \neq 0 \quad \text{if and only if} \quad \mathrm{kr}(-(8m/\lambda)K + 1 - (16/\lambda^2)W^2) \neq 0. \tag{39}$$

Moreover, if we multiply (38) by $\overline{f}$ and we integrate with respect to σ, using the self-adjointness of W we get

$$A(\lambda, f) = \|f\|^2_\sigma \qquad \text{for all } f \in \mathrm{kr}(-(8m/\lambda)K + 1 - (16/\lambda^2)W^2). \tag{40}$$

Using that W is invertible and that K is positive, it is easy to show that $A(\lambda-\epsilon, f) > A(\lambda, f)$ for all $f \neq 0$ and all $0 < \epsilon < \lambda$. In particular, $A(\lambda - \epsilon, f) > \|f\|^2_\sigma$ for all $0 \neq f \in \mathrm{kr}(-(8m/\lambda)K + 1 - (16/\lambda^2)W^2)$, which easily implies that $\lambda - \epsilon \leq \lambda_\Omega$ for all $0 < \epsilon < \lambda$ whenever $\mathrm{kr}(-(8m/\lambda)K + 1 - (16/\lambda^2)W^2) \neq 0$. Finally, applying (39) and taking $\epsilon \to 0$ we conclude that if $\mathrm{kr}(1/\lambda + C^m_\sigma) \neq 0$ then $\lambda \leq \lambda_\Omega$, and the proof of (*ii*) is complete.

Concerning (*iii*), if one repeats the arguments used to prove (*ii*) but on the assumtion that $\mathrm{kr}(1/\lambda + C^{-m}_\sigma) \neq 0$, one can show that there exists some $f \in L^2(\sigma)^2$ such that

$$0 = \Big(\frac{8m}{\lambda}K + 1 - \frac{16}{\lambda^2}W^2\Big)(f) = \Big(-\frac{8m}{|\lambda|}K + 1 - \frac{16}{|\lambda|^2}W^2\Big)(f),$$

since we are assuming $\lambda < 0$. Hence, we are reduced to the case treated in (38) but with the parameter $|\lambda|$. The rest of the proof follows the same lines, getting that $-\lambda = |\lambda| \leq \lambda_\Omega$. In particular, we also obtain that

$$\mathrm{kr}(1/\lambda + C^{-m}_\sigma) \neq 0 \quad \text{if and only if} \quad \mathrm{kr}((8m/\lambda)K + 1 - (16/\lambda^2)W^2) \neq 0. \tag{41}$$

Let us prove (*iv*). Since K is positive, $A(\lambda, f)$ is a non-increasing function of $\lambda > 0$ for all $f \in L^2(\sigma)^2$. By the definiton of λ_Ω, this monotony implies that (26) holds for all $\lambda \geq \lambda_\Omega$ and it is sharp for $\lambda = \lambda_\Omega$. It remains to be shown that if $\lambda_\Omega > 2\sqrt{2}$ then the equality is attained and that the minimizers give rise to functions in $\operatorname{kr}(1/\lambda_\Omega + C_\sigma^m)$ and vice versa. As we did in (25),

$$\int |W(f)|^2\,d\sigma = \int W(\sigma\cdot N)\{\sigma\cdot N, W\}(f)\cdot\overline{f}\,d\sigma + \frac{1}{4}\int |f|^2\,d\sigma$$

for all $f\in L^2(\sigma)^2$. Set

$$T = W(\sigma\cdot N)\{\sigma\cdot N, W\} = W^2 - \frac{1}{4}.$$

From Lemma 4.1 we see that T is self-adjoint and, since $\partial\Omega$ is $\mathcal{C}^\infty$, it is also compact by the same arguments that prove [1, Lemma 3.5]. Now, we can write

$$A(\lambda, f) - \|f\|_\sigma^2 = \int \left(\frac{16}{\lambda^2}T + \frac{8m}{\lambda}K\right)(f)\cdot\overline{f}\,d\sigma + \left(\frac{4}{\lambda^2} - 1\right)\int |f|^2\,d\sigma. \tag{42}$$

Let $\lambda > 2$. Then, (42) shows that

$$A(\lambda, f) \le \|f\|_\sigma^2 \quad \text{if and only if} \quad \int T_\lambda(f)\cdot\overline{f}\,d\sigma \le \int |f|^2\,d\sigma, \tag{43}$$

where we have set

$$T_\lambda = \frac{4}{\lambda^2 - 4}\left(4T + 2m\lambda K\right),$$

and the same holds replacing "$\le$" by "$=$" or "$>$" on both sides of (43). Observe that T_λ is also self-adjoint and compact (for all real $\lambda \neq \pm 2$), since T and K also are. In particular, by [5, Lemma (0.43)], there exists $0 \neq f_\lambda \in L^2(\sigma)^2$ such that $T_\lambda(f_\lambda) = \|T_\lambda\|_\sigma f_\lambda$ or $T_\lambda(f_\lambda) = -\|T_\lambda\|_\sigma f_\lambda$.

We are going to show that if $\lambda_\Omega > 2\sqrt{2}$ then we must have $T_{\lambda_\Omega}(f_{\lambda_\Omega}) = \|T_{\lambda_\Omega}\|_\sigma f_{\lambda_\Omega}$ with $\|T_{\lambda_\Omega}\|_\sigma = 1$. Using (42) we see that, if $\lambda > 2$,

$$A(\lambda, f) \ge 0 \quad \text{if and only if} \quad \int T_\lambda(f)\cdot\overline{f}\,d\sigma \ge -\frac{4}{\lambda^2 - 4}\int |f|^2\,d\sigma. \tag{44}$$

Since $A(\lambda, f) \ge 0$ for all $\lambda > 0$ because K is positive, from (44) we get that

$$\int T_\lambda(f)\cdot\overline{f}\,d\sigma > -\int |f|^2\,d\sigma \qquad \text{for all } 0 \neq f \in L^2(\sigma)^2 \text{ and all } \lambda > 2\sqrt{2}. \tag{45}$$

Combining (43) and (45), and using that T_λ is self-adjoint, we deduce that

$$\|T_\lambda\|_\sigma = \sup_{\|f\|_\sigma = 1}\left|\int T_\lambda(f)\cdot\overline{f}\,d\sigma\right| \le 1 \qquad \text{for all } \lambda \ge \lambda_\Omega \text{ if } \lambda_\Omega > 2\sqrt{2}. \tag{46}$$

Furthermore, the definition of λ_Ω and (43) imply that $\|T_\lambda\|_\sigma > 1$ for all $\lambda < \lambda_\Omega$. From this and (46) we get that if $\lambda_\Omega > 2\sqrt{2}$ then $\|T_{\lambda_\Omega}\|_\sigma = 1$, since $\|T_\lambda\|_\sigma$ depends continuously on λ for all $\lambda > 2$. In particular, we have seen that there exists $0 \neq f_{\lambda_\Omega} \in L^2(\sigma)^2$ such that $T_{\lambda_\Omega}(f_{\lambda_\Omega}) = f_{\lambda_\Omega}$ or $T_{\lambda_\Omega}(f_{\lambda_\Omega}) = -f_{\lambda_\Omega}$. However, if $\lambda_\Omega > 2\sqrt{2}$ then (45) shows that the case $T_{\lambda_\Omega}(f_{\lambda_\Omega}) = -f_{\lambda_\Omega}$ is not possible, thus $T_{\lambda_\Omega}(f_{\lambda_\Omega}) = f_{\lambda_\Omega}$ as claimed. From (43), we finally get $A(\lambda_\Omega, f_{\lambda_\Omega}) = \|f_{\lambda_\Omega}\|_\sigma^2$, which proves that the equality in (26) is attained for $\lambda = \lambda_\Omega$ on f_{λ_Ω}.

Concerning the minimizers of (26), assume that $A(\lambda_\Omega, f) = \|f\|_\sigma^2$ for some $f \neq 0$. Then (43) gives

$$\int T_{\lambda_\Omega}(f)\cdot\overline{f}\,d\sigma = \int |f|^2\,d\sigma. \tag{47}$$

Since T_{λ_Ω} is a compact self-ajoint operator (so it diagonalizes in an orthonormal basis of eigenvectors) and $\|T_{\lambda_\Omega}\|_\sigma = 1$ if $\lambda_\Omega > 2\sqrt{2}$, by (47) we must have $T_{\lambda_\Omega}(f) = f$. From the definitions of T_{λ_Ω} and T, we get

$$\left(16W^2 + 8m\lambda_\Omega K - {\lambda_\Omega}^2\right)(f) = 0,$$

which implies that $\mathrm{kr}(1/\lambda_\Omega + C_\sigma^m) \neq 0$ by (39) and that $\mathrm{kr}(-1/\lambda_\Omega + C_\sigma^{-m}) \neq 0$ by (41). On the contrary, if $\mathrm{kr}(1/\lambda_\Omega + C_\sigma^m) \neq 0$ then (39) and (40) show that there exists $f \neq 0$ such that $A(\lambda_\Omega, f) = \|f\|_\sigma^2$, and similarly for $\mathrm{kr}(-1/\lambda_\Omega + C_\sigma^{-m}) \neq 0$ using (41) and (40).

Regarding (v), one can check that the conclusions in (iv) also hold when one works with (27) instead of (26) since these quadratic form inequalities are equivalent (recall the computations carried out between (29) and (33)), we leave the details for the reader. The theorem is finally proved. □

Remark 4.4. Gathering (39) and (41), we get that

$$\mathrm{kr}(1/\lambda + C_\sigma^m) \neq 0 \quad \text{if and only if} \quad \mathrm{kr}(-1/\lambda + C_\sigma^{-m}) \neq 0. \tag{48}$$

This corresponds to the endpoint case $a = m$ in [2, Theorem 3.6], thanks to Proposition 3.1. The relevant fact here is that, despite that in [2, Theorem 3.6] we were assuming some invariance of σ with respect to reflections in order to obtain the antisymmetry property of the eigenvalues with respect to the potential, (48) holds without this assumption on σ.

Remark 4.5. The assumption $\lambda_\Omega > 2\sqrt{2}$ in Theorem 4.3(iv) can be weakened using essentially the same arguments as before. Roughly speaking, from (23) one sees that $16\|W\|_\sigma^2 W^2 \geq 1$, considering this as an inequality between operators in the sense of quadratic forms. Then

$$T_\lambda = \frac{4\,(4T + 2m\lambda K)}{\lambda^2 - 4} \geq \frac{16T}{\lambda^2 - 4} = \frac{16W^2 - 4}{\lambda^2 - 4} \geq \frac{1}{\lambda^2 - 4}\left(\frac{1}{\|W\|_\sigma^2} - 4\right). \tag{49}$$

The right hand side of (44) formally corresponds to the limiting case $\|W\|_\sigma = \infty$ in (49). Since the arguments in the proof of Theorem 4.3(iv) require that $T_{\lambda_\Omega} > -1$ in order to get $\|T_{\lambda_\Omega}\|_\sigma = 1$ and find the minimizers, in view of (49) one sees that a possible assumption is

$$\lambda_\Omega > 2\sqrt{2 - \frac{1}{4\|W\|_\sigma^2}}, \tag{50}$$

which is weaker than $\lambda_\Omega > 2\sqrt{2}$. Following the arguments in the forthcoming pages of this article until the proof of Theorem 1.1 but using (50) instead of $\lambda_\Omega > 2\sqrt{2}$, one can see that (3) can be weakened to

$$m\,\frac{\mathrm{Area}(\partial\Omega)}{\mathrm{Cap}(\Omega)} > \frac{1 - \frac{1}{4\|W\|_\sigma^2}}{4\sqrt{2 - \frac{1}{4\|W\|_\sigma^2}}}.$$

However, in what respects to the potential applications of Theorem 1.1 as an isoperimetric-type inequality, one may find bounded domains Ω with constant $m\mathrm{Area}(\partial\Omega)/\mathrm{Cap}(\Omega)$ but with $\|W\|_\sigma$ arbitrarily large, since this last quantity strongly depends on the abruptness of $\partial\Omega$. As a consequence, in general one has to assume (3) (or equivalently the limiting case $\lambda_\Omega > 2\sqrt{2}$) to make use of Theorem 1.1.

Corollary 4.6. *Let* λ^s_m *and* λ^i_{-m} *be as in* Corollary 3.3. *If* $\lambda_\Omega > 2\sqrt{2}$, *then*

$$\begin{aligned}\lambda_\Omega = \lambda^s_m &= -\lambda^i_{-m}\\ &= \sup\{\lambda\in\mathbb{R}:\, \mathrm{kr}(H+V_\lambda-a)\neq 0 \textit{ for some } a\in(-m,m)\}\\ &= -\inf\{\lambda\in\mathbb{R}:\, \mathrm{kr}(H+V_\lambda-a)\neq 0 \textit{ for some } a\in(-m,m)\}\\ &= \sup\{|\lambda|:\, \mathrm{kr}(H+V_\lambda-a)\neq 0 \textit{ for some } a\in(-m,m)\}\end{aligned}$$

and $4/\lambda_\Omega = \inf\{|\lambda|:\, \mathrm{kr}(H+V_\lambda-a)\neq 0$ *for some* $a\in(-m,m)\}$.

Proof. From Corollary 3.3 we already know that $\lambda^s_m > 0 > \lambda^i_{-m}$. Combining Theorem 4.3($ii$) and ($iv$), we easily see that

$$\lambda^s_m = \sup\{\lambda\in\mathbb{R}:\, \mathrm{kr}(1/\lambda + C^m_\sigma)\neq 0\} = \lambda_\Omega,$$

and similarly, using Theorem 4.3(iii) and (iv), we get that $\lambda^i_{-m} = -\lambda_\Omega$. Hence, the corollary follows directly from Corollary 3.3. Observe that the supremum and the infimum in the definitions of λ^s_m and λ^i_{-m} are a maximum and a minimum, respectively, if $\lambda_\Omega > 2\sqrt{2}$. □

Remark 4.7. Combining the methods used above one can also show that, if $\lambda_\Omega > 2\sqrt{2}$,

$$\begin{aligned}4/\lambda_\Omega &= \inf\{\lambda>0:\, \mathrm{kr}(H+V_\lambda-a)\neq 0 \text{ for some } a\in(-m,m)\}\\ &= -\sup\{\lambda<0:\, \mathrm{kr}(H+V_\lambda-a)\neq 0 \text{ for some } a\in(-m,m)\}.\end{aligned}$$

5. An isoperimetric-type inequality

For the sake of clarity, given a bounded open set $\Omega\subset\mathbb{R}^3$ with smooth boundary, we set

$$\mathrm{Vol}(\Omega)=\mu(\Omega) \quad\text{and}\quad \mathrm{Area}(\partial\Omega)=\sigma(\partial\Omega).$$

Furthermore, to stress the dependence of K and W on σ (that is, on $\partial\Omega$), we write K_Ω and W_Ω respectively.

5.1. A test to exclude constraints on Ω.

In the setting of bounded domains with smooth boundary, due to Theorem 4.3(i) we have

$$4\big(m\|K_\Omega\|_\sigma + \sqrt{m^2\|K_\Omega\|^2_\sigma + 1/4}\big) \leq \lambda_\Omega \leq 4\big(m\|K_\Omega\|_\sigma + \sqrt{m^2\|K_\Omega\|^2_\sigma + \|W_\Omega\|^2_\sigma}\big).$$

Since $\|W_\Omega\|^2_\sigma = 1/4$ if and only if $\partial\Omega$ is a sphere (recall [6]), one may be tempted to look for an isoperimetric-type inequality for $\|K_\Omega\|_\sigma$ so that the ball is a minimizer, and thus obtaining an inequality for λ_Ω. In order to do so, one may impose some constraint on the admissible domains because of the rescaling properties of $\|K_\Omega\|_\sigma$ under dilations; if

$$\Omega_t = \{tx:\, x\in\Omega\} \quad \text{for } t>0$$

and σ_t is the surface measure on $\partial\Omega_t$ then $\|K_{\Omega_t}\|_{\sigma_t} = O(t)$ but $\|W_{\Omega_t}\|_{\sigma_t} = \|W_\Omega\|_\sigma = O(1)$. We are going to present a simple and classical method to test possible constraints that do not permit the existence of domains that minimize $\|K_\Omega\|_\sigma$. Roughly speaking, the method is based on the splitting of a domain into two suitable copies of itself. In particular, it allows us to prove that "*there is no bounded domain with smooth boundary that attains the infimum of* $\|K_\Omega\|_\sigma$ *over all bounded domains* Ω *with smooth boundary and constant volume*". The same holds replacing "volume" by "area of the boundary".

For $t>0$ and $z\in\mathbb{R}^3$, we set $\Omega_{t,z} = \Omega_t\cup(\Omega_t+z)$ and we denote by $\sigma_{t,z}$ the surface measure on $\partial\Omega_{t,z}$. We assume that $|z|$ is big enough, so $\Omega_t\cap(\Omega_t+z)=\emptyset$.

Lemma 5.1. *Given* $\Omega\subset\mathbb{R}^3$ *and* $t>0$, *if* $|z|$ *is big enough then*

$$\big|\|K_{\Omega_{t,z}}\|_{\sigma_{t,z}} - t\|K_\Omega\|_\sigma\big| \leq \frac{\sigma(\partial\Omega_{t,z})}{2\pi\,\mathrm{dist}(\partial\Omega_t, \partial\Omega_t+z)}.$$

Proof. Since K_{Ω_t} is positive and self-adjoint, a change of variables easily yields

$$\begin{aligned}\|K_{\Omega_t}\|_{\sigma_t} &= \sup_{f\neq 0}\frac{1}{\|f\|^2_{\sigma_t}}\int K_{\Omega_t}(f)\cdot\overline{f}\,d\sigma_t = \sup_{f\neq 0}\frac{1}{\|f\|^2_{\sigma_t}}\iint\frac{f(x)\cdot\overline{f(y)}}{4\pi|x-y|}\,d\sigma_t(x)\,d\sigma_t(y)\\ &= \sup_{f\neq 0}\frac{t}{\int|f(tx)|^2\,d\sigma(x)}\iint\frac{f(tx)\cdot\overline{f(ty)}}{4\pi|x-y|}\,d\sigma(x)\,d\sigma(y) = t\|K_\Omega\|_\sigma.\end{aligned}\tag{51}$$

Given $f\in L^2(\sigma_{t,z})^2$, set

$$I(f) = \iint\frac{f(x)\cdot\overline{f(y)}}{4\pi|x-y|}\,d\sigma_{t,z}(x)\,d\sigma_{t,z}(y).$$

Since $\partial\Omega_{t,z} = \partial\Omega_t\cup(\partial\Omega_t+z)$, using Fubini's theorem we can decompose

$$\begin{aligned}I(f) &= \left(\iint_{\partial\Omega_t\times\partial\Omega_t}+\iint_{(\partial\Omega_t+z)\times(\partial\Omega_t+z)}+2\mathrm{Re}\iint_{\partial\Omega_t\times(\partial\Omega_t+z)}\right)\frac{f(x)\cdot\overline{f(y)}}{4\pi|x-y|}\,d\sigma_{t,z}(x)\,d\sigma_{t,z}(y)\\ &=: I_1(f)+I_2(f)+I_3(f).\end{aligned}\tag{52}$$

Note that $\|K_{\Omega_t+z}\|_{\sigma_t+z} = \|K_{\Omega_t}\|_{\sigma_t}$ and $\|f\|^2_{\sigma_t}+\|f\|^2_{\sigma_t+z} = \|f\|^2_{\sigma_{t,z}}$, thus using (51) we get

$$\begin{aligned}I_1(f)+I_2(f) &= \|f\|^2_{\sigma_t}\frac{I_1(f)}{\|f\|^2_{\sigma_t}}+\|f\|^2_{\sigma_t+z}\frac{I_2(f)}{\|f\|^2_{\sigma_t+z}}\\ &\leq \|f\|^2_{\sigma_t}\|K_{\Omega_t}\|_{\sigma_t}+\|f\|^2_{\sigma_t+z}\|K_{\Omega_t+z}\|_{\sigma_t+z} = \|f\|^2_{\sigma_{t,z}}t\|K_\Omega\|_\sigma.\end{aligned}\tag{53}$$

Moreover, by Hölder's inequality,

$$|I_3(f)|\leq\frac{\int_{\partial\Omega_t}|f|\,d\sigma_{t,z}\int_{\partial\Omega_t+z}|f|\,d\sigma_{t,z}}{2\pi\mathrm{dist}(\partial\Omega_t,\partial\Omega_t+z)}\leq\frac{\sigma(\partial\Omega_{t,z})\|f\|^2_{\sigma_{t,z}}}{2\pi\mathrm{dist}(\partial\Omega_t,\partial\Omega_t+z)}.\tag{54}$$

Dividing by $\|f\|^2_{\sigma_{t,z}}$ and taking the supremum over all $f\neq 0$ in (52), and using (53) and (54), we finally obtain

$$\begin{aligned}\|K_{\Omega_{t,z}}\|_{\sigma_{t,z}} &= \sup_{f\neq 0}\frac{I(f)}{\|f\|^2_{\sigma_{t,z}}}\leq\sup_{f\neq 0}\frac{I_1(f)+I_2(f)}{\|f\|^2_{\sigma_{t,z}}}+\sup_{f\neq 0}\frac{I_3(f)}{\|f\|^2_{\sigma_{t,z}}}\\ &\leq t\|K_\Omega\|_\sigma+\frac{\sigma(\partial\Omega_{t,z})}{2\pi\mathrm{dist}(\partial\Omega_t,\partial\Omega_t+z)}.\end{aligned}\tag{55}$$

Given $f\in L^2(\sigma_t)^2$ and $g\in L^2(\sigma_t+z)^2$, set $h = f\chi_{\partial\Omega_t}/\|f\|_{\sigma_t}+g\chi_{\partial\Omega_t+z}/\|g\|_{\sigma_t+z}\in L^2(\sigma_{t,z})^2$ (we extended f to be identically zero in $\partial\Omega_t+z$ and analogously for g). Then $\|h\|^2_{\sigma_{t,z}} = 2$ and, using (52) and (54) on h, we get

$$\begin{aligned}\frac{I_1(f)}{\|f\|^2_{\sigma_t}}+\frac{I_2(g)}{\|g\|^2_{\sigma_t+z}} &= I_1(h)+I_2(h)\leq 2\frac{I(h)}{\|h\|^2_{\sigma_{t,z}}}+|I_3(h)|\\ &\leq 2\|K_{\Omega_{t,z}}\|_{\sigma_{t,z}}+\frac{\sigma(\partial\Omega_{t,z})}{\pi\mathrm{dist}(\partial\Omega_t,\partial\Omega_t+z)}.\end{aligned}\tag{56}$$

Taking the supremum over all $f\neq 0$ and $g\neq 0$ in (56), and since K_{Ω_t} and K_{Ω_t+z} are positive and self-adjoint, using (51) we see that

$$2t\|K_\Omega\|_\sigma = \|K_{\Omega_t}\|_{\sigma_t}+\|K_{\Omega_t+z}\|_{\sigma_t+z}\leq 2\|K_{\Omega_{t,z}}\|_{\sigma_{t,z}}+\frac{\sigma(\partial\Omega_{t,z})}{\pi\mathrm{dist}(\partial\Omega_t,\partial\Omega_t+z)}.\tag{57}$$

The lemma follows from (55) and (57). □

With Lemma 5.1 at our disposal, we can easily prove that "*there is no bounded open set with smooth boundary that attains the infimum of* $\|K_\Omega\|_\sigma$ *over all bounded open sets* Ω *with smooth boundary and constant volume*", and that the same holds replacing "volume" by "area of the boundary". Let Ω be a bounded open set with smooth boundary. If $|z|$ is big enough, $\mathrm{Vol}(\Omega_{2^{-1/3},z}) = 2\mathrm{Vol}(\Omega_{2^{-1/3}}) = \mathrm{Vol}(\Omega)$, and Lemma 5.1 shows that

$$\|K_{\Omega_{2^{-1/3},z}}\|_{\sigma_{2^{-1/3},z}} \leq 2^{-1/3}\|K_\Omega\|_\sigma + \frac{\sigma(\partial\Omega_{2^{-1/3}})}{\pi \mathrm{dist}(\partial\Omega_{2^{-1/3}}, \partial\Omega_{2^{-1/3}} + z)} < \|K_\Omega\|_\sigma,$$

thus given Ω we have constructed another bounded domain $\Omega_{2^{-1/3},z}$ with smooth boundary, with the same volume as Ω, but with a strictly smaller norm of the associated operator K. Hence, there can not exists a minimizer. In case that the constraint concerns "constant area of the boundary", one only needs to argue with $\Omega_{2^{-1/2},z}$ instead of $\Omega_{2^{-1/3},z}$.

Finally, under the assumption of connectedness, the statement "*there is no bounded domain with smooth boundary that attains the infimum of* $\|K_\Omega\|_\sigma$ *over all bounded domains* Ω *with smooth boundary and constant volume*" can be proven with the same arguments as before but connecting, in a smooth way, the two connected components of $\Omega_{t,z}$ (once t and z are properly chosen) by a thin tube and showing that the contribution of the tube in $\|K_{\Omega_{t,z}}\|_{\sigma_{t,z}}$ is as small as we want by taking the tube thin enough, essentially because the kernel k is locally integrable with respect to surface measure. We leave the details for the reader.

5.2. **The relation with the Newtonian capacity.** Given a compact set $E \subset \mathbb{R}^3$, the *Newtonian capacity* of E (sometimes referred in the literature as *electrostatic* or *harmonic capacity*) is defined by

$$\mathrm{Cap}(E) = \left(\inf_\nu \iint \frac{d\nu(x)\, d\nu(y)}{4\pi|x-y|}\right)^{-1},$$

where the infimum is taken over all probability Borel measures ν supported in E. Sometimes in the literature, the 4π appearing in the definition of $\mathrm{Cap}(E)$ is changed by another precise constant. For the case of open sets $U \subset \mathbb{R}^3$, one defines

$$\mathrm{Cap}(U) = \sup\{\mathrm{Cap}(E) : E \subset U,\, E \text{ compact}\}.$$

The Newtonian capacity has a number of distinguished properties which we state as a lemma for future applications (see [7, Chapters 9 and 11] or [8], for example).

Lemma 5.2. *Let* Ω *be a bounded open set with smooth boundary. Then,*

(*i*) $\mathrm{Cap}(\Omega) = \mathrm{Cap}(\overline{\Omega}) = \mathrm{Cap}(\partial\Omega)$,

(*ii*) *Pólya-Szegö inequality: let* Ω^* *be the closed ball centered at the origin such that* $\mathrm{Vol}(\Omega^*) = \mathrm{Vol}(\Omega)$. *Then* $\mathrm{Cap}(\Omega) \geq \mathrm{Cap}(\Omega^*)$. *Moreover, the equality holds if and only if* Ω *is a ball.*

(*iii*) $\mathrm{Cap}(\Omega) = 2(6\pi^2)^{1/3}\mathrm{Vol}(\Omega)^{1/3}$ *if* Ω *is a ball.*

Regarding the uniqueness of the minimizer in Lemma 5.2(*ii*), it is important to impose some restriction on $\partial\Omega$ (such as regularity) in order to avoid sets of Newtonian capacity zero.

Lemma 5.3. *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain with smooth boundary. Then*

$$\lambda_\Omega \geq 4\left(m\,\frac{\mathrm{Area}(\partial\Omega)}{\mathrm{Cap}(\Omega)} + \sqrt{m^2\left(\frac{\mathrm{Area}(\partial\Omega)}{\mathrm{Cap}(\Omega)}\right)^2 + \frac{1}{4}}\right), \tag{58}$$

and the equality holds if and only if Ω *is a ball.*

Proof. Since K_Ω is a positive self-adjoint operator, we have

$$\begin{aligned}\|K_\Omega\|_\sigma &= \sup_{f\neq 0}\frac{1}{\|f\|_\sigma^2}\int K_\Omega(f)\cdot\overline{f}\,d\sigma \geq \frac{1}{\sigma(\partial\Omega)}\int K_\Omega\left(\begin{pmatrix}1\\0\end{pmatrix}\right)\cdot\overline{\begin{pmatrix}1\\0\end{pmatrix}}\,d\sigma\\ &= \sigma(\partial\Omega)\iint\frac{1}{4\pi|x-y|}\frac{d\sigma(x)}{\sigma(\partial\Omega)}\frac{d\sigma(y)}{\sigma(\partial\Omega)} \geq \frac{\text{Area}(\partial\Omega)}{\text{Cap}(\Omega)},\end{aligned} \tag{59}$$

where we also used Lemma 5.2(i) in the last inequality above. Gathering (59) and Theorem 4.3(i), we get (58).

Assume that Ω is a ball of radius $r>0$ centered at the origin. Then, for any $x,y\in\partial\Omega$,

$$\begin{aligned}(\sigma\cdot N(x))(\sigma\cdot(x-y)) &= \frac{1}{r}(\sigma\cdot x)(\sigma\cdot(x-y)) = \frac{1}{r}(r^2-(\sigma\cdot x)(\sigma\cdot y))\\ &= -\frac{1}{r}(-r^2+(\sigma\cdot x)(\sigma\cdot y)) = -\frac{1}{r}(\sigma\cdot(x-y))(\sigma\cdot y)\\ &= -(\sigma\cdot(x-y))(\sigma\cdot N(y)).\end{aligned}$$

This identity easily yields $\{\sigma\cdot N, W_\Omega\}=0$ and, by Lemma 4.2, $\|W_\Omega\|_\sigma = 1/2$. Therefore, from Theorem 4.3(i) we get that

$$\lambda_\Omega = 4\big(m\|K_\Omega\|_\sigma + \sqrt{m^2\|K_\Omega\|_\sigma^2+1/4}\big) \tag{60}$$

if Ω is a ball. Let $e_3=(0,0,1)\in\mathbb{R}^3$, and we identify the matrix k with its scalar version. Following [5, Generalized Young's Inequality (0.10)] and since $\partial\Omega$ is invariant under rotations, it is easy to see that

$$\|K_\Omega\|_\sigma \leq \|k(\cdot - re_3)\|_{L^1(\sigma)} = \frac{1}{\sigma(\partial\Omega)}\iint\frac{d\sigma(y)\,d\sigma(x)}{4\pi|re_3-y|} = \frac{1}{\sigma(\partial\Omega)}\iint\frac{d\sigma(y)\,d\sigma(x)}{4\pi|x-y|}.$$

In particular, this shows that the first inequality in (59) is an equality if Ω is a ball. It is well-known that the infimum in the definition of $\text{Cap}(\Omega)$ is attained on the normalized surface measure $\sigma/\sigma(\partial\Omega)$ when Ω is a ball (the Newtonian potential of $\sigma/\sigma(\partial\Omega)$ corresponds to the harmonic function that takes the constant value 1 on $\partial\Omega$, zero at infinity and minimizes the exterior Dirichlet energy), thus the second inequality in (59) is also an equality in this case. Therefore, $\|K_\Omega\|_\sigma = \text{Area}(\partial\Omega)/\text{Cap}(\Omega)$ if Ω is a ball, which combined with (60) proves that (58) is an equality in this case.

On the contrary, assume that (58) is an equality. From (59) and Theorem 4.3(i) we see that the second inequality in (59) must be an equality, which means that the infimum in the definition of $\text{Cap}(\Omega)$ is attained on $\sigma/\sigma(\partial\Omega)$. In the literature, the probability measure that gives the minimum in $\text{Cap}(\Omega)$ is referred as *equilibrium distribution*. Let us recall Gruber's conjecture (see [6, Section 4.1]): "the equilibrium distribution of $\overline{\Omega}$ is $c\sigma$ for some $c>0$ if and only if Ω is a ball". In [11] and [12], the author shows that Gruber's conjecture holds in the case of $\mathcal{C}^{2,\epsilon}$-domains. Putting all together, we see that if Ω is a bounded and smooth domain such that the equality in (58) holds, the equilibrium distribution of $\overline{\Omega}$ is $\sigma/\sigma(\partial\Omega)$, which implies that Ω is a ball by Gruber's conjecture. The lemma is finally proved. ☐

Despite (58) is sharp, it may not be a completely satisfactory inequality in the sense that the right hand side involves some "obscure" term, namely $\text{Cap}(\Omega)$, from a measure theoretic point of view. It would be interesting to derive some related inequality such that the right hand side only involves $\text{Area}(\partial\Omega)$ and/or $\text{Vol}(\Omega)$. This is precisely the purpose of the following corollary, where and isoperimetric-type inequality for the product $\lambda_\Omega\text{Cap}(\Omega)$ is derived.

Corollary 5.4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with smooth boundary. Then*
$$\lambda_\Omega \mathrm{Cap}(\Omega) \geq 4\left(m\mathrm{Area}(\partial\Omega) + \sqrt{m^2\mathrm{Area}(\partial\Omega)^2 + 6^{2/3}\pi^{4/3}\mathrm{Vol}(\Omega)^{2/3}}\right),$$
and the equality holds if and only if Ω is a ball.

Proof. From Lemma 5.2(*ii*) and (*iii*), we get
$$2(6\pi^2)^{1/3}\mathrm{Vol}(\Omega)^{1/3} = 2(6\pi^2)^{1/3}\mathrm{Vol}(\Omega^*)^{1/3} = \mathrm{Cap}(\Omega^*) \leq \mathrm{Cap}(\Omega),$$
and the equality holds if and only if Ω is a ball. The corollary follows from this and (58). □

Proof of Theorem 1.1*.* This is a straightforward application of Corollary 4.6 and Lemma 5.3, just observe that if $4\sqrt{2}m\mathrm{Area}(\partial\Omega)/\mathrm{Cap}(\Omega) > 1$ then $\lambda_\Omega > 2\sqrt{2}$. □

Remark 5.5*.* Combining Remark 4.7 with Corollary 4.6 and Lemma 5.3, we see that Theorem 1.1 also holds replacing
$$\sup\{|\lambda| : \ldots\} \text{ by } \sup\{\lambda > 0 : \ldots\} \text{ or } -\inf\{|\lambda < 0| : \ldots\}, \text{ and}$$
$$\inf\{|\lambda| : \ldots\} \text{ by } \inf\{\lambda > 0 : \ldots\} \text{ or } -\sup\{\lambda < 0 : \ldots\}.$$

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