ON THE BERNOULLI PROPERTY OF PLANAR HYPERBOLIC BILLIARDS

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Abstract. We consider billiards in non-polygonal domains of the plane with boundary consisting of curves of three different types: straight segments, strictly convex inward curves and strictly convex outward curves of a special kind. The billiard map for these domains is known to have non-vanishing Lyapunov exponents a.e. provided that the distance between the curved components of the boundary is sufficiently large, and the set of orbits having collisions only with the flat part of the boundary has zero measure. Under a few additional conditions, we prove that there exists a full measure set of the billiard phase space such that each of its points has a neighborhood contained up to a zero measure set in one Bernoulli component of the billiard map. Using this result, we show that there exists a large class of planar hyperbolic billiards that have the Bernoulli property. This class includes the billiards in convex domains bounded by straight segments and strictly convex inward arcs constructed by Donnay.

1. Introduction

A planar billiard is the mechanical system consisting of a point-particle moving freely inside a bounded domain $\Omega \subset \mathbb{R}^2$ with piecewise differentiable boundary, and being reflected off $\partial\Omega$ so that the angle of reflection equals the angle of incidence. This paper concerns hyperbolic billiards, i.e., billiards for which the corresponding map has no vanishing Lyapunov exponents. Maps with this property are not necessarily uniformly hyperbolic, but exhibit a weak form of hyperbolicity called non-uniform hyperbolicity [27].

The study of hyperbolic billiards was initiated by Sinai. In his seminal paper [28], he proved that billiards in 2-dimensional toral domains containing finitely many obstacles with strictly convex outward boundary are hyperbolic and K-mixing. In fact, Sinai billiards enjoy the Bernoulli property as well [17].

Later on, Bunimovich proved that also billiards in domains with boundary formed by strictly convex inward arcs and straight segments can be hyperbolic [1, 2]. The most celebrated example of such a domain

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is the stadium, the region bounded by two semi-circles connected by two parallel segments. The only strictly convex inward arcs allowed in Bunimovich billiards were arcs of circles. This limitation was eventually overcome by several researchers. Using new techniques for establishing the positivity of Lyapunov exponents [23, 32], Wojtkowski, Markarian and Donnay proved independently that besides arcs of circles many other strictly convex inward arcs can be used to construct hyperbolic billiards [16, 23, 25, 32]. Similar results were obtained by Bunimovich for a class of strictly convex arcs related to those of Donnay [5]. All these results showed that billiards in non-polygonal planar domains are hyperbolic if three conditions are fulfilled: B1) the strictly convex boundary components of the domain are of a special type, B2) the distance between these components and the other curved boundary components is sufficiently large, and B3) orbits having only collisions with straight segments of the boundary form a set of zero measure. A precise formulation of these conditions is given in Section 5.

In this paper, we address the question whether a hyperbolic billiard has the Bernoulli property (for short, 'it is Bernoulli'), i.e., whether it is isomorphic to a Bernoulli shift. The Bernoulli property is the strongest among the ergodic properties: it implies K-property, mixing and ergodicity. The Bernoulli property was proved for several billiards, including the Sinai billiards, the Bunimovich billiards, the Wojtkowski billiards and other special hyperbolic billiards [4, 11, 12, 13, 22, 24, 30]. Despite that, there remain many planar hyperbolic billiards for which not even the ergodicity has been proved. Notably, among them, there are the billiards constructed by Donnay [16]. The goal of this paper is to fill this gap: we show that there exists a large class of hyperbolic billiards, including the Donnay billiards, that have the Bernoulli property.

The key ingredient in the proof of this result is a local ergodic theorem for hyperbolic planar billiards. Roughly speaking, our local ergodic theorem states that if a planar billiard satisfies conditions B1-B3 and the extra condition B4 (see Section 5), then there exists a full measure set H in the billiard phase space with the property that each element of H has a neighborhood contained (mod 0) in one ergodic component of the billiard.

Condition B4 regards the singular set of the billiard, the set formed by the elements where the billiard map is not defined or is not twice-differentiable. This set corresponds to the trajectories that hit a corner of the billiard domain or have a tangential collision with its boundary. Condition B4 requires the elements of the singular set whose trajectories have eventually collisions only with straight segments to form a negligible subset (in the measure theoretical sense) of the singular set.

As a matter of fact, the neighborhood in the conclusion of the local ergodic theorem belongs (mod 0) to a single Bernoulli component of the billiard map (for the definition of a Bernoulli component, see Theorem A.6 in the appendix). As a consequence, every Bernoulli component of a billiard satisfying B1-B4 is open (mod 0). We stress, that although this is a remarkable property, it is not enough to yield the Bernoulli property.

Our local ergodic theorem for billiards is derived from a local ergodic theorem for general hyperbolic symplectomorphisms with singularities [14], which is a generalization of a theorem of Liverani and Wojtkowski [22]. Whereas in [22], it is assumed the existence of a cone field that is continuous everywhere on the billiard phase space, in [14], the existence of such a cone field is required only on an open subset of the billiard phase space. The billiards considered in this paper admit in general only an invariant piecewise-continuous cone field, and so the local theorem in [22] does not apply to all them. The proof of both [14] and [22] rely on the method developed by Sinai to prove the ergodicity of dispersing billiards [28]. Refinements of Sinai's method were also obtained in [9, 21, 29].

We also mention that the trick used in [30] to prove the ergodicity of Wojtkowski billiards and based on rescaling a certain semi-metric does not work with the majority of our billiards. It works with Wojtkowski billiards, because their invariant cones have the property that $G_y^- = G_y^+ = d(x)$ (for the meaning of these symbols, see Section 4), but this property is not satisfied in general by our billiards.

The paper is organized as follows. In Section 2, we recall the definition of the billiard map and its main properties. In Section 3, we give the definition of the focusing times. We also give the definition of a focusing arc introduced by Donnay, and collect the main properties of these arcs. Section 4 is concerned with the theory of invariant cone fields, and their construction for billiards. In this section, we also recall further results on focusing arcs. In Section 5, we give a detailed description of the hyperbolic billiards considered, and state the main results of this paper. In Section 6, we prove Conditions L1-L3 of the local ergodic theorem. Section 7 is entirely devoted to the proof of Condition L4. In Section 8, we introduce the class of billiards in polygons with pockets and bumps, and prove that they have the Bernoulli property. As a corollary, we obtain that the Donnay billiards have the Bernoulli property. In the Appendix, we state the local ergodic theorem for general hyperbolic symplectomorphisms with singularities from [14] together with relevant concepts and observations.

Sections 2, 3 and the beginning of Section 4 are meant to be an introduction to basic concepts on billiards and geometric optics, Donnay arcs and the theory of invariant cone fields. The reader already familiar with these notions may read quickly or skip these sections, and move directly to the second part of Section 4.

2. Generalities on billiards

A billiard system can be described either by a flow or a map. In this paper, we focus on the billiard map. For the relation between the billiard map and billiard flow, we refer the reader to the book [10]. In this section, we define the billiard map for a 2-dimensional domain, and single out its basic properties: regularity, natural invariant probability and singular sets.

2.1. **Billiard domain.** Let $0 \leq k \leq \infty$. A subset $\Gamma \subset \mathbb{R}^2$ is called an *arc of class* C^k if Γ is the image of a C^k embedding $\gamma\colon [0,1] \to \mathbb{R}^2$. The boundary of Γ is the set $\partial\Gamma = \gamma(0) \cup \gamma(1)$. A subset $\Gamma \subset \mathbb{R}^2$ is called a *closed curve of class* C^k if Γ is C^k diffeomorphic to the unit circle S^1. For a closed curve, we set $\partial\Gamma := \emptyset$.

Let Ω be an open bounded connected subset of $\mathbb{R}^2$ with boundary $\partial\Omega$ consisting of finitely many arcs and closed curves $\Gamma_1, \ldots, \Gamma_n$ of class C^3. The set Ω may have a finite number of holes (obstacles), i.e., it does not have to be simply connected. Let ℓ_i be the length of Γ_i, and consider the parametrization $\gamma_i : [0, \ell_i] \to \mathbb{R}^2$ of Γ_i by arc-length with the property that the interior of Ω remains on the left of the tangent vector $\gamma_i'(s)$ for $s \in [0, \ell_i]$. We assume that the curvature of each Γ_i computed with respect to the parametrization γ_i is either strictly negative, or strictly positive, or identically zero at every point of Γ_i (even at its boundary points). If Γ_i has zero curvature, then it is just a straight segment. We also assume that $\Gamma_i \cap \Gamma_j \subset \partial\Gamma_i \cap \partial\Gamma_j$ for $i \neq j$.

The sets $\Gamma_1, \ldots, \Gamma_n$ are called the *components* of $\partial\Omega$. The union of all components with positive curvature (focusing components), negative curvature (dispersing components) and zero curvature (flat components) are denoted by Γ^+, Γ^- and Γ^0, respectively. A point of $\bigcup_{i=1}^n \partial\Gamma_i$ is called a *corner* of Ω.

2.2. **Billiard phase space.** For each $i = 1, \ldots, n$, define

$$M_i = [0, \ell_i] \times [0, \pi]$$

with the elements $(0, \theta)$ and (ℓ_i, θ) identified if Γ_i is a closed curve. Hence M_i is either a rectangle or a cylinder. The billiard phase space M is the disjoint union of $M_1, \ldots, M_n$. An element $x \in M$ specifies a unit vector of $\mathbb{R}^2$ with base point at $s \in \Gamma_i$ and forming an angle θ with $\gamma_i'(s)$. It corresponds to the position and the velocity of the point-particle immediately after a collision with $\partial\Omega$. For this reason, the elements [1] of M are called *collisions*. We denote by M^+, M^-, M^0 the subsets of M obtained by taking the disjoint unions of sets M_i with Γ_i belonging to Γ^+, Γ^- and Γ^0, respectively.

[1]There is no one-to-one correspondence between the elements of M and the unit vectors of $\mathbb{R}^2$ attached to $\partial\Omega$. Indeed, two distinct elements $x_1 = (s_1, \theta_1) \in \Gamma_i$ and $x_2 = (s_2, \theta_2) \in \Gamma_j$ of M may correspond to the same unit vector of $\mathbb{R}^2$. However, this can happen only if $\gamma_i(s_1) = \gamma_j(s_2)$ is a corner point of $\partial\Omega$.

The set M is a smooth manifold with boundary $\partial M = \bigsqcup_{i=1}^{n} \partial M_i$. We equip M with the Riemannian metric $g = \{g_x\}_{x\in M}$ and the symplectic form $\omega = \{\omega_x\}_{x\in M}$ given by $g_x = ds^2 + d\theta^2$ and $\omega_x = \sin\theta(x) ds \wedge d\theta$. The norm generated by g is denoted by $\|\cdot\|$. The Riemannian metric g induces in the usual way a distance d on each M_i, which can be extended to the entire set M by setting $d(x,y) = \bar{d}$ whenever $x \in M_i$ and $y \in M_j$ with $i \neq j$, where $\bar{d} > 0$ is a sufficiently large constant depending on $\partial\Omega$. Let m be the volume generated by g. Then $\mu = (2\ell)^{-1} \sin\theta(x) m$ is the probability measure generated by ω with $\ell := \sum_{i=1}^{n} \ell_i$ being the total length of $\partial\Omega$.

Finally, we observe that there exists a natural involution $\mathcal{I}\colon M \to M$ given by $\mathcal{I}(s,\theta) = (s, \pi - \theta)$ whenever $(s,\theta) \in M_i$.

2.3. **Billiard map and singular sets.** Given $x = (s,\theta) \in M_i$, let $q(x) = \gamma_i(s) \in \Gamma_i$, and let $v(x)$ be the unit vector of $\mathbb{R}^2$ forming an angle θ with $\gamma_i'(s)$. Let $\rho(x) = \{\tau > 0\colon (q(x), q(x)+\tau v(x)) \subset \Omega\}$, where $(q(x), q(x)+\tau v(x))$ is the open segment $\{q(x) + tv(x)\colon 0 < t < \tau(x)\}$. Then, define $t(x) = 0$ if $\rho(x)$ is empty, and $t(x) = \sup \rho(x)$ otherwise.

Now, define

$$q_1(x) = q(x) + t(x)v(x),$$

and

$$M' = \{x \in M\colon q_1(x) \text{ is not a corner of } \Omega\}.$$

If $x \in M'$, then there exists a unique i such that $q_1(x)$ belongs to the interior of Γ_i, and we set $s_1(x) = \gamma_i^{-1}(q_1(x))$. Now, let

$$v_1(x) = -v(x) + 2\langle \gamma_i'(s_1(x)), v(x)\rangle \gamma_i'(s_1(x)),$$

where $\langle\cdot,\cdot\rangle$ denotes the usual scalar product in $\mathbb{R}^2$. In other words, $v_1(x)$ is the unit vector obtained by reflecting $v(x)$ about the direction $\gamma_i'(s_1(x))$. Let $\theta_1(x) \in [0,\pi]$ be the oriented angle between $\gamma_i'(s_1(x))$ and $v_1(x)$. The *billiard map* for the domain Ω is the transformation $T\colon M' \to M$ given by

$$Tx = (s_1(x), \theta_1(x)), \qquad x \in M'.$$

The regularity (continuity, differentiability, etc.) of T depends on the regularity of the boundary of Ω. To clarify this point, we define:

$$\begin{aligned}
A_1 &= \{x \in M\colon q(x) \text{ is a corner of } \Omega\},\\
A_2 &= \{x \in M\colon \theta(x) \in \{0,\pi\}\},\\
A_3 &= \{x \in M\colon q_1(x) \text{ is a corner of } \Omega\},\\
A_4 &= \{x \in M'\colon Tx \in A_2\}.
\end{aligned}$$

Let $S_1^+ = A_3 \cup A_4$, and let $S_1^- = \mathcal{I}(S_1^+)$. Both S_1^+ and S_1^- are compact sets. Next, define

$$R_1^+ = \partial M \cup S_1^+,$$

and for all $j \geq 1$, define iteratively

$$R_{j+1}^{+} = R_j^{+} \cup T^{-1} R_j^{+} \quad \text{and} \quad R_j^{-} = \mathcal{I}(R_j^{+}).$$

The billiard map T is a local C^{k-1} diffeomorphism at $x \in M \setminus R_1^{+}$ [20, Theorem 4.1]. Thus, the points where T is not continuous or more generally is not C^{k-1} are contained in R_1^{+}. Analogously, R_j^{+} (resp. R_j^{-}) contains the points where T^j (resp. T^{-j}) is not C^{k-1}. We call R_j^{+} (resp. R_j^{-} the *singular set* of T^j (resp. T^{-j}).

The map $T\colon M \setminus R_1^{+} \to M \setminus R_1^{-}$ is a C^{k-1} diffeomorphism preserving the symplectic form ω and the probability measure μ provided that $k > 1$ (see [20, Corollaries 4.1 and 4.4, Part V]). Under proper conditions on the components $\Gamma_1, \ldots, \Gamma_n$, which are satisfied by the billiards considered in this paper[2], the sets R_j^{+} and R_j^{-} are unions of finitely many arcs of class C^2 [15]. Hence, $\mu(R_j^{+}) = \mu(R_j^{-}) = 0$ for every $j \geq 1$. Finally, we observe that T is time-reversible, which means that $\mathcal{I} \circ T = T^{-1} \circ \mathcal{I}$ on $M \setminus R_1^{+}$.

3. Focusing times and focusing components

We now introduce the concept of focusing times. This notion is borrowed from geometrical optics, and permits to obtain an intuitive description of the action of the derivative of the billiard map on the projective line.

3.1. **Focusing times.** Let $x \in M$ and $u \in T_x M \setminus \{0\}$. There exists a differentiable curve $\varphi\colon [-\delta, \delta] \to M$ with $\delta > 0$ such that $\varphi(0) = x$ and $\varphi'(0) = u$. This curve defines a 1-parameter family of lines γ_u in $\mathbb{R}^2$, which is a differentiable variation of the line $l(x)$ through $q(x)$ and parallel to $v(x)$. In linear approximation, all lines of γ_u focus at the point $q^{+} \in l(x)$. If γ_u consists in linear approximation of parallel lines, then $q^{+} = \infty$. The point q^{+} depends only on u, and not on the curve φ. By reflecting the lines of γ_u about the boundary of Ω, we obtain another differentiable 1-parameter family of lines, which is generated by the curve $\mathcal{I}(\varphi)$. Denote by q^{-} the point where the lines of this family focus in linear approximation.

The *forward focusing time* $\tau^{+}(x, u)$ of the vector u is equal to length of $(q^{+} - q(x))$ multiplied by 1 or -1 depending on whether the vectors $(q^{+} - q(x))$ and $v(x)$ have the same direction or opposite directions. The *backward focusing time* $\tau^{-}(x, u)$ of the vector u is defined similarly with q^{+} and $v(x)$ replaced by q^{-} and $v(\mathcal{I}(x))$, respectively. Note that $\tau^{+}(x, u)$ and $\tau^{-}(x, u)$ may be negative numbers.

Define $\kappa_i(x)$ to be the curvature of Γ_i for every $x = (s, \theta) \in M_i$ such that $s \in (0, \ell_i)$. Then extend κ_i to the entire M_i by continuity. Finally, let $\kappa\colon M \to \mathbb{R}$ be the function given by $\kappa(x) = \kappa_i(x)$ if $x \in M_i$.

[2] More precisely, for billiards satisfying Conditions B1 and B2 in Section 5.

Let $x \in M$ and $u \in T_xM \setminus \{0\}$. Write $u = (u_s, u_\theta) \in \mathbb{R}^2$ in coordinates (s, θ). Let $m(u) = u_\theta/u_s \in \mathbb{R} \cup \{\infty\}$. A straightforward computation (for example, see [32, Section 2]) shows that

$$\tau^{\pm}(x,u) = \begin{cases} \frac{\sin\theta(x)}{\kappa(x) \pm m(u)} & \text{if } m(u) \neq \mp\kappa(x), \\ \infty & \text{otherwise.} \end{cases} \tag{1}$$

Let $d(x) = \sin\theta(x)/\kappa(x)$. From (1), one can easily derive the well known *Mirror Equation* of geometrical optics, relating the focusing times $\tau^+(x,u)$ and $\tau^-(x,u)$:

$$\frac{1}{\tau^+(x,u)} + \frac{1}{\tau^-(x,u)} = \frac{2}{d(x)}. \tag{2}$$

Definition 3.1. *Let $x \in M \setminus R_k^+$ for some $k \in \mathbb{N}$. Then define $\tau_k^{\pm}(x,u) = \tau^{\pm}(T^k x, D_x T^k u)$ for all $u \in T_xM \setminus \{0\}$.*

3.2. **Focusing components.** In this paper, we assume that the focusing components of $\partial\Omega$ are arcs of the type introduced by Donnay [16], which for simplicity will be called Donnay arcs.

The first property that a Donnay arc Γ_i must satisfy is the following:

$$\int_0^{\ell_i} \kappa(s(x))ds \leq \pi. \tag{3}$$

This condition amounts to saying that the tangents to Γ_i at its endpoints form an angle that is not larger than π. In particular, no Donnay arc can be a closed curve. Hence, no billiard orbit is trapped by Γ_i, except possibly the periodic orbit of period 2 whose trajectory coincides with the segment joining the endpoints of Γ_i [16, Lemma 1.1].

Definition 3.2. *Define $n \colon M^+ \to \{0, 1, 2, \ldots\} \cup \{+\infty\}$ by*

$$n(x) = \begin{cases} \sup\left\{j \geq 0 \colon T^k x \in M_i \quad \forall 0 \leq k \leq j\right\}, & x \in M_i \setminus A_2 \subset M^+, \\ +\infty, & x \in M^+ \cap A_2. \end{cases}$$

The number $n(x)$ is the number of consecutive collisions of x with Γ_i before either leaving it or hitting one of its endpoints.

Definition 3.3. *For every $\Gamma_i \subset \Gamma^+$, define*

$$E_i = \{x \in M_i \colon n(\mathcal{I}(x)) = 0\}.$$

Also, define $E^+ = \bigcup_{i \colon \Gamma_i \subset \Gamma^+} E_i$.

Equivalently, $x \in E_i$ if and only if $\mathcal{I}(x)$ either leaves Γ_i or hits one of the endpoints of Γ_i. Note that if $x \in E_i$, then $\{x, Tx, \ldots, T^{n(x)}x\} \subset M_i$ is the maximal sequence of consecutive collisions with Γ_i that cannot be extended forward or backward.

Definition 3.4. *Let $x \in E_i$, and denote by $u_x \in T_xM$ the tangent vector such that $\tau^-(x, u_x) = \infty$ or equivalently $m(x) = \kappa(x)$. In other words, the variation associated to u_x consists of lines parallel to $v(x)$.*

Definition 3.5. *We say that* $x \in E_i$ *is* focused *by* Γ_i *if*

(1) $0 < \tau_i^+(x, u_x) < t(T^i x)$ *for* $0 \leq i < n(x)$,
(2) $0 < \tau_{n(x)}^+(x, u_x) < +\infty$.

The concept of a focused collision is key in the definition of a Donnay arc. In words, x is focused by Γ_i if the infinitesimal family of parallel trajectories focuses between every two consecutive collisions with Γ_i and after the last one.

Definition 3.6. *A focusing component* Γ_i *is a Donnay arc if i)* Γ_i *is of class* C^∞, *ii)* Γ_i *satisfies* (3), *and iii) if every* $x \in E_i$ *is focused by* Γ_i.

The definition of a Donnay arc is similar to that of an *absolutely focusing* arc introduced by Bunimovich [3, 5]. The relation between the two definitions is discussed in [5].

Billiards in convex regions bounded by Donnay arcs connected by straight segments sufficiently long have non-zero Lyapunov exponents almost everywhere [16, Theorem 2]. In this paper, we consider billiard domains that are more general than those considered by Donnay: our billiard domains are not necessarily convex, and their boundaries are allowed to have components with negative curvature.

The key property of Donnay arcs used in the construction of hyperbolic billiards is the uniform boundedness of the focusing time $\tau_{n(x)}^+(x, u_x)$ for $x \in E_i$ [16, Theorem 4.4]. For this property to hold, the arc does not have to be smooth as required by Donnay. Indeed, Wojtkowski and Markarian found C^4 curves satisfying that property [23, 32].

Examples of Donnay arcs are arcs of circles, arcs of cardioids, arcs of logarithmic spirals and elliptical arcs. An example of a Donnay arc that is not one of the arcs discovered by Wojtkowski and Markarian is the half-ellipse $\{(x, y) \in \mathbb{R}^2 \colon x^2/a^2 + y^2/b^2 = 1 \text{ and } x \geq 0\}$ with $a/b < \sqrt{2}$ [16, Theorem 7.1].

We mention two interesting facts concerning the existence and robustness under perturbation of Donnay arcs. The first fact is that given a smooth arc Γ with positive curvature, every sufficiently short arc contained in Γ is a Donnay arc [16, Theorem 3]. A similar conclusion when Γ is only of class C^4, but satisfies also the condition $d^2(\kappa^{-1/3})/ds^2 > 0$ was obtained by Markarian [23]. The second fact we want to mention is that given a Donnay arc Γ, every sufficiently small perturbation of Γ in the C^6 topology is still a Donnay arc [16, Theorem 4].

4. Cone fields for billiards

In this section, we recall the notions of an invariant cone field and a monotone quadratic form. We restrict ourselves to the 2-dimensional setting, since we are dealing only with planar billiards in this paper. Next, we introduce a specific family of cone fields for planar billiards.

In Section 5, we will show that these cone fields are eventually strictly invariant if the billiards satisfy certain geometric conditions. The main reference for Subsections 4.1 and 4.2 is [22].

4.1. **Cone fields.** Let V be a 2-dimensional vector space with a symplectic form α. Given two linearly independent vectors X_1 and X_2 of V, we can write $v = v_1 + v_2$ for every $v \in V$ with $v_i \in X_i$ for $i = 1, 2$. We call the *cone* generated by X_1 and X_2 the set

$$C(X_1, X_2) := \{v \in V : \alpha(v_1, v_2) \geq 0\}.$$

The cone $C'(X_1, X_2) := C(X_2, X_1)$ is called the *complementary cone* of $C(X_1, X_2)$. The set $\operatorname{int} C(X_1, X_2) := \{v \in V : \alpha(v_1, v_2) > 0\}$ is called the *interior* of $C(X_1, X_2)$. To our knowledge, the description of a cone using symplectic forms first appeared in [31].

Now, consider a billiard in a domain $\Omega \subset \mathbb{R}^2$ with phase space M, billiard map T and the symplectic form $\omega = \{\omega_x\}_{x \in M}$ as in Section 2. Let U be an open set of M, and let X_1 and X_2 be two measurable vector fields on U such that $X_1(x)$ and $X_2(x)$ are linearly independent for all $x \in U$. The *cone field* C on $U \subset M$ generated by X_1 and X_2 is the family of cones $\{C(x)\}_{x \in U}$ with

$$C(x) := C(X_1(x), X_2(x)) = \{u \in T_xM : \omega_x(u_1, u_2) \geq 0\} \subset T_xM$$

for every $x \in U$, and it is denoted by (U, C).

The cone field (U, C) is called *continuous* if X_1 and X_2 are continuous. The cone field (U, C) is called *invariant* (resp. *strictly invariant*) if $x \in U$ and $T^k x \in U$ with $k > 0$ implies that $D_x T^k C(x) \subset C(T^k x)$ (resp. $D_x T^k C(x) \subset \operatorname{int} C(T^k x) \cup \{0\}$). The cone field (U, C) is called *eventually strictly invariant* if it is invariant, and for a.e. $x \in U$, there exists an integer $k(x) > 0$ such that $T^{k(x)} x \in U$ and $D_x T^{k(x)} C(x) \subset \operatorname{int} C(T^{k(x)}(x)) \cup \{0\}$.

A crucial property of the 2-dimensional cone $C(x)$ is that it can be identified with a closed interval of the projective space $\mathbb{P}(T_xM)$. Therefore, since m, τ^-, τ^+ are all projective coordinates, each of the following provides an alterative definition of the cone field (U, C): for all $x \in U$,

$$\begin{aligned}
C(x) &= \{u \in T_xM \setminus \{0\} : m(u) \in I(x)\} \cup \{0\}, \\
C(x) &= \{u \in T_xM \setminus \{0\} : \tau^+(x, u) \in I_+(x)\} \cup \{0\}, \\
C(x) &= \{u \in T_xM \setminus \{0\} : \tau^-(x, u) \in I_-(x)\} \cup \{0\}
\end{aligned}$$

with $I(x), I_-(x), I_+(x)$ being proper closed intervals of the extended real line $\hat{\mathbb{R}}$.

4.2. **Quadratic forms.** Consider a cone field (U, C) generated by the vector fields X_1 and X_2. The quadratic form $Q_C = \{Q_C(x, \cdot) : x \in U\}$ associated to (U, C) is defined by

$$Q_C(x, u) = \omega_x(u_1, u_2) \qquad \text{for } x \in U \text{ and } u \in T_xM.$$

The form Q_C is *monotone* (resp. strictly monotone) if $Q_C(T^k x, D_x T^k u) \geq Q_C(x,u)$ (resp. $Q_C(T^k x, D_x T^k u) > Q_C(x,u)$) for all $u \in T_x M \setminus \{0\}$ whenever $x \in U$ and $T^k x \in U$ with $k > 0$. The form Q_C is *eventually strictly monotone* if it is monotone, and for a.e. $x \in U$, there exists an integer $k(x) > 0$ such that $Q_C(T^{k(x)} x, D_x T^{k(x)} u) > Q_C(x,u)$ for all $u \in T_x M \setminus \{0\}$.

Following [22], to measure the expansion generated by the action of $D_x T^k$ with $k > 0$ on vectors of C_x, we define

$$\sigma_C(D_x T^k) = \inf_{u \in \operatorname{int} C(x)} \sqrt{\frac{Q_C(T^k x, D_x T^k u)}{Q_C(x,u)}},$$

and

$$\sigma_C^*(D_x T^k) = \inf_{u \in \operatorname{int} C(x)} \frac{\sqrt{Q_C(T^k x, D_x T^k u)}}{\|u\|}.$$

We denote by (U, C') the family of cones $\{C'(x)\}_{x \in U}$ with $C'(x)$ being the complementary cone of $C(x)$. We observe that C is invariant (resp. strictly invariant) if and only if C' is invariant (resp. strictly invariant) with respect to T^{-1}. The relation between the expansion of $D_x T^k$ on C and the expansion of $D_{T^k x} T^{-k}$ on C' is given by $\sigma_{C'}(D_{T^k x} T^{-k}) = \sigma_C(D_x T^k)$.

The cone field (U, C) is invariant (resp. strictly invariant) if and only if the quadratic form Q_C is monotone (resp. strictly monotone), and (U, C) is eventually strictly invariant if and only if Q_C is eventually strictly monotone. Furthermore, $D_x T^k C(x) \subset C(T^k x)$ (resp. $D_x T^k C(x) \subset \operatorname{int} C(T^k x) \cup \{0\}$) for $x \in U$ such that $T^k x \in U$ with $k > 0$ is equivalent to $\sigma_C(D_x T^k) \geq 1$ (resp. $\sigma_C(D_x T^k) > 1$).

4.3. **Cone fields for billiards.** Let $E^- = \bigcup_{i \colon \Gamma_i \subset \Gamma^-} M_i$, and let $E = E^+ \cup E^-$ (for the definition of E^+, see Definition 3.3). In this section, we introduce a family of continuous cone fields for billiards. The cones are defined separately for focusing and dispersing components, and to make their geometrical meaning more transparent, several equivalent definitions are provided. Rather than considering a single continuous cone field, as it is common in the literature on hyperbolic billiards, we consider a family of continuous cone fields $\{(U_x, C_x) \colon x \in E\}$. The reason for using $\{(U_x, C_x) \colon x \in E\}$ instead of a single continuous cone field on E is that for most of the billiards considered in this paper, we do not know whether such a cone field exists[3].

As a consequence of this choice, we will be able to apply the Local Ergodic Theorem (Theorem A.13) only to U_x with $x \in E$. This is not a serious limitation, because the hypotheses imposed on our billiards imply that almost every element of M visits E, and so knowing that

[3]A single continuous cone field on E exists for Bunimovich and Wojtkowski billiards [1, 31, 32], but these form a subset of the billiards considered here.

each U_x belongs (mod 0) to an ergodic component of T suffices to conclude that the same holds true for almost every point of M.

4.3.1. *Dispersing components.* Let $x \in M_i \subset M^-$ for some i. Then, define $U_x = M_i$, and for every $y \in U_x$,

$$C_x(y) = \{u \in T_y M_i \setminus \{0\} : m(u) \leq \kappa(y)\} \cup \{0\}.$$

The resulting cone field (U_x, C_x) is clearly continuous. Each cone $C_x(y)$ consists of tangent vectors focusing inside the disk tangent to Γ_i at $q(y)$ of radius $1/(4|\kappa(y)|)$. This can be easily checked by using (1): if $u \in C_x(y) \setminus \{0\}$, then $d(y) \leq 2\tau^+(y,u) \leq 0$ and $0 \leq \tau^-(y,u) \leq +\infty$.

4.3.2. *Focusing components.* To define (U_x, C_x) for every $x \in E^+$, we first recall a result of Donnay.

For every $x \in M^+$ and every real number $g < \kappa(x)$, let

$$\mathcal{D}(x,g) = \{u \in T_x M \setminus \{0\} : g \leq m(u) \leq \kappa(x)\}.$$

Theorem 4.1. *Suppose that Γ_i is a Donnay arc. Then there exist positive constants $a_i, m_i, \theta_i, t_i^-, t_i^+$, and for every $x \in E_i$, there exist a neighborhood $U_x \subset M_i \setminus A_2$ of x and a continuous function $g_x : U_x \to \mathbb{R}$ with $|g_x| < \kappa|_{U_x}$ such that if $x \in E_i$, $y \in U_x$, $j \in \{0, \ldots, n(y)\}$, $y_j := T^j y$, $u \in \mathcal{D}(y, g_x(y))$ and $u_j := D_y T^j u$, then*

(1) $-\kappa(y_j) + m_i \leq m(u_j) \leq \kappa(y_j)$,

(2) $\min\{\theta(y), \pi - \theta(y)\} < \theta_i \Longrightarrow |m(u_j)| \leq a_i \cdot \min\{\theta(y), \pi - \theta(y)\}$,

(3) $j < n(y) \Longrightarrow d(y_j) \leq 2\tau^+(y_j, u_j) < 2t(y_j) - d(Ty_j)$,

and if $x \in E_i$ and $y \in U_x$, then

(4) $\inf_{u \in \mathcal{D}(y, g_x(y))} \tau^-(y,u) \leq t_i^-$,

(5) $\sup_{u \in \mathcal{D}(y, g_x(y))} \tau^+_{n(y)}(y,u) \leq t_i^+$.

Proof. For the proof of Parts (1) and (2), see the proofs of [16, Theorems 4.4 and 5.6], whereas for the proof of Parts (3)-(5), see [16, Proposition 4.1 and Theorem 4.4]. □

Theorem 4.1 has the following geometrical interpretation. Part (1) states that the iterates along consecutive collisions with Γ_i of tangent vectors with initial slope between $g_x(y)$ and $\kappa(y)$ have slopes uniformly bounded for all $x \in E_i$ and all $y \in U_x$. This conclusion is strengthened in Part (2) provided that $\theta(y)$ is uniformly close to 0 or π. In particular, in this case, the iterates of the vectors remain uniformly close to the horizontal direction. Parts (3)-(5) implies the every $y \in U_x$ is focused (c.f. 3.5), and that the backward and forward focusing times of the tangent vectors considered are uniformly bounded by constants depending only on Γ_i. In fact, the constants t_i^- and t_i^+ are continuous functions of Γ_i with respect to the C^6 topology [16, Theorem 4.4].

Suppose that $x \in E^+$. Let U_x be the neighborhood of x and g_x be the continuous function as in Theorem 4.1. For every $y \in U_x$, define

$$C_x(y) = \mathcal{D}(y, g_x(y)) \cup \{0\}.$$

Remark 4.2. *The following are alternative descriptions of the cone $C_x(y)$ for every $x \in E$ in terms of the projective coordinates τ^+ and τ^-:*

$$\begin{aligned} C_x(y) &= \big\{u \in T_yM_i \setminus \{0\} \colon d(y)/2 \le \tau^+(y,u) \le G_x^+(y)\big\} \cup \{0\} \\ &= \big\{u \in T_yM_i \setminus \{0\} \colon G_x^-(y) \le \tau^-(y,u) \le +\infty\big\} \cup \{0\}, \end{aligned}$$

where $G_x^{\pm}(y) := \sin\theta(y)/(\kappa(y) \pm g_x(y))$ for all $y \in U_x$ if $x \in E^+$, and $G_x^{\pm}(y) := 0$ for all $y \in U_x$ if $x \in E^-$. Note that for $x \in E^+$, the numbers $G_x^+(y)$ and $G_x^-(y)$ are, respectively, the forward and backward focusing times of the vectors with slope equal to $g_x(y)$.

5. Results

In this section, we give a detailed description of the billiards that we want to study, and formulate the main results of the paper.

The billiards in question are characterized by four conditions called B1-B4. It is well known that B1-B3 are sufficient to guarantee the hyperbolicity of the billiard map T (see Proposition 5.3). This together with the Spectral Theorem implies that T has at most countably many ergodic components of positive measure with respect to μ, with each ergodic component further decomposed into finitely many Bernoulli components cyclically permuted by T (see Theorem A.6).

The main result of this paper is the following: if in addition to B1-B3, a billiard satisfies also Condition B4, then there exists a measurable set $H \subset M$ of full measure such that for every $x \in H$, there is a neighborhood of x in M contained up to a set of zero measure in a single Bernoulli component of T (see Theorem 5.6). This result implies immediately that every Bernoulli component of T is open up to a set of zero measure. Results of this type are often called Local Ergodic Theorems.

Local ergodicity alone is not enough to conclude that T is Bernoulli. This is obtained by imposing on T some extra conditions. It is not an easy task to formulate these conditions for the generality of the billiards considered in this paper. Therefore, rather than trying to formulate the optimal condition for the Bernoulli property of hyperbolic billiards, we limit ourselves here to give a simple condition, called B5, that yields the Bernoulli property of interesting subclasses of hyperbolic billiards (i.e., billiards with domains without straight boundaries, i.e., $\Gamma^0 = \emptyset$). It is unfortunate that B5 does not hold for Donnay billiards. Nevertheless, in Section 8, we prove that these billiards and some generalizations (billiards with pockets and bumps) are Bernoulli, using a proof that does not require B5.

5.1. **Important sets.** Next, we introduce several sets involved in the formulation of Conditions B1-B5. Recall that $R_k^\pm$ are the singular sets defined in Section 2. Define

- $R_\infty^\pm = \bigcup_{k\geq 1} R_k^\pm$,
- $R = R_\infty^- \cap R_\infty^+$,
- $N^\pm = \{x \in M \setminus R_\infty^\pm : \exists k > 0 \text{ s.t. } T^{(\pm)n}x \in M_0 \text{ for all } n \geq k\}$,
- $N = N^- \cap N^+$,
- $N' = (R_\infty^- \cap N^+) \cup (R_\infty^+ \cap N^-)$,
- $H = M \setminus (R \cup N \cup N')$.

The geometric meaning of these sets is the following: R_∞^+ (resp. R_∞^-) is the set of collisions with finite positive (resp. negative) semi-orbit; R is the set of collisions with finite orbit; N^+ (resp. N^-) is the set of collisions with positive (resp. negative) semi-orbit visiting eventually only flat components of $\partial\Omega$; N is the set of collisions with both semi-orbits visiting eventually only flat components of $\partial\Omega$; N' is the set of collisions with one semi-orbit being finite and the other semi-orbit visiting eventually only flat components of $\partial\Omega$; H is the set of collisions with one semi-orbit visiting the curved components of $\partial\Omega$ infinitely many times.

5.2. **Hyperbolic billiards.** We are ready to formulate Conditions B1-B5.

B1 (Non-polygonal domain): The domain Ω is not a polygon, and its boundary components can only be of the following type: straight segments, dispersing arcs of class C^3 and focusing arcs.

B2 (Distance between boundary components): For each curved component Γ_i, we define $\lambda_i^\pm = 0$ if Γ_i is dispersing, and $\lambda_i^\pm = t_i^\pm$ with $t_i^\pm$ as in Theorem 4.1 if Γ_i is focusing. Given two curved components Γ_i and Γ_j, denote by $t_{ij} \geq 0$ the infimum of the Euclidean length of all finite billiard orbits $\{x_0, \ldots, x_n\}$ with $n > 0$ such that $x_0 \in M_i$ and $x_n \in M_j$. We assume that

(1) there exists $\lambda > 0$ such that if Γ_i or Γ_j is focusing, then
$$t_{ij} \geq \lambda_i^- + \lambda_j^+ + \lambda,$$
(2) the distance between each focusing component Γ_i and the set of corners of $\partial\Omega$ formed by two straight segments is greater than λ_i^-.

B3 (Neutral orbits): We assume that $\mu(N^-) = 0$.

B4 (Singular-Neutral orbits): We assume $m_-(S_1^- \cap N^+) = 0$, where m_- is the measure induced by the Riemann metric g on S_1^-.

B5 (Connectedness): The set $H \cap M_i$ is connected for every component Γ_i of $\partial\Omega$.

Condition B2 has a couple of obvious consequences for the geometry of Ω: i) the internal angle between a focusing component and an adjacent curved component is greater than π, and ii) the internal angle between a focusing component and an adjacent flat component is greater than $\pi/2$. Also, note that Conditions B1-B4 allow $\partial\Omega$ to have cusps formed by two dispersing components or a dispersing and a flat component.

Remark 5.1. *Since $N^+ = \mathcal{I}(N^-)$ and $S_1^+ \cap N^- = \mathcal{I}(S_1^- \cap N^+)$, Conditions B3 and B4 imply that $\mu(N^+) = 0$ and $m_+(S_1^+ \cap N^-) = 0$, where m_+ is the measure induced by the Riemann metric g on S_1^+.*

From Conditions B1 and B2, it follows that cone field $\{(U_x, C_x)\}_{x\in E}$ is strictly invariant along a piece of an orbit connecting two elements of E. The next lemma is proved in [15, Lemma 5.2].

Lemma 5.2. *Suppose that the billiard in Ω satisfies Conditions B1 and B2. Also, suppose that there exist $x_1, x_2 \in E$ and $y \in E \cap U_{x_1} \setminus R_k^+$ for some $k > 0$ such that $T^k y \in E \cap U_{x_2}$. Then*

$$D_y T^k C_{x_1}(y) \subset \operatorname{int} C_{x_2}(T^k y) \cup \{0\}.$$

From the previous lemma, one obtains the hyperbolicity of the billiard map T provided that Conditions B1-B3 are satisfied. This is well known fact, but we give its proof for completeness.

Proposition 5.3. *If a billiard in a domain Ω satisfies Conditions B1-B3, then the Lyapunov exponents of the map T are non-zero a.e. on M.*

Proof. Let E' be the subset of $E \setminus (R_\infty^+ \cup N^+)$ defined by

$$E' = \left\{x \in E \setminus (R_\infty^+ \cup N^+) \colon \exists n_k \nearrow +\infty \quad \text{s.t.} \quad T^{n_k} x \in V_x \quad \forall k > 0\right\},$$

where (U_x, C_x) is the cone field associated to x. Since (U_x, C_x) is strictly invariant for every $x \in E'$ by Lemma 5.2, results of Wojtkowski [32] imply that the Lyapunov exponents of T are non-vanishing at every point of E'. By $\mu(R_\infty^+) = \mu(N^+) = 0$ and the Poincaré Recurrence Theorem, we obtain that $\mu(E') = \mu(E) > 0$. This fact together with $\mu(R_\infty^+) = \mu(N^+) = 0$ gives that the orbit of a.e. point of M visits E'. Since the Lyapunov exponents are constant along orbits, we can finally conclude that the Lyapunov exponents of T are non-vanishing at a.e. point of M. $\square$

Conditions B1-B3 are sufficient for the hyperbolicity of the map T. In fact, even part (2) of B2 can be dropped if we are only interested in the hyperbolicity of T. The extra Condition B4 is required to prove that T is locally ergodic. This condition is related to the Sinai-Chernov Ansatz (c.f. Condition L3 of Theorem A.13). Note also that B3 is a necessary condition for the hyperbolicity of T.

Remark 5.4. *We do not know whether or not, for a domain Ω satisfying B1 and B2, B3 or B4 is automatically satisfied. We also do not know whether B3 and B4 are independent. These questions are strictly related to the problem of understanding the distribution of orbits in polygonal billiards.*

Lemma 5.5. *We have $\mu(H) = 1$ provided that B1 and B3 and are satisfied.*

Proof. Since $\mu(R_1^+) = 0$ for billiards satisfying B1 (see the end of Subsection 2.3), we trivially obtain $\mu(R) = \mu(N') = 0$. From B3, we obtain immediately $\mu(N) = 0$. $\square$

5.3. **Main results.** The central result of this paper is the following theorem. Its proof is given in Section 6. For the definition of a Bernoulli component of T, see Appendix A.

Theorem 5.6. *If a billiard in a domain Ω satisfies Conditions B1-B4, then every point of H has a neighborhood contained (mod 0) in a Bernoulli component of T.*

We now prove that the map T is Bernoulli if it also satisfies Condition B5. As already explained in the introduction to this section, B5 applies only to a small subclass of billiards satisfying B1-B4 (see Theorem 8.7). We could have weakened considerably B5 for it to include many more hyperbolic billiards, but at the price of a much more technical formulation. Instead of attempting to give the weakest formulation of B5, we opted for a strong condition but with a simple formulation that allows for a relatively simple proof of the Bernoulli property for billiards.

Corollary 5.7. *If a billiard in a domain Ω satisfies Conditions B1-B4, then every Bernoulli component of T is open (mod 0).*

Proof. Let B be a Bernoulli component. Since $\mu(B) > 0$, we have $\mu(B \cap H) > 0$. Let $x \in B \cap H$, and let U be the neighborhood of x as in Theorem 5.6. The set $V := \bigcup_{n\in\mathbb{Z}} T^n U$ is open. Moreover, since V is invariant and contained (mod 0) in B, it follows that $B = V$ (mod 0). $\square$

Corollary 5.8. *If the billiard in a domain Ω satisfies Conditions B1-B5, then the map T is Bernoulli.*

Proof. By Theorem 5.6, every point of H has a neighborhood contained up to a set of zero measure in a Bernoulli component of T. The same is true for every connected component of H, and so for every $M_i \cap H$ such that $M_i \subset M^- \cup M^+$ by the first part of Condition B5. Since $\mu(H) = 1$ (see Remark 5.1), we conclude that every set $M_i \subset M^- \cup M^+$ is contained (mod 0) in a single Bernoulli component of T.

We now show that if Γ_i and Γ_j intersect, then M_i and M_j are contained in the same Bernoulli component of T. First, note that $S_1^- \setminus (R_\infty^+ \cup N^+)$ is contained in H. Next, since $S_1^- \cap R_k^+$ is finite for every $k > 0$ (see [15, Propositions 6.17-6.19]), it follows that $S_1^- \cap R_\infty^+$ is countable. This together with B4 implies that m_--a.e. element of S_1^- is contained in H. Hence, if $p \in \Gamma_i \cap \Gamma_j$ is a corner of $\partial\Omega$, then we can find $x \in H \cap S_1^-$ such that the ray emerging from $\mathcal{I}(x)$ is arbitrarily close to ℓ, the line bisecting p. Now, let U be the neighborhood of x contained (mod 0) in one Bernoulli component of T as stated by Theorem 5.6. It is clear that $T^{-1}U$ is contained (mod 0) in the same Bernoulli component. But, if $\mathcal{I}(x)$ is sufficiently close to ℓ, then $M_i \cap T^{-1}U$ and $M_j \cap T^{-1}U$ are non-empty open sets so that M_i and M_j must belong (mod 0) to the same Bernoulli component.

The previous conclusion implies that $q^{-1}(\Sigma)$ is contained (mod 0) in a Bernoulli component for every connected component Σ of $\partial\Omega$. Since Ω is connected, it follows that all sets M_i belongs to the same Bernoulli component, i.e., T is Bernoulli. □

6. Local ergodicity

In this section, we prove Theorem 5.6 by applying Theorem A.13 to the billiard map T. Theorem A.13 is a version of a Local Ergodic Theorem for hyperbolic symplectomorphisms with singularities in arbitrary dimension [14] specialized to planar billiards. Because of its length, Theorem A.13 is given in the Appendix together with all the definitions required for its formulation. Its main hypotheses are Conditions L1-L4. This section is devoted to the proof of Conditions L1-L3, whereas Section 7 is entirely devoted to the proof of Condition L4.

Proof of Theorem 5.6. The wanted conclusion follows at once by applying Theorem A.13 to points of H. Accordingly, we show that each point of H is sufficient (see Definition A.1) and satisfies Conditions L1-L4. This is achieved in Corollary 6.3, Propositions 6.4–6.6 and Proposition 7.30. □

6.1. **Sufficient points.**

We now prove that every point of H is sufficient (see Definition A.1).

Given $x \in E$ and $k \in \mathbb{N}$ such that $T^k x \in E$, define

$$\widetilde{\sigma}_x(D_y T^k) = \inf_{u \in \operatorname{int} C_x(y)} \sqrt{\frac{Q_{C_{T^k x}}(T^k y, D_y T^k u)}{Q_{C_x}(y,u)}}$$

and

$$\widetilde{\sigma}^*_x(D_y T^k) = \inf_{u \in \operatorname{int} C_x(y)} \frac{\sqrt{Q_{C_{T^k x}}(T^k y, D_y T^k u)}}{\|u\|}$$

for every $y \in U_x \cap T^{-k} U_{T^k x}$. The quantity $\widetilde{\sigma}_x$ (resp. $\widetilde{\sigma}^*_x$) is a generalization of σ_C (resp. σ^*_C), because if $U = U_x = U_{T_k x}$ and $C = C_x = C_{T^k x}$, then $\widetilde{\sigma}_x = \sigma_C$ (resp. $\widetilde{\sigma}^*_x = \sigma^*_C$).

Under the previous hypotheses $x, T^k x \in E$, also define

$$t(x, T^k x) = \sum_{i=0}^{m-1} t(T^i x).$$

In other words, $t(x, T^k x)$ is the sum of the length of the segments $[q(x), q(Tx)], \ldots, [q(T^{k-1}x), q(T^k x)]$. Furthermore, let $\lambda > 0$ be the constant in Condition B2.

Lemma 6.1. *There exists a constant c depending only on the billiard table Ω such that if $x \in E$, $T^m x \in E$ for some $m \in \mathbb{N}$ and $T^i x \notin E$ for every $0 < i < m$, then*

$$\widetilde{\sigma}_x(D_x T^m) \geq \sqrt{1 + c\delta} + \sqrt{c\delta},$$

where $\delta = \lambda$ if $x \in M^+$ or $T^m x \in M^+$, and $\delta = t(x, T^m x)$ if $x \in M^-$ and $T^m x \in M^-$.

Proof. The hypotheses imply that if $x \in M^+$ and $m > n(x) + 1$, then $T^i x \in M^0$ for every $n(x) < i < m$. Similarly, if $x \in M^-$ and $m > 1$, then $T^i x \in M^0$ for every $1 < i < m$. Define $\overline{m} = n(x)$ if $x \in M^+$, and $\overline{m} = 0$ if $x \in M^-$.

The cones $C_{T^m x}(T^m x)$ and $D_x T^m C_x(x)$ can be written as follows

$$\begin{aligned} C_{T^m x}(T^m x) &= \big\{u \in T_{T^m x} M \setminus \{0\} \colon a \leq \tau^-(T^m x, u) \leq b\big\} \cup \{0\}, \\ D_x T^m C_x(x) &= \big\{u \in T_{T^m x} M \setminus \{0\} \colon \bar{a} \leq \tau^-(T^m x, u) \leq \bar{b}\big\} \cup \{0\}, \end{aligned}$$

where

$$\begin{aligned} a &= G^-_{T^m x}(T^m x), \quad b = +\infty, \\ \bar{a} &= t(x, T^m x) - \sup_{u \in C_x(x) \setminus \{0\}} \tau^+_{\overline{m}}(x, u), \\ \bar{b} &= t(x, T^m x) - \inf_{u \in C_x(x) \setminus \{0\}} \tau^+_{\overline{m}}(x, u). \end{aligned}$$

Note that the collisions $\{T^{\overline{m}} x, \ldots, T^{m-1} x\}$ with the neutral boundary Γ^0 only affect the quantity $t(x, T^m x)$.

By Condition B2, we have

$$\sup_{u \in C_x(x) \setminus \{0\}} \tau^+_{\overline{m}}(x, u) \leq \lambda^+_j.$$

Next, directly from the definition of the cone $C_x(x)$ when Γ_j is dispersing, and from Part (3) of Theorem 4.1 when Γ_j is focusing, it follows that

$$\frac{d(T^{\overline{m}} x)}{2} \leq \inf_{u \in C_x(x) \setminus \{0\}} \tau^+_{\overline{m}}(x, u).$$

Finally, note that $G^-_{T^m x}(T^m x) \le \lambda^-_k$. By the previous observations,

$$a \le \lambda^-_k, \qquad t(x, T^m x) - \lambda^+_j \le \bar{a}, \qquad \bar{b} \le t(x, T^m x) - \frac{d(T^{\overline{m}} x)}{2}. \quad (4)$$

By Lemma 5.2, $D_x T^m C_x(x) \subset \operatorname{int} C_{T^m x}(T^m x) \cup \{0\}$. Hence, the expansion $\widetilde{\sigma}_x(D_x T^m)$ can be estimated by using a formula proved by Wojtkowski [32, Lemma A.4 and Appendix B]. We obtain[4]

$$\widetilde{\sigma}_x(D_x T^m) = \sqrt{1+w} + \sqrt{w}, \qquad \text{where } w = \frac{\bar{a} - a}{\bar{b} - \bar{a}}. \quad (5)$$

From (4), it follows that

$$w \ge \frac{t(x, T^m x) - \lambda^+_j - \lambda^-_k}{\lambda^+_j - d(T^{\overline{m}} x)/2}.$$

It is easy to see that there exists a constant $c > 0$ depending only on the billiard domain Ω such that

$$0 < \lambda^+_j - \frac{d(T^{\overline{m}} x)}{2} < \frac{1}{c}.$$

Hence,

$$w \ge c\left(t(x, T^m x) - \lambda^+_j - \lambda^-_k\right).$$

Now, the wanted conclusion follows from (5) once we have observed that $t(x, T^m x) - \lambda^+_j - \lambda^-_k \ge \lambda$ by Condition B2 if Γ_j or Γ_k is focusing, and $\lambda^+_j = \lambda^-_k = 0$ if Γ_j and Γ_k are dispersing. □

If $x \in M \setminus (R^+_\infty \cup N^+)$, the positive semi-orbit of x visits E infinitely many times. That is, there exists a strictly increasing sequence $\{r_k\}_{k \in \mathbb{N}}$ of non-negative integers such that for every $k \in \mathbb{N}$, we have $T^{r_k} x \in E$, and $T^i x \notin E$ for all $r_k < i < r_{k+1}$. Note that $r_1 = 0$ if and only if $x \in E$. We call $\{r_k\}_{k \in \mathbb{N}}$ the sequence of the *positive return times to E of x*.

Similarly, if $x \in M \setminus (R^-_\infty \cup N^-)$, there exists a strictly decreasing sequence of non-positive integers $\{r_k\}_{k \in \mathbb{N}}$ such that for every $k \in \mathbb{N}$, we have $T^{r_k} x \in E$ and $T^i x \notin E$ for all $r_{k+1} < i < r_k$. In this case, we call $\{r_k\}_{k \in \mathbb{N}}$ the sequence of the *negative return times to E of x*.

Proposition 6.2. *Suppose that $x \in M \setminus (R^+_\infty \cup N^+)$, and let $\{r_k\}$ be the sequence of the positive return times to E of x. Then*

$$\lim_{k \to +\infty} \widetilde{\sigma}^*_{T^{r_1} x}(D_{T^{r_1} x} T^{r_k - r_1}) = \lim_{k \to +\infty} \widetilde{\sigma}_{T^{r_1} x}(D_{T^{r_1} x} T^{r_k - r_1}) = +\infty.$$

Similarly, suppose $x \in M \setminus (R^-_\infty \cup N^-)$, and let $\{r_k\}$ be the sequence of the negative return times to E of x. Then

$$\lim_{k \to +\infty} \widetilde{\sigma}^*_{T^{r_k} x}(D_{T^{r_k} x} T^{r_1 - r_k}) = \lim_{k \to +\infty} \widetilde{\sigma}_{T^{r_k} x}(D_{T^{r_k} x} T^{r_1 - r_k}) = +\infty.$$

[4] In general, we have $\rho = \frac{b - \bar{b}}{b - a} \cdot \frac{\bar{a} - a}{\bar{b} - \bar{a}}$. Here, we have used $b = +\infty$.

Proof. We prove the proposition only for $x \in M \setminus (R_\infty^+ \cup N^+)$. For the other case, the proof is similar.

Define $x_k = T^{r_k} x$ and $m_k = r_{k+1} - r_k$ for all $k \in \mathbb{N}$. By Lemma 6.1, $D_{x_1} T^{r_2 - r_1} C_{x_1}(x_1) \subset \operatorname{int} C_{x_2}(x_2) \cup \{0\}$, which in turn implies

$$\widetilde{\sigma}^*_{x_1}(D_{x_1} T^{r_2 - r_1}) > 0. \tag{6}$$

Next, one can easily show that

$$\widetilde{\sigma}^*_{x_1}(D_{x_1} T^{r_k - r_1}) \geq \widetilde{\sigma}^*_{x_1}(D_{x_1} T^{r_2 - r_1}) \cdot \widetilde{\sigma}_{x_2}(D_{x_2} T^{r_k - r_2}), \tag{7}$$

and

$$\widetilde{\sigma}_{x_1}(D_{x_1} T^{r_k - r_1}) \geq \prod_{i=1}^{k-1} \widetilde{\sigma}_{x_i}(D_{x_i} T^{m_i}). \tag{8}$$

It follows from Lemma 6.1 that the sequence $\prod_{i=1}^{k-1} \widetilde{\sigma}_{x_i}(D_{x_i} T^{m_i})$ is strictly increasing in k. It does not diverge only if x_i for every i sufficiently large, and $\lim_{i \to +\infty} t(x_i, x_{i+1}) = 0$. This means that the positive semi-trajectory of x is eventually trapped inside a cusp formed by two dispersing components or a dispersing and a flat component. However, every trajectory entering a cusp leaves it after finitely many collisions [7, Appendix A1.3]. Hence, $\prod_{i=1}^{k-1} \widetilde{\sigma}_{x_i}(D_{x_i} T^{m_i}) \to +\infty$ as $k \to +\infty$. By (8), we conclude that

$$\lim_{k \to +\infty} \widetilde{\sigma}_{x_1}(D_{x_1} T^{r_k - r_1}) = +\infty.$$

The same argument applied to x_2 shows that $\widetilde{\sigma}_{x_2}(D_{x_2} T^{r_k - r_2}) \to +\infty$ as $k \to +\infty$. This together with (6) and (7) gives

$$\lim_{k \to +\infty} \widetilde{\sigma}^*_{x_1}(D_{x_1} T^{r_k - r_1}) = +\infty.$$

□

Corollary 6.3. *Every element $x \in H$ is sufficient, and has a quadruple (l, N, O, K) such that $O \cup T^{-N} O \subset E$.*

Proof. The set H can be decomposed as follows:

$$H = \left(M \setminus (R_\infty^+ \cup N^+)\right) \cup \left(M \setminus (R_\infty^- \cup N^-)\right).$$

So if $x \in H$, then $x \in M \setminus (R_\infty^+ \cup N^+)$ or $x \in M \setminus (R_\infty^- \cup N^-)$. We prove the corollary only for the case $x \in M \setminus (R_\infty^+ \cup N^+)$. The other case can be proved similarly.

Let $\{r_k\}_{k \in \mathbb{N}}$ be the sequences of the positive return times to E of x. Define $x_k = T^{r_k} x$ for all $k \in \mathbb{N}$. Note that although $x \in S_1^-$ is allowed, nevertheless $x_k \notin S_1^-$ for all $k > 1$. It follows that x_k is an interior point of E for every $k > 1$.

Proposition 6.2 applied to x_2 implies that there exists $k > 2$ such that $\widetilde{\sigma}_{x_2}(D_{x_2} T^{r_k - r_2}) > 3$. Set $l = r_k$ and $N = r_k - r_2$. We need to find the neighborhood O of x_k and the invariant continuous cone field K on $O \cup T^{-N} O$ as in Definition A.1.

Since T^N is a local diffeomorphism at x_2, and the cone fields (U_{x_2}, C_{x_2}) and (U_{x_k}, C_{x_k}) are continuous, the function $y \mapsto \widetilde{\sigma}_{x_2}(D_y T^N)$ is continuous at x_2. Hence, there exists a neighborhood V of x_2 such that i) $V \subset U_{x_2} \setminus R_N^+$ and $T^N V \subset U_{x_k}$, ii) $V \cap T^N V = \emptyset$ if $x_2 \neq x_k$, iii) $\widetilde{\sigma}_{x_2}(D_y T^N) > 3$ for every $y \in V$. Since x_2 and x_k are interior points of E, we can further choose V so that $V \cup T^N V \subset E$. It is easy to see that a similar choice can be made for the case $x \in M \setminus (R_\infty^- \cup N^-)$.

To construct the cone field $(O \cup T^{-N} O, K)$, we consider separately the cases $x_2 \neq x_k$ and $x_2 = x_k$. If $x_2 \neq x_k$, then define $O = T^N V$ and

$$K(y) = \begin{cases} C_{x_2}(y) & \text{if } y \in T^{-N} O, \\ C_{x_k}(y) & \text{if } y \in O. \end{cases}$$

Note that $O \cap T^{-N} O = V \cap T^N V = \emptyset$, and that C_{x_2} and C_{x_k} are continuous. This remark and Lemma 5.2 imply that the cone field K is continuous, invariant and satisfies

$$\sigma_K(D_y T^N) = \widetilde{\sigma}_{x_2}(D_y T^N) > 3 \qquad \text{for every } y \in T^{-N} O.$$

If $x_2 = x_k$ (i.e., x is periodic), then let $O = T^N V$. Since $U_{x_2} = U_{x_k}$, we have $O \cup T^{-N} O = V \cup T^N V \subset U_{x_2}$. Now, define K to be the restriction of C_{x_2} to $O \cup T^{-N} O$. As for the precious case, we see that the cone field K is continuous, invariant and satisfies $\sigma_K(D_y T^N) = \widetilde{\sigma}_{x_2}(D_y T^N) > 3$ for every $y \in T^{-N} O$. $\square$

We can now proceed to prove Conditions L1-L3.

6.2. **Proof of Conditions L1-L3.** From now on, for every sufficient point $x \in H$, we will always take as a quadruple (l, N, O, K) the one constructed in the proof of Corollary 6.3. The choice of this quadruple is of crucial importance for the proof of Conditions L1-L4, because the cone $(O \cup T^N O, K)$ is obtained directly from $\{(U_y, C_y)\}_{y \in E}$, and the neighborhood O has the property that $O \cup T^N O \subset E$.

We proved a number of properties concerning the geometry of the singulars sets R_k^- and R_k^+ in [15]. We will use several of these results in the proof of the following propositions.

Proposition 6.4. *Condition L1 is satisfied.*

Proof. For billiards satisfying Conditions B1 and B2, the regularity of the singular sets R_k^- and R_k^+ is proved in [15, Theorem 2.2]. $\square$

Proposition 6.5. *Every element of H satisfies Condition L2.*

Proof. This proposition follows from [15, Proposition 6.2]. For the convenience of the reader, we give here a more direct proof. We prove only the first part of Condition L2, because the second one can be proved similarly.

Let $x \in H$, and let $(O \cup T^N O, K)$ be the cone field for x as in the proof of Corollary 6.3. Suppose that Σ is a component of $R_k^- \cap T^{-N} O$

for some $k \in \mathbb{N}$. Without loss of generality, we can assume $\Sigma \not\subset R^-_{k-1}$. Let $\Sigma' = \Sigma \setminus \partial\Sigma$. Recall that $\partial\Sigma$ denotes the set of the endpoints of Σ.

The map $T^{-k+1} \colon \Sigma' \to T^{-k+1}\Sigma'$ is a diffeomorphism, and the family of rays emerging from the points of $\mathcal{I}(T^{-k}\Sigma')$ focuses in linear approximation at a corner of ∂Q or at a point lying on a dispersing component of ∂Q. Hence, if $y \in \Sigma'$ and $u \in T_y\Sigma \setminus \{0\}$, then

$$\tau^-(T^{-k+1}y, D_yT^{-k+1}u) = t(\mathcal{I}(T^{-k+1}y)). \tag{9}$$

Let $v \in K'(y) \setminus \{0\}$. By construction, we have $K'(y) = C_{x_2}(y)$ for some $x_2 \in T^{-N}O$ (see the proof of Corollary 6.3). There exists $1 \le i < k$ such that $T^{-i}\Sigma' \subset E$, and $T^{-j}\Sigma' \subset M^0$ for every $i < j < k$. Hence, $D_yT^{-i}v \in \operatorname{int} C'_{T^{-i}y}(T^{-i}y)$ by Lemma 5.2. Now, Condition B2 implies

$$\tau^-(T^{-i}y, D_yT^{-i}v) < \sum_{j=i}^{k-1} t(\mathcal{I}(T^{-j}y)).$$

To obtain this conclusion, we use Part (2) of Condition B2 whenever the family of rays emerging from the points of $\mathcal{I}(T^{-k}\Sigma')$ focuses at a corner of ∂Q. The previous inequality together with the fact that $T^{-j}\Sigma' \subset M^0$ for every $i < j < k$ implies

$$\begin{aligned}\tau^-(T^{-k+1}y, D_yT^{-k+1}v) &= \tau^-(T^{-i}y, D_yT^{-i}v) - \sum_{j=i}^{k-2} t(\mathcal{I}(T^{-j}y)) \\ &< t(\mathcal{I}(T^{-k+1}y)).\end{aligned}$$

By comparing the last conclusion with (9), we infer that $u \in \operatorname{int} K(y)$.

Finally, let $y \in \partial\Sigma$, and let $u \in T_y\Sigma \setminus \{0\}$. Since Σ is a C^2 arc, there exist two sequence $y_n \in \Sigma'$ and $u_n \in T_{y_n}\Sigma$ such that $y_n \to y$ and $u_n \to u$ as $n \to +\infty$. Since $u_n \in \operatorname{int} K(y)$ and K is a continuous cone field, it follows that $u \in K(y)$. This completes the proof. □

Proposition 6.6. *Every element of H satisfies Condition L3.*

Proof. We only proof the part of Condition L3 concerning S_1^-, because the proof of the other part is similar.

By [15, Propositions 6.17-6.19], the set $S_1^- \cap R^+_\infty$ is at most countable. This combined with Condition B4 gives $m_-(S_1^- \cap (N^+ \cup R^+_\infty)) = 0$. Hence, we first need to prove that each element of $S_1^- \setminus (N^+ \cup R^+_\infty)$ is u-essential.

Let $y \in S_1^- \setminus (N^+ \cup R^+_\infty)$, and let $\{r_k\}_{k\in\mathbb{N}}$ be the sequence of the positive return times to E of y, which exists by Proposition 6.2. Define $y_k = T^{r_k}y$ for every $k \in \mathbb{N}$.

Since T^{r_1} is a local diffeomorphism around y, there exist a neighborhood U of y and a constant $c > 0$ such that

$$\frac{\|D_yT^{r_1}u\|}{\|u\|} \ge c \tag{10}$$

for all $y \in U$ and for all $u \in T_yM \setminus \{0\}$. Now, let $\alpha > 0$. By Proposition 6.2, there is $k > 1$ such that

$$\widetilde{\sigma}^*_{y_1}(D_{y_1}T^{r_k - r_1}) \geq \alpha/c. \tag{11}$$

The argument that proves the continuity of $y \mapsto \widetilde{\sigma}_{x_2}(D_yT^N)$ at x_2 in the proof of Corollary 6.3 also proves the continuity of $z \mapsto \widetilde{\sigma}^*_{y_1}(D_zT^{r_k-r_1})$ at y_1. Next, since $y \in S^-_1 \setminus (N^+ \cup R^+_\infty)$, the point y is not periodic, and in particular, $y \neq y_k$. These observations imply that there is a neighborhood V of y_1 such that i) $V \subset U_{y_1} \setminus (R^-_{r_1} \cup R^+_{r_k-r_1})$ and $T^{r_k-r_1}V \subset U_{y_k}$, ii) $T^{-r_1}V \subset U$, iii) $T^{-r_1}V \cap T^{r_k-r_1}V = \emptyset$, and iv) $\widetilde{\sigma}^*_{y_1}(D_zT^{r_k-r_1}) > \alpha/c$ for every $z \in V$.

Now, define $O_{y,\alpha} = T^{-r_1}V$ and

$$K_{y,\alpha} = \begin{cases} D_{T^{r_1}z}T^{-r_1}C_{y_1}(T^{r_1}z) & \text{if } z \in O_{y,\alpha}, \\ C_{y_k}(z) & \text{if } z \in T^{r_k}O_{y,\alpha}. \end{cases}$$

Note that $O_{y,\alpha} \cap T^{r_k}O_{y,\alpha} = T^{-r_1}V \cap T^{r_k-r_1}V = \emptyset$, the cone fields C_{y_1} and C_{y_k} are continuous, and T^{r_1} is a diffeomorphism on $O_{y,\alpha}$. From the previous remark, Lemma 5.2 and the construction of $K_{y,\alpha}$, it follows that $K_{y,\alpha}$ is continuous and invariant. Set $n_{y,\alpha} = r_k$. For every $z \in O_{y,\alpha}$, we have

$$\begin{aligned}
\sigma_{K_{y,\alpha}}(D_zT^{n_{y,\alpha}}) &= \inf_{u \in \operatorname{int} K_{y,\alpha}(z)} \frac{\sqrt{Q_{K_{y,\alpha}}(T^{n_{y,\alpha}}z, D_zT^{n_{y,\alpha}}u)}}{\|u\|} \\
&= \inf_{u \in \operatorname{int} K_{y,\alpha}(z)} \frac{\|D_zT^{r_1}u\|}{\|u\|} \cdot \frac{\sqrt{Q_{C_{y_k}}(T^{n_{y,\alpha}}z, D_zT^{n_{y,\alpha}}u)}}{\|D_zT^{r_1}u\|} \\
&\geq \inf_{u \in \operatorname{int} K_{y,\alpha}(z)} \frac{\|D_zT^{r_1}u\|}{\|u\|} \cdot \widetilde{\sigma}^*_{y_1}(D_{T^{r_1}z}T^{r_k-r_1}),
\end{aligned}$$

where the last inequality is a consequence of $D_zT^{r_1}K_{y,\alpha}(z) = C_{y_1}(T^{r_1}z)$ and $T^{r_1}z \subset U_1$. By (10) and (11), we obtain

$$\sigma_{K_{y,\alpha}}(D_zT^{n_{y,\alpha}}) \geq c \cdot \widetilde{\sigma}^*_{y_1}(D_{T^{r_1}z}T^{r_k-r_1}) \geq c \cdot \frac{\alpha}{c} = \alpha.$$

We conclude that y is an u-essential point (see Definition A.2).

To prove that the cone field $(O \cup T^{-N}O, K)$ for $x \in H$ and the cone field $(O_{y,\alpha} \cup T^{n_{y,\alpha}}O_{y,\alpha}, K_{y,\alpha})$ just constructed are jointly invariant, we observe that K is a restriction of the cone fields C_{x_2} and C_{x_k} for some $x_2, x_k \in E$, and that $K_{y,\alpha}$ is obtained by taking the preimage of C_{y_1} and the restriction of C_{y_k} for some $y_1, y_k \in E$. The joint invariance then follows from Lemma 5.2. □

7. Proof of Condition L4

In Corollary 6.3, we proved that every $x \in H$ is a sufficient point with a quadruple (l, N, O, K) having the property that $O \cup T^{-N}O \subset E$. We recall that $l \in \mathbb{Z}$, $N \in \mathbb{N}$, O is a neighborhood of T^lx, and K is a continuous invariant cone field on $O \cup T^{-N}O$. We also recall that $\{(U_y, C_y)\}_{y\in E}$ is the family of cone fields introduced in Section 4.3.

In this section, we prove Condition L4 for every $x \in H$. This condition requires the existence of stable and unstable manifolds a.e. on the neighborhood O, which is guaranteed by Proposition A.3. We derive L4 from the following property:

Non-contraction: there exists $\beta' > 0$ such that if $z \in E \setminus R^+_m$ and $T^m z \in E \cup \mathcal{I}(E)$ for some $m \in \mathbb{N}$, then

$$\|D_z T^m v\| \geq \beta' \|v\| \qquad \text{for every } v \in C_z(z), \tag{12}$$

Since E is the set of the collisions 'entering' non-flat boundary components, $\mathcal{I}(E)$ is the set of the collisions 'leaving' non-flat boundary components.

To prove this property, we decompose the billiard orbits into special blocks, and study (12) separately for each block. This analysis will be carried out using certain semi-norms defined in terms of transversal Jacobi fields.

7.1. **Jacobi fields and semi-norms.** Jacobi fields for billiards are special vectors fields defined along trajectories of the billiard flow. For the definitions of a billiard flow and a Jacobi field for billiards, we refer the reader to [10, 20] and [16, 33], respectively. Here, we limit ourselves to recall the main properties of the Jacobi fields for billiards.

The billiard flow Φ^t on the domain Ω acts on the unit tangent bundle $T_1\overline{\Omega} \subset T_1\mathbb{R}^2$ of the closure of Ω. Let $\pi\colon T_1\overline{\Omega} \to \overline{\Omega}$ be the natural projection. A *billiard trajectory* ξ is a piecewise linear curve $\xi\colon (a,b) \to \overline{\Omega}$ with $-\infty \leq a < 0 < b \leq +\infty$ such that $\xi(t) = \pi(\Phi^t(z))$ for all $t \in (a,b)$ and for some $z \in T_1\overline{\Omega}$. We say that $\xi(t)$ is a *collision* for some $t \in (a,b)$ if $\xi(t) \in \partial\Omega$, or equivalently, if there exists $y \in M$ such that $\xi(t) = q(y)$. A collision is called *non-tangential* provided that $y \notin A_2$.

A *transversal Jacobi field* $\mathcal{J}$ along a billiard trajectory ξ is a piecewise smooth map $\mathcal{J}\colon (a,b) \to \mathbb{R}^2$ such that $\mathcal{J}(t)$ and its time time derivative $\mathcal{J}'(t)$ are perpendicular to $\xi'(t)$ every $t \in (a,b)$, and satisfy further properties for which we refer the reader to [16, 33]. It turns out that for any fixed t, the field $\mathcal{J}$ is uniquely determined by the values $\mathcal{J}(t)$ and $\mathcal{J}'(t)$. Let $\xi^\perp(t)$ denote the unit vector of $\mathbb{R}^2$ orthogonal to $\xi'(t)$ such that $\{\xi'(t), \xi^\perp(t)\}$ is a positively oriented basis of $\mathbb{R}^2$. Since the vectors $\mathcal{J}(t)$ and $\mathcal{J}'(t)$ are always collinear to $\xi^\perp(t)$, we can identify $\mathcal{J}(t)$ and $\mathcal{J}'(t)$ with the corresponding orthogonal projections onto $\xi^\perp(t)$, and think of $\mathcal{J}(t)$ and $\mathcal{J}'(t)$ as real numbers.

Suppose that there exists $0 < \delta < b$ such that $\xi(t)$ is not a collision for all $t \in (0,\delta)$, and $\xi(\delta) = q(y)$ is a non-tangential collision for some $y \in M \setminus A_2$. If $\mathcal{J}$ is a Jacobi field along ξ, then the evolution of the pair $(\mathcal{J}, \mathcal{J}')$ is governed by the following equations:

$$\begin{pmatrix}\mathcal{J}(t)\\ \mathcal{J}'(t)\end{pmatrix} = F(t)\begin{pmatrix}\mathcal{J}(0)\\ \mathcal{J}'(0)\end{pmatrix} \qquad \text{for } 0 \leq t < \delta,$$
$$\begin{pmatrix}\mathcal{J}(\delta)\\ \mathcal{J}'(\delta)\end{pmatrix} = R(y)\lim_{t\to\delta^-}\begin{pmatrix}\mathcal{J}(t)\\ \mathcal{J}'(t)\end{pmatrix},$$

where

$$F(t) := \begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix} \qquad \text{and} \qquad R(y) := \begin{pmatrix}-1 & 0\\ \frac{2}{d(y)} & -1\end{pmatrix}. \tag{13}$$

For every $y \in M \setminus A_2$ and every $u = (u_s, u_\theta) \in T_yM$, define

$$J(u) = \sin\theta(y)u_s, \qquad J'(u) = -u_\theta - \kappa(y)u_s, \tag{14}$$

and observe that the transformation $u \mapsto (J(u), J'(u))$ is a bijection between T_yM and $\mathbb{R}^2$. In particular, the pair (J, J') is a system of coordinates for T_yM. Note that J' is just a symbol and not the time derivative of J. Suppose that $\xi(0) = q(y)$ is a non-tangential collision for some $y \in M \setminus A_2$, and let $u \in T_yM$. Our previous observation combined with the fact that the entire transversal Jacobi field $\mathcal{J}$ along ξ is determined by the choice $\mathcal{J}(0) = J(u)$ and $\mathcal{J}'(0) = J'(u)$ shows that there is a bijective correspondence between transversal Jacobi fields along ξ and elements of T_yM, and so we can identify elements of T_yM with transversal Jacobi fields. Now, in addition to the previous assumption, suppose that there exists $0 < \delta < b$ such that $\xi(0)$ and $\xi(\delta)$ are consecutive non-tangential collisions. This means that $\delta = t(y)$ and $\xi(\delta) = q(Ty)$. Moreover, if $u \in T_yM$ and $u_1 \in T_{Ty}M$ are the tangent vectors corresponding to $(\mathcal{J}(0), \mathcal{J}'(0))$ and $(\mathcal{J}(\delta), \mathcal{J}'(\delta))$, respectively, one can show that $D_yTu = u_1$. It follows that the matrix of D_yT in coordinates (J, J') is given by

$$R(Ty)F(t(y)) = \begin{pmatrix}-1 & -t(y)\\ \frac{2}{d(Ty)} & -1 + \frac{2t(y)}{d(Ty)}\end{pmatrix}. \tag{15}$$

The matrices (13) and (15) have determinant equal to 1, and so D_yT preserves the standard symplectic form $J \wedge J'$. Finally, from (14), it follows immediately that the forward focusing time of a tangent vector $u \in T_yM$ in terms of $J(u)$ and $J'(u)$ is given by

$$\tau^+(y, u) = \begin{cases} -\frac{J(u)}{J'(u)} & \text{if } J'(u) \neq 0,\\ \infty & \text{if } J'(u) = 0.\end{cases} \tag{16}$$

We now introduce two semi-norms, which will help us prove the Non-contraction property.

Definition 7.1. *For every $y \in M$ and every $u \in T_yM$, define*

$$\|u\|_J = \sqrt{J^2(u) + J'^2(u)} \quad \textit{and} \quad |u|_{J'} = |J'(u)|.$$

In the rest of this subsection, we prove that $\|\cdot\|$ and $\|\cdot\|_J$ are equivalent on a certain subset of the tangent bundle TM.

Lemma 7.2. *There exists a constant $\alpha_1 > 1$ depending only on Ω and the family of cone fields $\{(U_y, C_y)\}_{y\in E^+}$ such that if $y \in E^+$, $z \in U_y$ and $0 \le k \le n(z)$, then*

$$|D_z T^k u|_{J'} \le \|D_z T^k u\|_J \le \alpha_1 |D_z T^k u|_{J'} \qquad \text{for } u \in C_y(z).$$

Proof. Let y, z, k, u be as in the hypotheses of the lemma. It is clear that $|D_z T^k u|_{J'} \le \|D_z T^k u\|_J$. By Part (3) of Theorem 4.1, there exists a constant $f > 0$ depending only on $\{(U_y, C_y)\}_{y\in E}$ such that

$$\frac{d(T^k z)}{2} \le \tau_k^+(z,u) \le f.$$

By (16), we then have $|J(D_z T^k u)| \le f|J'(D_z T^k u)|$, which gives the remaining inequality with $\alpha_1 = (1+f^2)^{1/2}$. □

Lemma 7.3. *There exists $\alpha_2 > 1$ such that if $y \in M \setminus \partial M$, then*

$$\|u\|_J \le \alpha_2 \|u\| \qquad \text{for } u \in T_y M.$$

Proof. A straightforward computation gives the wanted inequality with $\alpha_2 = 2^{1/2}(1+\kappa_1^2)^{1/2}$, where $\kappa_1 = \max_{z\in M} |\kappa(z)|$. □

Lemma 7.4. *There exists $\alpha_3 > 0$ such that if $y \in M^-$ and $z \in U_y$, then*

$$\|u\| \le \alpha_3 \|u\|_J \qquad \text{for } u \in C_y(z).$$

Proof. Let $\kappa_2 = \min_{z\in M^-} |\kappa| > 0$. By the definition of $C_y(z)$, we have $u_\theta/u_s \le \kappa(z) < 0$ for all $0 \ne u \in C_y(z)$. Hence,

$$\begin{aligned}\|u\|_J^2 \ge |J(u)|^2 &= \kappa^2(z)u_s^2 + u_\theta^2 + 2\kappa(z)u_s u_\theta \\ &\ge \kappa_2^2 u_s^2 + u_\theta^2 \ge \frac{\kappa_2^2}{1+\kappa_2^2}(u_s^2+u_\theta^2) = \frac{\kappa_2^2}{1+\kappa_2^2}\|u\|^2.\end{aligned}$$

□

Lemma 7.5. *There exists $\alpha_4 > 0$ such that if $y \in E^+$, $z \in U_y$ and $u \in C_y(z)$, then*

$$\|D_z T^k u\| \le \alpha_4 \|D_z T^k u\|_J \qquad \text{for } 0 \le k \le n(z).$$

Proof. Denote by κ_3 the maximum of κ on M^+, and denote by $\overline{m}$ the smallest m_i associated to focusing components of $\partial\Omega$ as in Theorem 4.1. Let $0 \ne u \in C_y(z)$, and write $(u_{k,s}, u_{k,\theta})$ for the vector $D_z T^k u$. By Theorem 4.1, if $z \in M_i \subset M^+$, then $-\kappa(T^k z) + m_i \le m(u_k) \le \kappa(T^k z)$

for $0 \le k \le n(z)$. In particular, $|u_{k,\theta}| \le \kappa(T^k z)|u_{k,s}|$. Therefore,

$$\begin{aligned}\|D_z T^k u\|_J^2 &\ge \left(\kappa(T^k z)u_{k,s} + u_{k,\theta}\right)^2 = \left(\kappa(T^k z) + m(u_k)\right)^2 u_{k,s}^2 \\ &\ge \left(\kappa(T^k z) - \kappa(T^k z) + m_i\right)^2 u_{k,s}^2 \ge \frac{m_i^2}{1+\kappa^2(T^k z)}\|D_z T^k u\|^2 \\ &\ge \frac{\overline{m}^2}{1+\kappa_3^2}\|D_z T^k u\|^2.\end{aligned}$$

□

Lemma 7.6. *There exist constants $\epsilon_0 > 0$ and $0 < \theta_0 < \pi/2$ such that $\theta(M^0 \cap S_1^-(\epsilon_0)) \subset (\theta_0, \pi - \theta_0)$.*

Proof. The lemma is an immediate consequence of the following easy-to-check fact. Consider a flat component Γ_i of $\partial\Omega$. If the line containing Γ_i contains also a corner p or is tangent to a dispersing component Γ_j, then no ray emerging from the elements of $M_i \cap S_1^+$ contains p or is tangent to Γ_j. □

Lemma 7.7. *Let ϵ_0 be the constant in Lemma 7.6. There exists $\alpha_5 > 0$ such that if $y \in M^0 \cap S_1^-(\epsilon_0)$, then*

$$\|u\| \le \alpha_5 \|u\|_J \qquad \text{for } u \in T_y M.$$

Proof. Since $\kappa(y) = 0$, we have $\|u\|_J^2 \ge \sin^2\theta_0 u_s^2 + u_\theta^2 > \sin^2\theta_0 \|u\|^2$. □

Corollary 7.8. *There exist two constants $0 < A_1 < A_2$ such that if $y \in E$, $z \in U_y$, $u \in C_y(z)$ or $y \in M^0 \cap S_1^-(\epsilon_0)$, $u \in T_y M$, then*

$$A_1\|u\|_J \le \|u\| \le A_2\|u\|_J.$$

Proof. The claim follows from Lemmas 7.3-7.7. □

Remark 7.9. *It can be easily proved that Corollary 7.8 remains valid if $\|\cdot\|$ is replaced by $|\cdot|_{J'}$, thus showing that the semi-norms $\|\cdot\|$, $\|\cdot\|_J$, $|\cdot|_{J'}$ are equivalent in the sense specified in the corollary. However, this general equivalence is not needed for the proof of L4.*

7.2. **Block decomposition.** The Non-contraction property is just a slight modification of a condition with the same name introduced in [22]. Roughly speaking, this property states that the expansion of the iterations of vectors from the cones in $\{U_y, C_y\}_{y\in E}$ is uniformly bounded from below along orbits starting at E and ending at $E \cup \mathcal{I}(E)$. This property implies that there is no loss of expansion along arbitrarily long sequences of consecutive collisions with focusing components or between flat components.

We now show that every sequence of collisions $\{z, \ldots, T^m z\}$ as in the non-contraction property can be decomposed into a finite number of special subsequences called blocks.

Definition 7.10. *Let $z \in M \setminus R_m^+$ with $m > 0$. A sequence of consecutive collisions $\{z, \ldots, T^m z\}$ is called a* block of type $i \in \{1, \ldots, 4\}$ *if the corresponding condition (i) below is satisfied:*

(1) $z \in E^+$, $\{T^{n(z)+1}z, \ldots, T^{m-1}z\} \subset M^0$ and $T^m z \in M^-$,
(2) $z \in M^-$, $\{Tz, \ldots, T^{m-1}z\} \subset M^0 \cup M^-$ and $T^m z \in M^-$,
(3) $z \in M^-$, $\{Tz, \ldots, T^{m-1}z\} \subset M^0$ and $T^m z \in E^+$,
(4) z and $T^m z$ belong to E^+.

Definition 7.11. *A block is called* minimal *if it does not contain any other block of the same type. A block included in a sequence of consecutive collisions φ is called* maximal in φ *if it is not contained in any other block of the same type in φ.*

We observe that blocks of type 1 and 3 are always maximal and minimal. Also, we observe that every block of type 2 and 4 is a concatenation of finitely many minimal blocks of the type 2 and 4, respectively.

Proposition 7.12. *Let $\varphi = \{z, \ldots, T^m z\}$ be a sequence of collisions such that both z and $T^m z$ belong to E. Then φ is the concatenation of $\varphi_1, \ldots, \varphi_n$ with $1 \leq n \leq 5$ and $\varphi_1, \ldots, \varphi_n$ being maximal blocks of type 1-4. Moreover, this decomposition is unique.*

Proof. First, suppose that φ does not contain blocks of type 2 and 4. Then we see that either φ is a block of type 1 or 3, or φ is the concatenation of two blocks: a block of type 3 followed by a block of type 1.

Now, suppose that φ contains blocks of type 2, but does not contain blocks of type 4. A maximal block of type 2 can only be preceded by a block of type 1 and followed by a block of type 3. Hence, φ can contain at most two maximal blocks of type 2. Moreover, if φ contains exactly two maximal blocks of type 2, then φ is the concatenation of a maximal block of type 2, a block of type 1, a block of type 3 and a maximal block of type 2, following one another in this order. Instead, if φ_1 contains only a single maximal block of type 2, then there are three possibilities: φ is a block of type 2, φ is the concatenation of a maximal block of type 2 and a block of type 1 or 3, and φ is the concatenation of a block of type 1, a maximal block of type 2 and a block of type 3, following one another in this order.

Finally, suppose that φ contains blocks of type 4. From the definition of these blocks, we see that there is only one single maximal block of type 4 in φ. Such a block can only be preceded by a block of type 3 and followed by a block of type 1. Moreover, blocks of type 1 and 3 in φ can only be adjacent to blocks of type 2 or 4. Since there is only one maximal block of type 4 in φ, besides this block, φ can contain at most a block of type 1, at most a block of type 3 and at most two maximal blocks of type 2. The blocks of type 1 and 3 are concatenated to the block of type 4, whereas one block of type 2 is concatenated to the block of type 1, and the second block of type two is concatenated to the block of type 3.

It is not difficult to see that the decomposition of φ into blocks of type 1-4 we have just obtained is unique, because the blocks forming it are maximal. $\square$

7.3. **Proving the Non-contraction property.** In view of Proposition 7.12, it suffices to show that (12) holds along each block of type 1-4 with β' depending only on the block type. We start with some preliminary results.

Lemma 7.13. *Let $0 \leq m_1 < m_2$, and suppose that $z \in E \setminus R^+_{m_2}$ and $v \in C_z(z) \setminus \{0\}$. Then*

$$\frac{|D_z T^{m_2} v|_{J'}}{|D_z T^{m_1} v|_{J'}} = \prod_{k=m_1+1}^{m_2} \left| \frac{\tau_k^-(z,v)}{\tau_k^+(z,v)} \right|.$$

Proof. Define $z_k = T^k z$ and $v_k = D_z T^k v$ for $0 \leq k \leq n(z)$. By the definition of C_z and its invariance, it is not difficult to see using Theorem 4.1 that $J'(v_k) \neq 0$ for $0 \leq k \leq n(z)$. Using (15) and Condition B2, one can further show that $J(v_k) \neq 0$ for $1 \leq k \leq n(z)$. Therefore,

$$\frac{|v_m|_{J'}}{|v_{m_1}|_{J'}} = \prod_{k=m_1+1}^{m} \left| \frac{J'(v_k)}{J(v_k)} \right| \cdot \left| \frac{J(v_k)}{J'(v_{k-1})} \right|.$$

Now, the wanted equality follows from $\tau^+(z_k, v_k) = -J(v_k)/J'(v_k)$ and $\tau^-(z_k, v_k) = J(v_k)/J'(v_{k-1})$. The first expression for $\tau^+(z_k, v_k)$ is just (16), whereas the one for $\tau^-(z_k, v_k)$ can be easily derived from (15) and (16). $\square$

The following proposition provides an upper bound on the number of consecutive collisions $n(x)$ along a Donnay arc provided that the initial angle $\theta(x)$ is sufficiently close to 0 or π. For its proof see [16, Corollary 5.3 and Part (1) of Proposition 6.1].

Proposition 7.14. *Suppose that Γ_i is a Donnay arc, and let θ_i be the corresponding constant as in Theorem 4.1. There exist a real number $c_i > 0$ and a function $N_i \colon (0, \pi/2) \to \mathbb{N}$ such that if $x \in M_i \setminus A_2$, then $n(x) \leq c_i / \min\{\theta(x), \pi - \theta(x)\}$ whenever $\theta(x) \in (0, \theta_i) \cup (\pi - \theta_i, \pi)$, and $n(x) < N_i(\theta(x))$ whenever $\theta(x) \in [\theta_i, \pi - \theta_i]$.*

Proposition 7.15. *There exists $\gamma_1 > 0$ such that if $z \in E^+$ and $0 \leq m_1 < m_2 \leq n(z)$, then*

$$\|D_z T^{m_2} v\| \geq \gamma_1 \|D_z T^{m_1} v\| \qquad \text{for } v \in C_z(z).$$

Proof. In virtue of Lemmas 7.2, 7.3 and 7.5, we can prove the proposition with the norm $\|\cdot\|$ replaced by the semi-norm $|\cdot|_{J'}$. Let $z \in E_i \subset E^+$ for some i, and pick $v \in C_z(z)$. Of course, it is enough to prove the proposition with γ_1 depending on the focusing component

Γ_i. By taking the minimum of such γ_1's over all focusing components, we obtain the proposition in its general form. Let θ_i, a_i, c_i, N_i be as in Theorem 4.1 and Proposition 7.14. We define $z_k = T^k z$, $v_k = D_z T^k v$ for $0 \le k \le n(z)$.

Suppose first that $\theta = \theta(z) \in (0, \theta_i) \cup (\pi - \theta_i, \pi)$. We consider only the case when $\theta \in (0, \theta_i)$, because the argument for $\theta \in (\pi - \theta_i, \pi)$ is similar. By Theorem 4.1, we have $m(v_k) \ge -a_i \theta$. Let $R_i = \max_{y \in M_i} 1/\kappa(y)$. Using (1), we easily obtain

$$\frac{\tau^-(z_k, v_k)}{\tau^+(z_k, v_k)} \ge \frac{1 - R_i a_i \theta}{1 + R_i a_i \theta}$$

By Proposition 7.14, we have $n(z) \le c_i/\theta$, and so Lemma 7.13 implies that

$$\frac{|v_{m_2}|_{J'}}{|v_{m_1}|_{J'}} \ge \left(\frac{1 - R_i a_i \theta}{1 + R_i a_i \theta} \right)^{\frac{c_i}{\theta}}.$$

Since the right hand-side of the previous inequality converges to $e^{-2a_i c_i R_i} > 0$ as $\theta \to 0^+$, there exists $0 < \bar{\theta} < \theta_i$ such that

$$\frac{|v_{m_2}|_{J'}}{|v_{m_1}|_{J'}} \ge \frac{1}{2} e^{-2a_i c_i R_i} \qquad \text{for } \theta \in (0, \bar{\theta}). \tag{17}$$

Now, suppose that $\theta(z) \in [\bar{\theta}, \pi - \bar{\theta}]$. By Part (3) of Theorem 4.1, there exists a constant $f_i > 0$ such that $d(z_k)/2 \le \tau^{\pm}(z_k, v_k) \le f_i$ for $1 \le k \le m_2$. Let $r_i = \min_{y \in M_i} 1/\kappa(y)$, and let $d_i = r_i \sin \theta_i$. Then

$$\frac{\tau^-(z_k, v_k)}{\tau^+(z_k, v_k)} \ge \frac{d(z_k)}{2 f_i} \ge \frac{d_i}{2 f_i} \qquad \text{for } 1 \le k \le m_2.$$

Using Lemma 7.13 and Proposition 7.14, we conclude that

$$\frac{|v_{m_2}|_{J'}}{|v_{m_1}|_{J'}} \ge \min \left\{ 1, \left(\frac{d_i}{2 f_i} \right)^{N_i(\bar{\theta})} \right\}. \tag{18}$$

The wanted conclusion now follows from (17) and (18). □

Let λ be as in B2, and let f_i be as in the proof of Proposition 7.15. Define f to be the maximum of all f_i's.

Lemma 7.16. *Consider a sequence of collisions $\{z, \dots, T^m z\}$ with $m > 0$ such that $z, T^m z \in E$, and $T^k z \in M^0$ for $n(z) < k < m$. Then, for every $v \in C_z(z) \setminus \{0\}$ and every $n(z) \le k < m$, we have*

$$|D_z T^m v|_{J'} > \delta |D_z T^k v|_{J'},$$

where $\delta = \lambda / f$ if $T^m z \in M^+$, and $\delta = 1$ if $T^m z \in M^-$.

Proof. Let $v \in C_z(z) \setminus \{0\}$, and define $z_k = T^k z$ and $v_k = D_z T^k v$ for $0 \le k \le m$. From $\{z_{n(z)+1}, \dots, z_{m-1}\} \subset M^0$, it follows that $|v_k|_{J'} = |v_{m-1}|_{J'}$ for all $n(z) \le k < m$. Then, by Lemma 7.13, we have

$$\frac{|v_m|_{J'}}{|v_k|_{J'}} = \frac{|v_m|_{J'}}{|v_{m-1}|_{J'}} = \frac{\tau^-(z_m, v_m)}{\tau^+(z_m, v_m)} \qquad \text{for } n(z) \le k < m.$$

If $z_m \in M^+$, then $\tau^-(z_m, v_m) = t(z_{n(z)}, z_m) - \tau^+(z_{n(z)}, v_{n(z)}) > \lambda$ by Condition B2. Since $0 < \tau^+(z_m, v_m) \leq f$ (see the proof of Proposition 7.15), we obtain $\tau^-(z_m, v_m)/\tau^+(z_m, v_m) > \lambda/f$. If $z_m \in M^-$, then we use the Mirror Formula (2) to relate the focusing times $\tau^-(z_m, v_m)$ and $\tau^+(z_m, v_m)$. A simple computation using $d(z_m) < 0$ shows that $\tau^-(z_m, v_m) < \tau^+(z_m, v_m) < 0$, and so $\tau^-(z_m, v_m)/\tau^+(z_m, v_m) > 1$. □

Lemma 7.17. *Consider a sequence of collisions $\{z, \ldots, T^m z\}$ with $m > 0$ such that $z \in M^-, T^m z \in E$ and $T^k z \in M^0$ for $0 < k < m$. Then, for every $v \in C_z(z) \setminus \{0\}$ and every $0 \leq k < m$, we have*

$$|J(D_z T^m v)| > |J(D_z T^k v)|.$$

Proof. Let $v \in C_z(z) \setminus \{0\}$. By the definition of C_z (see Subsection 4.3), it is easy to see that $J(v)J'(v) \geq 0$ and $J'(v) \neq 0$. Since $\{Tz, \ldots, T^{m-1}z\} \subset M^0$, we have $|J'(D_z T^k v)| = |J'(v)|$ for $0 < k < m$. Using (15), we obtain

$$|J(D_z T^k v)| = |J(v)| + l(z, T^k z)|J'(v)| \qquad \text{for } 0 \leq k \leq m,$$

where $t(z, T^k z)$ is the length of the piece of trajectory starting at z and ending at $T^k z$ (see the beginning of Subsection 6.1). The wanted conclusion follows immediately from the previous equality. □

Lemma 7.18. *There exists a constant $\delta_1 > 0$ such that for every sequence of consecutive collisions $\{z, \ldots, T^m z\}$ with $z \in E^+$, $T^m z \in E$ and $T^k z \in M^0$ for $n(z) < k < m$, we have*

$$\|D_z T^m v\| \geq \delta_1 \|D_z T^{n(z)} v\| \qquad \text{for } v \in C_z(z).$$

Proof. If we replace $\|\cdot\|$ with $|\cdot|_{J'}$, then the wanted conclusion with $\delta_1 = \delta$ follows from Lemma 7.16. To obtain the actual conclusion, use the obvious fact that $\|\cdot\|_J \geq |\cdot|_{J'}$, apply Lemma 7.3 to $\|D_z T^m v\|_J$, and finally apply Lemmas 7.2 and 7.5 to $|D_z T^{n(z)} v|_{J'}$ and $\|D_z T^{n(z)} v\|_J$, respectively. □

Lemma 7.19. *There exists $\beta'_1 > 0$ such that every sequence of consecutive collisions $\{z, \ldots, T^m z\}$ such that $z \in E^+$, $\{T^{n(z)+1}z, \ldots, T^{m-1}z\} \subset M^0$ and $T^m z \in E$ satisfies* (12) *with $\beta' = \beta'_1$. In particular, the previous conclusion is true for every block of type 1.*

Proof. Let $\varphi = \{z, \ldots, T^m z\}$ be a block of type 1, and choose $0 \neq v \in C_z(z)$. By Proposition 7.15, we have $\|D_z T^{n(z)} v\| \geq \gamma_1 \|v\|$. To complete the proof, use Lemma 7.18. □

Lemma 7.20. *There exists $\beta'_2 > 0$ such that every block of type 2 satisfies* (12) *with $\beta' = \beta'_2$.*

Proof. Note first that every block of type 2 consists of finitely many minimal blocks of type 2. Next, suppose that $\{z, \ldots, T^m z\}$ is a minimal block of type 2, and let $v \in C_z(z) \setminus \{0\}$. In this case, we have $T^k z \in M^0$ for $1 \leq k \leq m - 1$. It follows that we have $|J'(D_z T^m v)| > |J'(v)|$

by Lemma 7.16, and $|J(D_zT^mv)| > |J(v)|$ by Lemma 7.17. Therefore, $\|D_zT^mv\|_J > \|v\|_J$. Of course, the same conclusion extends to a general block of type 2. To complete the proof, apply Lemmas 7.3 and 7.4 to $\|D_zT^mv\|_J$ and $\|v\|_J$, respectively. □

Lemma 7.21. *There exists $\beta'_3 > 0$ such that every block of type 3 satisfies* (12) *with $\beta' = \beta'_3$.*

Proof. Suppose that $\{z, \dots, T^mz\}$ is a block of type 3, and let $v \in C_z(z) \setminus \{0\}$. Since $T^kz \in M^0$ for $1 \le k < m$, we have $|J'(D_zT^mv)| > |J'(v)|\lambda/f$ by Lemma 7.16, and $|J(D_zT^mv)| > |J(v)|$ by Lemma 7.17. Hence, $\|D_zT^mv\|_J > \min\{\lambda/f, 1\}\|v\|_J$. To complete the proof, apply Lemmas 7.3 and 7.4 to $\|D_zT^mv\|_J$ and $\|v\|_J$, respectively. □

Let $\{z, \dots, T^mz\}$ be a block of type 4. As before, we define $z_j = T^jz$ for all $0 \le j \le m$. We will use this notation throughout the rest of this section. Note that every block of type 4 consists of finitely many minimal blocks of type 4. In other words, there exist $N > 0$ integers $0 = i_0 < \cdots < i_N = m$ such that $\{z_{i_k}, \dots, z_{i_{k+1}}\}$ is a minimal block of type 4 for each $0 \le k \le N-1$ (see Fig. 1). Now, recall that $n(z_{i_k})$ denotes the number of consecutive collisions of z_{i_k} with the focusing component Γ_i before leaving it (see Definition 3.2). Define $j_k = n(z_{i_k}) + i_k$. Then

$$D_zT^m = D_{z_{j_{N-1}}}T^{i_N - j_{N-1}} \circ D_{z_{j_0}}T^{j_{N-1}-j_0} \circ D_{z_0}T^{j_0}.$$

Proposition 7.22. *If $N > 1$, then the matrix of $D_{z_{j_0}}T^{j_{N-1}-j_0}$ in coordinates (J, J') is given by*

$$D_{z_{j_0}}T^{j_{N-1}-j_0} = F_{N-1}^{-1}A_{N-1}B_{N-2}\cdots B_1A_1B_0F_0,$$

where A_k, B_k, F_k are matrices of the form

$$A_k = \pm\begin{pmatrix} 1/\zeta_k & 0 \\ \eta_k & \zeta_k \end{pmatrix}, \quad B_k = \pm\begin{pmatrix} 1+a_k & b_k \\ c_k & 1+d_k \end{pmatrix}, \quad F_k = \begin{pmatrix} 1 & f_k \\ 0 & 1 \end{pmatrix} \tag{19}$$

with $\zeta_k, \eta_k, f_k > 0$, $a_k, c_k, d_k \ge 0$ and $b_k > \lambda$.

Proof. For every $0 \le k \le N-1$, define

$$s_k^- = G^-_{z_{i_k}}(z_{i_k}) \quad \text{and} \quad s_k^+ = \sup\left\{\tau^+_{j_k - i_k}(z_{i_k}, v) \colon 0 \ne v \in C_{z_{i_k}}(z_{i_k})\right\},$$

where $G^{\pm}_{z_{i_k}}$ are defined in Remark 4.2. Recall the definitions of the matrices F and R in (13). If we define

$$\begin{aligned} A_k &= F(s_k^+)D_{z_{i_k}}T^{j_k - i_k}R(z_{i_k})F(s_k^-), \\ B_k &= F^{-1}(s_k^-)R^{-1}(z_{i_{k+1}})D_{z_{j_k}}T^{i_{k+1}-j_k}F^{-1}(s_k^+), \\ F_k &= F(s_k^+), \end{aligned}$$

then we easily see that

$$D_{z_{j_k}}T^{j_{k+1}-j_k} = F_{k+1}^{-1}A_{k+1}B_kF_k.$$

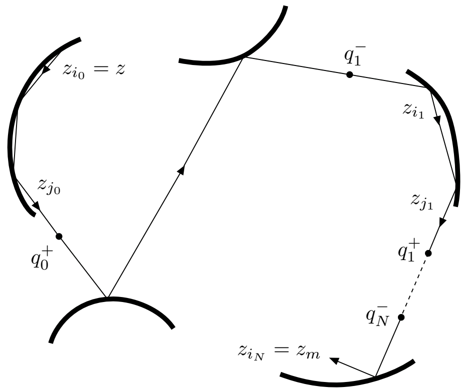

FIGURE 1. Decomposition of a block of type 4. The collisions $z_{i_k} \in E^+$ and $z_{j_k} \in \mathcal{I}(E^+)$ are the first and the last in a sequence of consecutive collisions with a focusing arc, respectively. The sequence of collisions $\{z_{i_k}, \ldots, z_{i_{k+1}}\}$ is a minimal block of type 4. The dashed piece of trajectory between the points q_1^+ and q_N^- represents a sequence of collisions with finitely many boundary components of the billiard domain.

Therefore,

$$D_{z_{j_0}} T^{j_{N-1}-j_0} = \prod_{k=0}^{N-2} D_{z_{j_k}} T^{j_{k+1}-j_k} = F_{N-1}^{-1} A_{N-1} B_{N-2} \cdots B_1 A_1 B_0 F_0.$$

The form of the matrices F_k can be immediately derived from (13). Since $f_k = s_k^+$, the positivity of f_k follows from $s_k^+ > 0$. We now determine the form of the matrices A_k and B_k. Let $v_k \in T_{z_{i_k}} M$ such that $\tau^+(z_{i_k}, v_k) = G^+_{z_{i_k}}(z_{i_k})$, and denote by q_k^- and q_k^+ the points along the trajectory of z_{i_k} where the vector v_k focuses backward, and the vector $D_{z_{i_k}} T^{j_k - i_k} v_k$ focuses forward, respectively. The matrix A_k governs the dynamics of transversal Jacobi fields along the trajectory starting at q_k^- and ending at q_k^+ (see Fig. 1). In other words, if a transversal Jacobi field is equal to (J, J') at q_k^-, then the same field is equal to $A_k(J, J')^T$ at q_k^+.

By construction of the billiard cone field (see Theorem 4.1), every transversal Jacobi field focusing at q_k^- focuses again at q_k^+. Since τ^+ is a strictly decreasing function of τ^- by the Mirror Formula, it follows that every transversal Jacobi field obtained from a parallel variation of the billiard trajectory passing through q_k^- focuses before reaching q_k^+. Thus, there exist $\xi_k, \zeta_k, \eta_k > 0$ such that

$$A_k \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \pm \begin{pmatrix} \xi_k \\ \eta_k \end{pmatrix}, \qquad A_k \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \pm \begin{pmatrix} 0 \\ \zeta_k \end{pmatrix}.$$

Since A_k is product of finitely many matrices as in (13), we have $\det A_k = 1$, and so $\xi_k = \zeta_k^{-1}$. We conclude that

$$A_k = \pm \begin{pmatrix} 1/\zeta_k & 0 \\ \eta_k & \zeta_k \end{pmatrix}.$$

To derive the form of B_k, we argue similarly. The matrix B_k governs the dynamics of transversal Jacobi fields along the piece of the billiard trajectory starting at q_k^+ and ending at q_{k+1}^- (see Fig. 1). By the definition of $q_k^\pm$, we see that along this piece of trajectory, there are only collisions with flat or dispersing components. Hence, it easily follows from (15) that B_k is a product of finitely many matrices of the form

$$P = \pm \begin{pmatrix} 1+a & b \\ c & 1+d \end{pmatrix} \tag{20}$$

with $a, b, c, d \geq 0$ and $\det P = 1$. All the matrices P forms a semigroup with respect to the standard matrix multiplication. For this reason, there are $a_k, b_k, c_k, d_k \geq 0$ such that

$$B_k = \pm \begin{pmatrix} 1+a_k & b_k \\ c_k & 1+d_k \end{pmatrix}.$$

Since F and R are upper and lower triangular, respectively, we see that the entry b_k cannot be smaller than the length of the piece of the trajectory connecting q_k^+ and q_{k+1}^-. This observation combined with Condition B2 gives $b_k > \lambda$. □

The notation used in the following proposition is as in Proposition 7.22 and the paragraph before it.

Proposition 7.23. *There exists* $\beta_4'' > 0$ *such that*

$$|F_{N-1} D_z T^{j_{N-1}} v|_{J'} \geq \beta_4'' |F_0 D_z T^{j_0} v|_{J'} \qquad \textit{for } v \in C_z(z).$$

Proof. The inequality holds trivially if $N = 1$. Therefore, we assume that $N > 1$. Given a matrix L, denote by $|L|$ the matrix obtained by replacing each entry of L with its absolute value. Given two square matrices L_1 and L_2 of the same order, we write $L_1 \geq L_2$ if each entry of L_1 is greater than or equal to the corresponding entry of L_2. Let $\zeta = \zeta_1 \cdots \zeta_{N-1} > 0$. By the properties the matrices A_k and B_k, it follows that

$$\begin{aligned} |A_{N-1} B_{N-2} \cdots A_1 B_0| &= |A_{N-1}||B_{N-2}| \cdots |A_1||B_0| \\ &\geq |A_{N-1}||A_{N-2}| \cdots |A_1||B_0| \\ &\geq \begin{pmatrix} 1/\zeta_{N-1} & 0 \\ 0 & \zeta_{N-1} \end{pmatrix} \cdots \begin{pmatrix} 1/\zeta_1 & 0 \\ 0 & \zeta_1 \end{pmatrix} \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1/\zeta & \lambda/\zeta \\ 0 & \zeta \end{pmatrix}. \end{aligned}$$

Let $v \in C_z(z)$. Also, define $v_k = D_zT^kv$ for $0 \le k \le m$, $w_0 = F_0 v_{j_0}$ and $w_1 = F_{N-1}v_{j_{N-1}}$. By Proposition 7.22, we have

$$w_1 = A_{N-1}B_{N-2}\cdots B_1A_1B_0w_0.$$

From the construction of the cone field $\{(U_y, C_y)\}_{y\in E}$ and the definition of F_k, we easily see that $0 \le J(w_i)/J'(w_i) \le f$ for $i = 0, 1$, where f is the positive constant in the proof of Lemma 7.2. The fact that $J(w_i)$ and $J'(w_i)$ have the same sign implies that

$$|J(w_1)| \ge \frac{1}{\zeta}|J(w_0)| + \frac{\lambda}{\zeta}|J'(w_0)| \quad \text{and} \quad |J'(w_1)| \ge \zeta|J'(w_0)|. \tag{21}$$

From $0 \le J(w_i)/J'(w_i) \le f$ and the inequality on the left hand-side of (21), we obtain

$$|J'(w_1)| \ge \frac{1}{f}|J(w_1)| \ge \frac{\lambda}{\zeta f}|J'(w_0)|,$$

which together with the inequality on the right-hand sides of (21) gives

$$|J'(w_1)| \ge \frac{1}{2}\left(\zeta + \frac{\lambda}{\zeta f}\right)|J'(w_0)|.$$

But $x + c^2/x \ge 2c$ for every $x > 0$ and every $c \in \mathbb{R}$, and so

$$|J'(w_1)| \ge \left(\frac{\lambda}{f}\right)^{1/2}|J'(w_0)|.$$

The wanted inequality holds with $\beta_4'' = (\lambda/f)^{1/2}$. □

Let $z, \ldots$ as before Proposition 7.22.

Lemma 7.24. *Let $\beta_4'' > 0$ be the constant in Proposition 7.23. Then*

$$\|D_zT^{j_{N-1}}v\| \ge \frac{\beta_4''}{\alpha_1\alpha_2\alpha_4}\|D_zT^{j_0}v\| \qquad \text{for } v \in C_z(z).$$

Proof. By Lemmas 7.2, 7.3 and 7.5, it suffices to prove the inequality with $\|\cdot\|$ replaced by $|\cdot|_{J'}$. From the form of the matrices F_k, we see at once that $|v_{j_0}|_{J'} = |w_0|_{J'}$ and $|v_{j_{N-1}}|_{J'} = |w_1|_{J'}$. The notation here is as in the proof of Proposition 7.23. The wanted conclusion now follows from Proposition 7.23. □

Proposition 7.25. *There exists $\beta_4' > 0$ such that every block of type 4 satisfies* (12) *with $\beta' = \beta_4'$.*

Proof. The notation is an in the proof of Proposition 7.22 and the paragraph before it. Accordingly, a block φ of type 4 is given by $\varphi = \{z_{i_0}, \ldots, z_{i_N}\}$. We write $\varphi = \varphi_1 \cup \varphi_2$, where $\varphi_1 = \{z_{i_0}, \ldots, z_{j_{N-1}}\}$ and $\varphi_2 = \{z_{j_{N-1}}, \ldots, z_{i_N}\}$. It is easy to see that there are 2 integers $j_{N-1} < k_1 \le k_2 \le i_N$ such that $\varphi_2 = \psi_1 \cup \psi_2 \cup \psi_3$ with $\psi_1 = \{z_{j_{N-1}}, \ldots, z_{k_1}\}$ being an orbit starting at $z_{j_{N-1}} \in \mathcal{I}(E^+)$ and ending at M^- after a sequence of collisions with M^0, $\psi_2 = \{z_{k_1}, \ldots, z_{k_2}\}$ being a block of type 2, and $\psi_3 = \{z_{k_2}, \ldots, z_{i_N}\}$ being a block of type 3.

Proposition 7.15 and Lemma 7.24 give $\|v_{j_0}\| \geq \gamma_1 \|v_{i_0}\|$ and $\|v_{j_{N-1}}\| \geq \beta_4'' \|v_{j_0}\|$, respectively. By Lemma 7.18, we obtain $\|v_{k_1}\| \geq \delta_1 \|v_{j_{N-1}}\|$ with δ_1 independent of ψ_1 and v_0. Since ψ_2 and ψ_3 are blocks of type 2 and 3, we have $\|v_{k_2}\| \geq \beta_2' \|v_{k_1}\|$ and $\|v_{i_N}\| \geq \beta_3' \|v_{k_2}\|$ by Lemmas 7.20 and 7.21. Putting all together, we obtain the wanted conclusion with $\beta_4' = \beta_2' \beta_3' \beta_4'' \gamma_1 \delta_1 / (\alpha_1 \alpha_2 \alpha_4)$. □

Corollary 7.26. *If $T^m z \in E$, then* (12) *is satisfied.*

Proof. If $T^m x \in E$, then (12) follows from Lemmas 7.19-7.21 and Proposition 7.25. □

Proposition 7.27. *There exists a constant $\gamma_2 > 0$ such that if $z \in E \setminus R_m^+$ with $m > 1$, $\{T^{n(z)+1} z, \ldots, T^{m-1} z\} \subset M^0$, $T^j z \in S_1^-(\epsilon_0)$ for some $n(z) < j < m$, and $T^m z \in E$, then*

$$\|D_z T^m v\| \geq \gamma_2 \|D_z T^j v\| \quad \textit{for } v \in C_z(z).$$

Proof. Let $0 \neq v \in C_z(z)$, and define z_k and v_k for $0 \leq k \leq m$ as in the proof of Lemma 7.13. We study separately the two cases: $J(v_j) J'(v_j) \geq 0$ and $J(v_j) J'(v_j) < 0$. By Lemmas 7.3 and 7.7, it is enough to prove the desired inequality with $\| \cdot \|$ replaced by $\| \cdot \|_J$.

Suppose that $J(v_j) J'(v_j) \geq 0$. By Lemma 7.16, we have $|J'(v_m)| > \delta |J'(v_j)|$, and by arguing as in the proof of Lemma 7.17, one can easily prove that $|J(v_m)| \geq |J(v_j)|$. It follows at once that there exists $c > 0$ independent of the sequence $\{z, \ldots, T^m z\}$ as in the hypotheses of the proposition and $v \in C_z(z)$ such that $\|v_m\|_J \geq c \|v_j\|_J$.

Suppose now that $J(v_j) J'(v_j) < 0$. This means that v_j is focusing ($\tau^+(z_j, v_j) > 0$), and so $z \in M^+$. By Theorem 4.1, we have $\tau^+(z_{n(z)}, v_{n(z)}) \leq f$. Recall that $f = \max_i f_i$ (see the paragraph before Lemma 7.16). From $z_k \in M^0$ for $n(z) < k \leq j$, it follows that $|J'(v_j)| = |J'(v_{n(z)})|$, and it is not difficult to see that $|J(v_j)| \leq |J(v_{n(z)})| \leq f |J'(v_j)|$. Thus $\|v_j\|_J \leq \delta_1 |v_j|_{J'}$, where $\delta_1 = (1+f^2)^{1/2}$. On the other hand, by applying Lemma 7.16 to $\{z, \ldots, T^m z\}$, we obtain $|v_m|_{J'} \geq \delta |v_j|_{J'}$. Therefore, using the trivial fact $\|v_m\|_J \geq |v_m|_{J'}$, we obtain $\|v_m\|_J \geq \delta \delta_1^{-1} \|v_j\|_J$. This completes the proof, because δ and δ_1 do not depend on the sequence $\{z, \ldots, T^m z\}$ and $v \in C_z(z)$. □

Proposition 7.28. *The Non-contraction property is satisfied.*

Proof. Let $\varphi = \{z, \ldots, T^m z\}$ with $z \in E$ and $T^m z \in \mathcal{I}(E)$. Since $M^- \subset E$ and $\mathcal{I}(M^-) = M^-$, it is enough to prove the lemma for the case $T^m z \in \mathcal{I}(E^+)$. If $T^m z \in E^+ \cap \mathcal{I}(E^+)$, then the Non-contraction property is proved in Corollary 7.26. So, suppose that $T^m z \notin E^+$. There are two cases: $\varphi \subset M_i \subset M^+$ for some i, and $\varphi \not\subset M_i$ for every i such that $M_i \subset M^+$. In the first case, we have $z \in E_i$ and $m = n(z)$. By Proposition 7.15, it follows that (12) is satisfied with $\beta' = \gamma_1$. In the second case, we can write $\varphi = \varphi_1 \cup \varphi_2$, where $\varphi_1 = \{z, \ldots, T^k z\}$ with $z, T^k z \in E$, and $\varphi_2 = \{T^k z, \ldots, T^m z\}$ with $T^k z \in E^+$ and $T^m z \in$

$\mathcal{I}(E^+)$. Since φ_2 is an orbit of the type considered in the first case, we have $\|D_{T^kz}T^{m-k}v\| \geq \gamma_1\|v\|$ for $v \in C_{T^kz}(T^kz)$. Note that φ_1 is a piece of orbit as in (12) so that by Corollary 7.26, there exists $\beta_5' > 0$ independent of φ_1 such that $\|D_zT^kv\| \geq \beta_5'\|v\|$ for $v \in C_z(z)$. Hence, we see again that (12) is satisfied with constant $\beta_5'\gamma_1$. This together with the fact that β_5' and γ_1 do not depend on φ completes the proof. $\square$

7.4. **Conclusion of the proof of Condition L4.** Since the billiard map T is time-reversible, it is easy to check that the stable part of L4 for x is equivalent to the unstable part of L4 but with O replaced by $\mathcal{I}(O)$. Also, note that by Proposition 5.3, the invariant set $\Lambda \subset M$ where T admits a local stable and an unstable manifold has full measure. For the definition of these manifolds, see Proposition A.3. The local stable manifold (resp. local unstable manifold) of $y \in \Lambda$ is denoted by $V^s(y)$ (resp. $v^u(y)$).

In view of the time-reversibility of T, we have $\mathcal{I}(V^s(y)) = V^u(\mathcal{I}(y))$ and $\mathcal{I}(V^u(y)) = V^s(\mathcal{I}(y))$ for every $y \in \Lambda$. Hence $\Lambda = \mathcal{I}(\Lambda)$. From these considerations, it follows that the unstable part of L4 with O replaced by $E \cup \mathcal{I}(E)$, and Λ_x replaced by Λ implies the full L4 (stable and unstable parts).

We will also need the following technical lemma. Its proof is exactly as the one of [14, Lemma 3.6]. The sets $W^s(y)$ and $W^u(y)$ are defined in Definition A.5.

Lemma 7.29. *The set Λ can be chosen so that it satisfies the following property: if $y \in \Lambda \cap E$ and $z \in W^u(y)$ (resp. $z \in W^s(y)$), then $T_zW^u(y) \subset C_y(y)$ (resp. $T_zW^u(y) \subset C_y'(y)$), where $\{(U_y, C_y)\}_{y\in E}$ is the family of cone fields defined in Subsection 4.3.*

Proposition 7.30. *Every point $x \in H$ satisfies Condition L4.*

Proof. Suppose that $x \in H$, and let (l, N, O, K) be the quadruple of x as specified at the beginning of Subsection 6.2. Also, let ϵ_0 as in Lemma 7.6. We will prove the unstable part of L4 with O replaced by $E \cup \mathcal{I}(E)$, Λ_x replaced by Λ and $\epsilon = \epsilon_0$. As explained at the beginning of this subsection, this implies L4.

Suppose that $y \in \Lambda \cap (E \cup \mathcal{I}(E))$, $z \in (E \cup \mathcal{I}(E)) \cap W^u(y) \cap T^kS_1^-(\epsilon_0)$ for some $k > 0$. We study separately the cases $T^{-k}z \in E$ and $T^{-k}z \notin E$. For simplifying the notation, we will write $D_{i,j}$ for the restriction of $D_{T^iz}T^j$ to the tangent subspace of $W^u(T^iy)$ at T^iz with $i, j \in \mathbb{Z}$.

Before continuing with the proof, we need to make a remark. First, that the cone field (O, K) is equal to (U_y, C_y) for a proper $y \in E$. Then, note that by Lemma 7.29 and the invariance of the cone fields $\{(U_y, C_y) \colon y \in E\}$, the tangent space of any unstable manifold considered in this proof at a point $z \in E$ is always contained in the cone $C_z(z)$. This property is essential for applying Propositions 7.27 and

7.28, and will be used implicitly in this proof every time that one of the two propositions is applied.

We can now resume the proof. First, suppose that $T^{-k}z \in E$. Using Proposition 7.28, we immediately obtain that $\|D_0^{-k}\| \leq 1/\beta'$, where β' is the constant appearing in (12).

Now, suppose that $T^{-k}z \notin E$. Then, there are 2 possibilities, either $T^{-k}z \in M^0 \cap S_1^-(\epsilon_0)$ or $T^{-k}z \in M^+ \setminus E$. We analyze each possibility individually. First, define $m_\mp = \inf\{i > 0 \colon T^{-k\mp i}z \in E\}$. Assume that $T^{-k}z \in M^0 \cap S_1^-(\epsilon_0)$. Applying Proposition 7.27 to $T^{-k-m_-}z$ with $m = m_- + m_+$ and $j = m_-$, we obtain $\|D_{-k+m_+,-m_+}\| \leq 1/\gamma_2$. Also, note that $\|D_{0,-k-m_+}\|$ is trivially equal to 1 if $m_+ = k$, and is not larger than $1/\beta'$ if $m_+ < k$ by Proposition 7.28. Combining the previous observations, we conclude that $\|D_{0,-k}\| \leq 1/(\gamma_2 \min\{1, \beta'\})$.

Finally, assume that $T^{-k}z \in M^+ \setminus E$. Let $n = n(T^{-k}z)$. By breaking the sequence $\{-k, \dots, 0\}$ into $\{-k, \dots, -k+n\}$, $\{-k+n, \dots, -k+m_+\}$ and $\{-k+m_+, \dots, 0\}$, we see that

$$D_{0,-k} = D_{-k+n,-n} \circ D_{-k+m_+,-m_++n} \circ D_{0,-k+m_+}.$$

By Proposition 7.28, Lemma 7.18 and Proposition 7.15, we then obtain $\|D_0^{-k+m_+}\| \leq 1/\beta'$, $\|D_{-k+m_+,-m_++n}\| \leq 1/\delta_1$ and $\|D_{-k+n,-n}\| \leq 1/\gamma_1$. Combining these inequalities, we conclude that $\|D_{0,-k}\| \leq 1/(\beta'\gamma_1\delta_1)$.

To complete the proof, we observe that the upper bounds of $\|D_{0,-k}\|$ for the different cases studied above do not depend on the data y, z, k of Condition L4. □

8. Donnay billiards and their generalizations

We now apply Theorem 5.6 to a large class of hyperbolic billiards. The purpose of this section is to provide concrete examples of new billiards with the Bernoulli property. For the definition of Bernoulli component, see Appendix A.

Definition 8.1. *A polygon with pockets and bumps is a planar domain Ω obtained from a polygon P by replacing a sufficiently small neighborhood of each vertex with internal angle less than π with a focusing arc (pocket) or a dispersing arc (bump), and of each vertex with internal angle greater than π by a dispersing arc so that the vertex lies outside the closure of Ω (see Fig. 2).*

Stadium-like domains with either parallel or non-parallel segments can be considered degenerate polygons with pockets. These domains correspond to quadrilaterals with a pocket or a bump replacing two vertexes instead of one. The results proved in this subsection apply also to billiards in stadium-like domains.

Note that polygons with pockets and bumps satisfy B1 by definition. In the rest of this section, we will assume that billiards in polygons

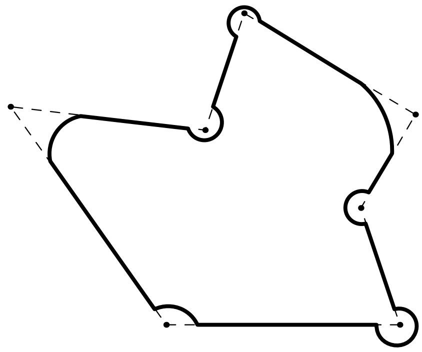

FIGURE 2. Polygon with pockets and bumps (solid curve) and original polygon (dashed curve).

with pockets and bumps satisfies Condition B2. This can achieved, for example, by using sufficiently short focusing arcs as pockets.

Proposition 8.2. *If a billiard in a polygon with pockets and bumps satisfies Condition B2, then it satisfies Conditions B3 and B4 as well. In particular, T has non-zero Lyapunov exponents a.e. on M.*

Proof. A general result for polygonal billiards states that every semi-orbit of a polygon billiard is either periodic or accumulates at least at one vertex of the polygon [18]. From this result and the assumption on the geometry of Ω, it follows that if $x \in N^- \cup N^+$, then x has to be periodic. Hence, $N^- = N^+ = N$ consists of periodic orbits. In a polygonal billiard, every periodic orbit is contained in a family of parallel orbits having the same period. This family consists of finitely many segments contained in horizontal segments ($\theta = const$) with endpoint belonging to R. Since the number of distinct strips in a polygonal billiard is countable, we immediately obtain $\mu(N) = 0$. Hence, B3 is satisfied. The fact that N consists of periodic orbits implies also that $N' = \emptyset$. Thus, $S_1^- \cap N^+ \subset N' = \emptyset$, and so Condition B4 is satisfied as well.

The second part of the proposition follows from Proposition 5.3. □

Chernov and Troubetzkoy proved that a billiard in convex polygons with pockets Ω is ergodic if the pockets are arcs of circles with the full circles contained in Ω [11]. Here, we consider the more general situation of polygons with pockets and bumps. The condition in Definition 8.1 that each vertex with internal angle greater than π is replaced by a dispersing arc so that the vertex lies outside the closure of the polygon plays a crucial role in the proof of the next theorem.

Theorem 8.3. *The map T of a billiard in a polygon with pockets and bumps that satisfies Condition B2 has the Bernoulli property.*

Proof. Conditions B3 and B4 are satisfied by Proposition 8.2. By Theorem 5.6, every point of H has neighborhood contained (mod 0) in one

Bernoulli component. Now, suppose that $M_i \subset M^- \cup M^+$. Since R is countable (see [15, Propositions 6.17-6.19]), $N' = \emptyset$ and $N \cap M_i = \emptyset$, the set $H \cap M_i$ is connected. Hence, every set $M_i \subset M^- \cup M^+$ is contained (mod 0) in one Bernoulli component. We cannot claim the same for sets $M_i \subset M^0$, because in this case[5], N may disconnect $H \cap M_i$. Thus, to prove that T is Bernoulli, we cannot use Corollary 5.8.

The alternative approach that we take is quite simple: we prove that the entire set $M^- \cup M^+$ is contained (mod 0) in one Bernoulli component. This together with $\mu(N) = 0$ implies that T is Bernoulli. Indeed, by Theorem A.6, the number of Bernoulli components of T is equal to the minimum integer $n > 0$ such that $T^n B = B$ for some Bernoulli component B. By the geometry of Ω, it is always possible to find two distinct curved components Γ_i and Γ_j such that $\mu(M_j \cap T M_i) > 0$. Thus, if B is the Bernoulli component containing $M^- \cup M^+$, then $TB = B$, and so $n = 1$.

To prove that $M^- \cup M^+$ is contained (mod 0) in a single Bernoulli component, we start by observing that the polygon P decomposes into finitely many triangles with pairwise disjoint interiors. The fact that this decomposition is not unique is irrelevant for our purposes. Let Δ be one of triangles of the decomposition. By construction of Ω, every vertex with internal angle greater than π lies outside Ω. Then, we claim that the union of the sets M_i corresponding to the pockets and bumps attached to the vertexes of Δ are contained (mod 0) in the same Bernoulli component. In fact, let Γ_i, Γ_j, Γ_k be the pockets or bumps attached at the vertexes of Δ. If B is the Bernoulli component containing M_i, then by Theorem A.6 states that there exists a Bernoulli component B' such that $TB = B'$. It is easy to see that $\mu(M_j \cap TM_i) > 0$ and $\mu(M_k \cap TM_i) > 0$. Thus, M_j and M_k must belong to B'. By symmetry of the configuration, also M_i and M_j belong the same Bernoulli component. We can then conclude that M_i, M_j, M_k belong to the same Bernoulli component. In a stadium-like domain, the previous argument does not work, because there are only 2 curved components Γ_i and Γ_j attached to the 3 vertexes of Δ. However, in this case, we have i) $\mu(M_j \cap TM_i) > 0$ and $\mu(M_j \cap TM_i) > 0$, and ii) $\mu(M_j \cap T^2 M_i) > 0$. Claim i) is obvious. Claim ii) follows from the fact that there is a positive measure set of orbits starting at M_i and reaching M_j after one collision with flat components of $\partial\Omega$. Since each M_i and M_j is contained in a Bernoulli component, and all the Bernoulli components are cyclically permuted with the same period $n > 0$ by Theorem A.6, claim i) implies that $n \leq 2$, whereas claim ii) implies that n is either 3 or one of its divisors. Hence $n = 1$. The same conclusion is clearly true for all the sets M_i corresponding to pockets and bumps attached to vertexes of two adjacent triangles Δ_1

[5]This is indeed the case whenever the polygon P has two parallel sides facing each other.

and Δ_2. Hence, the entire set $M^- \cup M^+$ belongs (mod 0) to a Bernoulli component. □

Donnay billiards are billiards in convex polygons with pockets [16]. The next result then follows directly from Theorem 8.3.

Corollary 8.4. *Donnay billiards have the Bernoulli property.*

The conclusion of Theorem 8.3 is robust for sufficiently small perturbations of the pockets and bumps.

Definition 8.5. *Two polygons with pockets and bumps Ω_1 and Ω_2 are ϵ-close if Ω_2 is obtained from Ω_1 by replacing each pocket (resp. bump) with another pocket (resp. bump) of the same length, ϵ-close in the C^6-topology (resp. C^3-topology) and ϵ-close in the Hausdorff distance.*

Proposition 8.6. *Suppose that the billiard in a polygon with pockets and bumps Ω_1 satisfies B2. Then there exists $\epsilon > 0$ such that if Ω_2 is ϵ-close to Ω_1, then the billiard in Ω_2 is Bernoulli.*

Proof. Condition B2 is an open condition. Thus, if ϵ is sufficiently small, then the billiard in Ω_2 satisfies B2, and so it is Bernoulli by Theorem 8.3. □

We conclude this section with a result concerning hyperbolic billiards with domains for which $\Gamma^0 = \emptyset$, i.e., without straight boundary components. For these billiards, the Bernoulli property follows quite directly from Corollary 5.8. We observe that Sinai billiards [28] – those with domains bounded only by strictly convex outwards arcs – belong to this class of billiards. For them, the Bernoulli property was first proved in [17].

Theorem 8.7. *Let Ω be a billiard domain without straight boundary components, and suppose that its map T satisfies Conditions B1-B4. Then T is Bernoulli.*

Proof. For billiards satisfying the hypotheses of the theorem, the set R is countable (see [15, Propositions 6.17-6.19]), and we trivially have $N = NR = \emptyset$. Thus, each set $H \cap M_i$ is connected, and B5 is satisfied. The wanted conclusion now follows from Corollary 5.8. □

Appendix A. Local Ergodic Theorem

We state the Local Ergodic Theorem proved in [14], and recall the relevant definitions. The formulation of this theorem in its general form requires a series of technical definitions, which are not needed for 2-dimensional billiards. Thus, to avoid unnecessary technicalities, we specialize the presentation of the Local ergodic Theorem to 2-dimensional billiards. Accordingly, T will denote the billiard map for some planar domain Ω throughout this appendix.

The definition of a cone field and related notions are given in Section 4.

Definition A.1. *A point $x \in M \setminus \partial M$ is called* sufficient *if there exist*

i) an integer l such that T^l is a local diffeomorphism at x,

ii) a neighborhood O of $T^l x$ and an integer $N > 0$ such that O and R^-_N are disjoint,

iii) an invariant continuous cone field K on $O \cup T^{-N}O$ such that $\sigma_K(D_y T^N) > 3$ for every $y \in T^{-N}O$.

Every time we need to emphasize the role of the data l, N, O, K in this definition, we will write that x is a sufficient point with quadruple (l, N, O, K).

Definition A.2. *A point $x \in M \setminus \partial M$ is called* u-essential *if for every $\alpha > 0$, there exist $n_{x,\alpha} \in \mathbb{N}$, a neighborhood $O_{x,\alpha}$ of x with $O_{x,\alpha} \cap R^+_{n_{x,\alpha}} = \emptyset$ and a continuous invariant cone field $(O_{x,\alpha} \cup T^{n_{x,\alpha}} O_{x,\alpha}, K_{x,\alpha})$ such that $\sigma^*_{K_{x,\alpha}}(D_y T^{n_{x,\alpha}}) > \alpha$ for every $y \in O_{x,\alpha}$. Analogously, a point $x \in M \setminus \partial M$ is called* s-essential *if, in the previous definition, T and $R^+_{n_{x,\alpha}}$ are replaced by T^{-1} and $R^-_{n_{x,\alpha}}$, respectively.*

The cone field (O, K) in Definition A.1 is eventually strictly invariant. By a well-known result [23, 31, 32], it follows that all the Lyapunov exponents of T are non-zero a.e. on the set $\bigcup_{k \in \mathbb{Z}} T^k O$. This fact combined with the Katok-Strelcyn theory [20] gives Proposition A.3 below (Part (3) is proved in [14, Proposition 5.3]). For the definition of absolute continuity of a foliation, we refer the reader to [10, 20].

Proposition A.3. *Let $x \in M \setminus \partial M$ be a sufficient point with quadruple (l, N, K, O). Then there exist an invariant set $\Lambda_x \subset \bigcup_{k \in \mathbb{Z}} T^k O$ with $\mu(\Lambda_x) = \mu(\bigcup_{k \in \mathbb{Z}} T^k O) > 0$ and two families $V^s = \{V^s(y)\}_{y \in \Lambda_x}$ and $V^u = \{V^u(y)\}_{y \in \Lambda_x}$ consisting of C^2 submanifolds such that for every $y \in \Lambda_x$, we have*

(1) $V^s(y) \cap V^u(y) = \{y\}$,

(2) $V^s(y)$ and $V^u(y)$ are embedded open intervals,

(3) $T_y V^s(y) \subset K'(y)$ and $T_y V^u(y) \subset K(y)$ provided that $y \in O \cup T^{-N}O$,

(4) $TV^s(y) \subset V^s(Ty)$ and $T^{-1}V^u(y) \subset V^u(T^{-1}y)$,

(5) $d(T^n y, T^n z) \to 0$ exponentially as $n \to +\infty$ for every $z \in V^s(y)$, and the same is true as $n \to -\infty$ for every $z \in V^u(y)$,

(6) $V^s(y)$ and $V^u(y)$ vary measurably with $y \in \Lambda_x$,

(7) the families V^s and V^u have the absolute continuity property.

Definition A.4. *The submanifolds forming the families V^s and V^u are called* local stable manifolds *and* local unstable manifolds*, respectively.*

Definition A.5. *Let x be a sufficient point of $M \setminus \partial M$, and let Λ_x be the set in Proposition A.3. For every $y \in \Lambda_x$, we denote by $W^u(y)$ the connected component of*

$$\bigcup_{k \geq 0} T^k V^u(T^{-k} y)$$

containing y. Analogously, denote by $W^s(y)$ the set obtained by replacing T with T^{-1} and V^u with V^s in the definition of $W^u(y)$.

We now recall the Spectral Decomposition Theorem. It applies to a larger class of hyperbolic system with singularities than billiards, but here we formulate it only for billiards. For its proof, one has to combine two results: [20, Theorem 13.1, Part II], which extends Pesin'result for smooth systems [27] to systems with singularities, and [9, Theorem 3.1]. See also [26], for results similar to those of [9].

Theorem A.6. *Suppose that the billiard map T has non-vanishing Lyapunov exponents a.e. on M. Then there exist at most countably many pairwise disjoint measurable subsets $E_0, E_1, \dots$ of M such that*

(1) $M = \bigcup_i E_i$,
(2) $\mu(E_0) = 0$, and $\mu(E_i) > 0$ for every i,
(3) $TE_i = E_i$, and $(T|_{E_i}, \mu|_{E_i})$ is ergodic for every i,
(4) for every i, there exist $m_i \in \mathbb{N}$ pairwise disjoint measurable subsets $B_{i,1}, \dots, B_{i,m_i}, B_{i,m_i+1} = B_{i,1}$ of M such that $E_i = \bigcup_{j=1}^{m_i} B_{i,j}$, $TB_{i,j} = B_{i,j+1}$ and $(T^{m_i}|_{B_{i,j}}, \mu|_{B_{i,j}})$ is Bernoulli for every $j = 1, \dots, m_i$.

The sets E_i and $B_{i,j}$ are called an *ergodic component* of T and a *Bernoulli component* of T, respectively. These sets are uniquely defined up to a set of zero measure.

We need one last definition before formulating the Local Ergodic Theorem.

Definition A.7. *Let (O_1, C_1) and (O_2, C_2) be two cone fields. We say that (O_1, C_1) and (O_2, C_2) are* jointly invariant *if $D_xT^kC_1(x) \subset C_2(T^kx)$ for every $x \in O_1$ and $k > 0$ such that $T^kx \in O_2$, and $D_xT^kC_2(x) \subset C_1(T^kx)$ for every $x \in O_2$ and $k > 0$ such that $T^kx \in O_1$.*

Note that in the previous definition, we neither require that the sets O_1 and O_2 are disjoint nor that the cone fields C_1 and C_2 are invariant. However, it is easy to see that C_1 and C_2 are invariant in the following sense: if $x \in O_1$ and $k_2 > k_1 > 0$ such that $T^{k_1}x \in O_2$ and $T^{k_2}x \in O_1$, then $D_xT^{k_2}C_1(x) \subset C_1(T^{k_2}x)$. The same is true for C_2, once O_1 has been replaced by O_2.

Definition A.8. *A subset $\Sigma \subset M$ is called* regular *if it is a union of finitely many arcs $\Sigma_1, \dots, \Sigma_k$ of class C^2 that can only intersect at their boundaries. The arcs $\Sigma_1, \dots, \Sigma_k$ are called the* components *of Σ.*

Definition A.9 (Regularity). *We say that T satisfies Condition L1 if the singular sets R_k^+ and R_k^- are regular for every $k > 0$.*

In the rest of this subsection, we assume that $x \in M \setminus \partial M$ is a sufficient point with quadruple (l, N, O, K). Let Λ_x be the subset associated to x as in Proposition A.3.

Definition A.10 (Alignment)**.** *We say that x satisfies Condition L2 if the sets $O\cap R_k^+$ and $O\cap R_k^-$ are regular for every $k>0$, and the tangent subspace*[6] *$T_y\Sigma$ is contained in $K(y)$ (resp. $K'(y)$) for every $k>0$, every component Σ of $O\cap R_k^-$ (resp. $O\cap R_k^+$) and every $y\in\Sigma\cap T^{-N}O$ (resp. $\Sigma\cap O$).*

Let m_+ and m_- be the 1-dimensional Riemannian volume on S_1^+ and S_1^-), respectively.

Definition A.11 (Ansatz)**.** *We say that x satisfies Condition L3 if the set of u-essential points of S_1^- (resp. s-essential points of S_1^+) has full m_--measure (resp. m_+-measure), and if y is any of such points, then the cone fields (O,K) and $(O_{y,\alpha},K_{y,\alpha})$ are jointly invariant for every $\alpha>0$.*

Given $A\subset M$ and $\epsilon>0$, we call the set $A(\epsilon)=\{x\in M : d(x,A)<\epsilon\}$ the *ϵ-neighborhood* of A.

Definition A.12 (Contraction)**.** *We say that x satisfies Condition L4 if there exist $\beta>0$ and $\epsilon>0$ such that*

$$\left\|D_zT^{-k}|_{T_zW^u(y)}\right\|\le\beta\quad\big(\textit{resp. }\left\|D_zT^{k}|_{T_zW^s(y)}\right\|\le\beta\big)$$

for every $y\in O\cap\Lambda_x$ and every $z\in O\cap W^u(y)\cap T^kS_1^-(\epsilon)$ (resp. $O\cap W^s(y)\cap T^{-k}S_1^+(\epsilon)$) with $k>0$.

Theorem A.13 (Local Ergodic Theorem)**.** *Suppose that L1 is satisfied, and that $x\in M\setminus\partial M$ is a sufficient point satisfying L2-L4. Then there exists a neighborhood of x contained up to a set of zero μ-measure in a Bernoulli component of T.*

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