Wijsman Strongly p-Cesàro Summability and Wijsman Statistical Convergence of Order α for Double Set Sequences

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Abstract

The concept of statistical convergence was introduced by Steinhaus (1951) and Fast (1951), and later reintroduced by Schoenberg (1959) independently. Then, many researchers have studied on this concept until recently (see Connor 1988; Fridy 1985; Šalát 1980; Tripathy 1998). The order of statistical convergence of a single sequence of numbers was given by Gadjiev and Orhan (2001). Then, the concepts of statistical convergence of order α and strongly p-Cesàro summability of order α were studied by Çolak (2010) and Çolak and Bektaş (2011).

In 1900, Pringsheim introduced the concept of convergence for double sequences. Recently, Mursaleen and Edely (2003) extended this concept to statistical convergence. More developments on double sequences can be found in Çakallı and Savaş (2010), Mohiuddine et al. (2012) and Bhunia et al. (2012). Very recently, the concepts of statistical convergence of order α and strongly p-Cesàro summability of order α for double sequences were studied by Savaş (2013) and Çolak and Altın (2013).

The concept of convergence for number sequences was transferred to the concepts of convergence for set sequences by many authors. In this study, the concept of Wijsman convergence which is one of these transfers is considered (see Baronti and Papini 1986; Beer 1985, 1994; Wijsman 1964). Nuray and Rhoades (2012) extended the concept of Wijsman convergence to statistical convergence for set sequences and gave some basic theorems. Very recently, the concept of Wijsman $\mathcal{I}$-statistical convergence of order α was studied by Savaş (2015) and Şengül and Et (2017).

Nuray et al. (2014) introduced the concepts of Wijsman convergence and Wijsman strongly p-Cesàro summability for double set sequences. Also, the concept of Wijsman statistical convergence was studied by Nuray et al. (2019).

In this study, we introduce the concepts of Wijsman strongly p-Cesàro summability of order α and Wijsman statistical convergence of order α for double set sequences. Also, we investigate some properties of these concepts and examine the relationship between them.

Keywords: Statistical convergence, Cesàro summability, double sequence, order α, Wijsman convergence, Set sequences.

Öz

İstatistiksel yakınsaklık kavramı Steinhaus (1951) ve Fast (1951) tarafından tanıtılmış, ve daha sonra bağımsız olarak Schoenberg (1959) tarafından yeniden tanımlanmıştır. Yakın zamana kadar pek çok araştırmacı da bu kavram üzerine çalışmıştır (bkz Connor 1988; Fridy 1985; Šalát 1980; Tripathy 1998). Reel sayı dizilerinin istatistiksel yakınsaklık mertebesi Gadjiev ve Orhan (2001) tarafından verilmiştir. Daha sonra Çolak (2010) ve Çolak ve Bektaş (2011) tarafından α. mertebeden istatistiksel yakınsaklık ve α. mertebeden kuvvetli p-Cesàro toplanabilirlik kavramları çalışılmıştır.

Pringsheim 1900 de çift diziler için yakınsaklık kavramını tanıtmıştır. Mursaleen ve Edely (2003) bu kavramı istatistiksel yakınsaklığa genişletmiştir. Çift diziler üzerine yapılan pek çok çalışma Çakallı ve Savaş (2010), Mohiuddine vd. (2012) ve Bhunia vd. (2012) de bulunabilir. Son zamanlarda, çift diziler için α. mertebeden istatistiksel yakınsaklık ve α. mertebeden kuvvetli p-Cesàro toplanabilirlik kavramları Savaş (2013) ve Çolak ve Altın (2013) tarafından çalışılmıştır.

Sayı dizileri için yakınsaklık kavramı pek çok araştırmacı tarafından küme dizileri için yakınsaklık kavramlarına aktarılmıştır. Bu çalışmada, küme dizileri için Wijsman yakınsaklık kavramı ele alınmıştır (bkz Baronti ve Papini 1986; Beer 1985, 1994; Wijsman 1964). Nuray ve Rhoades (2012) küme dizileri için Wijsman istatistiksel yakınsaklık kavramını çalışmış ve bazı temel teoremleri vermiştir. Son zamanlarda, α. mertebeden Wijsman I-istatistiksel yakınsaklık kavramı Savaş (2015) ve Şengül ve Et (2017) tarafından çalışılmıştır.

Nuray vd. (2014) çift küme dizileri için Wijsman yakınsaklık ve Wijsman kuvvetli p-Cesàro toplanabilirlik kavramlarını tanıtmıştır. Ayrıca, Wijsman istatistiksel yakınsaklık kavramı Nuray vd. (2019) tarafından incelenmiştir.

Bu çalışmada, çift küme dizileri için α. mertebeden Wijsman kuvvetli p-Cesàro toplanabilirlik ve α. mertebeden Wijsman istatistiksel yakınsaklık kavramları tanıtılmıştır. Ayrıca, bu kavramların bazı özellikleri araştırılmış ve bunlar arasındaki ilişki incelenmiştir.

Anahtar Kelimeler: İstatistiksel yakınsaklık, Cesàro toplanabilirlik, çift dizi, α. mertebe, Wijsman yakınsaklık, küme dizisi.

Introduction and Basic Concepts

Firstly, we recall the basic concepts that need for a good understanding of our study (see Baronti and Papini 1986; Beer 1985; Mursaleen and Edely 2003; Nuray et al. 2014, 2019, Pringsheim 1900, Wijsman 1964).

A double sequence (x_{ij}) is said to be convergent to L in Pringsheim's sense if for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{ij} - L| < \varepsilon$, whenever $i, j > N_\varepsilon$.

A double sequence (x_{ij}) is said to be statistically convergent to L if for every $\varepsilon > 0$,

$$\lim_{m,n\to\infty} \frac{1}{mn} \left|\left\{(i,j): i \leq m, j \leq n, \quad |x_{ij} - L| \geq \varepsilon\right\}\right| = 0.$$

Let X be any non-empty set. The function $f: \mathbb{N} \to P(X)$ is defined by $f(i) = U_i \in P(X)$ for each $i \in \mathbb{N}$, where $P(X)$ is power set of X. The sequence $\{U_i\} = (U_1, U_2, \dots)$, which is the range's elements of f, is said to be set sequences.

Let (X, d) be a metric space. For any point $x \in X$ and any non-empty subset U of X, the distance from x to U is defined by

$$\rho(x, U) = \inf_{u\in U} d(x, u).$$

Let (X, d) be a metric space and U, U_{ij} be any non-empty closed subsets of X. A double sequence $\{U_{ij}\}$ is said to be Wijsman convergent to U if for each $x \in X$,

$$\lim_{i,j\to\infty} \rho(x, U_{ij}) = \rho(x, U).$$

Let (X, d) be a metric space and U, U_{ij} be any non-empty closed subsets of X. A double sequence $\{U_{ij}\}$ is said to be Wijsman statistically convergent to U if for every $\varepsilon > 0$ and each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{mn} |\{(i,j): i \leq n, j \leq m, \quad |\rho(x, U_{ij}) - \rho(x, U)| \geq \varepsilon\}| = 0.$$

Let (X, d) be a metric space and U, U_{ij} be any non-empty closed subsets of X. A double sequence $\{U_{ij}\}$ is said to be Wijsman Cesàro summable to U if for each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} \rho(x, U_{ij}) = \rho(x, U).$$

Let $0 < p < \infty$, (X, d) be a metric space and U, U_{ij} be any non-empty closed subsets of X. A double sequence $\{U_{ij}\}$ is said to be Wijsman strongly p-Cesàro summable to U if for each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} |\rho(x, U_{ij}) - \rho(x, U)|^p = 0.$$

From now on, for short, we use $\rho_x(U)$ and $\rho_x(U_{ij})$ instead of $\rho(x, U)$ and $\rho(x, U_{ij})$, respectively.

Method

In the proofs of the theorems obtained from this study, the following proof methods, which are frequently used in mathematics, have been used as needed:

ii. Direct proof method,
ii. Inverse situation proof method,

iii. Method of non-finding (contradiction),

iv. Induction method.

Results

In this section, we introduce the concepts of Wijsman strongly p-Cesàro summability of order α and Wijsman statistical convergence of order α for double set sequences. Also, we investigate some properties of these concepts and examine the relationship between them.

Definition:

Let $0 < \alpha \leq 1$. A double sequence $\{U_{ij}\}$ is said to be Wijsman Cesàro summable of order α to U or $W(C_2^\alpha)$-convergent to U if for each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{(mn)^\alpha} \sum_{i,j=1,1}^{m,n} \rho_x(U_{ij}) = \rho_x(U).$$

In this case, we write $U_{ij} \overset{W(C_2^\alpha)}{\longrightarrow} U$ or $U_{ij} \longrightarrow U(W(C_2^\alpha))$.

The class of all $W(C_2^\alpha)$-convergent sequences will be denoted by simply $W(C_2^\alpha)$.

Definition:

Let $0 < \alpha \leq 1$ and $0 < p < \infty$. A double sequence $\{U_{ij}\}$ is said to be Wijsman strongly p-Cesàro summable of order α to U or $W[C_2^\alpha]^p$-convergent to U if for each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{(mn)^\alpha} \sum_{i,j=1,1}^{m,n} |\rho_x(U_{ij}) - \rho_x(U)|^p = 0.$$

In this case, we write $U_{ij} \overset{W[C_2^\alpha]^p}{\longrightarrow} U$ or $U_{ij} \longrightarrow U(W[C_2^\alpha]^p)$.

The class of all $W[C_2^\alpha]^p$-convergent sequences will be denoted by simply $W[C_2^\alpha]^p$.

If $p = 1$, then a double sequence $\{U_{ij}\}$ is simply said to be Wijsman strongly Cesàro summable of order α to U and we write $U_{ij} \overset{W[C_2^\alpha]}{\longrightarrow} U$ or $U_{ij} \longrightarrow U(W[C_2^\alpha])$.

Example:

Let $X = \mathbb{R}^2$ and a double sequence $\{U_{ij}\}$ be defined as following:

$$\{U_{ij}\} := \begin{cases} \left\{(x,y) \in \mathbb{R}^2 : x^2 + (y-1)^2 = \frac{1}{ij}\right\} & , \text{ if } i \text{ and } j \text{ are sequare integer,} \\ \{(1,0)\} & , \text{ otherwise.} \end{cases}$$

Then, the double sequence $\{U_{ij}\}$ is Wijsman strongly Cesàro summable of order α to the set $U = \{(1,0)\}$.

For $\alpha = 1$, the concepts of $W(C_2^\alpha)$-convergence and $W[C_2^\alpha]^p$-convergence coincides with the concepts of Wijsman Cesàro summable ($W(C_2)$-convergence) and Wijsman strongly p-Cesàro summable ($W[C_2]^p$-convergence) for double set sequences in Nuray (2014), respectively.

Theorem:

If $0 < \alpha \leq \beta \leq 1$, then $W[C_2^\alpha]^p \subset W[C_2^\beta]^p$.

If we take $\beta = 1$ in above Theorem, then we get $W[C_2^\alpha]^p \subset W[C_2]^p$.

Definition:

Let $0 < \alpha \leq 1$. A double sequence $\{U_{ij}\}$ is said to be Wijsman statistically convergent of order α to U or $W(S_2^\alpha)$-convergent to U if for every $\varepsilon > 0$ and each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{(mn)^\alpha} |\{(i,j): i \leq m, j \leq n, \quad |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon\}| = 0.$$

In this case, we write $U_{ij} \overset{W(S_2^\alpha)}{\longrightarrow} U$ or $U_{ij} \longrightarrow U(W(S_2^\alpha))$.

The class of all $W(S_2^\alpha)$-convergent sequences will be denoted by simply $W(S_2^\alpha)$.

For $\alpha = 1$, the concept of $W(S_2^\alpha)$-convergence coincides with the concept of Wijsman statistical convergence for double set sequences in Nuray et al. (…).

Theorem:

Let $0 < \alpha \leq 1$ and $0 < p < \infty$. If a double sequence $\{U_{ij}\}$ is Wijsman strongly p-Cesàro summable of order α to U, then the sequence is Wijsman statistically convergent of order α to U.

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