ASYMPTOTICALLY LACUNARY $\mathcal{I}$-INVARIANT EQUIVALENCE OF SEQUENCES DEFINED BY A MODULUS FUNCTION

NİMET P. AKIN, ERDİNÇ DÜNDAR AND UĞUR ULUSU

Abstract. In this paper, we introduce the concepts of strongly asymptotically lacunary ideal invariant equivalence, f-asymptotically lacunary ideal invariant equivalence, strongly f-asymptotically lacunary ideal invariant equivalence and asymptotically lacunary ideal invariant statistical equivalence for sequences. Also, we investigate some relationships among them.

1. Introduction

Throughout the paper $\mathbb{N}$ denotes the set of all natural numbers and $\mathbb{R}$ the set of all real numbers. The concept of convergence of a real sequence has been extended to statistical convergence independently by Fast [1], Schoenberg [24] and studied by many authors. The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [2] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of $\mathbb{N}$.

Several authors including Raimi [17], Schaefer [23], Mursaleen and Edely [7], Mursaleen [9], Savaş [18, 19], Nuray and Savaş [11], Pancaroğlu and Nuray [13] and some authors have studied invariant convergent sequences. The concept of strongly σ-convergence was defined by Mursaleen [8]. Savaş and Nuray [20] introduced the concepts of σ-statistical convergence and lacunary σ-statistical convergence and gave some inclusion relations. Nuray et al. [12] defined the concepts of σ-uniform density of a subset A of the set $\mathbb{N}$, $\mathcal{I}_\sigma$-convergence and investigated relationships between $\mathcal{I}_\sigma$-convergence and invariant convergence also $\mathcal{I}_\sigma$-convergence and $[V_\sigma]_p$-convergence. Pancaroḡlu and Nuray [13] studied Statistical lacunary invariant summability. Recently, Nuray and Ulusu [25] investigated lacunary $\mathcal{I}$-invariant convergence and lacunary $\mathcal{I}$-invariant Cauchy sequence of real numbers.

Marouf [6] peresented definitions for asymptotically equivalent and asymptotic regular matrices. Patterson [14] presented asymptotically statistical equivalent sequences for nonnegative summability matrices. Patterson and Savaş [15, 22] introduced asymptotically lacunary statistically equivalent sequences and also asymptotically $\sigma\theta$-statistical equivalent sequences. Ulusu [26, 27] studied asymptotically ideal invariant equivalence and asymptotically lacunary $\mathcal{I}_\sigma$-equivalence.

Modulus function was introduced by Nakano [10]. Maddox [5], Pehlivan [16] and many authors used a modulus function f to define some new concepts and inclusion theorems. Kumar and Sharma [3] studied lacunary equivalent sequences by ideals and modulus function.

Now, we recall the basic concepts and some definitions and notations (See [2, 4, 5, 6, 12, 14, 16]).

Let σ be a mapping of the positive integers into itself. A continuous linear functional φ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ mean, if and only if,

(1) $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,
(2) $\phi(e) = 1$, where $e = (1, 1, 1...)$,
(3) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in \ell_\infty$.

The mappings ϕ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the mth iterate of the mapping σ at n. Thus ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$. In case σ is translation mappings $\sigma(n) = n + 1$, the σ mean is often called a Banach limit and V_σ, the set of

2000 *Mathematics Subject Classification.* 40A99, 40A05.
Key words and phrases. Asymptotically equivalence, Lacunary invariant equivalence, $\mathcal{I}$-equivalence, Modulus function.

bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences. By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout the paper, we let θ a lacunary sequence.

The sequence $x = (x_k)$ is $S_{\sigma\theta}$-convergent to L, if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r}\Big|\big\{k \in I_r : |x_{\sigma^k(n)} - L| \geq \varepsilon\big\}\Big| = 0, \text{ uniformly in n=1,2,... .}$$

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if

(i) $\emptyset \in \mathcal{I}$, (ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, (iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

An ideal is called nontrivial if $\mathbb{N} \notin \mathcal{I}$ and nontrivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$. Throughout the paper we let $\mathcal{I}$ be an admissible ideal.

Let $A \subseteq \mathbb{N}$ and

$$s_m = \min_n \Big|A \cap \big\{\sigma(n), \sigma^2(n), ..., \sigma^m(n)\big\}\Big| \text{ and } S_m = \max_n \Big|A \cap \big\{\sigma(n), \sigma^2(n), ..., \sigma^m(n)\big\}\Big|.$$

If the limits $\underline{V}(A) = \lim_{m\to\infty} \frac{s_m}{m}$ and $\overline{V}(A) = \lim_{m\to\infty} \frac{S_m}{m}$ exist then, they are called a lower σ-uniform density and an upper σ-uniform density of the set A, respectively. If $\underline{V}(A) = \overline{V}(A)$, then $V(A) = \underline{V}(A) = \overline{V}(A)$ is called the σ-uniform density of A.

Denote by $\mathcal{I}_\sigma$ the class of all $A \subseteq \mathbb{N}$ with $V(A) = 0$.

A sequence $x = (x_k)$ is said to be $\mathcal{I}_\sigma$-convergent to L if for every $\varepsilon > 0$, the set $A_\varepsilon = \big\{k : |x_k - L| \geq \varepsilon\big\}$ belongs to $\mathcal{I}_\sigma$, i.e., $V(A_\varepsilon) = 0$. It is denoted by $\mathcal{I}_\sigma - \lim x_k = L$.

Let $\theta = \{k_r\}$ be a lacunary sequence, $A \subseteq \mathbb{N}$ and

$$s_r = \min_n \Big\{\big|A \cap \{\sigma^m(n) : m \in I_r\}\big|\Big\} \text{ and } S_r = \max_n \Big\{\big|A \cap \{\sigma^m(n) : m \in I_r\}\big|\Big\}.$$

If the limits $\underline{V_\theta}(A) = \lim_{r\to\infty} \frac{s_r}{h_r}$ and $\overline{V_\theta}(A) = \lim_{r\to\infty} \frac{S_r}{h_r}$ exist then, they are called a lower lacunary σ-uniform density and an upper lacunary σ-uniform density of the set A, respectively. If $\underline{V_\theta}(A) = \overline{V_\theta}(A)$, then $V_\theta(A) = \underline{V_\theta}(A) = \overline{V_\theta}(A)$ is called the lacunary σ-uniform density of A.

Denoted by $\mathcal{I}_{\sigma\theta}$ the class of all $A \subseteq \mathbb{N}$ with $V_\theta(A) = 0$.

A sequence (x_k) is said to be lacunary $\mathcal{I}_\sigma$-convergent or $\mathcal{I}_{\sigma\theta}$-convergent to L if for every $\varepsilon > 0$, $A_\varepsilon = \big\{k : |x_k - L| \geq \varepsilon\big\} \in \mathcal{I}_{\sigma\theta}$, i.e., $V_\theta(A_\varepsilon) = 0$. It is denoted by $\mathcal{I}_{\sigma\theta} - \lim x_k = L$.

The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if $\lim_k \frac{x_k}{y_k} = 1$ (denoted by $x \sim y$).

The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are strongly asymptotically lacunary invariant equivalent of multiple L if $\lim_r \frac{1}{h_r} \sum_{k\in I_r} \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| = 0$, uniformly in m (denoted by $x \overset{N_{\sigma\theta}}{\sim} y$) and strongly simply asymptotically lacunary invariant equivalent if $L = 1$.

The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically lacunary invariant statistical equivalent of multiple L if for every $\varepsilon > 0$,

$$\lim_r \frac{1}{h_r}\left|\left\{k \in I_r : \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| \geq \varepsilon\right\}\right| = 0, \text{ uniformly in m}$$

(denoted by $x \overset{S_{\sigma\theta}}{\sim} y$) and simply asymptotically lacunary invariant statistical equivalent if $L = 1$.

The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly asymptotically equivalent of multiple L with respect to the ideal $\mathcal{I}$ if for every $\varepsilon > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\left|\frac{x_k}{y_k} - L\right| \geq \varepsilon\right\} \in \mathcal{I}$$

(denoted by $x_k \overset{\mathcal{I}(\omega)}{\sim} y_k$) and simply strongly asymptotically equivalent with respect to the ideal $\mathcal{I}$, if $L = 1$.

The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly asymptotically lacunary equivalent of multiple L respect to the ideal $\mathcal{I}$ provided that for every $\varepsilon > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \in \mathcal{I}$$

(denoted by $x_k \overset{[\mathcal{I}(N_\theta)]}{\sim} y_k$) and simply strongly asymptotically lacunary $\mathcal{I}$-equivalent with respect to the ideal $\mathcal{I}$, if $L = 1$.

The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically lacunary statistical equivalent of multiple L with respect to the ideal $\mathcal{I}$ provided that for every $\varepsilon > 0$ and $\gamma > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| \geq \gamma \right\} \in \mathcal{I}$$

(denoted by $x_k \overset{\mathcal{I}(\mathcal{S}_\theta)}{\sim} y_k$) and simply asymptotically lacunary $\mathcal{I}$-statistical equivalent if $L = 1$.

The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically $\mathcal{I}_\sigma$-equivalent of multiple L if for every $\varepsilon > 0$, $A_\varepsilon = \left\{ k \in \mathbb{N} : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \in \mathcal{I}_\sigma$, i.e., $V(A_\varepsilon) = 0$. It is denoted by $x_k \overset{[\mathcal{I}_\sigma^L]}{\sim} y_k$.

The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically $\mathcal{I}_{\sigma\theta}$-equivalent of multiple L if for every $\varepsilon > 0$, $A_\varepsilon = \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \in \mathcal{I}_{\sigma\theta}$, i.e., $V_\theta(A_\varepsilon) = 0$. It is denoted by $x_k \overset{[\mathcal{I}_{\sigma\theta}^L]}{\sim} y_k$.

A function $f : [0, \infty) \to [0, \infty)$ is called a modulus if

(1) $f(x) = 0$ if and if only if $x = 0$,
(2) $f(x + y) \leq f(x) + f(y)$,
(3) f is increasing,
(4) f is continuous from the right at 0.

A modulus may be unbounded (for example $f(x) = x^p$, $0 < p < 1$) or bounded (for example $f(x) = \frac{x}{x+1}$).

Let f be modulus function. The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be f-asymptotically equivalent of multiple L with respect to the ideal $\mathcal{I}$ provided that, for every $\varepsilon > 0$,

$$\left\{ k \in \mathbb{N} : f\left(\left| \frac{x_k}{y_k} - L \right| \right) \geq \varepsilon \right\} \in \mathcal{I}$$

(denoted by $x_k \overset{\mathcal{I}(f)}{\sim} y_k$) and simply f-asymptotically $\mathcal{I}$-equivalent if $L = 1$.

Let f be modulus function. The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly f-asymptotically equivalent of multiple L with respect to the ideal $\mathcal{I}$ provided that, for every $\varepsilon > 0$

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} f\left(\left| \frac{x_k}{y_k} - L \right| \right) \geq \varepsilon \right\} \in \mathcal{I}$$

(denoted by $x_k \overset{\mathcal{I}(\omega_f)}{\sim} y_k$)) and simply strongly f-asymptotically $\mathcal{I}$-equivalent if $L = 1$.

Let f be a modulus function. The two nonnegative $x = (x_k)$ and $y = (y_k)$ are said to be strongly f-asymptotically lacunary equivalent of multiple L with respect to the ideal $\mathcal{I}$ provided that for every $\varepsilon > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} f\left(\left| \frac{x_k}{y_k} - L \right| \right) \geq \varepsilon \right\} \in \mathcal{I}$$

(denoted by $x_k \overset{[\mathcal{I}(N_\theta^f)]}{\sim} y_k$)) and simply strongly f-asymptotically lacunary $\mathcal{I}$-equivalent if $L = 1$.

The sequences $x=(x_k)$ and $y=(y_k)$ are said to be strongly asymptotically $\mathcal{I}$-invariant equivalent of multiple L if for every $\varepsilon>0$,

$$\left\{n\in\mathbb{N}:\frac{1}{n}\sum_{k=1}^{n}\left|\frac{x_k}{y_k}-L\right|\geq\varepsilon\right\}\in\mathcal{I}_\sigma$$

(denoted by $x_k \overset{[\mathcal{I}_\sigma^L]}{\sim} y_k$) and simply strongly asymptotically $\mathcal{I}$-invariant equivalent if $L=1$.

Let f be a modulus function. The sequences $x=(x_k)$ and $y=(y_k)$ are said to be f-asymptotically $\mathcal{I}$-invariant equivalent of multiple L if for every $\varepsilon>0$,

$$\left\{k\in\mathbb{N}:f\left(\left|\frac{x_k}{y_k}-L\right|\right)\geq\varepsilon\right\}\in\mathcal{I}_\sigma$$

(denoted by $x_k \overset{\mathcal{I}_\sigma^L(f)}{\sim} y_k$) and simply f-asymptotically $\mathcal{I}$-invariant equivalent if $L=1$.

Let f be a modulus function. The sequences $x=(x_k)$ and $y=(y_k)$ are said to be strongly f-asymptotically $\mathcal{I}$-invariant equivalent of multiple L if for every $\varepsilon>0$,

$$\left\{n\in\mathbb{N}:\frac{1}{n}\sum_{k=1}^{n}f\left(\left|\frac{x_k}{y_k}-L\right|\right)\geq\varepsilon\right\}\in\mathcal{I}_\sigma$$

(denoted by $x_k \overset{[\mathcal{I}_\sigma^L(f)]}{\sim} y_k$)) and simply strongly f-asymptotically $\mathcal{I}$-invariant equivalent if $L=1$.

The sequences x_k and y_k are said to be asymptotically $\mathcal{I}$-invariant statistical equivalent of multiple L if for every $\varepsilon>0$ and each $\gamma>0$,

$$\left\{n\in\mathbb{N}:\frac{1}{n}\left|\left\{k\leq n:\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}}-L\right|\geq\varepsilon\right\}\right|\geq\gamma\right\}\in\mathcal{I}_\sigma$$

(denoted by $x_k \overset{\mathcal{I}(\mathcal{S}_\sigma)}{\sim} y_k$) and simply asymptotically $\mathcal{I}$-invariant statistical equivalent if $L=1$.

Lemma 1. [16] *Let f be a modulus and $0<\delta<1$. Then, for each $x\geq\delta$ we have $f(x)\leq 2f(1)\delta^{-1}x$.*

2. Main Results

Definition 2.1. *The sequences $x=(x_k)$ and $y=(y_k)$ are said to be strongly asymptotically lacunary $\mathcal{I}$-invariant equivalent of multiple L, if for every $\varepsilon>0$*

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}\left|\frac{x_k}{y_k}-L\right|\geq\varepsilon\right\}\in\mathcal{I}_{\sigma\theta}$$

(denoted by $x_k \overset{[\mathcal{I}_{\sigma\theta}^L]}{\sim} y_k$) and simply strongly asymptotically lacunary $\mathcal{I}$-invariant equivalent if $L=1$.

Definition 2.2. *Let f be a modulus function. The sequences $x=(x_k)$ and $y=(y_k)$ are said to be f-asymptotically lacunary $\mathcal{I}$-invariant equivalent of multiple L, if for every $\varepsilon>0$*

$$\left\{k\in\mathbb{N}:f\left(\left|\frac{x_k}{y_k}-L\right|\right)\geq\varepsilon\right\}\in\mathcal{I}_{\sigma\theta},$$

(denoted by $x_k \overset{\mathcal{I}_{\sigma\theta}^L(f)}{\sim} y_k$) and simply f-asymptotically lacunary $\mathcal{I}$-invariant equivalent if $L=1$.

Definition 2.3. *Let f be a modulus function. The sequences $x=(x_k)$ and $y=(y_k)$ are said to be strongly f-asymptotically lacunary $\mathcal{I}$-invariant equivalent of multiple L, if for every $\varepsilon>0$*

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}f\left(\left|\frac{x_k}{y_k}-L\right|\right)\geq\varepsilon\right\}\in\mathcal{I}_{\sigma\theta}$$

(denoted by $x_k \overset{[\mathcal{I}_{\sigma\theta}^L(f)]}{\sim} y_k$)) and simply strongly f-asymptotically lacunary $\mathcal{I}$-invariant equivalent if $L=1$.

Theorem 2.1. *Let f be a modulus function. Then,*

$$x_k \overset{[\mathcal{I}^L_{\sigma\theta}]}{\sim} y_k \Rightarrow x_k \overset{[\mathcal{I}^L_{\sigma\theta}(f)]}{\sim} y_k.$$

Proof. Let $x_k \overset{[\mathcal{I}^L_{\sigma\theta}]}{\sim} y_k$ and $\varepsilon > 0$ be given. Choose $0 < \delta < 1$ such that $f(t) < \varepsilon$ for $0 \le t \le \delta$. Then, for $m = 1, 2, \ldots$, we can write

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right) &= \frac{1}{h_r}\sum_{\substack{k\in I_r\\ \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}}-L\right|\le\delta}} f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right)\\ &\quad+\frac{1}{h_r}\sum_{\substack{k\in I_r\\ \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}}-L\right|>\delta}} f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right)\end{aligned}$$

and so by Lemma 1

$$\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right) < \varepsilon + \left(\frac{2f(1)}{\delta}\right)\frac{1}{h_r}\sum_{k\in I_r}\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|$$

uniformly in m. Thus, for each any $\gamma > 0$

$$\left\{r\in\mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right) \ge \gamma\right\} \subseteq \left\{r\in\mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r}\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| \ge \frac{(\gamma-\varepsilon)\delta}{2f(1)}\right\},$$

uniformly in m. Since $x_k \overset{[\mathcal{I}^L_{\sigma\theta}]}{\sim} y_k$, it follows the later set and hence, the first set in above expression belongs to $\mathcal{I}_{\sigma\theta}$. This proves that $x_k \overset{[\mathcal{I}^L_{\sigma\theta}(f)]}{\sim} y_k$. □

Definition 2.4. *The sequences x_k and y_k are said to be asymptotically lacunary $\mathcal{I}$-invariant statistical equivalent of multiple L if for every $\varepsilon > 0$ and each $\gamma > 0$,*

$$\left\{r\in\mathbb{N} : \frac{1}{h_r}\left|\left\{k\in I_r : \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\}\right| \ge \gamma\right\} \in \mathcal{I}_{\sigma\theta}$$

(denoted by $x_k \overset{\mathcal{I}(\mathcal{S}_{\sigma\theta})}{\sim} y_k$) and simply asymptotically lacunary $\mathcal{I}$-invariant statistical equivalent if $L = 1$.

Theorem 2.2. *Let f be a modulus function. Then,*

$$x_k \overset{[\mathcal{I}^L_{\sigma\theta}(f)]}{\sim} y_k \Rightarrow x_k \overset{\mathcal{I}(\mathcal{S}_{\sigma\theta})}{\sim} y_k.$$

Proof. Assume that $x_k \overset{[\mathcal{I}^{\mathcal{L}}_{\sigma\theta}(f)]}{\sim} y_k$ and $\varepsilon > 0$ be given. Since for $m = 1, 2, \ldots$,

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right) &\ge \frac{1}{h_r}\sum_{\substack{k\in I_r\\ \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}}-L\right|\ge\varepsilon}} f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right)\\ &\ge f(\varepsilon).\frac{1}{h_r}\left|\left\{k\in I_r : \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| \ge \varepsilon\right\}\right|\end{aligned}$$

it follows that for any $\gamma > 0$,

$$\left\{r\in\mathbb{N} : \frac{1}{h_r}\left|\left\{k\in I_r : \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| \ge \varepsilon\right\}\right| \ge \gamma\right\} \subseteq \left\{r\in\mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right) \ge \gamma f(\varepsilon)\right\},$$

uniformly in m. Since $x_k \overset{[\mathcal{I}^L_{\sigma\theta}(f)]}{\sim} y_k$, the last set belongs to $\mathcal{I}_{\sigma\theta}$ and so by the definition of an ideal, the first set belongs to $\mathcal{I}_{\sigma\theta}$. Therefore, $x_k \overset{\mathcal{I}(\mathcal{S}_{\sigma\theta})}{\sim} y_k$. □

References

[1] H. Fast, *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241–244.
[2] P. Kostyrko, T. Šalát, W. Wilczyński, *$\mathcal{I}$-Convergence*, Real Anal. Exchange, **26**(2) (2000), 669–686.
[3] V. Kumar, A. Sharma, *Asymptotically lacunary equivalent sequences defined by ideals and modulus function*, Mathematical Sciences. **6**(23) (2012), 5 pages.
[4] G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math. **80** (1948), 167–190.
[5] I. J. Maddox, *Sequence spaces defined by a modulus*, Math. Proc. Camb. Phil. Soc. **100** (1986), 161–166.
[6] M. Marouf, *Asymptotic equivalence and summability*, Int. J. Math. Math. Sci. **16**(4) (1993), 755-762.
[7] M. Mursaleen, O. H. H. Edely, *On the invariant mean and statistical convergence*, Appl. Math. Lett. **22** (2009), 1700–1704.
[8] M. Mursaleen, *Matrix transformation between some new sequence spaces*, Houston J. Math. **9** (1983), 505–509.
[9] M. Mursaleen, *On finite matrices and invariant means*, Indian J. Pure and Appl. Math. **10** (1979), 457–460.
[10] H. Nakano, *Concave modulars*,J. Math. Soc.Japan, **5** (1953), 29-49.
[11] F. Nuray, E. Savaş, *Invariant statistical convergence and A-invariant statistical convergence*, Indian J. Pure Appl. Math. **10** (1994), 267–274.
[12] F. Nuray, H. Gök, U. Ulusu, *$\mathcal{I}_\sigma$-convergence*, Math. Commun. **16** (2011), 531–538.
[13] N. Pancaroğlu, F. Nuray, *Statistical lacunary invariant summability*, Theoretical Mathematics and Applications, **3**(2) (2013), 71–78.
[14] R. F. Patterson, *On asymptotically statistically equivalent sequences*, Demostratio Mathematica, **36**(1) (2003), 149–153.
[15] R. F. Patterson and E. Savaş, *On asymptotically lacunary statistically equivalent sequences*, Thai J. Math. **4**(2) (2006), 267–272.
[16] S. Pehlivan, B. Fisher, *Some sequences spaces defined by a modulus*, Mathematica Slovaca, **45** (1995), 275-280.
[17] R. A. Raimi, *Invariant means and invariant matrix methods of summability*, Duke Math. J. **30** (1963), 81–94.
[18] E. Savaş, *Some sequence spaces involving invariant means*, Indian J. Math. **31** (1989), 1–8.
[19] E. Savaş, *Strong σ-convergent sequences*, Bull. Calcutta Math. **81** (1989), 295–300.
[20] E. Savaş, F. Nuray, *On σ-statistically convergence and lacunary σ-statistically convergence*, Math. Slovaca, **43**(3) (1993), 309–315.
[21] E. Savaş, *On $\mathcal{I}$-asymptotically lacunary statistical equivalent sequences*, Adv. Differ. Equ. **2013**(111) (2013), doi:10.1186/1687-1847-2013-111.
[22] E. Savaş and R. F. Patterson, *σ-asymptotically lacunary statistical equivalent sequences,* Central European Journal of Mathematics, 4(4) (2006), 648-655.
[23] P. Schaefer, *Infinite matrices and invariant means*, Proc. Amer. Math. Soc. **36** (1972), 104–110.
[24] I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, **66** (1959), 361–375.
[25] U. Ulusu and F. Nuray, *Lacunary $\mathcal{I}_\sigma$-convergence*, (Under Comminication)
[26] U. Ulusu, *Asymptotoically Lacunary $\mathcal{I}_\sigma$-equivalence*, Afyon Kocatepe University Journal of Science and Engineering, **17**(2017), 031301, 899-905.
[27] U. Ulusu, *Asymptotoically ideal invariant equivalence*, Creat. Math. nform., **27**(2018), 215-220.

Deparment of Mathematics and Science Education , Afyon Kocatepe University, 03200, Afyonkarahisar, Turkey

E-mail address: npancaroglu@aku.edu.tr, edundar@aku.edu.tr, ulusu@aku.edu.tr