Colored Anchored Visibility Representations in 2D and 3D space⋆,⋆⋆

Carla Binucci[a,*], Emilio Di Giacomo[a], Seok-Hee Hong[b], Giuseppe Liotta[a], Henk Meijer[c], Vera Sacristán[d], Stephen Wismath[e]

[a] *Università degli Studi di Perugia, Italy*
[b] *University of Sydney, Australia*
[c] *University College Roosevelt*
[d] *Universitat Politècnica de Catalunya, Spain,*
[e] *University of Lethbridge, Canada*

Abstract

In a *visibility representation* of a graph G, the vertices are represented by non-overlapping geometric objects, while the edges are represented as segments that only intersect the geometric objects associated with their end-vertices. Given a set P of n points, an *Anchored Visibility Representation* of a graph G with n vertices is a visibility representation such that for each vertex v of G, the geometric object representing v contains a point of P. We prove positive and negative results about the existence of anchored visibility representations under various models, both in 2D and in 3D space. We consider the case when the mapping between the vertices and the points is not given and the case when it is only partially given.

1. Introduction

A *visibility representation* (VR) of a graph G maps the vertices of G to non-overlapping geometric objects and the edges of G to segments, called *visibilities*, that only intersect the geometric objects associated with their end-vertices. Various models of visibility representations have been studied in the plane using different types of objects to represent the vertices and different rules to represent the edges. Some examples are: *Bar Visibility Representations* (BVRs) [29, 45, 50, 51, 54], where the vertices are horizontal segments and the

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*Corresponding author

Email addresses: `carla.binucci@unipg.it` (Carla Binucci), `emilio.digiacomo@unipg.it` (Emilio Di Giacomo), `seokhee.hong@sydney.edu.au` (Seok-Hee Hong), `giuseppe.liotta@unipg.it` (Giuseppe Liotta), `h.meijer@ucr.nl` (Henk Meijer), `vera.sacristan@upc.edu` (Vera Sacristán), `wismath@uleth.ca` (Stephen Wismath)

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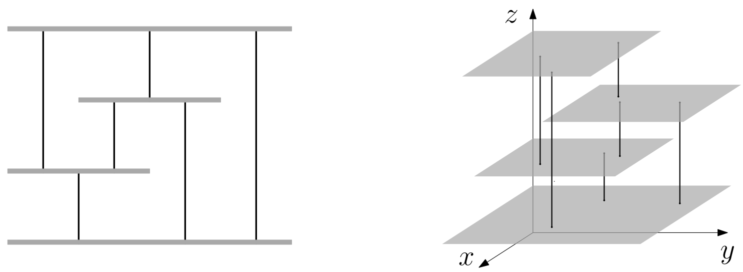


(a) (b)

Figure 1: (a) A BVR of K_4. (b) A ZPR of K_4.

edges are vertical segments (see Fig. 1(a)), *Rectangle Visibility Representations* (RVRs) [8, 11, 19, 21, 38, 48], which use axis-aligned rectangles and segments to represent vertices and edges, respectively, and *Orthopolygon Visibility Representations* [23, 41, 42], which generalize RVRs by using general orthogonal polygons.

Visibility representations in the three-dimensional space have also been considered. One of the first 3D models is the so-called *Z-parallel Visibility Representations* (ZPRs) [2, 12, 47], where vertices are represented by axis-aligned rectangles parallel to the xy-plane and edges are segments parallel to the z-axis (see Fig. 1(b)). Fekete and Meijer [33] considered the *Box Visibility Representations* where vertices are 3D boxes and visibilities are parallel to one of the three axis. Recently, *2.5D box visibility representations* have been proposed [3, 4]; in this model vertices are 3D boxes whose bottom faces lie in the plane $z = 0$ and visibilities are parallel to the x- and y-axis. A similar 2D variant where vertices are horizontal bars whose left end points all have the same x-coordinate have been studied by Cobos et al. [17] and by Felsner and Massow [35].

We remark that each visibility model can be studied in two variants; in the *strong* variant (see, e.g., [19, 20, 21, 33, 38, 50, 54]) two vertices are adjacent if and only if the corresponding objects are visible (i.e. they can be connected by a visibility segment); in the *weak* variant (see, e.g., [8, 13, 23, 32, 41, 46]) visibilities between objects representing non-adjacent vertices may exist.

In this paper we study weak visibility representations with additional constraints on the "positions" of the vertices. More precisely, given a set P of n points with distinct coordinates along one of the directions parallel to the visibilities, an *Anchored Visibility Representation* (AVR) of a graph G with n vertices is a VR such that for each vertex v of G, the geometric object representing v contains a point of P. In particular, we consider *Anchored Bar Visibility Representations* (ABVRs) and *Anchored Z-parallel Visibility Representations* (AZPRs) (see Fig. 2). AVRs can be studied in different variants depending on whether the mapping between the vertices and the points is given or not. It is also possible that this mapping is only partially specified. We capture all these

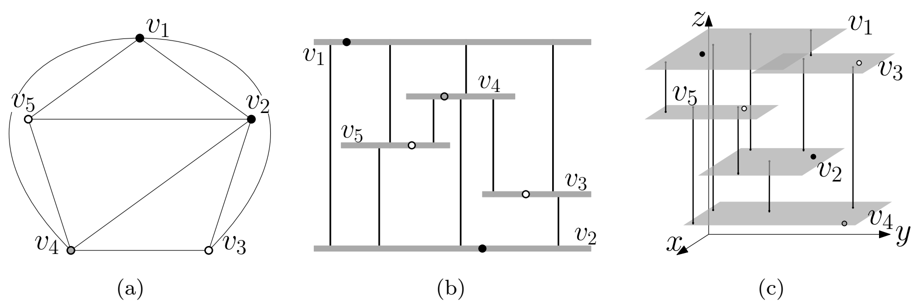


Figure 2: (a) A 3-colored graph G. (b) A 3-colored ABVR of G on a given set of points. (c) A 3-colored AZPR of G on a given set of points.

variants within a unique framework, described in terms of colors. Given a graph whose vertices are colored with k colors and a set of points also colored with k colors, a *k-colored AVR of G on P* is an AVR such that each point $p \in P$ belongs to a geometric object representing a vertex with the same color as p (see Fig. 2 for an example with three colors). With this framework if $k = n$ we have a complete mapping; if $k = 1$ there is no mapping; for any value of k between 1 and n we have a partial mapping. A similar framework based on colors has been used in the study of point-set embedding, where one wants to compute a 2D polyline drawing of a graph such that the vertices are represented by the points of a given point set [5, 25, 26, 27, 37]. We also remark that the problem of computing drawings with constraints on vertex positions is a classical subject in Graph Drawing (see, e.g., [9, 10, 39, 40, 43]). In particular, Chaplick et al. [14, 16] have studied the problem of deciding whether a given graph $G = (V, E)$ admits a BVR when the bars representing a subset $V' \subset V$ are given as a part of the input. They prove that the problem is $\mathcal{NP}$-complete in general [14, 16] and it is polynomially-time solvable if $V' = V$ [14]. The BVRs studied in this paper, where the bars representing the vertices are not fixed but are constrained to include the given points, can be considered as a relaxation of those considered by Chaplick et al. An even more relaxed version where only the y-coordinate of each bar is given has also been considered [14].

The contributions of this paper are the following:

- We first study AVRs in 2D space and prove that every 1-colored sub-Hamiltonian graph (i.e., a subgraph of a planar Hamiltonian graph) admits a 1-colored ABVR on every set of points in the plane. We show that the converse is also true if the points are collinear. As a consequence, deciding whether a graph admits a 1-colored ABVR on a given set of points is $\mathcal{NP}$-complete. We also show that an ABVR always exists for every planar graph if all the points have distinct x- and y-coordinates.

- Concerning 1-colored AVRs in 3D we first show that every 1-colored graph with page number 4 admits a 1-colored AZPR on every given set of points.

This extends the previous results on 1-colored sub-Hamiltonian graphs since these are exactly the graphs that have page number 2. We then show that K_n has a 1-colored ZPR on every set of points if and only if it admits a ZPR. This, together with known results about ZPRs of K_n, implies that K_n admits an AZPR on any set of points if $n \leq 22$ while it does not admit an AZPR if $n \geq 51$. We finally prove that every 1-colored graph that is 3-connected 1-planar or that has geometric thickness 2 admits a 1-colored AZPR on any given point set P.

- Still in 3D, we consider the 2-colored version of the problem and prove that every properly 2-colored tree T admits a 2-colored AZPR on any given 2-colored point set P.

The rest of the paper is organized as follows. Preliminary definitions are given in Section 2. The results about AVRs in the plane are presented in Section 3, while those in 3D are in Section 4. In particular, Subsection 4.1 is about 1-colored AZPRs, and Subsection 4.2 contains results about 2-colored trees. Conclusions and open problems can be found in Section 5.

2. Preliminaries

A *Bar Visibility Representation* (*BVR*) of a graph G is a 2D visibility representation where the vertices of G are mapped to horizontal segments, called *bars*, while visibilities are vertical segments. A *Z-parallel Visibility Representation* (*ZPR*) is a 3D visibility representation where vertices are mapped to axis-aligned rectangles belonging to planes parallel to the xy-plane, while visibilities are parallel to the z-axis.

In the rest of the paper, we will often transform BVRs into ZPRs; to keep the direction of the visibilities consistent between BVRs and ZPRs, we assume that a BVR is realized in the yz-plane with visibilities parallel to the z-axis. See, e.g., Fig. 3(c). Given a BVR (respectively a ZPR) Γ, the (partial) order of the bars (respectively of the rectangles), along the vertical direction is called the *z-ordering of* Γ. Throughout the paper we adopt the so-called *weak visibility model* where visibilities between bar/rectangles representing non-adjacent vertices may exist.

Let G be a graph with n vertices and let P be a set of n points in $\mathbb{R}^2$ or $\mathbb{R}^3$ with distinct z-coordinates (recall that in 2D we use the yz-plane). For each type of visibility representation defined above (BVR and ZPR), we define a constrained version in which for each vertex v of G, the object representing v contains a point of P. We will refer to these constrained versions as *Anchored Bar Visibility Representations* (ABVRs) and *Anchored Z-parallel Visibility Representations* (AZPRs). We require that the points in P have different z-coordinates because bars or rectangles with the same z-coordinate cannot be connected by a visibility segment (recall that visibilities are parallel to the z-axis). This avoids instances that trivially do not admit anchored visibility representations. For example, the complete graph K_n does not admit such a representation if at least two points have the same z-coordinate.

Let $G = (V, E)$ be a graph with n vertices. A *k-coloring of G* is a partition $\{V_1, \ldots, V_k\}$ of V where integers $\{1, \ldots, k\}$ are called *colors*. A graph G with a k-coloring is called a *k-colored graph*. A graph is *properly k-colored* if it is a k-colored graph and no two vertices of the same color are adjacent. Let P be a set of n points in $\mathbb{R}^2$ or $\mathbb{R}^3$ with distinct z-coordinates. A *k-coloring of P* is a partition $\{P_1, \ldots, P_k\}$ of P. A set of points P with a k-coloring is called a *k-colored set*. A k-colored set P is *compatible with* a k-colored graph G if $|V_i| = |P_i|$ for every i. Let G be a k-colored graph and let P be a k-colored set of points compatible with G ($k \geq 1$). A *k-colored ABVR of G on P* is an ABVR of G on P such that for each vertex v of G, the bar representing v contains a point of P with the same color as v. Analogous definitions hold for k-colored AZPRs. The assumption that the points of P have distinct z-coordinates avoids straightforward negative instances where adjacent vertices of G are forced to be mapped to points with the same z-coordinate.

A *k-colored sequence* λ is a sequence of (possibly repeated) colors c_1, ..., c_n such that $c_j \in \{1, 2, \ldots, k\}$ for every $j \in \{1, 2, \ldots, n\}$. Let G be a k-colored graph and let λ be a k-colored sequence; λ is *compatible with* G if the number of elements in λ colored i is equal to $|V_i|$, for every $i = 1, 2, \ldots, k$. A total order ρ of the vertices of G is *consistent with* λ if the sequence of the colors defined by ρ coincides with λ. Given a k-colored point set P, we denote by $\lambda(P)$ the sequence of colors of the points of P according to their order along the z-direction (this order is a total order because the points of P have distinct z-coordinates). Finally, given a set of n points p_1, p_2, ..., p_n we denote by $x(p_i)$, $y(p_i)$, and $z(p_i)$ the x-, y-, and z-coordinate of point $p_i \in P$ and by x_m and x_M the values $\min_{i=1}^{n}\{x(p_i)\}$ and $\max_{i=1}^{n}\{x(p_i)\}$, respectively. We analogously define y_m, y_M, z_m, and z_M.

3. Anchored Visibility Representations in 2D

A *book embedding* of a graph $G = (V, E)$ consists of a total order ρ of V and a partition of E into k disjoint sets, called *pages*, such that no two edges in the same page *cross*; that is, there are no two edges (u_1, v_1) and (u_2, v_2) in the same page with $u_1 <_\rho u_2 <_\rho v_1 <_\rho v_2$ (see Fig 3(a)). The minimum k for which a graph G admits a book embedding with k pages is the *page number* of G. A graph has page number one if and only if it is outerplanar [7]. This also means that the graph induced by each page of a k-page book embedding is an outerplanar graph. A graph is *Hamiltonian* if it has a simple cycle that contains all its vertices. A graph is *sub-Hamiltonian* if it is a subgraph of a planar Hamiltonian graph. A graph has page number two if and only if it is sub-Hamiltonian [7]. A *semi-bar visibility representation* of a planar graph G is a BVR of G such that the left endpoints of all the bars representing the vertices of G belong to a vertical line. Cobos et al. [17] proved that a graph has a semi-bar visibility representation if and only if it is outerplanar.

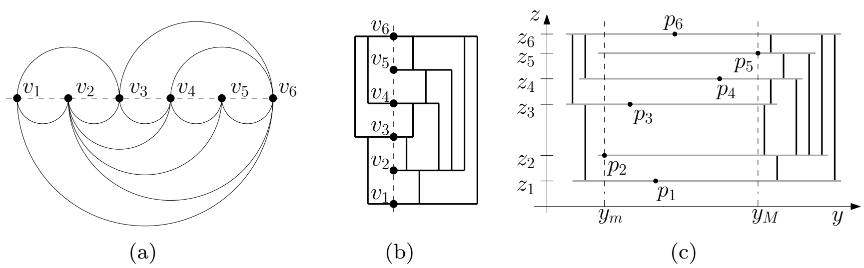


Figure 3: (a) A 2-page book embedding of a (sub-Hamiltonian) graph $G = (V, E)$; (b) The same book embedding with pages represented as semi-bar visibility; (c) An ABVR of G on a set P of $n = |V|$ points with distinct z-coordinates.

Theorem 1. *Let G be a 1-colored sub-Hamiltonian graph and let P be a 1-colored point set in $\mathbb{R}^2$; G has a 1-colored ABVR on P.*

Proof. Since G is sub-Hamiltonian, it admits a book embedding γ with two pages [7]. Let $p_1, p_2, \ldots, p_n$ be the points of P in the order as they appear along the z-axis and let $v_1, v_2, \ldots, v_n$ be the vertices of G according to the total order ρ of γ. The bar b_i representing v_i is drawn as a segment parallel to the y-axis with z-coordinate $z(p_i)$, with minimum y-coordinate less than y_m and maximum y-coordinate greater than y_M. This guarantees that b_i contains the point p_i (refer to Fig. 3 for an illustration). The amount of the extension of each b_i in the half-planes $y < y_m$ and $y > y_M$ is chosen to realize the visibilities that represent the edges. In other words, we realize in each of the half-planes $y < y_m$ and $y > y_M$ two semi-bar visibility representations, one for each page of γ. Such semi-bar visibility representations exist by the result of Cobos et al. [17]. For completeness, we give a detailed description of the construction, which will be also useful to extend the result in the 3D scenario.

Denote by b_i^- (resp. b_i^+) the length of the portion of b_i that lies in the half-plane $y < y_m$ (resp. $y > y_M$). For each edge (v_i, v_j), the *span* of (v_i, v_j) in γ is $|j - i|$. The length b_i^- (resp. b_i^+) is chosen equal to the maximum span of an edge incident to v_i in the first (resp. second) page. With this choice every pair of adjacent vertices v_i and v_j are visible. Suppose, as a contradiction, that v_i and v_j, with $i < j$, are not visible, i.e., there exists a bar b_k with $i < k < j$ such that $b_i^-, b_j^- < b_k^-$ and $b_i^+, b_j^+ < b_k^+$. Without loss of generality assume that (v_i, v_j) is in the first page in γ. This implies that $b_i^-, b_j^- \geq |j - i|$. On the other hand, $b_k^- > b_i^-, b_j^-$ implies that there is an edge (v_k, v_h) in the first page with $|h - k| > |j - i|$, but this implies that $v_i <_\rho v_k <_\rho v_j <_\rho v_h$ or $v_h <_\rho v_i <_\rho v_k <_\rho v_j$, which is impossible because (v_i, v_j) and (v_k, v_h) are in the same page in γ. $\square$

If the points are aligned in the z-direction the converse of Theorem 1 holds.

Theorem 2. *Let G be a 1-colored planar graph and let P be a 1-colored point set in $\mathbb{R}^2$ such that each point in P has the same y-coordinate; G admits a 1-colored ABVR on P if and only if G is sub-Hamiltonian.*

Proof. The proof of sufficiency follows from Theorem 1. Consider now the necessity. Let Γ be an ABVR of G on P and let $b_1, b_2, \ldots, b_n$ be the bars of Γ according to the z-ordering of Γ. Since all points have the same y-coordinate $\bar{y}$, every b_i sees b_{i+1} along the line $y = \bar{y}$. Thus, we can add the visibilities (if not already present) between b_i and b_{i+1} ($i = 1, 2, \ldots, n$). Moreover, we can add a visibility between b_1 and b_n. To this aim, let y_{min} be the minimum y-coordinate of a bar in Γ; we extend b_1 and b_n in the half-plane $y < y_{min}$ and add a visibility between them. The resulting representation is an ABVR of a planar Hamiltonian supergraph G' of G and therefore G is sub-Hamiltonian. □

We remark that sub-Hamiltonian graphs include various sub-families of planar graphs, such as 2-trees (and therefore series-parallel graphs, outerplanar graphs and trees) [44], Halin graphs [18], 4-connected planar graphs [52], and planar graphs with maximum vertex degree 4 [6]. On the other hand, testing a graph for sub-Hamiltonicity is $\mathcal{NP}$-complete [53], and therefore a consequence of Theorem 2 is that testing whether a given planar graph admits an ABVR on a given set of collinear points is $\mathcal{NP}$-complete.

Corollary 1. *The problem of deciding whether a 1-colored planar graph admits an ABVR on a given 1-colored point set is $\mathcal{NP}$-complete.*

In contrast to the $\mathcal{NP}$-hardness result about vertically aligned points, if the points of P are not vertically nor horizontally aligned, then an ABVR exists for every planar graph G. The next theorem is a consequence of a paper by Felsner [34] about floorplans (see also [15] for a similar proof). A *generic floorplan* is a partition of a rectangle into a finite set of interiorly disjoint rectangles that have no point where four rectangles meet. Two floorplans F and F' are *weakly equivalent* if there exist a bijection ϕ_H between the horizontal segments and ϕ_V between the vertical segments, such that a horizontal (resp. vertical) segment s has an endpoint on a vertical (resp. horizontal) segment t in F if and only if $\phi_H(s)$ (resp. $\phi_V(s)$) has an endpoint on $\phi_V(t)$ (resp. $\phi_H(t)$) in F'. A set P of points in $\mathbb{R}^2$ is *generic* if no two points from P have the same x- or y-coordinate. The following theorem has been proved by Felsner [34].

Theorem 3. [34] *If P is a generic set of k points in a rectangle R and F is a generic floorplan with $n > k$ segments and S is a prescribed subset of the segments of F having size k, then there exists a generic floorplan F' that is weakly equivalent to F and such that every segment of F' that corresponds to a segment of S contains exactly one point of P and no point is contained in two segments.*

Theorem 4. *Let G be a 1-colored planar graph and let P be a generic 1-colored point set in $\mathbb{R}^2$; G admits a 1-colored ABVR on P.*

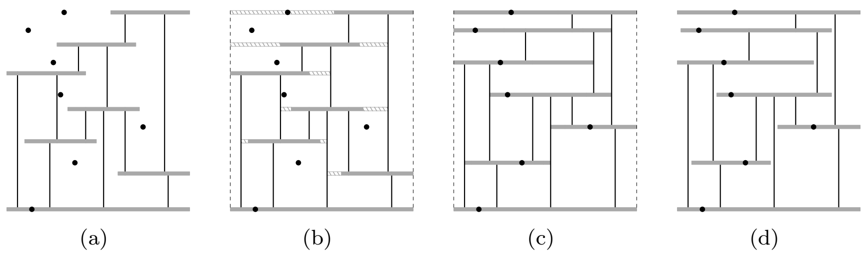


Figure 4: (a) A generic point set P and a BVR Γ of a planar graph G; (b) A floorplan F obtained from Γ; (c) A floorplan F' weakly equivalent to F and covering P; (d) An ABVR of G on P.

Proof. Let Γ be a BVR of G. We now construct a generic floorplan starting from Γ (see Fig. 4(a)). Let b_b and b_t be the bottommost and the topmost bars in Γ; we extends b_b and b_t so that their left endpoints and their right endpoints can be connected by two vertical segments s_l and s_r. The four segments b_b, b_t, s_l and s_r are the boundary of a rectangle. Extending every bar of Γ until its endpoints touch a vertical segment, we obtain a floorplan F (see Fig. 4(b)). Let S be the set of horizontal segments of F. By Theorem 3, there exists a floorplan F' that is weakly equivalent to F and such that every point of P belongs to exactly one horizontal segment (see Fig. 4(c)). Since F' and F are weakly equivalent the adjacencies between vertical and horizontal segments are the same. We shorten the horizontal segments so to remove the adjacencies that are in F but not in Γ and we also remove the two segments s_l and s_r. We thus obtain a new BVR Γ' of G (see Fig. 4(d)). Moreover, since no point of P belongs to two segments of F', every point of P is an internal point of a horizontal segment; this means that we can shorten the horizontal segments in such a way that they still contain the points of P. This implies that Γ' is in fact an ABVR of G on P. □

We conclude this section with a result about k-colored ABVR, for $2 \leq k \leq n$, that is an immediate consequence of Theorem 2. By Theorem 2 every non-sub-Hamiltonian graph G does not admit an ABVR on a given set P of vertically aligned points. This implies that any k-colored graph G' that has G as a monochromatic subgraph does not admit an ABVR on any set of points P' that contains P as a monochromatic subset (with the color of the vertices of G only occurring in P).

Corollary 2. *For every $1 \leq k \leq n$ there exists a k-colored planar graph G and a k-colored point set P such that G does not admit a k-colored ABVR on P.*

4. Anchored Visibility Representations in 3D

In this section we study anchored visibility representations in 3D, in particular AZPRs, and present results about the 1-colored version (Subsection 4.1) and the 2-colored version of the problem (Subsection 4.2). We start with two lemmas that will be useful in both subsections.

Lemma 1. *Let G be a graph with n vertices and let P be a set of n points in $\mathbb{R}^3$ with distinct z-coordinates. If G has a ZPR Γ such that each rectangle representing a vertex has nonempty intersection with the z-axis, then G admits an AZPR on P whose z-ordering is the same as Γ.*

PROOF. We show that Γ can be modified to an AZPR Γ' on P with the same z-ordering as Γ. Let $p_1, p_2, \dots, p_n$ be the sequence of the points of P ordered by increasing z-coordinate; let $r_1, r_2, \dots, r_n$ be the sequence of the rectangles ordered according to the z-ordering of Γ. First, we translate the rectangles $r_1, r_2, \dots, r_n$ so that r_i has z-coordinate $z(p_i)$. Since the order of the rectangles in the z-direction is not changed, the visibilities of Γ are preserved. Denote by $x'(r_i)$ and $y'(r_i)$ the maximum x- and y-coordinate of r_i, respectively and by $x''(r_i)$ and $y''(r_i)$ the minimum x- and y-coordinate of r_i, respectively. Note that $(x'(r_i), y'(r_i), z(p_i))$ and $(x''(r_i), y''(r_i), z(p_i))$ are two opposite corners of r_i.

In order to obtain an AZPR of G on P we extend each rectangle r_i in such a way that the coordinates of its two opposite corners become $(x_m + x''(r_i), y_m + y''(r_i), z(p_i))$ and $(x_M + x'(r_i), y_M + y'(r_i), z(p_i))$, respectively. Moreover, each visibility segment s_j whose x- and y-coordinate are $x(s_j)$ and $y(s_j)$, respectively, is translated as follows. If $x(s_j) \geq 0$, then s_j is translated so that its x-coordinate is $x_M + x(s_j)$, while if $x(s_j) < 0$, then s_j is translated so that its x-coordinate is $x_m + x(s_j)$. Analogously, if $y(s_j) \geq 0$, then s_j is translated so that its y-coordinate is $y_M + y(s_j)$, while if $y(s_j) < 0$, then s_j is translated so that its y-coordinate is $y_m + y(s_j)$. See Fig. 5 for an illustration. Let Γ' be the resulting representation. We denote by r'_i the rectangle of Γ' obtained by extending the rectangle r_i of Γ; analogously, we denote by s'_j the segment of Γ' obtained by translating the visibility segment s_j of Γ.

We now prove that Γ' is a valid AZPR of G on P. Clearly, point p_i belongs to r'_i because it is contained in the rectangle with opposite corners $(x_M, y_M, z(p_i))$ and $(x_m, y_m, z(p_i))$, which is contained in r'_i. Also, each segment s'_j of Γ' is a valid visibility segment. Namely, assume that $x(s_j) \geq 0$ and $y(s_j) \geq 0$ (the other cases are analogous). The coordinates of s'_j are $x_M + x(s_j)$ and $y_M + y(s_j)$. If s'_j is not a valid visibility segment in Γ', then there exists a rectangle r'_k that intersect s'_j at an interior point $(x_M + x(s_j), y_M + y(s_j), z(p_k))$. This implies that $x_M + x'(r_k) \geq x_M + x(s_j)$ and $y_M + y'(r_k) \geq y_M + y(s_j)$, i.e., $x'(r_k) \geq x(s_j)$ and $y'(r_k) \geq y(s_j)$. But this means that s_j intersects r_k in Γ, contradicting the fact that s_j is a valid visibility segment in Γ. □

The next lemma explains how to transform a BVR into a ZPR with the additional properties that it is completely contained in the region of space with $x \geq 0$ and $y \geq 0$ and such that all rectangles representing vertices have a corner on the z-axis. Any such ZPR will be called *cornered ZPR*.

Lemma 2. *Let G be a graph that has a BVR Γ where vertices have distinct z-coordinates; G admits a cornered ZPR whose z-ordering is the same as Γ.*

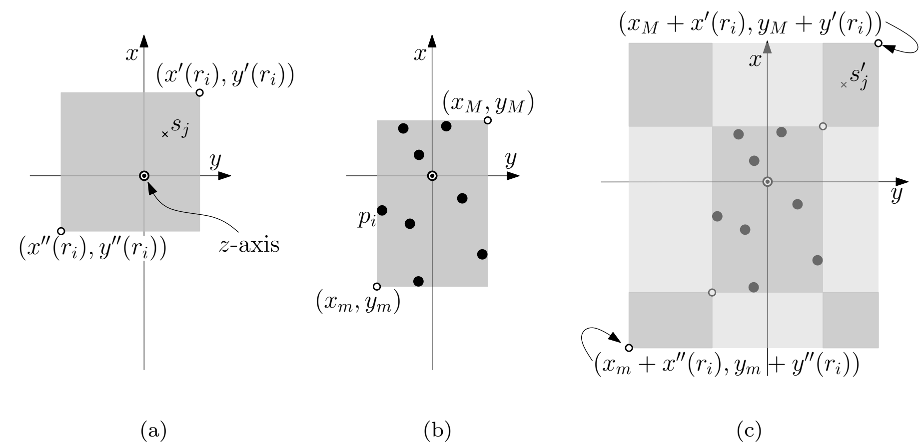
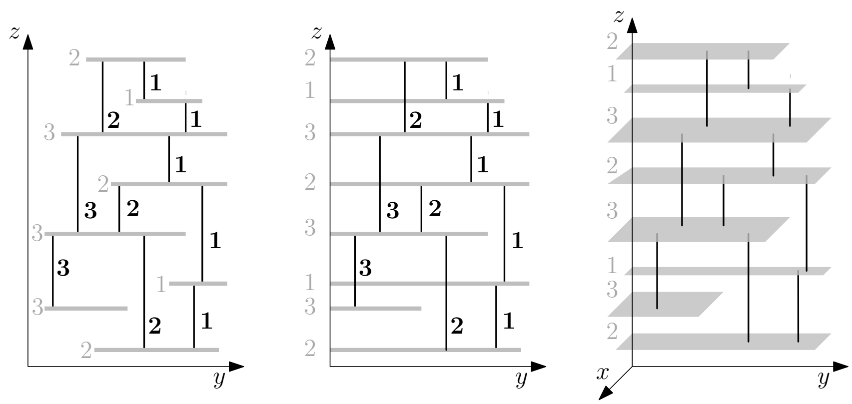


Figure 5: Illustration for the proof of Lemma 1. (a) The projection of a rectangle r_i on the xy-plane. (b) The projection on the xy-plane of the bounding box of the point set P. (c) The projection of the rectangle r_i' on the xy-plane.

Proof. By possibly translating it, we can assume that Γ is contained in the first quadrant of the yz-plane (see Fig. 6). Let e_i be an edge of G, we denote by s_i the visibility segment representing e_i in Γ. We enumerate the visibility segments from right to left and we assign to the segments integer numbers that increase from right to left. More precisely, we assign to each visibility segment s_i a number $n(s_i) \in \mathbb{N}^+$ so that $n(s_i) < n(s_j)$ if there exists a y-parallel straight line that intersects both s_i and s_j, and s_i is to the right of s_j. We assign to each bar b_i a number $n(b_i) \in \mathbb{N}^+$ equal to the maximum number of a visibility segment incident to b_i.

We now extend each bar b_i of Γ so that it touches the z-axis. Let Γ' be the resulting representation. We denote by b_i' the bar obtained by extending b_i and we set $n(b_i') = n(b_i)$. Observe that the visibility segments of Γ' are the same as those of Γ. In Γ' bars can intersect the visibility segments. However, if a bar b_i' intersects a visibility segment s_j, then $n(b_i') < n(s_j)$. Namely, since b_i did not intersect s_j before the extension, every point of b_i has a y-coordinate larger than the y-coordinate of s_j; hence any visibility segment s_k incident to b_i has a y-coordinate greater than the y-coordinate of s_j and therefore $n(s_k) < n(s_j)$. Since $n(b_i)$ is equal to the maximum number of a visibility segment incident to b_i and $n(b_i') = n(b_i)$, we have $n(b_i') < n(s_j)$.

In order to construct a ZPR with the desired properties, we transform the bars representing the vertices into rectangles by extending them in the positive x-direction. In particular, a bar b_i' is transformed into a rectangle r_i with a side coincident with b_i' and whose dimension in the x-direction is equal to $n(b_i')$. We also translate the visibility segments in the x-direction. A visibility segment s_i is moved so that its x-coordinate is $n(s_i)$. Denote by Γ'' the resulting representation. By construction, all the rectangles of Γ'' are in the region of



(a) (b) (c)

Figure 6: (a) A BVR Γ where each visibility s_i (each bar b_i) is associated with a number in black (gray) according to the partial order $\prec$. (b) The drawing Γ' obtained from Γ by extending each bar so that it touches the z-axis. (c) A cornered ZPR Γ'' of Γ'.

space with $x \geq 0$ and $y \geq 0$ and have a corner on the z-axis. Furthermore, no rectangle r_i intersects a visibility segment s_j. Namely, if b'_i (i.e. the bar that has been extended to create r_i) and s_j did not intersect each other in Γ', then they do not intersect in Γ''. If b'_i and s_j intersected in Γ', then $n(b'_i) < n(s_j)$, and it follows that in Γ'' s_j has a x-coordinate $n(s_j)$ while the maximum x-coordinate of r_i is $n(b'_i)$, which implies that r_i and s_j do not intersect. □

4.1. 1*-colored AZPRs*

We start with a theorem that is the 3D counterpart of Theorem 1 and that can be proven similarly.

Theorem 5. *Let G be a* 1*-colored graph with page number four and let P be a* 1*-colored point set in $\mathbb{R}^3$; G admits a* 1*-colored AZPR on P.*

Proof. Let $p_1, p_2, \ldots, p_n$ be the points of P in the order they appear along the z-axis and let $v_1, v_2, \ldots, v_n$ be the vertices of G according to the total order ρ of a given 4-page book embedding of G. Vertex v_i will be represented by a rectangle r_i parallel to the xy-plane whose z-coordinate is $z(p_i)$ and such that its minimum x-coordinate (respectively y-coordinate) is less than x_m (respectively y_m) and its maximum x-coordinate (respectively y-coordinate) is greater than x_M (respectively y_M). This guarantees that r_i contains the point p_i. The visibilities to represent the edges in each page are realized by choosing the amount of the extension of each r_i in the half-planes $x < x_m$, $x > x_M$, $y < y_m$ and $y > y_M$, analogously to what done in the proof of Theorem 1. □

In Section 3 we showed (Theorem 2) that if a graph admits an ABVR on a set of collinear points, then it is sub-Hamiltonian and therefore has page number two. In other words, when restricted to collinear points the converse of Theorem 1 holds. We now show that this is not the case for Theorem 5. Indeed, using results from [12] and [47] we can prove that K_n, which has page number $\lceil \frac{n}{2} \rceil$, admits an AZPR on any given set of points (even if collinear or coplanar) for $n \leq 22$.

Theorem 6. *The complete graph K_n admits a 1-colored AZPR on any given set of 1-colored points if and only if it admits a ZPR.*

Proof. The only-if direction is trivial. Suppose then that K_n admits a ZPR Γ; as observed by Bose et al. [12] if a ZPR of K_n exists, then all rectangles intersect a line parallel to the visibilities. By possibly translating Γ we can assume that this line is the z-axis and therefore by Lemma 1 K_n admits an AZPR on any given set of points. □

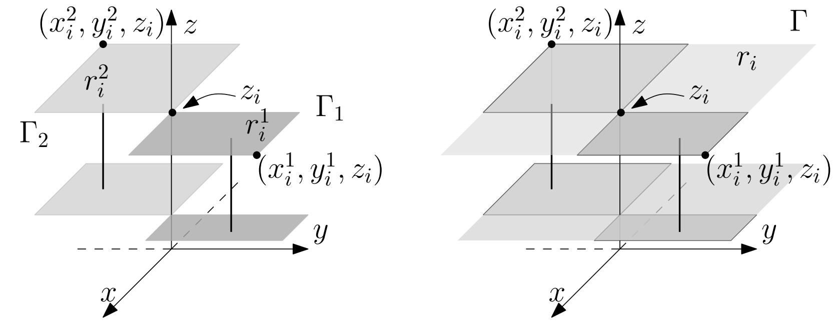
Bose et al. [12] construct a ZPR of K_{22} and prove that a ZPR cannot exist for K_n with $n \geq 56$. Afterwards, Štola [47] lowered the upper bound to 51. Hence we have the following.

Corollary 3. *Let P be a set of 1-colored points in $\mathbb{R}^3$. If $n \leq 22$, then K_n admits a 1-colored AZPR on P; if $n \geq 51$ a 1-colored AZPR of K_n on P does not exist.*

In the rest of this section we will describe two families of graphs that admit a 1-colored AZPR on every set of points, namely the 3-*connected* 1-*planar graphs* and the *graphs with geometric thickness two*. A graph is 1-planar if it has a drawing where each edge is crossed at most once; a graph has geometric thickness two if it has a straight-line drawing whose edges can be partitioned into two sets and no two edges in the same set cross. This latter family includes the RAC graphs [28] (i.e. graphs that have a straight-line drawing where each edge crossing forms a right angle) and the graphs with maximum vertex-degree 4 [30]. Both the results are proved using a common approach based on the fact that a graph G belonging to the families above can be decomposed into two planar graphs. The idea is to combine two cornered ZPRs of the two planar graphs whose union is G to create an AZPR of G on a given set of points P. The next lemma explains how to achieve this, provided that the z-orderings of the two cornered ZPRs is the same.

Lemma 3. *Let G be a 1-colored graph that is the union of two planar graphs G_1 and G_2 with the same vertex set and let P be a 1-colored point set in $\mathbb{R}^3$. If G_1 and G_2 admit two BVRs whose z-ordering is the same, then G admits an AZPR on P.*

Proof. Consider two BVRs of G_1 and G_2 with the same z-ordering and denote by ρ this ordering. By Lemma 2, G_1 and G_2 admit two cornered ZPRs Γ_1 and Γ_2, respectively, whose z-ordering is ρ. We now explain how to combine Γ_1 and Γ_2 to obtain a ZPR of G such that all rectangles representing vertices have



(a) (b)

Figure 7: (a) A cornered ZPR Γ_1 in the region $x \geq 0$, $y \geq 0$ (grey) and a cornered ZPR Γ_2 in the region $x \geq 0$, $y \geq 0$ (lightgrey). (b) A ZPR Γ of G obtained by combining Γ_1 and Γ_2 while preserving visibilities.

nonempty intersection with the z-axis. By Lemma 1 this implies that G has an AZPR on P.

Let r_i^1 and r_i^2 be the rectangles representing a vertex v_i in Γ_1 and Γ_2, respectively. We first translate the rectangles of Γ_2 so that r_i^1 has the same z-coordinate z_i of r_2^i, for every $i = 1, 2, \ldots, n$. Since the z-ordering of Γ_1 and Γ_2 is the same, this translation does not change the z-ordering of Γ_2. Next, we rotate Γ_2 by 180° around the z-axis. In this way, Γ_2 is completely contained in the region $x \leq 0$ and $y \leq 0$ and all its rectangles have a corner on the z-axis. Rectangles r_i^1 and r_i^2 (after the rotation) only share the point $(0, 0, z_i)$. Let (x_i^1, y_i^1, z_i) be the corner of r_i^1 opposite to $(0, 0, z_i)$ and let (x_i^2, y_i^2, z_i) be the corner of r_i^2 opposite to $(0, 0, z_i)$. Vertex v_i is represented in Γ by a rectangle whose opposite corners are (x_i^1, y_i^1, z_i) and (x_i^2, y_i^2, z_i) (see Fig. 7). Γ is a ZPR of G because the visibilities of Γ_1 and Γ_2 have not been destroyed: those of Γ_1 still exist in the region $x \geq 0$ and $y \geq 0$, while those of Γ_2 have been moved to the region $x \leq 0$ and $y \leq 0$ by the rotation. □

Let Γ be a drawing of a graph G. Γ has *thickness* $k \geq 1$ if the edges of Γ can be colored with k colors so that no two edges of the same color cross in Γ. Let G be a directed graph; an *upward planar drawing* of G is a planar drawing where the edges are monotonically increasing in the vertical direction (the z-direction in our case).

Lemma 4. *Let G be a graph that admits a drawing with thickness two that can be oriented to become an upward drawing; G admits an AZPR on any given set of points P in $\mathbb{R}^3$.*

Proof. Let Γ be a drawing of G with thickness two that can be oriented to become an upward drawing $\overrightarrow{\Gamma}$. Since Γ has thickness two, the edges of $\overrightarrow{\Gamma}$ can be partitioned to obtain two upward planar drawings $\overrightarrow{\Gamma}_1$ and $\overrightarrow{\Gamma}_2$ with the same vertex set. The graph G_i represented by $\overrightarrow{\Gamma}_i$ ($i = 1, 2$) admits a BVR whose z-ordering coincides with the vertical order ρ of the vertices in Γ. Namely, $\overrightarrow{\Gamma}_i$ can be augmented to an upward planar drawing $\overrightarrow{\Gamma}'_i$ of a supergraph G'_i of G_i that is a *planar st-graph* (i.e., a planar digraph with a single source s and a single sink t, embedded so that s and t are on the external face) [22, Chapter 6]; the order ρ is an *st-numbering* of G'_i (i.e., a total order $v_1, v_2, \ldots, v_n$ of the vertices of G'_i such that $s = v_1$, $t = v_n$, and each vertex v_j other than s and t is adjacent to at least two vertices v_h and v_k such that $h < j < k$). It is known that it is possible to compute a BVR of a graph whose z-ordering is a given st-numbering [49, 54]. Thus, it is possible to compute a BVR of G'_i whose z-ordering is ρ. Since G_i is a spanning subgraph of G'_i, the computed BVR contains a BVR of G_i with z-ordering ρ. In other words G is the union of two planar graphs that admit two BVRs whose z-ordering is the same and therefore, by Lemma 3, G admits an AZPR on any given set of points P in $\mathbb{R}^3$. □

We are now ready to prove the following.

Theorem 7. *Let G be an n-vertex 1-colored graph and let P be a 1-colored point set of size n in $\mathbb{R}^3$. If G is 3-connected 1-planar or has geometric thickness two, then G admits a 1-colored AZPR on P.*

Proof. By Lemma 4 it is sufficient to prove that G admits a drawing with thickness two that can be oriented to become upward.

For the case of 3-connected 1-planar graphs, Alam et al. [1] proved that every 3-connected 1-planar graph $G = (V, E)$ admits a 1-planar drawing Γ where all edges are straight-line except one edge that has one bend. Since each edge crosses at most one other edge, the edges of Γ can be partitioned into two sets E_1 and E_2 such that the edges in each set do not cross. By possibly rotating Γ we can guarantee that all vertices have distinct z-coordinates and that the edge with one bend is monotone in the vertical direction. By orienting each edge from the vertex with lower z-coordinate to the vertex with higher z-coordinate we obtain the desired upward drawing.

In the case where $G = (V, E)$ has geometric thickness two, it admits a straight-line drawing Γ such that E can be partitioned into two sets E_1 and E_2 each containing non-intersecting edges. By possibly rotating Γ we can guarantee that each vertex of Γ has a distinct z-coordinate. Also in this case we obtain the desired upward drawing by orienting each edge of Γ from the vertex with lower z-coordinate to the vertex with higher z-coordinate. □

4.2. 2-colored AZPRs

In this section we study 2-colored AZPRs of properly 2-colored trees. The idea is to first compute a BVR whose z-ordering is consistent with $\lambda(P)$ and then to use Lemmas 1 and 2 to obtain an AZPR on P.

Let T be a properly 2-colored tree and let λ be a 2-colored sequence compatible with T. We construct a BVR of T whose z-ordering is consistent with λ. To this aim we first define a mapping of the vertices of T to the elements of

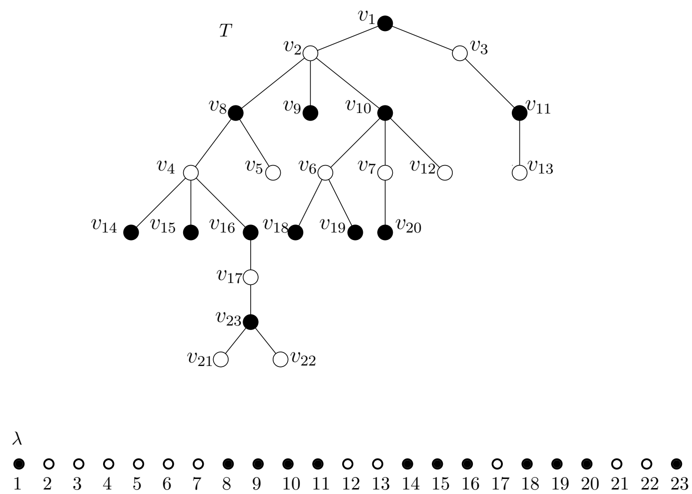


Figure 8: A properly 2-colored tree T and a 2-colored sequence λ compatible with T. The color of the root is equal to the first element of λ. The labels associated with the vertices are determined by the mapping.

λ. Root T at any vertex whose color is equal to the first element of λ and arbitrarily order the children of each node from left to right. We visit the vertices of T level by level starting from the root and at each level we visit the vertices from left to right. The current vertex v is mapped to the first element of λ with the same color as v that has not yet been used. The resulting ordering of the vertices ρ is consistent with λ by construction and its first element is the root of T (see Fig. 8).

We now explain how to use the defined mapping to construct a BVR Γ of T. First of all, we assign the z-coordinates to the vertices according to the ordering $\rho = \langle v_1, v_2, \ldots, v_n \rangle$. More precisely, we assign to vertex v_j the z-coordinate $z(v_j) = j$. This implies that the z-ordering of Γ will be ρ and therefore consistent with λ. The children of v_j whose z-coordinates are less than $z(v_j)$ are called *backward children*, the others are called *forward children*. Observe that the grandchildren of a vertex v_j (which have the same color as v_j because T is properly 2-colored) have a z-coordinate larger than that of v_j; this property will be used to guarantee that there will be no crossings between bars and visibilities.

In order to actually construct Γ, we consider the vertices level by level (starting from the root) and from left to right within each level. At each step we draw a set of vertices having the same parent, which has already been drawn because it is on the previous level (clearly the root is the first vertex to be drawn). We call a region of plane delimited by two straight lines parallel to the z-axis a *strip* and we say that a bar b *crosses* a strip σ if b intersects σ and both endpoints of b are outside σ. For example, in Fig. 9(a) the bar b_k crosses the strip

$\sigma(b_j)$ (shown in gray), while the bar b_j does not. We assume that, during the construction, the following invariant holds. For each bar b_j corresponding to a vertex v_j whose children have not yet been drawn, there exists a strip $\sigma(b_j)$ such that:

(P1) The endpoint of b_j with minimum y-coordinate lies in the interior of $\sigma(b_j)$, while the other endpoint lies outside. Furthermore, the visibility between b_j and its parent is outside $\sigma(b_j)$. See also Fig. 9(a).

(P2) If $\sigma(b_j)$ is crossed by a bar b_l, then $z(v_l) < z(v_j)$.

(P3) Let b_k be the bar with the maximum z-coordinate among those that cross $\sigma(b_j)$. If v_k is not the parent of v_j, then the children of v_j are forward children.

Intuitively, the strip $\sigma(b_j)$ is a sort of a "tunnel" where the visibility between b_j and its children is guaranteed.

The root of T is drawn as a bar of arbitrary length not touching the z-axis. Clearly the invariant holds. Let v_j be the parent of the vertices to be drawn at the generic step. We process the children of v_j from left to right. If some backward child exists, then, by property **P3**, either $\sigma(b_j)$ is not crossed by any other bar or the bar crossing it with the maximum z-coordinate represents the parent v_k of v_j. Since the children of v_j have the same color as v_k (because the tree is properly colored), the backward children of v_j have a z-coordinate greater than $z(v_k)$ (because they appear after v_k in ρ). Thus, the bars representing backward children can be drawn inside $\sigma(b_j)$ so that they are visible from b_j (refer also to Fig. 9(b)). More precisely, let $v_{j_1}, \dots, v_{j_\alpha}$ be the backward children of v_j ordered according to ρ (thus $z(v_{j_i}) < z(v_{j_{i+1}})$). The bars $b_{j_1}, \dots, b_{j_\alpha}$ are drawn inside $\sigma(b_j)$ in such a way that $y''(b_{j_i}) < y''(b_{j_{i+1}}) < y''(b_j) < y'(b_{j_{i+1}}) < y'(b_{j_i})$, where $y''(b_{j_i})$ and $y'(b_{j_i})$ represent the minimum and maximum y-coordinate of b_{j_i}, respectively.

Since each bar b_{j_i} has a maximum y-coordinate larger than the maximum y-coordinate of the bars between it and b_j, b_{j_i} can be connected with a visibility to b_j. The strip $\sigma(b_{j_i})$ of b_{j_i} is determined as follows. Let ℓ_{j_i}, for $i = 1, 2, \dots, \alpha-1$, be a z-parallel line having a y-coordinate between $y''(b_{j_i})$ and $y''(b_{j_{i+1}})$, and let ℓ_{j_α} be a z-parallel line between $y''(b_{j_\alpha})$ and $y''(b_j)$. The strip $\sigma(b_{j_i})$ with $1 < i \leq \alpha$ is contained in the region of the yz-plane delimited by the two lines ℓ_{j_i} and $\ell_{j_{i-1}}$. Further, $\sigma(b_{j_1})$ is contained in the region between ℓ_{j_1} and a z-parallel line having a y-coordinate between the minimum y-coordinate of $\sigma(b_j)$ and $y''(b_{j_1})$.

We now explain how to draw the forward children (if necessary). Refer to Fig. 9(c) for an illustration. Let $v_{k_1}, v_{k_2}, \dots, v_{k_\beta}$ be the forward children of v_j ordered according to ρ (thus $z(v_{k_i}) < z(v_{k_{i+1}})$). Since these vertices have z-coordinate larger than $z(b_j)$, by property **P2**, the bars representing forward children can be drawn inside $\sigma(b_j)$ so that they are visible from b_j. More precisely, the bars $b_{k_1}, b_{k_2}, \dots, b_{k_\beta}$ are drawn inside $\sigma(b_j)$ in such a way that $y''(b_j) < y''(b_{k_i}) < y'(b_{k_i}) < y''(b_{k_{i+1}}) < y'(b_{k_{i+1}})$. With this construction each

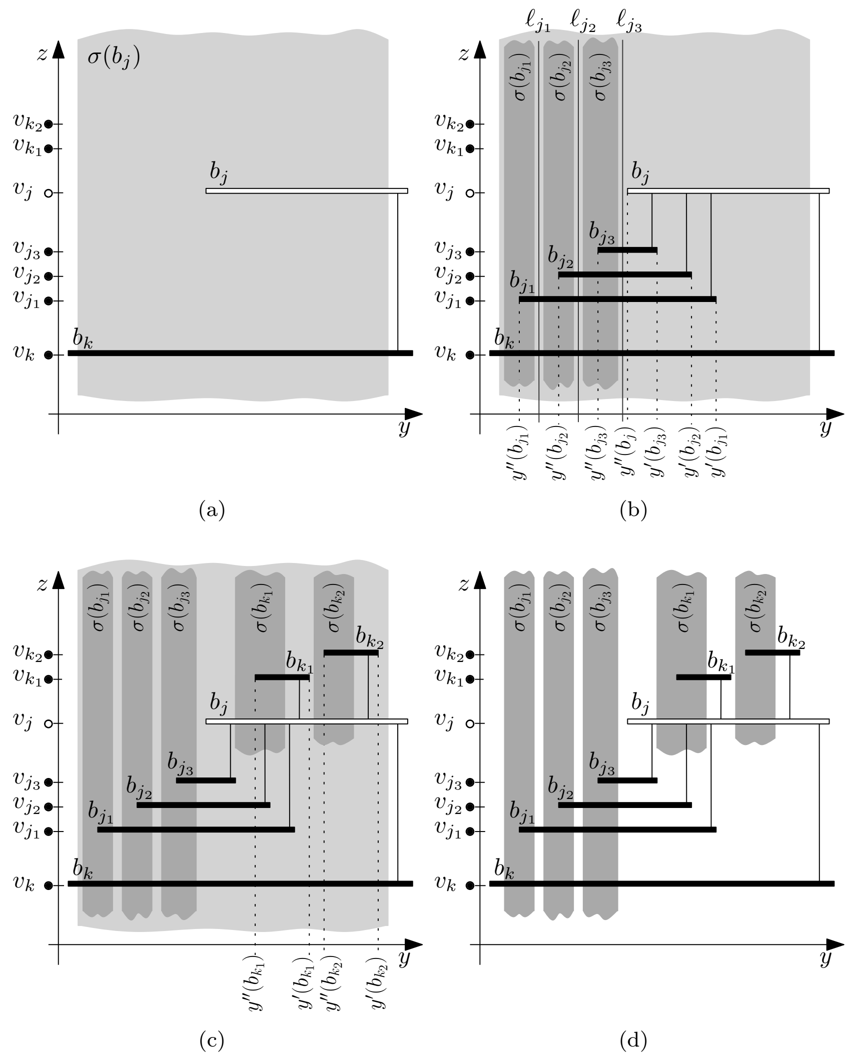


Figure 9: The generic step of the construction of a BVR Γ of a tree T. (a) Vertex v_j and its parent v_k have been already drawn; the strip $\sigma(v_j)$ is shown in gray; (b) the backward children $v_{j_1}, v_{j_2}, v_{j_3}$ are drawn inside $\sigma(b_j)$ and their strips (dark grey) are defined; (c) the forward children v_{k_1} and v_{k_2} of v_j are drawn inside $\sigma(b_j)$ and their strips (dark grey) are defined; (d) The strips of the children of v_j satisfy the properties **P1**–**P3**.

bar b_{k_i} can be connected with a visibility to b_j. The strip $\sigma(b_{k_i})$ is contained in the region of the yz-plane delimited by two z-parallel lines, the first having a y-coordinate between $y'(b_{k_{i-1}})$ and $y''(b_{k_i})$ (if $i = 1$, between $y''(b_j)$ and $y''(b_{k_i})$) and the second having a y-coordinate between $y''(b_{k_i})$ and the y-coordinate of the visibility between b_{k_i} and b_j.

Fig. 10 shows a BVR of the tree T of Fig. 8 computed by the described algorithm. The next Lemma proves the correctness of the algorithm.

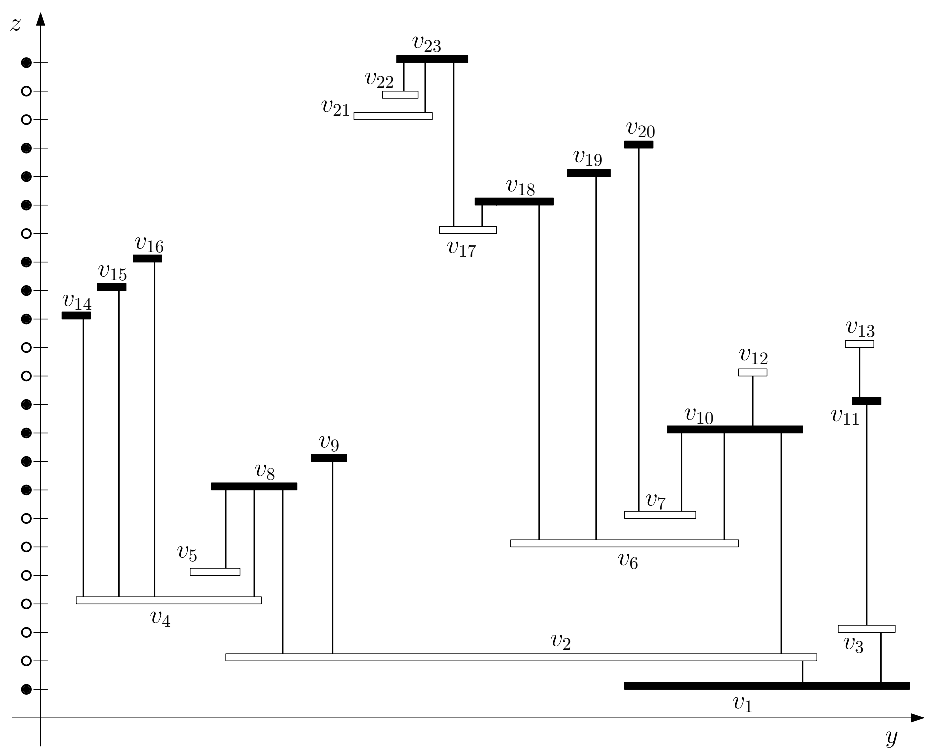


Figure 10: A BVR of tree T of Fig. 8 whose z-ordering is consistent with λ.

Lemma 5. *Let T be a properly 2-colored tree, and let λ be a 2-colored sequence compatible with T; T admits a BVR whose z-ordering is consistent with λ.*

Proof. Compute a BVR Γ of T by using the described algorithm. We prove that, during the construction, the strips defined for the newly drawn bars satisfy properties **P1**–**P3** (see also Fig. 9(d)). Let b_{j_i} be a bar representing a backward child of b_j. The strip $\sigma(b_{j_i})$ satisfies **P1** by construction. For **P2**, we observe that $\sigma(b_{j_i})$ is completely contained inside $\sigma(b_j)$ and the only bars that intersect $\sigma(b_j)$ and have a z-coordinate larger than the z-coordinate of b_{j_i} are those representing the backward children b_{j_l} of b_j with $l > i$ and the forward children of b_j. By construction all these bars have minimum y-coordinate larger than

ℓ_{j_i} and therefore they do not cross $\sigma(b_{j_i})$. For property **P3**, we observe that since b_{j_i} is a backward child, it does not have backward children. Namely, its children have the same color as its parent v_j and therefore they have a z-coordinate larger than $z(v_j)$ (because they appear after v_j in ρ). It follows that property **P3** holds.

Let b_{k_i} be a bar representing a forward child of b_j. Also in this case, the strip $\sigma(b_{k_i})$ satisfies **P1** by construction. For **P2**, $\sigma(b_{k_i})$ is completely contained in $\sigma(b_j)$ and the only bars that intersect $\sigma(b_j)$ and have a z-coordinate larger than the z-coordinate of b_{k_i} are those representing the forward children b_{k_l} of b_j with $l > i$. By construction all these bars have minimum y-coordinate larger than $y'(b_{k_i})$ and therefore they do not cross $\sigma(b_{k_i})$. For property **P3**, b_{k_i} can have backward children. On the other hand, by construction, the bar with the largest z-coordinate that crosses $\sigma(b_{k_i})$ is b_j (the bars representing the forward children b_{k_l} of b_j with $l < i$ do not cross $\sigma(b_{k_i})$). Thus property **P3** holds also in this case. □

We have the following theorem.

Theorem 8. *Let T be a properly 2-colored tree and let P be a 2-colored point set in $\mathbb{R}^3$ compatible with G; G admits a 2-colored AZPR on P.*

PROOF. By Lemma 5, T admits a BVR whose z-ordering is consistent with $\lambda(P)$. By Lemma 2 T admits a cornered ZPR whose z-ordering is consistent with $\lambda(P)$. Finally, by Lemma 1 T admits an AZPR on P whose z-ordering is consistent with $\lambda(P)$, i.e., a 2-colored AZPR on P. □

5. Conclusions and Open Problems

In this paper we introduced and studied colored anchored visibility representations in 2D and in 3D space. We used a framework based on colors to describe different variants concerning how the mapping of the vertices to the points is specified. In 2D we have proved that a 1-colored ABVR always exists for sub-Hamiltonian graphs with no restriction on the point set and that only sub-Hamiltonian graphs admit an ABVR on set of vertically aligned points. This implies that the problem of deciding whether a planar graph admits an ABVR is $\mathcal{NP}$-complete. If we restrict the set of points to be generic (i.e., all the points have distinct x- and y-coordinates) then a 1-colored ABVR exists for every planar graph. The case when not all points are vertically aligned but not all have distinct y-coordinate remains open.

As for the version with more than one color, we have used the results above to show that for every $k > 1$ there exists a k-colored planar graph that does not admit a k-colored ABVR on every set of k-colored points in the plane. A question arising from Theorem 1 and Corollary 2 is whether for $k > 1$ all k-colored sub-Hamiltonian graphs admit a k-colored ABVR on any given k-colored set of points.

The result proving the existence of ABVRs of sub-Hamiltonian graphs has been extended in 3D to prove the existence of AZPRs of graphs with page number four. We have also shown that an AZPR of K_n exists when $n \leq 22$ and does not exist for $n \geq 51$. These results derive from analogous results about ZPRs because, as stated by Theorem 6, K_n has an AZPR if and only if it has a ZPR. Hence, the longstanding open problem of investigating whether K_n for $22 < n < 51$ admits a ZPR or not, is of interest also for AZPRs.

We have also proven the existence of an AZPR on any set of given points for specific families of graphs both in the 1-colored case (3-connected 1-planar graphs and thickness-two graphs) and in the 2-colored case (properly 2-colored trees). It would be interesting to prove analogous results for other families of graphs. In particular, can we extend our results to general 1-colored 1-planar graphs and to general 2-colored trees? What about more than two colors?

Concerning the last question, we give a preliminary result for the case when the number of colors is equal to the number of vertices. A *z-assignment* of $G = (V, E)$ is a one-to-one mapping $\phi : V \to \{1, 2, \dots, |V|\}$. G is *unlabeled level planar* (ULP) if for any given z-assignment ϕ, it admits a planar straight-line drawing with $z(v) = \phi(v)$ for every $v \in V$ [24, 31, 36].

Theorem 9. *Let G be an n-colored n-vertex graph that is the union of two ULP graphs with the same vertex set and let P be an n-colored point set in $\mathbb{R}^3$; G admits an n-colored AZPR on P.*

Proof. Since both G and P are n-colored, $\lambda(P)$ defines a total order and therefore a z-assignment of G. Let G_1 and G_2 be the two ULP graphs whose union is G. Since each G_i $(i = 1, 2)$ is ULP then it admits a planar straight-line drawing Γ_i such that $z(v) = \phi(v)$ for every $v \in V$. By orienting each edge of both Γ_1 and Γ_2 from the end-vertex with lower z-coordinate to the end-vertex with higher z-coordinate we obtain two upward planar drawings of G_1 and G_2 with the same order ρ of the vertices in the vertical direction. Moreover, ρ is consistent with $\lambda(P)$. By Lemma 3, G has an AZPR Γ on P whose z-ordering is ρ. Since ρ is consistent with $\lambda(P)$, Γ is an n-colored AZPR of G on P. □

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