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ASYMPTOTIC BEHAVIOR OF A CAHN-HILLIARD/ALLEN-CAHN SYSTEM WITH TEMPERATURE

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Abstract. The main goal of this paper is to study the asymptotic behavior of a coupled Cahn-Hilliard/Allen-Cahn system with temperature. The work is divided into two parts: In the first part, the heat equation is based on the usual Fourier law. In the second one, it's based on the type III heat conduction law. In both parts, we prove the existence of exponential attractors and, therefore, of finite-dimensional global attractors.

1. **Introduction.** J. Cahn and A. Novick-Cohen introduced, in [4], the following system:

$$\frac{\partial u}{\partial t} = h^2\Delta(f(u+v) + f(u-v) - h^2\Delta u),$$
$$\frac{\partial v}{\partial t} = -f(u+v) + f(u-v) - \alpha v + h^2\Delta v,$$

where u is the concentration of one of the components and it is a conserved quantity, v is an order parameter, h is a (positive) parameter which represents the lattice spacing, α is a parameter that reflects the location of the system within the phase diagram (it may be either positive or negative), and the nonlinear term f is the derivative of a double-well potential F.

The system models simultaneous order-disorder and phase separation in binary

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alloys on a BCC lattice in the neighborhood of the triple point.
We further note that it is a gradient flow in $\left(H^1\right)' \times L^2$ for the free energy

$$J(u,v)=\int_\Omega \left\{F(u+v)+F(u-v)+\frac{\alpha}{2}v^2+\frac{1}{2}h^2(|\nabla u|^2+|\nabla v|^2)\right\}dx,$$

These equations, endowed with Neumann boundary conditions, have been studied in [2] by A. Novick-Cohen, D. Brochet, and D. Hilhorst who proved the well-posedness and the existence of maximal attractors and inertial sets (i.e., exponential attractors) for the usual cubic nonlinear term $f(s) = s^3 - \beta s$ in three space dimensions. These results were improved in [24]: taking initial conditions in $H^2(\Omega)$ allowed the authors to prove the existence of exponential attractors (and, thus, of the finite-dimensional global attractor) for a large class of nonlinear terms containing polynomials of arbitrary odd degree with a strictly positive leading coefficient in three space dimensions. This model has been also studied in [25], where an exponential attractor for singular potentials was found, and by consequence a global attractor of finite dimension.

A similar system, with a non-constant mobility, was treated in [6] where the authors proved the existence of weak solutions for the Neumann problem for a degenerate parabolic system consisting of a fourth-order and a second-order equations with singular lower-order terms in one space dimension. In addition, asymptotics for a similar system with a non-constant mobility, proposed as a diffuse interface model for simultaneous order-disorder and phase separation, was studied in [28]. This work was extended in [29], where the authors studied the partial wetting case, and their analysis accounts for motion in three space dimensions.

We also mention that numerical methods to solve coupled AC/CH systems were studied in, e.g. [19, 38, 40, 41, 42]. Furthermore, a NKS method for the implicit solution of a coupled AC/CH system was proposed in [43].

In this work, we study two systems of three simultaneous equations: a Cahn-Hilliard equation, an Allen-Cahn equation and a heat equation. In the first part of the paper, the heat equation is based on the usual Fourier law. We are able to find exponential attractors hence the global attractor associated to the system. In the second part, the heat equation is based on the type III law of thermoelasticity (note that type I corresponds to the usual Fourier law, while type II yields a purely hyperbolic (and nondissipative) equation for the temperature). There, to find exponential attractors, we were obliged to decompose the system into the sum of two systems to overcome the difficulties created by a second-order derivative term.
Not much work has been done in similar cases. However, in a recent paper, [18], the authors studied a coupled AC/CH system with temperature and long-time oscillating properties were found.

It is important to note that an exponential attractor is expected to be more robust than a global attractor under perturbations. And that's because the rate of attraction of the global attractor is slow and it's very hard to estimate it with respect to the physical parameters of the problem in general. Therefore, global attractors may change drastically under small perturbations. However, the rate of attraction of exponential attractors is considerably fast (an exponential rate) and that gives them some kind of resilience. We refer the reader to [7] and [27] for more details on this subject.

Throughout this paper, the same letter c (and, sometimes, c', c'', and c''') denotes constants which may change from line to line, or even in a same line. Similarly, the same letter Q denotes monotone increasing (with respect to each argument) functions which may change from line to line, or even in the same line.

2. **Part I: The Classical Fourier Law.** In this part, as mentioned before, the generalized heat equation is based on the classical Fourier law for heat conduction. Indeed, we can rewrite this equation as

$$\frac{\partial H}{\partial t} = -\text{div } q,$$

where q is the thermal flux vector and, assuming the Fourrier law

$$q = -\nabla\theta.$$

The function H is the enthalpy defined by

$$H = v + \theta,$$

and θ is the relative temperature.

2.1. **Setting of the Problem.** We take $h = 1$ and $\alpha = 0$ for simplicity, and consider what follows

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta(f(u+v) + f(u-v)) = 0, \tag{1}$$

$$\frac{\partial v}{\partial t} - \Delta v + f(u+v) - f(u-v) = \theta, \tag{2}$$

$$\frac{\partial \theta}{\partial t} - \Delta\theta = -\frac{\partial v}{\partial t}, \tag{3}$$

$$u = \Delta u = v = \theta = 0 \text{ on } \Gamma, \tag{4}$$

$$u_{|t=0} = u_0, v_{|t=0} = v_0, \theta_{|t=0} = \theta_0, \tag{5}$$

where Ω is a bounded domain of $\mathbb{R}^N$ (N = 1, 2, or 3) with smooth boundary Γ.

As far as the nonlinear term is concerned, we make the following assumptions:

$$f \text{ is of class } C^2,\ f(0) = 0, \tag{6}$$

$$f'(s) \geq -c, c > 0, s \in \mathbb{R}, \tag{7}$$

$$\text{and } f(s)s \geq cF(s) - c',\ F(s) \geq -c'',\ c > 0,\ c',\ c'' \geq 0,\ s \in \mathbb{R}, \tag{8}$$

where $F(s) = \int_0^s f(\xi)d\xi$.

We denote by $||.||$ the usual L^2-norm, $((.,.))$ its associated scalar product, $||.||_{-1} = ||(-\Delta)^{\frac{-1}{2}}||$, and $||.||_X$ is the norm in the Banach space X.

2.2. **Existence and Uniqueness of Solutions.** We rewrite equation (1) in the equivalent form

$$(-\Delta)^{-1}\frac{\partial u}{\partial t} - \Delta u + f(u+v) + f(u-v) = 0. \tag{9}$$

We multiply (9) by $\frac{\partial u}{\partial t}$, (2) by $\frac{\partial v}{\partial t}$, and (3) by θ and then integrate by parts over Ω, we obtain

$$\left|\left|\frac{\partial u}{\partial t}\right|\right|_{-1}^2 + \frac{1}{2}\frac{d}{dt}||\nabla u||^2 + ((f(u+v) + f(u-v), \frac{\partial u}{\partial t})) = 0, \tag{10}$$

$$\left|\left|\frac{\partial v}{\partial t}\right|\right|^2 + \frac{1}{2}\frac{d}{dt}||\nabla v||^2 + ((f(u+v) - f(u-v), \frac{\partial v}{\partial t})) = ((\theta, \frac{\partial v}{\partial t})), \tag{11}$$

$$\frac{1}{2}\frac{d}{dt}||\theta||^2 + ||\nabla\theta||^2 = ((-\frac{\partial v}{\partial t}, \theta)). \tag{12}$$

Summing (10), (11), and (12), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(||\nabla u||^2 + ||\nabla v||^2 + ||\theta||^2 + 2\int_\Omega [F(u+v) + F(u-v)]dx\right) + ||\nabla\theta||^2 \\ + \left|\left|\frac{\partial u}{\partial t}\right|\right|_{-1}^2 + \left|\left|\frac{\partial v}{\partial t}\right|\right|^2 = 0.\end{aligned} \tag{13}$$

Based on (13), we have the following result

Theorem 2.1. *Assume that $(u_0, v_0, \theta_0) \in H_0^1(\Omega)^3$. Then, (1)-(5) possesses at least one solution (u, v, θ) such that*
$(u, v, \theta) \in L^\infty(\mathbb{R}^+; H_0^1(\Omega)^2 \times L^2(\Omega)) \cap L^2_{loc}(\mathbb{R}^+; H^2(\Omega)^3)$, $\theta \in L^\infty(\mathbb{R}^+; L^2(\Omega) \cap H_0^1(\Omega))$ and $(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial \theta}{\partial t}) \in L^2(\mathbb{R}^+; H^{-1} \times L^2(\Omega)^2)$.

Proof. The proof of existence (as well as the above and the subsequent a priori estimates) are based, e.g. on a classical Galerkin scheme. Let A denote the minus Laplace operator associated with Dirichlet boundary conditions. This operator is a bounded, selfadjoint and strictly positive operator with compact inverse from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$. There is a set of eigenvectors $\{\phi_i,\ i \geq 1\}$ for this operator, associated with the eigenfunctions $0 < \lambda_1 \leq \lambda_2 \leq ...$, such that it is orthonormal relative to the inner product in $L^2(\Omega)$ and orthogonal relative to the one in $H_0^1(\Omega)$. Setting $V_m = Span\{\phi_1, ..., \phi_m\}$, we consider the following approximating problem, written in the fuctional form:

$$\frac{du_m}{dt} + A^2 u_m + A(f(u_m + v_m) + f(u_m - v_m)) = 0, \tag{14}$$

$$\frac{dv_m}{dt} + Av_m + f(u_m + v_m) - f(u_m - v_m) = \theta_m, \tag{15}$$

$$\frac{d\theta_m}{dt} + A\theta_m = -\frac{dv_m}{dt}, \tag{16}$$

together with suitable initial conditions, namely,

$$u_{m|t=0} = P_m u_0,\ v_{m|t=0} = P_m v_0,\ \theta_{m|t=0} = P_m \theta_0,$$

where P_m is the orthogonal projector from $L^2(\Omega)$ onto V_m (for the L^2-metric).

This is equivalent to the following problem:

$$\frac{d}{dt}((u_m,p))-((\Delta u_m,\Delta p))+((\nabla f(u_m+v_m),\nabla p))+((\nabla f(u_m-v_m),\nabla p))=0,$$

$$\frac{d}{dt}((v_m,q))+((\nabla v_m,\nabla q))+((f(u_m+v_m),q))-((f(u_m-v_m),q))=((\theta_m,q)),$$

$$\frac{d}{dt}((\theta_m,r))+((\nabla\theta_m,\nabla r))=-\frac{d}{dt}((v_m,r)),$$

$\forall\ p,q,r\in V_m$, together with the above initial conditions. The proof of existence of a local (in time) solution to the approximating problem is standard (indeed, one has to solve a continuous system of ODEs).

Furthermore, we can write the equivalent of the previous and the subsequent estimates (with u, v and θ replaced by u_m, v_m, and θ_m respectively); this is now fully justified and no longer formal. Then we can deduce from (13) that this solution is actually global. And, the passage to the limit is based on classical (Aubin-Lions type) compactness results. Indeed, we have, in particular, u_m bounded in $L^\infty(0,T;H^1_0(\Omega))$ and $\frac{du_m}{dt}$ bounded in $L^2(0,T;H^{-1}(\Omega))$, independently of m, which yields that (at least for a subsequence which we do not relabel) u_m converges strongly to, say, u in $C([0,T];H^{1-\delta}(\Omega)),\forall\ \delta>0$. In addition, v_m is bounded in $L^\infty(0,T;H^1_0(\Omega))$ and $\frac{dv_m}{dt}$ is bounded in $L^2(0,T;L^2(\Omega))$, independently of m, which also yields the strong convergence of v_m to, say, v in $C([0,T];H^{1-\delta}(\Omega)),\forall\ \delta>0$.

We also note that it follows from (13) that $(u,v)\in L^\infty(\mathbb{R}^+;H^1_0(\Omega)^2)$, $\theta\in L^\infty(\mathbb{R}^+;L^2(\Omega))\cap L^2(\mathbb{R}^+;H^1_0(\Omega))$ and that $(\frac{\partial u}{\partial t},\frac{\partial v}{\partial t})\in L^2(\mathbb{R}^+;H^{-1}(\Omega)\times L^2(\Omega))$.

We now multiply (9) by $-\Delta u$, (2) by $-\Delta v$, (3) by $-\Delta\theta$, and integrate over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}||u||^2+||\Delta u||^2+((f'(u+v)\nabla(u+v),\nabla u))+((f'(u-v)\nabla(u-v),\nabla u))=0, \tag{17}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\nabla v||^2+||\Delta v||^2+((f'(u+v)\nabla(u+v),\nabla v))-((f'(u-v)\nabla(u-v),\nabla v))\\=((\theta,-\Delta v)),\end{aligned} \tag{18}$$

and

$$\frac{1}{2}\frac{d}{dt}||\nabla\theta||^2+||\Delta\theta||^2=((\frac{\partial v}{\partial t},\Delta\theta)). \tag{19}$$

Summing (17), (18), and (19) and using (7), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(||u||^2+||\nabla v||^2+||\nabla\theta||^2)+||\Delta u||^2+c'(||\Delta v||^2+||\Delta\theta||^2)\\ \leq c(||\nabla u||^2+||\nabla v||^2)+c\left|\left|\frac{\partial v}{\partial t}\right|\right|^2.\end{aligned} \tag{20}$$

Hence, we can deduce from (20) that $(u,v,\theta)\in L^2_{loc}(\mathbb{R}^+;H^2(\Omega)^3)$.

We finally multiply (3) by $\frac{\partial\theta}{\partial t}$ and get

$$\frac{d}{dt}||\nabla\theta||^2 + \left|\left|\frac{\partial\theta}{\partial t}\right|\right|^2 \leq \left|\left|\frac{\partial v}{\partial t}\right|\right|^2. \tag{21}$$

Then $\frac{\partial\theta}{\partial t} \in L^2(\mathbb{R}^+; L^2(\Omega))$ which finishes the proof of the regularity of the solution. □

Lemma 2.2. *The solution of (1)-(5) in Theorem 2.1 verifies formally the following inequalities:*

- *a)*
$$E_2(t) \leq e^{-ct}E_2(0) + c', \quad \forall\ t \geq 0, \quad c > 0, \tag{22}$$
where
$$E_1 = \epsilon||u||^2_{-1} + \epsilon||v||^2 + ||\nabla u||^2 + ||\nabla v||^2 + ||\theta||^2 + 2\int_\Omega [F(u+v) + F(u-v)]dx$$
and $E_2 = E_1 + \epsilon'||\nabla\theta||^2$,
- *b) For every $r > 0$,*
$$\int_t^{t+r}\left(\left|\left|\frac{\partial u}{\partial t}\right|\right|^2_{-1} + \left|\left|\frac{\partial v}{\partial t}\right|\right|^2 + \left|\left|\frac{\partial\theta}{\partial t}\right|\right|^2\right)d\tau \leq ce^{-c't}\Big(||\nabla u_0||^2 + ||\nabla v_0||^2 + ||\nabla\theta_0||^2 + \int_\Omega [F(u_0+v_0) + F(u_0-v_0)]dx\Big) + c''(r), \tag{23}$$
- *c) There exists $T_0 = T_0(||u_0||_{H^2(\Omega)}, ||v_0||_{H^2(\Omega)}, ||\theta_0||_{H^1(\Omega)})$ such that*
$$\left|\left|\frac{\partial u}{\partial t}(t)\right|\right|^2_{-1} + \left|\left|\frac{\partial v}{\partial t}(t)\right|\right|^2 \leq \frac{1}{t}e^{c't}Q(||u_0||_{H^2(\Omega)}, ||v_0||_{H^2(\Omega)}, ||\theta_0||_{H^1(\Omega)}), \quad t \in (0, T_0], \tag{24}$$
and
$$\left|\left|\frac{\partial u}{\partial t}(t)\right|\right|^2_{-1} + \left|\left|\frac{\partial v}{\partial t}(t)\right|\right|^2 \leq e^{ct}Q(||u_0||_{H^2(\Omega)}, ||v_0||_{H^2(\Omega)}, ||\theta_0||_{H^1(\Omega)}), \quad t \geq T_0, \tag{25}$$
- *d)*
$$||u(t)||^2_{H^2(\Omega)} + ||v(t)||^2_{H^2(\Omega)} + ||\theta(t)||^2_{H^1(\Omega)} \leq e^{ct}Q(||u_0||_{H^2(\Omega)}, ||v_0||_{H^2(\Omega)}, ||\theta_0||_{H^1(\Omega)}), \quad t \geq 0, \tag{26}$$
and we can say that (u, v, θ) belongs a priori to $L^\infty(0, T; H^2(\Omega)^3)$, $T > 0$.
- *e)*
$$\frac{d}{dt}||\Delta\theta||^2 + ||\nabla\frac{\partial\theta}{\partial t}||^2 \leq ||\nabla\frac{\partial v}{\partial t}||^2, \tag{27}$$
hence $\frac{\partial\theta}{\partial t} \in L^2(0, T; H^1_0(\Omega))$ a priori.

Proof. • a)

We start by multiplying (9) by u, (2) by v, sum them together and use (8) to obtain

$$\frac{1}{2}\frac{d}{dt}(||u||_{-1}^2+||v||^2)+||\nabla u||^2+c||\nabla v||^2+c'\int_\Omega[F(u+v)+F(u-v)]dx \leq c''+c||\theta||^2. \tag{28}$$

Then, we sum (13) and ϵ(28), where ϵ is small enough so that

$$\frac{d}{dt}\Big(\epsilon||u||_{-1}^2+\epsilon||v||^2+||\nabla u||^2+||\nabla v||^2+||\theta||^2+2\int_\Omega[F(u+v)+F(u-v)]dx\Big) + c\Big(\epsilon||\nabla u||^2+||\nabla v||^2+||\nabla\theta||^2+\int_\Omega[F(u+v)+F(u-v)]dx\Big) + \Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|_{-1}^2+\Big|\Big|\frac{\partial v}{\partial t}\Big|\Big|^2 \leq c''. \tag{29}$$

We set

$$E_1=\epsilon||u||_{-1}^2+\epsilon||v||^2+||\nabla u||^2+||\nabla v||^2+||\theta||^2+2\int_\Omega[F(u+v)+F(u-v)]dx.$$

Whence, we deduce from (29) the inequality

$$\frac{dE_1}{dt}+c\Big(E_1+\Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|_{-1}^2+\Big|\Big|\frac{\partial v}{\partial t}\Big|\Big|^2+||\nabla\theta||^2\Big)\leq c'. \tag{30}$$

Next, we add ϵ'(21) and (30), we obtain

$$\frac{dE_2}{dt}+c\Big(E_2+\Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|_{-1}^2+\Big|\Big|\frac{\partial v}{\partial t}\Big|\Big|^2+\Big|\Big|\frac{\partial \theta}{\partial t}\Big|\Big|^2\Big)\leq c', \tag{31}$$

where $E_2=E_1+\epsilon'||\nabla\theta||^2$.

Applying Gronwall's lemma to (31) yields

$$E_2(t)\leq e^{-ct}E_2(0)+c',\quad \forall\ t\geq 0,\quad c>0.$$

• b) We can deduce from (31) and (22) that, for every $r>0$,

$$\int_t^{t+r}\Big(\Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|_{-1}^2+\Big|\Big|\frac{\partial v}{\partial t}\Big|\Big|^2+\Big|\Big|\frac{\partial \theta}{\partial t}\Big|\Big|^2\Big)d\tau\leq ce^{-c't}\Big(||\nabla u_0||^2+||\nabla v_0||^2+||\nabla\theta_0||^2 + \int_\Omega[F(u_0+v_0)+F(u_0-v_0)]dx\Big)+c''(r).$$

• c) We differentiate equations (2) and (9) with respect to time, we obtain

$$(-\Delta)^{-1}\frac{\partial}{\partial t}\frac{\partial u}{\partial t}-\Delta\frac{\partial u}{\partial t}+f'(u+v)(\frac{\partial u}{\partial t}+\frac{\partial v}{\partial t})+f'(u-v)(\frac{\partial u}{\partial t}-\frac{\partial v}{\partial t})=0, \tag{32}$$

$$\frac{\partial}{\partial t}\frac{\partial v}{\partial t}-\Delta\frac{\partial v}{\partial t}+f'(u+v)(\frac{\partial u}{\partial t}+\frac{\partial v}{\partial t})-f'(u-v)(\frac{\partial u}{\partial t}-\frac{\partial v}{\partial t})=\frac{\partial\theta}{\partial t}, \tag{33}$$

with

$$\frac{\partial u}{\partial t}=\frac{\partial v}{\partial t}=\frac{\partial \theta}{\partial t}=0 \text{ on } \Gamma. \tag{34}$$

We then multiply (32) by $t\frac{\partial u}{\partial t}$, (33) by $t\frac{\partial v}{\partial t}$ and we sum the resulting equations, we obtain

$$\frac{1}{2}\frac{d}{dt}\Big(t\Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|_{-1}^2+t\Big|\Big|\frac{\partial v}{\partial t}\Big|\Big|^2\Big)+ct\Big|\Big|\nabla\frac{\partial u}{\partial t}\Big|\Big|^2+t\Big|\Big|\nabla\frac{\partial v}{\partial t}\Big|\Big|^2+$$
$$((f'(u+v)(\frac{\partial u}{\partial t}+\frac{\partial v}{\partial t}),\frac{\partial u}{\partial t}+\frac{\partial v}{\partial t}))+((f'(u-v)(\frac{\partial u}{\partial t}-\frac{\partial v}{\partial t}),\frac{\partial u}{\partial t}-\frac{\partial v}{\partial t}))$$
$$=t\Big(\Big(\frac{\partial\theta}{\partial t},\frac{\partial v}{\partial t}\Big)\Big)+\frac{1}{2}\Big(\Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|_{-1}^2+\Big|\Big|\frac{\partial v}{\partial t}\Big|\Big|^2\Big),$$

which yields, owing to (7),

$$\frac{1}{2}\frac{d}{dt}\Big(t\Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|_{-1}^2+t\Big|\Big|\frac{\partial v}{\partial t}\Big|\Big|^2\Big)+ct\Big|\Big|\nabla\frac{\partial u}{\partial t}\Big|\Big|^2+t\Big|\Big|\nabla\frac{\partial v}{\partial t}\Big|\Big|^2$$
$$\leq c't\Big(\Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|^2+\Big|\Big|\frac{\partial v}{\partial t}\Big|\Big|^2+\Big|\Big|\frac{\partial\theta}{\partial t}\Big|\Big|^2\Big)+\frac{1}{2}\Big(\Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|_{-1}^2+\Big|\Big|\frac{\partial v}{\partial t}\Big|\Big|^2\Big).$$

Employing the interpolation inequality

$$\Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|^2\leq c'\Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|_{-1}\Big|\Big|\nabla\frac{\partial u}{\partial t}\Big|\Big|,\ c'>0,$$

and the Young's inequality, we deduce that

$$\frac{1}{2}\frac{d}{dt}\Big(t\Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|_{-1}^2+t\Big|\Big|\frac{\partial v}{\partial t}\Big|\Big|^2\Big)+ct\Big|\Big|\nabla\frac{\partial u}{\partial t}\Big|\Big|^2+t\Big|\Big|\nabla\frac{\partial v}{\partial t}\Big|\Big|^2$$
$$\leq c't\Big(\Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|_{-1}^2+\Big|\Big|\frac{\partial v}{\partial t}\Big|\Big|^2+\Big|\Big|\frac{\partial\theta}{\partial t}\Big|\Big|^2\Big)+\frac{1}{2}\Big(\Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|_{-1}^2+\Big|\Big|\frac{\partial v}{\partial t}\Big|\Big|^2\Big). \tag{35}$$

We now apply Gronwall's lemma and use (23) to obtain

$$\Big|\Big|\frac{\partial u}{\partial t}(t)\Big|\Big|_{-1}^2+\Big|\Big|\frac{\partial v}{\partial t}(t)\Big|\Big|^2\leq\frac{1}{t}e^{c't}Q(||u_0||_{H^2(\Omega)},||v_0||_{H^2(\Omega)},||\theta_0||_{H^1(\Omega)}),$$
$$t\in(0,T_0].$$

Moreover, summing (32) times $\frac{\partial u}{\partial t}$ and (33) times $\frac{\partial v}{\partial t}$, then using (7) and an interpolation inequality, we find

$$\frac{d}{dt}\Big(\Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|_{-1}^2+\Big|\Big|\frac{\partial v}{\partial t}\Big|\Big|^2\Big)+\Big|\Big|\nabla\frac{\partial u}{\partial t}\Big|\Big|^2+\Big|\Big|\nabla\frac{\partial v}{\partial t}\Big|\Big|^2\leq c\Big(\Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|_{-1}^2+\Big|\Big|\frac{\partial v}{\partial t}\Big|\Big|^2\Big)+c'\Big|\Big|\frac{\partial\theta}{\partial t}\Big|\Big|^2. \tag{36}$$

Applying Gronwall's lemma and using (23) and (24), we obtain

$$\Big|\Big|\frac{\partial u}{\partial t}(t)\Big|\Big|_{-1}^2+\Big|\Big|\frac{\partial v}{\partial t}(t)\Big|\Big|^2\leq e^{ct}Q(||u_0||_{H^2(\Omega)},||v_0||_{H^2(\Omega)},||\theta_0||_{H^1(\Omega)}),$$
$$t\geq T_0.$$

- d) We multiply (9) by $-\Delta\frac{\partial u}{\partial t}$, (2) by $-\Delta\frac{\partial v}{\partial t}$, and (3) by $-\Delta\theta$ and sum the resulting equations. Then using (6) and the continuous embedding $H^2(\Omega) \subset C(\overline{\Omega})$, we obtain

$$\frac{1}{2}\frac{d}{dt}(||\Delta u||^2 + ||\Delta v||^2 + ||\nabla\theta||^2) + c(||\nabla\frac{\partial v}{\partial t}||^2 + \left|\left|\frac{\partial u}{\partial t}\right|\right|^2 + ||\Delta\theta||^2) \leq Q(||u||_{H^2(\Omega)}, ||v||_{H^2(\Omega)}). \tag{37}$$

Setting $y = ||\Delta u||^2 + ||\Delta v||^2 + ||\nabla\theta||^2$, we deduce from (37) a differential inequality of the form

$$y' \leq Q(y).$$

Let z be the solution of the ordinary differential equation $z' = Q(z)$ with $z(0) = y(0)$. It follows from the comparison principle that there exists $T_0 = T_0(||u_0||_{H^2(\Omega)}, ||v_0||_{H^2(\Omega)}, ||\theta_0||_{H^1(\Omega)})$ belonging to, say, $(0, \frac{1}{2})$ such that

$$y(t) \leq z(t), t \in [0, T_0],$$

whence

$$||u(t)||^2_{H^2(\Omega)} + ||v(t)||^2_{H^2(\Omega)} + ||\theta(t)||^2_{H^1(\Omega)} \leq Q(||u_0||_{H^2(\Omega)}, ||v_0||_{H^2(\Omega)}, ||\theta_0||_{H^1(\Omega)}), \quad t \leq T_0. \tag{38}$$

We now rewrite equations (2) and (9) in the following forms

$$-\Delta u + f(u+v) + f(u-v) = h_u(t), \tag{39}$$

$$-\Delta v + f(u+v) - f(u-v) = h_v(t), \tag{40}$$

$$\text{with} \quad u = \Delta u = v = \theta = 0 \ on \ \Gamma, \tag{41}$$

for $t \geq T_0$ fixed, where

$$h_u(t) = -(-\Delta)^{-1}\frac{\partial u}{\partial t},$$

and

$$h_v(t) = -\frac{\partial v}{\partial t} + \theta,$$

satisfy, owing to (22) and (25)

$$||h_u(t)|| \leq e^{ct}Q(||u_0||_{H^2(\Omega)}, ||v_0||_{H^2(\Omega)}, ||\theta_0||_{H^1(\Omega)}), \quad t \geq T_0, \tag{42}$$

and

$$||h_v(t)|| \leq e^{ct}Q(||u_0||_{H^2(\Omega)}, ||v_0||_{H^2(\Omega)}, ||\theta_0||_{H^1(\Omega)}), \quad t \geq T_0. \tag{43}$$

We multiply (39) by u and (40) by v and we sum the result. Noting then that $f(s)s \geq -c, \ c \geq 0$, we obtain

$$||\nabla u||^2 + ||\nabla v||^2 \leq c(||h_u(t)||^2 + ||h_v(t)||^2) + c'. \tag{44}$$

Next, we multiply (39) by $-\Delta u$ and (40) by $-\Delta v$, we sum the resulting equations and obtain, using (7),

$$||\Delta u(t)||^2 + ||\Delta v(t)||^2 \leq c(||h_u(t)||^2 + ||h_v(t)||^2 + ||\nabla u(t)||^2 + ||\nabla v(t)||^2). \tag{45}$$

We thus deduce from (42)-(45) that

$$||u(t)||^2_{H^2(\Omega)} + ||v(t)||^2_{H^2(\Omega)} \leq e^{ct}Q(||u_0||_{H^2(\Omega)}, ||v_0||_{H^2(\Omega)}, ||\theta_0||_{H^1(\Omega)}), \quad t \geq T_0. \tag{46}$$

Therefore,

$$||u(t)||^2_{H^2(\Omega)} + ||v(t)||^2_{H^2(\Omega)} \le e^{ct}Q(||u_0||_{H^2(\Omega)}, ||v_0||_{H^2(\Omega)}, ||\theta_0||_{H^1(\Omega)}), \quad t \ge 0.$$

Finally, we can say that (u, v) belongs a priori to $L^\infty(0, T; H^2(\Omega)^2)$, $T > 0$.

- e) We note from (19) that we have

$$\frac{d}{dt}||\nabla\theta||^2 + ||\Delta\theta||^2 \le \left|\left|\frac{\partial v}{\partial t}\right|\right|^2. \tag{47}$$

Integrating (47) between T_0 and t, then using (23) and (38), we obtain

$$||\nabla\theta(t)||^2 \le e^{ct}Q(||u_0||_{H^2}, ||v_0||_{H^2}, ||\theta_0||_{H^1}). \tag{48}$$

Combining (38) with (48), we get

$$\begin{aligned} &||u(t)||^2_{H^2(\Omega)} + ||v(t)||^2_{H^2(\Omega)} + ||\theta(t)||^2_{H^1(\Omega)} \\ &\qquad \le e^{ct}Q(||u_0||_{H^2(\Omega)}, ||v_0||_{H^2(\Omega)}, ||\theta_0||_{H^1(\Omega)}), \\ &\qquad t \ge 0. \end{aligned} \tag{49}$$

Multiplying then (3) by $-\Delta\frac{\partial\theta}{\partial t}$, we obtain

$$\frac{d}{dt}||\Delta\theta||^2 + ||\nabla\frac{\partial\theta}{\partial t}||^2 \le ||\nabla\frac{\partial v}{\partial t}||^2.$$

Hence $\theta \in L^\infty(0, T; H^2(\Omega))$ and $\frac{\partial\theta}{\partial t} \in L^2(0, T; H^1_0(\Omega))$ a priori.

□

Lemma 2.3. *The same solution also verifies:*

- *a)*

$$\begin{aligned} ||u(t)||^2 + ||\nabla v(t)||^2 + ||\nabla\theta(t)||^2 \le e^{c't}Q\Big(&||\nabla u_0||^2 + ||\nabla v_0||^2 + ||\nabla\theta_0||^2 \\ &+ \int_\Omega [F(u_0 + v_0) + F(u_0 - v_0)]dx\Big). \end{aligned} \tag{50}$$

- *b)*

$$\begin{aligned} ||u(t)||^2_{H^2(\Omega)} + ||v(t)||^2_{H^2(\Omega)} + ||\theta(t)||^2_{H^2(\Omega)} \le e^{-ct}Q(&||u_0||_{H^2(\Omega)}, ||v_0||_{H^2(\Omega)}, \\ &||\theta_0||_{H^1(\Omega)}) + c'(r), \ c > 0, \ r > 0. \end{aligned} \tag{51}$$

Proof. • a) We start by recalling the inequality (20)

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(||u||^2 + ||\nabla v||^2 + ||\nabla\theta||^2) + ||\Delta u||^2 + c(||\Delta v||^2 + ||\Delta\theta||^2) \\ &\qquad \le c(||\nabla u||^2 + ||\nabla v||^2) + c''\left|\left|\frac{\partial v}{\partial t}\right|\right|^2. \end{aligned} \tag{52}$$

Using the interpolation inequality

$$||u||^2_{H^1(\Omega)} \le c||u|| \ ||u||_{H^2(\Omega)}, \quad c \ge 0,$$

and then the Young's inequality, we obtain

$$\begin{aligned} &\frac{d}{dt}(||u||^2 + ||\nabla v||^2 + ||\nabla\theta||^2) + c(||\Delta u||^2 + ||\Delta v||^2 + ||\Delta\theta||^2) \\ &\qquad \le c'(||u||^2 + ||\nabla v||^2) + c\left|\left|\frac{\partial v}{\partial t}\right|\right|^2. \end{aligned} \tag{53}$$

Applying now Gronwall's lemma and using(23), we find

$$||u(t)||^2 + ||\nabla v(t)||^2 + ||\nabla \theta(t)||^2 \leq e^{c't} Q\Big(||\nabla u_0||^2 + ||\nabla v_0||^2 + ||\nabla \theta_0||^2 + \int_\Omega [F(u_0+v_0) + F(u_0-v_0)]dx \Big).$$

- b) It follows from (53) after using (23) and (50) that

$$\int_0^1 (||u||^2_{H^2(\Omega)} + ||v||^2_{H^2(\Omega)} + ||\theta||^2_{H^2(\Omega)})dt \leq Q\Big(||\nabla u_0||^2 + ||\nabla v_0||^2 + ||\nabla \theta_0||^2 + \int_\Omega [F(u_0+v_0) + F(u_0-v_0)]dx \Big). \tag{54}$$

Hence, there exists $T \in (0,1)$ such that

$$||u(T)||^2_{H^2(\Omega)} + ||v(T)||^2_{H^2(\Omega)} + ||\theta(T)||^2_{H^2(\Omega)} \leq Q\Big(||\nabla u_0||^2 + ||\nabla v_0||^2 + ||\nabla \theta_0||^2 + \int_\Omega [F(u_0+v_0) + F(u_0-v_0)]dx \Big). \tag{55}$$

Repeating the estimates leading to (49), but starting from $t = T$ instead of $t = 0$, we have

$$||u(t)||^2_{H^2(\Omega)} + ||v(t)||^2_{H^2(\Omega)} + ||\theta(t)||^2_{H^1(\Omega)} \leq e^{ct} Q\Big(||u(T)||^2_{H^2(\Omega)} + ||v(T)||^2_{H^2(\Omega)} + ||\theta(T)||^2_{H^1(\Omega)} \Big).$$

Then using (55), we obtain

$$||u(1)||^2_{H^2(\Omega)} + ||v(1)||^2_{H^2(\Omega)} + ||\theta(1)||^2_{H^1(\Omega)} \leq Q\Big(||\nabla u_0||^2 + ||\nabla v_0||^2 + ||\nabla \theta_0||^2 + \int_\Omega [F(u_0+v_0) + F(u_0-v_0)]dx \Big). \tag{56}$$

We now repeat the estimates leading to (56), and since our equations are autonomous, we can make a translation in time. We obtain, for $t \geq 1$,

$$||u(t)||^2_{H^2(\Omega)} + ||v(t)||^2_{H^2(\Omega)} + ||\theta(t)||^2_{H^1(\Omega)} \leq Q\Big(||\nabla u(t-1)||^2 + ||\nabla v(t-1)||^2 + ||\nabla \theta(t-1)||^2 + \int_\Omega [F(u(t-1)+v(t-1)) + F(u(t-1)-u(t-1))]dx \Big), \tag{57}$$

which yields, owing to (22),

$$||u(t)||^2_{H^2(\Omega)} + ||v(t)||^2_{H^2(\Omega)} + ||\theta(t)||^2_{H^1(\Omega)} \leq e^{-ct} Q\Big(||u_0||^2_{H^1(\Omega)} + ||v_0||^2_{H^1(\Omega)} + ||\theta_0||^2_{H^1(\Omega)} + \int_\Omega [F(u_0+v_0) + F(u_0-v_0)]dx \Big) + c', \ c>0, \quad t \geq 1. \tag{58}$$

Combining the above estimate with (49) from 0 to 1, we obtain

$$\begin{aligned}||u(t)||^2_{H^2(\Omega)} + ||v(t)||^2_{H^2(\Omega)} + ||\theta(t)||^2_{H^1(\Omega)} \leq e^{-ct}Q\Big(||u_0||^2_{H^2(\Omega)} + ||v_0||^2_{H^2(\Omega)} \\ + ||\theta_0||^2_{H^1(\Omega)}\Big) + c',\ c>0,\quad t\geq 0.\end{aligned} \tag{59}$$

Furthermore, we recall the equation (27)

$$\frac{d}{dt}||\Delta\theta||^2 + \left|\left|\nabla\frac{\partial\theta}{\partial t}\right|\right|^2 \leq \left|\left|\nabla\frac{\partial v}{\partial t}\right|\right|^2. \tag{60}$$

Noting that it follows from (23), and (77) that

$$\begin{aligned}\int_t^{t+r}\left(\left|\left|\nabla\frac{\partial u}{\partial t}\right|\right|^2 + \left|\left|\nabla\frac{\partial v}{\partial t}\right|\right|^2\right)dx \leq e^{-ct}Q\Big(||\nabla u_0||^2 + ||\nabla v_0||^2 + ||\nabla\theta_0||^2 \\ + \int_\Omega [F(u_0+v_0) + F(u_0-v_0)]dx\Big) + c'(r),\ c>0,\ r>0,\end{aligned} \tag{61}$$

and from (47), (58) and (61) that

$$\begin{aligned}\int_t^{t+r}||\Delta\theta||^2 dx \leq e^{-ct}Q\Big(||\nabla u_0||^2 + ||\nabla v_0||^2 + ||\nabla\theta_0||^2 + \int_\Omega [F(u_0+v_0) + F(u_0-v_0)]dx\Big) \\ + c'(r),\ c>0,\quad r>0.\end{aligned} \tag{62}$$

We deduce from (60)-(62) and the uniform's Gronwall lemma, (see, e.g. [39]), that

$$\begin{aligned}||\theta(t)||^2_{H^2(\Omega)} \leq e^{-ct}Q\Big(||\nabla u_0||^2 + ||\nabla v_0||^2 + ||\nabla\theta_0||^2 + \int_\Omega [F(u_0+v_0) + F(u_0-v_0)]dx\Big) \\ + c'(r),\ c>0,\ r>0.\end{aligned} \tag{63}$$

Collecting (59) and (63), we obtain

$$\begin{aligned}||u(t)||^2_{H^2(\Omega)} + ||v(t)||^2_{H^2(\Omega)} + ||\theta(t)||^2_{H^2(\Omega)} \leq e^{-ct}Q(||u_0||_{H^2(\Omega)}, ||v_0||_{H^2(\Omega)}, \\ ||\theta_0||_{H^1(\Omega)}) + c'(r),\ c>0,\ r>0.\end{aligned}$$

□

Theorem 2.4. *Let (u,v,θ) be the solution to the problem with initial data (u_0,v_0,θ_0) obtained in Theorem 2.1. If $(u_0,v_0,\theta_0) \in (H^2(\Omega)\cap H^1_0(\Omega))^2 \times H^1_0(\Omega)$, then there exists a unique solution $(u,v,\theta) \in L^\infty_{loc}(\mathbb{R}^+;(H^2(\Omega)^3)$, and $(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial \theta}{\partial t}) \in L^\infty_{loc}(\mathbb{R}^+;H^{-1}(\Omega)\times L^2(\Omega)^2)\cap L^2_{loc}(\mathbb{R}^+;H^1_0(\Omega)^3)$.*

Proof. The proof of the existence (as well as the above a priori estimates) are based on a classical Galerkin scheme as in the previous section and mainly on the estimates (49) and (27).

We now turn our attention to prove the uniqueness:
We consider two solutions (u_1,v_1,θ_1) and (u_2,v_2,θ_2) to the problem with initial data $(u_{0,1},v_{0,1},\theta_{0,1})$ and $(u_{0,2},v_{0,2},\theta_{0,2})$ respectively.
We set $(u,v,\theta) = (u_1,v_1,\theta_1) - (u_2,v_2,\theta_2)$ and $(u_0,v_0,\theta_0) = (u_{0,1},v_{0,1},\theta_{0,1}) - (u_{0,2},v_{0,2},\theta_{0,2})$.

We then have the following system

$$(-\Delta)^{-1}\frac{\partial u}{\partial t} - \Delta u + f(u_1+v_1) - f(u_2+v_2) + f(u_1-v_1) - f(u_2-v_2) = 0, \quad (64)$$

$$\frac{\partial v}{\partial t} - \Delta v + f(u_1+v_1) - f(u_2+v_2) - f(u_1-v_1) + f(u_2-v_2) = \theta, \quad (65)$$

$$\frac{\partial \theta}{\partial t} - \Delta \theta = -\frac{\partial v}{\partial t}, \quad (66)$$

$$\text{with} \quad u = \Delta u = v = \theta = 0 \text{ on } \Gamma, \quad (67)$$

$$u_{|t=0} = u_0,\ v_{|t=0} = v_0,\ \theta_{|t=0} = \theta_0. \quad (68)$$

We multiply (64) by $\frac{\partial u}{\partial t}$, (65) by $\frac{\partial v}{\partial t}$, and (66) by θ. We then sum the result to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(||\nabla u||^2 + ||\nabla v||^2 + ||\theta||^2\Big) + \Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|_{-1}^2 + \Big|\Big|\frac{\partial v}{\partial t}\Big|\Big|^2 + ||\nabla\theta||^2 \\ + ((f(u_1+v_1) - f(u_2+v_2), \frac{\partial}{\partial t}(u+v))) \\ + ((f(u_1-v_1) - f(u_2-v_2), \frac{\partial}{\partial t}(u-v))) = 0.\end{aligned} \quad (69)$$

Furthermore,

$$\begin{aligned}&\Big(\Big(f(u_1+v_1) - f(u_2+v_2), \frac{\partial}{\partial t}(u+v)\Big)\Big) \\ &= \Big|\Big(\Big((-\Delta)^{\frac{1}{2}}\big(f(u_1+v_1) - f(u_2+v_2)\big), (-\Delta)^{\frac{-1}{2}}\frac{\partial}{\partial t}(u+v)\Big)\Big)\Big| \\ &\le c\Big|\Big|\frac{\partial}{\partial t}(u+v)\Big|\Big|_{-1} ||\nabla(f(u_1+v_1) - f(u_2+v_2))||,\ c>0,\end{aligned} \quad (70)$$

and similarly

$$\begin{aligned}&\Big(\Big(f(u_1-v_1) - f(u_2-v_2), \frac{\partial}{\partial t}(u-v)\Big)\Big) \\ &\le c'\Big|\Big|\frac{\partial}{\partial t}(u-v)\Big|\Big|_{-1} ||\nabla(f(u_1-v_1) - f(u_2-v_2))||,\ c'>0.\end{aligned} \quad (71)$$

Therefore,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(||\nabla u||^2 + ||\nabla v||^2 + ||\theta||^2\Big) + c\Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|_{-1}^2 + c'\Big|\Big|\frac{\partial v}{\partial t}\Big|\Big|^2 + ||\nabla\theta||^2 \\ &\le c(||\nabla(f(u_1+v_1) - f(u_2+v_2))||^2 + ||\nabla(f(u_1-v_1) - f(u_2-v_2))||^2).\end{aligned} \quad (72)$$

Owing to (51), we can see that

$$\begin{aligned}&||\nabla\big(f(u_1+v_1)-f(u_2+v_2)\big)||\\ &=||\nabla\Big(\int_0^1 f'(u_1+v_1+s(u_2+v_2-u_1-v_1))ds(u-v)\Big)||\\ &\leq||\int_0^1 f'(u_1+v_1+s(u_2+v_2-u_1-v_1))ds\nabla(u-v)||\\ &\quad+||(u-v)\int_0^1 f''(u_1+v_1+s(u_2+v_2-u_1-v_1))\big(\nabla(u_1+v_1)\\ &\quad+s\nabla(u_2+v_2-u_1-v_1)\big)ds||\\ &\leq Q\big(||\nabla(u-v)||+||\ |u-v|\ |\nabla(u_1+v_1)|\ ||+||\ |u-v|\ |\nabla(u_2+v_2)|\ ||\big)\\ &\leq Q(||\nabla u||+||\nabla v||).\end{aligned} \tag{73}$$

In the same way,

$$||\nabla\big(f(u_1-v_1)-f(u_2-v_2)\big)||\leq Q(||\nabla u||+||\nabla v||), \tag{74}$$

where

$Q=Q(||u_{0,1}||_{H^2(\Omega)},||u_{0,2}||_{H^2(\Omega)},||v_{0,1}||_{H^2(\Omega)},||v_{0,2}||_{H^2(\Omega)},||\theta_{0,1}||_{H^2(\Omega)},||\theta_{0,2}||_{H^2(\Omega)})$.

We deduce from (69)-(74) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(||\nabla u||^2+||\nabla v||^2+||\theta||^2\Big)+\Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|_{-1}^2+\Big|\Big|\frac{\partial v}{\partial t}\Big|\Big|^2+||\nabla\theta||^2\\ &\qquad\leq Q(||\nabla u||^2+||\nabla v||^2).\end{aligned} \tag{75}$$

Now using Gronwall's lemma, we obtain

$$||\nabla u(t)||^2+||\nabla v(t)||^2+||\theta(t)||^2\leq e^{Qt}(||\nabla u_0||^2+||\nabla v_0||^2+||\theta_0||^2), \tag{76}$$

whence the uniqueness (taking $(u_0,v_0,\theta_0)=(0,0,0)$), as well as the continuous dependence with respect to the initial data.

□

2.3. **Global and Exponential attractors.** We set $\mathcal{E}=(H^2(\Omega)\cap H_0^1(\Omega))^3$. Note that it follows from Theorem 2.2 that we can define the semigroup

$$S(t):\mathcal{E}\longrightarrow\mathcal{E}$$

$$(u_0,v_0,\theta_0)\longrightarrow(u(t),v(t),\theta(t)),$$

where (u,v,θ) is the unique solution to our system.

Theorem 2.5. *The semigroup $S(t)$ associated with (1)-(5) possesses a bounded absorbing set $\mathcal{B}_0$ in $\mathcal{E}$ such that, for every bounded set $B\subset\mathcal{E}$, there exists $t_0=t_0(B)\geq 0$ such that $t\geq t_0$ implies $S(t)B\subset\mathcal{B}_0$.*

It comes directly from (22) and (51).

Remark 1. We can assume, without loss of generality, that $\mathcal{B}_0$ is positively invariant by $S(t)$, i.e., $S(t)\mathcal{B}_0\subset\mathcal{B}_0,\forall\ t\geq 0$.

Theorem 2.6. *The semigroup $S(t)$ possesses an exponential attractor $\mathcal{M}\subset\mathcal{B}_0$, i.e.,*
(i) $\mathcal{M}$ is compact in $H^1(\Omega)^2\times L^2(\Omega)$;
(ii) $\mathcal{M}$ is positively invariant, which means that $S(t)\mathcal{M}\subset\mathcal{M},\ \forall\ t\geq 0$;
(iii) $\mathcal{M}$ has a finite fractal dimension in $H^1(\Omega)^2\times L^2(\Omega)$;

(iv) $\mathcal{M}$ *attracts exponentially fast the bounded subsets of* $\mathcal{E}:\ \forall\ B\subset\mathcal{E}$ *bounded,* $dist_{H^1(\Omega)^2\times L^2(\Omega)}(S(t)B,\mathcal{M})\leq Q(||B||_{\mathcal{E}})e^{-ct}, c>0,\ t\geq 0,$ *where the constant* c *is independent of* B *and* $dist_{H^1(\Omega)^2\times L^2(\Omega)}$ *denotes the Hausdorff semidistance between sets defined by*

$$dist_{H^1(\Omega)^2\times L^2(\Omega)}(A,B)=\sup_{a\in A}\inf_{b\in B}||a-b||_{H^1(\Omega)^2\times L^2(\Omega)}.$$

Proof. Here, we assume that the initial conditions are in the bounded absorbing set $\mathcal{B}_0$. To complete the proof, we need an asymptotic smoothing property on the difference of two solutions, a Hölder estimate with respect to space and time, and a compactness estimate of the solution. These are the key tools to construct exponential attractors (see [8]-[10], [11], [26], and [27]).

The Hölder estimate is as follows

$$\begin{aligned}
&||u(t_1)-u(t_2)||_{H^1(\Omega)}+||v(t_1)-v(t_2)||_{H^1(\Omega)}+||\theta(t_1)-\theta(t_2)||_{L^2(\Omega)}\\
&\qquad\leq c(||\nabla(u(t_1)-u(t_2))||+||\nabla(v(t_1)-v(t_2))||+||(\theta(t_1)-\theta(t_2))||)\\
&\qquad\leq c\left(\left|\left|\int_{t_1}^{t_2}\nabla\frac{\partial u}{\partial t}d\tau\right|\right|+\left|\left|\int_{t_1}^{t_2}\nabla\frac{\partial v}{\partial t}d\tau\right|\right|+\left|\left|\int_{t_1}^{t_2}\frac{\partial\theta}{\partial t}d\tau\right|\right|\right)\\
&\qquad\leq c|t_1-t_2|^{\frac{1}{2}}\left|\int_{t_1}^{t_2}\left(\left|\left|\nabla\frac{\partial u}{\partial t}\right|\right|^2+\left|\left|\nabla\frac{\partial v}{\partial t}\right|\right|^2+\left|\left|\frac{\partial\theta}{\partial t}\right|\right|^2\right)d\tau\right|^{\frac{1}{2}}.
\end{aligned}$$

We now differentiate equations (2) and (9) with respect to time, then we multiply the resulting equations by $\frac{\partial u}{\partial t}$ and $\frac{\partial v}{\partial t}$ respectively, we use (7) and an interpolation inequality to find

$$\frac{d}{dt}\left(\left|\left|\frac{\partial u}{\partial t}\right|\right|_{-1}^2+\left|\left|\frac{\partial v}{\partial t}\right|\right|^2\right)+\left|\left|\nabla\frac{\partial u}{\partial t}\right|\right|^2+\left|\left|\nabla\frac{\partial v}{\partial t}\right|\right|^2\leq c\left(\left|\left|\frac{\partial u}{\partial t}\right|\right|_{-1}^2+\left|\left|\frac{\partial v}{\partial t}\right|\right|^2\right)+c'\left|\left|\frac{\partial\theta}{\partial t}\right|\right|^2. \tag{77}$$

We note that it follows from (23), (24), (77) and the fact that the initial conditions are in a bounded absorbing set that

$$\int_{t_1}^{t_2}\left(\left|\left|\nabla\frac{\partial u}{\partial t}\right|\right|^2+\left|\left|\nabla\frac{\partial v}{\partial t}\right|\right|^2\right)d\tau\leq c, \tag{78}$$

where c only depends on $\mathcal{B}_0$ and $T\geq T_0$ such that $t_1,\ t_2\ \in[T_0,T]$.

Moreover, it follows from (75) and (76) that

$$\int_0^t\left(\left|\left|\frac{\partial u}{\partial t}\right|\right|_{-1}^2+\left|\left|\frac{\partial v}{\partial t}\right|\right|^2+||\nabla\theta||^2\right)dx\leq ce^{Qt}(||\nabla u_0||^2+||\nabla v_0||^2+||\theta_0||^2)\leq c'', \tag{79}$$

where c'' only depends on $\mathcal{B}_0$.

Plus, it follows from (21), (50), and (79) that

$$\int_{t_1}^{t_2}\left|\left|\frac{\partial\theta}{\partial t}\right|\right|^2dx\leq c, \tag{80}$$

where c only depends on $\mathcal{B}_0$.

Therefore, we have

$$||u(t_1)-u(t_2)||_{H^1(\Omega)}+||v(t_1)-v(t_2)||_{H^1(\Omega)}+||\theta(t_1)-\theta(t_2)||_{L^2(\Omega)} \leq c|t_1-t_2|^{\frac{1}{2}}, \quad (81)$$

where c only depends on $\mathcal{B}_0$, and $t_1, t_2 \in [T_0, T]$, where $T \in \mathbb{R}^+$.

We now want to find a compactness estimate:

First, we differentiate (64) and (65) with respect to time, we multiply the resulting equations by $(t-T_0)\frac{\partial u}{\partial t}$ and $(t-T_0)\frac{\partial v}{\partial t}$ respectively, where T_0 is the same as before and we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left((t-T_0)\left|\left|\frac{\partial u}{\partial t}\right|\right|_{-1}^2 + (t-T_0)\left|\left|\frac{\partial v}{\partial t}\right|\right|^2\right) + (t-T_0)\left|\left|\nabla\frac{\partial u}{\partial t}\right|\right|^2 \\
&\quad + (t-T_0)\left|\left|\nabla\frac{\partial v}{\partial t}\right|\right|^2 \leq c(t-T_0)\left(\left|\left|\frac{\partial u}{\partial t}\right|\right|^2 + \left|\left|\frac{\partial v}{\partial t}\right|\right|^2\right) \\
&\quad + \frac{1}{2}\left|\left|\frac{\partial u}{\partial t}\right|\right|_{-1}^2 + \frac{1}{2}\left|\left|\frac{\partial v}{\partial t}\right|\right|^2 - (t-T_0)\left(\left(\frac{\partial \theta}{\partial t}, \frac{\partial v}{\partial t}\right)\right) \\
&\quad + (t-T_0)\int_\Omega \left(|f'(u_1+v_1) - f'(u_2+v_2)|\left|\frac{\partial}{\partial t}(u+v)\right|\left|\frac{\partial}{\partial t}(u_2+v_2)\right|\right)dx \\
&\quad + (t-T_0)\int_\Omega \left(|f'(u_1-v_1) - f'(u_2-v_2)|\left|\frac{\partial}{\partial t}(u+v)\right|\left|\frac{\partial}{\partial t}(u_2-v_2)\right|\right)dx.
\end{aligned}$$

Noting that

$$\begin{aligned}
&\int_\Omega \left(|f'(u_1+v_1) - f'(u_2+v_2)|\left|\frac{\partial}{\partial t}(u+v)\right|\left|\frac{\partial}{\partial t}(u_2+v_2)\right|\right)dx \\
&\qquad \leq c\int_\Omega \left(|u+v|\left|\frac{\partial}{\partial t}(u+v)\right|\left|\frac{\partial}{\partial t}(u_2+v_2)\right|\right)dx \qquad (82)\\
&\qquad \leq c||\nabla(u+v)||\left|\left|\nabla\frac{\partial}{\partial t}(u+v)\right|\right| \left|\left|\frac{\partial}{\partial t}(u_2+v_2)\right|\right|,
\end{aligned}$$

using Hölder inequality and the continuous embeddings $H^1(\Omega) \subset L^3(\Omega)$ and $H^1(\Omega) \subset L^6(\Omega)$.

Similarly,

$$\begin{aligned}
&\int_\Omega \left(|f'(u_1-v_1) - f'(u_2-v_2)|\left|\frac{\partial}{\partial t}(u+v)\right|\left|\frac{\partial}{\partial t}(u_2-v_2)\right|\right)dx \\
&\qquad \leq c||\nabla(u-v)||\left|\left|\nabla\frac{\partial}{\partial t}(u+v)\right|\right| \left|\left|\frac{\partial}{\partial t}(u_2-v_2)\right|\right|.
\end{aligned} \qquad (83)$$

Owing then to a proper interpolation inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big((t-T_0)\Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|_{-1}^2+(t-T_0)\Big|\Big|\frac{\partial v}{\partial t}\Big|\Big|^2\Big)+c(t-T_0)\Big(\Big|\Big|\nabla\frac{\partial u}{\partial t}\Big|\Big|^2\\
&\quad+\Big|\Big|\nabla\frac{\partial v}{\partial t}\Big|\Big|^2\Big)\le c(t-T_0)\Big(\Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|_{-1}^2+\Big|\Big|\frac{\partial v}{\partial t}\Big|\Big|^2+\Big|\Big|\frac{\partial \theta}{\partial t}\Big|\Big|^2\Big)\\
&\quad+\frac{1}{2}\Big(\Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|_{-1}^2+\Big|\Big|\frac{\partial v}{\partial t}\Big|\Big|^2\Big)+c'(t-T_0)||\nabla(u+v)||^2\Big|\Big|\frac{\partial}{\partial t}(u_2+v_2)\Big|\Big|^2\\
&\quad+c'(t-T_0)||\nabla(u-v)||^2\Big|\Big|\frac{\partial}{\partial t}(u_2-v_2)\Big|\Big|^2.
\end{aligned}\tag{84}$$

Noting also that it follows from (23), (77), and (25) that

$$\int_{T_0}^t\Big(\Big|\Big|\nabla\frac{\partial u}{\partial t}\Big|\Big|^2+\Big|\Big|\nabla\frac{\partial v}{\partial t}\Big|\Big|^2\Big)dx\le ce^{c't},\ t\ \ge\ T_0,$$

where the constants only depend on $\mathcal{B}_0$. Hence we deduce that

$$\int_{T_0}^t\Big(\Big|\Big|\frac{\partial u_2}{\partial t}\Big|\Big|^2+\Big|\Big|\frac{\partial v_2}{\partial t}\Big|\Big|^2\Big)dx\le ce^{c't},\ t\ \ge\ T_0,\tag{85}$$

for $(u,v)=(u_2,v_2)$ and the constants only depend on $\mathcal{B}_0$.

Furthermore, applying Gronwall's lemma on (84) over (T_0,t) and owing to (76), (79), (80) and (85), we obtain

$$\Big|\Big|\frac{\partial u}{\partial t}(t)\Big|\Big|_{-1}^2+\Big|\Big|\frac{\partial v}{\partial t}(t)\Big|\Big|^2\le ce^{c't}(||u_0||^2_{H^1(\Omega)}+||v_0||^2_{H^1(\Omega)}+||\theta_0||^2),\quad t\ge 1,\tag{86}$$

where the constants only depend on $\mathcal{B}_0$.

Next, we rewrite equations (64) and (65) in the following forms

$$-\Delta u=\tilde{h}_u(t),\tag{87}$$

and

$$-\Delta v=\tilde{h}_v(t),\tag{88}$$

where $u=\Delta u=v=\theta=0$ on Γ, for $t\ge 1$ fixed,

and

$$\tilde{h}_u(t)=-(-\Delta)^{-1}\frac{\partial u}{\partial t}-\Big(f(u_1+v_1)-f(u_2+v_2)\Big)-\Big(f(u_1-v_1)-f(u_2-v_2)\Big),\tag{89}$$

$$\tilde{h}_v(t)=-\frac{\partial v}{\partial t}-\Big(f(u_1+v_1)-f(u_2+v_2)\Big)-\Big(f(u_2-v_2)-f(u_1-v_1)\Big)+\theta,\tag{90}$$

satisfy, owing to (76) and (86),

$$||\tilde{h}_u(t)||^2\le ce^{c't}(||u_0||^2_{H^1(\Omega)}+||v_0||^2_{H^1(\Omega)}+||\theta_0||^2),\ t\ge 1,\tag{91}$$

and

$$||\tilde{h}_v(t)||^2\le ce^{c't}(||u_0||^2_{H^1(\Omega)}+||v_0||^2_{H^1(\Omega)}+||\theta_0||^2),\ t\ge 1,\tag{92}$$

where the constants only depend on $\mathcal{B}_0$.

Multiplying now (87) by $-\Delta u$ and (88) by $-\Delta v$, we obtain

$$||\Delta u(t)|| \leq ||\tilde{h}_u(t)||, \ t \geq 1,$$

and

$$||\Delta v(t)|| \leq ||\tilde{h}_v(t)||, \ t \geq 1,$$

whence

$$\begin{aligned} ||u(t)||^2_{H^2(\Omega)} + ||v(t)||^2_{H^2(\Omega)} \leq ce^{c't}(||u_0||^2_{H^1(\Omega)} + ||v_0||^2_{H^1(\Omega)} + ||\theta_0||^2), \\ t \geq 1, \end{aligned} \tag{93}$$

where the constants only depend on $\mathcal{B}_0$.

We also multiply (66) by $-(t-T_0)\Delta\theta$ and find

$$\frac{d}{dt}((t-T_0)||\nabla\theta||^2) + (t-T_0)||\Delta\theta||^2 \leq c(t-T_0)\Big(\Big|\Big|\nabla\frac{\partial v}{\partial t}\Big|\Big|^2 + ||\nabla\theta||^2\Big). \tag{94}$$

We find combining (84) and (94) then applying Gronwall's lemma (applied over (T_0, t); note that $T_0 \leq 1$) and using (76)-(80), and (85), we find

$$\begin{aligned} \Big|\Big|\frac{\partial u}{\partial t}(t)\Big|\Big|^2_{-1} + \Big|\Big|\frac{\partial v}{\partial t}(t)\Big|\Big|^2 + ||\nabla\theta(t)||^2 \leq ce^{c't}(||u_0||^2_{H^1(\Omega)} + ||v_0||^2_{H^1(\Omega)} + ||\theta_0||^2), \\ t \geq 1, \end{aligned} \tag{95}$$

where the constants only depend on $\mathcal{B}_0$.

At the end, we can see from (93) and (95), that

$$\begin{aligned} ||u(t)||^2_{H^2(\Omega)} + ||v(t)||^2_{H^2(\Omega)} + ||\theta(t)||^2_{H^1(\Omega)} \leq ce^{c't}(||u_0||^2_{H^1(\Omega)} + ||v_0||^2_{H^1(\Omega)} + ||\theta_0||^2), \\ t \geq 1, \end{aligned} \tag{96}$$

And the result follows from (76), (81), and (96). □

Moreover, we can deduce from Theorem 4.1 and standard results the

Corollary 1. *The semigroup $S(t)$ possesses the finite dimensional global attractor $\mathcal{A} \subset \mathcal{B}_0$.*

Remark 2. The global attractor $\mathcal{A}$ is the smallest (for the inclusion) compact set of the phase space which is invariant by the flow (i.e., $S(t)\mathcal{A} = \mathcal{A}, \ \forall t \geq 0$) and attracts all bounded sets of initial data as time goes to infinity; that's why, it's important in the study of the asymptotic behavior of the system. Furthermore, the finite dimensionality means, roughly speaking, that, even though the initial phase space is infinite dimensional, the reduced dynamics can be described by a finite number of parameters. We refer the reader to [1], [3], [39], and [27] for more details and discussions on this topic.

3. **Part II: The Type III Law.** The classical Fourier law presented in the previous section has one essential drawback, that is, it predicts that thermal signals propagate with an infinite speed, which violates causality (the so-called 'paradox of heat conduction', see, e.g., [5]). That is why, several modifications of this law have been proposed in the literature to correct this unrealistic feature, leading to a second order in time equation for the temperature.

In particular, in [23], the authors considered in the place of the Fourier law, the Maxwell-Cattaneo law

$$\left(1+\eta\frac{\partial}{\partial t}\right)q=-\nabla\theta,\ \ \eta>0,$$

which leads to

$$\eta\frac{\partial^2\theta}{\partial t^2}+\frac{\partial\theta}{\partial t}-\Delta\theta=-\eta\frac{\partial^2 v}{\partial t^2}-\frac{\partial v}{\partial t},$$

(see also [16], [17]).

On the other hand, Green and Naghdi proposed in [12]-[15] an alternative treatment for a thermomechanical theory of deformable media which presents an entropy balance rather than the usual entropy inequality. However, if we restrict our attention to the heat conduction, we recall that three different theories, labelled as type I, type II and type III, were proposed. In particular, the Fourier law is found when type I is linearized. The linearized versions of the two other theories are described by the following constitutive equations (knowing that we are going to study only the type III in what follows)

$$q=-k\nabla\alpha,\ \ k>0,\ \ \text{(Type II)} \tag{97}$$

and

$$q=-k\nabla\alpha-k^*\nabla\theta,\ \ k,\ \ k^*>0,\ \ \text{(Type III)}, \tag{98}$$

where

$$\alpha(t)=\int_{t_0}^t\theta(\tau)d\tau+\alpha_0,\quad \left(\theta=\frac{\partial\alpha}{\partial t}\right)$$

is called the thermal displacement variable. These theories were well studied in the recent years and, particularly, a special interest was given to the qualitative study of the solutions (see e.g. [30]-[35] for studies concerned with linear thermoelastic theories). In addition, non-linear acceleration waves have been studied for types II and III non-linear thermoelasticity [36] and fluids without energy dissipation [37].

Adding equations (97) and (98) to the equation

$$\frac{\partial H}{\partial t}+\operatorname{div}\,q=0, \tag{99}$$

we obtain the following equations

$$\frac{\partial^2\alpha}{\partial t^2}-k\Delta\alpha=-\frac{\partial v}{\partial t}$$

for type II and

$$\frac{\partial^2\alpha}{\partial t^2}-k^*\frac{\partial}{\partial t}\Delta\alpha-k\Delta\alpha=-\frac{\partial v}{\partial t}$$

for type III.

3.1. **Setting of the New Problem.** We consider the following initial and boundary value problem (for simplicity, we take $k = k^* = 1$):

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta(f(u+v) + f(u-v)) = 0, \tag{100}$$

$$\frac{\partial v}{\partial t} - \Delta v + f(u+v) - f(u-v) = \frac{\partial \alpha}{\partial t}, \tag{101}$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial v}{\partial t}, \tag{102}$$

$$u = \Delta u = v = \alpha = 0 \text{ on } \Gamma, \tag{103}$$

$$u_{|t=0} = u_0,\ v_{|t=0} = v_0,\ \alpha_{|t=0} = \alpha_0,\ \frac{\partial \alpha}{\partial t}\Big|_{t=0} = \alpha_1, \tag{104}$$

where Ω is a bounded domain of $\mathbb{R}^N$ (N = 1, 2, or 3) with smooth boundary Γ.

We assume that f is of class C^2 and satisfies

$$-c \le F(s) \le f(s)s,\ c \ge 0,\ s \in \mathbb{R}, \tag{105}$$

where

$$F(s) = \int_0^s f(\tau)d\tau.$$

We also assume that

$$f(0) = 0, \quad f'(s) \ge -c', \quad s \in \mathbb{R},\ c' \ge 0. \tag{106}$$

3.2. **Global and Exponential Attractors.**

Lemma 3.1. *The solution of (100)-(104) verifies formally the following:*

- *a)*

$$\begin{aligned} &\frac{d}{dt}\Big(||\nabla u||^2 + ||\nabla v||^2 + 2\int_\Omega [F(u+v) + F(u-v)]dx + \Big|\Big|\frac{\partial \alpha}{\partial t}\Big|\Big|^2 + ||\nabla \alpha||^2\Big) \\ &\quad + 2\Big(\Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|_{-1}^2 + \Big|\Big|\frac{\partial v}{\partial t}\Big|\Big|^2 + \Big|\Big|\nabla \frac{\partial \alpha}{\partial t}\Big|\Big|^2\Big) = 0, \end{aligned} \tag{107}$$

- *b)*

$$\begin{aligned} E_3(t) \le ce^{-c't}\Big(&||u_0||_{-1}^2 + ||v_0||^2 + ||\alpha_0||^2 + ||\nabla u_0||^2 + ||\nabla v_0||^2 + ||\nabla \alpha_0||^2 + ||\alpha_1||^2 \\ &+ \int_\Omega [F(u_0+v_0) + F(u_0-v_0)]dx\Big),\ c' > 0,\ t \ge 0, \end{aligned} \tag{108}$$

where

$$\begin{aligned} E_3 = \epsilon_1||u||_{-1}^2 + \epsilon_1||v||^2 + ||\nabla u||^2 + ||\nabla v||^2 + 2\int_\Omega [F(u+v) + F(u-v)]dx \\ + (1+\epsilon_2)||\nabla \alpha||^2 + 2\epsilon_2((\frac{\partial \alpha}{\partial t}, \alpha)) + \Big|\Big|\frac{\partial \alpha}{\partial t}\Big|\Big|^2, \end{aligned}$$

- *c)*

$$\left\|\frac{\partial u}{\partial t}(t)\right\|_{-1}^2 + \left\|\frac{\partial v}{\partial t}(t)\right\|^2 + \left\|\frac{\partial \alpha}{\partial t}(t)\right\|^2 + ||\nabla\alpha(t)||^2 \le e^{ct}Q\Big(||u_0||^2_{H^1(\Omega)} + ||v_0||^2_{H^1(\Omega)} + ||\alpha_0||^2_{H^1(\Omega)} + ||\alpha_1||^2 + \int_\Omega [F(u_0+v_0)+F(u_0-v_0)]dx\Big), \tag{109}$$

$t > 0$.

- *d)*

$$\frac{1}{2}\frac{d}{dt}\left(||\Delta\alpha||^2 + \left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2\right) + c\left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2 \le \left\|\nabla\frac{\partial v}{\partial t}\right\|^2, \tag{110}$$

- *e)*

$$||u(t)||^2_{H^2(\Omega)} + ||v(t)||^2_{H^2(\Omega)} + ||\alpha(t)||^2_{H^2(\Omega)} + \left\|\frac{\partial\alpha}{\partial t}(t)\right\|^2_{H^1(\Omega)} \le e^{ct}Q(||u_0||_{H^2(\Omega)}, ||v_0||_{H^2(\Omega)}, ||\alpha_0||_{H^2(\Omega)}, ||\alpha_1||_{H^1(\Omega)}), \quad t \ge 0. \tag{111}$$

Proof. • a) We rewrite (100) in the equivalent form

$$(-\Delta)^{-1}\frac{\partial u}{\partial t} - \Delta u + f(u+v) + f(u-v) = 0 \tag{112}$$

We multiply (112) by $\frac{\partial u}{\partial t}$, (101) by $\frac{\partial v}{\partial t}$ and have, summing the results,

$$\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + \left\|\frac{\partial v}{\partial t}\right\|^2 + \frac{d}{dt}\left(\frac{1}{2}||\nabla u||^2 + \frac{1}{2}||\nabla v||^2 + \int_\Omega [F(u+v)+F(u-v)]dx\right) = \left(\left(\frac{\partial\alpha}{\partial t}, \frac{\partial v}{\partial t}\right)\right). \tag{113}$$

We then multiply (102) by $\frac{\partial\alpha}{\partial t}$ to obtain

$$\frac{1}{2}\frac{d}{dt}\left(\left\|\frac{\partial\alpha}{\partial t}\right\|^2 + ||\nabla\alpha||^2\right) + \left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2 = -\left(\left(\frac{\partial\alpha}{\partial t}, \frac{\partial v}{\partial t}\right)\right). \tag{114}$$

Thus, summing (113) and (114), we find

$$\frac{d}{dt}\left(||\nabla u||^2 + ||\nabla v||^2 + 2\int_\Omega [F(u+v)+F(u-v)]dx + \left\|\frac{\partial\alpha}{\partial t}\right\|^2 + ||\nabla\alpha||^2\right) + 2\left(\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + \left\|\frac{\partial v}{\partial t}\right\|^2 + \left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2\right) = 0.$$

- b) We multiply (112) by u, (101) by v and sum the results to have, owing to (105),

$$\frac{d}{dt}(||u||^2_{-1}+||v||^2)+||\nabla u||^2+||\nabla v||^2+\int_\Omega [F(u+v)+F(u-v)]dx \le c\left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2. \tag{115}$$

Then, we multiply (102) by α and obtain

$$\frac{d}{dt}\left(||\nabla\alpha||^2 + 2(\frac{\partial\alpha}{\partial t}, \alpha)\right) + c||\nabla\alpha||^2 \le c'\left\|\frac{\partial v}{\partial t}\right\|^2 + c''\left\|\frac{\partial\alpha}{\partial t}\right\|^2. \tag{116}$$

We sum (107), ϵ_1(115) and ϵ_2(116), where ϵ_1 and $\epsilon_2 > 0$ are chosen small enough so that

$$\left|\left|\frac{\partial\alpha}{\partial t}\right|\right|^2 + \epsilon_2\Big(||\nabla\alpha||^2 + 2(\frac{\partial\alpha}{\partial t},\alpha)\Big) \geq c\Big(\left|\left|\frac{\partial\alpha}{\partial t}\right|\right|^2 + ||\nabla\alpha||^2\Big),\ c>0, \tag{117}$$

$$2 - \epsilon_1 c - \epsilon_2 c' > 0, \tag{118}$$

$$2 - \epsilon_2 c'' > 0, \tag{119}$$

and have an inequality of the form

$$\frac{dE_3}{dt} + E_3 + \left|\left|\frac{\partial u}{\partial t}\right|\right|^2_{-1} + \left|\left|\frac{\partial v}{\partial t}\right|\right|^2 + \left|\left|\nabla\frac{\partial\alpha}{\partial t}\right|\right|^2 + ||\nabla\alpha||^2 \leq 0, \tag{120}$$

where

$$E_3 = \epsilon_1||u||^2_{-1} + \epsilon_1||v||^2 + ||\nabla u||^2 + ||\nabla v||^2 + 2\int_\Omega [F(u+v)+F(u-v)]dx$$
$$+ (1+\epsilon_2)||\nabla\alpha||^2 + 2\epsilon_2((\frac{\partial\alpha}{\partial t},\alpha)) + \left|\left|\frac{\partial\alpha}{\partial t}\right|\right|^2,$$

satisfies

$$E_3 \geq c\Big(||\nabla u||^2 + ||\nabla v||^2 + \int_\Omega [F(u+v)+F(u-v)]dx + ||\nabla\alpha||^2 + \left|\left|\frac{\partial\alpha}{\partial t}\right|\right|^2\Big) - c', \quad c>0. \tag{121}$$

In particular, we deduce from (120) the following estimate

$$E_3(t) \leq ce^{-c't}\Big(||u_0||^2_{-1} + ||v_0||^2 + ||\alpha_0||^2 + ||\nabla u_0||^2 + ||\nabla v_0||^2 + ||\nabla\alpha_0||^2 + ||\alpha_1||^2 + \int_\Omega [F(u_0+v_0)+F(u_0-v_0)]dx\Big),\ c'>0,\ t\geq 0.$$

Furthermore, for every $r>0$,

$$\int_t^{t+r}\Big(\left|\left|\frac{\partial u}{\partial t}\right|\right|^2_{-1} + \left|\left|\frac{\partial v}{\partial t}\right|\right|^2 + \left|\left|\nabla\frac{\partial\alpha}{\partial t}\right|\right|^2 + ||\nabla\alpha||^2\Big)dx \leq ce^{-c't}\Big(||u_0||^2_{-1} + ||v_0||^2 + ||\alpha_0||^2 + ||\nabla u_0||^2 + ||\nabla v_0||^2 + ||\nabla\alpha_0||^2 + ||\alpha_1||^2 + \int_\Omega [F(u_0+v_0)+F(u_0-v_0)]dx\Big) + c''(r), \quad c'>0,\ t\geq 0. \tag{122}$$

- c) We multiply (100) by $\frac{\partial u}{\partial t}$, (101) by $-\Delta\frac{\partial v}{\partial t}$, and (102) by $-\Delta\frac{\partial\alpha}{\partial t}$, we obtain

$$\frac{d}{dt}||\Delta u||^2 + \left|\left|\frac{\partial u}{\partial t}\right|\right|^2 \leq Q(||u||^2_{H^2(\Omega)} + ||v||^2_{H^2(\Omega)}), \tag{123}$$

$$\frac{d}{dt}||\Delta v||^2 + \left|\left|\nabla\frac{\partial v}{\partial t}\right|\right|^2 \leq Q(||u||^2_{H^2(\Omega)} + ||v||^2_{H^2(\Omega)}) + \left|\left|\nabla\frac{\partial\alpha}{\partial t}\right|\right|^2, \tag{124}$$

and

$$\frac{d}{dt}\Big(||\Delta\alpha||^2 + \left|\left|\nabla\frac{\partial\alpha}{\partial t}\right|\right|^2\Big) + c\left|\left|\Delta\frac{\partial\alpha}{\partial t}\right|\right|^2 \leq \left|\left|\nabla\frac{\partial v}{\partial t}\right|\right|^2, \tag{125}$$

Summing (123), (124), and (125) yields

$$\frac{d}{dt}\left(||\Delta u||^2+||\Delta v||^2+||\Delta\alpha||^2+\left|\left|\nabla\frac{\partial\alpha}{\partial t}\right|\right|^2\right)+\left|\left|\frac{\partial u}{\partial t}\right|\right|^2+c\left|\left|\Delta\frac{\partial\alpha}{\partial t}\right|\right|^2 \leq Q(||u||^2_{H^2(\Omega)}+||v||^2_{H^2(\Omega)}). \tag{126}$$

In particular, we set

$$y=||\Delta u||^2+||\Delta v||^2+||\Delta\alpha||^2+\left|\left|\nabla\frac{\partial\alpha}{\partial t}\right|\right|^2.$$

Thus, we deduce from (126) a differential inequality of the form

$$y'\leq Q(y).$$

Let z be the solution of the ordinary differential equation

$$z'=Q(z),$$

with $z(0)=y(0)$. It follows from the comparison principle that there exists $T_0=T_0(||u_0||_{H^2(\Omega)},||v_0||_{H^2(\Omega)},||\alpha_0||_{H^2(\Omega)},||\alpha_1||_{H^1(\Omega)})$ belonging to, say, $(0,\frac{1}{2})$ such that

$$y(t)\leq z(t),\quad t\in[0,T_0],$$

whence

$$||u(t)||^2_{H^2(\Omega)}+||v(t)||^2_{H^2(\Omega)}+||\alpha(t)||^2_{H^2(\Omega)}+\left|\left|\frac{\partial\alpha}{\partial t}(t)\right|\right|^2_{H^1(\Omega)} \leq Q(||u_0||_{H^2(\Omega)},||v_0||_{H^2(\Omega)},||\alpha_0||_{H^2(\Omega)},||\alpha_1||_{H^1(\Omega)}),\quad t\leq T_0. \tag{127}$$

Therefore $(u,v,\alpha,\frac{\partial\alpha}{\partial t})\in L^\infty(0,T_0;H^2(\Omega)^3\times H^1(\Omega))$ a priori.

We now differentiate (101) and (112) with respect to time to find, owing to (102)

$$(-\Delta)^{-1}\frac{\partial}{\partial t}\frac{\partial u}{\partial t}-\Delta\frac{\partial u}{\partial t}+f'(u+v)(\frac{\partial u}{\partial t}+\frac{\partial v}{\partial t})+f'(u-v)(\frac{\partial u}{\partial t}-\frac{\partial v}{\partial t})=0, \tag{128}$$

$$\frac{\partial}{\partial t}\frac{\partial v}{\partial t}-\Delta\frac{\partial v}{\partial t}+f'(u+v)(\frac{\partial u}{\partial t}+\frac{\partial v}{\partial t})-f'(u-v)(\frac{\partial u}{\partial t}-\frac{\partial v}{\partial t})=-\frac{\partial v}{\partial t}+\Delta\frac{\partial\alpha}{\partial t}+\Delta\alpha, \tag{129}$$

$$\text{with}\quad \frac{\partial u}{\partial t}=\frac{\partial v}{\partial t}=\frac{\partial\alpha}{\partial t}=0 \text{ on } \Gamma. \tag{130}$$

Multiplying then (128) by $t\frac{\partial u}{\partial t}$, (129) by $t\frac{\partial v}{\partial t}$, and (102) by $t\frac{\partial\alpha}{\partial t}$ to obtain, summing the three resulting inequalities and using (106) and an interpolation inequality,

$$\frac{1}{2}\frac{d}{dt}\left(t\left|\left|\frac{\partial u}{\partial t}\right|\right|^2_{-1}+t\left|\left|\frac{\partial v}{\partial t}\right|\right|^2+t\left|\left|\frac{\partial\alpha}{\partial t}\right|\right|^2+t||\nabla\alpha||^2\right)+ct\left(\left|\left|\nabla\frac{\partial u}{\partial t}\right|\right|^2+\left|\left|\nabla\frac{\partial v}{\partial t}\right|\right|^2 +\left|\left|\nabla\frac{\partial\alpha}{\partial t}\right|\right|^2\right)\leq\frac{1}{2}\left|\left|\frac{\partial u}{\partial t}\right|\right|^2_{-1}+\frac{1}{2}\left|\left|\frac{\partial v}{\partial t}\right|\right|^2+\frac{1}{2}\left|\left|\frac{\partial\alpha}{\partial t}\right|\right|^2+\frac{1}{2}||\nabla\alpha||^2 +c'\left(t\left|\left|\frac{\partial u}{\partial t}\right|\right|^2_{-1}+t\left|\left|\frac{\partial v}{\partial t}\right|\right|^2+t\left|\left|\frac{\partial\alpha}{\partial t}\right|\right|^2+t||\nabla\alpha||^2\right). \tag{131}$$

We apply Gronwall's lemma to (131) and we use (122), we get

$$\left|\left|\frac{\partial u}{\partial t}(t)\right|\right|_{-1}^2+\left|\left|\frac{\partial v}{\partial t}(t)\right|\right|^2+\left|\left|\frac{\partial \alpha}{\partial t}(t)\right|\right|^2+||\nabla\alpha(t)||^2\leq\frac{1}{t}Q\Big(||u_0||^2_{H^1(\Omega)}$$
$$+||v_0||^2_{H^1(\Omega)}+||\alpha_0||^2_{H^1(\Omega)}+||\alpha_1||^2+\int_\Omega[F(u_0+v_0)+F(u_0-v_0)]dx\Big),$$
$$t\in(0,T_0]. \tag{132}$$

Summing (128) times $\frac{\partial u}{\partial t}$, (129) times $\frac{\partial v}{\partial t}$, and (102) times $\frac{\partial \alpha}{\partial t}$ and interpolating yields

$$\frac{1}{2}\frac{d}{dt}\Big(\left|\left|\frac{\partial u}{\partial t}\right|\right|_{-1}^2+\left|\left|\frac{\partial v}{\partial t}\right|\right|^2+\left|\left|\frac{\partial \alpha}{\partial t}\right|\right|+||\nabla\alpha||^2\Big)+c\left|\left|\nabla\frac{\partial u}{\partial t}\right|\right|^2$$
$$+\left|\left|\nabla\frac{\partial v}{\partial t}\right|\right|^2+c'\left|\left|\nabla\frac{\partial \alpha}{\partial t}\right|\right|^2\leq c''\Big(\left|\left|\frac{\partial u}{\partial t}\right|\right|_{-1}^2+\left|\left|\frac{\partial v}{\partial t}\right|\right|^2+\left|\left|\frac{\partial \alpha}{\partial t}\right|\right|^2+||\nabla\alpha||^2\Big). \tag{133}$$

Applying then Gronwall's lemma to (133) from t to T_0 and using (132), we obtain

$$\left|\left|\frac{\partial u}{\partial t}(t)\right|\right|_{-1}^2+\left|\left|\frac{\partial v}{\partial t}(t)\right|\right|^2+\left|\left|\frac{\partial \alpha}{\partial t}(t)\right|\right|^2+||\nabla\alpha(t)||^2\leq e^{ct}Q\Big(||u_0||^2_{H^1(\Omega)}$$
$$+||v_0||^2_{H^1(\Omega)}+||\alpha_0||^2_{H^1(\Omega)}+||\alpha_1||^2+\int_\Omega[F(u_0+v_0)+F(u_0-v_0)]dx\Big), \tag{134}$$
$$t\geq T_0.$$

Combining (132) and (134) yields

$$\left|\left|\frac{\partial u}{\partial t}(t)\right|\right|_{-1}^2+\left|\left|\frac{\partial v}{\partial t}(t)\right|\right|^2+\left|\left|\frac{\partial \alpha}{\partial t}(t)\right|\right|^2+||\nabla\alpha(t)||^2\leq e^{ct}Q\Big(||u_0||^2_{H^1(\Omega)}$$
$$+||v_0||^2_{H^1(\Omega)}+||\alpha_0||^2_{H^1(\Omega)}+||\alpha_1||^2+\int_\Omega[F(u_0+v_0)+F(u_0-v_0)]dx\Big),$$
$$t>0.$$

- d) We rewrite (101) and (112) in the following forms

$$-\Delta u+f(u+v)+f(u-v)=h_u(t), \tag{135}$$
$$-\Delta v+f(u+v)-f(u-v)=h_v(t), \tag{136}$$

where $u=\Delta u=v=0$ on Γ, $t\geq T_0$,

and

$$h_u(t)=-(-\Delta)^{-1}\frac{\partial u}{\partial t} \tag{137}$$
$$h_v(t)=\frac{\partial \alpha}{\partial t}-\frac{\partial v}{\partial t}, \tag{138}$$

satisfy, owing to (134),

$$||h_u(t)||\leq e^{ct}Q(||u_0||_{H^2(\Omega)},||v_0||_{H^2(\Omega)},||\alpha_0||_{H^1(\Omega)},||\alpha_1||),\ t\geq T_0, \tag{139}$$

and

$$||h_v(t)||\leq e^{ct}Q(||u_0||_{H^2(\Omega)},||v_0||_{H^2(\Omega)},||\alpha_0||_{H^1(\Omega)},||\alpha_1||),\ t\geq T_0. \tag{140}$$

Next, we multiply (135) by u, (136) by v and sum the result. Then, noting that $f(s)s \geq -c,\ c \geq 0$, we obtain

$$||\nabla u||^2 + ||\nabla v||^2 \leq c(||h_u(t)||^2 + ||h_v(t)||^2) + c'. \tag{141}$$

Multiplying now (135) by $-\Delta u$ and (136) by $-\Delta v$, summing the result, then noting that $f'(s) \geq -c'$, we obtain

$$||\Delta u(t)||^2 + ||\Delta v(t)||^2 \leq c(||h_u(t)||^2 + ||h_v(t)||^2 + ||\nabla u(t)||^2 + ||\nabla v(t)||^2). \tag{142}$$

We finally deduce that

$$\begin{gathered}||u(t)||^2_{H^2(\Omega)} + ||v(t)||^2_{H^2(\Omega)} \leq ce^{c't}Q(||u_0||_{H^2(\Omega)}, ||v_0||_{H^2(\Omega)}, ||\alpha_0||_{H^1(\Omega)},\\ ||\alpha_1||), \quad t \geq T_0.\end{gathered} \tag{143}$$

Moreover, we multiply (102) by $-\Delta \frac{\partial \alpha}{\partial t}$ and find

$$\frac{1}{2}\frac{d}{dt}\left(||\Delta \alpha||^2 + \left|\left|\nabla \frac{\partial \alpha}{\partial t}\right|\right|^2\right) + c\left|\left|\Delta \frac{\partial \alpha}{\partial t}\right|\right|^2 \leq \left|\left|\nabla \frac{\partial v}{\partial t}\right|\right|^2.$$

- e) Integrating (133) between t and T_0 and using (134), we obtain

$$\begin{aligned}\int_t^{T_0} \left(\left|\left|\nabla \frac{\partial u}{\partial t}\right|\right|^2 + \left|\left|\nabla \frac{\partial v}{\partial t}\right|\right|^2 + \left|\left|\nabla \frac{\partial \alpha}{\partial t}\right|\right|^2\right) dx \leq e^{ct}Q\Big(||u_0||^2_{H^1(\Omega)} + ||v_0||^2_{H^1(\Omega)}\\ + ||\alpha_0||^2_{H^1(\Omega)} + ||\alpha_1||^2 + \int_\Omega [F(u_0+v_0) + F(u_0 - v_0)]dx\Big),\ t \geq T_0.\end{aligned} \tag{144}$$

Applying then Gronwall's lemma to (110) and using (144), we find

$$\begin{aligned}||\Delta \alpha(t)||^2 + \left|\left|\nabla \frac{\partial \alpha}{\partial t}(t)\right|\right|^2 \leq ce^{c't}Q\Big(||u_0||^2_{H^1(\Omega)} + ||v_0||^2_{H^1(\Omega)} + ||\alpha_0||^2_{H^1(\Omega)}\\ + ||\alpha_1||^2 + \int_\Omega [F(u_0+v_0) + F(u_0-v_0)]dx\Big),\ t \geq T_0.\end{aligned} \tag{145}$$

Thus, we deduce from (127), (143), and (145) that

$$\begin{aligned}||u(t)||^2_{H^2(\Omega)} + ||v(t)||^2_{H^2(\Omega)} + ||\alpha(t)||^2_{H^2(\Omega)} + \left|\left|\frac{\partial \alpha}{\partial t}(t)\right|\right|^2_{H^1(\Omega)}\\ \leq e^{ct}Q(||u_0||_{H^2(\Omega)}, ||v_0||_{H^2(\Omega)}, ||\alpha_0||_{H^2(\Omega)}, ||\alpha_1||_{H^1(\Omega)}), \quad t \geq 0.\end{aligned}$$

□

Lemma 3.2. *The solution also satisfies the following inequality:*

$$\begin{aligned}||u(t)||^2_{H^2(\Omega)} + ||v(t)||^2_{H^2(\Omega)} + ||\alpha(t)||^2_{H^2(\Omega)} + \left|\left|\frac{\partial \alpha}{\partial t}(t)\right|\right|^2_{H^1(\Omega)} \leq e^{-ct}Q(||u_0||_{H^2(\Omega)},\\ ||v_0||_{H^2(\Omega)}, ||\alpha_0||_{H^2(\Omega)}, ||\alpha_1||_{H^1(\Omega)}),\ c > 0,\ t \geq 0.\end{aligned} \tag{146}$$

Proof. We start by multiplying (112) by $-\Delta u$, (101) by $-\Delta v$, and (102) by $\frac{\partial \alpha}{\partial t}$, we obtain, after summing the result and using (106),

$$\frac{1}{2}\frac{d}{dt}\Big(||u||^2+||\nabla v||^2+||\nabla\alpha||^2+\Big|\Big|\frac{\partial\alpha}{\partial t}\Big|\Big|^2\Big)+||\Delta u||^2+c||\Delta v||^2+c'\Big|\Big|\nabla\frac{\partial\alpha}{\partial t}\Big|\Big|^2 \leq c''(||\nabla u||^2+||\nabla v||^2)+\Big|\Big|\frac{\partial v}{\partial t}\Big|\Big|^2. \tag{147}$$

Using an interpolation inequality and Young's inequality, we have

$$||u||^2_{H^1(\Omega)} \leq c||u||\ ||u||_{H^2(\Omega)} \leq \frac{1}{2}||\Delta u||^2+c||u||^2,\ c\geq 0.$$

Therefore

$$\frac{d}{dt}\Big(||u||^2+||\nabla v||^2+||\nabla\alpha||^2+\Big|\Big|\frac{\partial\alpha}{\partial t}\Big|\Big|^2\Big)+c\Big(||\Delta u||^2+||\Delta v||^2+\Big|\Big|\nabla\frac{\partial\alpha}{\partial t}\Big|\Big|^2\Big) \leq c'(||u||^2+||\nabla v||^2)+\Big|\Big|\frac{\partial v}{\partial t}\Big|\Big|^2. \tag{148}$$

It follows from (148), using (122) and (109), that

$$\int_0^1(||u||^2_{H^2(\Omega)}+||v||^2_{H^2(\Omega)}+\Big|\Big|\frac{\partial\alpha}{\partial t}\Big|\Big|^2_{H^1(\Omega)})dt \leq Q\Big(||u_0||^2_{H^1(\Omega)}+||v_0||^2_{H^1(\Omega)} +||\alpha_0||^2_{H^1(\Omega)}+||\alpha_1||^2+\int_\Omega[F(u_0+v_0)+F(u_0-v_0)]dx\Big). \tag{149}$$

Hence, there exists $T\in(0,1)$ such that

$$||u(T)||^2_{H^2(\Omega)}+||v(T)||^2_{H^2(\Omega)}+\Big|\Big|\frac{\partial\alpha}{\partial t}(T)\Big|\Big|^2_{H^1(\Omega)} \leq Q\Big(||u_0||^2_{H^1(\Omega)}+||v_0||^2_{H^1(\Omega)} +||\alpha_0||^2_{H^1(\Omega)}+||\alpha_1||^2+\int_\Omega[F(u_0+v_0)+F(u_0-v_0)]dx\Big). \tag{150}$$

Repeating now the estimates leading to (111) from $t=T$ instead of $t=0$, we have

$$||u(1)||^2_{H^2(\Omega)}+||v(1)||^2_{H^2(\Omega)}+\Big|\Big|\frac{\partial\alpha}{\partial t}(1)\Big|\Big|^2_{H^1(\Omega)} \leq Q\Big(||u_0||^2_{H^1(\Omega)}+||v_0||^2_{H^1(\Omega)} +||\alpha_0||^2_{H^1(\Omega)}+||\alpha_1||^2+\int_\Omega[F(u_0+v_0)+F(u_0-v_0)]dx\Big). \tag{151}$$

Similarly, repeating the estimates leading to (151) also, we have, for $t\geq 1$,

$$||u(t)||^2_{H^2(\Omega)}+||v(t)||^2_{H^2(\Omega)}+\Big|\Big|\frac{\partial\alpha}{\partial t}(t)\Big|\Big|^2_{H^1(\Omega)} \leq Q\Big(||u(t-1)||^2_{H^1(\Omega)}+||v(t-1)||^2_{H^1(\Omega)} +||\alpha(t-1)||^2_{H^1(\Omega)}+\Big|\Big|\frac{\partial\alpha}{\partial t}(t-1)\Big|\Big|^2+\int_\Omega[F(u(t-1)+v(t-1)) +F(u(t-1)-v(t-1))]dx\Big). \tag{152}$$

Owing to (108), the above estimate yields

$$||u(t)||^2_{H^2(\Omega)} + ||v(t)||^2_{H^2(\Omega)} + \left|\left|\frac{\partial\alpha}{\partial t}(t)\right|\right|^2_{H^1(\Omega)} \leq e^{-ct}Q\Big(||u_0||^2_{-1} + ||v_0||^2 + ||\nabla u_0||^2 + ||\nabla v_0||^2 + ||\nabla\alpha_0||^2 + \int_\Omega [F(u_0+v_0) + F(u_0-v_0)]dx\Big),\ c>0,\ t\geq 1. \tag{153}$$

Combining (153) with (111) for $t=0$, we get

$$||u(t)||^2_{H^2(\Omega)} + ||v(t)||^2_{H^2(\Omega)} + \left|\left|\frac{\partial\alpha}{\partial t}(t)\right|\right|^2_{H^1(\Omega)} \leq e^{-ct}Q(||u_0||_{H^2(\Omega)}, ||v_0||_{H^2(\Omega)}, ||\alpha_0||_{H^2(\Omega)}, ||\alpha_1||_{H^1(\Omega)}),\ c>0,\ t\geq 0. \tag{154}$$

Moreover, we note that it follows from (126) and (154) that

$$\frac{d}{dt}\Big(||\Delta u||^2 + ||\Delta v||^2 + ||\Delta\alpha||^2 + \left|\left|\nabla\frac{\partial\alpha}{\partial t}\right|\right|^2\Big) + \left|\left|\frac{\partial u}{\partial t}\right|\right|^2 + \left|\left|\Delta\frac{\partial\alpha}{\partial t}\right|\right|^2 \leq e^{-ct}Q(||u_0||_{H^2(\Omega)}, ||v_0||_{H^2(\Omega)}, ||\alpha_0||_{H^2(\Omega)}, ||\alpha_1||_{H^1(\Omega)}). \tag{155}$$

Multiplying then (102) by $-\Delta\alpha$, we have

$$\frac{d}{dt}\Big(||\Delta\alpha||^2 + 2((\Delta\alpha, \frac{\partial\alpha}{\partial t}))\Big) + c||\Delta\alpha||^2 \leq c\left|\left|\frac{\partial v}{\partial t}\right|\right|^2 + 2\left|\left|\nabla\frac{\partial\alpha}{\partial t}\right|\right|^2. \tag{156}$$

We now sum (120) and ϵ_3(148), where $\epsilon_3 > 0$ is small enough so that

$$\frac{dE_4}{dt} + c(E_4 + \left|\left|\frac{\partial u}{\partial t}\right|\right|^2_{-1} + \left|\left|\frac{\partial v}{\partial t}\right|\right|^2) + c'||\Delta u||^2 + \epsilon_3||\Delta v||^2 + c''\left|\left|\nabla\frac{\partial\alpha}{\partial t}\right|\right|^2 \leq 0, \tag{157}$$

where

$$E_4 = E_3 + \epsilon_3(||u||^2 + ||\nabla v||^2 + c||\nabla\alpha||^2 + \left|\left|\frac{\partial\alpha}{\partial t}\right|\right|^2).$$

We also sum (155), ϵ_4(156), and (157), where $\epsilon_4 > 0$ is small enough so that

$$\left|\left|\frac{\partial\alpha}{\partial t}\right|\right|_{H^2(\Omega)} + \epsilon_4(||\Delta\alpha||^2 + 2((\Delta\alpha, \frac{\partial\alpha}{\partial t}))) \geq c(\left|\left|\frac{\partial\alpha}{\partial t}\right|\right|_{H^2(\Omega)} + ||\Delta\alpha||^2),$$

and

$$\frac{dE_5}{dt} + c\Big(E_5 + \left|\left|\frac{\partial u}{\partial t}\right|\right|^2_{-1} + \left|\left|\frac{\partial v}{\partial t}\right|\right|^2 + \left|\left|\frac{\partial\alpha}{\partial t}\right|\right|^2_{H^2(\Omega)}\Big) \leq e^{-c't}Q(||u_0||_{H^2(\Omega)}, ||v_0||_{H^2(\Omega)}, ||\alpha_0||_{H^2(\Omega)}, ||\alpha_1||_{H^1(\Omega)}),\quad c>0,\ t\geq 0, \tag{158}$$

where

$$E_5 = E_4 + ||\Delta u||^2 + ||\Delta v||^2 + (1+\epsilon_4)||\Delta\alpha||^2 + \epsilon_4((\Delta\alpha, \frac{\partial\alpha}{\partial t})) + \left|\left|\nabla\frac{\partial\alpha}{\partial t}\right|\right|^2.$$

We finally deduce from (158) the inequality

$$||u(t)||^2_{H^2(\Omega)} + ||v(t)||^2_{H^2(\Omega)} + ||\alpha(t)||^2_{H^2(\Omega)} + \left|\left|\frac{\partial\alpha}{\partial t}(t)\right|\right|^2_{H^1(\Omega)} \leq e^{-ct}Q(||u_0||_{H^2(\Omega)}, ||v_0||_{H^2(\Omega)}, ||\alpha_0||_{H^2(\Omega)}, ||\alpha_1||_{H^1(\Omega)}),\ c>0,\ t\geq 0.$$

□

Theorem 3.3. *For every* $(u_0, v_0, \alpha_0, \alpha_1) \in (H^2(\Omega) \cap H^1_0(\Omega))^3 \times H^1_0(\Omega)$, *(100)-(104) possesses a unique solution* $(u, v, \alpha, \frac{\partial \alpha}{\partial t})$ *such that* $(u, v, \alpha) \in L^\infty(\mathbb{R}^+; \mathrm{H}^1_0(\Omega)^3)$ $\cap\, L^\infty_{loc}(\mathbb{R}^+; \mathrm{H}^2(\Omega)^3)$, $(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}) \in L^\infty(\mathbb{R}^+; \mathrm{H}^{-1}(\Omega) \times L^2(\Omega)) \cap L^2_{loc}(\mathbb{R}^+; \mathrm{H}^1_0(\Omega)^2)$ *and* $\frac{\partial \alpha}{\partial t} \in L^\infty(\mathbb{R}^+; \mathrm{H}^1_0(\Omega)) \cap L^2(\mathbb{R}^+; \mathrm{H}^2(\Omega))$.

Proof. The proof of existence is based on the a priori estimates mentioned in the previous lemmas and on, e.g., a standard Galerkin scheme similar to the proof of Theorem 2.1 based mainly on (107) and (111). Therefore, we will only be proving the uniqueness.

Let $(u^{(1)}, v^{(1)}, \alpha^{(1)}, \frac{\partial \alpha}{\partial t}^{(1)})$ and $(u^{(2)}, v^{(2)}, \alpha^{(2)}, \frac{\partial \alpha}{\partial t}^{(2)})$ be two solutions of (100)-(104) with initial data $(u_0^{(1)}, v_0^{(1)}, \alpha_0^{(1)}, \alpha_1^{(1)})$ and $(u_0^{(2)}, v_0^{(2)}, \alpha_0^{(2)}, \alpha_1^{(2)})$ respectively.

We set

$$(u, v, \alpha) = (u^{(1)}, v^{(1)}, \alpha^{(1)}, \frac{\partial \alpha}{\partial t}^{(1)}) - (u^{(2)}, v^{(2)}, \alpha^{(2)}, \frac{\partial \alpha}{\partial t}^{(2)}),$$

and

$$(u_0, v_0, \alpha_0) = (u_0^{(1)}, v_0^{(1)}, \alpha_0^{(1)}, \alpha_1^{(1)}) - (u_0^{(2)}, v_0^{(2)}, \alpha_0^{(2)}, \alpha_1^{(2)}).$$

Hence (u, v, α) satisfy

$$(-\Delta)^{-1}\frac{\partial u}{\partial t} - \Delta u + f(u^{(1)} + v^{(1)}) - f(u^{(2)} + v^{(2)}) + f(u^{(1)} - v^{(1)}) - f(u^{(2)} - v^{(2)}) = 0, \tag{159}$$

$$\frac{\partial v}{\partial t} - \Delta v + f(u^{(1)} + v^{(1)}) - f(u^{(2)} + v^{(2)}) - f(u^{(1)} - v^{(1)}) + f(u^{(2)} - v^{(2)}) = \frac{\partial \alpha}{\partial t}, \tag{160}$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial v}{\partial t}, \tag{161}$$

$$u = \Delta u = v = \alpha = 0 \text{ on } \Gamma, \tag{162}$$

$$u_{|t=0} = u_0, \; v_{|t=0} = v_0, \; \alpha_{|t=0} = \alpha_0, \; \frac{\partial \alpha}{\partial t}\Big|_{t=0} = \alpha_1. \tag{163}$$

First, we multiply (159) by $\frac{\partial u}{\partial t}$, (160) by $\frac{\partial v}{\partial t}$, (161) by $\frac{\partial \alpha}{\partial t}$, and we sum the resulting equations to obtain, using the continuous embedding $H^2(\Omega) \subset C(\overline{\Omega})$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(||\nabla u||^2 + ||\nabla v||^2 + ||\nabla \alpha||^2 + \Big|\Big|\frac{\partial \alpha}{\partial t}\Big|\Big|^2\Big) + \Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|^2_{-1} + \Big|\Big|\frac{\partial v}{\partial t}\Big|\Big|^2 \\ &\quad + \Big|\Big|\nabla \frac{\partial \alpha}{\partial t}\Big|\Big|^2 \le Q\Big(||u_0^{(1)}||_{H^2(\Omega)}, ||u_0^{(2)}||_{H^2(\Omega)}, ||v_0^{(1)}||_{H^2(\Omega)}, ||v_0^{(2)}||_{H^2(\Omega)}, \\ &\quad ||\alpha_0^{(1)}||_{H^2(\Omega)}, ||\alpha_0^{(2)}||_{H^2(\Omega)}, ||\alpha_1^{(1)}||_{H^1(\Omega)}, ||\alpha_1^{(2)}||_{H^1(\Omega)}\Big). \end{aligned} \tag{164}$$

It thus follows from (164) and Gronwall's lemma that

$$\begin{aligned} ||u(t)||^2_{H^1(\Omega)} + ||v(t)||^2_{H^1(\Omega)} + ||\alpha(t)||^2_{H^1(\Omega)} + \Big|\Big|\frac{\partial \alpha}{\partial t}(t)\Big|\Big|^2 \le e^{Qt}\Big(||u_0||^2_{H^1(\Omega)} \\ + ||v_0||^2_{H^1(\Omega)} + ||\alpha_0||^2_{H^1(\Omega)} + ||\alpha_1||^2\Big), \; t \ge 0, \end{aligned} \tag{165}$$

hence the uniqueness, as well as the continuity (with respect to the $H^1_0(\Omega)^3 \times L^2(\Omega)$-norm) with respect to the initial data. □

It follows from the previous theorem that we can define the family of solving operators

$$S(t) : \mathcal{E}' \longrightarrow \mathcal{E}'$$

$$(u_0, v_0, \alpha_0, \alpha_1) \longrightarrow (u(t), v(t), \alpha(t), \frac{\partial \alpha}{\partial t}(t)),\ t \geq 0,$$

where $(u, v, \alpha, \frac{\partial \alpha}{\partial t})$ is the unique solution to our system and $\mathcal{E}' = (H^2(\Omega)^3 \times H_0^1(\Omega))$.

Furthermore, this family of solving operators forms a continuous semigroup (for the $H^1(\Omega)^3 \times L^2(\Omega)$-topology), i.e. $S(0) = Id$ and $S(t+\tau) = S(t) \circ S(\tau), \forall\, t, \tau \geq 0$.

Theorem 3.4. *The semigroup associated with (100)-(104) possesses a bounded absorbing set $\mathcal{B}_1$ in $\mathcal{E}'$.*

The result follows directly from (146).

We will be searching now for exponential attractors:

The term $\frac{\partial^2 \alpha}{\partial t^2}$ creates difficulties in the calculations. This is the reason why, as in [21] and [22], we decompose the solution $(u, v, \alpha, \frac{\partial \alpha}{\partial t})$ to our system with initial data $(u_0, v_0, \alpha_0, \alpha_1)$ into the sums

$$u(t) = w_1(t) + w_2(t), \tag{166}$$

$$v(t) = q_1(t) + q_2(t), \tag{167}$$

and

$$\alpha(t) = r_1(t) + r_2(t), \tag{168}$$

where w_1, q_1, and r_1 are solutions to

$$(-\Delta)^{-1} \frac{\partial w_1}{\partial t} - \Delta w_1 = 0, \tag{169}$$

$$\frac{\partial q_1}{\partial t} - \Delta q_1 = \frac{\partial r_1}{\partial t}, \tag{170}$$

$$\frac{\partial^2 r_1}{\partial t^2} - \Delta \frac{\partial r_1}{\partial t} - \Delta r_1 = -\frac{\partial q_1}{\partial t}, \tag{171}$$

$$w_1 = \Delta w_1 = q_1 = r_1 = 0 \text{ on } \Gamma, \tag{172}$$

$$w_{1|t=0} = u_0,\ q_{1|t=0} = v_0,\ r_{1|t=0} = \alpha_0,\ \left.\frac{\partial r_1}{\partial t}\right|_{t=0} = \alpha_1. \tag{173}$$

and w_2, q_2, and r_2 are solutions to

$$(-\Delta)^{-1} \frac{\partial w_2}{\partial t} - \Delta w_2 + f(u^{(1)} + v^{(1)}) - f(u^{(2)} + v^{(2)}) + f(u^{(1)} - v^{(1)}) - f(u^{(2)} - v^{(2)}) = 0, \tag{174}$$

$$\frac{\partial q_2}{\partial t} - \Delta q_2 + f(u^{(1)} + v^{(1)}) - f(u^{(2)} - v^{(2)}) - f(u^{(1)} - v^{(1)}) + f(u^{(2)} - v^{(2)}) = \frac{\partial r_2}{\partial t}, \tag{175}$$

$$\frac{\partial^2 r_2}{\partial t^2} - \Delta \frac{\partial r_2}{\partial t} - \Delta r_2 = -\frac{\partial q_2}{\partial t}, \tag{176}$$

$$w_2 = \Delta w_2 = q_2 = r_2 = 0 \text{ on } \Gamma, \tag{177}$$

$$w_{2|t=0} = 0,\ q_{2|t=0} = 0,\ r_{2|t=0} = 0,\ \left.\frac{\partial r_2}{\partial t}\right|_{t=0} = 0. \tag{178}$$

Repeating the same calculations leading to (158), but considering now (169)-(173), where $f=0$, we obtain

$$\frac{dE_6}{dt} + cE_6 \leq 0,\ \ c \geq 0, \tag{179}$$

where

$$\begin{aligned} E_6 = & ||\Delta w_1||^2 + ||\Delta q_1||^2 + (1+\epsilon_4)||\Delta r_1||^2 + \epsilon_1||w_1||^2_{-1} + \epsilon_1||q_1||^2 + ||\nabla w_1||^2 \\ & + (1+\epsilon_3)||\nabla q_1||^2 + (1+\epsilon_2+c\epsilon_3)||\nabla r_1||^2 + 2\epsilon_2((\frac{\partial r_1}{\partial t}, r_1)) + (1+\epsilon_3)\left|\left|\frac{\partial r_1}{\partial t}\right|\right|^2 \\ & + ((\Delta r_1, \frac{\partial r_1}{\partial t}))| + \epsilon_3||w_1||^2 + \left|\left|\nabla\frac{\partial r_1}{\partial t}\right|\right|^2. \end{aligned} \tag{180}$$

Using Gronwall's lemma on (179), we get

$$E_6(t) \leq e^{-ct}E_6(0). \tag{181}$$

On the other hand,

$$\begin{aligned} & ||u(t_1)-u(t_2)||_{H^1(\Omega)} + ||v(t_1)-v(t_2)||_{H^1(\Omega)} + ||\alpha(t_1)-\alpha(t_2)||_{H^1(\Omega)} \\ & \quad + \left|\left|\frac{\partial\alpha}{\partial t}(t_1) - \frac{\partial\alpha}{\partial t}(t_2)\right|\right| \leq c\Big(||\nabla(u(t_1)-u(t_2))|| + ||\nabla(v(t_1)-v(t_2))|| \\ & \quad + ||\nabla(\alpha(t_1)-\alpha(t_2))|| + \left|\left|\frac{\partial\alpha}{\partial t}(t_1) - \frac{\partial\alpha}{\partial t}(t_2)\right|\right|\Big) \\ & \quad \leq c\Big(\left|\left|\int_{t_1}^{t_2}\nabla\frac{\partial u}{\partial t}d\tau\right|\right| + \left|\left|\int_{t_1}^{t_2}\nabla\frac{\partial v}{\partial t}d\tau\right|\right| + \left|\left|\int_{t_1}^{t_2}\nabla\frac{\partial \alpha}{\partial t}d\tau\right|\right| + \left|\left|\int_{t_1}^{t_2}\frac{\partial^2 \alpha}{\partial t^2}d\tau\right|\right|\Big) \\ & \quad \leq c|t_1-t_2|^{\frac{1}{2}}\left|\int_{t_1}^{t_2}\Big(\left|\left|\nabla\frac{\partial u}{\partial t}\right|\right|^2 + \left|\left|\nabla\frac{\partial v}{\partial t}\right|\right|^2 + \left|\left|\nabla\frac{\partial \alpha}{\partial t}\right|\right|^2 + \left|\left|\frac{\partial^2 \alpha}{\partial t^2}\right|\right|^2\Big)d\tau\right|^{\frac{1}{2}}. \end{aligned} \tag{182}$$

In addition, it follows from (122), (133), (134) and the fact that the initial data are in a bounded absorbing set, that

$$\int_{t_1}^{t_2}\Big(\left|\left|\nabla\frac{\partial u}{\partial t}\right|\right|^2 + \left|\left|\nabla\frac{\partial v}{\partial t}\right|\right|^2 + \left|\left|\nabla\frac{\partial \alpha}{\partial t}\right|\right|\Big)^2 \leq c, \tag{183}$$

where c only depends on $\mathcal{B}_1$ and $T \geq T_0$ such that $t_1, t_2 \in [T_0, T]$..

Moreover, looking at the equation (161), we can see that

$$\left|\left|\frac{\partial^2\alpha}{\partial t^2}\right|\right| \leq \left|\left|\Delta\frac{\partial\alpha}{\partial t}\right|\right| + \left|\left|\frac{\partial v}{\partial t}\right|\right| + ||\Delta\alpha||. \tag{184}$$

Thus, it follows from (110), (111), (183), and (184) that

$$\int_{t_1}^{t_2}\left|\left|\frac{\partial^2\alpha}{\partial t^2}\right|\right| \leq c, \tag{185}$$

where c only depends on $\mathcal{B}_1$.

Whence, we have

$$||u(t_1)-u(t_2)||_{H^1(\Omega)}+||v(t_1)-v(t_2)||_{H^1(\Omega)}+||\alpha(t_1)-\alpha(t_2)||_{H^1(\Omega)} + \left|\left|\frac{\partial\alpha}{\partial t}(t_1)-\frac{\partial\alpha}{\partial t}(t_2)\right|\right| \le c'|t_1-t_2|^{\frac{1}{2}}. \tag{186}$$

It is important to note that w_2, q_2, and r_2 verify the same system as u, v, and α so they satisfy the same estimations found in the previous sections.

Furthermore, we differentiate (174) and (175) with respect to time and we multiply the resulting equations by $(t-T_0)\frac{\partial w_2}{\partial t}$ and $(t-T_0)\frac{\partial q_2}{\partial t}$, respectively. Then, we sum the result with $(t-T_0)\frac{\partial r_2}{\partial t}$ times (176), and similarly to (84), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Bigg((t-T_0)\left|\left|\frac{\partial w_2}{\partial t}\right|\right|^2_{-1}+(t-T_0)\left|\left|\frac{\partial q_2}{\partial t}\right|\right|^2+(t-T_0)\left|\left|\frac{\partial r_2}{\partial t}\right|\right|^2+(t-T_0)||\nabla r_2||^2\Bigg)\\
&\quad+c(t-T_0)\Bigg(\left|\left|\nabla\frac{\partial w_2}{\partial t}\right|\right|^2+\left|\left|\nabla\frac{\partial q_2}{\partial t}\right|\right|^2+\left|\left|\nabla\frac{\partial r_2}{\partial t}\right|\right|^2\Bigg)\\
&\quad\le c'(t-T_0)\Bigg(\left|\left|\frac{\partial w_2}{\partial t}\right|\right|^2_{-1}+\left|\left|\frac{\partial q_2}{\partial t}\right|\right|^2+||\nabla r_2||^2\Bigg)+c''(t-T_0)\Bigg(\left|\left|\frac{\partial u}{\partial t}\right|\right|^2\\
&\quad+\left|\left|\frac{\partial v}{\partial t}\right|\right|^2\Bigg)+\frac{1}{2}\Bigg(\left|\left|\frac{\partial w_2}{\partial t}\right|\right|^2_{-1}+\left|\left|\frac{\partial q_2}{\partial t}\right|\right|^2+\left|\left|\frac{\partial r_2}{\partial t}\right|\right|^2+||\nabla r_2||^2\Bigg)\\
&\quad+c'''(t-T_0)||\nabla(u+v)||\left|\left|\nabla\frac{\partial w_2}{\partial t}\right|\right|\,\left|\left|\frac{\partial}{\partial t}(u_2+v_2)\right|\right|\\
&\quad+c'''(t-T_0)||\nabla(u+v)||\left|\left|\nabla\frac{\partial q_2}{\partial t}\right|\right|\,\left|\left|\frac{\partial}{\partial t}(u_2+v_2)\right|\right|\\
&\quad+c'''(t-T_0)||\nabla(u-v)||\left|\left|\nabla\frac{\partial w_2}{\partial t}\right|\right|\,\left|\left|\frac{\partial}{\partial t}(u_2-v_2)\right|\right|\\
&\quad+c'''(t-T_0)||\nabla(u-v)||\left|\left|\nabla\frac{\partial q_2}{\partial t}\right|\right|\,\left|\left|\frac{\partial}{\partial t}(u_2-v_2)\right|\right|.
\end{aligned} \tag{187}$$

It follows from (133), (134) that

$$\int_{T_0}^t\Bigg(\left|\left|\nabla\frac{\partial u}{\partial t}\right|\right|^2+\left|\left|\nabla\frac{\partial v}{\partial t}\right|\right|^2+\left|\left|\nabla\frac{\partial\alpha}{\partial t}\right|\right|^2\Bigg)\le ce^{c't},\ t\ge T_0, \tag{188}$$

where the constants only depend on $\mathcal{B}_1$.

We can thereby deduce that

$$\int_{T_0}^t\Bigg(\left|\left|\frac{\partial u_2}{\partial t}\right|\right|^2+\left|\left|\frac{\partial v_2}{\partial t}\right|\right|^2\Bigg)dx\le ce^{c't},\ t\ge T_0, \tag{189}$$

for $(u,v)=(u_2,v_2)$ and the constants only depend on $\mathcal{B}_1$.

It also follows from (164) and (165) that

$$\int_0^t\Bigg(\left|\left|\frac{\partial u}{\partial t}\right|\right|^2_{-1}+\left|\left|\frac{\partial v}{\partial t}\right|\right|^2+\left|\left|\nabla\frac{\partial\alpha}{\partial t}\right|\right|^2\Bigg)dx\le c(||u_0||^2_{H^1(\Omega)}+||v_0||^2_{H^1(\Omega)} + ||\alpha_0||^2_{H^1(\Omega)}+||\alpha_1||^2),\ t\ge 0. \tag{190}$$

Applying Gronwall's lemma to (187) over (T_0, t) and owing to (165)and (188)-(190), we obtain

$$\left|\left|\frac{\partial w_2}{\partial t}(t)\right|\right|_{-1}^2 + \left|\left|\frac{\partial q_2}{\partial t}(t)\right|\right|^2 + \left|\left|\frac{\partial r_2}{\partial t}(t)\right|\right|^2 + ||\nabla r_2(t)||^2 \leq ce^{c't}(||u_0||^2_{H^1(\Omega)}+ ||v_0||^2_{H^1(\Omega)} + ||\alpha_0||^2_{H^1(\Omega)} + ||\alpha_1||^2),\ t \geq 1. \tag{191}$$

where the constants only depend on $\mathcal{B}_1$.

We now rewrite equations (174) and (175) in the following forms

$$-\Delta w_2 = \tilde{h}_{w_2}(t), \tag{192}$$

and

$$-\Delta q_2 = \tilde{h}_{q_2}(t), \tag{193}$$

where $w_2 = \Delta w_2 = q_2 = 0$ on Γ, for $t \geq 1$ fixed, and

$$\tilde{h}_{w_2}(t) = -(-\Delta)^{-1}\frac{\partial w_2}{\partial t} - \Big(f(u^{(1)}+v^{(1)}) - f(u^{(2)}+v^{(2)})\Big) - \Big(f(u^{(1)}-v^{(1)}) - f(u^{(2)}-v^{(2)})\Big), \tag{194}$$

$$\tilde{h}_{q_2}(t) = -\frac{\partial q_2}{\partial t} + \frac{\partial r_2}{\partial t} - \Big(f(u^{(1)}+v^{(1)}) - f(u^{(2)}+v^{(2)})\Big) + \Big(f(u^{(1)}-v^{(1)}) - f(u^{(2)}-v^{(2)})\Big), \tag{195}$$

satisfy, owing to (165) and (191),

$$||\tilde{h}_{w_2}(t)||^2 \leq ce^{Qt}(||u_0||^2_{H^1(\Omega)} + ||v_0||^2_{H^1(\Omega)} + ||\alpha_0||^2_{H^1(\Omega)} + ||\alpha_1||^2),\ t \geq 1, \tag{196}$$

and

$$||\tilde{h}_{q_2}(t)||^2 \leq ce^{Qt}(||u_0||^2_{H^1(\Omega)} + ||v_0||^2_{H^1(\Omega)} + ||\alpha_0||^2_{H^1(\Omega)} + ||\alpha_1||^2),\ t \geq 1. \tag{197}$$

where the constants only depend on $\mathcal{B}_1$.

Multiplying now (192) by $-\Delta w_2$ and (193) by $-\Delta q_2$, we obtain

$$||\Delta w_2(t)|| \leq ||\tilde{h}_{w_2}(t)||,\ t \geq 1,$$

and

$$||\Delta q_2(t)|| \leq ||\tilde{h}_{q_2}(t)||,\ t \geq 1.$$

Whence

$$||w_2(t)||^2_{H^2(\Omega)} + ||q_2(t)||^2_{H^2(\Omega)} \leq ce^{Qt}(||u_0||^2_{H^1(\Omega)} + ||v_0||^2_{H^1(\Omega)} + ||\alpha_0||^2_{H^1(\Omega)} + ||\alpha_1||^2),\ t \geq 1. \tag{198}$$

Next, we multiply (176) by $-(t-T_0)\Delta\frac{\partial r_2}{\partial t}$ and find

$$\frac{1}{2}\frac{d}{dt}\left((t-T_0)\left|\left|\nabla\frac{\partial r_2}{\partial t}\right|\right|^2 + (t-T_0)||\Delta r_2||^2\right) + (t-T_0)\left|\left|\Delta\frac{\partial r_2}{\partial t}\right|\right|^2 \leq c(t-T_0)\left|\left|\nabla\frac{\partial r_2}{\partial t}\right|\right|^2 + c(t-T_0)\left|\left|\nabla\frac{\partial q_2}{\partial t}\right|\right|^2 + \frac{1}{2}\left|\left|\nabla\frac{\partial r_2}{\partial t}\right|\right|^2 + \frac{1}{2}||\Delta r_2||^2. \tag{199}$$

Applying then Gronwall's lemma over (T_0, t) and using (145) and (188), we obtain

$$\left|\left|\nabla\frac{\partial r_2}{\partial t}(t)\right|\right|^2 + ||\Delta r_2(t)||^2 \leq ce^{c't}. \tag{200}$$

Finally, summing (198) and (200), we obtain

$$||w_2(t)||^2_{H^2(\Omega)} + ||q_2(t)||^2_{H^2(\Omega)} + ||r_2(t)||^2_{H^2(\Omega)} + \left|\left|\frac{\partial r_2}{\partial t}(t)\right|\right|^2_{H^1(\Omega)} \le ce^{Qt}(||u_0||^2_{H^1(\Omega)} + ||v_0||^2_{H^1(\Omega)} + ||\alpha_0||^2_{H^1(\Omega)} + ||\alpha_1||^2),\ t \ge 1, \tag{201}$$

where the constants only depend on $\mathcal{B}_1$.

The existence of exponential attractors then follows from ((165)), (181), (186), and (201) (see [7]- [8]). Therefore, we have

Theorem 3.5. *The semigroup $S(t)$ possesses an exponential attractor $\mathcal{M}' \subset \mathcal{B}_1$, i.e.,*
(i) $\mathcal{M}'$ is compact in $H^1(\Omega)^3 \times L^2(\Omega)$;
(ii) $\mathcal{M}'$ is positively invariant, $S(t)\mathcal{M} \subset \mathcal{M}'$, $\forall\ t \ge 0$;
(iii) $\mathcal{M}'$ has a finite fractal dimension in $H^1(\Omega)^3 \times L^2(\Omega)$;
(iv) $\mathcal{M}'$ attracts exponentially fast the bounded subsets of $\mathcal{E}'$:

$\forall\ B \subset \mathcal{E}'$ bounded, $dist_{H^1(\Omega)^3\times L^2(\Omega)}(S(t)B, \mathcal{M}') \le Q(||B||_{\mathcal{E}'})e^{-ct}$, $c > 0$, $t \ge 0$, where the constant c is independent of B and $dist_{H^1(\Omega)^3\times L^2(\Omega)}$ denotes the Hausdorff semidistance between sets defined by

$$dist_{H^1(\Omega)^3\times L^2(\Omega)}(A, B) = \sup_{a\in A} \inf_{b\in B} ||a - b||_{H^1(\Omega)^3\times L^2(\Omega)}.$$

Consequently, we deduce from standard results the

Corollary 2. *The semigroup $S(t)$ possesses the finite dimensional global attractor $\mathcal{A}'' \subset \mathcal{B}_1$.*

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