On the Computation of Poisson Probabilities

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1 Introduction

The Poisson distribution is a distribution commonly used in statistics and in operations research (Haight, 1967; Johnson et al., 2005; Krishnamoorthy, 2016). It also plays a central role in the analysis of the transient behavior of continuous-time Markov chains (see, e.g., (Trivedi, 2011)). Let $\lambda > 0$ and $\mathbb{N} := \{0, 1, \dots\}$. A random variable X is said to have a Poisson distribution with parameter λ if

$$\Pr[X = n] = \frac{\lambda^n}{n!}\,\mathrm{e}^{-\lambda}, n \in \mathbb{N}\,. \tag{1}$$

In the following, we will use the notation $P_n(\lambda) := (\lambda^n/n!)\,\mathrm{e}^{-\lambda}$. We will also assume that all non-integer computations will be performed using IEEE 754 floating-point arithmetic and using the `binary64` format with rounding mode round to nearest even (IEEE, 2008) (see also (Muller et al., 2010)). With that format, the smallest normal number that can be represented is $\tau = 2^{-1\,022} \approx 2.2 \cdot 10^{-308}$ and the largest number that can be represented is $\Omega = (2 - 2^{-52}) \cdot 2^{1\,023} \approx 1.8 \cdot 10^{307}$. A number x will be said to underflow if $x < \tau$ and will be said to overflow if $x > \Omega$. Also of interest is the roundoff unit of the format, which for rounding mode round to nearest even is 2^{-53}, meaning, approximately, that the number of correct decimal digits that the format can guarantee when performing elementary arithmetic operations is $-\log_{10} 2^{-53} \approx 16$.

Direct use of (1) easily leads to numerical underflow or overflow even for moderate values of λ and n. Consequently, there have been published several methods for the computation of $P_n(\lambda)$. These are, to the best of the authors' knowledge, the ones described in (Whittlesey, 1963; Knüsel, 1986; Fox and Glynn, 1988; Kemp and Kemp, 1991; Johnson et al., 2005; Press et al., 2007; Forbes et al., 2011; Krishnamoorthy, 2016). Broadly speaking, these methods fall into two classes: methods intended for the computation of a whole set of probabilities for the same value of the parameter λ (Fox and Glynn, 1988; Kemp and Kemp, 1991; Forbes et al., 2011) and methods intended for the computation of a single probability $P_n(\lambda)$ (Whittlesey, 1963; Knüsel, 1986; Johnson et al., 2005; Press et al., 2007; Krishnamoorthy, 2016). In this paper, we develop a new method for the computation of a single probability $P_n(\lambda)$. As we will illustrate, the method seems to be more accurate than and as fast as any of the methods described in (Whittlesey, 1963; Knüsel, 1986; Johnson et al., 2005; Press et al., 2007; Krishnamoorthy, 2016).

The rest of the paper is organized as follows. In section 2, we describe the published methods and analyze the accuracy of those intended for the computation of a single probability $P_n(\lambda)$. In section 3, we develop the new method. In section 4, we assess the method in terms of accuracy and computation time and compare it with the methods described in (Whittlesey, 1963; Knüsel, 1986; Press et al., 2007). Finally, in section 5 we present our conclusions.

2 Previous work

In this section, we will review the methods described in (Whittlesey, 1963; Knüsel, 1986; Fox and Glynn, 1988; Kemp and Kemp, 1991; Johnson et al., 2005; Press et al., 2007; Forbes et al., 2011; Krishnamoorthy, 2016) and discuss briefly the accuracy of those intended for the computation of a single probability $P_n(\lambda)$.

We review first the methods intended for the computation of whole set of probabilities for the same value of the parameter λ. Let $\lfloor x \rfloor$ denote the integer part of x. The method described in (Fox and Glynn, 1988) determines, using normal bounds for the tails of the Poisson distribution, nonnegative integers L, R such that $\sum_{n=L}^{R} P_n(\lambda) \geq 1-\varepsilon$, where ε is an error control parameter, next computes weights w_n, $L \leq n \leq R$, by setting $w_{\lfloor \lambda \rfloor} = \Omega/(10^{10}(R-L))$ and using the recurrence

$$w_n = \frac{\lambda}{n} w_{n-1}, n \geq 1 \,,$$

and finally computes upper bounds for $P_n(\lambda)$, $L \leq n \leq R$, as $w_n / \sum_{i=L}^{R} w_i$. The bounds are tight in the sense that

$$0 < \frac{w_n}{\sum_{i=L}^{R} w_i} - P_n(\lambda) \leq \frac{1}{\sum_{n=L}^{R} P_n(\lambda)} - 1 \leq \frac{\varepsilon}{1-\varepsilon} \,.$$

The methods described in (Kemp and Kemp, 1991; Forbes et al., 2011) are based on computing a starting probability and next obtaining the probabilities of interest using the recurrence

$$P_n(\lambda) = \frac{\lambda}{n} P_{n-1}(\lambda), n \geq 1 \,.$$

In (Kemp and Kemp, 1991, sect. 3), the starting probability is $P_{\lfloor \lambda+0.5 \rfloor}(\lambda)$, which is approximated using formulas developed in (Kemp, 1988b). In (Forbes et al., 2011, sect. 35.1), the starting probability is $P_0(\lambda)$.

We review next the methods intended for the computation of a single probability $P_n(\lambda)$ described in (Whittlesey, 1963; Knüsel, 1986; Johnson et al., 2005; Press et al., 2007;

Krishnamoorthy, 2016). All these methods save the one described in (Johnson et al., 2005, sect. 4.5) are based on computing the natural logarithm of $P_n(\lambda)$. Therefore, for ease of exposition we will start by reviewing the method proposed in (Johnson et al., 2005, sect. 4.5) and next will review the remaining methods starting with the least accurate ones.

In (Johnson et al., 2005, sect. 4.5), three strategies are considered for the computation of $P_n(\lambda)$. The first one is based on the central limit theorem and consists in using the approximation

$$P_n(\lambda) \approx \frac{1}{\sqrt{2\pi}} \int_{K_-}^{K_+} \mathrm{e}^{-u^2/2} \, \mathrm{d}u \,,$$

where $K_- = (n - \lambda - \frac{1}{2})/\sqrt{\lambda}$ and $K_+ = (n - \lambda + \frac{1}{2})/\sqrt{\lambda}$. This strategy is reasonably accurate only if λ is large (say $\geq 10^6$) and therefore does not seem appropriate to compute $P_n(\lambda)$ in the general case.

Let $\Gamma(x)$ denote the gamma function. The second strategy consists in using $n! = \Gamma(n+1) = n\Gamma(n)$ in (1) and next estimating $\Gamma(n)$ using its Laplace expansion (see, e.g., (Wang, 2016, p. 571)). This yields the approximation

$$P_n(\lambda) \approx \frac{\mathrm{e}^{n-\lambda}}{\sqrt{2\pi n}} \Big(\frac{\lambda}{n}\Big)^n \Big(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{571}{2488320n^4} + \cdots\Big)^{-1} . \tag{2}$$

The third strategy proposed in (Johnson et al., 2005) consists in replacing the expression within parenthesis in (2) by a polynomial approximation developed in (Kemp, 1988a). The result is the approximation

$$P_n(\lambda) \approx \frac{\mathrm{e}^{n-\lambda}}{\sqrt{2\pi n}} \Big(\frac{\lambda}{n}\Big)^n \Bigg(1 - \frac{\frac{1}{12}}{n + \frac{1}{24} + \frac{293}{8640n}}\Bigg) . \tag{3}$$

When $n = \lambda$, this strategy can be very advantageous since, in that case,

$$P_n(\lambda) \approx \frac{1}{\sqrt{2\pi n}} \Big(1 - \frac{1/12}{n + 1/24 + 293/(8640n)}\Big) ,$$

an expression that is likely to be fast in terms of computation time because it does not require the evaluation of the exponential function nor of powers of n. However, when $n \gg 1$ the factor $(\lambda/n)^n$ can easily overflow if $\lambda > n$ and can easily underflow if $\lambda < n$, and the factor $\mathrm{e}^{n-\lambda}$ can easily underflow if $\lambda \gg n$ and can easily overflow if $\lambda \ll n$. Therefore, neither (2) nor (3) seem to be appropriate to compute $P_n(\lambda)$ in the general case.

The methods described in (Whittlesey, 1963; Knüsel, 1986; Press et al., 2007; Krishnamoorthy, 2016) are all based on

$$P_n(\lambda) = \mathrm{e}^{-\lambda + n \log \lambda - \log \Gamma(n+1)} , \tag{4}$$

which follows immediately by $n!$ by $\Gamma(n+1)$ in (1) and next taking logarithms. We will review first the methods described in (Press et al., 2007; Krishnamoorthy, 2016), which are very similar and, we will argue, are the least accurate. Next, we will review the method described in (Whittlesey, 1963), which can be regarded as an improvement of the previous two methods. Finally, we will review the method described in (Knüsel, 1986), which in turn can be regarded as an improvement of the method described in (Whittlesey, 1963).

In the method described in (Press et al., 2007, sect. 6.4.13), the probability $P_n(\lambda)$ is approximated by combining (4) with an approximation for $\ln\Gamma(n+1)$ based on formulas derived in (Lanczos, 1964) (see (Press et al., 2007, sect. 6.1)). In the method described in (Krishnamoorthy, 2016, sect. 5.13), the probability $P_n(\lambda)$ is approximated by combining (4) with an approximation for $\ln\Gamma(n+1)$ based on a continuous fraction derived in (Hart et al., 1968) (see (Krishnamoorthy, 2016, sect. 1.8)). Both methods suffer from severe cancellations when both λ and n are large (Knüsel, 1986) because, then, $\lambda + \ln\Gamma(n+1) \approx n\ln\lambda$. As an example, for $\lambda = 10^6$ and $n = \lambda + \sqrt{\lambda} = 1\,001\,000$, we have $\lambda + \ln\Gamma(n+1) \approx 1.3829334 \times 10^7$, $n\ln\lambda \approx 1.3829326 \times 10^7$, and $|\ln P_n(\lambda)| \approx 8.32703$. This suggests a loss of about seven decimal digits of accuracy. To verify it, we computed tight rigorous bounds for $P_n(\lambda)$ using the multiprecision interval arithmetic library MPFI (Revol and Rouillier, 2005) in order to have accurate estimates of $P_n(\lambda)$ and, using them, computed an accurate estimate for

$$d(\lambda, n) := -\log_{10}\left|\frac{P_n(\lambda) - \widetilde{P}_n(\lambda)}{\widetilde{P}_n(\lambda)}\right|, \tag{5}$$

where $\widetilde{P}_n(\lambda)$ denotes the approximation. The result was $d = 8.80$. This confirms that for this particular example, the methods proposed in (Press et al., 2007; Krishnamoorthy, 2016) lose around $16.0 - 8.80 \approx 7$ decimal digits of accuracy.

The method proposed in (Whittlesey, 1963, sect. 2) tackles this loss of accuracy by introducing a scaled gamma function

$$G(n) = \frac{\Gamma(n+1)}{\left(\frac{n}{\mathrm{e}}\right)^n},\ n \geq 1,$$

in order not to have to deal with $\ln\Gamma(n+1)$. Clearly,

$$\ln G(n) = \ln\Gamma(n+1) - n\ln n + n \tag{6}$$

and, therefore, using (4),

$$P_n(\lambda) = \mathrm{e}^{n-\lambda+n\ln(\lambda/n)-\ln G(n)}. \tag{7}$$

In the method, the $\ln G(n)$ is approximated as follows. By combining $\ln\Gamma(n+1) = \ln n + \ln\Gamma(n)$ with the Stirling series for $\ln\Gamma(n)$ (Abramowitz and Stegun, 1965, 6.1.41), we obtain

$$\ln\Gamma(n+1) \approx \left(n + \frac{1}{2}\right)\ln n - n + \frac{1}{2}\ln(2\pi) + I(m, n),\ n > 0, \tag{8}$$

with

$$I(m,n) := \sum_{j=1}^{m} T(j,n), \tag{9}$$

$$T(j,n) := \frac{B_{2j}}{2j(2j-1)n^{2j-1}}\,, \tag{10}$$

where B_k denotes the kth Bernoulli number. Then, by (6) and (8),

$$\ln G(n) \approx \frac{1}{2}\ln(2\pi n) + I(m,n)\,. \tag{11}$$

Finally, using (7),

$$\mathrm{e}^{(n-\lambda)+n\ln(\lambda/n)-\ln G(n)} \approx \mathrm{e}^{(n-\lambda)+n\ln(\lambda/n)-\ln(2\pi n)/2-I(m,n)} := E(m,\lambda,n)$$

and the $P_n(\lambda)$ is approximated as

$$P_n(\lambda) \approx \begin{cases} E(7,\lambda,n) & \text{if } n \geq 8 \\ E(7,\lambda,8)\times 8\times 7\cdots(n+1)\times\lambda^{n-8} & \text{if } n < 8 \end{cases}.$$

Continuing with the previous example, for $\lambda = 10^6$, $n = 1\,001\,000$ we have $n - \lambda = 1\times 10^3$, $n\ln(\lambda/n) - (1/2)\ln(2\pi n) - I(7,n) \approx -1.083\times 10^3$, and $|\ln P_n(\lambda)| \approx 8.32703$. This suggests losing only about three decimal digits of accuracy. However, in this case $d = 10.5$. This implies that the number of decimal digits of accuracy actually lost is about $16.0 - 10.5 = 5.5$, suggesting that there is another source of numerical inaccuracy.

The second source of numerical inaccuracy turns out to be the computation of $\ln(\lambda/n)$ when n is close to λ. The reason is that, since the condition number of $\ln x$ is $1/|\ln x|$, the error incurred in the actual computation of λ/n tends to result in a relative error in $\ln(\lambda/n)$ of the order of that in λ/n divided by $|\ln(\lambda/n)|$, which can be very large if $\lambda \approx n$. The method proposed in (Knüsel, 1986, sect. 5) tackles this second source of error by introducing a shifted logarithm function

$$\ln x = \ln(1+x)\,,$$

which has a condition number of one for $x = 0$ and is therefore much less sensitive to errors in x when $x \approx 0$. Let

$$h(\lambda,n) = \begin{cases} \operatorname{lns}\left(\dfrac{n-\lambda}{\lambda}\right) & \text{if } n \geq \lambda \\ -\operatorname{lns}\left(\dfrac{\lambda-n}{n}\right) & \text{if } n < \lambda \end{cases}. \tag{12}$$

Then, by (7), (11), (12),

$$\begin{aligned} P_n(\lambda) &= \mathrm{e}^{n-\lambda-n\,h(\lambda,n)-\ln G(n)},\ n \geq 1, & (13)\\ &\approx \mathrm{e}^{(n-\lambda)-n\,h(\lambda,n)-(1/2)\ln(2\pi n)-I(m,n)},\ n \geq 1\,. & (14) \end{aligned}$$

λ	$d_{\mathrm{m}}(\lambda, n)$	$d_{\mathrm{M}}(\lambda, n)$	$\overline{d(\lambda, n)}$	$\langle d(\lambda, n) \rangle$
10^0	2.30	15.4	13.6	4.40
10^1	2.30	16.6	13.8	15.1
10^2	2.30	17.9	13.9	15.6
10^3	12.4	16.6	13.7	15.1
10^4	12.0	17.1	13.2	14.6
10^5	11.7	16.2	12.7	14.2
10^6	11.0	15.8	12.2	13.6
10^7	10.7	15.9	11.7	13.1
10^8	10.2	15.1	11.2	12.6
10^9	9.57	13.7	10.7	12.1
10^{10}	9.19	13.4	10.2	11.7
10^{11}	8.72	12.6	9.71	11.1
10^{12}	8.17	11.9	9.25	10.6
10^{13}	7.68	12.1	8.75	10.1
10^{14}	7.25	11.4	8.22	9.65
10^{15}	6.62	11.2	7.71	9.13

Table 1: Accuracy of the method proposed in (Knüsel, 1986, sect. 5).

In the method proposed in (Knüsel, 1986, sect. 5), the probability is approximated using (14) with the recommended choice $m = 7$. Continuing with the example previously considered, the computation of $P_n(\lambda)$ for $\lambda = 10^6$ and $n = 1\,001\,000$ using this method results in $d = 13.2$. For this particular example, then, the accuracy is much better than it was in the method proposed in (Whittlesey, 1963, sect. 2).

In order to more thoroughly assess the accuracy of the method proposed in (Knüsel, 1986, sect. 5), we performed the following experiment. First, we chose a representative set of values for the λ parameter, namely $10^0, \ldots, 10^{15}$. Next, for each value of λ we obtained, using the MPFI library, the set of n values $\{n_{\mathrm{l}}, \ldots, n_{\mathrm{r}}\}$ for which $P_n(\lambda)$ does not underflow. Finally, for each set $\{\max\{n_{\mathrm{l}}, 1\}, \ldots, n_{\mathrm{r}}\}$, we chose $\min\{1\,000, n_{\mathrm{r}} - \max\{1, n_{\mathrm{l}}\} + 1\}$ values of n as described later and for each such n computed tight rigorous bounds for $P_n(\lambda)$ using the MPFI library in order to have accurate estimates of $P_n(\lambda)$ and, using them, computed an accurate estimate for (5) $d(\lambda, n)$. With $\Delta := (n_{\mathrm{r}} - \max\{1, n_{\mathrm{l}}\})/(\min\{999, n_{\mathrm{r}} - \max\{1, n_{\mathrm{l}}\})$, the values of n considered for each value of λ, $\mathcal{N}(\lambda)$, were obtained by rounding $\max\{1, n_{\mathrm{l}}\} + k\Delta$, $k = 0, 1, \ldots \min\{999, n_{\mathrm{r}} - \max\{1, n_{\mathrm{l}}\}\}$ to nearest ties to away. The results are summarized in table 1, where we show the minimum accuracy $d_{\mathrm{m}}(\lambda, n) := \min_{n \in \mathcal{N}(\lambda)} d(\lambda, n)$, the maximum accuracy $d_{\mathrm{M}}(\lambda, n) := \max_{n \in \mathcal{N}(\lambda)} d(\lambda, n)$, the average accuracy $\overline{d(\lambda, n)} := \sum_{n \in \mathcal{N}(\lambda)} d(\lambda, n)/n$ and the weighted average accuracy $\langle d(\lambda, n) := \sum_{n \in \mathcal{N}(\lambda)} P_n(\lambda) d(\lambda, n) / \sum_{n \in \mathcal{N}(\lambda)} P_n(\lambda)$. The results were obtained on a workstation equipped with Intel® Xeon® E7-8837 microprocessors, using only one core. As we can see, for λ small (say $\leq 10^2$), the method exhibits poor accuracy for some values of n, and for λ large (say $\geq 10^9$), the minim and average accuracy are substantially smaller than 16, which is, we recall, the approximate number of correct decimal digits that the `binary64` format with rounding mode round to nearest even can guarantee when performing elementary arithmetic operations. In the following section we will argue that this loss of accuracy can be explained by the fact that when both λ and n are large and n is close to λ but different from it, there can be cancellations in (14).

3 Proposed Method

3.1 Development

As previously mentioned, direct use of (1) easily leads to numerical overflow or underflow. However, trivially, $P_0(\lambda) = \mathrm{e}^{-\lambda}$, $\lambda > 0$. Besides, taking into account that $n!$, $0 \le n \le 22$, can be computed exactly using the `binary64` format (Press et al., 2007), that, if $\lambda \ge 2^{-43}$, then $\lambda^{22}/22! \ge 2^{-1\,022}$, and that, if $\lambda \le 2^9$, then $\mathrm{e}^{-\lambda} \ge 2^{-1\,022}$ and $\lambda^{22} \le (2 - 2^{-52})2^{1\,023}$, it turns out that $P_n(\lambda)$ can be safely computed using (1) for the set of (λ, n) pairs, $\lambda > 0$, $n \in \mathbb{N}$, with n satisfying $n = 0$ and the set of (λ, n) pairs, $\lambda > 0$, $n \in \mathbb{N}$, with λ satisfying $2^{-43} \le \lambda \le 2^9$ and n satisfying $0 < n \le 22$. This set will be referred to as set $\mathcal{S}_1$.

For $(\lambda, n) \notin \mathcal{S}_1$, it seems reasonable to turn our attention to (14). As we have commented, that equation can be inaccurate when λ is small and when both λ and n are large and n is close to λ but different from it. When λ is small, so is n because otherwise the probability would underflow. And when n is small, the approximation for $\ln G(n)$ that results from taking $m = 7$ in (9) can be inaccurate. This may make the method inaccurate when λ is small. We argue next that in the case in which both λ and n are large and n is close to λ but different from it, there can be cancellations in (14). Let us start analyzing the case $n \ge \lambda$. Define

$$f(x) := -x + (x+1)\operatorname{lns} x\,. \tag{15}$$

Then, using (14), (12),

$$\begin{aligned}\ln P_n(\lambda) + \frac{1}{2}\ln(2\pi n) &\approx -\Big(\lambda - n + n \operatorname{lns}\Big(\frac{n-\lambda}{\lambda}\Big) + I(m,n)\Big)\\ &= -\big(\lambda(-t + (t+1)\operatorname{lns}(t)) + I(m,n)\big)\\ &= -\big(\lambda\, f(t) + I(m,n)\big)\,,\end{aligned} \tag{16}$$

where $t = (n-\lambda)/\lambda$. The Maclaurin's series of $\operatorname{lns}(t)$ truncated after the second term gives

$$\operatorname{lns}(t) \approx t - \frac{t^2}{2}\,. \tag{17}$$

Combining (16), (15), and (17) yields

$$\ln P_n(\lambda) + \frac{1}{2}\ln(2\pi n) \approx -\Big(\lambda\Big(-t + \Big(t + \frac{t^2}{2} - \frac{t^3}{2}\Big)\Big) + I(m,n)\Big)\,.$$

The above equation suggests that if $t = (n-\lambda)/\lambda$ is positive and small, i.e., if $0 < n - \lambda \ll \lambda$, which holds if both λ and n are large and n is larger than λ but close to it, then there can be cancellations in (14).

Consider now the case $n < \lambda$. Define

$$g(x) := \frac{x - \operatorname{lns} x}{1+x} \,. \tag{18}$$

Using again (14), (12),

$$\begin{aligned}\ln P_n(\lambda) + \frac{1}{2}\ln(2\pi n) &\approx n - \lambda + n \operatorname{lns}\Big(\frac{\lambda - n}{n}\Big) - I(m,n) \\ &= -\Big(\lambda\Big(\frac{v}{1+v} - \frac{1}{1+v}\operatorname{lns}(v)\Big) + I(m,n)\Big) \\ &\approx -\big(\lambda\, g(v) + I(m,n)\big)\,, \end{aligned} \tag{19}$$

where $v = (\lambda - n)/n$. Combining (19), (18), (17) gives

$$\ln P_n(\lambda) + \frac{1}{2}\ln(2\pi n) \approx -\Big(\lambda\frac{v - (v - v^2/2)}{1+v} + I(m,n)\Big)\,.$$

Again, the above equation suggests that if $v = (\lambda - n)/n$ is small and positive, i.e., if $0 < \lambda - n \ll n$, which holds if both λ and n are large and n is smaller than λ but close to it, then there can be cancellations in (14).

After performing several numerical experiments, we found that the cancellations in (14) are not significant provided that either $(n-\lambda)/\lambda > 0.5$, $n-\lambda = 0$, or $(\lambda-n)/n > 0.5$, i.e., if either $n > 1.5\lambda$, $n = \lambda$, or $n < \lambda/1.5$. Accordingly, in the new method, the (λ, n) pairs, $\lambda > 0$, $n \in \mathbb{N}$, not belonging to $\mathcal{S}_1$ will be partitioned into three additional sets labeled $\mathcal{S}_2$, $\mathcal{S}_3$, and $\mathcal{S}_4$, and the probability will be approximated differently in each of them.

Set $\mathcal{S}_2$ will include all (λ, n) pairs, $\lambda > 0$, $n \in \mathbb{N}$, with λ satisfying $\lambda < 2^{-43}$ or $\lambda > 2^9$ and n satisfying $0 < n \le 22$, all pairs with n satisfying $n > \max\{22, 1.5\lambda\}$, all pairs with λ and n satisfying $n = \lambda > 22$, and all pairs with n satisfying $22 < n < \lambda/1.5$. Set $\mathcal{S}_3$ will include all (λ, n) pairs, $\lambda > 0$, $n \in \mathbb{N}$, with n satisfying $\max\{\lambda, 22\} < n \le 1.5\lambda$. Finally, set $\mathcal{S}_4$ will include all (λ, n) pairs, $\lambda > 0$, $n \in \mathbb{N}$, with n satisfying $\max\{\lambda/1.5, 23\} \le n < \lambda$.

For the (λ, n) pairs in set $\mathcal{S}_2$ with n satisfying $0 < n \le 22$, the $P_n(\lambda)$ will be computed using

$$P_n(\lambda) = \frac{1}{\sqrt{2\pi n}} \mathrm{e}^{(n-\lambda) - n\,h(\lambda,n) - \ln(G(n)/\sqrt{2\pi n})}\,. \tag{20}$$

The values for $\ln(G(n)/\sqrt{2\pi n})$, $0 < n \le 22$ are computed accurately beforehand using the MPFI library and stored.

For the (λ, n) pairs in set $\mathcal{S}_2$ with n satisfying $n > 22$, the $P_n(\lambda)$ will be approximated using (14) after factoring the term $\ln(2\pi n)/2$ out of the exponential. This gives

$$P_n(\lambda) \approx P_{m,n}(\lambda) := \frac{1}{\sqrt{2\pi n}} \mathrm{e}^{(n-\lambda) - n\,h(\lambda,n) - I(m,n)}\,. \tag{21}$$

The value of m will be chosen so that the approximation relative error is $\leq 2^{-53}$. To that end, we note that the difference between $\ln G(n)$ and $\ln(2\pi n)/2 + I(m,n)$ has the same sign as $T(m+1,n)$ (positive if m is even and negative otherwise) and is upper bounded in absolute value by $|T(m+1,n)|$ (Abramowitz and Stegun, 1965, 6.1.42), i.e., that, if m is even, then

$$0 \leq \ln G(n) - \frac{1}{2}\ln(2\pi n) - I(m,n) \leq T(m+1,n) \tag{22}$$

and otherwise,

$$0 \leq -\ln G(n) + \ln(2\pi n)/2 + I(m,n) \leq -T(m+1,n)\,. \tag{23}$$

Therefore, if m is even, using (13), (21), and (22),

$$0 \leq \frac{P_{m,n}(\lambda) - P_n(\lambda)}{P_{m,n}(\lambda)} \leq 1 - \mathrm{e}^{-T(m+1,n)}\,,$$

and, if m is odd, using (13), (21), and (23),

$$0 \leq \frac{P_n(\lambda) - P_{m,n}(\lambda)}{P_{m,n}(\lambda)} \leq \mathrm{e}^{-T(m+1,n)} - 1\,.$$

We then proceed as follows to determine appropriate values for m. Consider, for a given m, the minimum nonnegative integer q such that $1 - \mathrm{e}^{-T(m+1,2^q)} \leq 2^{-53}$ if m is even and $\mathrm{e}^{-T(m+1,2^q)} - 1 \leq 2^{-53}$ otherwise. Since the exponential function is monotone and by (10), $|T(m+1,n)|$ decreases on n, both the differences $1 - \mathrm{e}^{-T(m+1,n)}$ and $\mathrm{e}^{-T(m+1,n)} - 1$ will decrease on n as well, implying that for all $n \geq 2^q$ we will have $1 - \mathrm{e}^{-T(m+1,n)} \leq 2^{-53}$ if m is even and $\mathrm{e}^{-T(m+1,n)} - 1 \leq 2^{-53}$ otherwise. Then, using the MPFI library to ensure that we computed accurate upper bounds for $1 - \mathrm{e}^{-T(m+1,n)}$ in case m is even and for $\mathrm{e}^{-T(m+1,n)} - 1$ otherwise, we solved

$$q = \min_{r \geq 0}\{1 - \mathrm{e}^{-T(m+1,2^r)} \leq 2^{-53}\} \tag{24}$$

for $m = 0$, 2, 4, and 6, obtaining, respectively, $q = 50$, 9, 5, and 4, and solved

$$q = \min_{r \geq 0}\{\mathrm{e}^{-T(m+1,2^r)} - 1 \leq 2^{-53}\} \tag{25}$$

for $m = 1$, 3, 5, and 7, obtaining, respectively, $q = 15$, 7, 4, and 4. Therefore, taking $m = 5$ if $2^4 \leq n < 2^5$, $m = 4$ if $2^5 \leq n < 2^7$, $m = 3$ if $2^7 \leq n < 2^9$, $m = 2$ if $2^9 \leq n < 2^{15}$, $m = 1$ if $2^{15} \leq n < 2^{50}$, and $m = 0$ if $n \geq 2^{50}$ will ensure that the approximation relative error $|P_n(\lambda)/P_{m,n}(\lambda) - 1|$ is $\leq 2^{-53}$. These results are summarized in table 2.

Consider now the set $\mathcal{S}_3$. Combining (13), (12) with $n \geq \lambda$, and (15), we obtain

$$P_n(\lambda) = \mathrm{e}^{-(\lambda f((n-\lambda)/\lambda) + \ln G(n))}\,. \tag{26}$$

n	$[2^4, 2^5)$	$[2^5, 2^7)$	$[2^7, 2^9)$	$[2^9, 2^{15})$	$[2^{15}, 2^{50})$	$[2^{50}, \infty)$
m	5	4	3	2	1	0

Table 2: Value of m in $P_{m,n}(\lambda)$ as a function of $n \in \mathbb{N}$.

The term $\ln G(n)$ will be approximated using (11) with m chosen appropriately. The function $f(x)$ will be approximated by means of a truncated series. The starting point will be the series (Abramowitz and Stegun, 1965, 4.1.29)

$$\operatorname{lns} x = 2 \sum_{k \geq 0} \frac{z^{2k+1}}{2k+1}, \; z = \frac{x}{2+x}\,, \tag{27}$$

which for x real and > -1 is convergent. Combining the series with (15) gives

$$f(x) = \frac{x^2}{2+x} + \frac{2(x+1)x}{2+x} \sum_{k=1}^{\infty} \frac{(z^2)^k}{2k+1}, \; z = \frac{x}{2+x}\,. \tag{28}$$

By truncating the series at $k = l$, we obtain

$$f_l(x) = \frac{x^2}{2+x} + \frac{2(x+1)x}{2+x} \sum_{k=1}^{l} \frac{(z^2)^k}{2k+1}, \; z = \frac{x}{2+x}\,. \tag{29}$$

The $P_n(\lambda)$ will then be approximated by

$$P_{l,m,n}(\lambda) := \frac{1}{\sqrt{2\pi n}} \mathrm{e}^{-(\lambda\, f_l((n-\lambda)/\lambda) + I(m,n))}\,. \tag{30}$$

We describe next how m, l will be chosen so that the relative approximation error

$$\operatorname{err}_{l,m}(\lambda, n) := \left| \frac{P_n(\lambda)}{P_{l,m,n}(\lambda)} - 1 \right| \tag{31}$$

is $\leq 2^{-53}$. We begin by combining (26), (30), and (31), obtaining

$$\operatorname{err}_{l,m}(\lambda, n) = \left| \mathrm{e}^{-(\lambda(f((n-\lambda)/\lambda) - f_l((n-\lambda)/\lambda)) + \ln G(n) - I(m,n) - \ln(2\pi n)/2)} - 1 \right|. \tag{32}$$

Eq. (22) provides a bound for $|\ln G(n) - I(m,n) - \frac{1}{2}\ln(2\pi n)|$ if m is even and (23) provides a bound if m is odd. To obtain bounds for $f(x) - f_l(x)$, we note that, if $x > 0$, which implies $z = x/(2+x) < 1$,

$$\sum_{k=l+1}^{\infty} \frac{(z^2)^k}{2k+1} \leq \frac{1}{2l+3} \sum_{k=l+1}^{\infty} (z^2)^k = \frac{(z^2)^{l+1}}{2l+3} \sum_{k=0}^{\infty} (z^2)^k = \frac{(z^2)^{l+1}}{(2l+3)(1-z^2)}\,. \tag{33}$$

Therefore, using (28) and (29),

$$0 \leq f(x) - f_l(x) \leq \frac{2(x+1)z^{2l+3}}{(2l+3)(1-z^2)}, \; z = \frac{x}{2+x}\,. \tag{34}$$

n	$[2^4, 2^7)$	$[2^7, 2^{15})$	$[2^{15}, \infty)$
m	5	3	1

Table 3: Value of m in $P_{l,m,n}(\lambda)$ as a function of $n \in \mathbb{N}$.

To simplify the notation, let

$$\omega_k(x) = \frac{2x^k}{k(1-x^2)}\,. \tag{35}$$

With that notation, (34) becomes

$$0 \le f(x) - f_l(x) \le (1+x)\omega_{2l+3}\Big(\frac{x}{x+2}\Big)\,. \tag{36}$$

If m is even, by (22) the factor $\ln G(n) - I(m,n) - \ln(2\pi n)/2$ will be nonnegative and upper bounded by $T(m+1,n) > 0$. Therefore, using (32) and (36),

$$\begin{aligned}\mathrm{err}_{l,m}(\lambda, n) &= 1 - \mathrm{e}^{-(\lambda(f((n-\lambda)/\lambda)-f_l((n-\lambda)/\lambda))+\ln G(n)-I(m,n)-\ln(2\pi n)/2)}\\ &\le 1 - \mathrm{e}^{-\lambda(1+t)\omega_{2l+3}(t/(2+t))-T(m+1,n)},\ t = \tfrac{n-\lambda}{\lambda}\,.\end{aligned}$$

If m is odd, by (23) the factor $\ln G(n) - I(m,n) - \frac{1}{2}\ln(2\pi n)$ will be nonpositive and lower bounded by $T(m+1,n) < 0$. Then, using again (32) and (36),

$$\mathrm{err}_{l,m}(\lambda, n) \le \max\big\{1 - \mathrm{e}^{-\lambda(1+t)\omega_{2l+3}(t/(2+t))}, \mathrm{e}^{-T(m+1,n)} - 1\big\},\ t = \tfrac{n-\lambda}{\lambda}\,. \tag{37}$$

The bound (37) will be used to select the parameters m, l for the relative approximation error (31) to be $\le 2^{-53}$.

To determine appropriate values for m in $P_{l,m,n}(\lambda)$, we solved (25) for $m = 1$, 3, 5, and 7, obtaining $q = 15$, 7, 4, and 4, respectively. Therefore, taking $m = 5$ if $2^4 \le n < 2^7$, $m = 3$ if $2^7 \le n < 2^{15}$, and $m = 1$ if $n \ge 2^{15}$ will ensure $\mathrm{e}^{-T(m+1,n)} - 1 \le 2^{-53}$. These results are summarized in table 3.

The parameter l is computed on the fly. By imposing $1 - \mathrm{e}^{-\lambda(1+t)\omega_{2l+3}(t/(2+t))} \le 2^{-53}$, $t = (n-\lambda)/\lambda$ and using (35), we obtain

$$\frac{z^{2l}}{2l+1} \le -\frac{1-z^2}{2z^3}\,\frac{2l+3}{2l+1}\,\frac{\ln(1-2^{-53})}{\lambda(1+t)},\ t = \frac{n-\lambda}{\lambda},\ z = \frac{t}{2+t}\,. \tag{38}$$

From a computational point of view, we find it convenient to replace the quantity $-\ln(1-2^{-53})$ in (38) by a lower bound that can be computed exactly. Thus, using the inequality $-\ln(1-x) > x$, $x < 1$ (Abramowitz and Stegun, 1965, 4.1.34) we have $-\ln(1-2^{-53}) > 2^{-53}$. Then, in the method the factor $f_l\left((n-\lambda)/\lambda\right)$ is computed using (29) starting with $l = 1$ and increasing l until it holds that

$$\frac{z^{2l}}{2l+1} \le \frac{1-z^2}{2z^3}\,\frac{2l+3}{2l+1}\,\frac{2^{-53}}{\lambda(1+t)},\ t = \frac{n-\lambda}{\lambda},\ z = \frac{t}{2+t}\,. \tag{39}$$

It remains to consider the set $\mathcal{S}_4$. Combining (13), (12) with $n < \lambda$, and (18), we obtain

$$P_n(\lambda) = \mathrm{e}^{-(\lambda\, g((\lambda-n)/n)+\ln G(n))}\,. \tag{40}$$

The factor $\ln G(n)$ will be approximated using (11) with m chosen appropriately. The function $g(x)$ will be approximated by a truncated series.

Using (18), (27),

$$g(x) = \frac{x^2}{(1+x)(2+x)} - \frac{2x}{(1+x)(2+x)}\sum_{k=1}^{\infty}\frac{(z^2)^k}{2k+1},\; z = \frac{x}{2+x}\,, \tag{41}$$

which, truncating the infinite series at $k = l$, becomes

$$g_l(x) = \frac{x^2}{(1+x)(2+x)} - \frac{2x}{(1+x)(2+x)}\sum_{k=1}^{l}\frac{(z^2)^k}{2k+1},\; z = \frac{x}{2+x}\,. \tag{42}$$

The $P_n(\lambda)$ will be approximated by

$$P'_{l,m,n}(\lambda) := \frac{1}{\sqrt{2\pi n}}\mathrm{e}^{-(\lambda\, g_l((\lambda-n)/n)+I(m,n))}\,. \tag{43}$$

The parameters m, l will be chosen so that the relative approximation error

$$\mathrm{err}'_{l,m}(\lambda,n) := \Big|\frac{P_n(\lambda)}{P'_{l,m,n}(\lambda)} - 1\Big| \tag{44}$$

is $\leq 2^{-53}$. To that end, we start by combining (40), (43), and (44). This yields

$$\mathrm{err}'_{l,m}(\lambda,n) = \Big|\mathrm{e}^{-\left(\lambda\left(g((\lambda-n)/n)-g_l((\lambda-n)/n)\right)+\ln G(n)-I(m,n)-\ln\sqrt{2\pi n}\right)} - 1\Big|\,. \tag{45}$$

Eq. (22) provides a bound for $|\ln G(n) - I(m,n)\ln(2\pi n)/2|$ if m is even and (23) provides a bound if m is odd. Using (41), (42), (33), and (35) we can obtain the bounds for $g_l(x) - g(x)$

$$0 \leq g_l(x) - g(x) \leq \frac{\omega_{2l+3}\left(x/(2+x)\right)}{1+x}\,. \tag{46}$$

Then, combining (46), (22), (23), and (45), gives, if m is even,

$$\mathrm{err}'_{l,m}(\lambda,n) \leq \max\left\{\mathrm{e}^{\lambda\omega_{2l+3}(v/(2+v))/(1+v)} - 1, 1 - \mathrm{e}^{-T(m+1,n)}\right\},\; v = \tfrac{\lambda-n}{n}\,, \tag{47}$$

and, if m is odd,

$$\mathrm{err}'_{l,m}(\lambda,n) \leq \mathrm{e}^{\lambda\omega_{2l+3}(v/(2+v))/(1+v)-T(m+1,n)} - 1,\; v = \tfrac{\lambda-n}{n}\,.$$

The bound (47) will be used to select the parameters m, l for the relative approximation error (44) to be $\leq 2^{-53}$.

n	$[2^4, 2^5)$	$[2^5, 2^9)$	$[2^9, 2^{50})$	$[2^{50}, \infty)$
m	6	4	2	0

Table 4: Value of m in $P'_{l,m,n}(\lambda)$ as a function of $n \in \mathbb{N}$.

To determine appropriate values for m in (43) $P'_{l,m,n}(\lambda)$, we solved (24) for $m = 0$, 2, 4, and 6, obtaining $q = 50$, 9, 5, and 4, respectively. Therefore, taking $m = 6$ for $2^4 \le n < 2^5$, $m = 4$ for $2^5 \le n < 2^9$, $m = 2$ for $2^9 \le n < 2^{50}$, and $m = 0$ for $n \ge 2^{50}$ will ensure $1 - \mathrm{e}^{-T(m+1,n)} \le 2^{-53}$. These results are summarized in table 4.

The parameter l is computed on the fly. Imposing $\mathrm{e}^{\lambda\omega_{2l+3}(v/(2+v))/(1+v)} - 1 \le 2^{-53}$, $v = (\lambda - n)/n$, gives, using (35),

$$\frac{z^{2l}}{2l+1} \le \frac{1-z^2}{2z^3}\frac{2l+3}{2l+1}\frac{1+v}{\lambda}\ln(1+2^{-53}),\ v = \frac{\lambda - n}{n},\ z = \frac{v}{2+v}\,. \tag{48}$$

From a computational point of view, we find it convenient to replace the quantity $\ln(1+2^{-53})$ in (48) by a lower bound that can be computed exactly. Using the inequality $\ln(1+x) > x/(1+x)$, $x > -1$ (Abramowitz and Stegun, 1965, 4.1.33), we get

$$\begin{aligned}\ln(1+2^{-53}) &> 2^{-53}/(1+2^{-53}) = 1/(2^{53}+1) = (2^{53}-1)/\left((2^{53}-1)(2^{53}+1)\right)\\ &= (2^{53}-1)/(2^{106}-1) > (2^{53}-1)2^{-106}\,.\end{aligned}$$

Then, in the method the factor $g_l((\lambda - n)/n)$ is computed using (42) starting with $l = 1$ and increasing l until it holds that

$$\frac{z^{2l}}{2l+1} \le \frac{1-z^2}{2z^3}\frac{2l+3}{2l+1}\frac{1+v}{\lambda}(2^{53}-1)2^{-106},\ v = \frac{\lambda - n}{n},\ z = \frac{v}{2+v}\,. \tag{49}$$

3.2 Summary

In the method, the set of (λ, n) pairs, $\lambda > 0$, $n \in \mathbb{N}$, is partitioned into four sets and the probability is approximated differently in each set. The first set, which we labeled $\mathcal{S}_1$, consists of all pairs with n satisfying $n = 0$ and all pairs with λ satisfying $2^{-43} \le \lambda \le 2^9$ and n satisfying $0 < n \le 22$. For $(\lambda, n) \in \mathcal{S}_1$, the $P_n(\lambda)$ is computed using (1) with $\lambda^0 = 1$, $0! = 1$, evaluating λ^n if $n > 0$ as $\lambda^n = \lambda \times \lambda \times \cdots \times \lambda$ and evaluating $n!$ if $n > 0$ as $n! = n \times (n-1) \times \cdots \times 2$.

The second set, labeled $\mathcal{S}_2$, consists of all pairs with λ satisfying $\lambda < 2^{-43}$ or $\lambda > 2^9$ and n satisfying $0 < n \le 22$, all pairs with n satisfying $n > \max\{22, 1.5\lambda\}$, all pairs with λ and n satisfying $n = \lambda > 22$, and all pairs with n satisfying $22 < n < \lambda/1.5$. For $(\lambda, n) \in \mathcal{S}_2$, if $0 < n \le 22$ the $P_n(\lambda)$ is computed using (20), and if $n > 22$, the $P_n(\lambda)$ is approximated using (21) with m given in table 2. In the case $0 < n \le 22$, the values for $\ln\left(G(n)/\sqrt{2\pi n}\right)$, $0 < n \le 22$ are computed accurately beforehand using the MPFI library and stored.

The third set, labeled $\mathcal{S}_3$, consists of all pairs with n satisfying $\max\{\lambda, 22\} < n \le 1.5\lambda$. For $(\lambda, n) \in \mathcal{S}_3$, the $P_n(\lambda)$ is approximated using (30). The value of m is given in table 3.

The factor $f_l\left((n-\lambda)/\lambda\right)$ is computed using (29) with the minimum $l \geq 1$ such that (39) holds.

The fourth set, labeled $\mathcal{S}_4$, consists of all pairs with n satisfying $\max\{\lambda/1.5, 23\} \leq n < \lambda$. For $(\lambda, n) \in \mathcal{S}_4$, the $P_n(\lambda)$ is approximated using (43). The value of m is given in table 4. The factor $g_l((\lambda - n)/n)$ is computed using (42) with the minimum $l \geq 1$ such that (49) holds.

For the sake of clarity, in figures 1 and 2 we give a pseudo-code for the proposed method.

4 Numerical Results

In order to compare the proposed method with the methods described in (Whittlesey, 1963; Knüsel, 1986; Press et al., 2007; Krishnamoorthy, 2016), we performed with each method the experiment described in sect. 2. In addition, for each method we estimated the average CPU time in microseconds required to compute $P_n(\lambda)$ as a function of λ by measuring the overall time required to obtain 10 000 times the $P_n(\lambda)$ for all the values of n considered for each value of λ. The results are summarized in table 5, where we give $\overline{d(\lambda, n)} = \sum_{n\in\mathcal{N}(\lambda)} d(\lambda, n)/n$, $\langle d(\lambda, n)\rangle = \sum_{n\in\mathcal{N}(\lambda)} P_n(\lambda) d(\lambda, n)/\sum_{n\in\mathcal{N}(\lambda)} P_n(\lambda)$, and the average CPU time in microseconds, t_{CPU}.[1] The results were obtained on a workstation equipped with Intel® Xeon® E7-8837 microprocessors, using only one core.

[1] The methods described in (Press et al., 2007; Krishnamoorthy, 2016) perform very similarly and we only give results for the former.

```
Require: Inputs $\lambda > 0$, $n \in \mathbb{N}$; precomputed values for $\ln(G(n)/\sqrt{2\pi n})$, $1 \leq n \leq 22$
Ensure: In exact arithmetic, $|(P_n(\lambda) - \widetilde{P}_n(\lambda))/\widetilde{P}_n(\lambda)| \leq 2^{53}$
  if $n = 0$ or $(2^{-53} \leq \lambda \leq 2^9$ and $0 < n \leq 22)$ then        ▷ Set $\mathcal{S}_1$
    $\widetilde{P}_n(\lambda) := \dfrac{\lambda^n}{n!}\mathrm{e}^{-\lambda}$
  else if $(\lambda < 2^{-43}$ or $\lambda > 2^9)$ and $0 < n \leq 22$ then        ▷ Set $\mathcal{S}_2$
    $\widetilde{P}_n(\lambda) := \dfrac{1}{\sqrt{2\pi n}}\mathrm{e}^{(n-\lambda)-n\,h(\lambda,n)-\ln(G(n)/\sqrt{2\pi n})}$
  else if $n > \max\{22, 1.5\lambda\}$ or $n = \lambda > 22$ or $22 < n \leq \lambda/1.5$ then
    if $n > \max\{22, 1.5\lambda\}$ then
      $h(\lambda, n) := \operatorname{lns}\left(\dfrac{n-\lambda}{\lambda}\right)$
    else if $n = \lambda > 22$ then
      $h(\lambda, n) := 0$
    else
      $h(\lambda, n) := -\operatorname{lns}\left(\dfrac{\lambda-n}{n}\right)$
    end if
    Select $m$ using table 2
    $I(m,n) := \displaystyle\sum_{j=1}^{m} \frac{B_{2j}}{2j(2j-1)n^{2j-1}}$
    $\widetilde{P}_n(\lambda) := \dfrac{1}{\sqrt{2\pi n}}\mathrm{e}^{(n-\lambda)-n\,h(\lambda,n)-I(m,n)}$
  else if $\max\{\lambda, 22\} < n \leq 1.5\lambda$ then        ▷ Set $\mathcal{S}_3$
    $h(\lambda, n) := \operatorname{lns}\left(\dfrac{n-\lambda}{\lambda}\right)$
        ▷ Compute $f_l((n-\lambda)/\lambda)$
    $t := \dfrac{n-\lambda}{\lambda}$
    $z := \dfrac{t}{2+t}$
    $l := 1$
    $aux := \dfrac{z^{2l}}{2l+1}$
    $sum := aux$
    $bound := \dfrac{1-z^2}{2z^3}\,\dfrac{2l+3}{2l+1}\,\dfrac{2^{-53}}{\lambda(1+t)}$
    while $aux > bound$ do
      $l := l + 1$
      $aux := \dfrac{z^{2l}}{2l+1}$
      $sum := sum + aux$
      $bound := \dfrac{1-z^2}{2z^3}\,\dfrac{2l+3}{2l+1}\,\dfrac{2^{-53}}{\lambda(1+t)}$
    end while
```

Figure 1: Pseudo-code for the proposed method.

```
    $f_l\Big(\frac{n-\lambda}{\lambda}\Big) := \frac{t^2}{2+t} + \frac{2(t+1)t}{2+t} \times sum$
    Select $m$ using table 3
    $I(m,n) := \sum_{j=1}^{m} \frac{B_{2j}}{2j(2j-1)n^{2j-1}}$
    $\widetilde{P}_n(\lambda) := \frac{1}{\sqrt{2\pi n}} \mathrm{e}^{-(\lambda\, f_l((n-\lambda)/\lambda)+I(m,n))}$
else if $\max\{\lambda/1.5, 23\} \le n < \lambda$ then                    $\triangleright$ Set $\mathcal{S}_4$
    $h(\lambda,n) := -\mathrm{lns}\Big(\frac{\lambda-n}{n}\Big)$
        $\triangleright$ Compute $g_l((\lambda-n)/n)$
    $v := \frac{\lambda-n}{n}$
    $z := \frac{v}{2+v}$
    $l := 1$
    $aux := \frac{z^{2l}}{2l+1}$
    $sum := aux$
    $bound := \frac{1-z^2}{2z^3}\frac{2l+3}{2l+1}\frac{1+v}{\lambda}(2^{-53}-1)2^{-106}$
    while $aux > bound$ do
        $l := l+1$
        $aux := \frac{z^{2l}}{2l+1}$
        $sum := sum + aux$
        $bound := \frac{1-z^2}{2z^3}\frac{2l+3}{2l+1}\frac{1+v}{\lambda}(2^{-53}-1)2^{-106}$
    end while
    $g_l\Big(\frac{\lambda-n}{n}\Big) := \frac{v^2}{(1+v)(2+v)} - \frac{2v}{(1+v)(2+v)} \times sum$
    Select $m$ using table 4
    $I(m,n) := \sum_{j=1}^{m} \frac{B_{2j}}{2j(2j-1)n^{2j-1}}$
    $\widetilde{P}_n(\lambda) := \frac{1}{\sqrt{2\pi n}} \mathrm{e}^{-(\lambda\, g_l((\lambda-n)/n)+I(m,n))}$
end if
return $\widetilde{P}_n(\lambda)$
```

Figure 2: Pseudo-code for the proposed method. (Cont'd.)

λ	Press et al. (2007)			Whittlesey (1963)			Knüsel (1986)			proposed		
	$\overline{d(\lambda,n)}$	$\langle d(\lambda,n)\rangle$	t_{CPU}	$\overline{d(\lambda,n)}$	$\langle d(\lambda,n)\rangle$	t_{CPU}	$\overline{d(\lambda,n)}$	$\langle d(\lambda,n)\rangle$	t_{CPU}	$\overline{d(\lambda,n)}$	$\langle d(\lambda,n)\rangle$	t_{CPU}
10^0	13.9	15.9	0.27	13.8	15.2	0.19	13.6	4.40	0.18	14.1	16.5	0.12
10^1	13.6	15.0	0.28	13.9	15.5	0.19	13.8	15.1	0.18	14.0	16.3	0.12
10^2	13.1	13.5	0.27	13.9	14.5	0.18	13.9	15.6	0.19	14.1	16.0	0.13
10^3	12.5	12.6	0.28	13.4	13.6	0.19	13.7	15.1	0.19	14.0	16.0	0.14
10^4	11.2	11.1	0.28	12.5	12.5	0.18	13.2	14.6	0.17	14.0	16.2	0.18
10^5	10.2	10.2	0.27	11.6	11.6	0.18	12.7	14.2	0.17	14.0	16.1	0.15
10^6	9.20	9.12	0.27	10.6	10.6	0.17	12.2	13.6	0.17	14.1	16.2	0.14
10^7	7.99	7.91	0.27	9.53	9.39	0.17	11.7	13.1	0.17	14.0	16.1	0.13
10^8	6.85	6.88	0.27	8.55	8.54	0.17	11.2	12.6	0.17	14.0	16.1	0.13
10^9	5.91	5.96	0.27	7.53	7.43	0.17	10.7	12.1	0.17	14.0	16.2	0.13
10^{10}	4.92	4.90	0.27	6.54	6.52	0.17	10.2	11.7	0.17	14.0	16.1	0.12
10^{11}	3.69	3.74	0.28	5.54	5.52	0.18	9.71	11.1	0.18	14.1	16.2	0.12
10^{12}	2.80	2.71	0.28	4.54	4.46	0.18	9.25	10.6	0.17	14.0	16.2	0.13
10^{13}	1.82	1.81	0.28	3.54	3.52	0.17	8.75	10.1	0.18	14.0	16.1	0.12
10^{14}	0.51	0.51	0.27	2.54	2.56	0.17	8.22	9.65	0.17	14.1	16.1	0.11
10^{15}	0.10	0.08	0.27	1.52	1.44	0.17	7.71	9.13	0.17	14.1	16.2	0.11

Table 5: Comparison of the methods described in (Whittlesey, 1963; Knüsel, 1986; Press et al., 2007) with the method proposed in the paper.

As we can see, the method developed in the paper seems to be more accurate and almost always faster than the remaining methods.

Next, we illustrate that the accuracy of the method seems to be quite good if we take into account what can be expected by using IEEE 754 floating-point arithmetic. Let y denote the result of evaluating a given algebraic expression using exact arithmetic and let $\mathrm{fl}(y)$ denote the result of evaluating it using IEEE 754 floating-point arithmetic. We can expect the relative error in $\mathrm{fl}(y)$ to be at least equal to 2^{-53}, i.e., that $\mathrm{fl}(y) = y(1+\delta_y)$, $|\delta_y| \geq 2^{-53}$. Now, consider $\mathrm{fl}(x \cdot \mathrm{e}^y)$ assuming, optimistically, that multiplication can be performed exactly and that the exponential function can be evaluated exactly. We have

$$\mathrm{fl}(x \cdot \mathrm{e}^y) = \mathrm{fl}(x) \cdot \mathrm{e}^{\mathrm{fl}(y)} = x(1+\delta_x)\mathrm{e}^{y(1+\delta_y)} = x(1+\delta_x)\mathrm{e}^y \mathrm{e}^{y\delta_y} .$$

Assuming, realistically, that $|y\delta_y| \ll 1$, so that we can approximate $\mathrm{e}^{y\delta_y}$ by $1 + y\delta_y$, and that $|\delta_x\delta_y| \ll |\delta_y|$, so that we can approximate $y\delta_y + y\delta_y\delta_x$ by $y\delta_y$,

$$x(1+\delta_x)\mathrm{e}^y\mathrm{e}^{y\delta_y} \approx x(1+\delta_x)\mathrm{e}^y(1+y\delta_y) \approx x\mathrm{e}^y(1+y\delta_y+\delta_x) .$$

Therefore, if $|\delta_x| \geq 2^{-53}$, $|\delta_y| \geq 2^{-53}$,

$$\begin{aligned}
-\log_{10}\Big|\frac{x\cdot \mathrm{e}^y - \mathrm{fl}(x\cdot\mathrm{e}^y)}{\mathrm{fl}(x\cdot \mathrm{e}^y)}\Big| &\approx -\log_{10}|y\delta_y+\delta_x| + \log_{10}|1+y\delta_y+\delta_x| \\
&= -\log_{10}\big(|y|\big|\delta_y + \frac{\delta_x}{y}\big|\big) + \log_{10}|1+y\delta_y+\delta_x| \\
&\leq -\log_{10}|y| + 53\log_{10}2 - \log\left(\Big|1+\frac{1}{y}\Big|\right) \\
&\quad + \log_{10}|1 + y\times 2^{-53} - 2^{-53}| \\
&= -\log_{10}|y| + 53\log_{10}2 - \log\left(\Big|1+\frac{1}{y}\Big|\right) \\
&\quad + \log_{10}|(2^{53}-1)\times 2^{-53} + y\times 2^{-53}| . \qquad (50)
\end{aligned}$$

Using (50) with $x = 1/\sqrt{2\pi n}$ and $y = \ln(\sqrt{2\pi n}\,P_n(\lambda))$, we obtain

$$\begin{aligned}
&-\log_{10}\left|\frac{P_n(\lambda) - \mathrm{fl}\left((1/\sqrt{2\pi n})\exp\left(\ln(\sqrt{2\pi n})\,P_n(\lambda)\right)\right)}{\mathrm{fl}\left((1/\sqrt{2\pi n})\exp\left(\ln(\sqrt{2\pi n})\,P_n(\lambda)\right)\right)}\right| \\
&\leq -\log_{10}\left|\ln(\sqrt{2\pi n}\,P_n(\lambda))\right| + 54\log_{10}2 - \log_{10}\left|1+\frac{1}{\ln(\sqrt{2\pi n}P_n(\lambda))}\right| \\
&\quad + \log_{10}\left|(2^{53}-1)\times 2^{-53} + \ln(\sqrt{2\pi n}\,P_n(\lambda))\times 2^{-53}\right| \\
&:= d_{\mathrm{b64}}(\lambda, n) . \qquad (51)
\end{aligned}$$

The quantity $d_{\mathrm{b64}}(\lambda, n)$ can be regarded as the accuracy we can expect if we compute $P_n(\lambda)$ as

$$P_n(\lambda) = \mathrm{fl}\left\{\frac{1}{\sqrt{2\pi n}}\right\}\mathrm{e}^{\mathrm{fl}\left\{\ln(\sqrt{2\pi n})\,P_n(\lambda)\right\}}$$

λ	$d_{\mathrm{m}}(\lambda,n)$	$d_{\mathrm{M}}(\lambda,n)$	$\overline{d(\lambda,n)}$	$d_{\mathrm{m}}^{\mathrm{b64}}(\lambda,n)$	$d_{\mathrm{M}}^{\mathrm{b64}}(\lambda,n)$	$\overline{d^{\mathrm{b64}}(\lambda,n)}$
10^0	12.9	18.1	14.1	13.1	17.2	13.7
10^1	12.8	17.0	14.0	13.1	17.3	13.8
10^2	12.7	17.2	14.1	13.1	16.2	14.0
10^3	12.4	17.2	14.0	13.1	15.9	13.9
10^4	12.3	18.1	14.0	13.1	15.7	13.9
10^5	12.5	17.7	14.0	13.1	15.6	13.9
10^6	12.5	17.7	14.1	13.1	15.5	13.9
10^7	12.5	17.6	14.0	13.1	15.4	13.9
10^8	12.5	17.6	14.0	13.1	15.3	13.9
10^9	12.6	17.4	14.0	13.1	15.3	13.9
10^{10}	12.6	17.7	14.0	13.1	15.2	13.9
10^{11}	12.6	18.1	14.1	13.1	15.2	13.9
10^{12}	12.4	17.3	14.0	13.1	15.2	13.9
10^{13}	12.6	17.6	14.0	13.1	15.1	13.8
10^{14}	12.6	17.7	14.1	13.1	15.1	13.8
10^{15}	12.6	19.4	14.1	13.1	15.0	13.8

Table 6: Measured accuracy for the proposed method and expected accuracy.

with $\mathrm{fl}(1/\sqrt{2\pi n}) = (1/\sqrt{2\pi n})(1+\delta)$, $|\delta| \approx 2^{-53}$ and $\mathrm{fl}\{\ln(\sqrt{2\pi n})\,P_n(\lambda)\} = (\ln(\sqrt{2\pi n})\,P_n(\lambda))(1+\delta')$, $|\delta'| \approx 2^{-53}$. Since in the proposed method the approximation for $P_n(\lambda)$ is mostly of the form $P_n(\lambda) \approx (1/\sqrt{2\pi n})\mathrm{e}^y$ (cf. (20), (21), (30), (43)), it makes sense to compare the accuracy measured for the proposed method with the expected accuracy. This is done in table 6, where we show the minimum, maximum, and average accuracy measured for the proposed method, $d_{\mathrm{m}}(\lambda,n)$, $d_{\mathrm{M}}(\lambda,n)$, and $\overline{d(\lambda,n)}$, and the corresponding values obtained using instead $d_{\mathrm{b64}}(\lambda,n)$ defined in (51). As we can observe, the accuracy of the proposed method compares very favorably with the expected one.

5 Conclusions

In this paper, we have reviewed published methods for the computation of Poisson probabilities. Restricting ourselves to methods aimed at the computation of a single probability, we have shown that neither of them is completely satisfactory in terms of accuracy. With that motivation, we have developed a new method for the computation of Poisson probabilities. The method is intended for computing a single probability. Unlike previous methods, the new method comes with guaranteed approximation relative error. Numerical experimentation shows that the method seems to be more accurate and almost always faster than published methods.

References

John RB Whittlesey. Incomplete gamma functions for evaluating Erlang process probabilities. *Mathematics of Computation*, pages 11–17, 1963.

Cornelius Lanczos. A precision approximation of the gamma function. *Journal of the Society for Industrial and Applied Mathematics, Series B: Numerical Analysis*, 1(1): 86–96, 1964.

Milton Abramowitz and Irene A Stegun. *Handbook of mathematical functions: with formulas, graphs, and mathematical tables*, volume 55. Courier Corporation, 1965.

Frank Avery Haight. *Handbook of the Poisson distribution.* John Wiley & Sons, 1967.

J. F. Hart, E. W. Cheney, C. L. Lawson, H. J. Maehly, H. J. Mesztenyi, J. R. Rice, H. G. Thacher Jr., and C. Witzgall. *Computer Approximations.* John Wiley, 1968.

Leo Knüsel. Computation of the chi-square and Poisson distribution. *SIAM Journal on Scientific and Statistical Computing*, 7(3):1022–1036, 1986.

Bennett L Fox and Peter W Glynn. Computing Poisson probabilities. *Communications of the ACM*, 31(4):440–445, 1988.

Adrienne W Kemp. A note on Stirling's expansion for factorial n. *Statistics & probability letters*, 7(2):139–143, 1988a.

Adrienne W Kemp. Simple algorithms for the Poisson modal cumulative probability. *Communications in Statistics-Simulation and Computation*, 17(4):1495–1508, 1988b.

CD Kemp and Adrienne W Kemp. Poisson random variate generation. *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, 40(1):143–158, 1991.

Norman L Johnson, Adrienne W Kemp, and Samuel Kotz. *Univariate discrete distributions*, volume 444. John Wiley & Sons, 3rd edition, 2005.

Nathalie Revol and Fabrice Rouillier. Motivations for an arbitrary precision interval arithmetic and the MPFI library. *Reliable Computing*, 11(4):275–290, Aug 2005.

William H Press, Saul A Teukolsky, William T Vetterling, and Brian P Flannery. *Numerical recipes: The art of scientific computing.* Cambridge University Press, third edition, 2007.

IEEE. *Standard IEEE Std-754 for floating-point arithmetic.* Institute of Electrical and Electronics Engineers, 2008.

Jean-Michel Muller, Nicolas Brisebarre, Florent De Dinechin, Claude-Pierre Jeannerod, Vincent Lefevre, Guillaume Melquiond, Nathalie Revol, Damien Stehlé, Serge Torres, et al. Handbook of floating-point arithmetic. 2010.

Catherine Forbes, Merran Evans, Nicholas Hastings, and Brian Peacock. *Statistical distributions*. John Wiley & Sons, fourth edition, 2011.

Kishore S Trivedi. *Probability & Statistics with Reliability, Queuing and Computer Science Applications*. PHI Learning Pvt. Limited, 2011.

Kalimuthu Krishnamoorthy. *Handbook of statistical distributions with applications*. Chapman and Hall/CRC, 2016.

Weiping Wang. Unified approaches to the approximations of the gamma function. *Journal of Number Theory*, 163:570–595, 2016.