Harmonic functions on metric graphs under the anti–Kirchhoff law

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Abstract

When does an infinite metric graph allow nonconstant bounded harmonic functions under the anti–Kirchhoff transition law? We give a complete answer to this question in the cases where Liouville's Theorem holds, for trees, for graphs with finitely many essential ramification nodes and for generalized lattices. It turns out that the occurrence of nonconstant bounded harmonic functions under the anti–Kirchhoff law differs strongly from the one under the classical continuity and the Kirchhoff flow condition.

Keywords: harmonic functions, Liouville's Theorem, infinite graphs, metric graphs, quantum graphs, anti–Kirchhoff law, generalized lattices.
MSC (2010): 35R02, 35J25, 34B45, 05C50, 05C10, 05C63, 31C05.

Among the classical and often considered transition conditions at the nodes of a metric graph we find the continuity condition at the ramification nodes

$$\forall v_i \in V_{\mathrm{r}} \ : \ k_j \cap k_s = \{v_i\} \implies u_j(v_i) = u_s(v_i), \tag{1}$$

and Kirchhoff's flow law, by which, at each node v_i, all the incident outer normal derivatives sum up to 0:

$$\sum_j d_{ij} \partial_j u_j(v_i) = 0. \tag{2}$$

We shall cite both conditions (1) and (2) together as (CK). They have been treated by many authors, including generalizations as weighted Kirchhoff conditions and dynamical ones in connection with differential operators on the edges, and are of interest in many settings and applications.

However, when treating wave dispersion problems on graphs e.g., or as in many other cases, the (CK)–condition is not suitable and should be replaced by its *orthogonal* condition, the so–called *anti–Kirchhoff condition* (KC), see [6] and the

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†Supported by MINECO grants MTM2014-52402-C3-1-P and MTM2017-84214-C2-1-P and part of the Catalan research group 2017 SGR 1392.

references therein. It is given by the continuity of the outer normal derivatives at the ramification nodes (4) and by vanishing potential sums at all vertices (3). Mathematically it stems from the self–adjoint orthogonal boundary condition in the sense of the Y–boundary conditions associated to corresponding Bochner–spaces, see [6].

The present paper deals with the validity of Liouville's Theorem on infinite uniformly locally finite metric graphs under the anti–Kirchhoff law, or more generally, with the multiplicity of 0 as an eigenvalue of the edge Laplacian under (KC) in $L^\infty(G)$. It turns out that the (KC)–condition enforces a behaviour of bounded harmonic functions that differs strongly from the one under the (CK)–condition. Denoting the edges of the graph by k_j, at each vertex v_i we impose the following transition condition

$$\sum_{v_i \in k_j} u_j(v_i) = 0, \tag{3}$$

$$k_j \cap k_s = \{v_i\} \implies d_{ij}\partial_j u_j(v_i) = d_{is}\partial_s u_s(v_i). \tag{4}$$

Conceivably, we shall cite both conditions (3) and (4) together as (KC). Note that (3) reduces to the 0–Dirichlet condition at boundary vertices.

The presentation is organized as follows. After some prerequisites, graph theoretical preliminaries and basic facts about harmonic functions in Section 1, Section 2 deals with the finite case. It turns out that a finite metric graph Γ under (KC) is a Liouville graph if and only if Γ is a tree or a non bipartite unicyclic graph. Section 3 presents some basic tools for the infinite case, especially the distinctive impact of circuits on the occurrence of nonconstant bounded harmonic functions. These results lead among others to the characterization of infinite Liouville graphs and of trees without two–sided unbounded paths in Section 4. In particular it will be shown that a graph Γ containing an one–sided unbounded path Γ_0 of infinite total length is a Liouville graph under (KC) if and only if it is a sole Γ_0 with finite trees attached to its vertices, see Theorem 4.1.

In Sections 5 and 6 the occurrence of bounded harmonic functions on infinite graphs with finitely many essential ramification nodes, the so–called medusae, and infinite trees will be treated. In the first case, optimal upper and lower bounds for the dimension $m(0;\Gamma;KC)$ of the vector space of bounded harmonic functions under (KC) in dependence of the smallest connected finite graph containing the essential ramification nodes will be established, see Theorems 5.1 and 5.2. As for infinite trees it will be shown that a tree T satisfies $m(0;\Gamma;KC) = \infty$ if and only if its reduced tree has infinitely many essential ramification nodes, see Theorem 6.2. In contrast to the (CK)–condition, no geometrical restriction to the edges is necessary here.

In Section 7 graphs with exactly one independent bounded harmonic function and generalized lattices, the so–called periodic graphs will be treated. It turns out that all periodic graphs containing circuits fulfill $m(0;\Gamma;KC) = \infty$, even with inseparable eigenspaces, while periodic trees are precisely those periodic graphs fulfilling $m(0;\Gamma;KC) = 1$. Here, the behaviour under (KC) is in total contrast to the one under (CK) where every periodic graph is a Liouville graph.

A final section containing remarks and supplements is added.

1 Metric graphs and harmonic functions

For any graph $\Gamma = (V, E, \in)$, the vertex set is denoted by $V = V(\Gamma)$, the edge set by $E = E(\Gamma)$ and the incidence relation by $\in \subset V \times E$. The *valency* or *degree* of each vertex v is denoted by

$$\gamma(v) = \gamma(v; \Gamma) = \#\{k \in E(\Gamma) \mid v \in k\}.$$

The adjacency of two vertices $v, w \in V(\Gamma)$ will be denoted by $v \sim w$.

Unless otherwise stated, all graphs considered in this paper are assumed to be nonempty, countable, connected and uniformly locally finite. The last property means that

$$\max_{v \in V(\Gamma)} \gamma(v) =: \gamma_{\max} < \infty.$$

Accordingly, the vertices will be numbered by v_i with $i \in \mathcal{I} \subset \mathbb{N}$, the respective valencies by $\gamma_i = \gamma(v_i)$, and the edges by k_j with $j \in \mathcal{J} \subset \mathbb{N}$. The *boundary vertices* $V_{\text{b}} = \{v \in V(\Gamma) | \gamma(v) = 1\}$ will be distinguished from the *ramification nodes* $V_{\text{r}} = \{v \in V(\Gamma) | \gamma(v) \geq 2\}$ and the *essential ramification nodes* $V_{\text{ess}} = \{v \in V(\Gamma) | \gamma(v) \geq 3\}$. By definition, a *circuit* is a connected and regular graph of valency 2. This includes the infinite circuit in the form of the two–sided unbounded path Γ_1. A finite *path* is a connected graph with two distinct vertices of valency 1 while the other vertices are all of valency 2.

The sequences or column vectors with constant entries equal to 1 are denoted by $\mathbf{e}$ and $\mathbf{e}^n$ if the dimension n is to be noted. The n–vectors of the canonical basis are denoted by $\mathbf{e}_k^n = (\delta_{ik})_{n \times 1}$. For a subgraph $\Theta \leq \Gamma$ let $\bar{\Theta} = (V(\Theta), E(\bar{\Theta}), \in)$ denote the subgraph of Γ spanned by the vertices in Θ with

$$E(\bar{\Theta}) = \{e | \, e \in E(\Gamma), \; e \cap V(\Gamma) \subset V(\Theta)\}.$$

The subgraph Θ is called *induced* if $\bar{\Theta} = \Theta$, i.e. Θ contains all edges in Γ that have their vertices in Θ. Two subgraphs are called *vertex independent* if they have no vertex in common, and *essentially or almost disjoint* if they have only a finite number of vertices in common. For further graph theoretical terminology we refer to [7, 11].

Moreover, we consider each graph as a connected *topological graph* in $\mathbb{R}^m$, i.e. $V(\Gamma) \subset \mathbb{R}^m$ and the edge set consists in a collection of Jordan curves

$$E(\Gamma) = \{\pi_j : [0, \ell_j] \to \mathbb{R}^m \mid j \in \mathcal{J}\}$$

with the following properties: Each support $k_j := \pi_j\left([0, \ell_j]\right)$ has its endpoints in the set $V(\Gamma)$, any two vertices in $V(\Gamma)$ can be connected by a path with arcs in $E(\Gamma)$, and any two edges $k_j \neq k_h$ satisfy $k_j \cap k_h \subset V(\Gamma)$ and $\#(k_j \cap k_h) \leq 1$. The arc length parameter of an edge k_j is denoted by x_j. Unless otherwise stated, we identify the graph $\Gamma = (V, E, \in)$ with its associated *metric graph*, *network* or *quantum graph*

$$G = \bigcup_{j \in \mathcal{J}} \pi_j\left([0, \ell_j]\right),$$

especially each edge π_j with its support k_j. Throughout it will be assumed that all π_j are at least Lipschitz continuous. Thus, endowed with the induced topology G is a connected space in $\mathbb{R}^m$. Throughout, we shall denote the *total graph length* by

$$L(\Gamma) = L(G) = \sum_{j\in\mathcal{J}} \ell_j.$$

The orientation of the graph Γ is given by the *incidence matrix* $\mathcal{D}(\Gamma) = (d_{ik})_{\mathcal{I}\times\mathcal{J}}$ with

$$d_{ij} = \begin{cases} 1 & \text{if } \pi_j(\ell_j) = v_i, \\ -1 & \text{if } \pi_j(0) = v_i, \\ 0 & \text{otherwise.} \end{cases}$$

This allows a refinement of the valency notion by defining the *outdegree* $\gamma^-(v)$ and *indegree* $\gamma^+(v)$ of a vertex v by

$$\gamma^-(v;\Gamma) = \#\{k_j \in E(\Gamma) \mid d_{ij} = -1\},$$
$$\gamma^+(v;\Gamma) = \#\{k_j \in E(\Gamma) \mid d_{ij} = 1\}.$$

The *corank* of the graph Γ is defined by

$$\operatorname{corank}(\Gamma) = \dim\ker\mathcal{D}(\Gamma),$$

that in the finite case amounts to $\#E(\Gamma) - \#V(\Gamma) + 1$ and, thereby, reduces to the value 1 if and only if Γ is unicyclic. In general, the *circuit space* being defined as $\Pi(\Gamma) = \left\langle c \in \ker\mathcal{D}(\Gamma) \,\middle|\, \#\operatorname{supp}(c) < \infty \right\rangle$ is only a subspace of $\ker\mathcal{D}(\Gamma)$ and not identical with it. However, it holds

$$\dim\Pi(\Gamma) = \dim\ker\mathcal{D}(\Gamma),$$

see e.g. [4]. In the context of the anti–Kirchhoff law, the *generalized unsigned circuit space* defined by

$$\Pi^+(\Gamma) = \left\{ c \in \mathbb{R}^{E(\Gamma)} \,\middle|\, \forall v_i \in V(\Gamma) : \sum_{j\in\mathcal{J}} d_{ij}^2 c_j = 0 \right\}$$

plays a crucial role and is in fact just the kernel of the unsigned incidence operator given by the matrix $(|d_{ij}|)_{\mathcal{I}\times\mathcal{J}}$. In particular in the finite not necessarily connected case, we note that

$$\dim\Pi^+(\Gamma) = \#E(\Gamma) - \#V(\Gamma) + c^+(\Gamma),$$

where $c^+(\Gamma)$ stands for the number of bipartite connected components of Γ.

For vectors of functions $u := (u_j)_{j\in\mathcal{J}}$ defined by their edge components $u_j : [0,\ell_j] \to \mathbb{C}$ we use the abbreviations

$$u_j(v_i) := u_j(\pi_j^{-1}(v_i)), \quad \partial_j u_j(v_i) := \frac{\partial}{\partial x_j} u_j(x_j)\Big|_{\pi_j^{-1}(v_i)} \quad \text{etc.}$$

Endowed with a usual product norm we introduce for $k \in \mathbb{N}$ and $p \in [1, \infty]$

$$\mathcal{V}^k(G) = \prod_{j \in \mathcal{J}} \mathcal{C}^k[0, \ell_j] \qquad L^p(G) = \prod_{j \in \mathcal{J}} L^p(0, \ell_j).$$

The validity of the anti–Kirchhoff conditions (3) and (4) in a function space will be indicated by the subscript (KC), the one of (CK) by the subscript CK correspondingly.

Of course, harmonic functions $u := (u_j)_{j \in \mathcal{J}}$ on a metric graph G are defined as solutions of $\Delta u = 0$ under suitable transition conditions. But, as these solutions are affine linear on each edge, we can define a harmonic function intrinsically without higher regularity assumptions on the arc length parametrizations than the rectifiability condition. Conceivably, a function $u := (u_j)_{j \in \mathcal{J}}$ with $u_j \in \mathcal{C}[0, \ell_j]$ is called *harmonic* if on each edge e_j it is of the form $u_j(x_j) = u_j(0) + \alpha_j x_j$ and satisfies the anti–Kirchhoff condition (KC). The vector space of all harmonic functions on G satisfying (KC) will be denoted by $\mathcal{H}_{KC}(G)$. Correspondingly, we shall write $\mathcal{H}_{CK}(G)$ etc. Moreover, set

$$\begin{aligned} \mathcal{H}^\infty_{KC}(G) &= \mathcal{H}_{KC}(G) \cap L^\infty(G), \\ \Pi^+_\infty(\Gamma) &= \Pi^+(\Gamma) \cap \ell^\infty\left(E(\Gamma)\right). \end{aligned}$$

Definition 1.1 $\qquad m(0; \Gamma; KC) = \dim \mathcal{H}^\infty_{KC}(G).$

If G is a $\mathcal{C}^2$–metric graph, then clearly

$$m(0; \Gamma; KC) = m_a(0; \Delta^{KC}; \mathcal{V}^2(G) \cap L^\infty(G)) = \dim E_0(\Delta^{KC}; \mathcal{V}^2(G) \cap L^\infty(G)).$$

Definition 1.2 A metric graph G is called a *Liouville graph* under (KC), if each harmonic bounded function on G fulfilling (3) and (4) is constant, i.e. if

$$m(0; \Gamma; KC) = 0.^*$$

Unless otherwise stated, the harmonic functions to be considered will be supposed to fulfill (KC).

Note that any $u = (u_j)_{j \in \mathcal{J}} \in \mathcal{H}_{KC}(G)$ has edge restrictions of the form $u_j(x_j) = u_j(0) + \alpha_j x_j$ with slopes

$$\alpha_j = \frac{u_j\left(\ell_j\right) - u_j(0)}{\ell_j} = \partial_j u_j\left(\pi_j(\ell_j)\right). \tag{5}$$

Moreover, by (4), at each vertex v_i all incident normal derivatives $d_{ij}\partial_j u_j(v_i)$ coincide. Conceivably, we can set

$$\nu(v_i) = \nu_i = d_{ij}\partial_j u_j(v_i)$$

*In order to be consistent with Liouville's Theorem in Riemannian or complex manifolds, the authors prefer the present definition of the Liouville property instead of defining it by the multiplicity being equal to 1 as in the (CK)–case. In the latter one this is conceivable, since all constant functions are solutions. But, under (KC), the only constant solution is the trivial one and, thereby, the solution characterizing the Liouville property.

for some incident edge k_j with v_i. Thus, by connectedness, there is a constant *slope factor* $\nu = \nu(u)$ such that

$$\forall i \in \mathcal{I}: \ |\nu_i| = |\nu|, \tag{6}$$

and

$$v_i \sim v_h \ \Rightarrow \ \nu_h = -\nu_i. \tag{7}$$

This yields immediately the

Corollary 1.3 *If* Γ *is not bipartite, then* $\Pi^+_\infty(\Gamma) \cong \mathcal{H}^\infty_{KC}(G)$.

But in the bipartite case, there can be bounded harmonic functions with nonvanishing slope factor ν, see Sections 5 and 6. Of course, (7) does not mean that $m(0;\Gamma;KC) \leq \dim \Pi^+_\infty + 1$, see Section 5.

2 The finite case

Suppose Γ is a finite connected graph with n vertices and N edges. In the case of all edge lengths equal to 1, it has been shown in [6] that 0 is not an eigenvalue of Δ^{KC} on trees or on non bipartite unicyclic graphs, since its algebraic multiplicity amounts to $N-n+1$ or to $N-n$, respectively. The proof given there is readily extended to arbitrary edge lengths. First, we note that a harmonic function under (KC) on Γ is constant on each edge, i.e. $\nu = 0$, since

$$\begin{aligned} 0 = \sum_{j=1}^{N} \int_0^{\ell_j} \left(\partial_j^2 u_j\right) u_j dx_j &= -\sum_{j=1}^{N} \int_0^{\ell_j} \left(\partial_j u_j\right)^2 dx_j + \sum_{i=1}^{n} \nu_i \sum_{v_i \in k_j} u_j(v_i) \\ &= -\sum_{j=1}^{N} \int_0^{\ell_j} \left(\partial_j u_j\right)^2 dx_j . \end{aligned}$$

Thus, the eigenspace belonging to 0 of the Laplacian under (KC) satisfies

$$E_0(\Gamma;\Delta^{KC}) \cong \Pi^+(\Gamma).$$

Introduce

$$\mathcal{M}(\Gamma) = \left\{ M \mid M = (m_{ih})_{n\times n} \, , \ \forall i,h \in \{1,\ldots,n\} : \ v_i \not\sim v_h \Rightarrow m_{ih} = 0 \right\}$$

and

$$\mathcal{M}^+(\Gamma) = \left\{ M \in \mathcal{M}(\Gamma) \mid M^* = M, \, M\mathbf{e} = 0 \right\}.$$

The Hadamard multiplication with the length adjacency matrix is an isomorphism of $\mathcal{M}(\Gamma)$, while the dimension of $\mathcal{M}^+(\Gamma)$ has been determined in [1, Section 5]. This leads to the following result.

Lemma 2.1 *If* Γ *is a finite connected graph, then*

$$m(0;\Gamma;KC) = \dim \mathcal{M}^+(\Gamma) = \begin{cases} N-n+1 & \textit{if } \Gamma \textit{ is bipartite,} \\ N-n & \textit{if } \Gamma \textit{ is not bipartite.} \end{cases}$$

In particular, Γ *is a Liouville graph under* (KC) *if and only if* Γ *is a tree or a non bipartite unicyclic graph.*

3 Basic facts

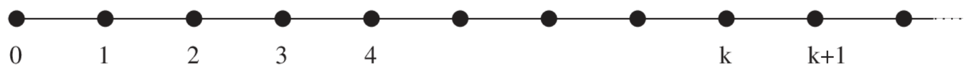

Figure 1: The one–sided unbounded path Γ_0.

First, we consider the smallest infinite graph.

Example 3.1 The one–sided unbounded path Γ_0 consists in the vertex set $\mathbb{N}$ with edges defined by the natural incidences

$$d_{ii} = -1 \quad \text{and} \quad d_{i+1,i} = 1.$$

Clearly, $\Pi^+_\infty(\Gamma_0) = \{0\}$. Suppose $u \in \mathcal{H}^\infty_{KC}(\Gamma_0)$ satisfies $u_0'(0) = \nu$, and $u_0(0) = 0$. Set $y_k = u_k(0)$ and get recursively

$$\forall k \in \mathbb{N}: \ y_{k+1} = -\left(y_k + (-1)^k \nu \ell_k\right) = -y_k + (-1)^{k+1}\nu\ell_k$$

and

$$\forall k \in \mathbb{N}^*: \ u_k(0) = (-1)^k \nu \sum_{i=0}^{k-1} \ell_i.$$

Thus, u is bounded if and only if either $\nu = 0$, and thereby $u = 0$, or $\nu \neq 0$ and $L(\Gamma_0) < \infty$. Thus, Γ_0 is Liouville w.r. to (KC) if and only if $L(\Gamma_0) = \infty$.

Next, we show three useful lemmata. If Θ is a subgraph of the graph Γ then by zero extension, $\Pi^+(\Theta)$ and $\Pi^+_\infty(\Theta)$ are isomorphic to subspaces of $\Pi^+(\Gamma)$ and $\Pi^+_\infty(\Gamma)$, respectively. In particular,

$$\Theta \leq \Gamma \ \Rightarrow \ \dim \Pi^+_\infty(\Theta) \leq \dim \Pi^+_\infty(\Gamma). \tag{8}$$

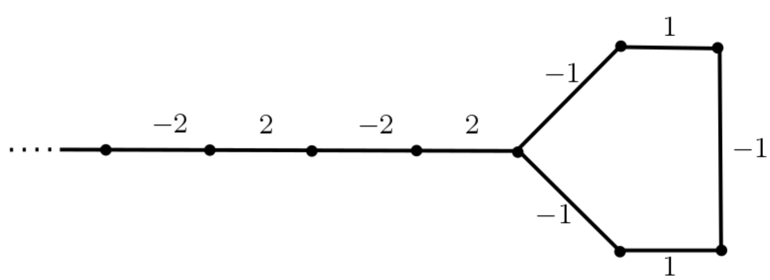

Figure 2: An infinite non bipartite graph with a nonvanishing element belonging to $\Pi^+_\infty(\Gamma)$.

Infinite and even finite circuits ζ fulfill $\dim \Pi_\infty^+(\zeta) = 1$. The same holds for a *dumbbell* δ composed by two odd finite circuits with at most one vertex in common or with a path joining them. Finally, a graph π composed by an odd finite circuit ζ and an one–sided unbounded path $N \cong \Gamma_0$ with $V(\zeta) \cap V(N) = \{v_0\}$ and $\gamma(v_0; N) = 1$, satisfies $\dim \Pi_\infty^+(\pi) = 1$, see Fig. 2. Thus, we have shown the following

Lemma 3.2 *A graph Γ satisfies $\dim \Pi_\infty^+(\Gamma) \geq 1$ if it contains*

(a) either an infinite circuit,

(b) or a finite even circuit,

(c) or two finite odd circuits with at most one vertex in common,

(d) or a graph π composed by an odd finite circuit ζ and an one–sided unbounded path $N \cong \Gamma_0$ with $V(\zeta) \cap V(N) = \{v_0\}$ and $\gamma(v_0; N) = 1$.

In particular, Γ cannot be a Liouville graph with respect to (KC).

Lemma 3.3 *Consider a vertex v in some graph Γ, at which a finite tree T is attached such that $V(\Gamma) \cap V(T) = \{v\}$ and such that $\gamma^-(v;T) = 0 = \gamma^+(v;T) - 1$. Suppose that at all other vertices $w \in V(T)$ the outdegree satisfies $\gamma^-(w;T) = 1$. Let Σ denote the composed graph by Γ and T and suppose $u \in \mathcal{H}_{KC}(\Sigma)$. Then on each edge k_j of T, u satisfies*

$$\nu\left(\pi_j(\ell_j)\right) u_j(\ell_j) \geq 0.$$

Proof. For $\nu = 0$, the assertion is plain. W.l.o.g. we can assume $\nu = 1$. We always have

$$u_k(x_k) = u_k(0) - \nu\left(\pi(0)\right) x_k = u_k(0) + \nu\left(\pi(\ell_k)\right) x_k,$$

in particular

$$\nu\left(\pi(\ell_k)\right) u_k(\ell_k) = \nu\left(\pi_k(\ell_k)\right) u_k(0) + \ell_k = -\nu\left(\pi_k(0)\right) u_k(0) + \ell_k.$$

If $\pi_j(0)$ is a boundary vertex, then evidently $\nu\left(\pi_j(\ell_j)\right) u_j(\ell_j) = \ell_j > 0$. By recurrence on the distance to the boundary of T, we conclude

$$\begin{aligned} \nu\left(\pi_j(\ell_j)\right) u_j(\ell_j) &= \ell_j - \nu\left(\pi_j(0)\right) u_j(0) \\ &= \ell_j + \sum_{\pi_j(0)=\pi_k(\ell_k)} \nu\left(\pi_k(\ell_k)\right) u_k(\ell_k) > 0. \end{aligned}$$

■

Lemma 3.4 *Let Γ be a graph that contains as an induced subgraph Γ_0 such that at the vertices of the latter one finite induced trees in Γ are eventually attached. Label the vertices of Γ_0 in Γ by $\mathbb{N}$ as above by the natural incidences*

$$d_{ii} = -1, \quad d_{i+1,i} = 1.$$

Suppose that

$$\sum_{i=0}^{\infty} \ell_i = \infty. \tag{9}$$

Then Γ *has no bounded harmonic function with nonzero slopes.*

Proof. Reasoning by contradiction, we suppose that there exists $u \in \mathcal{H}_{KC}^{\infty}(G)$ with $\nu \neq 0$. W.l.o.g. suppose $\nu = 1$.

Let T_k denote the eventual finite tree at v_k. Let a_k denote the sum of the incoming values of u at v_k that do not belong to edges of Γ_0. As a harmonic function has to vanish on edges incident with boundary vertices, as T_k does not contain circuits, and as by choosing the orientation "from the boundary of T_k to Γ_0, the outdegree of each vertex in $V(T_k) \setminus V(\Gamma_0)$ amounts to 1, each a_k is uniquely determined. Moreover, by Lemma 3.3,

$$\forall k \in \mathbb{N} : \nu_k a_k \geq 0. \tag{10}$$

On each edge of Γ_0, u has the form

$$u_k(x_k) = u_k(0) - \nu_k x_k =: y_k - \nu_k x_k.$$

Then (KC) leads to the recurrence

$$y_{k+1} = -y_k - a_{k+1} + \nu_k \ell_k. \tag{11}$$

Multiply (11) by ν_{k+1} and get with $z_k = \nu_k y_k$ the recurrence

$$z_{k+1} = z_k - \nu_{k+1} a_{k+1} + \nu_k \nu_{k+1} \ell_k = z_k - \nu_{k+1} a_{k+1} - \ell_k,$$

in other words

$$z_k = z_0 - \sum_{i=1}^{k} (\nu_i a_i + \ell_{i-1})$$

Using (9) and (10), this shows that $|z_k| = |y_k| \to \infty$ as $|k| \to \infty$, in contradiction to the presumed bounded character of u. Thus, $\nu = 0$. ■

Example 3.5 The two–sided unbounded path $\mathbf{\Gamma_1}$ consists in the vertex set $\mathbb{Z}$ with edges defined by the natural incidences

$$d_{ii} = -1, \quad d_{i,i-1} = 1.$$

Clearly, $\dim \Pi_{\infty}^{+}(\Gamma_1) = 1$. Γ_1 is Liouville w.r. to (CK), but not w.r. to (KC), since by Lemma 3.4 or more directly, using that $u \in \mathcal{H}_{KC}(\Gamma_1)$ is completely determined by its restriction to an arbitrary fixed edge,

$$m(0; \Gamma_1; KC) = \begin{cases} 1 & \text{if } \sum_{k \in \mathbb{Z}} \ell_k = \infty \qquad (\nu = 0) \\[2ex] 2 & \text{if } \sum_{k \in \mathbb{Z}} \ell_k < \infty \qquad (\nu = 0 \,\&\, \nu \neq 0) \end{cases}$$

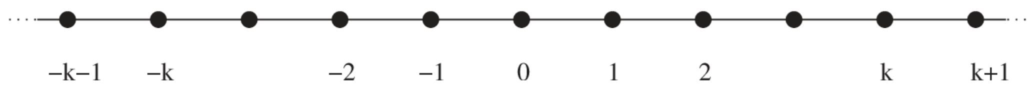

Figure 3: The two–sided unbounded path Γ_1.

4 Infinite Liouville graphs and infinite trees without two–sided unbounded paths

According to Lemma 2.1, the finite Liouville graphs under(KC) are exactly trees and unicyclic non bipartite graphs. In the infinite case, we can state the following

Theorem 4.1 *Let Γ be a graph that contains a copy of Γ_0 whose edges satisfy*

$$\sum_{e \in E(\Gamma_0)} \ell(e) = \infty. \tag{12}$$

Then Γ is a Liouville graph under (KC) if and only if Γ is a tree in the following list:

1. *The one–sided unbounded path Γ_0,*
2. *a sole Γ_0 attached at its boundary vertex to a finite tree.*
3. *a sole Γ_0 with infinitely many finite trees attached to its vertices.*

This applies in particular to the equal length case, or more generally, to the case $\inf\{\ell_i \,|\, i \in \mathbb{N}\} > 0$.

Proof. The sufficiency is plain with Lemma 3.4. Next, suppose that Γ is a Liouville graph. By Lemma 3.2, Γ must be a tree and cannot contain copies of Γ_1. By hypothesis and (12), $\dim \Pi^+_\infty(\Gamma) = 0$. Finally, the assertion follows with Lemma 3.4. ■

As already seen in Example 3.1, without Condition (12) the assertion is wrong, since Γ_0 admits bounded harmonic functions with $\nu \neq 0$ for $L\,(\Gamma_0) < \infty$. Does this also hold if to each vertex of Γ_0 finite trees are attached? Accordingly, suppose that T is a tree containing Γ_0, but not Γ_1 and fulfilling

$$L\,(\Gamma_0) = \sum_{e \in E(\Gamma_0)} \ell(e) < \infty. \tag{13}$$

Number the vertices of Γ_0 by $\mathbb{N}$ and choose the natural incidence

$$d_{ii} = -1, \quad d_{i,i-1} = 1.$$

on Γ_0. Clearly, $\Pi^+_\infty(T) = \{0\}$. We want to know, whether bounded harmonic functions with nonvanishing slope factor occur or not. Let T_k denote the finite wood incident with v_k, but having no edges with Γ_0 in common. Let $u \in \mathcal{H}_{KC}(T)$ allow $\nu = \nu(u) > 0$ and, as above, a_k denote the sum of the incoming values of u at v_k on T that do not belong to edges of Γ_0. Recall that each a_k is uniquely determined for given ν. W.l.o.g. choose $\nu = 1$ and

$$\nu_k = -(-1)^k,$$

set $y_k := u_k(0)$ and get $u_k(x_k) = y_k + (-1)^k x_k$. Then (KC) defines the linear recurrence

$$y_0 = -a_0, \quad \forall k \in \mathbb{N}^* : y_{k+1} = -y_k - (-1)^k \ell_k - a_{k+1}. \tag{14}$$

The solution of (14) takes the form

$$y_k = (-1)^k \left(\sum_{i=0}^{k-1} \ell_i - a_0 + \sum_{i=1}^{k} (-1)^{i+1} a_i \right) = (-1)^k \left(\sum_{i=0}^{k} \nu_i a_i + \sum_{i=0}^{k-1} \ell_i \right), \tag{15}$$

Lemma 3.3 shows that all $\nu_k a_k$ are nonnegative. Thus, under (13)

$$u \in \mathcal{H}^\infty_{KC}(T) \iff \sum_{k=0}^{\infty} |a_k| < \infty. \tag{16}$$

This yields examples of trees T fulfilling (13) that are not Liouville graphs. Take $\nu = 1$ and Γ_0 with edge lengths just fulfilling (13). Or add to at each vertex v_i of Γ_0 just one additional edge of length $\frac{1}{(k+1)^2}$ with its boundary vertex as initial node. Then

$$|y_k| \le \frac{\pi^6}{6} + \sum_{i=0}^{\infty} \ell_i < \infty.$$

Thus, u defines a bounded harmonic function with nonvanishing slope factor.

But (16) yields also examples that even under (13), T can be a Liouville graph. If each T_k is just an edge of length 1 with $a_k = 1$, see the graph on the l.h.s in Fig. 4, or if T_k is a path of length $k+1$ with edge lengths 1 with

$$\forall k \in \mathbb{N} : \; a_k = (-1)^k (k+1),$$

see see the graph on the r.h.s in Fig. 4, then Formulae (15) and (16) show that $\nu = 1$ is excluded for a bounded harmonic function. Thus, in both cases, $m(0; T; KC) = 0$.

Corollary 4.2 *Suppose the tree T consists in a copy Γ_0 whose edge lengths satisfy Condition* (13) *and the vertices v_k of which finite woods T_k are attached that have precisely the vertex v_k with Γ_0 in common. If*

$$\sum_{i=0}^{\infty} L(T_i) < \infty,$$

then $\mathcal{H}_{KC}(T)$ contains bounded elements with arbitrary nonvanishing slope factor ν. In particular, T is not a Liouville graph.

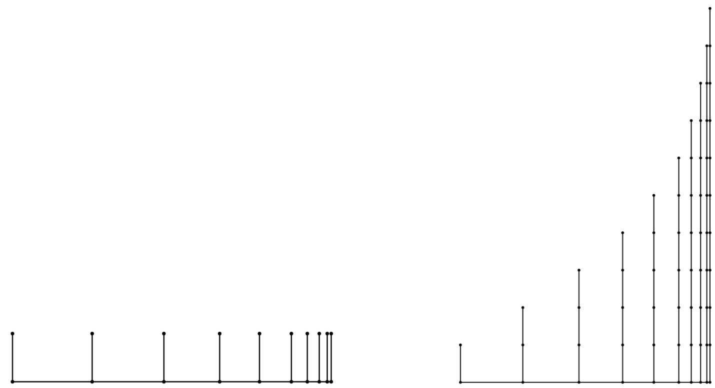

Figure 4: Two infinite Liouville trees with (13).

Proof. By recurrence on the number of edges of T_k it follows that

$$|a_k| \leq |\nu| \; L(T_k).$$

Now (16) permits to conclude that there exists $u \in \mathcal{H}^{\infty}_{KC}(T)$ with $\nu(u) = 1$, which, in turn, yields the assertion. ∎

5 Medusae

By definition, an infinite graph Γ is called a *medusa* if it has only finitely many essential ramification nodes,

$$\# V_{\text{ess}}(\Gamma) < \infty.$$

Under (CK), a medusa is a Liouville graph, see [3]. For such a graph Γ we shall adopt the following notation throughout this section. Let $B \leq \Gamma$ denote the smallest finite connected induced subgraph such that

$$V_{\text{ess}}(\Gamma) \subset V(B).$$

Moreover, let $\Gamma_0^1, \ldots, \Gamma_0^N \leq \Gamma$ denote the N disjoint subgraphs isomorphic to Γ_0. For each Γ_0^k let v_0^k denote its boundary vertex and e_0^k its incident edge in Γ_0^k. We can choose these nodes to fulfill for each $1 \leq k \leq N$

$$\left\{v_0^k\right\} = V(B) \cap V\left(\Gamma_0^k\right).$$

Then we can state

Theorem 5.1 *Suppose that Γ is a medusa that either fulfills $L(G) = \infty$, or that is non bipartite with $L(G) < \infty$. Then all slope factors ν of bounded harmonic functions on G vanish and*

$$\dim \Pi^+(B) + N - 1 \leq m(0; \Gamma; KC) \leq \dim \Pi^+(B) + N.$$

Proof. Necessarily $\nu = 0$ by (7). The lower bound follows readily with Lemma 3.2, first applied to $\Pi^+(B)$ considered as a subspace of $\Pi^+_\infty(\Gamma)$. Secondly, each pair Γ_0^k and Γ_0^j forms an infinite circuit $\Gamma_1^{k,j}$ isomorphic to Γ_1. Among those exactly $N-1$, say $\Gamma_1^{2,1}, \ldots, \Gamma_1^{N,1}$, are the supports of linearly independent elements belonging to $\Pi^+_\infty(\Gamma)$. As each $\varphi \in \Pi^+_\infty(\Gamma)$ with support $\Gamma_1^{k,j}$ can be written as a linear combination of two such elements with supports $\Gamma_1^{1,k}$ and $\Gamma_1^{1,j}$, the lower bound is shown.

As for the upper bound, suppose first that $N = 1$. If there is $\varphi \in \Pi^+_\infty(\Gamma)$ such that $\varphi(v_0^1) = 1$, then each harmonic function can be written as

$$u = \underbrace{u - u\left(v_0^1\right)\varphi}_{\in \Pi^+(B)} + \underbrace{u\left(v_0^1\right)\varphi}_{\in\langle\varphi\rangle}.$$

As the restriction of u to Γ_0^1 is uniquely determined by $u\left(v_0^1\right)$, $u - u\left(v_0^1\right)\varphi$ belongs to $\Pi^+(B)$. This shows $m(0;\Gamma;KC) \leq \dim \Pi^+(B) + 1$.

If there is no $\varphi \in \Pi^+_\infty(\Gamma)$ such that $\varphi(v_0^1) = 1$, then all elements of $\Pi^+_\infty(\Gamma)$ have their supports in B and $m(0;\Gamma;KC) = \dim \Pi^+(B)$. Thus, the assertion is shown for $N = 1$.

For $N \geq 2$, there exists always $\varphi \in \Pi^+_\infty(\Gamma)$ such that $\varphi(v_0^N) = 1$. Let $\tilde{\Gamma}$ denote the graph by removing the edges of Γ_0^N from Γ while keeping $v_0^N \in V\left(\tilde{\Gamma}\right)$. Again, as the restriction to Γ_0^N of each bounded harmonic function u is uniquely determined by $u\left(v_0^N\right)$, u reads

$$u = \underbrace{u - u\left(v_0^N\right)\varphi}_{=:w} + \underbrace{u\left(v_0^N\right)\varphi}_{\in\langle\varphi\rangle},$$

where w vanishes on Γ_0^N and can be considered as an element of $\mathcal{H}^\infty_{KC}\left(\tilde{G}\right)$ by restriction to $\tilde{\Gamma}$. By recurrence on N, $m(0;\Gamma;KC) \leq \dim \Pi^+(B) + N - 1 + 1$. ∎

The examples at the end of this section will show that both bounds in Theorem 5.1 are optimal. However, in the bipartite case the upper bound is always attained when the total graph length is finite.

Theorem 5.2 *Suppose Γ is a bipartite medusa with $L(G) < \infty$. Then there are bounded harmonic functions on G with nonvanishing slope factor and*

$$m(0;\Gamma;KC) = \dim \Pi^+(B) + N.$$

Proof. **(a)** Let us show first that for each $0 \neq \nu \in \mathbb{R}$, there exists $u \in \mathcal{H}^\infty_{KC}(G)$ such that $\nu(u) = \nu$. Clearly, w.l.o.g. we can choose $\nu = 1$. As Γ is bipartite, we can endow it with the sink–source–orientation, i.e. each vertex v of Γ either fulfills $\gamma(v) = \gamma^-(v)$ or $\gamma(v) = \gamma^+(v)$. Thus, a presumed bounded harmonic function u is on each edge of B the form

$$u_j(x_j) = c_j + x_j$$

with some $c_j \in \mathbb{R}$ for $1 \leq j \leq m := \#E(B)$. The m–column vector $c = (c_j)_{m\times 1}$ satisfies the following $n := \#V(B)$ equations. Let α_0^k denote the presumed value of u on e_0^k in Γ_0^k. At each vertex v_i set

(17) $$s_i = \sum_{v_i \in k_j} \ell_j.$$

Then at each source $v_i \in V(B)$ with $\nu_i = -1$,

$$\sum_{v_i = \pi_j(0)} c_j = \begin{cases} 0 & \text{if } v_i \notin \left\{ v_0^k | \, 1 \leq k \leq N \right\}, \\ -\alpha_0^k & \text{if } v_i = v_0^k, \end{cases}$$

and at each sink $v_i \in V(B)$ with $\nu_i = 1$,

$$\sum_{v_i = \pi_j(\ell_j)} c_j = \begin{cases} -s_i & \text{if } v_i \notin \left\{ v_0^k | \, 1 \leq k \leq N \right\}, \\ \\ -\alpha_0^k - s_i & \text{if } v_i = v_0^k. \end{cases}$$

In other words, if $\mathcal{D}^+$ denotes the unsigned $n \times m$–incidence matrix of B, then

(18) $$\mathcal{D}^+ c = b := -s^+ - \sum_{v_i = v_0^k} \alpha_0^k \mathbf{e}_i^n$$

with

$$s^+ = \left(s_i^+\right)_{n \times 1}, \qquad s_i^+ = \begin{cases} s_i & \text{if } v_i \text{ is a sink,} \\ 0 & \text{otherwise.} \end{cases},$$

admitting the $N \leq n$ parameters α_0^k. But $\operatorname{rank}(\mathcal{D}^+) = n - 1$. Thus, it remains to show that the parameters α_0^k can be chosen such that the image of $\mathcal{D}^+$ and the affine subspace $-s^+ - \bigoplus_{v_i = v_0^k} \mathbb{R}\mathbf{e}_i^n$ are not parallel.

But, it suffices to show the case $N = 1$, since eventual additional Γ_0^k, $k \geq 2$, allow $\alpha_0^k = 0$ and $\partial_0 u_0^k(0) = \pm\nu$ together with the shown case $N = 1$. However, $\alpha_0^1 = 0$ is excluded, since B alone does not allow bounded harmonic functions with $\nu \neq 0$ by Section 2. W.l.o.g. we can assume that $v_1 = v_0^1$ and $\nu_1 = -1$. Then the r.h.s. in (18) reduces to $b = -\alpha_0^1 \mathbf{e}_1^n - s^+$. As $\operatorname{rank} \mathcal{D}^+(B) = n - 1$, we find $h \in \mathbb{R}^n$ such that

$$\mathcal{D}^+ \left(\mathbb{R}^m\right) = \langle h \rangle^\perp \quad \text{and} \quad (h, h)_2 = 1,$$

where $(\cdot, \cdot)_2$ denotes the Euclidean scalar product in $\mathbb{R}^n$. Lemma 8.2 guarantees that $(s^+, h)_2 \neq 0$, since s^+ cannot belong to $\operatorname{Im} \mathcal{D}^+(B)$. If $(h, \mathbf{e}_1^n)_2 \neq 0$, then for

$$\alpha_0^1 = -\frac{(s^+, h)_2}{(h, \mathbf{e}_1^n)_2}$$

$b \in \mathcal{D}^+ \left(\mathbb{R}^m\right)$, and its preimage defines the desired vector $c = \left(c_j\right)_{m \times 1}$.

Finally, if $(h, \mathbf{e}_1^n)_2 = 0$, then $\mathbf{e}_1^n \in \mathcal{D}^+ \left(\mathbb{R}^m\right)$, which is excluded by (8.2). Thus, the existence of $u \in \mathcal{H}_{KC}^\infty(G)$ with $\nu(u) = 1$ is well established.

Next we show the formula for $m(0;\Gamma;KC)$ and suppose first that

$$N = 1.$$

(b) There is no $u \in \mathcal{H}^{\infty}_{KC}(\Gamma)$ such that $\nu(u) \neq 0$ and $u\left(v_0^1\right) = 0$. Otherwise, by (6), the restriction of u to B would constitute an element of $\mathcal{H}_{KC}(B)$ with nonzero slope, which is excluded in the finite case as above, see Section 2.

(c) By (a) there exists $u \in \mathcal{H}^{\infty}_{KC}(\Gamma)$ with nonzero slope factor $\nu(u)$. Thus, there is no $\varphi \in \Pi^+_{\infty}(\Gamma)$ such that $\varphi(v_0^1) = 1$. Otherwise, each u as above could be written in the form

$$u = \underbrace{u - u\left(v_0^1\right)\varphi}_{=:w} + \underbrace{u\left(v_0^1\right)\varphi}_{\in \Pi^+_{\infty}(\Gamma)}$$

with $w \in \mathcal{H}^{\infty}_{KC}(\Gamma)$ and $\nu(w) = \nu(u) \neq 0$ and $v\left(w_0^1\right) = 0$, in contradiction to (b). Conclusion: All elements of $\Pi^+_{\infty}(\Gamma)$ have their supports in B and

$$\Pi^+_{\infty}(\Gamma) \cong \Pi^+(B).$$

(d) Each element $u \in \mathcal{H}^{\infty}_{KC}(\Gamma)$ with $\nu(u) \neq 0$ is uniquely determined by $u\left(v_0^1\right)$. Thus, if $\tilde{u} \in \mathcal{H}^{\infty}_{KC}(\Gamma)$ with $\nu(\tilde{u}) = \nu(u)$ and $u\left(v_0^1\right) = \tilde{u}\left(v_0^1\right)$, then $u - \tilde{u}$ vanishes on Γ_0. If u and $\tilde{u}$ were different, then $u - \tilde{u}$ would constitute a bounded harmonic function on B with slope factor $\nu(u) \neq 0$, which is excluded by Section 2. Thus,

$$\dim \left\langle u \in \mathcal{H}^{\infty}_{KC}(\Gamma) \,|\, \nu(u) \neq 0 \right\rangle = 1,$$

and, thereby with (c), $m(0;\Gamma;KC) = \dim \Pi^+(B) + 1$.

(e) For

$$N \geq 2,$$

again there exists always $\varphi \in \Pi^+_{\infty}(\Gamma)$ such that $\varphi(v_0^N) = 1$. Thus, each $u \in \mathcal{H}^{\infty}_{KC}(G)$ can be written as

$$u = \underbrace{u - u\left(v_0^N\right)\varphi}_{=:w} + \underbrace{u\left(v_0^N\right)\varphi}_{\in\langle\varphi\rangle}.$$

As above, let $\tilde{\Gamma}$ denote the graph by removing the edges of Γ_0^N from Γ while keeping $v_0^N \in V(\tilde{\Gamma})$. The slope factor of the restriction of w to $\tilde{\Gamma}$ determines uniquely the one of u on the whole medusa and, thereby, $\left\{w \in \mathcal{H}^{\infty}_{KC}(G) \,|\, w(v_0^N) = 0\right\}$ is isomorphic to a subspace of $\mathcal{H}^{\infty}_{KC}(\tilde{G})$. By recurrence,

$$m(0;\Gamma;KC) = \dim \Pi^+(B) + N - 1 + 1.$$

■

The proofs of Theorems 5.1 and 5.2 illustrate very well that the case $N = 1$ is more delicate than higher numbers N of "tentacles" in the medusa. We illustrate this also with the aid of some examples.

The minimal medusa is just Γ_0 with $\Pi^+_{\infty}(\Gamma_0) = \{0\}$. If $L(\Gamma_0) < \infty$, then $m(0;\Gamma_0;KC) = 1$, and the only nontrivial bounded harmonic functions are the ones with $u_0(0) = 0$ and $\nu(u) \neq 0$.

Example 5.3 If B is tree and $L(G)=\infty$, then $\dim \Pi^{+}(B)=0$ and Γ is a Liouville graph for $N=1$ by Theorem 4.1. For $N\geq 1$, $m(0;\Gamma;KC)=N-1$. See e.g. the infinite star graph in Fig. 9. This shows that, in general, the upper bound $\dim \Pi^{+}(B)+N$ in Theorem 5.1 is not attained. But for $L(G)<\infty$, the upper bound is well attained since $m(0;\Gamma;KC)=N$ by Theorem 5.2. Here we find exactly one additional independent bounded harmonic function with $\nu\neq 0$, since $\mathrm{corank}(\Gamma)=0$.

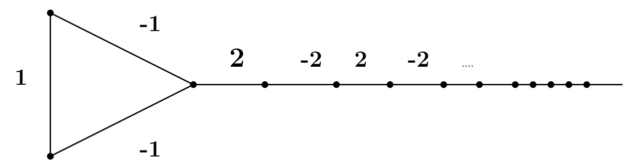

Figure 5: A non bipartite medusa displaying a nonvanishing bounded harmonic function stemming from the generalized unsigned circuit space.

Example 5.4 Compose a medusa Γ by joining an odd circuit C and one copy of Γ_0, see Fig. 5. Clearly $\nu=0$, and independently of $L(\Gamma_0)$, $\Pi^{+}(C)\cong\{0\}\cong\Pi^{+}_{\infty}(\Gamma_0)$, but $m(0;\Gamma;KC)=1$, since $\dim \Pi^{+}_{\infty}(\Gamma)=1$. This example also displays that, in general, the vector space Π^{+}_{∞} cannot be generated by the ones of covering induced subgraphs. Mutatis mutandis, for any non bipartite unicyclic graph B, we have $\Pi^{+}(B)\cong\{0\}\cong\Pi^{+}_{\infty}(\Gamma_0)$, but

$$m(0;\Gamma;KC)=N.$$

A basis of $\mathcal{H}^{\infty}_{KC}(G)$ is readily obtained by choosing for each Γ_0^k an element as in Fig.5 and by extending it trivially to the remaining edges of Γ.

Example 5.5 Compose a medusa Γ by joining an even circuit C and one copy of Γ_0. For $L(\Gamma_0)=\infty$, $m(0;\Gamma;KC)=1$, while for $L(\Gamma_0)<\infty$, $m(0;\Gamma;KC)=2$, since on the one side $\dim \Pi^{+}_{\infty}(C)=1$ and each $\varphi\in\Pi^{+}_{\infty}(\Gamma)$ has its support in C, while on the other there is one independent $u\in\mathcal{H}^{\infty}_{KC}(G)$ with $\nu(u)=1$, that is well displayed in Fig. 6. Note that $\bullet$ stands for a vertex with $\nu_i>0$, while $\circ$ stands for $\nu_i<0$.

Example 5.6 Compose a medusa Γ by attaching two copies of Γ_0 with $L(\Gamma_0)<\infty$ to a circuit C with 4 vertices as depicted above. Then $m(0;\Gamma;KC)=4$, since on the one side $\dim \Pi^{+}_{\infty}(\Gamma)=2$, while on the other there are two independent $u\in\mathcal{H}^{\infty}_{KC}(G)$ with $\nu(u)=1$, that are displayed in the Fig. 7, where again $\bullet$ stands for a vertex with $\nu_i>0$, while $\circ$ stands for $\nu_i<0$. Note that two incident values at one of the nodes of degree 3 determine u completely.

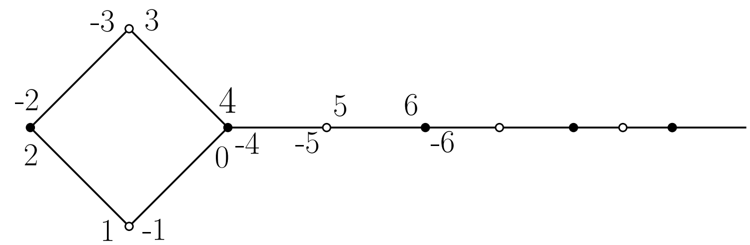

Figure 6: A bipartite medusa allowing the slope factor 1.

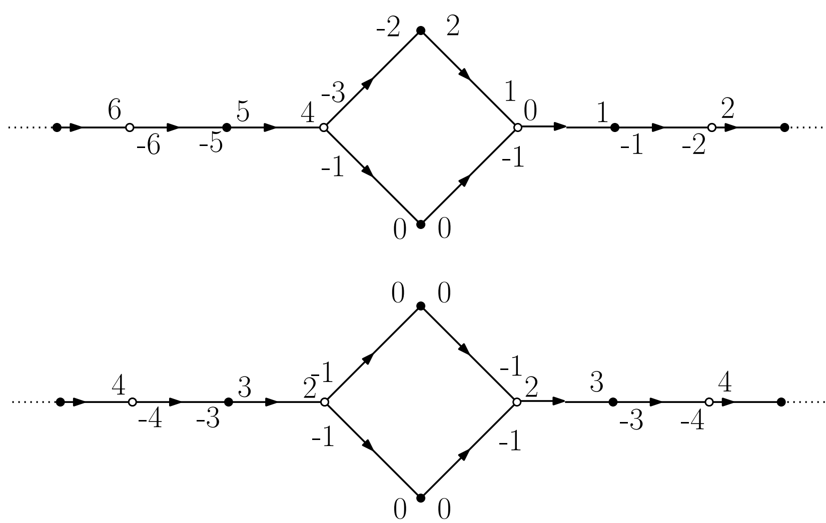

Figure 7: Two independent bounded harmonic functions with slope factor 1.

6 Trees

Under (CK), the condition

$$\inf\{\ell_j \,|\, j \in \mathbb{N}\} > 0. \tag{19}$$

plays a key role for the multiplicity of independent bounded harmonic functions. Moreover, the equivalence in Theorem 6.2 requires a supplementary condition on the edge length ratios. Under (KC), no restriction is needed, only the notion of the reduced graph. Let us recall its construction. By definition, a *viaduct* in a graph Γ is a path π of length at least 2 in Γ joining two distinct vertices u and v such that all other vertices in $V(\pi)\backslash\{u,v\}$ have the valency 2 in Γ too.

Definition 6.1 The *reduced graph* Γ_{red} of a given graph Γ is constructed as follows. Introduce the operations

(I) Withdraw all edges in Γ incident to boundary vertices.

(II) Withdraw each one–sided unbounded path π in Γ whose ramification nodes $V_r(\pi)$ are all nodes of valency 2 in Γ.

(III) Replace any viaduct π in Γ by a single edge of length l, where l is the sum of the lengths of all edges of π.

Repeat (I) and (II) until there are no more vertices of valency 1 and no more one–sided unbounded paths as in (II) in the remaining graph. Then apply (III) such that there are no more vertices of valency 2. The resulting graph is called the reduced graph Γ_{red} of Γ.

Note that Γ_{red} can reduce to a single vertex without edges, e.g. for $\Gamma = \Gamma_1$, or for any tree containing at most one copy of Γ_1 up to translation. As for the (CK)–condition, denote the minimal valency of a graph Γ by $\gamma_{\min}(\Gamma)$ and introduce

$$L(\Gamma) = \inf\left\{ \frac{\text{arc length of } e}{\text{arc length of } f} \,\middle|\, e, f \in E(\Gamma),\, e \cap f \neq \emptyset \right\}.$$

Then we can cite the following

Theorem 6.2 *[3, Theorem 5.5] Suppose that T is a tree satisfying* (19) *such that*

$$L(T_{\text{red}}) > \frac{1}{\gamma_{\min}(T_{\text{red}}) - 1}. \tag{20}$$

Then

$$m(0;T;CK) = \infty \iff \#V(T_{\text{red}}) = \infty. \tag{21}$$

In particular, the tree T is a Liouville under (CK) tree if and only if $\#V(T_{\text{red}}) < \infty$.

Condition (20) is optimal. Already for 3–regular trees with $L(T) = \frac{1}{2}$ the assertion can be wrong, see [3, Example 5.9]. Note that for $\gamma_{\min} \geq 3$, Condition (20) reduces $L(T) > \frac{1}{\gamma_{\min}(T)-1}$.

As for the anti–Kirchhoff law, the situation is quite different. First note the following fact.

Lemma 6.3 *A tree T satisfies $\#V_{\text{ess}}(T_{\text{red}}) = \infty$ if and only if its reduced tree contains a copy of the 3–regular tree T_3.*

Theorem 6.4 *Let T be a tree. Then $m(0;T;KC) = \infty$ if and only if*

$$\#V_{\text{ess}}(T_{\text{red}}) = \infty.$$

Proof. If $\#V_{\text{ess}}(T_{\text{red}}) = \infty$, then T contains infinitely many finally independent copies of Γ_1. Each of them gives rise to a nonvanishing element of $\Pi_\infty^+(T)$. Conversely, $\#V_{\text{ess}}(T_{\text{red}}) < \infty$ implies that T contains only finitely many copies of Γ_1 that are almost disjoint. As the supports of nonvanishing harmonic functions contain at least Γ_1, $m(0;T;KC) < \infty$. ∎

But, bounded harmonics with nonzero slope ν can occur in trees containing the 3–regular tree T_3.

Example 6.5 Choose $\nu = 1$. Consider the 3–regular tree T_3 with equal edge lengths 1. Endow T_3 with the 1 in, 2 out orientation. Choose any edge k_0 in T_3 and number the edges in the $\log_2(\cdot)$ backwards genealogical way. First, consider the vertex $\pi_0(0)$ as a root of the tree T_0 containing k_0 and all edges of T_3 being connected to $\pi_0(1)$ without passing by $\pi_0(0)$. Define a harmonic function $u \in \mathcal{H}_{KC}(T)$ as follows. Set

$$u_0(0) = 0, \quad u_0(1) = 1, \quad u_1(0) = -\frac{1}{2} = u_2(0).$$

Recursively in the K–th generation with $K \geq 1$, with an edge k_j with slope 1,

$$2^{K-1} + 1 \leq j + 2 \leq 2^K,$$

and

$$u_j(0) = \frac{2^{2K} - 1}{2^{2K}},$$

on the edges k_{j+1} and k_{j+2} with

$$\pi_j(1) = \pi_{j+1}(0) = \pi_{j+2}(0)$$

set

$$(a) \qquad u_j(1) = \frac{2^{2K+1} - 1}{2^{2K}},$$

$$(b) \qquad u_{j+1}(0) = u_{j+2}(0) = -\frac{2^{2K+1} - 1}{2^{2K+1}},$$

$$(c) \qquad u_{j+1}(1) = u_{j+2}(1) = -\frac{2^{2(K+1)} - 1}{2^{2K+1}},$$

$$(d) \qquad u_{j+3}(0) = \ldots = u_{j+6}(0) = \frac{2^{2(K+1)} - 1}{2^{2(K+1)}}.$$

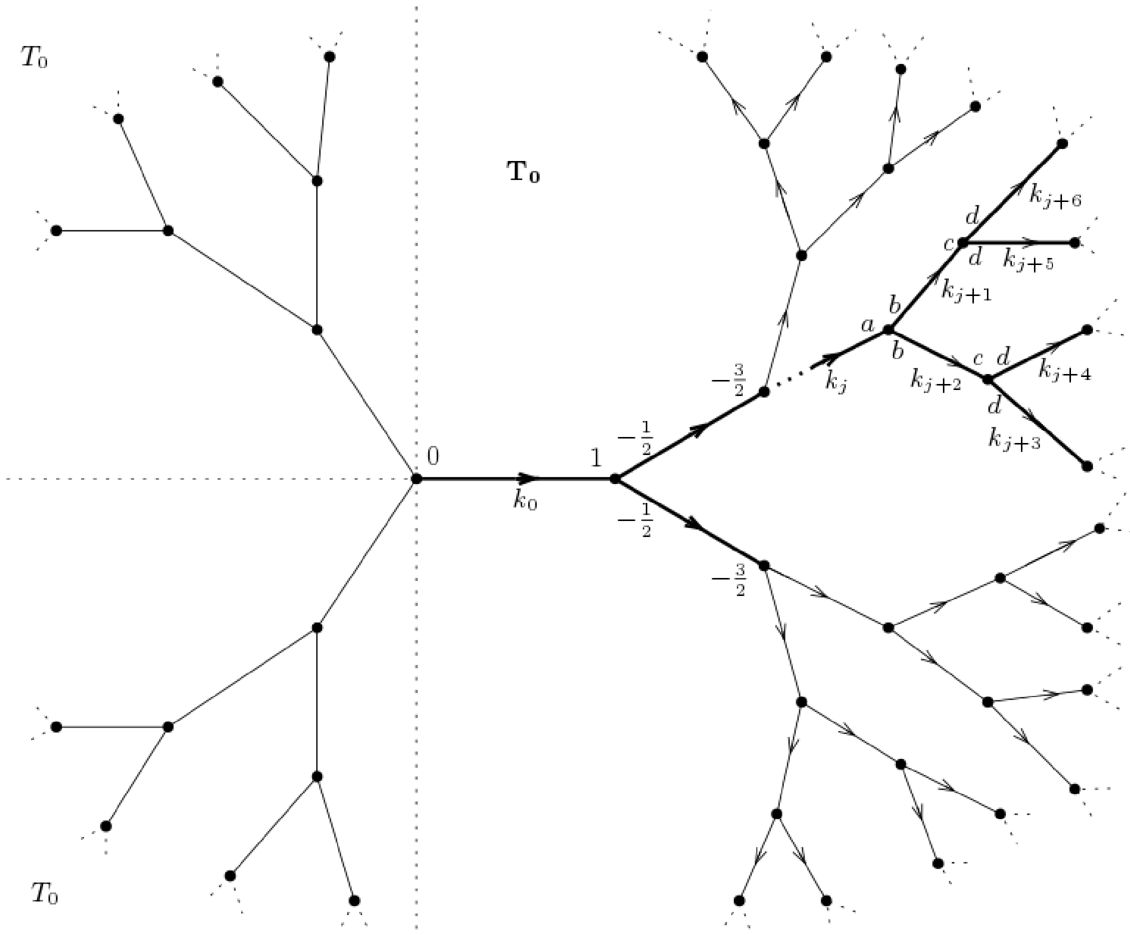

Figure 8: The 3–regular tree displaying a harmonic bounded function with $\nu = 1$.

The letters a, b, c, d stand for the corresponding values on the edges as indicated in Fig. 8. On the two copies of T_0 at $\pi_0(0)$ repeat the same construction. Then

$$u \in L^\infty(T) \cap \mathcal{H}(T).$$

Clearly, at all the nodes, $\nu_i = \pm 1$.

Under (CK), without (19), any finite multiplicity $m(0; T; CK)$ occurs. But, under (19), there is only the following alternative.

Theorem 6.6 *[5] A tree T satisfying* (19) *either is a Liouville graph under* (CK) *or satisfies $m(0; T; CK) = \infty$.*

Under (KC) this is no longer true, even under (19). Take the infinite star Σ with equal edge lengths, $V_{\text{ess}}(\Sigma) \subset \{v_0\}$ and $N \geq 1$ copies of Γ_0 intersecting precisely in v_0. Then

$$\dim \Pi^+_\infty(\Sigma) = N - 1 = m(0; \Sigma; KC).$$

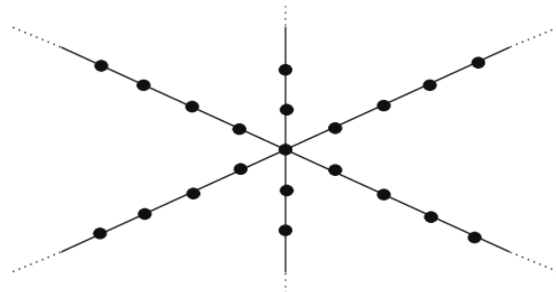

Figure 9: The infinite star with equal edge lengths.

7 Graphs fulfilling $\mathbf{m(0;\Gamma;KC) = 1}$ and periodic graphs

First, we characterize graphs with one independent bounded harmonic function satisfying (KC).

Theorem 7.1 *Let Γ be a graph that contains a copy of Γ_0 with $L(\Gamma_0) = \infty$. Then $m(0;\Gamma;KC) = 1$ if and only if Γ is a graph of the following list:*

1. *a sole induced one–sided unbounded path Γ_0 attached at its boundary vertex to a finite unicyclic graph,*
2. *a graph of Type 1 with finite trees attached to the vertices of Γ_0,*
3. *the two–sided unbounded path Γ_1 or Γ_1 with finite trees attached to its vertices.*

In all these cases, there is no bounded harmonic function with nonzero slope factor.

Proof. Suppose $m(0;\Gamma;KC) = 1$. By hypothesis, Γ can neither contain two finally disjoint copies of Γ_1, nor Γ_1 and a finite circuit, nor two finite circuits. Thus, if Γ contains Γ_1, then it is a tree and contains exactly one copy of Γ_1 up to translation, at the vertices v_k of which there might be finite trees T_k, which leads to the third case.

If Γ contains an odd circuit η, then there cannot be a bounded harmonic function with nonzero slope by (7), and there is exactly one independent bounded harmonic function belonging to $\Pi^+_\infty(\Gamma)$ as defined in Fig. **??** and extended by 0 outside the circuit η and outside Γ_0.

If Γ contains an even circuit ζ, then $\Pi^+_\infty(\Gamma) \cong \Pi^+_\infty(\zeta)$, and there cannot be a bounded harmonic function with nonzero slope. All bounded harmonic functions belonging to $\Pi^+_\infty(\Gamma)$ have their supports in ζ.

In both of the latter two cases, Γ_0 must be induced and can allow finite trees at its vertices. This leads to the first and the second case, since Lemma 3.4 excludes nonzero slopes of bounded harmonic functions and since each such a function cannot vanish on the sole circuit.

Conversely, if Γ is of Type 1 or 2, then Lemma 3.4 permits to conclude that $\Pi_\infty^+(\Gamma) \cong \mathcal{H}_{KC}^\infty(G)$. Using the same constructions as above it then follows that $m(0;\Gamma;KC) = 1$.

If Γ is of Type 3, then Lemma 3.4 again permits to conclude that $\Pi_\infty^+(\Gamma) \cong \mathcal{H}_{KC}^\infty(G)$. As a harmonic function has to vanish on the eventual finite trees attached to the vertices of Γ_1, it follows that $m(0;\Gamma;KC) = 1$. ∎

Theorem 7.1 applies in particular to periodic trees. Let us recall the following definition.

Definition 7.2 A graph Γ is called a *generalized lattice* or *periodic of rank* $m \geq 1$ with *translation group*

$$G = \bigoplus_{i=1}^{m} \mathbb{Z} b_i \leq \mathrm{Aut}(\Gamma),$$

with *kernel* N and with *cell* F, if the following conditions hold:

(a) Γ is connected.

(b) N and F are finite connected subgraphs of Γ.

(c) $V(N)^G = V(\Gamma)$.

(d) $\{b_i \,|\, 1 \leq i \leq m\}$ is a basis of the free abelian group G.

(e) $F = \overline{N \cup \bigcup_{i=1}^m N^{b_i}}$ and $E(F)^G = E(\Gamma)$.

(f) $\forall g, h \in G: \quad g \neq h \quad \Longrightarrow \quad V(N^g) \cap V(N^h) = \emptyset$.

For more details see [2, 8, 12] and [3, 4] for the spectral aspects with respect to Δ^{CK}. Clearly, periodic graphs cannot be a Liouville graph under (KC) by Lemma 3.2, since they contain copies of Γ_1. Note that it is conceivable to assume $m \geq 1$ here, since $m = 0$ corresponds to the case of a finite graph. Furthermore, recall that an eigenvalue of an operator is called a *black hole eigenvalue* if its eigenspace contains a subspace isomorphic to the inseparable Banach space $\ell^\infty(\mathbb{N})$.

Theorem 7.3 *Let Γ be a periodic graph.*

1. *If* $\mathrm{rank}\Gamma \geq 2$, *then* $m(0;\Gamma;KC) = \infty$ *and* 0 *is a black hole eigenvalue under*(KC).
2. *The same holds in the case* $\mathrm{rank}\ \Gamma = 1$, *if* Γ *contains circuits.*
3. *Finally,* $m(0;\Gamma;KC) < \infty$ *if and only if* Γ *is a periodic tree, i.e. a band without circuits. In that case,* $m(0;\Gamma;KC) = 1$, *and no nonzero slope factor is possible.*

Proof. If a periodic graph contains an even circuit, then 0 is a black hole eigenvalue since it possesses eigenfunctions of compact support. If the periodic graph contains odd circuits, then either the dumbbell construction, or a kernel enlargement leading to even circuits [2] lead to the same conclusion. If a periodic graph does not contain circuits, then it contains exactly one copy of Γ_1, up to translation. And at each vertex of Γ_1 an eventual finite tree is attached in a certain periodic way. Moreover, by periodicity, Condition (12) is fulfilled, and Theorem 7.1 permits to conclude. ∎

Example 7.4 Let T be the infinite comb with all $\ell_j = 1$. Thus, T consists in the path Γ_1 to the vertices of which a sole edge is attached. Denote the vertices and edges of the path Γ_1 by $\mathbb{Z}$ and by the incidence $d_{ii} = -1$ and $d_{i,i-1} = 1$. Let us illustrate that there is no bounded harmonic function u on T with $\nu = 1$ and that $m(0;T;KC) = 1$. Choose

$$\nu_i = -(-1)^i.$$

Then

$$\forall i \in \mathbb{Z}: \; u_{2(i+1)}(0) = u_{2i}(0) + 4 \quad \& \quad u_{2i+1}(0) = u_{2i-1}(0) - 4,$$

which clearly shows that u has to be unbounded.

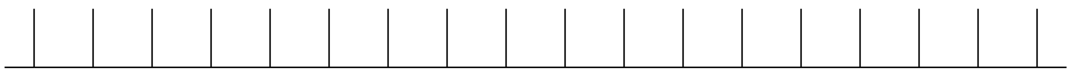

Figure 10: The infinite comb.

Example 7.5 Add to the foregoing example just one edge in a suitable kernel and get the periodic band B generated by squares or just by one vertical edge. Again, we suppose that all $\ell_j = 1$. Then $m(0;B;KC) = \infty$ and 0 is in fact a black hole eigenvalue of Δ^{KC} with

$$\mathcal{H}^\infty_{KC}(B) \cong \Pi^+_\infty(B) \cong \ell^\infty(\mathbb{Z}).$$

This readily follows by associating to each sequence $(x_k)_{k\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ a unique element in $\Pi^+_\infty(B)$ as indicated in Fig. 11.

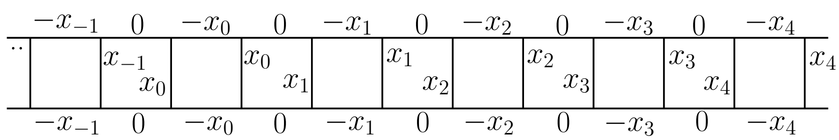

Figure 11: The periodic band B with $\mathcal{H}^\infty_{KC}(B) \cong \ell^\infty(\mathbb{Z})$.

8 Remarks and supplements

8.1 Length adjacency and anti–Kirchhoff law

For the sake of simplicity we restrict ourselves to simple graphs, i.e. we exclude loops and multiple edges. In the (CK) case, harmonic functions are closely related to a normalized adjacency operator. To be more specific, introduce

$$\mathcal{L}(\Gamma) = (\ell_{ih})_{i,h\in\mathcal{I}} : \mathbb{R}^{V(\Gamma)} \longrightarrow \mathbb{R}^{V(\Gamma)}, \qquad \mathcal{P}(\Gamma) = \left(\ell_{ih}^{-1}\right)_{i,h\in\mathcal{I}} : \mathbb{R}^{V(\Gamma)} \longrightarrow \mathbb{R}^{V(\Gamma)}$$

$$\ell_{ih} = \begin{cases} \ell_{s(i,h)} & \text{if } v_i \sim v_h, \\ 0 & \text{otherwise,} \end{cases} \qquad s(i,h) = \begin{cases} s & \text{if } e_s \cap V = \{v_i, v_h\}, \\ 1 & \text{otherwise.} \end{cases}$$

Then a continuous function u on G with $\mathbf{x} = (x_i)_{i\in\mathcal{I}}$, $x_i = u(v_i)$ belongs to $\mathcal{H}_{CK}(G)$ if and only if

$$\mathrm{Diag}_i\,(\mathcal{P}(\Gamma)\mathbf{e})^{-1}\,\mathcal{P}(\Gamma)\,\mathbf{x} = \mathbf{x}. \tag{22}$$

In particular for equal edge lengths, (22) reduces to the classical mean value property

$$\forall i \in \mathcal{I} \;:\; x_i = \frac{1}{\gamma_i}\sum_{v_h\sim v_i} x_h.$$

For the anti–Kirchhoff law $u \in \mathcal{H}_{KC}(G)$

$$\begin{aligned} 0 &= \sum_{v_i\in k_j} u_j\,(v_i) = \sum_{v_i=\pi_j(0)} u_j\,(0) + \sum_{v_i=\pi_j(\ell_j)} u_j\,(\ell_j) \\ &= \sum_{v_i=\pi_j(0)} (u_j\,(\ell_j) - \ell_j\alpha_j) + \sum_{v_i=\pi_j(\ell_j)} (u_j\,(0) + \ell_j\alpha_j) \\ &= \sum_{v_i\in k_j\ni v_h\neq v_i} u_j\,(v_h) + \nu_i \sum_{v_i\in k_j} \ell_j = \sum_{v_i\sim v_h} u_{ih}\,(v_h) + \nu_i \sum_{v_i\in k_j} \ell_j, \end{aligned}$$

where we have used the adjacency setting. and $\mathbf{U} = \left(u_{s(i,h)}\right)_{i,h\in\mathcal{I}}$. Thus, we have shown

Lemma 8.1 *A harmonic function under (KC) satisfies*

$$\mathbf{U}(1)\mathbf{e} = \mathbf{U}(0)^*\mathbf{e} = -\,(\nu_i\delta_{ik})_{\mathcal{I}\times\mathcal{I}}\,\mathcal{L}(\Gamma)\mathbf{e}.$$

For $\nu = 0$, in particular for non bipartite graphs, Lemma 8.1 is just equivalent with the defining relation for $\Pi^+_\infty(\Gamma)$. But in the bipartite case, it imposes restrictions. If all edge lengths amount to 1 e.g., then we have a very specific mean value property of the form

$$\forall i \in \mathcal{I} \;:\; \nu_i = -\frac{1}{\gamma_i}\sum_{v_i\sim v_h} u_{ih}\,(v_h)\,.$$

8.2 The unsigned incidence matrix in the finite case

In view of applications in Section 5, we present some details for the unsigned incidence matrix $\mathcal{D}^+ = \mathcal{D}^+(B) = (|d_{ij}|)_{n\times m}$ of a finite connected simple graph B with n vertices and m edges. Recall that in this case

$$\dim \Pi^+(B) = \dim\ker \mathcal{D}^+(B) = \begin{cases} m-n+1 & \text{if } B \text{ is bipartite,} \\ m-n & \text{if } B \text{ is not bipartite.} \end{cases} \tag{23}$$

As each edge is incident with exactly two vertices by simplicity, any $\varphi = (\varphi_j)_{m\times 1}$ fulfills

$$\sum_{j=1}^{m} \varphi_j = \sum_{i=1}^{n}\sum_{d_{ij}=-1} \varphi_j = \sum_{i=1}^{n}\sum_{d_{ij}=1} \varphi_j. \tag{24}$$

In particular, if B is bipartite and thereby endowable with the sink–source –orientation, (24) takes the form

$$(25)\qquad \sum_{v_i \text{ sink}} \left(\mathcal{D}^+(B)\varphi\right)_i = \sum_{v_i \text{ sink}} \sum_{v_i\in k_j} \varphi_j = \sum_{v_i \text{ source}} \sum_{v_i\in k_j} \varphi_j = \sum_{v_i \text{ source}} \left(\mathcal{D}^+(B)\varphi\right)_i .$$

Thus, in the bipartite case, a vector of the canonical basis $\mathbf{e}_i^n$ of $\mathbb{R}^n$ can never belong to the image of $\mathcal{D}^+(B)$. This follows also from the fact that

$$\forall j \in \{1,\dots,m\} : \mathcal{D}^+\mathbf{e}_j^m = \mathbf{e}_i^n + \mathbf{e}_k^n \text{ with } v_i, v_k \in k_j.$$

Finally, for application in Section 5, we can state the following

Lemma 8.2 *Let B be a finite bipartite graph, endowed with the sink–source –orientation. Set*

$$(26)\qquad s = (s_i)_{n\times 1}, \qquad s^+ = \left(s_i^+\right)_{n\times 1}, \qquad s^- = \left(s_i^-\right)_{n\times 1}, \qquad s_i = \sum_{v_i\in k_j} \ell_j,$$

$$s_i^+ = \begin{cases} s_i & \text{if } v_i \text{ is a sink,} \\ 0 & \text{otherwise,} \end{cases} \qquad s_i^- = \begin{cases} s_i & \text{if } v_i \text{ is a source,} \\ 0 & \text{otherwise,} \end{cases}$$

Then

$$s \in \operatorname{Im} \mathcal{D}^+(B), \quad \textit{but} \quad s^+, s^- \notin \operatorname{Im} \mathcal{D}^+(B).$$

Proof. Clearly $\mathcal{D}^+$ applied to the vector $\ell = (\ell_j)_{m\times 1}$ yields the vector $s = s^- + s^+$. Thus, it remains to show that $s^+ \notin \operatorname{Im}\mathcal{D}^+$. But, if there were φ with $\mathcal{D}^+\varphi = s^+$, (25) would lead to the contradiction

$$L(B) = \sum_{v_i \text{ sink}} \left(\mathcal{D}^+(B)\varphi\right)_i = \sum_{v_i \text{ source}} \left(\mathcal{D}^+(B)\varphi\right)_i = 0.$$

■

Acknowledgments

Joachim von Below is grateful to the research group GREDPA at UPC Barcelona for the invitation in 2018. José A. Lubary is grateful to the LMPA Joseph Liouville at ULCO in Calais for the invitation in 2018.

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