Title	A note on retracts of polynomial rings in three variables
Author(s)	Nagamine, Takanori
Citation	代数幾何学シンポジウム記録 (2019), 2019: 107-107
Issue Date	2019
URL	http://hdl.handle.net/2433/245724
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

A note on retracts of polynomial rings in three variables

Takanori Nagamine (Niigata University)

Key words and phrases

Retracts of polynomial rings, Zariski's cancellation problem

Abstract

For retracts of the polynomial ring, in [Cos77], Costa asks us whether every retract of $k[x_1 \ldots, x_n]$ is also the polynomial ring or not, where k is a field. We call it the *polynomial retraction problem* (PRP).

In this paper, we give an affirmative answer to PRP in the case where k is a field of characteristic zero and $n = 3$ ([Nag19]). Also, we state relations between PRP and Zariski's cancellation problem.

Definition (retracts of a commutative ring)

B: commutative ring,
$A \subset B$: subring of B.
We say A is a **retract** of B if
$\exists$ an ideal $I \subset B$ such that $B \cong A \oplus I$ as A-modules,
$\Leftrightarrow \exists\ \varphi : B \to A$ such that the following splits:

$$0 \to \ker \varphi \to B \xrightarrow{\varphi} A \to 0,$$

$\Leftrightarrow \exists\ \varphi : B \to A$ such $\varphi|_A = \mathrm{id}_A$.

Example

$B = k[x, y, z]$: polynomial ring in three variables.
Then,

- k, $k[x]$, $k[x, y]$ and $k[x, y, z]$ are retracts of B.
- $k[xz, yz]$ is a retract of B.
  $\because$ Define $\varphi : B \to k[xz, yz]$ by $x \mapsto xz,\ \ y \mapsto yz,\ \ z \mapsto 1$.
  Then $\varphi|_{k[xz,yz]} = \mathrm{id}_{k[xz,yz]}$.
- $k[x, xz + y^2]$ is NOT a retract of B.

Polynomial Retraction Problem (PRP)

Is every retract of $k[x_1, \ldots, x_n]$ the polynomial ring?

dimension n	char $k = 0$	char $k > 0$
$n = 1$	YES	YES
$n = 2$	YES ([Cos77])	YES ([Cos77])
$n = 3$	**YES (Main Theorem)**	???
$n \geq 4$	???	NO ([Gup14a], [Gup14b])

Zariski's Cancellation Problem (ZCP)

$X \times \mathbb{A}^1_k \cong_k \mathbb{A}^{n+1}_k \Longrightarrow X \cong_k \mathbb{A}^n_k$?

dimension n	char $k = 0$	char $k > 0$
$n = 1$	YES	YES
$n = 2$	YES ([Fuj79], [MS80])	YES ([Rus81])
$n = 3$	???	NO ([Gup14a])
$n \geq 4$	???	NO ([Gup14b])

Proposition (PRP vs ZCP)

Let $n \geq 1$. Then the affirmative answer to PRP for n implies the affirmative answer to ZCP for $n - 1$.

Proof of Proposition

Suppose that PRP holds true for $n \geq 1$.
Let $X = \mathrm{Spec}\,(A)$ such that $X \times \mathbb{A}^1_k \cong_k \mathbb{A}^n_k$.
Then $A[t] = k[x_1, \ldots, x_n]$.
Define $\varphi : A[t] \to A$ by $\varphi(f(t)) = f(0)$.
Then A is a retract of $k[x_1, \ldots, x_n]$.
Therefore $A = k[y_1, \ldots, y_{n-1}]$, hence $X \cong_k \mathbb{A}^{n-1}_k$. $\square$

Main theorem (N. 2019)

k: field of characteristic zero.
$k[x_1, \ldots, x_n]$: polynomial ring in $n \geq 3$ variables.
$A \subset k[x_1, \ldots, x_n]$: sub k-algebra.
Assume that A is a retract of $k[x_1, \ldots, x_n]$ of dimension d.
If $0 \leq d \leq 2$ or $d = n$, then $A = k[y_1, \ldots, y_d]$.

Corollary (the answer to PRP)

k: field of characteristic zero.
Every retract of $k[x, y, z]$ is the polynomial ring.

Outline of the proof

k: field of characteristic zero.
$B = k[x_1, \ldots, x_n]$: polynomial ring in n variables.
$A \subset B$: retract of B.

- $\mathrm{tr.deg}_k\, A = 0, n \Rightarrow$ easy to show that A is the polynomial ring.
- $\mathrm{tr.deg}_k\, A = 1 \Rightarrow A = k[t]$ (follows from [Cos77]).

Suppose that $\mathrm{tr.deg}_k\, A = 2$.
Due to [Kam75], we may assume that k is algebraically closed.

By combing results in [Eak72], [Cos77] and [Iit77], we have:

- A is a UFD, finitely generated over k, and $A^* = k^*$,
- $X = \mathrm{Spec}\,(A)$ is a smooth affine surface over k,
- the logarithmic Kodaira dimension of X is $-\infty$.

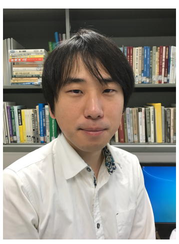
By combing results in [Miy75], [Fuj79] and [MS80], we have $X \cong_k \mathbb{A}^2_k$.
This implies that $A = k[s, t]$. $\square$

[Cos77] D. Costa, J. Algebra, 1977.
[Eak72] P. Eakin, Proc. Amer. Math. Soc., 1972.
[Fuj79] T. Fujita, Proc. Japan Acad., Ser. A, 1979.
[Gup14a] N. Gupta, Invent. Math., 2014.
[Gup14b] N. Gupta, Adv. Math., 2014.
[Iit77] S. Iitaka, Complex Analysis and Algebraic Geometry, 1977.
[Kam75] T. Kambayashi, J. Algebra, 1975.
[Miy75] M. Miyanishi, J. Math. Kyoto Univ., 1975.
[MS80] M. Miyanishi and T. Sugie, J. Math. Kyoto Univ., 1980.
[Nag19] T. Nagamine, J. Algebra, 2019.
[Rus81] P. Russell, Math. Ann., 1981.

Takanori Nagamine (長峰 孝典)

Graduate School of Science and Technology,
Niigata University,
Japan

Email: t.nagamine14@m.sc.niigata-u.ac.jp



Research of the author was partially supported by Grant-in-Aid for JSPS Fellows (No. 18J10420) from Japan Society for the Promotion of Science