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*Inventiones mathematicae*


# Random matrices, amalgamated free products and subfactors of the von Neumann algebra of a free group, of noninteger index *

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We introduce in this paper a noncommutative probability approach (in the sense considered by D. Voiculescu in [28]) to the algebras that are associated to certain amalgamated free products. In this way we find that the type $II_1$ factors associated to the free, noncommutative groups $F_N, N \geq 2$ have a rich lattice of irreducible subfactors of noninteger index. Our main result states that many index values for irreducible subfactors of the hyperfinite $II_1$ factor are also index values for irreducible subfactors of $\mathscr{L}(F_N)$. This answers a question raised by V.F.R. Jones in [13].

**Theorem** *Let $\mathcal{C} = (A \supseteq B \supseteq D; A \supseteq C \supseteq D)$ be a commuting square ([20]) of finite dimensional algebras, which is irreducible (i.e. the centers of $A, B$ and respectively $C, D$ have trivial intersection) and $\lambda$-Markov ([21],[31]), (i.e. there exists a $\lambda$-Markov trace, in the sense of Jones ([11]) for $C \subseteq A$, which restricts to a $\lambda$-Markov trace for $D \subseteq B$). Let $N \geq 2$ be any natural number.*

*Then there exists a subfactor $\mathcal{A} \subseteq \mathscr{L}(F_N)$ of index $\lambda^{-1}$ with relative commutant $B \cap C'$. In addition*

$$\mathcal{A} \cong \mathscr{L}(F_{(N-1)\lambda^{-1}+1}) \cong \left[\mathscr{L}(F_N)\right]_{\lambda^{1/2}}.$$

We note that the same statement holds true if $\mathcal{C}$ is one of the atomic infinite dimensional commuting squares considered by U. Haagerup and J. Schou in [10], [26] (see Remark 5.4).

Here $\mathscr{L}(F_N), N \in \mathbb{N}, N \geq 2$ is the type $II_1$ factor associated to a free group $F_N$, i.e. the weak closure of the group algebra $\mathbb{C}(F_N)$ acting on the Hilbert space

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$l^2(F_N)$ by left translation. These factors were first considered by F.J. Muray and J. von Neumann in [16]. They proved that they are not isomorphic to the hyperfinite $II_1$ factor (since they do not have the property $\Gamma$). Recently, it appeared ([28]) that these factors are, in a certain sense, the natural algebra of observables, if one starts with Wigner's point of view ([32]) in which one models the observables by random matrices of very large size.

If $\mathscr{C}$ is any commuting square such that the iterated basic construction of $\mathscr{C}$ gives an irreducible subfactor of the hyperfinite $II_1$ factor ([11],[12], [31],[17], [20],[10],[26]), for instance if $\mathscr{C}$ is one of the commuting squares that are associated with the Jones' subfactors $R_\lambda \subseteq R$ for $\lambda^{-1} \in \{4\cos^2 \pi/n | n \geq 3\}$, one obtains:

**Corollary** *For any $N \in (1,\infty]$ and for any index value $\lambda^{-1}$ of an irreducible subfactor of the hyperfinite $II_1$ factor that may be obtained by iteration of Jones' basic construction from a commuting square (for instance if $\lambda^{-1} \in \{4\cos^2 \pi/n | n \geq 3\}$) there exists an irreducible subfactor $\mathscr{M} \subseteq \mathscr{L}(F_N)$ of index $[\mathscr{L}(F_N) : \mathscr{M}] = \lambda^{-1}$.*

*Moreover $\mathscr{M}$ is isomorphic to $\mathscr{L}(F_{(N-1)\lambda^{-1}+1}) \cong [\mathscr{L}(F_N)]_{\lambda^{1/2}}$.*

Recall that for a type $II_1$ factor $\mathscr{M}$ with trace $\tau$ one denotes by $\mathscr{M}_t$ the isomorphism class of the algebra $e\mathscr{M}e$ (with unit $e$), where $e$ is any projection in $\mathscr{M}$ of trace $\tau(e) = t$.

The higher relative commutant for the algebras in the Jones' tower ([11]) of the above inclusion $\mathscr{M} \subseteq \mathscr{L}(F_N)$ coincide with the ones for the inclusion of hyperfinite type $II_1$ factors $P_\infty \subseteq Q_\infty$ that is associated with the commuting square $\mathscr{C}$ (we owe this observation to S. Popa).

Due to our theorem, we are able to compute for $\mathscr{L}(F_\infty)$ one of the invariants that was introduced by V. F. R. Jones in connection with his Galois type classification of subfactors. Recall that for an arbitrary type $II_1$ factor $M$ this invariant is $\mathscr{I}(M)$, the set of all possible values of indices of subfactors of $M$. Our result gives that

**Corollary** *The set $\mathscr{I}(\mathscr{L}(F_\infty))$ of all possible values of indices of subfactors of $\mathscr{L}(F_\infty)$ is $\{4\cos^2(\pi/n) | n \geq 3\} \cup [4,\infty)$ (i.e the same set as for $R$, the hyperfinite factor of type $II_1$).*

Note that the only other type $II_1$ factors $M$ for which anything is known about the invariant $\mathscr{I}(M)$ are the Connes' property T factors, ([5], [1]) for which, by the results of M.Pimsner and S.Popa [19], this set is countable.

The fact that the continuous line $[4,\infty)$ is in $\mathscr{I}(\mathscr{L}(F_\infty))$ is due to the fact [23] that $\mathscr{F}(\mathscr{L}(F_\infty)) = \mathbb{R}_+/\{0\}$ (by Jones' remark in [11]). Recall that the fundamental group $\mathscr{F}(M)$ of a type $II_1$ factor $M$ is the multiplicative subgroup of $\mathbb{R}_+ \setminus \{0\}$ defined by

$$\mathscr{F}(M) = \{t > 0 | M_t \cong M\}.$$

Note that by a well known theorem of A. Connes in [3], this group is again countable for property $T$ factors.

By a theorem in group theory ([15]), an index $k$ subgroup $G$ of $F_N$ is isomorphic to $F_{(N-1)k+1}$. At the group algebra level this corresponds to the fact that there exists a subfactor $\mathscr{L}(F_{(N-1)k+1}) \cong \mathscr{L}(G) \subseteq \mathscr{L}(F_N)$ of index $k$ (since by [11], the index $[\mathscr{L}(F_N) : \mathscr{L}(G)]$ is the group index $[F_N : G]$). Thus our main theorem extends in a certain sense the preceding (group theoretic) result to the case of noninteger index.

To give a meaning for the isomorphism class of the subfactors that we find by the construction in the first theorem we introduce a real continuation $\mathscr{L}(F_r)$, $r > 1$

with type $II_1$ factors for the sequence of algebras $\mathcal{L}(F_N)$, $N \in \mathbb{N}, N > 1$, although there is no meaning for $F_r$ as a group (at least for nonrational values of $r$).

Such a continuous series appears in a natural way if one tries to find the analogue of Voiculescu's formula

$$\mathcal{L}(F_N) \cong M_k(\mathbb{C}) \otimes \mathcal{L}(F_{(N-1)k^2+1})$$

or equivalently

(0.1) $$\mathcal{L}(F_N)_{1/k} \cong \mathcal{L}(F_{(N-1)k^2+1}) \text{ for all } k \in \mathbb{N}, N \in \mathbb{N}, N \geq 2$$

for numbers other than $1/k$.

A first sign that such a continuation may exist, is the fact ([24]) that for $1/\sqrt{k}$ instead of $1/k$ in (0.1) (in which case both terms in (0.1) still make sense as group algebras) the isomorphism:

(0.1)′ $$\mathcal{L}(F_N)_{1/\sqrt{k}} \cong \mathcal{L}(F_{(N-1)k+1}) \text{ for all } k \in \mathbb{N}, N \in \mathbb{N}, N \geq 2$$

is valid.

The series $\mathcal{L}(F_r)_{r\in\mathbb{R},r>1}$ is thus uniquely defined as a real continuation of the sequence of factors associated to free groups and it is subject to the following conditions:

(0.2) $$(\mathcal{L}(F_r))_t \cong \mathcal{L}(F_{(r-1)t^{-2}+1}), \text{for all } t > 0, r > 1, r, t \in \mathbb{R},$$

(0.3) $$\mathcal{L}(F_r) * \mathcal{L}(F_{r'}) \cong \mathcal{L}(F_{r+r'}) \text{ for all } r, r' > 1.$$

A similar series with the properties (0.2), (0.3) was considered by K. Dykema in [8].

Formula (0.2) shows in particular (see [23], [28]) that the type $II_1$ factors $\mathcal{L}(F_N)$, $N \in \mathbb{N}$, $N \geq 2$ are stably isomorphic (by tensoring with $B(H)$, the algebra of all bounded linear operators on a separable infinite dimensional Hilbert space $H$). Note that Voiculescu's formula (0.1) implies in particular that $\mathcal{L}(F_2)$ and $\mathcal{L}(F_5)$ are stably isomorphic but, it does not imply, for example, that $\mathcal{L}(F_2)$ and $\mathcal{L}(F_3)$ are stably isomorphic (as formula (0.1)′ shows).

Note that (by (0.2)) the algebras $\mathcal{L}(F_N)$, $N > 1$ are also stably isomorphic in the sense that

$$\mathcal{L}(F_\infty) \otimes \mathcal{L}(F_N) \cong \mathcal{L}(F_\infty) \otimes \mathcal{L}(F_M) \text{ for all } N, M > 1.$$

(we owe this observation to G. Skandalis). This is due to the (general) fact that for all type $II_1$ factors $A, B$, we have

$$A_t \otimes B_{1/t} \cong A \otimes B$$

and thus by formula (0.2) we get

**Corollary** *The isomorphism class of* $\mathcal{L}(F_r) \otimes \mathcal{L}(F_s)$ *depends only on*

$$(r-1)(s-1) \text{ for all } r, s \in (1, \infty].$$

Moreover, a simple algebraic manipulation with both formulae (0.2), (0.3) will show that the fundamental group of $\mathcal{L}(F_r)$ is either $\mathbb{R}_+/\{0\}$ or $\{1\}$.

In fact we will prove a more precise statement

**Corollary** *One (and only one) of the following two statements holds true:*

*(i) For all finite $r > 1$ the type $II_1$ factor $\mathcal{L}(F_r)$ is isomorphic to $\mathcal{L}(F_\infty)$.*

*(ii) The type $II_1$ factors $\mathcal{L}(F_r)$ are mutually nonisomorphic for $r \in (1, \infty]$.*

*Note that (i) is equivalent to any of the following statements*

*(i)′ The isomorphism $\mathcal{L}(F_r) \otimes B(H) \cong \mathcal{L}(F_\infty) \otimes B(H)$ holds true for some (equivalently for all) finite $r > 1$.*

*(i)″ The fundamental group $\mathcal{F}(\mathcal{L}(F_r))$ is nontrivial for some (equivalently for all) finite $r > 1$.*

At this moment it is still unknown if the type $II_1$ factors

$$\mathcal{L}(F_N)_{N \in \mathbb{N} \cup \{\infty\}, N \geq 2}$$

may be isomorphic (this is an old question of R. V. Kadison in the early 50's ([14])), but we hope that the results above could serve as evidence for a positive answer to this question.

Amalgamated free products of operator algebras have been intensively studied in [30], from the point of view of noncommutative probability theory. They were introduced in the setting of type $II_1$ factors in [22], where the basic results on the existence of traces, factoriality and computation of relative commutants are obtained.

S. Popa realized that the problem of constructing irreducible subfactors of index $> 4$ in arbitrary factors naturally leads to the consideration of certain canonical subfactors coming from amalgamated free products involving an "initial algebra" $Q$ and the Jones' sequence of projections. He uses such subfactors to construct a series of irreducible inclusions of non-$\Gamma$ factors $N^s(Q) \subseteq M^s(Q)$, of any index $s \in \{4\cos^2 \pi/n | n \geq 3\} \cup [4, \infty)$.

Let $R_\lambda \subseteq R$ be the canonical Jones' inclusion of hyperfinite $II_1$ factors of index $\lambda^{-1} = s \in \{4\cos^2 \pi/n | n \geq 3\} \cup [4, \infty)$. The factors $N^s(Q) \subseteq M^s(Q)$ are subfactors in $(Q \otimes R_\lambda) *_{R_\lambda} R$ and they depend in a functorial way on the type $II_1$ factor $Q$.

In fact, when $\mathcal{C} = \mathcal{C}_\lambda$ is a commuting square of relative commutants in the tower of the Jones' basic construction for $R_\lambda \subseteq R$, $\lambda^{-1} = s \in \{4\cos^2 \pi/n | n \geq 3\}$ the inclusion described in the statement of our main theorem, coincides with a term in Jones' basic construction of Popa's inclusion $N^s(Q) \subseteq M^s(Q)$, for $Q = \mathcal{L}(F_\infty)$ (although our approach will be more direct, by making use of the finite depth properties of the inclusion $R_\lambda \subseteq R$).

In its most general form our construction depends also on an initial algebra $Q$ and also depends (rather then on a scalar $\lambda^{-1}$) on a nicely behaved finite dimensional commuting square $\mathcal{C}$, called $\lambda$-Markov (with Ind $\mathcal{C} = \lambda^{-1}$). We prove:

**Theorem** *Let $Q$ be a type $II_1$ factor and let*

$$\mathcal{C} = (A \supseteq B \supseteq D;\ A \supseteq C \supseteq D)$$

*be a commuting square [20] of finite dimensional algebras which is irreducible (i.e the centers of the algebras $A, B$ (respectively $C, D$) have trivial intersection) and $\lambda$-Markov ( i.e. $A$ is endowed with a $\lambda$-Markov trace for the inclusion $C \subseteq A$ (in the sense of Jones [11]), which restricts to a $\lambda$-Markov trace on $D \subseteq B$). Then*

$$N^{\mathcal{C}}(Q) = (Q \otimes D) *_D C \subseteq M^{\mathcal{C}}(Q) = (Q \otimes B) *_B A$$

*is an inclusion of type $II_1$ factors of index $\left[M^{\mathscr{C}}(Q) : N^{\mathscr{C}}(Q)\right] = \lambda^{-1}$ and relative commutant $B \cap C'$.*

In this paper (except when otherwise mentioned) by amalgamated free product of finite von Neumann algebras, we will always understand the reduced, von Neumann algebra, amalgamated free product that is obtained, via the G. N. S. construction, from the trace on the algebraic amalgamated free product of the algebras (see [22]).

The higher relative commutants invariants of the series $N^{\mathscr{C}}(Q) \subseteq M^{\mathscr{C}}(Q)$ may be computed (by Theorem 1.3, in this paper, which is due to S. Popa) and they coincide with the corresponding invariants of the inclusion of hyperfinite $II_1$ factors $P_\infty \subseteq Q_\infty$ that is canonically associated to the commuting square $\mathscr{C}$. This means that the standard invariants ([21]), or the paragroups ([17])

$$\mathscr{G}(N^{\mathscr{C}}(Q) \subseteq M^{\mathscr{C}}(Q))$$

and $\mathscr{G}(P_\infty \subseteq Q_\infty)$ coincide.

The main body of this paper is concerned with the case $Q = \mathscr{L}(F_\infty)$. Let $A \supseteq B$ be an irreducible inclusion of finite dimensional algebras and fix a faithful trace $\tau$ on $A$. We prove first that there exists a trace preserving embedding of $\mathscr{L}(F_\infty) \otimes B *_B A$ into $\mathscr{L}(F_\infty) * A$. This will allow to find a random matrix model (in the sense of D. Voiculescu) for the von Neumann algebra $(\mathscr{L}(F_\infty) \otimes B *_B A)$. The precise form of this model is as follows:

**Theorem** *Let $1 \in B \subseteq A$ be finite dimensional algebras. Let $\tau$ be a faithful, normalized trace on $A$. Assume that $B$ is abelian. Then there exist a (natural) trace preserving embedding of $(\mathscr{L}(F_\infty) \otimes B) *_B A$ into $\mathscr{L}(F_\infty) * A$. This embedding is realized explicitly as follows:*

*Let $(Y^s)_{s\in S}$ be an infinite semicircular family, let $(p_i)_{i=1}^{l}$ be the minimal projections in $B$ and let $\mathscr{B}$ be the von Neumann algebra free product*

$$\{(Y^s)_{s\in S}\}'' * A \cong \mathscr{L}(F_\infty) * A.$$

*Define*

$$X^s = \sum_i \tau(p_i)^{-1/2} p_i Y^s p_i \text{ for all } s \in S.$$

*Then $(X^s)_{s\in S}$ is a free semicircular family (hence $\{(X^s)_{s\in S}\}'' \cong \mathscr{L}(F_\infty)$) and $\{(X^s)_{s\in S} \cup A\}''$ is isomorphic to the algebra*

$$\left[\{(X^s)_{s\in S}\}'' \otimes B\right] *_B A \cong (\mathscr{L}(F_\infty) \otimes B) *_B A.$$

Finally we will use the above statement to prove that even if $B$ is not necessarily abelian, a reduced algebra of $(\mathscr{L}(F_\infty) \otimes B) *_B A$ is isomorphic to $\mathscr{L}(F_\infty)$. Since by [23] $\mathscr{L}(F_\infty) \cong e\mathscr{L}(F_\infty)e$ for any idempotent $e$ in $\mathscr{L}(F_\infty)$, it follows that $(\mathscr{L}(F_\infty) \otimes B) *_B A$ is isomorphic to $\mathscr{L}(F_\infty)$.

This gives the lines for the proof of the main theorem in the case of infinitely many generators (which is done in the first three paragraphs). To prove the general case for $\mathscr{L}(F_N)$ we follow the steps of the preceding proof and perform some supplementary computations. By the definition of the series $\mathscr{L}(F_r)_{r\in \mathbb{R}, r>1}$, we only have to count the elements in certain sets of generators (since we already proved the similar result for the case of infinitely many generators). In this way we obtain:

**Theorem** *Let $1_A \in B \subseteq A$ be an irreducible inclusion of finite dimensional algebras. Let $\tau$ be a normalized faithful trace on $A$, let $\Gamma$ be the inclusion matrix of $A \supseteq B$ and let $(s_k)_{k\in K}$ be the vector of the values of the trace $\tau$ on a set of representatives for the minimal projections in $A$.*

*For any real $N > 1$ or for $N = \infty$, we have:*

$$(\mathcal{L}(F_N) \otimes B) *_B A \cong \mathcal{L}(F_{1+N<(\Gamma\Gamma^t)s,s>-<s,s>}).$$

Clearly, the above theorem and the theorem on subfactors in amalgamated free products will complete the proof of the main result.

Finally we point out that an intriguing question that remains open is to describe the structure of the set of all possible values of irreducible subfactors of $\mathcal{L}(F_\infty)$ (this is an other invariant for type $II_1$ factors introduced by V.F.R. Jones in [11]). On one side this set should coincide with the similar set for the hyperfinite $II_1$ factor $R$ and on the other hand, the factors in Popa's series $N^s \subseteq M^s$ for $s \in [4, \infty)$ are irreducible, nonhyperfinite and have (by [2]) Haagerup's approximation property (and thus are close to $\mathcal{L}(F_\infty)$).

Another related open question is if one can prove unicity, modulo conjugacy, for the one parameter action of $\mathbb{P}$ on $\mathcal{L}(F_\infty)$ that was constructed in [25]. For the hyperfinite $II_\infty$ factor such an action is unique, modulo conjugacy, by the work of A. Connes ([4]) and U. Haagerup ([9]).

The paper is organized as follows: the next paragraph recalls the definitions we use. Paragraph 0 contains an outline of the proof and the first paragraph contains the proof of the facts concerning subfactors in $(Q \otimes B) *_B A$. In the second paragraph we construct a random matrix model for a reduced algebra of $(\mathcal{L}(F_\infty) \otimes B) *_B A$. We use this model in the third paragraph to give a direct proof of the main result in the case of a free group with infinitely many generators.

The fourth paragraph contains the definition of the real continuation for the sequence of algebras associated to free groups (and the consequences of the existence of such a continuation). In the last paragraph we use the results above to prove our main result.

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## Definitions

Let $H$ be a Hilbert space and let $B(H)$ be the space of all bounded linear operators acting on $H$. A weakly closed unital subalgebra $M$ of $B(H)$ is called a von Neumann algebra. When no mention is made of the underlying Hilbert space on which $M$ acts, then $M$ is called a $W^*$ -algebra. In this case the weak topology on $M$ comes from its predual $M_*$ ([27]).

An abelian von Neumann subalgebra of $M$ is called diffuse if it does not contain minimal projections. As usual if $B$ is a $W^*$ -algebra and $\sum$ is a selfadjoint subset of $B$, $\sum''$ will be the von Neumann subalgebra of $B$ generated by $\sum$ and $1 \in B$. If $B \subseteq B(H)$, we denote by $\sum'$ the commutant of $\sum$ in $B(H)$. For a subset $S$ in $B$ we denote by $\mathrm{Sp}(S)$ the linear span of the set $S$.

By $\mathscr{P}(M)$ we denote the set of all projections $p = p^2 = p^*$ in $M$ and by $\mathscr{Z}(M)$ we denote the center of $M$. For a projection $e$ in $M$ we denote by $eMe$ or $M_e$ the reduced algebra which acts on $eH$. In fact $M_e$ is a von Neumann subalgebra of $B(eH)$. If $M$ is a type $II_1$ factor (i.e. $\mathscr{Z}(M) = \mathbb{C}$ and there exists a normal finite faithful trace $\tau$ on $M$ with $\tau(1) = 1$) then we denote by $M_t$ the isomorphism class of $eMe$ for any projection $e$ in $M$ of trace $\tau(e)$ $t$.

If $t \geq 1$ then we consider an infinite dimensional separable Hilbert space $H$, and endow $B(H)$ with the usual faithful semifinite normal trace that takes value 1 when evaluated on projections of dimension 1. Let $\tau'$ be the corresponding tensor product trace on $M \otimes B(H)$ and let $e$ be any projection in $M \otimes B(H)$ with $\tau'(e) = t$. In this case, $M_t$ is the isomorphism class of $(M \otimes B(H))_e$. It is well known [6], [27] that the isomorphism class of $eMe$ does not depend on the choices made in the selection of the projection $e$.

Recall some definitions and facts from [28],[29]. Let $(A, \varphi)$ be a unital $C^*$-algebra endowed with a normalized trace $\varphi$. A family of subalgebras $1 \in A_i \subseteq A$ $(i \in I)$ is called a free family of subalgebras if $\varphi(a_1 a_2 .. a_n) = 0$ whenever $a_j \in A_{i_j}$, $\varphi(a_j) = 0$ for all $j = 1, 2 \ldots, n$ and $i_j \neq i_{j+1}$ for all $1 \leq j \leq n-1$.

A family $(\Omega_i)_{i \in I}$ of subsets of $A$ is free if the family of the subalgebras generated by the $\Omega_i$ and 1 is free. A family of elements $(f_i)_{i \in I} \subseteq A$ is free if the family $\{f_i\}_{i \in I}$ is free.

Moreover a free family $(f_i)_i$ is called semicircular if the elements $f_i$ are selfadjoint and the distributions of $f_i$ with respect to $\varphi$ are given by the semicircle law

$$\varphi(f_i^k) = 2/\pi \int_{-1}^{1} t^k (1 - t^2)^{1/2}\, \mathrm{d}t, k \in \mathbb{Z}, k \geq 0.$$

A family $(f_i)_{i \in I}$ is called circular if the family $\{x_i\}_i \cup \{y_i\}_i$ is semicircular, where $f_j = x_j + \sqrt{-1} y_j, j \in I$ is the decomposition of $f_j$ into real and imaginary part. By abuse of language, we will say that $(f_i)_{i \in I}$ is semicircular even if the constant $2/\pi$, in the formula above, is replaced by a strictly positive constant.

By Theorem 2.1 in [28] the von Neumann algebra generated by a free semicircular family $(Y^s)_{s \in S}$ is isomorphic to the von Neumann algebra $\mathscr{L}(F_{\mathrm{card}\ S})$ associated to a free group with the same number of generators as the cardinality of $S$.

We recall some facts from [22]. Let $A_1, A_2$ be two von Neumann algebras endowed with normal, faithful traces $\tau_1, \tau_2$ and let $B$ be a common von Neumann subalgebra containing the unit of $A_i$ for $i = 1, 2$. Assume that $\tau_1|_B = \tau_2|_B$ and let $E_B^{A_i}$ be the corresponding conditional expectation from $A_i$ onto $B$ for $i = 1, 2$. Let $\tau$ be the trace initially defined on the algebraic amalgamated free product $A_1 *_B A_2$ by the condition

$$\tau(a_1 a_2 \ldots, a_n) = 0$$

if $a_i \in A_{i_j}, E_B(a_i) = 0, j = 1, \ldots, k, i_1 \neq i_2, \ldots, i_{k-1} \neq i_k$. The reduced, von Neumann algebra, amalgamated free product $A_1 *_B A_2$ is the weakly closed subalgebra obtained via G.N.S. construction with respect to the trace $\tau$. Moreover $\tau$ extends to a faithful normal trace on $A_1 *_B A_2$ (see [22], Lemma 4.1).

We will review , for the convenience of the reader, Voiculescu's random matrix picture ([28], Theorem 2.2) for semicircular families. Let $(\Sigma, \nu)$ be a probability space and let $\mathscr{E}$ be the expectation value on $\Sigma$, i.e.

$$\mathscr{E}(f) = \int_\sigma f(\sigma) d\nu(\sigma) \text{ for all } f \in \bigcap_{p \geq 1} L^p(\Sigma, \nu).$$

Let $S$ be a nonvoid set and for each $s \in S$ let

$$Y(s,n) = (a(i,j,n,s))_{i,j=1}^n, n \in \mathbb{N}$$

be a selfadjoint $n \times n$ matrix, whose entries

$$a(i,j,n,s), i,j = 1, \ldots, n$$

are measurable functions with the property that the family

$$\{\operatorname{Re} a(i,j,n,s) | 1 \leq i \leq j \leq n \ , s \in S\} \cup$$

$$\{\operatorname{Im} a(i,j,n,s) | 1 \leq i < j \leq n \ , s \in S\},$$

is an independent gaussian family, for each $n \in \mathbb{N}$ and

$$\mathscr{E}(a(i,j,n,s)) = 0,$$

$$\mathscr{E}(|a(i,j,n,s)|^2) = 1/n, \text{for all } s \in S, 1 \leq i,j \leq n.$$

Let $D(n)_{n \in \mathrm{N}}$ be a family of diagonal, $n \times n$ matrices. Assume that

$$\sup ||D(n)|| < \infty$$

and that the family $D(n)_{n \in \mathrm{N}}$ has a limit distribution with respect to the normalized traces on $n \times n$ matrices, when $n$ tends to infinity. The elements in the family $D(n)_{n \in \mathrm{N}}$ will be identified with constant functions on $\Sigma$ with values in $M_n(\mathbb{C})$.

Let $\tau_{\mathscr{E}_n}$ be the trace on $\bigcap_{p \geq 1} L^p(\Sigma) \otimes M_n(\mathbb{C})$ obtained as the composition of the normalized trace on $M_n(\mathbb{C})$ with $\mathscr{E}$. Let $(X^s)_{s \in S} \cup \{D\}$ be undetermined variables and let $\mathbb{C}\left[\left[(X^s)_{s \in S}, D\right]\right]$ be the noncommutative ring over $\mathbb{C}$ in the above variables.

Let $\phi$ be the linear functional on $\mathbb{C}\left[\left[(X^s)_{s \in S}, D\right]\right]$ defined by

$$\phi((X^{s_1})^{p_1}(D)^{i_1} \ldots (X^{s_k})^{p_k}(D)^{i_k}) =$$

$$= \lim{}_n \tau_{\mathscr{E}_n}(((Y(s_1,n))^{p_1}(D(n))^{i_1} \ldots (Y(s_k,n))^{p_k}(D(n))^{i_k}),$$

for all $s_j \in S, i_j \in \mathbb{N}, p_j \in \mathbb{N}, j = 1, \ldots, k$.

Under these assumptions Voiculescu's theorem 2.2 in [29] asserts that the above limit exists. Moreover, with respect to $\phi$, $(X^s)_{s \in S} \cup \{D\}$ is a free family and the distribution of each $X^s, s \in S$, is given by a semicircular law.

Following Voiculescu's proof or using [7], one may drop the assumption that the matrices in $((D(n))_{n \in \mathrm{N}}$ are diagonal. This means that we assume only that the matrices in $((D(n))_{n \in \mathrm{N}}$ have a limit distribution when $n$ tends to infinity, $D(n)$ are constant matrix functions on $\Sigma$ and the minimum of the dimensions of the minimal projections in the algebras generated by $D(n)$, tends to infinity with $n$. Under these weaker assumptions Voiculescu's theorem still holds true.

The random matrix picture([29]) for semicircular free families was essential (in [28]) in getting a new view of the free group factors. The main device in D. Voiculescu's approach to the structure of the von Neumann algebras of free groups was a new representation for a free semicircular family $(X_s)_{s\in S}$ by embedding $(X_s)_{s\in S}$ into $M_n(\mathbb{C})\otimes B$, where $B$ is a $W^*$-algebra containing an infinite free family.

Explicitly, following [28], Proposition 2.6, let

$$\omega_1 = \{g(i,j,s) | 1 \le i,j \le n, s \in S\}$$

be a free circular family in $B$, let

$$\omega_2 = \{f(i,s) | 1 \le n, s \in S\}$$

be a free semicircular family in $B$, so that $\omega_1 \cup \omega_2$ is free and let $1 \in D \subseteq B$ be a commutative subalgebra of $B$ that is free with respect to $\omega_1 \cup \omega_2$. Let $(e_{ij})_{i,j=1}^n$ be the canonical matrix unit in $M_n(\mathbb{C})$, let $D_n$ be the diagonal algebra generated by $\{e_{11}, \dots, e_{nn}\}$ and let

$$X_s = \sum_{1\le i<j\le n} (g(i,j,s)\otimes e_{ij} + (g(i,j,s)^* \otimes e_{ji} + \sum_{1\le i\le n} f(i,s)\otimes e_{ii}, s \in S.$$

Then

$$(X_s)_{s\in S} \subseteq M_n(\mathbb{C})\otimes B$$

is a free semicircular family that is also free with respect to the algebra $D_n \otimes D$.

Using this, D. Voiculescu proved that there exists an infinite free semicircular family which generates the reduced algebra

$$(1\otimes e_{11})\{(X_s)_{s\in S}\}''(1\otimes e_{11}).$$

This shows that

$$\mathcal{L}(F_k)_{(1/N)} \cong \mathcal{L}(F_{kN^2-N^2+1}),$$

and hence that $\mathbb{Q} \subseteq \mathcal{F}(\mathcal{L}(F_\infty))$ ([28], Theorem 3.2).

## 0. Outline of the proof

In the first paragraph we prove general results about type $II_1$ factors $(Q\otimes B) *_B A$ and their subfactors. Let $\tau$ be a normalized faithful trace on $A$ and let $\mathcal{C}$ be an extremal commuting square

$$\mathcal{C} = (A \supseteq B \supseteq D;\ A \supseteq B \supseteq D)$$

of finite dimensional algebras. Recall that $\mathcal{C}$ is a commuting square ([20]) if the square whose arrows are the conditional expectations (with respect to $\tau$) between the algebras $A, B, C, D$, is commutative.

The extremality condition [21] means that $\mathcal{C}$ is non degenerate (i.e. $A = \mathrm{Sp}(BC) = \mathrm{Sp}(CD)$) and irreducible (i.e. the centers of $A, B$ and respectively $C, D$ have trivial intersection). Let $\Gamma$ be the inclusion matrix for $B \subseteq A$ and let $\lambda$ be $\|\Gamma^t\Gamma\|^{-1}$.

By [21] there exists a projection $f$ so that $A \subseteq <A, f>$ (and $f$) is the Jones' basic construction for $B \subseteq A$. Here $<A, f>$ is the (finite dimensional) algebra generated by $A$ and $f$. The fact that $A \subseteq <A, f>$ is the Jones' basic construction implies (by [11]) that $<A, f> = \mathrm{Sp}(AfA)$ and that there exists a trace (also denoted by $\tau$) extending the trace on $A$ and so that the conditional expectation $E_A$ from $<A, f>$ on $A$ (with respect to $\tau$) has the properties $E_A(f) = \lambda f$ and $fxf = E_A(x)f = fE_A(x)$ for all $x \in <A, f>$. Note that in this case $\lambda = \tau(f)$.

In addition, the extremality condition for $\mathscr{C}$ implies that $B \subseteq <B, f>$ is the Jones' basic construction for $D \subseteq B$ ($<B, f> = BfB$ is a subalgebra of $<A, f>$).

Such commuting squares appear, for instance, as relative commutants in the iterated basic construction associated to a finite depth inclusion of hyperfinite type $II_1$ factors.

For example if $R_\lambda \subseteq R$ is the Jones' inclusion corresponding to

$$\lambda^{-1} \in \{4\cos^2(\pi/n) | n \geq 3\},$$

$(e_1, e_2, \ldots,)$ are the corresponding Jones' projections ([11]) and $A_\lambda, B_\lambda, C_\lambda, D_\lambda$ are defined by

$$A_\lambda = A^1_{n+1} = \{e_1, e_2, \ldots, e_{n+1}\}'', \quad B_\lambda = A^2_{n+1} = \{e_2, \ldots, e_{n+1}\}''$$

$$C_\lambda = A^1_n = \{e_1, e_2, \ldots, e_n\}'', \quad D_\lambda = A^2_n = \{e_2, \ldots, e_n\}''$$

then

$$\mathscr{C}_\lambda = (A_\lambda \supseteq C_\lambda \supseteq D_\lambda;\ (A_\lambda \supseteq B_\lambda \supseteq D_\lambda)$$

is an extremal commuting square One may take here $f = e_{n+2}$.

The main result of the first paragraph is that

$$N^{\mathscr{C}}(Q) = (Q \otimes D) *_D C \subseteq M^{\mathscr{C}}(Q) = (Q \otimes B) *_B A.$$

is an inclusion of type $II_1$ factors of index

$$\left[M^{\mathscr{C}}(Q) : N^{\mathscr{C}}(Q)\right] = \lambda^{-1} = \|\Gamma^t \Gamma\|$$

and relative commutant $B \cap C'$ (in this paper, except when otherwise mentioned, by amalgamated free product for finite von Neumann algebras, we will always understand the reduced, von Neumann algebra, amalgamated free product).

The idea to prove this is to show that

$$M_1^{\mathscr{C}}(Q) = (Q \otimes \langle B, f\rangle) *_{\langle B, f\rangle} \langle A, f\rangle \text{ and } f$$

is the basic construction for $N^{\mathscr{C}}(Q) \subseteq M^{\mathscr{C}}(Q)$. To show the last equality we note that $A = \mathrm{Sp}(BC) = \mathrm{Sp}(CB)$, $[B, Q] = 0$ and thus

$$M^{\mathscr{C}}(Q) = \text{weak closure} \cup \mathrm{Sp}\{AQA \ldots AQA | n \text{ factors in } Q\}$$

$$= \text{weak closure } \cup \mathrm{Sp}\{CQC \ldots CQA | n \text{ factors in } Q\}.$$

A similar formula holds true for $M_1^{\mathscr{C}}(Q)$ and this in turn shows that

$$M_1^{\mathscr{C}}(Q) = M^{\mathscr{C}}(Q) f M^{\mathscr{C}}(Q),$$

$$f M^{\mathscr{C}}(Q) f \subseteq M^{\mathscr{C}}(Q) f,$$

$$E_{M^{\mathscr{C}}(Q)}(f) = \lambda = \|\Gamma^t\Gamma\|^{-1} = \tau(f).$$

By Lemma 1.5.2 in [22], these conditions are sufficient to show that the index of $M^{\mathscr{C}}(Q)$ in $M_1^{\mathscr{C}}(Q)$ is $\lambda^{-1}$.

In the next two paragraphs we will prove that $(\mathscr{L}(F_\infty) \otimes B) *_B A$ is isomorphic to $\mathscr{L}(F_\infty)$ if $B \subseteq A$ is an irreducible inclusion of finite dimensional algebras. We will do this in two steps. Clearly, since (by [23]) $h\mathscr{L}(F_\infty)h \cong \mathscr{L}(F_\infty)$ for all projections $h$ in $\mathscr{L}(F_\infty)$, it is sufficient to show that for a suitable projection $g = fe$ in $(\mathscr{L}(F_\infty) \otimes B) *_B A$ we have that

$$\left[(\mathscr{L}(F_\infty) \otimes B) *_B A\right]_g \cong \mathscr{L}(F_\infty).$$

We choose a maximal system $(f_l)_{l\in L}$ of nonequivalent minimal projections in $B$ and let $f = \Sigma f_l$. Since $B_f$ is abelian and since

$$[(Q \otimes B) *_B A]_f \cong (Q \otimes B_f) *_{B_f} A_f$$

it follows that we may assume (to establish the isomorphism) that $B$ is abelian and $B = \{f_l\}''$.

We now use the techniques from the papers of D. Voiculescu. We construct a trace preserving isomorphism from $(\mathscr{L}(F_\infty) \otimes B) *_B A$ onto a von Neumann subalgebra of the reduced free product $\mathscr{L}(F_\infty) * A$. This isomorphism will allow us to transfer the random matrix representation ([28]) of $\mathscr{L}(F_\infty) * A$ to the amalgamated free product.

The isomorphism is constructed as follows: Let $(Y^s)_{s\in S}$ be a free semicircular family that is free with respect to $A$. Clearly,

$$\mathscr{L}(F_\infty) * A \cong \{(Y^s)_{s\in S}, A\}''.$$

Let $(X^s)_{s\in S}$ be the family defined by:

$$X^s = \sum_l \tau(f_l)^{-1/2} f_l Y^s f_l \text{ for all } s \in S.$$

The $(X^s)_{s\in S}$ is a free semicircular family (that commutes with $B$) and the von Neumann algebra generated by $(X^s)_{s\in S}$ and $A$ is isomorphic to $(\mathscr{L}(F_\infty) \otimes B) *_B A$. Thus $(\mathscr{L}(F_\infty) \otimes B) *_B A$ is identified with a subfactor of $\mathscr{L}(F_\infty) * A$.

Let $(e_k)_{k\in K}$ be a maximal family of mutually orthogonal, nonequivalent, minimal projections in $A$, that commute with $(f_l)_{l\in L}$ and let $e = \sum e_i$. We will show that $e\left[(\mathscr{L}(F_\infty) \otimes B) *_B A\right]e$ is isomorphic to $\mathscr{L}(F_\infty)$. We outline here the proof of this statement.

Let $r_k$ be the central support of $e_k$ in $A$, for all $k \in K$ and let $\{e^k_{pq}\}_{pq=1}^{t_k}$ be a matrix unit for $Ar_k$ so that $e^k_{pp}$ commutes with $(f_l)_{l\in L}$ and $e^k_{11} = e_k$, for all $p = 1, \ldots, m_k$, $k \in K$ ($m_k$ is the dimension of $Ar_k$). Let $\Gamma = (a_{ij})_{i\in K, j\in L}$ be the inclusion matrix of $A \supseteq B$ and let $N$ be the reflexive, symmetric relation on $K^2$ defined by

$$N = \{(k, m) \in K \times K | \sum_{l\in L} a_{kl} a_{ml} \cdot \neq 0\}$$

Since the centers of $A$ and $B$ have trivial intersection, $N$ contains a single orbit.

With this notations, a system of generators for $e\left[(\mathscr{L}(F_\infty) \otimes B) *_B A\right]e$ is $A_e$ and the set

$$(0.1)\qquad \mathscr{X}' = \bigcup_{(m,k)\in N} \{e^k_{1p} Y^s e^m_{q1} | s \in S, p = 1,\ldots,t_k, q = 1,\ldots,t_m\}.$$

The random matrix picture of Voiculescu allows us to permute the matrix blocks in $\mathscr{X}'$ (we act as if the elements in $\mathscr{X}'$ were matrix blocks whose entries are in an independent gaussian family on a probability space). Thus a more convenient way to express the generators for the algebra $\left[(\mathscr{L}(F_\infty)\otimes B) *_B A\right]_e$ is given by the following procedure:

We start with an infinite semicircular family $(Z^s)_{s\in S}$ that is free with respect to $A_e$ and a symmetric reflexive relation $N$ on $K$ which contains a single orbit (the last condition is the factoriality condition for $e((\mathscr{L}(F_\infty)\otimes B) *_B A)e)$. The algebra $e((\mathscr{L}(F_\infty)\otimes B) *_B A)e$ is generated by $(X^s)_{s\in S}$ and $(e_k)_{k\in K}$, where $(X^s)_{s\in S}$ is now defined by

$$X^t = \sum_{(k,m)\in N} e_k Z^t e_m, t \in S.$$

One could see, by the above picture and by the definition of $\mathscr{L}(F_r), r > 1$, that $e((\mathscr{L}(F_\infty)\otimes B) *_B A)e$ is isomorphic to $\mathscr{L}(F_\infty)$. Instead of doing that, we will present (to make the task easier for the reader) in the third paragraph a direct proof of the fact that an algebra with such a system of generators is $\mathscr{L}(F_\infty)$.

The proof will consist of considering a reduced algebra of

$$((\mathscr{L}(F_\infty)\otimes B) *_B A)_e$$

by a projection $g$. This time the system of generators for the reduced algebra are obtained from an infinite free semicircular family by deleting blocks from "half" of the elements in the family. The fact that the family is infinite ( by the random matrix picture for semicircular families) makes it possible to fill the corresponding holes by pieces from the other "half" of the elements. This shows that there exists another infinite semicircular family which generates the reduced algebra $\left[(\mathscr{L}(F_\infty)\otimes B) *_B A\right]_{eg}$.

In the fourth paragraph we define a real continuation for the sequence $\mathscr{L}(F_N)_{N\geq 2, N\in\mathbb{N}}$ of the type $II_1$ factors associated to free groups. Let $(X^s)_{s\in S}$ be a free semicircular family, let $e_s, f_s$ be projections in $\{X^\sigma\}''$ which are either mutually orthogonal or equal and let

$$r = 1 + \sum k_s \tau(e_s)\tau(f_s),$$

where the above factor $k_s$ is 1 if $e_s = f_s$ and 2 elsewhere.

For real $r > 1$ we define $\mathscr{L}(F_r)$ to be the isomorphism class of

$$\mathscr{A} = \{X^\sigma, (e_s X^s f_s)_{s\in S/\{\sigma\}}\}'',$$

if the above projections $e_s, f_s$ are chosen with the additional property that the finite von Neumann algebra $\mathscr{A}$ is a factor.

The correctness of this definition and the fact that indeed, when $r \in \mathbb{N}$, we recover in this way the factors associated to free groups relies on elementary properties of free semicircular free families.

The properties (0.2), (0.3) are simple consequences of the definition. As an application of our definition we will prove that either all $\mathscr{L}(F_r)$ are isomorphic for $r \in (1,\infty]$ or that they are mutually nonisomorphic.

To do that assume that for some $s > 2$ we have $\mathcal{L}(F_2) \cong \mathcal{L}(F_s)$. An immediate consequence would be (by Eq. (0.3)) that $\mathcal{L}(F_2) \cong \mathcal{L}(F_s)$ for any finite $s > 2$. Let $\mathcal{L}(F_2)$ be represented as

$$\{X^{\sigma_0}, (p_i X^{\nu_i} p_i), (p_i X^{\sigma_i} p_i), i \in \mathbb{N}\}''.$$

We use our assumption to replace for each $i$ the generators

$$(p_i X^{\nu_i} p_i), (p_i X^{\sigma_i} p_i)$$

by a semicircular family $(Z^s)_{s \in S_i}$ . If card $S_i = N_i$ is big enough so that

$$\sum \tau(p_i)^2 N_i = \infty$$

then we would get that $\mathcal{L}(F_2)$ is isomorphic to $\mathcal{L}(F_\infty)$.

In the last paragraph we analyse the isomorphism class of the algebra $(\mathcal{L}(F_N) \otimes B) *_B A$. Here $1_A \in B \subseteq A$ is an irreducible inclusion of finite dimensional algebras and we fix a normalized faithful trace $\tau$ on $A$. Let $\Gamma = (a_{ij})_{i \in K, j \in L}$ be the inclusion matrix of $A \supseteq B$ and let $(s_k)_{k \in K}$ be the vector of the values of the trace $\tau$ over a system of representatives for the minimal projections in $A$.

Clearly the definition of

$$(\mathcal{L}(F_r))_{r \in \mathbb{R}, r > 1}$$

and the procedure that we used to show that

$$(\mathcal{L}(F_\infty) \otimes B) *_B A \cong \mathcal{L}(F_\infty)$$

reduces the analysis of the isomorphism class of the algebra $(\mathcal{L}(F_N) \otimes B) *_B A$ to counting the elements corresponding to $\mathcal{L}(F_N)$ in the set $\mathcal{X}$ defined by Eq. (0.1).

By this method we prove that $(\mathcal{L}(F_N) \otimes B) *_B A$ is isomorphic to $\mathcal{L}(F_M)$ if

$$M = 1 + N < (\Gamma\Gamma^t)s, s > - < s, s > .$$

Let $\mathcal{C} = (A \supseteq C \supseteq D;\ A \supseteq B \supseteq D)$ be a commuting square as in the first paragraph and let $(s_k)_{k \in K}, (t_l)_{l \in L}$ be the vectors of the values of the trace $\tau$ over a system of representatives for the minimal projections in $A$ and $C$ respectively. By the assumptions on $\mathcal{C}$ (i.e. the extremality condition in Definition 1.1) it follows that $< s, s >= \lambda < t, t >$. Let $\Gamma_1$ be the inclusion matrix for $D \subseteq C$.

For each $P > 1$, by the above formula, we obtain that

$$(\mathcal{L}(F_P) \otimes B) *_B A \cong \mathcal{L}(F_N)$$

if

$$N - 1 = P < (\Gamma\Gamma^t)s, s > - < s, s > .$$

The same argument as above shows that

$$\mathcal{A} = (\mathcal{L}(F_P) \otimes D) *_D C \cong \mathcal{L}(F_M),$$

if

$$M - 1 = P < (\Gamma_1\Gamma_1^t)t, t > - < t, t > .$$

Since $s, t$ are Perron -eigenvectors (i.e. $(\Gamma^t\Gamma)t = \lambda^{-1}t, (\Gamma_1^t\Gamma_1)s = \lambda^{-1}s$) it follows that

$$(M - 1)/(N - 1) = \lambda^{-1}$$

or, equivalently, that

$$M = (N-1)\lambda^{-1} + 1 = (N-1)||\Gamma^t\Gamma|| + 1.$$

Thus, by the first paragraph, $\mathcal{L}(F_N)$ has a subfactor $\mathcal{L}(F_M)$ of index $\lambda^{-1}$.

## 1. Subfactors in amalgamated free products over finite dimensional algebras

In this section we introduce a series of inclusions

$$N^{\mathcal{C}}(Q) \subseteq M^{\mathcal{C}}(Q),$$

that are canonically associated to a type $II_1$ factor $Q$ and a commuting square ([20])

$$\mathcal{C} = (A \supseteq B \supseteq D,\ A \supseteq C \supseteq D)$$

of finite dimensional algebras. The commuting square is assumed to have certain supplementary properties, as for the commuting squares coming from finite depth inclusions ([21], [17], [11]).

These series of inclusions may be viewed, by specialization to the case when $\mathcal{C}$ is the commuting square associated to a Jones' pair $R_\lambda \subseteq R$, for $\lambda^{-1} = s$ in the discrete series

$$\{4\cos^2 \pi/n \mid n \geq 3\},$$

as an extension of the series of inclusions $N^s(Q) \subseteq M^s(Q)$ introduced by S. Popa in [22].

In this case, the relation between the two series is that $N^{\mathcal{C}}(Q) \subseteq M^{\mathcal{C}}(Q)$ may be obtained from the inclusion $N^s(Q) \subseteq M^s(Q)$ in [22] by iteration of the basic construction.

The properties of the commuting squares $\mathcal{C}$ that we are working with are summarized in the next definition.

**Definition 1.1** *([20],[21]) Let $A \supseteq B \supseteq D$; $A \supseteq C \supseteq D$ be a commuting square of finite von Neumann algebras which are weakly separable. Let $\tau$ be a normalized, faithful, trace on $A$. The commuting square is extremal (for the value $\lambda^{-1}$) if the following conditions hold:*

*(i) Markov trace. There exists a trace $\tilde{\tau}$ on the basic construction*

$$\langle A, f\rangle = \text{weak closure } \operatorname{Sp} AfA$$

*for the inclusion $C \subseteq A$, that extends the trace $\tau$ on $A$ and so that $\tau(fa) = \lambda\tau(a)$ for all $a \in A$. Such a trace is called a $\lambda$-Markov trace ([31],[11]) for $C \subseteq A$. Moreover we assume that $\langle f, B\rangle''$ is the basic construction for $D \subseteq B$ with respect to the trace $\tilde{\tau}|_{<B,f>}$.*

*(ii) Nondegeneracy. We assume that*

$$A = \text{weak closure } \operatorname{Sp}(BC) = \text{weak closure } \operatorname{Sp}(CB).$$

*(iii) Irreducibility. We assume that the centers of the algebras $A, B$ and $C, D$ have trivial intersection: $\mathcal{Z}(A) \cap \mathcal{Z}(B) = \mathbb{C}1$ and $\mathcal{Z}(C) \cap \mathcal{Z}(D) = \mathbb{C}1$.*

Some of the above conditions may be redundant. In particular if $A$ is finite dimensional then the second condition is equivalent to the first ([21], Theorem 1.4). In addition

$$(\langle A, f\rangle \supseteq \langle B, f\rangle \supseteq B,\ \langle A, f\rangle \supseteq A \supseteq B)$$

is again an extremal commuting square.

Note that such commuting squares naturally appear in the context of finite depth inclusion of hyperfinite $II_1$ factors ([17], [12], [31]). In fact ([17],[20]) any inclusion of finite depth of hyperfinite $II_1$ factors may be obtained from an extremal commuting square of finite dimensional algebras, by iteration of the Jones' basic construction.

Starting with an extremal commuting square $\mathcal{C} = (A \supseteq B \supseteq D;\ A \supseteq C \supseteq D)$ and a type $II_1$ factor $Q$ we introduce the following inclusion of type $II_1$ factors:

$$N^{\mathcal{C}}(Q) = (Q \otimes D) *_D C \subseteq (Q \otimes B) *_B A = M^{\mathcal{C}}(Q).$$

Note that the trace $\tau$ on $A$ induces a trace $\tau_1$ on $Q \otimes B$ by taking the tensor product of $\tau|_B$ with the normalized trace on the type $II_1$ factor $Q$. Using the natural identification of $1_Q \otimes B$ with $B$, we clearly have that $\tau_1|_B = \tau|_B$. Hence, by the definition in [22], we may construct the von Neumann algebra, amalgamated free product $(Q \otimes B) *_B A$. The same is true for the trace $\tau|_C$ and hence we may also construct the von Neumann algebra, amalgamated free product $(Q \otimes D) *_D C$.

By the commuting square property, the trace on the algebraic amalgamated free product

$$(Q \otimes B) *_B A = M^{\mathcal{C}}(Q)$$

restricts to the trace on the algebraic free product

$$(Q \otimes D) *_D C = N^{\mathcal{C}}(Q).$$

Hence (in the corresponding G.N.S. representation) we obtain indeed an inclusion of type $II_1$ factors.

Note that $(Q \otimes B) *_B A$ is a factor since (by lemma 1.4.1 in [22])

$$((Q \otimes B) *_B A)' \cap ((Q \otimes B) *_B A) =$$

$$Q' \cap ((Q \otimes B) *_B A) \cap A' =$$

$$B \cap A' = \mathcal{Z}(A) \cap B = \mathcal{Z}(A) \cap \mathcal{Z}(B) = \mathbb{C}1.$$

With this definitions we have:

**Theorem 1.2** *Let $Q$ be a weakly separable type $II_1$ factor and let*

$$\mathcal{C} = (A \supseteq C \supseteq D;\ A \supseteq B \supseteq D)$$

*be an extremal commuting square (for the value $\lambda$) of finite, countably generated, von Neumann algebras. Let $\langle A, f\rangle$ be the basic construction for $A \supseteq C$ (so that $\langle B, f\rangle$ is the basic construction for $B \supseteq D$). Consider the following inclusion of type $II_1$ factors:*

$$N^{\mathcal{C}}(Q) = (Q \otimes D) *_D C \subseteq M^{\mathcal{C}}(Q) = (Q \otimes B) *_B A.$$

*Then*

$$M_1^{\mathcal{C}}(Q) = (Q \otimes \langle B, f\rangle) *_{\langle B, f\rangle} \langle A, f\rangle \text{ and } f$$

*is the basic construction for the inclusion $N^{\mathcal{C}}(Q) \subseteq M^{\mathcal{C}}(Q)$. In particular $N^{\mathcal{C}}(Q) \subseteq M^{\mathcal{C}}(Q)$ is an inclusion of type $II_1$ factors of index*

$$\left[M^{\mathcal{C}}(Q) : N^{\mathcal{C}}(Q)\right] = \lambda^{-1}.$$

*Moreover the relative commutant of $N^{\mathcal{C}}(Q)$ in $M^{\mathcal{C}}(Q)$ is $B \cap C'$.*

*Proof*. We will verify this by checking the conditions in lemma 1.5.1 from [22]. We have to prove that

$$M_1^{\mathcal{C}}(Q) = \text{weak closure } \left[\text{Sp}(M^{\mathcal{C}}(Q) f M^{\mathcal{C}}(Q))\right],$$

$$f M^{\mathcal{C}}(Q) f \subseteq M^{\mathcal{C}}(Q) f,$$
$$E_{M^{\mathcal{C}}(Q)}(f) = \lambda = \tau(f).$$

This conditions are sufficient to show that the index of $M^{\mathcal{C}}(Q)$ in $M_1^{\mathcal{C}}(Q)$ is $\lambda^{-1}$. To prove that the inclusion $M^{\mathcal{C}}(Q) \subseteq M_1^{\mathcal{C}}(Q)$ (together with $f$) is the Jones' basic construction for the inclusion $N^{\mathcal{C}}(Q) \subseteq M^{\mathcal{C}}(Q)$ we will have to check that also

$$\{f\}' \cap M^{\mathcal{C}}(Q) = N^{\mathcal{C}}(Q).$$

Since

$$A = \text{weak closure } \left[\text{Sp}(BC)\right] = \text{weak closure } \left[\text{Sp}(CB)\right]$$

and since $[B, Q] = 0$ it follows that

$$M^{\mathcal{C}}(Q) = \text{weak closure } \cup_n \text{ Sp}\{AQA \ldots AQA | n \text{ factors in } Q\}$$
$$= \text{weak closure } \cup_n \text{ Sp}\{CBQA \ldots AQA | n \text{ factors in } Q\}$$
$$= \text{weak closure } \cup_n \text{ Sp}\{CQBA \ldots AQA | n \text{ factors in } Q\}$$
$$= \text{weak closure } \cup_n \text{ Sp}\{CQA \ldots AQA | n \text{ factors in } Q\}$$
$$= \text{weak closure } \cup_n \text{ Sp}\{CQCB \ldots AQA | n \text{ factors in } Q\}$$
$$= \text{weak closure } \cup_n \text{ Sp}\{CQC \ldots CQA | n \text{ factors in } Q\}.$$

Thus

(1.1). $$M^{\mathcal{C}}(Q) = \text{weak closure } \cup_n \text{ Sp}\{CQC \ldots CQA | n \text{ factors in } Q\}$$

Similarly one may prove that

$$M_1^{\mathcal{C}}(Q) = \text{weak closure } \cup_n \text{ Sp}\{AQA \ldots AQ < A, f > | n \text{ factors in } Q\}.$$

Since $< A. f >=$ weak closure $\left[\text{Sp}(AfA)\right]$ it follows that

$$M_1^{\mathcal{C}}(Q) = \text{weak closure } \cup_n \text{ Sp}\{AQA \ldots AQAfA | n \text{ factors in } Q\}$$

and henceforth $M_1^{\mathcal{C}}(Q) =$ weak closure $\left[\text{Sp}(M^{\mathcal{C}}(Q) f M^{\mathcal{C}}(Q))\right]$.

Moreover by Eq. (1.1) and since $f$ commutes with $Q$ and $C$, while $fBf \subseteq Df$ and $A =$ weak closure $\left[\text{Sp}(CB)\right]$ it follows that

$$f M^{\mathcal{C}}(Q) f \subseteq \text{weak closure } \cup_n \text{ Sp}\{f(CQC \ldots CQCB)f\} | n \text{ factors in } Q\} =$$

$$= \text{weak closure } \cup_n \text{ Sp}\{CQC \ldots CQCfBf\} | n \text{ factors in } Q\} =$$

$$= \text{weak closure } \cup_n \text{ Sp}\{CQC\ldots CQCDf\}|n \text{ factors in } Q\} =$$
$$= [(Q\otimes D) *_D C]f = f(Q\otimes D *_D C)f.$$

Thus $fM^{\mathcal{C}}(Q)f \subseteq N^{\mathcal{C}}(Q)f \subseteq M^{\mathcal{C}}(Q)$. Finally we have to show that

$$E_{M^{\mathcal{C}}(Q)}(f) = \lambda.$$

This is equivalent to show that $\tau(f-\lambda)x) = 0$ for any $x \in M^{\mathcal{C}}(Q)$. Using Eq. (1.1) it is thus sufficient to prove that

$$\tau((f-\lambda)a_1q_1\ldots a_nq_na_{n+1}) = 0, \text{for any } a_i \in A, q_i \in Q.$$

Recall that the trace $\tau$ on

$$(Q\otimes < B, f >)*_{<B,f>} < A, f >$$

is defined by the requirement

(1.2)
$$\tau(a_0'q_1'a_1'\ldots q_n'a_{n+1}') = 0$$

for all $a_i' \in < A, f >, q_i' \in Q$ with $E_{<B,f>}(a_i') = 0, \tau(q_i') = 0$.

But $\tau(f-\lambda)a_1q_1\ldots a_nq_na_{n+1}$ is a sum of elements as in Eq. (1.2) plus terms of the form $\tau(f-\lambda)b$. The later terms are vanishing also since $E_B(f) = \lambda$ (because $< f, B >$ is the basic construction for $B \supseteq D$).

Finally it remains to prove that

(1.3)
$$\{f\}' \cap M^{\mathcal{C}}(Q) = N^{\mathcal{C}}(Q)$$

Since $f$ commutes with $Q$ and $C$ it is clear that

$$N^{\mathcal{C}}(Q) \subseteq \{f\}' \cap M^{\mathcal{C}}(Q).$$

The reverse inclusion is a consequence of the fact:

$$fM^{\mathcal{C}}(Q)f \subseteq N^{\mathcal{C}}(Q)f = fN^{\mathcal{C}}(Q),$$

that we already proved. Because of Eq. (1.3), and by using Lemma. 1.5.2 in [22], we obtain that

$$N_1 = \{f\}' \cap M^{\mathcal{C}}(Q)$$

is a type $II_1$ factor and that

$$fM^{\mathcal{C}}(Q)f = N_1f = fN_1.$$

Thus $N^{\mathcal{C}}(Q)f = N_1f$ and thus

$$N^{\mathcal{C}}(Q) = N_1 = \{f\}' \cap M^{\mathcal{C}}(Q)$$

as $f$ commutes with both $N_1, N^{\mathcal{C}}(Q)$.

To end the proof we also have to compute the relative commutant. We have (again by lemma 1.4.1 in [22])

$$[(Q\otimes D) *_D C]' \cap [(Q\otimes B) *_B A] = Q' \cap C' \cap [(Q\otimes B) *_B A] = B \cap C'.$$

This completes the proof of the theorem.

The following statement is due to S.Popa. It concerns the computation of the higher relative commutants of the above irreducible inclusion.

**Theorem 1.3** *(S. Popa). The higher relative commutants of the inclusion $N^{\mathcal{C}}(Q) \subseteq M^{\mathcal{C}}(Q)$ are isomorphic with the corresponding higher relative commutants of the inclusion $Q_\infty \subseteq P_\infty$, where $Q_\infty \subseteq P_\infty$ is obtained from the extremal commuting square $\mathcal{C}$ by iterating the basic construction. Equivalently the paragroup ([17] )*

$$\mathcal{G}(N^{\mathcal{C}}(Q) \subseteq M^{\mathcal{C}}(Q))$$

*coincides with the paragroup $\mathcal{G}(P_\infty \subseteq Q_\infty)$.*

*Moreover, the canonical representation of $B \subseteq A$ in $N^{\mathcal{C}}(Q) \subseteq M^{\mathcal{C}}(Q)$ can be extended to a representation of $Q_\infty \subseteq P_\infty$ such that the higher relative commutants of $N^{\mathcal{C}}(Q) \subseteq M^{\mathcal{C}}(Q)$ and $Q_\infty \subseteq P_\infty$ coincide.*

*Proof* . Let $A_n, B_n$ be the iterated steps of the Jones' basic construction for $C \subseteq A$ and respectively $D \subseteq B$, and let $f_n \in B_n, n \in \mathbb{N}$ be the corresponding projection ( with $f_1 = f$, $A_1 = < A, f >, B_1 = < B, f >$).

By Lemma 1.4.1 in [22], we have that

$$((Q \otimes B) *_B A)' \cap ((Q \otimes B_n) *_{B_n} A_n) =$$

$$Q' \cap ((Q \otimes B_n) *_{B_n} A_n) \cap A' = B_n \cap A'.$$

Similarly

$$((Q \otimes D) *_D C)' \cap ((Q \otimes B_n) *_{B_n} A_n) = B_n \cap C'.$$

Note that by the preceding lemma the $n-$th step of th basic construction of $N^{\mathcal{C}}(Q) \subseteq M^{\mathcal{C}}(Q)$ is

$$M_n^{\mathcal{C}}(Q) = (Q \otimes B_n) *_{B_n} A_n.$$

By [20], [21], [17], it follows that $P_\infty \subseteq Q_\infty$ has the same higher relative commutants as

$$N^{\mathcal{C}}(Q) \subseteq M^{\mathcal{C}}(Q).$$

Indeed one may check this as follows : Let $P_n$, respectively $Q_n$ be the $n-$th step of the Jones' tower of the inclusion $B \subseteq A$ and respectively $D \subseteq C$ with the convention that $P_0 = B, P_1 = A$ and $Q_0 = D, Q_1 = C$. Clearly we have the inclusion

$$P_0 \cap Q_1' = P_0 \cap Q_\infty' \subseteq P_\infty \cap Q_\infty',$$

as $Q_n$ is obtained from $Q_{n-1}$ by adding a projection that commutes with $Q_{n-2}$, $n \geq 2$.

Conversely, if $x \in P_\infty \cap Q_\infty'$, then $x$ is close to $x' \in Q_n' \cap P_n$ and to $x'' \in Q_{n+1}' \cap P_{n+1}$, for some large $n$. But as $n$ tends to infinity the angle between the finite dimensional vector spaces $Q_n' \cap P_n$, $Q_{n+1}' \cap P_{n+1}$ becomes stationary. Since $||x' - x''||_2$ is small it follows that there exists

$$x''' \in (Q_{n+1}' \cap P_{n+1}) \cap (Q_n' \cap P_n)$$

close to both $x', x''$, thus close to $x$. But $Q_{n+1}' \cap P_n = Q_1' \cap P_0$, so that $x$ is arbitrary close to an element in $Q_\infty \cap P_0$, i.e. $P_\infty \cap Q_\infty' \subseteq Q_\infty \cap P_0$.

The rest is trivial by our construction and [22]. This ends the proof.

## 2. The random matrix model for amalgamated free products

In this paragraph we introduce a random matrix model for the amalgamated free products $(\mathcal{L}(F_\infty) \otimes B) *_B A$, where $A \supseteq B$ are finite dimensional algebras. Let $\tau$ be a faithful, normalized trace on $A$.

First, we describe this model for commutative $B$; the general case is then handled by showing that a similar model is valid for a reduced algebra of $(\mathcal{L}(F_\infty)\otimes B)*_B A$.

For abelian $B$ we will construct an isomorphism from $(\mathcal{L}(F_\infty)\otimes B)*_B A$ onto a von Neumann subalgebra of the reduced free product $\mathcal{L}(F_\infty) * A$. This isomorphism will allow us to transfer the random matrix representation for $\mathcal{L}(F_\infty) * A$ ([28]) to the amalgamated free product.

We prove the above statement by showing that for a suitable system of generators $(Y^s)_{s\in S}$ for $\mathcal{L}(F_\infty)$ which is identified with a subalgebra in $\mathcal{L}(F_\infty) * A$, the conditional expectations $(X^s)_{s\in S}$ of $(Y^s)_{s\in S}$ onto the relative commutant of $B$ in $\mathcal{L}(F_\infty) * A$, have the property that $(X^s)_{s\in S}$ and $A$ generate a copy of $(\mathcal{L}(F_\infty) \otimes B) *_B A$ in $\mathcal{L}(F_\infty) * A$.

**Proposition 2.1** *Let $1 \in B \subseteq A$ be finite dimensional algebras. Let $\tau$ be a faithful, normalized trace on $A$. Assume that $B$ is abelian. Then there exist a (natural) trace preserving embedding of $(\mathcal{L}(F_\infty) \otimes B) *_B A$ into $\mathcal{L}(F_\infty) * A$. This embedding is realized explicitly as follows:*

*Let $(Y^s)_{s\in S}$ be an infinite semicircular family, let $\mathcal{B}$ be the free product*

$$\{(Y^s)_{s\in S}\}'' * A \cong \mathcal{L}(F_\infty) * A$$

*and let $(p_i)_{i=1}^{l}$ be the minimal projections in $B$. Define*

$$X^s = \sum_i \tau(p_i)^{-1/2} p_i Y^s p_i \text{ for all } s \in S.$$

*Then $(X^s)_{s\in S}$ is a free semicircular family (hence $\{(X^s)_{s\in S}\}'' \cong \mathcal{L}(F_\infty)$) and $\{(X^s)_{s\in S} \cup A\}'' \subseteq \mathcal{B}$ is isomorphic to the algebra*

$$\left[\{(X^s)_{s\in S}\}'' \otimes B\right] *_B A \cong (\mathcal{L}(F_\infty) \otimes B) *_B A.$$

*Remark* . Clearly the above statement is still valid under the weaker assumption that $A, B$ are finite type $I$ von Neumann algebras with discrete centers. This may easily be seen if one follows the lines of the proof of Proposition 2.1.

*Proof (of Proposition* 2.1). Recall that by theorem 2.3 in [28] we have that

$$\{\tau(p_k)^{-1/2}(p_k Y^s p_k)\}_{s\in S}$$

is a free semicircular family with respect to the induced trace $\tau_{p_k} = \tau(p_k)^{-1}\tau$ on $\mathcal{B}_{p_k}$, for each $k$. In particular

$$\tau(p_i(X^{s_1})^{k_1} \dots (X^{s_n})^{k_n}) =$$

$$\tau(p_i)\tau_{p_i}(\left[(p_i Y^{s_1} p_i)\tau(p_i)^{-1/2}\right]^{k_1} \dots \left[(p_i Y^{s_n} p_i)\tau(p_i)^{-1/2}\right]^{k_n}) =$$

(2.1)
$$\tau(p_i)\tau((Y^{s_1})^{k_1}\dots(Y^{s_n})^{k_n}).$$

Thus
$$\tau((X^{s_1})^{k_1}\dots(X^{s_n})^{k_n}) = \sum_i \tau(p_i(X^{s_1})^{k_1}\dots(X^{s_n})^{k_n})$$
$$= \sum \tau(p_i)\tau((Y^{s_1})^{k_1}\dots(Y^{s_n})^{k_n}) = \tau((Y^{s_1})^{k_1}\dots(Y^{s_n})^{k_n}).$$

In particular $(X^s)_{s\in S}$ is free and semicircular and by Eq. (2.1) we get
$$\tau(p_i(X^{s_1})^{k_1}\dots(X^{s_n})^{k_n}) =$$
$$\tau(p_i)\tau((Y^{s_1})^{k_1}\dots(Y^{s_n})^{k_n}) = \tau(p_i)\tau((X^{s_1})^{k_1}\dots(X^{s_n})^{k_n}).$$
Thus $\tau(p_i x) = \tau(p_i)\tau(x)$ for all $x \in (\{X^s\}_{s\in S})''$, $i = 1,..,l$.

Using the definition of the trace on the amalgamated free products ([22]) it follows that to complete the proof, we only have to check that

(2.2)
$$\tau(f_1a_1f_2a_2\dots f_na_{n+1}) = 0$$

for all $f_i \in (\{X^s\}_{s\in S})''$, $a_i \in A$ with $\tau(f_i) = E_B(a_i) = 0$ for $i = 1,2,\dots,n$.

We will also have to check the zero trace condition in Eq (2.2) for terms
$$f_1a_1f_2a_2\dots f_na_{n+1}$$
with $f_1 = 1$ or with $f_{n+1} = 1$; but the computations are the same).

To prove Eq. (2.2) we may assume (since $[f_j, p_i] = 0$ for all $i,j$) that there exist $i_1,\dots,i_n \in L$, so that
$$p_{i_j}a_jp_{i_{j+1}} = a_j \text{ for all } j = 1,\dots,n-1$$
and so that
$$\tau(a_j) = 0 \text{ for all } j = 1,\dots,n.$$
Note that the last condition is effective only when $i_j = i_{j+1}$.

As $\tau(f_j) = 0$, since
$$f_jp_{i_j} = p_{i_j}f_j = p_{i_j}f_jp_{i_j}$$
and since $p_{i_j}f_jp_{i_j}$ is a term of zero trace in $\{(Y^s)_{s\in S}, p_{i_j}\}''$ for each fixed $j$, it follows that $p_{i_j}f_jp_{i_j}$ may be written as a sum of terms of the form
$$(p_{i_j} - \tau(p_{i_j}))g_1^j(p_{i_j} - \tau(p_{i_j}))g_2^j(p_{i_j} - \tau(p_{i_j}))\dots g_{m_j}^j(p_{i_j} - \tau(p_{i_j}))$$
where each $g_i^j$ is an element of $\{(Y^s)_{s\in S}\}''$ of null trace. Moreover $p_{i_j}f_jp_{i_j}$ may contain similar terms starting (or ending) directly with $g_1^j$ (or respectively $g_{m_j}^j$). Since
$$\tau(p_{i_j} - \tau(p_{i_j}))a_j(p_{i_{j+1}} - \tau(p_{i_{j+1}}))) =$$
$$\tau((p_{i_j} - \tau(p_{i_j}))a_j) = \tau(a_j(p_{i_{j+1}} - \tau(p_{i_{j+1}}))) = \tau(a_j) = 0$$
it follows that we have proved that all the terms $f_1a_1f_2a_2\dots f_na_n$ in Eq. (2.2) have an expression as a sum of terms of the form
$$b_0g_1b_1\dots g_nb_{n+1}$$

where $g_i \in \{(Y^s)_{s\in S}\}''$, $b_i \in A$, and $\tau(g_i) = \tau(b_i) = 0$ for all possible $i$ with the the exception that $b_0, b_{n+1}$ may be equal to the unit 1. Since $(\{Y^s\}'')_{s\in S}, A$ is a free family of algebras it follows that (2.2) holds true. This ends the proof.

In general, if $B$ is an arbitrary finite dimensional algebra, we choose a maximal system $(f_l)_{l\in L}$ of mutually orthogonal, nonequivalent minimal projections in $B$ and let $f = \Sigma f_i$. Since $B_f$ is abelian and since

$$(2.3) \qquad [(Q \otimes B) *_B A\,]_f \cong (Q \otimes B_f) *_{B_f} A_f$$

it follows that we may use the random matrix picture from the preceding lemma to describe the reduced algebra $[(Q \otimes B) *_B A]_f$. This will be sufficient for our aims since we only want to prove that a reduced algebra of $(\mathcal{L}(F_\infty)\otimes B)*_B A$ is isomorphic to $\mathcal{L}(F_\infty)$. We first prove formula (2.3).

**Lemma 2.2** *Let $B \subseteq A$ be finite dimensional algebra and let $Q$ a type $II_1$ factor. Fix a normalized faithful trace $\tau$ on $A$. Let $\{f_k\}_{k\in K}$ be a maximal family of mutually orthogonal, nonequivalent minimal projections in $B$. Let $f = \sum f_k$.*

*Then there exists a trace preserving isomorphism from $(Q \otimes B_f) *_{B_f} A_f$ onto $(Q\otimes B)*_B A)_f$, where the later algebra is endowed with the normalized trace induced from the trace on $(Q \otimes B) *_B A)$.*

*Proof* . Let $g_i, i = 1, \ldots, m$ be a family of projections in $B$ with $\sum_i g_i = 1 - f$ and so that there exists partial isometries $v_i$ in $B$ with

$$v_i^* v_i = f_i \leq f,\ v_i v_i^* = g_i,\ \text{for } i = 1, \ldots, m.$$

This is always possible because of the choice of $f$.

Any element in $f((Q \otimes B) *_B A\,)f$ is a sum of products of elements of the form

$$x = f a_1\,[(1-f)q_1(1-f)]\,a_2\,[(1-f)q_2(1-f)]\ldots[(1-f)q_n(1-f)]\,a_{n+1} f,$$

where $q_i \in Q$, $a_i \in A$. Since $1 - f = \sum g_i$ and since $[q_i, g_j] = 0$ for all $i, j$ it follows that $x$ itself is a sum of terms of the form:

$$f a_1 (g_{i_1} q_1 g_{i_1}) a_2 (g_{i_2} q_2 g_{i_2}) \ldots (g_{i_n} q_n g_{i_n}) a_{n+1} f =$$

$$f a_1 (v_{i_1} v_{i_1}^* q_1 v_{i_1} v_{i_1}^*) a_2 (v_{i_2} v_{i_2}^* q_2 v_{i_2} v_{i_2}^*) \ldots (v_{i_n} v_{i_n}^* q_n v_{i_n} v_{i_n}^*) a_{n+1} f.$$

Since $v_i$ commutes with $q_j$ (as $v_i \in B$, $i \in I$ and $[Q, B] = 0$) this last term is also equal to

$$(f a_1 v_{i_1} v_{i_1}^* v_{i_1}) q_1 (v_{i_1}^* a_2 v_{i_2} v_{i_2}^* v_{i_2}) q_2 (v_{i_2}^* a_3 v_{i_3}) \ldots q_n (v_{i_n}^* a_{n+1} f) =$$

$$(f a_1 v_{i_1}) q_1 (v_{i_1}^* a_2 v_{i_2}) q_2 (v_{i_2}^* a_3 v_{i_3}) \ldots (v_{i_{n-1}}^* a_n v_{i_n}) q_n (v_{i_n}^* a_{n+1} f).$$

As $v_{i_j}^* a v_{i_{j+1}}$ belongs to $A_f$ (since $v_i^* = f v_i^*$, $v_i = v_i f$) it follows that any element in $f((Q \otimes B) *_B A)f$ is indeed a sum of elements of the form

$$a_0 q_1 a_1 \ldots q_n a_n, a_i \in A_f, q_i \in Q_f.$$

To conclude the proof of the lemma it remains to check that

$$\tau(a_0 q_1 a_1 \ldots q_n a_n) = 0, \text{for all } a_i \in A_f,\ q_i \in Q$$

with

$$\tau(q_i) = 0, \ E_{B_f}(a_i) = 0 \text{ for all } i = 0, \ldots, n,$$

with the exception of $a_0, a_n$ which may be equal to 1.

As $E_{B_f}(a) = E_B(a)$ for all $a \in A_f$, the last condition follows from the similar properties of the trace on $(Q \otimes B) *_B A$. This ends the proof.

We use the picture from Proposition 2.1 for the algebra $\left[(\mathcal{L}(F_\infty) \otimes B) *_B A\right]_f \cong (\mathcal{L}(F_\infty) \otimes B_f) *_{B_f} A_f$, when $B$ is no longer assumed to be abelian and $f$ is as in the statement of the preceding Proposition. Let $e$ be the sum of a maximal system $(e_k)_{k\in K}$ of mutually orthogonal, nonequivalent, minimal projections in $A_f$ that commute with $(f_l)_{l\in L}$.

We will construct a trace preserving isomorphism from $\left[(\mathcal{L}(F_\infty) \otimes B) *_B A\right]_{fe}$ onto a von Neumann subalgebra of $\mathcal{L}(F_\infty) * A_{fe}$. This will be essential in the next paragraph, when we show that $\mathcal{A} \cong \mathcal{L}(F_\infty)$.

**Corollary 2.3** *Let $A \supseteq B$ be finite dimensional algebras, let $\tau$ be a faithful normalized trace on $A$, and let $\Gamma = (a_{kl})_{l\in L, k\in K}$ the inclusion matrix of $A \supseteq B$. Let $(f_l)_{l\in L}$ be a maximal system of mutually orthogonal, nonequivalent, minimal projections in $B$ and let $(e_k)_{k\in K}$ be a maximal system of mutually orthogonal, nonequivalent, minimal projections in $A_f$ that commute with $(f_l)_{l\in L}$. Denote by $e, f$ the projections defined by $f = \Sigma f_l$ and $e = \Sigma e_k$.*

*Then there exists a (natural) trace preserving embedding*

$$\left[(\mathcal{L}(F_\infty) \otimes B) *_B A\right]_{fe} \subseteq \mathcal{L}(F_\infty) * (A_{fe}),$$

*which is realized as follows: Let $(Z^t)_{t\in T}$ be an infinite free semicircular family that is free with respect to the algebra $A_{fe}$, let*

$$N = \{(k,m) \in K \times K \mid \sum_{l\in L} a_{kl}a_{ml} \neq 0\}$$

*and let*

$$X^t = \sum_{(k,m)\in N} e_k Z^t e_m \text{ for all } t \in T.$$

*Then $\{(X^t)_{t\in T}, A_{fe}\}''$ is isomorphic to $\left[(\mathcal{L}(F_\infty) \otimes B) *_B A\right]_{fe}$ (and the isomorphism preserves the trace).*

*Proof*. Since $\Gamma$ is also the inclusion matrix for $B_f \subseteq A_f$ and since

$$[(Q \otimes B) *_B A]_f \cong (Q \otimes B_f) *_{B_f} A_f$$

for any type $II_1$ factor $Q$, we may assume that $f = 1$ and henceforth that $B$ is abelian and $B = \{f_l\}''_{l\in L}$.

Let $(Y^s)_{s\in S}$ be an infinite free semicircular that is free with $A$, let

$$\mathcal{B} = \{(Y^s)_{s\in S}, A\}'' \cong \mathcal{L}(F_\infty) * A$$

and let

$$X^s = \sum_{i\in L} \tau(f_i)^{-1/2} f_i Y^s f_i \text{ for all } s \in S.$$

By proposition 2.1, we conclude that:

$$\mathscr{S} = (\mathscr{L}(F_\infty) \otimes B) *_B A \cong \{(X^s)_{s\in S}, A\}''.$$

Let $r_k$ be the central support of $e_k$ in $A$, for $k \in K$ and let $\{e^k_{p,q}\}^{t_k}_{p,q=1}$ be a matrix unit for $A_{r_k}$ so that $e^k_{p,p}$ commutes with $(f_l)_{l\in L}$ and so that $e^k_{11} = e_k$ for all $p = 1,\ldots,t_k, k \in K$ ($t_k$ is the dimension of $A_{r_k}$).

By Lemma 1 in [25] (see also Lemma 3.1 in [28]) a system of generators for $\mathscr{S}_e$ is

$$\mathscr{X} = \bigcup_{(m,k)\in N} \{e^k_{1p} Y^s e^m_{q1} | s \in S, p = 1,\ldots,t_k, q = 1,\ldots,t_m\}$$

and $A_e$. A system of generators for $\mathscr{B}_e$ is $A_e$ and

$$\mathscr{Y} = \bigcup_{(m,k)\in K\times K} \{e^k_{1p} Y^s e^m_{q1} | s \in S, p = 1,\ldots,t_k, q = 1,\ldots,t_m\}.$$

In order to compute traces of monomials with variables in the sets $\mathscr{Y}$ and $A_e$ (and thus to be able to determine the isomorphism class of the algebra generated by $\mathscr{Y}$ and $A_e$) we will make use of the random matrix picture of D. Voiculescu for free semicircular families (see also [7]).

Assume that $(Y^s)_{s\in S}$ are represented as selfadjoint $m \times m$ matrices $(T^{m,s})_{s\in S}$ with random entries. This means that

$$T^{m,s} = (a^{m,s}_{i,j})_{1\le i,j\le m}$$

where $a^{m,s}_{i,j}$ are measurable functions on a probability space $(\Sigma, \nu)$ so that, for fixed $m$, the variables in the family:

$$\{\text{Re } a^{m,s}_{i,j} | 1 \le i \le j \le m, s \in S\} \cup \{\text{Im } a^{m,s}_{i,j} | 1 \le i < j \le m, s \in S\}$$

are an independent gaussian family of functions on $\Sigma$ with

$$\mathscr{E}(a^{s,m}_{i,j}) = 0, \ \mathscr{E}(|a^{s,m}_{i,j}|^2) = 1/m.$$

Recall from the introductory paragraph that $\mathscr{E}$ is the expectation value on $\Sigma$ given by

$$\mathscr{E}(f) = \int f \mathrm{d}\nu, f \in L^\infty(\Sigma, \nu).$$

Moreover $A_e$ is represented (asymptotically) by constant matrix functions on $\Sigma$ (we assume that the dimensions of the minimal projections in $A_e$ in the corresponding (asymptotic) representation on $m \times m$ matrices, tend to infinity with $m$). Let also $\tau_{\mathscr{E},m} : \bigcap_{p\ge1} L^p(\Sigma) \otimes M_m(\mathbb{C}) \to \mathbb{C}$ be the composition of the normalized trace on $m \times m$ matrix functions with the expectation value $\mathscr{E}$.

By ([29], see also [7]) the trace of any monomial whose variables belong to $(Y^s)_{s\in S}$ and $A_e$ is the limit (as the dimension $m$ tends to infinity) of the value of $\tau_{\mathscr{E},m}$ evaluated on the corresponding monomial with variables in $(T^{m,s})$ and $A_e$.

This shows that in order to compute traces for monomials with variables in the sets $\mathscr{Y}$ and $A_e$ we may act as if the elements $e^k_{1p} Y^s e^m_{q1}$ (with left support $e^k_1$, right support $e^l_1$) were matrix blocks of very large size and so that the collection of all the entries in the matrix blocks is an independent gaussian family as above (for fixed $m$).

Hence the elements in the set $\mathscr{Y}$ may be "glued" to another free semicircular system $(Z^t)_{t\in T}$ in $\mathscr{B}_e = \mathscr{C}$ that is free with $(e_k)_{k\in K}$ and so that

$$\{e_k Z^t e_m\}_{t\in T} = \{e_{1p}^k Y^s e_{q1}^m | s \in S, p = 1, \ldots, t_k, q = 1, \ldots, t_m\}$$

for all $k, m \in K$ (recall that $S$ is infinite).

Clearly this shows that the elements in $\mathscr{X}$ (which are together with $A_e$ a system of generators for the reduced algebra $\mathscr{D}_e$) may be obtained from the elements $Z^s$ by the procedure described in the statement (after another permutation of the matrix blocks in $(Z^s)_{s\in S}$). This ends the proof of Corollary 2.2.

*Remark 2.4* The statement of Corollary 2.3 still holds if one assumes the weaker hypothesis that $A, B$ are finite, type $I$, von Neumann algebras with discrete centers.

*Proof* . Indeed the only complication that occurs in the proof of Corollary 2.3 to this more general statement is the fact that this time $K$ may be infinite and therefore we cannot make a simultaneous use of Voiculescu's random matrix picture for all the elements in the set $\mathscr{Y}$ that appeared in the proof of Corollary 2.3.

On the other hand the fact that the elements in the set $\mathscr{Y}$ may be "glued" in a semicircular family is a fact that only concerns traces of monomials of elements in the set $\mathscr{Y}$ and $A_e$. Thus to show that the elements in $\mathscr{Y}$ may be "glued" in a semicircular family it is sufficient to show that the values of the traces of such monomials are exactly those which should be obtained from the freeness relations in the free semicircular family in which the elements will be "glued". As any monomial includes only a finite number of terms, we are clearly reduced to the case of finite $K$

In the proof of Corollary 2.3 we incidentally proved (and implicitly used) the following statement which is a straightforward consequence of Voiculescu's random matrix picture for semicircular families :

**Lemma 2.5** *Let $\mathscr{A}$ be a type $II_1$ factor and let $\{Y^s, 1\}''_{s\in S} \cup \{f_1, \ldots, f_k\}''$ be a free family of algebras in $\mathscr{A}$ so that $(Y^s)_{s\in S}$ is an infinite semicircular family and so that $f_1, \ldots, f_k$ are mutually orthogonal projections with $\sum f_i = 1$. Let $\{\sigma(i,j,s) | s \in S\}$ be permutations of $S$, $1 \le i \le j \le k$. Then*

$$Z^s = \sum_{i<j} (f_i Y^{\sigma(i,j,s)} f_j + f_j Y^{\sigma(i,j,s)} f_i) + \sum_i f_i Y^{\sigma(i,i,s)} f_i, s \in S$$

*is a free semicircular family and $\{Z^s, 1\}''_{s\in S} \cup \{f_1, \ldots, f_k\}''$ is again a free family of algebras.*

## 3. The isomorphism theorem

Let $1 \in B \subseteq A$ be finite dimensional algebras whose centers have trivial intersection (by [22], this is the factoriality condition for $(Q \otimes B) *_B A$ if $Q$ is a type $II_1$ factor). Assume that $A$ is endowed with a normalized faithful trace and let $f$ and $e$ be as in Corollary 2.3. In this paragraph we use the description for the reduced algebra $\left[(\mathscr{L}(F_\infty) \otimes B) *_B A\right]_{fe}$ that we obtained in Corollary 2.3, to prove that this algebra is isomorphic to $\mathscr{L}(F_\infty)$

Our strategy is to show that a reduced algebra of $\left[(\mathcal{L}(F_\infty)\otimes B)*_B A\right]_{fe}$ is isomorphic to $\mathcal{L}(F_\infty)$. By theorem 6 in [23] (which asserts that $g\mathcal{L}(F_\infty)g$ is isomorphic to $\mathcal{L}(F_\infty)$ for any projection $g$ in $\mathcal{L}(F_\infty)$) it will follow that

$$\mathcal{L}(F_\infty)\cong\left[(\mathcal{L}(F_\infty\otimes B)*_B A\right]_{fe}.$$

This will be proved by a similar technique to the one used by D. Voiculescu in [28] (see also [25]) for the isomorphism $\mathcal{L}(F_\infty)_{1/k}\cong\mathcal{L}(F_\infty)$.

The first result implies in particular that $\mathcal{L}(F_\infty)*A\cong\mathcal{L}(F_\infty)$ if $A$ is a finite dimensional algebra. This extends the fact ([28]) that $\mathcal{L}(F_\infty)*\mathcal{L}(G)\cong\mathcal{L}(F_\infty)$ if $G$ is a cyclic group.

**Lemma 3.1** *Let $\mathcal{B}$ be a type $II_1$ factor, let $\mathcal{P}=\{f_1,\dots,f_k\}$ be a partition of the unity with projections in $\mathcal{B}$ and let $(Y^s)_{s\in S}$ be an infinite free semicircular family that is also free with respect to the algebra generated by $\mathcal{P}$.*

*Fix $\sigma$ in $S$ and let $D$ be the diffuse, abelian von Neumann algebra generated by $\{f_1,\dots,f_k\}$ and $\{f_iY^\sigma f_i | i=1,\dots,k\}$. Let $(Y'^s)_{s\in S}$ be any free semicircular family that is obtained by "gluing" the remaining pieces of $(Y^s)_{s\in S}$, i.e. such that*

$$\{f_iY'^sf_j\}_{s\in S}=\{f_iY^sf_j\}_{s\in S}\text{ for all }1\le i<j\le k$$

*and*

$$\{f_iY'^sf_i\}_{s\in S}=\{f_iY^sf_i\}_{s\in S/\{\sigma\}}.$$

*Then $(Y'^s)_{s\in S}$ is also free with $D$ and*

$$\{(Y'^s)_{s\in S},D\}''=\{(Y^s)_{s\in S},\ f_1,\dots,f_k\}''\text{ for all }i=1,\dots,k.$$

*Proof* . The last equality is simply a consequence of the fact that both size have the same generators. The only thing to prove is the fact that $\{(Y'^s)_{s\in S},D\}''$ is free. Let $\widetilde{\mathcal{B}}$ be a larger type $II_1$ factor that contains $\mathcal{B}$ and that also contains a semicircular element $Z\in\widetilde{\mathcal{B}}$ so that $Z\cup(Y^s)_{s\in S}$is a free semicircular family which is also free with respect to $\{f_1,\dots,f_k\}''$. Let

$$Y^0=\sum_i f_iY^\sigma f_i+\sum_{i\neq j}f_iZf_j.$$

Lemma 2.5, then shows that $Y^0\cup(Y'^s)_{s\in S}$ is also a free semicircular family, that is also free with respect to $\{f_1,\dots,f_k\}''$. The associativity property for free family of algebras ([28]) implies that $\{Y^0\cup D\}''$ is free with respect to $(Y'^s)_{s\in S}$. The result now follows since $\{(f_iY^\sigma f_i)_{i=1}^k,D\}''$ is contained in $\{Y^0,\ f_1,\dots,f_k\}''$.

The following theorem will be used, together with the results in the first paragraph to prove our main result in the case of $\mathcal{L}(F_\infty)$. In the proof we will use the description of the reduced algebra $\left[(\mathcal{L}(F_\infty)\otimes B)*_B A\right]_e$ that we obtained in Corollary 2.3.

**Theorem 3.2** *Let $1\in B\subseteq A$ be finite dimensional algebras whose centers have trivial intersection. Then $(\mathcal{L}(F_\infty)\otimes B)*_B A$ is isomorphic to $\mathcal{L}(F_\infty)$.*

*Proof* . Let $f=\sum f_k$ be the sum over a maximal system of mutually orthogonal, nonequivalent, minimal projections in $B$.

By the preceding lemma and by Corollary 2.3 we may use the following description for a reduced algebra of $((\mathcal{L}(F_\infty)\otimes B)*_B A)_f$. Let $f=\sum f_k$ be the sum over a maximal system of mutually orthogonal, nonequivalent, minimal projections in $B$.

Let $(e_i)_{i\in K=\{1,..,k\}}$ be a maximal system of mutually orthogonal, nonequivalent, minimal projections in $A_f$, let $e=\sum e_i$ and let $N$ be the symmetric reflexive relation on $K$ as in the statement of Corollary 2.3. Let $D_0$ be the von Neumann algebra generated by $(e_i)_{i\in K=\{1,..,k\}}$.

Let $\mathcal{C}$ a type $II_1$ factor which is generated by a free family of algebras $\{Y^s\}''_{s\in S}\cup D$ where $D$ is a diffuse, abelian von Neumann algebra and $(Y^s)_{s\in S}$ is a free semicircular infinite family.

We choose $(Y^s)_{s\in S}$ and $D$ so that $D_0=\{e_1,\dots,e_k\}''$ is a subalgebra of $D$. Let

$$X^s=\sum_{(i,j)\in N} e_iY^se_j,\ s\in S,$$

then we have

$$\mathcal{B}=\{(X^s)_{s\in S}\cup D\}''\cong\left[((\mathcal{L}(F_\infty)\otimes B)*_B A\right]_{fe}.$$

Since $N$ contains a single orbit, as a consequence of the fact that the centers of $A$,$B$ have trivial intersection, we may assume that

$$(1,2),(2,3),\dots,(k-1,k)\in N.$$

By choosing $n$ big enough we may find a partition of the unity $\{g_1,\dots,g_n\}$ in $\mathcal{P}(D)$ so that $\tau(g_i)=1/n, i=1,\dots,n$ and so that the following inequality holds true:

$$\sum_{|i-j|\le 1} g_i\otimes g_j\le\sum_{|r-s|\le 1} e_r\otimes e_s.$$

Fix $\sigma$ in $S$ and let $v_i$ be the partial isometry from the polar decomposition $v_ib_i$ of

$$g_{i-1}Y^\sigma g_i=g_{i-1}X^\sigma g_i \text{ for all } i=2,\dots,n..$$

By Theorem 3.1 in [28] we have

$$v_iv_i^*=g_{i-1},\ v_i^*v_i=g_i,\ b_i=g_ib_ig_i \text{ for all } i=2,\dots,n.$$

Let $B_i$ be a semicircular element in $g_i\mathcal{C}g_i$ with

$$\{B_i\}''=\{b_i\}''\subseteq g_i\mathcal{C}g_i$$

for $i=2,\dots,n$, let $a_i$ be a semicircular generator of $g_iDg_i$ for $i=1,\dots,n$ and let $w_i=v_iv_{i+1}\dots v_n$. It is obvious that $w_iw_i^*=g_{i-1}$, $w_i^*w_i=g_n$, for $i=2,\dots,n$.

By Theorem 3.2 in [28] and its proof (see also lemma 2 in [23]) the elements in the following set are a free semicircular family generating $\mathcal{C}_{g_n}$:

(a) $$\omega_0=\{w_i^*Y^sw_i,|i=2,\dots,n,s\in S\}\cup$$

(b) $$\{w_i^*a_{i-1}w_i\}_{i=2}^n\cup\{a_n\}\cup$$

(c) $$\{\mathrm{Re}\,(w_i^*Y^sw_j),\ \mathrm{Im}\,(w_i^*Y^sw_j)|2\le j<i\le n, s\in S/\{\sigma\}\}\cup$$

(d) $\{\text{Re } (w_i^* Y^\sigma w_j),\ \text{Im}(w_i^* Y^\sigma w_j) | 2 \leq j < i \leq n, j \neq i-1\} \cup$

(e) $\cup \{w_i^* B_{i-1} w_i\}_{i=2}^n \cup \{g_n Y^s g_n\}_{s \in S}.$

We will describe the generators for the reduced algebra $\mathcal{A}_{g_n}$ with a less confusing notation. Let $\mathcal{B}$ be the type $II_1$ factor generated by $\omega_0$. Then the elements on the lines (a), (b), (e) are also elements of $\mathcal{A}_{g_n}$, while for the elements on the lines (b), (c), to get generators for $\mathcal{A}_{g_n}$, we have to delete certain blocks. Indeed, for $i, j = 2, \ldots n$ we have:

$$w_i^* X^s w_j = w_i^* g_{i-1} X^s g_{j-1} w_j = w_i^* \sum_{(r,s) \in N} (g_{i-1} e_r) Y^s (g_{j-1} e_s) w_j =$$

$$\sum_{(r,s) \in N} (w_i^* (g_{i-1} f_r) w_i)(w_i^* Y^s w_j)(w_j^* (g_{j-1} e_s) w_j).$$

By hypothesis

$$\sum_{|i-j| \leq 1} g_i \otimes g_j \leq \sum_{|r-s| \leq 1} e_r \otimes e_s$$

and henceforth $g_i f_r \neq 0$ for utmost two consecutive values of $r$ (for fixed $i$). Thus we proved that $w_i^* X^s w_j$ is obtained from $w_i^* Y^s w_j$ as a sum of blocks of the form $g w_i^* X^s w_j h$ where $g \in \{w_i^* a_{i-1} w_i\}''$, $h \in \{w_j^* a_{j-1} w_j\}''$.

The less confusing notation we promised above is now as follows: we assume that $\omega_0$ is the union of the infinite free semicircular family $(x_s)_{s \in S}$ with the infinite circular family $(y_s)_{s \in T}$. We assume that $S$ has the partition $S = \{\sigma_1, \ldots, \sigma_n\} \cup S_0$, where $x^{\sigma_1}, \ldots, x^{\sigma_n}$ correspond to the elements on the line (b) while $(y_s)_{s \in T}$ correspond to the elements on lines (c), (d) and $(x_s)_{s \in S_0}$ correspond to the remaining elements. Moreover we are given a finite family of projections $\{g_j\}_{j \in J}$, containing 0, 1 and so that each $g_j$ belongs to $\{x^{\sigma_{p_j}}\}''$ for some $\sigma_{p_j} \in \{\sigma_1, \ldots, \sigma_n\}$ for all $j \in J$.

The generators $\omega_1$ for $\mathcal{A}_{g_n}$ are now obtained from $\omega_0$ by the following procedure: for each $s \in T$ we are given $i_s, j_s$ in $J$ Consider the family

$$\omega_1 = (x_s)_{s \in S} \cup \{g_{i_s} y_s g_{j_s}\}.$$

We will prove that $\omega_1$ generates a factor which is isomorphic to $\mathcal{L}(F_\infty)$. To that end we consider a partition with infinite sets

$$S_0 = \cup_{(i,j) \in J^2} S_{i,j}$$

and show that for fixed $(i,j) \in J^2$ the algebra generated by

$$(x_s)_{s \in S_{i,j}} \cup \{x^{\sigma_{p_j}}, x^{\sigma_{p_i}}\} \cup \{g_i y_s g_j | i_s = i, j_s = j, s \in S\},$$

is isomorphic to $\mathcal{L}(F_\infty)$. Moreover we may choose this isomorphism so as to act identically on $\{x^{\sigma_{p_j}}, x^{\sigma_{p_i}}\}''$. Clearly this will complete the proof.

To prove the existence of such an isomorphism we may also assume that $x^{\sigma_{p_j}} = x^{\sigma_{p_i}}$. Indeed, let $w$ be any unitary in the type $II_1$ factor $\{x^{\sigma_{p_j}}, x^{\sigma_{p_i}}\}''$ so that $w g_i = g_i' w$ where

$$g_i' \in \mathcal{P}(\{x^{\sigma_{p_j}}\}'').$$

To complete the proof it remains to show that the algebra generated by

$$(x_s)_{s\in S_{ij}} \cup \{x^{\sigma p_j}, x^{\sigma p_i}\} \cup \{wg_i\dot{y}_s g_j\} =$$

$$(x_s)_{s\in S_{ij}} \cup \{x^{\sigma p_j}, x^{\sigma p_i}\} \cup \{g_i' w y_s g_j\}$$

is isomorphic to $\mathcal{L}(F_\infty)$. As the family $\{wy^s\}_{s\in T}$ is again circular and since now $g_i', g_j \in \{x^{\sigma p_j}\}''$ we are left to prove the following:

**Lemma 3.3** *Let $\{z_s\}_{s\in T_1\cup T_2} \cup \{x_0\}$ be a free family where $\{z_s\}_{s\in T_1\cup T_2}$ is a circular family and $x_0$ is a semicircular element. Assume that $T_1, T_2$ are infinite sets. Let $g, h$ be projections in $\{x_0\}''$.*

*Then*

$$\mathcal{C} = \{x_0, (y_t)_{t\in T_1}, (hy_tg)_{t\in T_2}\}''$$

*is isomorphic to $\mathcal{L}(F_\infty.)$*

*Proof* . We simply use the random matrix picture from [29]. Consequently we may represent (asymptotically) the elements in the set $\{\mathrm{Re}\, y_s, \mathrm{Im}\, y_s\}_{s\in T_1\cup T_2}$ by large matrices, with random entries (of size tending to infinity) and that $x_0$ is (asymptotically) represented by constant matrices. Clearly we may permute the blocks in the family $(y_t)_{t\in T1}$ so as to fill the holes in $y_t$ for $t \in T_2$ without changing the isomorphism class of the algebra.

As a consequence of Theorem 3.2 and of paragraph 1, we have now proved our main result for $\mathcal{L}(F_\infty)$. We state it separately.

**Theorem 3.4** *Let $\mathcal{C} = (A \supseteq B \supseteq C; A \supseteq C \supseteq D)$ be a commuting square ([20]) of finite dimensional algebras. Assume that $\mathcal{C}$ is irreducible (i.e. the centers of the algebras $A, B$ and respectively $C, D$ have trivial intersection) and that $\mathcal{C}$ is $\lambda$-Markov (i.e. there exists a $\lambda$-Markov trace ([11]) for $C \subseteq A$ which restricts to a $\lambda$-Markov trace for $D \subseteq B$).*

*Then there exists an subfactor $\mathcal{A} \subseteq \mathcal{L}(F_\infty)$ of index $\lambda^{-1}$ and the relative commutant $\mathcal{A}' \cap \mathcal{L}(F_\infty)$ is isomorphic to $B \cap C'$. In addition $\mathcal{A} \cong \mathcal{L}(F_\infty)$.*

In addition, if $\mathcal{C}$ is any commuting square which (by iteration of the Jones' basic construction) yields a subfactor of the hyperfinite factor, then the higher relative commutants of the inclusion $\mathcal{A} \subseteq \mathcal{L}(F_\infty)$ coincide with the ones for the inclusion of type $II_1$, hyperfinite factors $P_\infty \subseteq Q_\infty$ that is associated with the commuting square $\mathcal{C}$ (see Theorem 1.3).

## 4. A real continuation of the sequence $(\mathcal{L}(F_N))_{N\geq 2, N\in\mathbb{N}}$

In this section we introduce a continuous series of type $II_1$ factors $\mathcal{L}(F_r)$, $r \in \mathbb{R}$, $r > 1$ which extends the sequence of the type $II_1$ factors $(\mathcal{L}(F_N))_{N\geq 2, N\in\mathbb{N}}$ that are associated to the noncommutative free groups $F_N$.

This series appears naturally in the analysis of the isomorphism class of the algebras $\mathcal{L}(F_N) \otimes M_p(\mathbb{C})$. Indeed by Voiculescu's formula (see Theorem 3.2 in [28]) we have

$$(4.1) \qquad (\mathcal{L}(F_N)_{1/p} \cong \mathcal{L}(F_{(N-1)p^2+1}) \text{ for all } p > 1, p \in \mathbb{N}$$

or equivalently

$$\mathscr{L}(F_{(N-1)p^2+1}) \otimes M_p(\mathbb{C}) \cong \mathscr{L}(F_N).$$

Formula (4.1) suggests to define $\mathscr{L}(F_{(N-1)t^{-2}+1})$ as the isomorphism class of the reduced algebra $(\mathscr{L}(F_N))_t$, for all $t > 0$, $N \in \mathbb{N}, N \geq 2$.

The first indication that such a definition would be possible was the following generalization of Eq. (4.1) ([25], Th. 6 ):

(4.2) $$\mathscr{L}(F_N)_{1/\sqrt{k}} \cong \mathscr{L}(F_{(N-1)k+1}), k \in \mathbb{N}, N \in \mathbb{N}, N \geq 2.$$

This isomorphism shows in particular that the von Neumann algebras

$$\mathscr{L}(F_N)_{N\geq 2, N \in \mathbb{N}}$$

are stably isomorphic (i.e. that the isomorphism class of $\mathscr{L}(F_N) \otimes B(H)$ is independent of $N$ for all finite $N > 1$). Note that formula (4.1) implies in particular that $\mathscr{L}(F_2)$ and $\mathscr{L}(F_5)$ are stably isomorphic but does not imply that $\mathscr{L}(F_2)$ and $\mathscr{L}(F_3)$ are.

The construction used in [25], [28] suggest that a natural way to define $(\mathscr{L}(F_r))_{r\in\mathbb{R}, r>1}$ is to use infinite free semicircular families $(X^s)_{s\in S}$. Let $\sigma$ be a fixed element in $S$, let $(e_s, f_s)_{s\in S\setminus\{\sigma\}}$ be a family of projections in $\{X^\sigma\}''$ which are either mutually orthogonal or equal and let

$$r = 1 + \sum k_s \tau(e_s)\tau(f_s)$$

where $k_s = 1$ if $e_s = f_s$ and $k_s = 2$ if $e_s f_s = 0$.

Then $\mathscr{L}(F_r)$ is the isomorphism class of the algebra

$$\mathscr{A} = \{X^\sigma, (e_s X^s f_s)_{s\in S\setminus\{\sigma\}}\}',$$

if $e_s, f_s$ are chosen so that the algebra $\mathscr{A}$ is a factor.

With this definition we prove that the following formulae hold true:

(4.3) $$\mathscr{L}(F_r)_t \cong \mathscr{L}(F_{(r-1)t^{-2}+1}), \text{ for all } r > 1,\ t > 0$$

(4.4) $$\mathscr{L}(F_r) * \mathscr{L}(F_r') \cong \mathscr{L}(F_{r+r'}), \text{ for all } r, r' > 1.$$

A similar series was considered in [8], and the formulae (4.3), (4.4) were also proved there.

The direct consequence of the two formulae (4.3), (4.4) is the fact that the fundamental group $\mathscr{F}(\mathscr{L}(F_r))$ is either 1 or $\mathbb{R}_+ \setminus \{0\}$, independently on $r \in (1, \infty)$.

Thus the algebras $(\mathscr{L}(F_r))_{r\in\mathbb{R}, r>1}$ are either all isomorphic or they are mutually nonisomorphic. In fact the first situation occurs if for some (equivalently for all $r > 1$) $\mathscr{L}(F_r) \cong \mathscr{L}(F_\infty)$ (using [23]). In this paragraph we will prove that the above definition is independent on the choice of the projections $e_s, f_s$. After proving this result, the formulae (4.3), (4.4) will follow immediately.

Finally in the last part of the paragraph we will present a slight variation on the definition of $(\mathscr{L}(F_r))_{r\in\mathbb{R}, r>1}$, which will allow us to conclude that for all finite dimensional algebras $B \subseteq A$, whose centers have trivial intersection, the algebras $(\mathscr{L}(F_N) \otimes B) *_B A$ belong to the the series $(\mathscr{L}(F_r))_{r\in\mathbb{R}, r>1}$.

Our first result will be used to show that the isomorphism class of $\mathscr{L}(F_r)$ is independent on the choice of the projections $e_s, f_s$.

**Theorem 4.1** *Let $A, B, C$ be von Neumann subalgebras of a type $II_1$ factor $\mathcal{B}$ which are free with respect to the trace $\tau$ on $\mathcal{B}$. Assume that $A$ is a factor and that $B$ is generated by an infinite free semicircular family $(Y^s)_{s\in S}$. Let $e_s, f_s$ be projections in $A$ which are either mutually orthogonal or equal. Let $k_s = 2$ or $k_s = 1$ correspondingly.*

*Then the isomorphism class of*

$$\mathcal{A} = \{A, (e_s X^s f_s)_{s\in S}, C\}''$$

*depends only on $\sum_s k_s \tau(e_s)\tau(f_s)$*

*Proof* . The proof is done into two steps. First step is to consider another family of projections $(e'_s, f'_s)_{s\in S}$ (which are either mutually orthogonal or equal) with $\tau(e_s) = \tau(e'_s)$, $\tau(f_s) = \tau(f'_s)$ and so that $e_s = f_s$ iff. $e'_s = f'_s$.

**Step 1** *The algebra*

$$\mathcal{A}_1 = \{A, (e'_s X f'_s)_{s\in S}, C\}''$$

*is isomorphic to $\mathcal{A}$.*

*Proof* . Indeed by hypothesis, there exists unitaries $(w_s)_{s\in S}$ in $A$ with

$$e'_s w_s = w_s e_s, f'_s w_s = w_s f_s, s \in S.$$

Then the algebra $\mathcal{A}$ is generated by $A$,

$$w_s e_s X^s f_s w_s^* = e'_s (w_s X^s w_s^*) f'_s, \ \ s \in S,$$

and $C$. As $(w_s X^s w_s^*)_{s\in S}$ is again a free semicircular family and since

$$\{A\}, (\{w_s X^s w_s^*\}'')_{s\in S}, \{C\}$$

is a free family of algebras, it follows that

$$\mathcal{A} = \{A, e'_s (w_s X^s w_s^*) f'_s, C\}''$$

is isomorphic to $\mathcal{A}_1$.

In the second step we prove that if we change the "shape" of the projection $\sum_s e_s \otimes f_s$ (when $e_s \otimes f_s$ are mutually orthogonal), without modifying the "surface" $\sum_s k_s \tau(e_s)\tau(f_s)$, then the isomorphism class of the algebra $\mathcal{A}$ remains unchanged.

The content of the second step is summarized as follows:

**Step 2** *Let $S = \cup_{n\in I} S^n$ be a partition of $S$ with nonvoid sets and fix $\sigma_n$ in $S_n$. Let $S_1^n = \{s \in S^n | e_s = f_s\}$ and let $S_2^n = S^n \setminus S_1^n$. Assume that the projections in the family*

$$\{e_s \otimes f_s\}_{s\in S_1^n} \cup \{e_s \otimes f_s, f_s \otimes e_s \mid s \in S_2^n\}$$

*are mutually orthogonal.*

*Then $\mathcal{A}$ is isomorphic to the algebra*

$$\{A, (\sum_{s\in S_1^n} e_s X^s f_s + \sum_{s\in S_2^n} (e_s X^s f_s + f_s X^s e_s))_{n\in \mathbb{N}}, C\}''.$$

*Proof* . It is clear that it is sufficient to prove this isomorphism under the additional assumption that card $I = 1$. Moreover we may assume that card $S_n$ is finite. The case of finite $S_n$ is then simply a consequence of Lemma 2.5.

The reduction to the case of finite $S_n$ is a consequence of the so-convergence of the sum considered above, which in turn is a consequence of the finiteness of

$$\sum_{s\in S_1^n} ||e_sX^sf_s||_2^2 + \sum_{s\in S_2^n} 2||e_sX^sf_s||_2^2 =$$

$$\sum_{s\in S_1^n} \tau(e_s)\tau(f_s) + 2\sum_{s\in S_2^n} \tau(e_s)\tau(f_s) < \infty.$$

Clearly a (finite) repeated use of the steps 1 and 2 completes the proof of the theorem.

We are now able to state (and prove in the same time the correctness) the definition of $\mathcal{L}(F_\gamma)$.

**Definition 4.2** *Let $(X^s)_{s\in S}$ be an infinite free semicircular family. Fix a finite subset of distinct indices $\{\sigma_0, \sigma_1, \ldots \sigma_n\}$ in $S$. For each $s$ in $S_0 = S \setminus \{\sigma_0, \sigma_1, \ldots, \sigma_n\}$, let $e_s, f_s$ be projections in $\{X^{\sigma_{i_s}}\}''$ for some $i_s \in \{1, 2, \ldots, n\}$, which are either mutually orthogonal or equal. Let $h$ be a fixed nonzero projection in $\{Y^{\sigma_1}\}''$. Let*

$$\gamma = n + 2\tau(h)\tau(1-h) + \sum_s k_s\tau(e_s)\tau(f_s),$$

*where* $k_s = 1$ if $e_s = f_s$ and $k_s = 2$ if $e_sf_s = 0$.

*Then $\mathcal{L}(F_\gamma)$ is the isomorphism class of the type $II_1$ factor*

$$\{X^{\sigma_1}, X^{\sigma_2}, \ldots, X^{\sigma_n}, hX^{\sigma_0}(1-h), (e_sX^sf_s)_{s\in S_0}\}''.$$

Note that $\{X^{\sigma_1}, hX^{\sigma_0}(1-h)\}''$ is a factor. By the preceding theorem (in fact by the first step) we may assume in the above definition, without any restriction of generality, that $n = 1$, Thus we may assume that all of the projections $e_s, f_s$ are in $\{Y^{\sigma_1}\}''$. Hence an equivalent definition for $\mathcal{L}(F_\gamma)$ is the following:

**Definition 4.2 ′** *Let $(X^s)_{s\in S}$ be an infinite free semicircular family, let $\sigma_0, \sigma_1$ be two distinct elements in $S$ and for each $s \in S \setminus \{\sigma_1, \sigma_0\}$, let $e_s, f_s$ be projections in $\{X^{\sigma_1}\}''$, which are either mutually orthogonal or equal. Let $h$ be a nonzero projection in $\{X^{\sigma_1}\}''$. Let*

$$\gamma = 1 + 2\tau(h)\tau(1-h) + \sum_{s\in S\setminus\{\sigma_1,\sigma_0\}} k_s\tau(e_s)\tau(f_s),$$

*where* $k_s = 1$ if $e_s = f_s$ and $k_s = 2$ if $e_sf_s = 0$.

*Then $\mathcal{L}(F_\gamma)$ is the isomorphism class of the type $II_1$ factor:*

$$\{X^{\sigma_1}, hX^{\sigma_0}(1-h), (e_sX^sf_s)_{s\in S\setminus\{\sigma_1,\sigma_0\}}\}''.$$

One of the reasons for considering the more complicated Definition 4.2, (instead of the Definition 4.2′) is the fact that this definition makes trivial the proof of formula

(4.4). We now complete the proof of the correctness of Definition 4.2. Theorem 4.1 proves that $\mathcal{L}(F_\gamma)$ is well defined except for one point: to show that the relation

$$\mathcal{L}(F_\gamma) \cong \{X^{\sigma_1}, hX^{\sigma_0}(1-h)\}'' \text{ if } \gamma = 1 + 2\tau(h)\tau(1-h),$$

is consistent with the previous definition. This is done in the following lemma.

**Lemma 4.3** *Let $(Y^s)_{s\in S}$be a free semicircular family, let $\sigma_1, \sigma_2$ be two distinct elements in $S$ and let $h, g$ be two nonzero projections in $\{Y^{\sigma_1}\}''$. For all $s \in S\setminus\{\sigma_1,\sigma_2\}$ let $e_s, f_s$, be projections in $\{Y^{\sigma_1}\}''$ which are either mutually orthogonal or equal.*

*Assume that $\tau(h)\tau(1-h) = \tau(g)\tau(1-g) + \sum_s k_s\tau(e_s)\tau(f_s)$, with $k_s$ as above. Then the type $II_1$ factor $\{Y^{\sigma_1}, hY^{\sigma_2}(1-h)\}''$ is isomorphic to*

$$\{Y^{\sigma_1}, gY^{\sigma_2}(1-g), (e_sY^sf_s)_{s\in S\setminus\{\sigma_1,\sigma_2\}}\}''.$$

*Proof* . Clearly it is sufficient to prove the statement for $\tau(g)\tau(1-g)$ small enough. Let $h_1, h_2$ be nonzero projections in $\{Y^{\sigma_1}\}''$, with $h_1 \le h, h_2 \le 1-h$ and so that $h_1 \ne h$ and $h_2 \ne 1-h$.

It is also clear that $\{Y^{\sigma_1}, hY^{\sigma_2}(1-h) - h_1Y^{\sigma_2}h_2\}''$ is a factor. Hence we may use of Theorem 4.1, to show that an equivalent system of generators for

$$\mathcal{L} = \{Y^{\sigma_1}, hY^{\sigma_2}(1-h)\}''$$

is

$$\{Y^{\sigma_1}, hY^{\sigma_2}(1-h) - h_1Y^{\sigma_2}h_2, gY^{\sigma_3}(1-g)\}.$$

Fix $\sigma_3 \in S\setminus\{\sigma_1,\sigma_2\}$ and let $g$ be a nonzero projection in $\{Y^{\sigma_1}\}''$ with

$$\tau(g)\tau(1-g) = \tau(h_1)\tau(h_2).$$

By Theorem 4.1 this ends the proof of Lemma 4.3 and this completes the proof of the correctness of Definition 4.2.

The Definition 4.2 directly implies the following statement:

**Proposition 4.4** *For all $r, r' > 1$ the von Neumann algebra, reduced free product $\mathcal{L}(F_r) * \mathcal{L}(F_r')$ is isomorphic to $\mathcal{L}(F_{r+r'})$.*

Formula (4.3) is a little bit more difficult to prove than formula (4.4) but if we choose the right generators for the algebra $\mathcal{L}(F_r)$ then the computations for the isomorphism class of $\mathcal{L}(F_r)_t$ are less involved.

**Proposition 4.5** *The type $II_1$ factor $\mathcal{L}(F_r)_t$ is isomorphic to $\mathcal{L}(F_{(r-1)t^{-2}+1})$, for all $t > 0$, $r > 1$.*

*Proof* .Let $(Y^s)_{s\in S}$ be an infinite free semicircular family. Clearly it is sufficient to prove the statement for $t$ sufficiently close to 1. Assume that $\mathcal{L}(F_r)$ is generated as in Definition 4.2 by

$$\{Y^{\sigma_1}, hY^{\sigma_2}(1-h), (e_sY^sf_s)_{s\in S\setminus\{\sigma_1,\sigma_2\}}\}$$

where $e_s, f_s$ are projections in $\{Y^{\sigma_1}\}''$ which are either mutually orthogonal or equal and $h$ is a nonzero projection $\{Y^{\sigma_1}\}''$. We may also assume (by Theorem 4.1) that

$\tau(h) = t > 1/2$ and that $e_s, f_s \leq h$. It will be sufficient to find the isomorphism class of $h\mathcal{L}(F_r)h$.

Let $f$ be any projection in $\{Y^{\sigma_1}\}''$ with $f \leq h$, $\tau(f) = \tau(1-h)$. Let $w$ be the partial isometry from the polar decomposition

$$(1-h)Y^{\sigma_1}f = wb.$$

Note that by Theorem 2.5 in [28] we have $w = (1-h)wf$. Let $B = fBf$ be a semicircular element generating the same von Neumann algebra as $\{fbf\}''$and let $A = (1-h)A(1-h)$ be a semicircular generator for $\{(1-h)X^{\sigma_1}(1-h)\}''$.

Then $h\mathcal{L}(F_r)h$ is generated by

$$\{hY^{\sigma_1}h, B, (h-f)X^{\sigma_2}w, w^*Aw, e_sX^sf_s\}.$$

and let $k_s$ be as in Definition 4.2. By the Definition 4.2, the isomorphism class of the algebra $h\mathcal{L}(F_r)h$ is thus $\mathcal{L}(F_M)$ with

$$M = 1 + (\tau(h))^{-2}\left[2(\tau(1-h))^2 + 2\tau(1-h)\left[\tau(h) - (1-\tau(h)\right] + \gamma\right] =$$

$$1 + t^{-2}(2(1-t)^2 + 2(1-t)(2t-1) + \gamma) = 1 + t^{-2}(2t(1-t) + \gamma).$$

On the other hand $r = \gamma + 2t(1-t)$ since $t = \tau(h)$.

To complete the proof it remains to check the conditions in the statement of Definition 4.2. We have to show that there exists a semicircular family $(Z^t)_{t\in T}$ with

$$Z^t = hZ^th,\ t \in T,$$

containing $(hX^sh)_{s\in S}$ and so that there exist distinct elements

$$Z^{t_0}, Z^{t_1}, Z^{t_2} \in (Z^t)_{t\in T} \setminus (hX^sh)_{s\in S}$$

with

$$B = fZ^{t_0}f,\ (h-f)X^{\sigma_2}w = (h-f)Z^{t_2}f,\ w^*Aw = fZ^{t_1}f.$$

This is a direct consequence of Theorem 2.5 in [28] and of the proof of Theorem 3.2 in [28] once we observe that to prove the last statement, we may assume that $t = \tau(h) = 1/n$ for some $n \in \mathbb{N}$. In this form, the last statement to prove is a general fact that concerns only the semicircular family $(Y^s)_{s\in S}$. This completes the proof.

The consequence of the two formulae (4.3), (4.4) is the fact that the isomorphism class of $\mathcal{L}(F_r)$ is either the same for all finite $r > 1$ or that the algebras $(\mathcal{L}(F_r))_{r\in\mathbb{R}, r>1}$ are mutually nonisomorphic.

**Corollary 4.6** *The fundamental group of $\mathcal{L}(F_r)$, for finite $r > 1$ is either $\mathcal{R}_+/\{0\}$ or* 1, *independently of $r$.*

*Proof*. Clearly, if $x \geq y \geq r > 1$, then $\mathcal{L}(F_x) \cong \mathcal{L}(F_y)$ is equivalent to the fact that $\left[(y-1)/(x-1)\right]^{1/2}$ is in $\mathcal{F}(\mathcal{L}(F_r))$. Thus if $\mathcal{F}(\mathcal{L}(F_r))$ is non trivial then $\mathcal{L}(F_x) \cong \mathcal{L}(F_y)$ for some $x > y$ and thus by Proposition 4.4

$$\mathcal{L}(F_{x+a}) \cong \mathcal{L}(F_{y+a}), a \geq 0.$$

It follows that $\mathcal{F}(\mathcal{L}(F_r))$ contains a line and hence $\mathcal{F}(\mathcal{L}(F_r)) = \mathbb{R}_+/\{0\}$ (since $\mathcal{F}(\mathcal{L}(F_r))$ is a multiplicative group).

The next corollary gives a more precise statement than the previous one.

**Corollary 4.7** *One (and only one) of the following two statements holds true:*

*(i) The type $II_1$ factor $\mathscr{L}(F_r)$ is isomorphic to $\mathscr{L}(F_\infty)$ for all (equivalently for one) finite $r > 1$.*

*(ii) The type $II_1$ factors $\mathscr{L}(F_r)$, $r \in (1, \infty]$ are mutually non-isomorphic.*

*Note that statement in (i) is equivalent to the following weaker statement:*

*(i′) $\mathscr{L}(F_r) \otimes B(H) \cong \mathscr{L}(F_\infty) \otimes B(H)$ for some finite $r > 1$.*

*(i)″ The fundamental group $\mathscr{F}(\mathscr{L}(F_r))$ is nontrivial for some (equivalently for all) finite $r > 1$.*

*Recall that by Corollary 4.5, the algebras $\mathscr{L}(F_r) \otimes B(H)$ are all isomorphic for finite $r > 1$.*

*Proof* . By the preceding corollary, we only have to prove that the hypothesis that the type $II_1$ factors $\mathscr{L}(F_s)$ are all isomorphic for finite $s$ would imply that $\mathscr{L}(F_r)$ is isomorphic to $\mathscr{L}(F_\infty)$ for all $r$.

Let $(X^s)_{s \in S}$ be an infinite free semicircular family and fix a finite $r > 2$. Let

$$\{\sigma_0, \sigma_1, \dots\} \cup \{\nu_0, \nu_1, \dots\}$$

be an infinite subset of distinct indices in $S$ with infinite complement and let $(p_i)_{i=0,1,\dots}$ be projections in $(X^{\sigma_0})''$ with

$$r = 2 + 2 \sum_{i \geq 1} \tau(p_i)^2.$$

By definition $\mathscr{L}(F_r)$ is isomorphic to

$$\mathscr{A} = \{X^{\sigma_0}, X^{\nu_0}, p_i X^{\sigma_i} p_i, p_i X^{\nu_i} p_i, i \in \mathbb{N}\}''.$$

Let $S = \cup_{i \in \mathbb{N}} S_i'$ be a partition of $S$ with infinite sets and let $S_i$ be finite subsets of $S_i'$, of cardinality card $S_i = N_i$ and so that $\sigma_i, \nu_i \in S_i$ for all $i$. In this case our assumption (that all $\mathscr{L}(F_r)$ are isomorphic for finite $r > 1$) implies that there exists a semicircular family $(Z^s)_{s \in S_i} \subseteq \{(p_i X^{\sigma_i} p_i), (p_i X^{\nu_i} p_i)\}''$, with $Z^s = p_i Z^s p_i$, $s \in S_i$ and such that

$$\{(Z^s)_{s \in S_i}\}'' = \{(p_i X^{\sigma_i} p_i), (p_i X^{\nu_i} p_i)\}'',$$

for all $i$.

Since $(Z^s)_{s \in S_i'} \subseteq \{(p_i X^s p_i)_{s \in S_i}\}''$, the random matrix picture of Voiculescu (see the next lemma), shows that there exists a free semicircular family

$$(T^s)_{s \in \cup_n S_n} \subseteq (X^s)_{s \in S}$$

so that

$$Z^s = p_i T^s p_i \text{ for all } s \in S_i, i \in \mathbb{N}$$

and

$$T^{\sigma_0} = X^{\sigma_0}, T^{\nu_0} = X^{\nu_0}.$$

Thus

$$\mathscr{L}(F_r) = \{X^{\sigma_0}, X^{\nu_0}, (p_i X^{\sigma_i} p_i), (p_i X^{\nu_i} p_i), i \in \mathbb{N}\}'' =$$

$$\{X^{\sigma_0}, X^{\nu_0}, (Z^s)_{s \in S_i}, i \in \mathbb{N}\}'' =$$

$$\{T^{\sigma_0}, T^{\nu_0}, (p_i T^s p_i)_{s \in S_i}, i \in \mathbb{N}\}''.$$

By the Definition 4.2 we obtain that $\mathcal{L}(F_r)$ is isomorphic to $\mathcal{L}(F_{r'})$ where

$$r' > \sum N_i \tau(p_i)^2.$$

By choosing $N_i$ big enough, for each $i$, we get $r' = \infty$.

The procedure used in the construction of $(T^s)_{s \in \cup_n S_n}$ is explained in the following lemma

**Lemma 4.8** *Let $(x^s)_{s\in S}$ be an infinite free semicircular family. Fix a partition $S = S' \cup S''$ with infinite $S''$ and fix three distinct indices $\sigma_0, \sigma, \nu \in S'$ and a nonzero projection $p$ in $\{x^{\sigma_0}\}''$. Let $(z^t)_{t\in T}$ be a free semicircular family in $\{px^\sigma p, px^\nu p\}''$.*

*Then there exists a free semicircular family $(a_t)_{t\in T}$ in $\{(x^s)_{s\in S'\setminus\{\sigma_0\}}\}''$ which is free with respect to $\{(x^s)_{s\in S''}\}'' \cup \{x^{\sigma_0}\}''$ and so that $z^t = pa^t p$ for $t \in T$.*

*Proof* . It is clearly sufficient to prove the statement under the additional assumption that $S'$ is infinite. Also we may assume that $\tau(p) = 1/n$ (eventually by considering a semicircular family $(\tilde{x}^s)_{s\in S}$ and a projection $\tilde{p}$ in $\{\tilde{x}^{\sigma_0}\}''$ so that $x^s = \tilde{p}\tilde{x}^s\tilde{p}$ for all $s \in S$ and by proving first the similar result for the later family).

Fix $\theta$ in $S'/\{\sigma_0, \sigma, \nu\}$ and let $(e_i)_{i=1}^n$ be a partition of the unity in $\{x^{\sigma_0}\}''$, with $\tau(e_i) = 1/n$ for $i = 1, \ldots n$ and $e_1 = p$.

As in the proof of Theorem 3.2 in [28], let $w_{ij}$ be the matrix unit $w_{ij} = w_i^* w_j$, where $w_i$ is the partial isometry from the polar decomposition of

$$e_i x^\theta e_1 = w_i b_i, \ 2 \le i \le n$$

and let $w_1 = e_1$.

Let $B_i$ be a semicircular element generating the same von Neumann algebra as $b_i$ and let $d_i$ be a semicircular generator of $e_i\{x^{\sigma_0}\}''e_i$ for all $i = 1, \ldots, n$. Denote

$$\mathcal{E}_\theta = \{w_i^* x^\theta w_i | 2 \le i \le n\} \cup \{B_i | 1 \le i \le n\} \cup$$

$$\{w_i^*(\operatorname{Re} x^\theta) w_j, \ w_i^*(\operatorname{Im} x^\theta) w_j | 1 \le i < j \le n\}$$

and

$$\mathcal{E} = \mathcal{E}_\theta \cup \{w_i^*(\operatorname{Re} x^s) w_j, \ w_i^*(\operatorname{Im} x^s) w_j | 1 \le i < j \le n, s \in S/\{\theta, \sigma_0\}\} \cup$$

$$\{w_i^* x^s w_i | 1 \le i \le n, s \in S/\{\theta, \sigma_0\}\} \cup$$

$$\{w_i^* d_i w_i | 1 \le i \le n\}.$$

Also denote by $\mathcal{E}'$ the set obtained by replacing $S$ by $S'$, everywhere in the above definition of the set $\mathcal{E}$. Let $\mathcal{E}'' = \mathcal{E} \setminus \mathcal{E}''$.

By Theorem 2.6 in [28], $\mathcal{E}$ is a free semicircular family and thus so is

$$\mathcal{E}_0' = \left(\mathcal{E}' \cup \{z^t\}_{t\in T}\right) \setminus \{px^\sigma p, px^\nu p\}.$$

Moreover $\mathcal{E}_0' \cup \mathcal{E}''$ is a free semicircular family. To simplify the notation write

$$\{c^r \mid r \in R\} = \mathcal{E}_0' \setminus \left[\{w_i^* d_i w_i | 1 \le i \le n\} \cup \{z^t\}_{t\in T} \cup \mathcal{E}_\theta'\right]$$

and fix an injective map $\chi : T \times \{1, \ldots, n\}^2 \to R$.

We may now paste back the elements in $\mathcal{E}_0'$ in a new semicircular family:

$$a_t = \sum_{1\leq i<j\leq n} \left[w_i c^{\chi(i,j,t)} w_j^* + \sqrt{-1} w_i c^{\chi(j,i,t)} w_j^*\right]$$

$$+ \sum_{1\leq i<j\leq n} \left[w_j c^{\chi(i,j,t)} w_i^* - \sqrt{-1} w_j c^{\chi(j,i,t)} w_i^*\right]$$

$$+ \sum_{2\leq i\leq n} w_i c^{\chi(i,i,t)} w_i^* + p z^t p, t \in T.$$

The same argument as above (i.e. Theorem 2.6 in[28]) shows that $(a_t)_{t\in T}$ is free with respect to $(x^s)_{s\in S''} \cup \{x^{\sigma_0}, x^{\theta}\}$ (using also the fact that $\mathcal{E}_0'$ is free with respect to $\mathcal{E}''$). This ends the proof of the Lemma 4.8.

The construction of $T^s$ in Proposition 4.7 runs as follows : By induction, assume that we constructed up to a given $n \in \mathbb{N}$ the family

$$(T^s)_{s\in \cup_{i=1}^n S_i}$$

with the required properties, i.e. such that $(T^s)_{s\in \cup_{i=1}^n S_i}$ is semicircular and free with respect to $\{x^\sigma, x^\nu\}''$ and so that

$$p_i T^s p_i = Z^s, s \in S_i, i = 1, 2, \ldots, n.$$

We apply the above lemma with $x^{\sigma_0} = X^{\sigma_0}$,

$$\{x^s\}_{s\in S''} = \{X^{\nu_0}\} \cup (T^s)_{s\in\cup_{i=1}^n S_i} \cup \{X^s | s \in S_i', \ i = n+2, \ldots\},$$

$$\{x^s\}_{s\in S'} = \{X^{\sigma_0}\} \cup \{X^s\}_{s\in S'_{n+1}},$$

$$p = p_{n+1}, x^\sigma = X^{\sigma_{n+1}}, x^\nu = X^{\nu_{n+1}}$$

and

$$(z^t)_{t\in T} = (Z^s)_{s\in S_{n+1}}.$$

We take $(T^s)_{s\in S_{n+1}}$ be the family $(a^t)_{t\in T}$ given by Lemma 4.8. This completes the proof of Corollary 4.7.

Finally we state a slightly different form of the definition of $\mathcal{L}(F_\gamma)$, which will occur in the analysis of the algebras $(\mathcal{L}(F_\infty) \otimes B) *_B A$, where $B \subseteq A$ are finite dimensional algebras with disjoint centers.

**Lemma 4.9** . *Let $(X^s)_{s\in S}$ be an infinite free semicircular family, let $D_0$ be a discrete von Neumann abelian algebra with minimal projections $(e_i)_{i\in I}$ and let $\mathcal{B}$ be the von Neumann algebra free product $\{(X^s)_{s\in S}\}'' * D_0$.*

*Fix an element $\sigma$ in $S$ and let $p_s, h_s$ be projections in the diffuse abelian von Neumann algebra $D = \{D_0, (e_i X^\sigma e_i)_{i\in K}\}''$ which are either mutually orthogonal or equal and let $k_s = 1$* if $h_s = p_s$ and $k_s = 2$ if $p_s h_s = 0$.

*Assume that $N$ is a reflexive symmetric relation on $K \times K$ with a single orbit and let*

$$\gamma = 1 + \sum_s k_s \tau(h_s)\tau(p_s) + \sum_{(i,j)\in N} \tau(e_i)\tau(e_j) - \sum \tau(e_i)^2.$$

*Then*

$$\mathcal{A} = \{D_0, \sum_{(i,j)\in N} e_i X^\sigma e_j, (p_s X^s h_s)_{s\in S}\}''$$

*is isomorphic to* $\mathcal{L}(F_\gamma)$.

*Proof* . Let $k$ be the cardinality of $K$ ($k$ may be infinite). If $k = 1$ the statement is obvious. Thus we may assume that $k \geq 2$, $K = \{1, 2 \ldots, k\}$ and that

$$(1,2), (2,3), \ldots, (k-1, k) \in N.$$

We use the same kind of arguments as in Proposition 4.3.

Replacing $\mathcal{B}$ with eventually a larger type $II_1$ factor, we may assume that there exists semicircular elements $X^{\sigma_0}, X^{\sigma_1}, X^{\sigma_2}$ in $\mathcal{B}$, so that

$$(X^s)_{s\in S} \cup \{X^{\sigma_0}, X^{\sigma_1}, X^{\sigma_2}\}$$

is a free semicircular family.

For all $s \in S$ let $p'_s$, $h'_s \in \{X^{\sigma_0}\}''$ be projections that are either mutually orthogonal or equal and have the property that

$$\tau(p'_s) = \tau(p_s),\ \tau(h'_s) = \tau(h_s) \text{ and } p'_s h'_s = 0 \text{ iff. } p_s h_s = 0 \text{ for all } s \in S.$$

We choose a partition of the unity $(e'_i)_{i\in I}$ in $\{X^{\sigma_0}\}''$ with $\tau(e_i) = \tau(e'_i)$ and let $g_1, g_2$ be two non zero projections in $\{X^{\sigma_0}\}''$ with $g_i \leq e'_i$, $g_i \neq e'_i$ for $i = 1, 2$.

By Lemma 3.1 the algebra

$$\{X^{\sigma_0}, \sum_{|r-p|\geq 1, (r,p)\in N} e'_r X^{\sigma_1} e'_p, (p'_s X^s h'_s)_{s\in S/\{\sigma\}}\}''$$

is isomorphic to $\mathcal{A}$. Let $k_0$ be any projection in $\{X^{\sigma_0}\}''$ with $2\tau(k_0)\tau(1-k_0) = 2\tau(g_1)\tau(g_2)$.

Since

$$A = \{X^{\sigma_0}, \sum_{|r-p|\geq 1, (r,p)\in N} e'_r X^{\sigma_1} e'_p - g_1 X^{\sigma_1} g_2 - g_2 X^{\sigma_1} g_1\}''$$

is a factor (due to the choice for $g_1, g_2$), by Theorem 4.1, it follows that $\mathcal{A}$ is also isomorphic to the algebra

$$\{X^{\sigma_0}, \sum_{r\neq p} e'_r X^{\sigma_1} e'_p - g_1 X^{\sigma_1} g_2 - g_2 X^{\sigma_1} g_1, k_0 X^{\sigma_2}(1-k_0), (p'_s X^s h'_s)_{s\in S/\{\sigma\}}\}''.$$

The Definition 4.2 shows that the isomorphism class of $\mathcal{A}$ is thus $\mathcal{L}(F_r)$ with

$$r = 1 + \left[\sum_{r\neq p} \tau(e_r)\tau(e_p) - 2\tau(g_1)\tau(g_2)\right] + 2\tau(k_0)\tau(1-k_0) + \sum_s \tau(h_s)\tau(p_s).$$

This completes the proof.

## 5. Subfactors in $\mathcal{L}(F_N)$

In this paragraph we prove our main result for $\mathcal{L}(F_N)$, in the case of finite real $N > 1$. In fact due to Theorem 1.2, we only have to check that any $\mathcal{L}(F_N)$, for finite $N > 1$, is isomorphic to the amalgamated free product $(Q \otimes B) *_B A$ for some $Q = \mathcal{L}(F_M)$ depending on $N$. This is done in the first theorem.

The expression that we find for $M$ (as a function of $N$) will be used in the proof of our main result to determine the isomorphism class of the subfactors given by Theorem 1.2.

**Theorem 5.1** *Let $1_A \in B \subseteq A$ be an inclusion of finite dimensional algebras and assume that the centers of $A$ and $B$ have trivial intersection. Let $\tau$ be a normalized faithful trace on $A$, let $\Gamma$ be the inclusion matrix of $A \supseteq B$ and let $(s_k)_{k \in K}$ be the vector of the values of the trace $\tau$ on a set of representatives for the minimal projections in $A$. Then*

$$(\mathcal{L}(F_N) \otimes B) *_B A \cong \mathcal{L}(F_M),$$

*and*

$$M = 1 + N < (\Gamma\Gamma^t)s, s > - < s, s > .$$

*Remark 5.2* Note that the above statement holds true under the weaker assumptions that $A, B$ are finite type $I$ von Neumann algebras with discrete centers.

*Proof of Theorem* 5.1. Let $\{f_l\}_{l \in L}$ be a maximal set of mutually orthogonal, minimal, nonequivalent projections in $B$ and let $\{t_l\}_{l \in L}$ be the vector of the values of the trace $\tau$ on $\{f_l\}_{l \in L}$. Let $f = \sum f_j$. Clearly the inclusion matrix of $B_f \subseteq A_f$ is again $\Gamma$ and $B_f$ is abelian.

By the proof of Lemma 2.2 we have that that

$$\left[(\mathcal{L}(F_N) \otimes B) *_B A\right]_{\sum t_l} \cong (\mathcal{L}(F_N) \otimes B_f) *_{B_f} A_f.$$

Let $(e_k)_{k \in K}$ be a maximal set of mutually orthogonal, minimal, nonequivalent projections in $A_f$, that commute with $\{f_l\}_{l \in L}$ and let $e = \sum e_i$. Note that the relative traces $(\tau'(e_i))_{i \in K}$ of the projections $(e_i)_{i \in K}$ in the reduced algebra $A_{fe}$ are now given by the formula

$$\tau'(e_i) = (\sum t_l)^{-1}(\sum s_k)^{-1}\tau(e_i) =$$

(5.1) $$= (\sum t_l)^{-1}(\sum s_k)^{-1} s_i, i \in K.$$

We use the procedure described in the Proposition 3.2 to find a convenient system of generators for the von Neumann algebra

$$\left[(\mathcal{L}(F_N) \otimes B_f) *_{B_f} A_f\right]_e .$$

We will first prove the theorem under the additional assumption that $N \geq 2$ is an integer.

Let $(Y^s)_{s=1,2,\ldots,N}$ be a free semicircular family that is also free with $A_f$ and let

$$Z^s = \sum \tau(f_l)^{-1/2} f_l Y^s f_l, s = 1, 2, \ldots, N.$$

Recall the notations from Corollary 2.3: $r_k$ is the central support of $e_k$ in $A$ for $k \in K$ and $\{e^k_{pq}\}^{t_k}_{pq=1}$ is a matrix unit for $Ar_k$ with the property that $e^k_{pp}$ commutes with $(f_l)_{l\in L}$ and $e^k_{11} = e_k$ for all $p = 1, \dots, m_k, k \in K$ ($m_k$ is the dimension of $Ar_k$).

Let $N_0$ be the reflexive symmetric relation on $K^2$ defined by

$$N_0 = \{(k, m) \in K \times K | \sum_{l\in L} a_{kl}a_{ml} \neq 0\}.$$

Note that $N_0$ contains a single orbit, since the centers of $A$ and $B$ have trivial intersection.

Recall from Proposition 2.1 that $(Z^s)_{s=1,\dots,N}$ and $A_f$ is a system of generators for $(\mathcal{L}(F_N) \otimes B_f) *_{B_f} A_f$. Let

$$\mathcal{X}^{\cdot} = \bigcup_{(m,k)\in N_0} \{e^k_{1p}Z^s e^m_{q1} | s = 1, 2, \dots, N, p = 1, \dots, t_k, q = 1, \dots, t_m\}.$$

By Corollary 2.3 a system of generators for

$$((Q \otimes B) *_B A)_{fe} \cong ((Q \otimes B_f) *_{B_f} A_f)_e$$

(with $Q = \mathcal{L}(F_N)$) is $\mathcal{X}^{\cdot}$ and $A_{fe}$. If we fix $(k, m) \in K^2$ then there are exactly

$$\sum_{l\in L} a_{ml}a_{kl}$$

nonzero blocks of the form $e^k_{1p}Z^s e^m_{q1}$ in the set $\mathcal{X}^{\cdot}$ for each fixed $s = 1, 2, \dots, N$ (since $Z^s$ commutes with $f_l, l \in L$).

By the Lemma 4.9 and Eq. (5.1), if we sum the "area" of these elements, we obtain that $((\mathcal{L}(F_N) \otimes B_f) *_{B_f} A_f)_e$ is isomorphic to $\mathcal{L}(F_\nu)$ where $\nu$ is given by the formula

$$\nu = 1 + N \sum_{l\in L} \sum_{\imath,\jmath\in K^2} a_{\imath l}a_{\jmath l}\tau'(e_\imath)\tau'(e_\jmath) - \sum_{k\in K} \tau'(e_k)^2$$

$$= 1 + N < (\Gamma\Gamma^t)s, s > (\sum t_l)^{-2}(\sum s_k)^{-2} - < s, s > (\sum t_l)^{-2}(\sum s_k)^{-2}.$$

By Proposition 4.5 in the preceding paragraph (the reduction formula) we deduce that $(\mathcal{L}(F_N) \otimes B_f) *_{B_f} A_f$ is isomorphic to $\mathcal{L}(F_P)$ where $P$ is determined by

$$\mathcal{L}(F_P)_{s_1+\dots s_k} \cong \mathcal{L}(F_\nu).$$

Consequently

$$(P - 1)(\sum s_k)^{-2} = \nu - 1$$

or

$$P = 1 + N \left[< (\Gamma\Gamma^t)s, s > - < s, s >\right] (\sum t_l)^{-2}.$$

Hence

$$\left[(\mathcal{L}(F_N) \otimes B) *_B A\right]_{\sum t_l} \cong$$

$$\left[(\mathcal{L}(F_N) \otimes B) *_B A\right]_f \cong (\mathcal{L}(F_N) \otimes B_f) *_{B_f} A_f \cong \mathcal{L}(F_P),$$

with $P$ as above. The Proposition 4.5 and Lemma 2.2 implies again that $(\mathscr{L}(F_N)\otimes B)*_B A$ is isomorphic to $\mathscr{L}(F_M)$ where $M$ is determined by the relation

$$(M-1)(\sum t_l)^{-2}=P-1$$

whence the formula in the statement.

We now complete the proof of the theorem by dropping the assumption that $N$ is natural. Clearly it is sufficient to prove that $(\mathscr{L}(F_\nu)\otimes B)*_B A$ is isomorphic to $\mathscr{L}(F_{M(\nu)})$ with $M(\nu)$ an affine function depending on $\nu$.

Let $e,f$ be as above and let $\mathscr{B}$ be a type $II_1$ factor that contains $A_f$ and an infinite free semicircular family $(Y^s)_{s\in S}$ that is free with respect to $A_f$. Let $(S_l)_{l\in L},\{\sigma_l\}_{l\in L}$ be disjoint, nonvoid subsets of $S$ and let $g_s^l,h_s^l$ be projections (either mutually orthogonal or equal) in $\{(f_lY^{\sigma_l}f_l\}''$, $s\in S_l, l\in L$, so that

$$\tau(f_l)^2(\nu-1)=\sum_{s\in S_l}k_s\tau(g_s^l)\tau(h_s^l)\text{ for all } l\in L$$

and $k_s=2$ if $g_s^lh_s^l=0$ or $k_s=1$ if $g_s=h_s$. By Theorem 4.1, by Proposition 2.1 and by Lemma 4.9, a possible system of generators for

$$\mathscr{A}=(\mathscr{L}(F_\nu)\otimes B_f)*_{B_f}A_f$$

is

$$A_f,(f_lY^{\sigma_l}f_l)_{l\in L}\text{ and }(g_sY^sh_s)_{s\in S_l}.$$

Note that

$$\{f_lY^{\sigma_l}f_l,(g_s^lY^sh_s^l)_{s\in S_l}\}''$$

is a copy of $\mathscr{L}(F_\nu)$ for each $l$.

For each $l\in L$ fix a projection $e_{k_l}\in(e_k)_{k\in K}$ with $e_{k_l}\le f_l$. Since $\{A_{f_l},f_lY^{\sigma_l}f_l\}''$ is a factor, by Theorem 4.1, we do not change the isomorphism class of $\mathscr{A}$ by replacing for $s\in S_l$, the projections $g_s^l,h_s^l$ with projections $g_s',h_s'$ in $\{e_{k_l}Y^{\sigma_l}e_{k_l}\}''$ that are either mutually orthogonal or equal with

$$\tau(g_s')=\tau(g_s),\ \ \tau(h_s)=\tau(h_s'),\ \ g_s'h_s'=0\text{ iff.}g_sh_s=0\text{ for all } s\in S_l,l\in L$$

and

$$\tau(f_l)^2(\nu-1)=\sum_{s\in S_l}k_s\tau(g_s')\tau(h_s')\text{ for all } l\in L.$$

This time the generators for the reduced algebra $\mathscr{A}_e$ are: $(e_k)_{k\in K}$, the elements

$$\{g_s'Y^sh_s'|s\in S_l,l\in L\}$$

and a fixed set of generators $O\subseteq\mathscr{A}_e$ corresponding to $f_lX^{\sigma_l}f_l$, for all $l\in L$. Note that by hypothesis $O$ and $(e_k)_{k\in K}$ are generating a factor.

For each $k\in K$ fix a nonzero element of the form $e_kX^{\nu_{l_k}}e_k$ and let $D$ be the diffuse abelian von Neumann algebra generated by$(e_k)_{k\in K}$ and $e_kX^{\nu_{l_k}}e_k, k\in K$. We use again Theorem 4.1 and replace this time the projections $g_s',h_s'$ by similar projections $g_s'',h_s''\in\mathscr{P}(D)$ with

$$\tau(g_s')=\tau(g_s''),\ \ \tau(h_s')=\tau(h_s''),$$

and

$$g_s''h_s'' = 0 \text{ iff. } g_s h_s = 0 \text{ for all } s \in S_l, l \in L.$$

Thus $\mathcal{A}_e$ has the following system of generators : $D$, the set $O'$ (obtained from $O$ by deleting the elements used to construct $D$) and

$$\{g_s''Y^s h_s'' | s \in S_l, l \in L\}.$$

As $O'$ is independent on $\nu$ and since

$$\sum_{s \in S_l, l \in L} k_s \tau'(g_s'')\tau'(h_s'')$$

is an affine function of $\nu$ (recall that $\tau'$ is the induced trace on $\mathcal{A}_e$), the definition 4.2 shows that the isomorphism class of $\mathcal{A}_e$ is $\mathcal{L}(F_{N(\nu)})$ where $N(\nu)$ is an affine function of $\nu$. The reduction formula (4.4) shows that the same is true for $\mathcal{A}$ and thus for $(\mathcal{L}(F_\nu) \otimes B) *_B A$.

This completes the proof of our theorem. The proof of the Remark 5.2 now follows from the Remark 2.4.

By the above results and by Theorem 1.2 we get:

**Theorem 5.3** *Let $\mathcal{C} = (A \supseteq B \supseteq C; A \supseteq C \supseteq D)$ be a commuting square ([20]) of finite dimensional algebras. Assume that $\mathcal{C}$ is irreducible (i.e. the centers of $A, B$ and respectively $C, D$ have trivial intersection) and $\lambda$-Markov (i.e. there exists a $\lambda$-Markov trace ([11]) for $C \subseteq A$ which restricts to a $\lambda$-Markov trace for $D \subseteq B$). Let $N \geq 2$ be any natural number (or more generally let $N > 1$ be any real number).*

*Then there exists a subfactor $\mathcal{A} \subseteq \mathcal{L}(F_N)$ of index $\lambda^{-1}$. In addition the relative commutant $\mathcal{A}' \cap \mathcal{L}(F_N)$ is isomorphic to $B \cap C'$ and*

$$\mathcal{A} \cong \mathcal{L}(F_{(N-1)\lambda^{-1}+1}) \cong \left[\mathcal{L}(F_N)\right]_{\lambda^{1/2}}.$$

Before proving the theorem we note the following remark which shows that one may also construct subfactors for the von Neumann algebra of a free group starting from the infinite dimensional commuting squares considered by U. Haagerup and J. Schou ([10], [26]). We state it separately to make the task easier for the reader. The proof of this remark will be identical to the proof of Theorem 5.3, by making use of the Remark 5.2.

*Remark 5.4* The statement of Theorem 5.3 holds true for the infinite dimensional commuting squares considered by U. Haagerup and C. Schou ([26],[10]).

The assumptions are now that $A, B, C, D$ are finite type $I$ von Neumann algebras with discrete centers and that all the corresponding inclusion matrices have finite norm.

Moreover one assumes that there exists a normalized faithful trace $\tau$ on $A$ which is a $\lambda$-Markov trace for $C \subseteq A$ and that $\tau$ restricts to a $\lambda$-Markov trace for the inclusion $D \subseteq B$. In addition there exists $\nu > 0$ so that $\tau$ is a $\nu$-Markov trace for the other two inclusions.

*Proof* (*of Theorem* 5.3). Let $(s_k)_{k \in K}, (t_l)_{l \in L}$ be the vectors of the values of the trace $\tau$ over a system of representatives for the minimal projections in $A$ and $C$ respectively. Let $\Gamma$ be the inclusion matrix of $A \supseteq B$ and let $N > 1$ be any real number (possibly infinite). By the assumptions on $\mathcal{C}$ it follows that $< s, s > = \lambda < t, t >$.

Indeed if $X$ is the inclusion matrix for $C \subseteq A$ then the $\lambda$-Markov property of the trace implies ([12], [31]) that $s$ is eigenvector with eigenvalue $\lambda^{-1}$ for $X^tX$. Hence

$$< t,t >=< Xs, Xs >=< X^tXs, s >= \lambda^{-1} < s,s > .$$

By the preceding theorem, there exists a real $P > 1$ so that

$$(\mathscr{L}(F_P) \otimes B) *_B A \cong \mathscr{L}(F_N)$$

and

$$N = 1 + P < (\Gamma\Gamma^t)s, s > - < s,s > .$$

On the other hand, if the inclusion matrix for $C \supseteq D$ is $\Gamma_1$. then

$$\mathscr{C} = (\mathscr{L}(F_P) \otimes D) *_D C \cong \mathscr{L}(F_M),$$

and, by the same arguments as above, we have:

$$M = 1 + P < (\Gamma_1\Gamma_1^t)t, t > - < t,t > .$$

By Proposition 1.4 in [21], $s, t$ are Perron -eigenvectors for the same eigenvalue $\nu$, i.e.

$$(\Gamma_1\Gamma_1^t)t = \nu^{-1}t, (\Gamma\Gamma^t)s = \nu^{-1}s,$$

and $\nu^{-1} = ||\Gamma||^2 = ||\Gamma_1||^2$. Hence it follows that

$$(M - 1)/(N - 1) = \lambda^{-1}$$

or equivalently

$$M = (N - 1)\lambda^{-1} + 1.$$

This completes the proof.

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