Quantum Laplacians on Generalized Operators on Boson Fock Space

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Abstract

By adapting the white noise theory, the quantum analogues of the (classical) Gross Laplacian and Lévy Laplacian, so called the quantum Gross Laplacian and quantum Lévy Laplacian, respectively, are introduced as the Laplacians acting on the spaces of generalized operators. Then the integral representations of the quantum Laplacians in terms of quantum white noise derivatives are studied. Correspondences of the classical Laplacians and quantum Laplacians are studied. The solutions of heat equations associated with the quantum Laplacians are obtained from a normal-ordered white noise differential equation.

Keywords: Fock space, generalized operator, operator symbol, quantum white noise, quantum Gross Laplacian, quantum Lévy Laplacian, heat equation.

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1 Introduction

Infinite dimensional generalizations of the usual Laplacian were introduced by V. Volterra and studied by P. Lévy [29, 30] who introduced a different type of Laplacian producing the first example of an *essentially infinite dimensional differential operator* (i.e. a differential operator which is identically zero on all functions depending only on a finite number of variables (cylindrical functions)). Gross [15] initiated a systematic study of the Volterra Laplacian in the context of abstract Wiener spaces. L. Accardi and O. Smolyanov [1] introduced a countable hierarchy (Δ_n), of essentially infinite dimensional, Laplacians with the property that (Δ_0) is the usual Laplacian, (Δ_1) the Lévy and the domain of Δ_n is contained in the kernel of Δ_{n+1}.

Quantum extensions of the Lévy Laplacian, acting on generalized operators on appropriate Boson Fock spaces, were introduced by L. Accardi, H. Ouerdiane and O. Smolyanov [4] extended to the framwork of Hida distributions by U.C. Ji and N. Obata [23]. A different extension in the same direction is due to W. Arveson [8]

These Laplacians has been studied from various points of view by many authors, see [14, 35, 36, 37] and references cited therein. Interesting connections with different fields of mathematics have emerged, for example: infinite dimensional harmonic analysis [18, 33], transformation groups [10, 17], differential equations in infinite dimensional [10, 28],[26], stochastic processes [2, 5, 39, 41], Poisson noise functionals [40], infinite dimensional rotation group [31], and Cauchy Problem [12]. Applications to physic have emerged in connections with Yang–Mills and Maxwell equations [3, 27].

The integral representations of the Volterra–Gross and the Lévy Laplacian in terms of white noise operators were obtained by H.H. Kuo [25] and motivated the conjecture by Accardi, Lu and Volovich [7] that the Lévy Laplacian should be related to the square of quantum white noise just as the usual Laplacian is realted to the first order quantum white noise. This conjecture received recently a strong support by the result of Obata [34]).

On the other hand the main result of the paper [6] was the identification of the quantum Brownian motion (QBM) associated to the Lévy Laplacian with the QBM associated to the usual Volterra–Gross Laplacian whose initial space as well as the multiplicity space of the associated white noise coincide with the Cesàro Hilbert space. A consequence of this result is the identification of the Lévy Laplacian with the Gross Laplacian on an appropriate Fock space.

Our main goal in the present paper is to exploit this identification to find a new integral representation of the Lévy Laplacian in terms of white noise operators.

This result is new even in the classical case, but we will prove it directly in the quantum case. Since the above mentioned identification is heavily based on quantum probabilistic techniques, a pre–requisite for the achievement of this goal is the development of the analogue of these techniques in a white noise framework. This was done in the paper [23] by Ji, Obata and Ouerdiane, however in this paper the authors do not consider the problem of the integral representation of the Lévy Laplacian. Since our paper heavily relies on the results of [23], we will briefly recall these results.

This paper is organized as follows: In Section 2 we review the basic construction of nuclear riggings and characterization theorems in white noise theory following [20, 32, 38]. In Section 3, following [25, 26] we recall the definitions of classical Gross and Lévy Laplacian. In Section 4, following [21, 22], we introduce the quantum white noise derivatives and study their basic properties. In Section 5 we introduce the quantum Gross and Lévy Laplacians on generalized operators and study their properties. Our main result, i.e. the integral representations of the quantum Laplacians in terms of quantum white noise derivatives are obtained in Theorems (**??**), (5.10) respectively. In Section 6 we study correspondences of the classical Laplacians and the quantum laplacians. In Section 7 we investigate solutions of the Cauchy problems associated with the quantum Laplacians connecting with a normal-ordered white noise differential equation.

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2 Preliminaries

2.1 Standard Construction of Gel'fand triple

Let $H_{\mathbf{R}} = L^2(\mathbf{R}, dt)$ be the real Hilbert space with the norm $|\cdot|_0$ generated by the inner product $\langle\cdot,\,\cdot\rangle$ and $\mathcal{E}$ the Schwartz space of rapidly decreasing functions on $\mathbf{R}$. Note that $\mathcal{E}$ is a standard countable Hilbert (nuclear) space constructed from the Hilbert space $H_{\mathbf{R}}$ and the harmonic oscillator $A = 1 + t^2 - d^2/dt^2$, i.e.,

$$\mathcal{E} = \operatorname*{proj\,lim}_{p\to\infty} \mathcal{E}_p,$$

where $\mathcal{E}_p = \mathrm{Dom}\,(A^p)$ $(p \geq 0)$ is the Hilbert space corresponding to the domain of A^p, i.e., $\mathcal{E}_p = \{\xi \in H_{\mathbf{R}}\,;\, |\,\xi\,|_p \equiv |\,A^p\xi\,|_0 < \infty\}$. Defining $\mathcal{E}_{-p}$ to be the completion of $H_{\mathbf{R}}$ with respect to $|\cdot|_{-p} \equiv |\,A^{-p}\cdot|_0$ for $p \geq 0$, we obtain a chain of Hilbert spaces $\{\mathcal{E}_p\,;\, p \in \mathbf{R}\}$. By taking topological isomorphism:

$$\mathcal{E}^* \cong \operatorname*{ind\,lim}_{p\to\infty} \mathcal{E}_{-p},$$

and by identifying $H_{\mathbf{R}}$ with its dual space, we obtain a real Gel'fand triple:

$$\mathcal{E} \subset H_{\mathbf{R}} \subset \mathcal{E}^*, \tag{2.1}$$

where $\mathcal{E}$ and $\mathcal{E}^*$ are mutually dual spaces. Finally, by taking complexification we have a complex Gel'fand triple:

$$\mathcal{S} \subset H \subset \mathcal{S}^*, \tag{2.2}$$

where $\mathcal{S}$, H and $\mathcal{S}^*$ are the complexifications of $\mathcal{E}$, $H_{\mathbf{R}}$ and $\mathcal{E}^*$, respectively. The canonical $\mathbf{C}$-bilinear form on $\mathcal{S}^* \times \mathcal{S}$ which is compatible with the inner product of H is denoted by $\langle\cdot,\,\cdot\rangle$ again.

2.2 Hida–Kubo–Takenaka Space

For each $p \in \mathbf{R}$, let $\mathcal{S}_p$ be complexification of $\mathcal{E}_p$. The (Boson) Fock space over $\mathcal{S}_p$ is defined by

$$\Gamma(\mathcal{S}_p) = \left\{ \phi = (f_n)_{n=0}^{\infty}\,;\, f_n \in \mathcal{S}_p^{\widehat{\otimes} n},\ \|\,\phi\,\|_p^2 = \sum_{n=0}^{\infty} n!\,|\,f_n\,|_p^2 < \infty \right\}.$$

From a chain of Fock spaces $\{\Gamma(\mathcal{S}_p)\,;\, p \in \mathbf{R}\}$, by setting

$$(\mathcal{S}) = \operatorname*{proj\,lim}_{p\to\infty} \Gamma(\mathcal{S}_p) \quad \text{and} \quad (\mathcal{S})^* = \operatorname*{ind\,lim}_{p\to\infty} \Gamma(\mathcal{S}_{-p}),$$

we have a complex Gel'fand triple:

$$(\mathcal{S}) \subset \Gamma(H) \subset (\mathcal{S})^*$$

which is referred to as the *Hida–Kubo–Takenaka space* in the white noise theory [16, 25, 33]. It is known that $(\mathcal{S})$ is a countable Hilbert (nuclear) space. By definition the topology of $(\mathcal{S})$ is defined by the norms

$$\| \phi \|_p^2 = \sum_{n=0}^{\infty} n! \, | f_n |_p^2 , \qquad \phi = (f_n), \qquad p \in \mathbf{R}.$$

On the other hand, for each $\Phi = (F_n) \in (\mathcal{S})^*$ there exists $p \geq 0$ such that $\Phi \in \Gamma(\mathcal{S}_{-p})$ and

$$\| \Phi \|_{-p}^2 \equiv \sum_{n=0}^{\infty} n! \, | F_n |_{-p}^2 < \infty.$$

The canonical $\mathbf{C}$-bilinear form on $(\mathcal{S})^* \times (\mathcal{S})$ is denoted by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ and we have

$$\langle\!\langle \Phi, \phi \rangle\!\rangle = \sum_{n=0}^{\infty} n! \, \langle F_n, f_n \rangle , \qquad \Phi = (F_n) \in (\mathcal{S})^*, \quad \phi = (f_n) \in (\mathcal{S}).$$

An *exponential vector* (or *coherent vector*) associated with $x \in \mathcal{S}^*$ is defined by

$$\phi_x = \left(1, x, \frac{x^{\otimes 2}}{2!}, \dots, \frac{x^{\otimes n}}{n!}, \dots \right).$$

Obviously, $\phi_x \in (\mathcal{S})^*$ and $\phi_\xi \in (\mathcal{S})$ for all $\xi \in \mathcal{S}$. In particular, ϕ_0 is called the *vacuum vector*. The *S-transform* of an element $\Phi \in (\mathcal{S})^*$ is defined by

$$S\Phi(\xi) = \langle\!\langle \Phi, \phi_\xi \rangle\!\rangle , \qquad \xi \in \mathcal{S}.$$

Every element $\Phi \in (\mathcal{S})^*$ is uniquely specified by its S-transform $S\Phi$ since $\{\phi_\xi\,;\, \xi \in \mathcal{S}\}$ spans a dense subspace of $(\mathcal{S})$. A complex-valued function F on $\mathcal{S}$ is called a *U-functional* if F is Gâteaux-entire and there exist constants $C, K \geq 0$ and $p \geq 0$ such that

$$|F(\xi)| \leq C \exp\left(K \, | \xi |_p^2 \right), \qquad \xi \in \mathcal{S}.$$

Theorem 2.1 ([38]) *A $\mathbf{C}$-valued function F on $\mathcal{S}$ is the S-transform of an element in $(\mathcal{S})^*$ if and only if F is a U-functional.*

2.3 Characterizations

A continuous linear operator Ξ from $(\mathcal{S})$ into $(\mathcal{S})^*$ is called a *generalized operator*. Let $\mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$ denote the space of all generalized operators equipped with the topology of bounded convergence. The *Wick symbol* of $\Xi \in \mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$ is defined by

$$w\Xi(\xi, \eta) = \langle\!\langle \Xi\phi_\xi, \phi_\eta \rangle\!\rangle \, e^{-\langle \xi, \eta \rangle}, \qquad \xi, \eta \in \mathcal{S}.$$

Then we have the following characterization of Wick symbols which is a operator version of the characterization of S-transform (Theorem 2.1). For the proof, we refer to [32].

Theorem 2.2 *Let Θ be a* $\mathbf{C}$*-valued function on* $\mathcal{S}\times\mathcal{S}$*. Then* Θ *is the Wick symbol of an operator in* $\mathcal{L}((\mathcal{S}),(\mathcal{S})^*)$ *if and only if* Θ *is Gâteaux entire and satisfying that*

(O) *there exist constant numbers* $C\geq 0$, $a\geq 0$ *and* $p\geq 0$ *such that*

$$|\Theta(\xi,\eta)|\leq Ce^{a(|\xi|_p^2+|\eta|_p^2)},\qquad \xi,\eta\in\mathcal{S}.$$

Let $l,m\geq 0$ and $\kappa\in(\mathcal{S}^{\otimes(l+m)})^*$. Then by applying Theorem 2.2, we prove that there exists a unique $\Xi\in\mathcal{L}((\mathcal{S}),(\mathcal{S})^*)$ such that

$$w\Xi(\xi,\eta)=\left\langle\kappa,\,\eta^{\otimes l}\otimes\xi^{\otimes m}\right\rangle,\qquad \xi,\eta\in\mathcal{S}.$$

The operator Ξ is called an *integral kernel operator* and denoted by $\Xi_{l,m}(\kappa)$. In particular, for each $x\in\mathcal{S}^*$, the annihilation operator $A(x)$ and the creation operator $A^*(x)$ are defined by

$$A(x)=\Xi_{0,1}(x)\quad\text{and}\quad A^*(x)=\Xi_{1,0}(x),$$

respectively. For notational convenience, we write

$$a_t\equiv A(\delta_t),\qquad a_t^*\equiv A^*(\delta_t),\qquad t\in\mathbf{R}.$$

Then we sometimes use a formal integral expression:

$$\Xi_{l,m}(\kappa)=\int_{\mathbf{R}^{l+m}}\kappa(s_1,\cdots,s_l,t_1,\cdots,t_m)a_{s_1}^*\cdots a_{s_l}^*a_{t_1}\cdots a_{t_m}ds_1\cdots ds_l dt_1\cdots dt_m.$$

Every operator $\Xi\in\mathcal{L}((\mathcal{S}),(\mathcal{S})^*)$ admits the following expansion:

$$\Xi=\sum_{l,m}^{\infty}\Xi_{l,m}(\kappa_{l,m}),\qquad \kappa_{l,m}\in(\mathcal{S}^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}\tag{2.3}$$

which is called the *Fock expansion* of Ξ (see [32, 33]). In this case, we have

$$w\Xi(\xi,\eta)=\sum_{l,m=0}^{\infty}\left\langle\kappa_{l,m},\,\eta^{\otimes l}\otimes\xi^{\otimes m}\right\rangle,\qquad \xi,\eta\in\mathcal{S},\tag{2.4}$$

see [19].

Theorem 2.3 ([20]) *A Gâteaux-entire function* $F:\mathcal{S}^4\to\mathbf{C}$ *is expressed in the form*

$$F(\xi_1,\xi_2,\xi_3,\xi_4)=\langle\!\langle\Xi(\phi_{\xi_1}\otimes\phi_{\xi_2}),\phi_{\xi_3}\otimes\phi_{\xi_4}\rangle\!\rangle,$$

with $\Xi\in\mathcal{L}((\mathcal{S})^{\otimes 2},(\mathcal{S})^{*\otimes 2})$ *if and only if there exist constant numbers* $C\geq 0$ *and* $p\geq 0$ *such that*

$$|F(\xi_1,\xi_2,\xi_3,\xi_4)|^2\leq C\exp\left(\sum_{i=1}^{4}|\,\xi_i\,|_p^2\right),$$

for any $\xi_i\in\mathcal{S}$, $i=1,2,3,4$.

Theorem 2.4 ([20]) *A Gâteaux-entire function* $G:\mathcal{S}^4\to\mathbf{C}$ *is expressed in the form*

$$G(\xi_1,\xi_2,\eta_1,\eta_2)=\langle\!\langle\Xi(\phi_{\xi_1}\otimes\phi_{\xi_2}),\phi_{\eta_1}\otimes\phi_{\eta_2}\rangle\!\rangle,$$

with $\Xi\in\mathcal{L}((\mathcal{S})^{\otimes 2},(\mathcal{S})^{\otimes 2})$ *if and only if for any* $p\geq 0$ *and* $\epsilon>0$ *there exist* $C\geq 0$ *and* $q\geq 0$ *such that*

$$|G(\xi_1,\xi_2,\eta_1,\eta_2)|^2\leq C\exp\epsilon\left(\sum_{i=1}^{2}|\,\xi_i\,|_{p+q}^2+\sum_{j=1}^{2}|\,\eta_j\,|_{-p}^2\right),$$

for any $\xi_1,\xi_2,\eta_1,\eta_2\in\mathcal{S}$.

3 Laplacians on Fock Space

3.1 Gross Laplacian

Let τ be the trace on H, i.e., $\langle \tau,\, \xi\otimes\eta\rangle = \langle \xi,\, \eta\rangle$, $\xi,\eta\in\mathcal{S}$. Then $\tau\in(\mathcal{S}^{\otimes 2})^*$ and the integral kernel operator

$$\Delta_{\mathrm{G}} = \Xi_{0,2}(\tau) = \int_{\mathbf{R}^2} \tau(s,t) a_s a_t ds dt$$

is called the *Gross Laplacian*, see [15, 24, 25, 33]. It is known that Δ_{G} is continuous linear operator from $(\mathcal{S})$ into itself.

Let $\{e_n\}_{n=1}^{\infty}\subset\mathcal{E}$ be a complete orthonormal basis for $H_{\mathbf{R}}$. Then the Gross Laplacian is represented by

$$\Delta_{\mathrm{G}} = \sum_{n=1}^{\infty} A(e_n)A(e_n), \tag{3.1}$$

see [25].

Let $F\in C^2(\mathcal{S})$. Then for each $\xi\in\mathcal{S}$ there exist $F'(\xi)\in\mathcal{S}^*$ and $F''(\xi)\in(\mathcal{S}\otimes\mathcal{S})^*$ such that

$$F(\xi+\eta) = F(\xi) + \langle F'(\xi),\eta\rangle + \frac{1}{2}\langle F''(\xi),\eta\otimes\eta\rangle + o(|\,\eta\,|_p^2), \qquad \eta\in\mathcal{S} \tag{3.2}$$

for some $p\geq 0$. Moreover, the maps $\mathcal{S}\ni\xi\mapsto F'(\xi)\in\mathcal{S}^*$ and $\mathcal{S}\ni\xi\mapsto F''(\xi)\in(\mathcal{S}\otimes\mathcal{S})^*$ are continuous. For more study, we refer to [13]. By the kernel theorem we have the canonical isomorphism

$$(\mathcal{S}\otimes\mathcal{S})^* \cong \mathcal{L}(\mathcal{S},\mathcal{S}^*) \cong \mathcal{B}(\mathcal{S},\mathcal{S})$$

from which, for notational convenience, sometimes we write

$$\langle F''(\xi),\eta\otimes\eta\rangle = \langle F''(\xi)\eta,\eta\rangle = F''(\xi)(\eta,\eta).$$

Note that for each $\phi\in(\mathcal{S})$, $S\phi\in C^2(\mathcal{S})$ and

$$S(\Delta_{\mathrm{G}}\phi)(\xi) = \widetilde{\Delta}_{\mathrm{G}}(S\phi)(\xi) \equiv \sum_{n=1}^{\infty}\left\langle (S\phi)''(\xi),\, e_n\otimes e_n\right\rangle, \qquad \xi\in\mathcal{S} \tag{3.3}$$

and so the Gross Laplacian can be represented by

$$\Delta_{\mathrm{G}} = S^{-1}\widetilde{\Delta}_{\mathrm{G}}S,$$

see [25].

3.2 Lévy Laplacian

Let $\{\ell_k\}_{k=1}^{\infty}$ be a fixed infinite sequence in $\mathcal{E}$ and let $\Phi\in(\mathcal{S})^*$. If the limit

$$\widetilde{\Delta}_{\mathrm{L}}(S\Phi)(\xi) = \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\langle (S\Phi)''(\xi),\, \ell_k\otimes\ell_k\rangle$$

exists for all $\xi \in \mathcal{S}$ and the function $\widetilde{\Delta}_{\mathrm{L}}(S\Phi)$ is a U-functional, then the *Lévy Laplacian* Δ_{L} is defined by

$$\Delta_{\mathrm{L}}\Phi = S^{-1}\Big(\widetilde{\Delta}_{\mathrm{L}}(S\Phi)\Big).$$

For the given infinite sequence $\{\ell_k\}_{k=1}^{\infty}$, we denote $\mathbf{L}$ the set of all elements $x \in \mathcal{S}^*$ such that the limit

$$\langle x \otimes x \rangle_{\mathrm{L}} \equiv \lim_{N\to\infty} \frac{1}{N}\left(\sum_{k=1}^{N} \langle x,\, \ell_k\rangle^2\right)$$

exists. Then for each $x \in \mathbf{L}$ we have

$$\Delta_{\mathrm{L}}\phi_x = \langle x \otimes x \rangle_{\mathrm{L}}\phi_x,$$

i.e., ϕ_x is an eigenvector of Δ_{L} corresponding to the eigenvalue $\langle x \otimes x \rangle_{\mathrm{L}}$.

4 Quantum White Noise Derivatives

4.1 Creation and Annihilation Derivatives

Note that for each $\zeta \in \mathcal{S}$, $A(\zeta)$ can be extended to a continuous linear operator (denoted by the same symbol) from $(\mathcal{S})^*$ into itself and $A^*(\zeta)$ is a continuous linear operator from $(\mathcal{S})$ into itself. Therefore, for any generalized operator $\Xi \in \mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$ and $\zeta \in \mathcal{S}$ the commutators

$$[A(\zeta), \Xi] = A(\zeta)\Xi - \Xi A(\zeta), \qquad [A^*(\zeta), \Xi] = A^*(\zeta)\Xi - \Xi A^*(\zeta),$$

are well-defined, i.e., elements of $\mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$. Then we define

$$D_\zeta^+\Xi = [A(\zeta), \Xi], \qquad D_\zeta^-\Xi = -[A^*(\zeta), \Xi].$$

The generalized operators $D_\zeta^+\Xi$ and $D_\zeta^-\Xi$ are called the *creation derivative* and *annihilation derivative* of Ξ, respectively, and both together the *quantum white noise derivatives* of Ξ, see [21, 22]. Then it is obvious that $D_\zeta^{\pm}$ becomes a linear map from $\mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$ into itself. Moreover, the bilinear map $(\zeta, \Xi) \mapsto D_\zeta^{\pm}\Xi$ is continuous from $\mathcal{S} \times \mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$ into $\mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$. In particular, for any $\zeta \in \mathcal{S}$,

$$D_\zeta^{\pm} \in \mathcal{L}(\mathcal{L}((\mathcal{S}), (\mathcal{S})^*), \mathcal{L}((\mathcal{S}), (\mathcal{S})^*)).$$

For the proof, we refer to [22].

For $\kappa \in (\mathcal{S}^{\otimes n})^*$ and $f \in \mathcal{S}^{\otimes k}$ $(0 \le k \le n)$, the left and right k–contractions $f *^k \kappa$, $\kappa *_k f \in (\mathcal{S}^{\otimes(n-k)})^*$ are defined by

$$\left\langle f *^k \kappa,\, g\right\rangle = \langle \kappa,\, f \otimes g\rangle, \qquad \langle \kappa *_k f,\, g\rangle = \langle \kappa,\, g \otimes f\rangle,$$

where $g \in \mathcal{S}^{\otimes(n-k)}$ (see [33]).

Theorem 4.1 ([22]) *For each $\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$ and any $\zeta \in \mathcal{S}$ we have*

$$D_\zeta^-\Xi = \sum_{l,m=0}^{\infty} m\Xi_{l,m-1}(\kappa_{l,m} *_1 \zeta), \qquad D_\zeta^+\Xi = \sum_{l,m=0}^{\infty} l\Xi_{l-1,m}(\zeta *^1 \kappa_{l,m}).$$

4.2 Pointwise Creation and Annihilation Derivatives

We start with the following lemma for which we refer to [23], in which we can find a special case and a different proof.

Lemma 4.2 *Let $\zeta \in \mathcal{S}$, $n \in \mathbf{N}$ and $\Xi \in \mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$. Then we have*

$$w\left(\left(D_\zeta^+\right)^n \Xi\right)(\xi, \eta) = \left.\frac{d^n}{dz^n}\right|_{z=0} w\Xi(\xi, \eta + z\zeta), \tag{4.1}$$

$$w\left(\left(D_\zeta^-\right)^n \Xi\right)(\xi, \eta) = \left.\frac{d^n}{dz^n}\right|_{z=0} w\Xi(\xi + z\zeta, \eta) \tag{4.2}$$

Proof. We now prove only (4.1). Suppose that Ξ admits the Fock expansion $\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})$, see (2.3). Then, by Theorem 4.10 in [22], we have

$$\left(D_\zeta^+\right)^n \Xi = \sum_{l,m=0}^{\infty} \frac{(l+n)!}{l!} \Xi_{l,m}(\zeta^{\otimes n} *^n \kappa_{l+n,m}).$$

Therefore, by (2.4) we have

$$\begin{aligned} w\left(\left(D_\zeta^+\right)^n \Xi\right)(\xi, \eta) &= \sum_{l,m=0}^{\infty} \frac{(l+n)!}{l!} \left\langle \kappa_{l+n,m},\, \zeta^{\otimes n} \otimes \eta^{\otimes l} \otimes \xi^{\otimes m} \right\rangle \\ &= \sum_{l,m=0}^{\infty} \left.\frac{d^n}{dz^n}\right|_{z=0} \left\langle \kappa_{l+n,m},\, (\eta + z\zeta)^{\otimes(l+n)} \otimes \xi^{\otimes m} \right\rangle \\ &= \left.\frac{d^n}{dz^n}\right|_{z=0} w\Xi(\xi, \eta + z\zeta), \end{aligned}$$

which proves (4.1). ∎

Theorem 4.3 *Let $x \in \mathcal{S}^*$. Then*

(1) *D_x^+ is continuous operator from $\mathcal{L}((\mathcal{S}), (\mathcal{S}))$ into itself;*

(2) *D_x^- is continuous operator from $\mathcal{L}((\mathcal{S})^*, (\mathcal{S})^*)$ into itself.*

Moreover, if $x \in \mathcal{S}$, then $D_x^\pm$ are continuous operators from $\mathcal{L}((\mathcal{S})^*, (\mathcal{S}))$ into itself.

Proof. (1) Note that $\mathcal{L}((\mathcal{S}), (\mathcal{S})) \cong (\mathcal{S}) \otimes (\mathcal{S})^*$ by the kernel theorem, i.e., for any $\Xi \in \mathcal{L}((\mathcal{S}), (\mathcal{S}))$ and $\phi \in (\mathcal{S})$, $\Phi \in (\mathcal{S})^*$

$$\langle\langle \Xi\phi,\, \Phi \rangle\rangle = \langle\langle \Xi,\, \Phi \otimes \phi \rangle\rangle .$$

Therefore, we have

$$\begin{aligned} \mathcal{L}(\mathcal{L}((\mathcal{S}), (\mathcal{S})), \mathcal{L}((\mathcal{S}), (\mathcal{S}))) &\cong (\mathcal{S}) \otimes (\mathcal{S})^* \otimes ((\mathcal{S}) \otimes (\mathcal{S})^*)^* \\ &\cong (\mathcal{S}) \otimes (\mathcal{S})^* \otimes (\mathcal{S})^* \otimes (\mathcal{S}). \end{aligned} \tag{4.3}$$

On the other hand, by Lemma 4.2 we have

$$\begin{aligned}\left\langle\!\left\langle D_x^+(\phi_{\xi_1}\otimes\phi_{\xi_2}),\,\phi_{\xi_3}\otimes\phi_{\xi_4}\right\rangle\!\right\rangle e^{-\langle\xi_4,\,\xi_3\rangle} &= \left\langle\!\left\langle D_x^+(\phi_{\xi_1}\otimes\phi_{\xi_2})(\phi_{\xi_4}),\,\phi_{\xi_3}\right\rangle\!\right\rangle e^{-\langle\xi_4,\,\xi_3\rangle}\\ &= (\langle x,\,\xi_1\rangle-\langle x,\,\xi_4\rangle)\,e^{\langle\xi_2,\,\xi_4\rangle+\langle\xi_1,\,\xi_3\rangle-\langle\xi_4,\,\xi_3\rangle}.\end{aligned}$$

for any $\xi_i\in\mathcal{S}$, $i=1,2,3,4$. Therefore, for any $\xi_i\in\mathcal{S}$, $i=1,2,3,4$

$$\left\langle\!\left\langle D_x^+(\phi_{\xi_1}\otimes\phi_{\xi_2}),\,\phi_{\xi_3}\otimes\phi_{\xi_4}\right\rangle\!\right\rangle = (\langle x,\,\xi_1\rangle-\langle x,\,\xi_4\rangle)\,e^{\langle\xi_2,\,\xi_4\rangle+\langle\xi_1,\,\xi_3\rangle}.$$

Hence by applying Theorem 2.4 with (4.3) we prove that $D_x^+\in\mathcal{L}((\mathcal{S})\otimes(\mathcal{S})^*,(\mathcal{S})\otimes(\mathcal{S})^*)$.

(2) Similarly, $\mathcal{L}((\mathcal{S})^*,(\mathcal{S})^*)\cong(\mathcal{S})^*\otimes(\mathcal{S})$ and

$$\mathcal{L}(\mathcal{L}((\mathcal{S})^*,(\mathcal{S})^*),\mathcal{L}((\mathcal{S})^*,(\mathcal{S})^*))\cong(\mathcal{S})^*\otimes(\mathcal{S})\otimes(\mathcal{S})\otimes(\mathcal{S})^*.$$

Also, by Lemma 4.2 we have

$$\left\langle\!\left\langle D_x^-(\phi_{\xi_1}\otimes\phi_{\xi_2}),\,\phi_{\xi_3}\otimes\phi_{\xi_4}\right\rangle\!\right\rangle = (\langle x,\,\xi_2\rangle-\langle x,\,\xi_3\rangle)\,e^{\langle\xi_2,\,\xi_4\rangle+\langle\xi_1,\,\xi_3\rangle}.$$

Hence by applying Theorem 2.4 we prove that $D_x^-\in\mathcal{L}((\mathcal{S})^*\otimes(\mathcal{S}),(\mathcal{S})^*\otimes(\mathcal{S}))$.

Finally, if $x\in\mathcal{S}$, then by applying Theorem 2.4 we prove that $D_x^\pm$ are continuous operators from $\mathcal{L}((\mathcal{S})^*,(\mathcal{S}))$ into itself. ∎

From Theorem 4.3, for each $t\in\mathbf{R}$ the quantum white noise derivatives $D_{\delta_t}^+$ and $D_{\delta_t}^-$ are well-defined as continuous linear operators acting on $\mathcal{L}((\mathcal{S}),(\mathcal{S}))$ and $\mathcal{L}((\mathcal{S})^*,(\mathcal{S})^*)$, respectively. For simple notation, we write $D_t^\pm=D_{\delta_t}^\pm$ for any $t\in\mathbf{R}$. Then D_t^+ and D_t^- are called the *pointwise creation derivative* and *pointwise annihilation derivative*, respectively.

5 Quantum Laplacians

For each $F\in C^2(\mathcal{S}\times\mathcal{S})$, there exist $F_i'(\xi_1,\xi_2)\in\mathcal{S}^*$ and $F_{ij}''(\xi_1,\xi_2)\in(\mathcal{S}\otimes\mathcal{S})^*$ for any $\xi_1,\xi_2\in\mathcal{S}$ and $i,j=1,2$ such that

$$\begin{aligned}F(\xi_1+\eta_1,\xi_2+\eta_2) &= F(\xi_1,\xi_2)+\sum_{i=1}^{2}\langle F_i'(\xi_1,\xi_2),\,\eta_i\rangle\\ &\quad+\frac{1}{2}\sum_{i,j=1}^{2}\left\langle F_{ij}''(\xi_1,\xi_2)\eta_i,\,\eta_j\right\rangle+o(|\,\eta_1\,|_p^2+|\,\eta_2\,|_p^2)\end{aligned}$$

for some $p\geq 0$. For more study, we refer to [13].

5.1 Quantum Gross Laplacian

For each $\Xi\in\mathcal{L}((\mathcal{S}),(\mathcal{S})^*)$, $w\Xi\in C^2(\mathcal{S}\times\mathcal{S})$. Define

$$\widetilde{\Delta}_{\mathrm{G}}^{\mathrm{Q}}(w\Xi)(\xi_1,\xi_2)=\sum_{k=1}^{\infty}\langle(w\Xi)_{11}''(\xi_1,\xi_2),\,e_k\otimes e_k\rangle+\sum_{k=1}^{\infty}\langle(w\Xi)_{22}''(\xi_1,\xi_2),\,e_k\otimes e_k\rangle \tag{5.1}$$

if the limits exist. If $\widetilde{\Delta}_{\mathrm{G}}^{\mathrm{Q}}(w\Xi)$ is Gâteaux entire and satisfies the condition (O) in Theorem 2.2, then there exists a unique operator, denoted by $\Delta_{\mathrm{G}}^{\mathrm{Q}}\Xi$, in $\mathcal{L}((\mathcal{S}),(\mathcal{S})^*)$ such that

$$w(\Delta_{\mathrm{G}}^{\mathrm{Q}}\Xi)=\widetilde{\Delta}_{\mathrm{G}}^{\mathrm{Q}}(w\Xi). \tag{5.2}$$

Then $\Delta_{\mathrm{G}}^{\mathrm{Q}}$ is called the *quantum Gross Laplacian*. We denote by Dom $(\Delta_{\mathrm{G}}^{\mathrm{Q}})$ the set of all generalized operators $\Xi\in\mathcal{L}((\mathcal{S}),(\mathcal{S})^*)$ such that $\Delta_{\mathrm{G}}^{\mathrm{Q}}\Xi$ is well-defined as in (5.2).

Theorem 5.1 *Let $\Xi \in \mathcal{L}((\mathcal{S}),(\mathcal{S})^*)$. If the series*

$$\sum_{k=1}^{\infty} D_{e_k}^- D_{e_k}^- \Xi \quad \text{and} \quad \sum_{k=1}^{\infty} D_{e_k}^+ D_{e_k}^+ \Xi$$

exist in $\mathcal{L}((\mathcal{S}),(\mathcal{S})^*)$, then we have

$$\Delta_{\mathrm{G}}^{\mathrm{Q}} \Xi = \sum_{k=1}^{\infty} D_{e_k}^- D_{e_k}^- \Xi + \sum_{k=1}^{\infty} D_{e_k}^+ D_{e_k}^+ \Xi. \tag{5.3}$$

PROOF. By applying (4.1) and (4.2) we prove that

$$\begin{aligned} \langle (w\Xi)''_{11}(\xi_1,\xi_2),\, e_k \otimes e_k \rangle &= w(D_{e_k}^- D_{e_k}^- \Xi)(\xi,\eta), \\ \langle (w\Xi)''_{22}(\xi_1,\xi_2),\, e_k \otimes e_k \rangle &= w(D_{e_k}^+ D_{e_k}^+ \Xi)(\xi,\eta). \end{aligned}$$

Therefore, by definition we have

$$\begin{aligned} \widetilde{\Delta}_{\mathrm{G}}^{\mathrm{Q}} w\Xi(\xi_1,\xi_2) &= \sum_{k=1}^{\infty} w(D_{e_k}^- D_{e_k}^- \Xi)(\xi,\eta) + \sum_{k=1}^{\infty} w(D_{e_k}^+ D_{e_k}^+ \Xi)(\xi,\eta) \\ &= w\left(\sum_{k=1}^{\infty} D_{e_k}^- D_{e_k}^- \Xi + \sum_{k=1}^{\infty} D_{e_k}^+ D_{e_k}^+ \Xi \right)(\xi,\eta) \end{aligned}$$

which proves (5.3) by assumption. ∎

Proposition 5.2 *Let $\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}((\mathcal{S}),(\mathcal{S})^*)$ with $\kappa_{l,m} \in \mathcal{S}^{\widehat{\otimes} l} \otimes \mathcal{S}^{\widehat{\otimes} m}$ for any $l,m \geq 0$. Then we have*

$$\Delta_{\mathrm{G}}^{\mathrm{Q}} \Xi = \sum_{l,m=0}^{\infty} (m+2)(m+1)\Xi_{l,m}(\kappa_{l,m+2} * \tau) + \sum_{l,m=0}^{\infty} (l+2)(l+1)\Xi_{l,m}(\tau * \kappa_{l+2,m}). \tag{5.4}$$

PROOF. For any $l,m \geq 0$, in the sense of Theorem 4.1 we prove that the series

$$\sum_{k=1}^{\infty} D_{e_k}^- D_{e_k}^- \Xi_{l,m+2}(\kappa_{l,m+2}) \quad \text{and} \quad \sum_{k=1}^{\infty} D_{e_k}^+ D_{e_k}^+ \Xi_{l+2,m}(\kappa_{l+2,m})$$

exist in $\mathcal{L}((\mathcal{S}),(\mathcal{S})^*)$ and

$$\sum_{k=1}^{\infty} D_{e_k}^- D_{e_k}^- \Xi_{l,m+2}(\kappa_{l,m+2}) = (m+2)(m+1)\Xi_{l,m}(\kappa_{l,m+2} * \tau)$$

and

$$\sum_{k=1}^{\infty} D_{e_k}^+ D_{e_k}^+ \Xi_{l+2,m}(\kappa_{l+2,m}) = (l+2)(l+1)\Xi_{l,m}(\tau * \kappa_{l+2,m}).$$

Therefore, by Theorem 5.1 we prove the assertion. In fact, the convergence of the series in (5.4) can be proved by similar arguments in [33]. ∎

Theorem 5.3 *The quantum Gross Laplacian $\Delta_{\mathrm{G}}^{\mathrm{Q}}$ is a continuous linear operator from $\mathcal{L}((\mathcal{S})^*, (\mathcal{S}))$ into $\mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$.*

Proof. By applying Lemma 4.2 and the proof of Theorem 4.3, for any $n \geq 1$ and $\xi_i \in \mathcal{S}$, $i = 1, 2, 3, 4$,

$$\begin{aligned} \left\langle\!\left\langle D_{e_n}^{+} D_{e_n}^{+} (\phi_{\xi_1} \otimes \phi_{\xi_2}),\, \phi_{\xi_3} \otimes \phi_{\xi_4} \right\rangle\!\right\rangle &= (\langle e_n,\, \xi_1\rangle - \langle e_n,\, \xi_4\rangle)^2\, e^{\langle \xi_2,\, \xi_4\rangle + \langle \xi_1,\, \xi_3\rangle}, \\ \left\langle\!\left\langle D_{e_n}^{-} D_{e_n}^{-} (\phi_{\xi_1} \otimes \phi_{\xi_2}),\, \phi_{\xi_3} \otimes \phi_{\xi_4} \right\rangle\!\right\rangle &= (\langle e_n,\, \xi_2\rangle - \langle e_n,\, \xi_3\rangle)^2\, e^{\langle \xi_2,\, \xi_4\rangle + \langle \xi_1,\, \xi_3\rangle}. \end{aligned}$$

Therefore, by Theorem 5.1 we prove that for any $\xi_i \in \mathcal{S}$, $i = 1, 2, 3, 4$,

$$\left\langle\!\left\langle \Delta_{\mathrm{G}}^{\mathrm{Q}} (\phi_{\xi_1} \otimes \phi_{\xi_2}),\, \phi_{\xi_3} \otimes \phi_{\xi_4} \right\rangle\!\right\rangle = \left(\sum_{i=1}^{4} \langle \xi_i,\, \xi_i\rangle - 2\langle \xi_1,\, \xi_4\rangle - 2\langle \xi_2,\, \xi_3\rangle \right) e^{\langle \xi_2,\, \xi_4\rangle + \langle \xi_1,\, \xi_3\rangle}.$$

Since $\mathcal{L}((\mathcal{S})^*, (\mathcal{S})) \cong (\mathcal{S})^{\otimes 2}$ and $\mathcal{L}((\mathcal{S}), (\mathcal{S})^*) \cong (\mathcal{S})^{*\otimes 2}$, by applying Theorem 2.3 we prove the assertion. ∎

Now, motivated by (5.3) we define the quantum Gross Laplacian associated with the annihilation derivative and creation derivative by

$$\Delta_{\mathrm{G}}^{\mathrm{Q}+} = \sum_{k=1}^{\infty} D_{e_k}^{+} D_{e_k}^{+}, \qquad \Delta_{\mathrm{G}}^{\mathrm{Q}-} = \sum_{k=1}^{\infty} D_{e_k}^{-} D_{e_k}^{-}, \tag{5.5}$$

respectively.

Theorem 5.4 *$\Delta_{\mathrm{G}}^{\mathrm{Q}\pm}$ are continuous operators acting on $\mathcal{L}((\mathcal{S}), (\mathcal{S}))$ and $\mathcal{L}((\mathcal{S})^*, (\mathcal{S})^*)$, respectively.*

Proof. The proof is a simple modification of the proof of Theorem 5.3. ∎

Theorem 5.5 *The quantum Gross Laplacian admits the following integral representation:*

$$\Delta_{\mathrm{G}}^{\mathrm{Q}} = \int_{\mathbf{R}} \left(D_t^{+2} + D_t^{-2} \right) dt \tag{5.6}$$

on $\mathcal{L}((\mathcal{S}), (\mathcal{S})) \cap \mathcal{L}((\mathcal{S})^*, (\mathcal{S})^*)$.

Proof. By similar arguments used in the proof of Theorem 5.3 we can easily show that for any $\xi_i \in \mathcal{S}$, $i = 1, 2, 3, 4$,

$$\begin{aligned} \left\langle\!\left\langle \Delta_{\mathrm{G}}^{\mathrm{Q}+} (\phi_{\xi_1} \otimes \phi_{\xi_2}),\, \phi_{\xi_3} \otimes \phi_{\xi_4} \right\rangle\!\right\rangle &= (\langle \xi_1,\, \xi_1\rangle - 2\langle \xi_1,\, \xi_4\rangle + \langle \xi_4,\, \xi_4\rangle)\, e^{\langle \xi_2,\, \xi_4\rangle + \langle \xi_1,\, \xi_3\rangle} \\ &= \left\langle\!\left\langle \int_{\mathbf{R}} D_t^{+2} dt\, (\phi_{\xi_1} \otimes \phi_{\xi_2}),\, \phi_{\xi_3} \otimes \phi_{\xi_4} \right\rangle\!\right\rangle, \end{aligned}$$

similarly, we have

$$\left\langle\!\left\langle \Delta_{\mathrm{G}}^{\mathrm{Q}-} (\phi_{\xi_1} \otimes \phi_{\xi_2}),\, \phi_{\xi_3} \otimes \phi_{\xi_4} \right\rangle\!\right\rangle = \left\langle\!\left\langle \int_{\mathbf{R}} D_t^{-2} dt\, (\phi_{\xi_1} \otimes \phi_{\xi_2}),\, \phi_{\xi_3} \otimes \phi_{\xi_4} \right\rangle\!\right\rangle.$$

Therefore, we have

$$\Delta_{\rm G}^{\rm Q+} = \int_{\mathbf{R}} D_t^{+2} dt, \qquad \Delta_{\rm G}^{\rm Q-} = \int_{\mathbf{R}} D_t^{-2} dt$$

on $\mathcal{L}((\mathcal{S}), (\mathcal{S}))$ and $\mathcal{L}((\mathcal{S})^*, (\mathcal{S})^*)$, respectively, which proves (5.6). ∎

For any $x, y \in \mathcal{S}^*$, by applying Theorem 2.2, we prove that

$$\Xi(x, y) \equiv \sum_{l,m=0}^{\infty} \frac{1}{l!m!} \Xi_{l,m}(x^{\otimes l} \otimes y^{\otimes m}) \in \mathcal{L}((\mathcal{S}), (\mathcal{S})^*). \tag{5.7}$$

Then we have the following

Theorem 5.6 *For any $f, g \in H$, $\Xi(f, g)$ is an eigenvector of the quantum Gross Laplacian $\Delta_{\rm G}^{\rm Q}$ corresponding to the eigenvalue $\langle f,\, f\rangle + \langle g,\, g\rangle$, i.e.,*

$$\Delta_{\rm G}^{\rm Q}\Xi(f, g) = (\langle f,\, f\rangle + \langle g,\, g\rangle)\, \Xi(f, g). \tag{5.8}$$

Proof. For any $\xi, \eta \in \mathcal{S}$ we have

$$w\Xi(f, g)(\xi, \eta) = \exp\left(\langle f,\, \eta\rangle + \langle g,\, \xi\rangle\right).$$

Therefore, we obtain that

$$w\Big(\Delta_{\rm G}^{\rm Q}\Xi(f, g)\Big)(\xi, \eta) = \widetilde{\Delta}_{\rm G}^{\rm Q}\Big(w\Xi(f, g)\Big)(\xi, \eta) = (\langle f,\, f\rangle + \langle g,\, g\rangle)\, w\Xi(f, g)(\xi, \eta)$$

which proves (5.8). ∎

5.2 Quantum Lévy Laplacian

Let $\{\ell_k\}_{k=1}^{\infty}$ be a fixed infinite sequence in $\mathcal{E}$. For each $\Xi \in \mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$, $w\Xi \in C^2(\mathcal{S} \times \mathcal{S})$. Define

$$\widetilde{\Delta}_{\rm L}^{\rm Q-}(w\Xi)(\xi_1, \xi_2) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \langle (w\Xi)''_{11}(\xi_1, \xi_2),\, \ell_k \otimes \ell_k\rangle\,, \tag{5.9}$$

$$\widetilde{\Delta}_{\rm L}^{\rm Q+}(w\Xi)(\xi_1, \xi_2) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \langle (w\Xi)''_{22}(\xi_1, \xi_2),\, \ell_k \otimes \ell_k\rangle\,, \tag{5.10}$$

if the limits exist and then define

$$\widetilde{\Delta}_{\rm L}^{\rm Q}(w\Xi)(\xi_1, \xi_2) = \widetilde{\Delta}_{\rm L}^{\rm Q-}(w\Xi)(\xi_1, \xi_2) + \widetilde{\Delta}_{\rm L}^{\rm Q+}(w\Xi)(\xi_1, \xi_2).$$

If $\widetilde{\Delta}_{\rm L}^{\rm Q}(w\Xi)$ is Gâteaux entire and satisfies the condition (O) in Theorem 2.2, then there exists a unique operator, denoted by $\Delta_{\rm L}^{\rm Q}\Xi$, in $\mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$ such that

$$w(\Delta_{\rm L}^{\rm Q}\Xi) = \widetilde{\Delta}_{\rm L}^{\rm Q}(w\Xi). \tag{5.11}$$

Then $\Delta_{\rm L}^{\rm Q}$ is called the *quantum Lévy Laplacian*. We denote by Dom $(\Delta_{\rm L}^{\rm Q})$ the set of all generalized operators $\Xi \in \mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$ such that $\Delta_{\rm L}^{\rm Q}\Xi$ is well-defined as in (5.2).

For $\kappa_{l,m} \in (\mathcal{S}^{\otimes(l+m)})^*$ we define the *left Lévy-contraction* $\tau_L * \kappa_{l,m} \in (\mathcal{S}^{\otimes(l-2+m)})^*$ by

$$\begin{aligned}
&\langle \tau_L * \kappa_{l,m},\, \eta_1 \otimes \cdots \otimes \eta_{l-2} \otimes \xi_1 \otimes \cdots \otimes \xi_m \rangle \\
&\quad = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \langle \kappa_{l,m},\, \ell_k \otimes \ell_k \otimes \eta_1 \otimes \cdots \otimes \eta_{l-2} \otimes \xi_1 \otimes \cdots \otimes \xi_m \rangle ,
\end{aligned}$$

if the limit exists and is a continuous linear form on $\mathcal{S}^{\otimes(l-2+m)}$. Similarly, the *right Lévy-contraction* $\kappa_{l,m} * \tau_L \in (\mathcal{S}^{\otimes(l+m-2)})^*$ is defined.

Lemma 5.7 ([23]) *Let $\kappa_{l,m} \in (\mathcal{S}^{\otimes(l+m)})^*$ for which both $\tau_L * \kappa_{l,m}$ and $\kappa_{l,m} * \tau_L$ are defined. Then, $\Xi_{l,m}(\kappa_{l,m}) \in \mathrm{Dom}\,(\Delta_{\mathrm{L}}^{\mathrm{Q}})$ and*

$$\Delta_{\mathrm{L}}^{\mathrm{Q}} \Xi_{l,m}(\kappa_{l,m}) = l(l-1)\Xi_{l-2,m}(\tau_L * \kappa_{l,m}) + m(m-1)\Xi_{l,m-2}(\kappa_{l,m} * \tau_L).$$

Let $\mathbf{L}_{\mathrm{Q}} = \{\Xi(x,y)\,;\, x, y \in \mathbf{L}\}$, where $\mathbf{L}$ is defined in Subsection 3.2 and $\Xi(x,y)$ is given as in (5.7). Then we have the following

Theorem 5.8 *For any $x, y \in \mathbf{L}$, $\Xi(x,y)$ is an eigenvector of the quantum Lévy Laplacian $\Delta_{\mathrm{L}}^{\mathrm{Q}}$ corresponding to the eigenvalue $\langle x \otimes x \rangle_{\mathrm{L}} + \langle y \otimes y \rangle_{\mathrm{L}}$, i.e.,*

$$\Delta_{\mathrm{L}}^{\mathrm{Q}} \Xi(x,y) = \left(\langle x \otimes x \rangle_{\mathrm{L}} + \langle y \otimes y \rangle_{\mathrm{L}}\right) \Xi(x,y). \tag{5.12}$$

Proof. The proof is similar to the proof of Theorem 5.6. ∎

Theorem 5.9 *The quantum Lévy Laplacian admits the following representation:*

$$\Delta_{\mathrm{L}}^{\mathrm{Q}} = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} D_{\ell_k}^{-} D_{\ell_k}^{-} + \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} D_{\ell_k}^{+} D_{\ell_k}^{+} \tag{5.13}$$

on $\mathbf{L}_{\mathrm{Q}}$.

Proof. Since for any $\zeta \in \mathcal{S}$ the differential operators $D_\zeta^{\pm}$ are continuous from $\mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$ into $\mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$, by Theorem 4.1 we prove that for any $x, y \in \mathbf{L}$

$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} D_{\ell_k}^{-} D_{\ell_k}^{-} \Xi(x,y) &= \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \sum_{l,m=0}^{\infty} D_{\ell_k}^{-} D_{\ell_k}^{-} \left(\frac{1}{l!m!} \Xi_{l,m}(x^{\otimes l} \otimes y^{\otimes m}) \right) \\
&= \langle y \otimes y \rangle_{\mathrm{L}} \Xi(x,y)
\end{aligned}$$

and similarly

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} D_{\ell_k}^{+} D_{\ell_k}^{+} \Xi(x,y) = \langle x \otimes x \rangle_{\mathrm{L}} \Xi(x,y).$$

Therefore, by applying Theorem 5.8 we prove (5.13). ∎

A similar result given as in Theorem 5.9 can be fund in [23].

If the given sequence $\{\ell_k\}_{k=1}^{\infty} \subset \mathcal{E}$ is an orthonormal subset of H, then we denote P_N the orthogonal projection from H onto the subspace of $\mathcal{S}$ generated by $\{\ell_1, \cdots, \ell_N\}$, i.e., $P_N = \sum_{k=1}^{N} |\ell_k\rangle\langle\ell_k|$, where $|\ell_k\rangle\langle\ell_k| \,:\, H \ni x \mapsto \langle \ell_k,\, x\rangle\, \ell_k \in \mathcal{S}$. As is clearly seen, P_N can be extended to a continuous linear operator from $\mathcal{S}^*$ into $\mathcal{S}$.

Theorem 5.10 *Let* $\{\ell_k\}_{k=1}^\infty \subset \mathcal{E}$ *be an orthonormal subset of* H. *Then the quantum Lévy Laplacian admits the following integral representation:*

$$\Delta_{\mathrm{L}}^{\mathrm{Q}} = \lim_{N\to\infty} \int_{\mathbf{R}} \left(D^+_{Q_N(\delta_t)} D^+_{Q_N(\delta_t)} + D^-_{Q_N(\delta_t)} D^-_{Q_N(\delta_t)} \right) dt$$

on $\mathbf{L}_{\mathrm{Q}}$, *where* $Q_N = \frac{1}{\sqrt{N}} P_N$ *for* $N \geq 1$.

PROOF. By direct computation, for any $x, y \in \mathbf{L}$ we have

$$\begin{aligned}\left(\int_{\mathbf{R}} D^+_{Q_N(\delta_t)} D^+_{Q_N(\delta_t)} dt \right) \Xi(x,y) &= \left(\int_{\mathbf{R}} \langle \delta_t,\, Q_N(x) \rangle^2 \, dt \right) \Xi(x,y) \\ &= \langle Q_N(x),\, Q_N(x) \rangle \, \Xi(x,y) \\ &= \frac{1}{N} \sum_{k=1}^N D^+_{\ell_k} D^+_{\ell_k} \Xi(x,y).\end{aligned}$$

Similarly, we have

$$\left(\int_{\mathbf{R}} D^-_{Q_N(\delta_t)} D^-_{Q_N(\delta_t)} dt \right) \Xi(x,y) = \frac{1}{N} \sum_{k=1}^N D^-_{\ell_k} D^-_{\ell_k} \Xi(x,y).$$

Therefore, by applying Theorem 5.9 we prove the assertion. ∎

6 Quantum–Classical Correspondence

For $\Phi \in (\mathcal{S})^*$ we define a multiplication operator $M_\Phi \in \mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$ by

$$\langle\!\langle M_\Phi \phi,\, \psi \rangle\!\rangle = \langle\!\langle \Phi,\, \phi\psi \rangle\!\rangle\,, \qquad \phi, \psi \in (\mathcal{S}),$$

where $\phi\psi$ is the pointwise multiplication, see e.g., Ji–Obata [19]. Moreover, $\Phi \mapsto M_\Phi$ yields a continuous injection from $(\mathcal{S})^*$ into $\mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$, and obviously, $M_\Phi \phi_0 = \Phi$. Moreover, if $\phi \in (\mathcal{S})$, then M_ϕ belongs to $\mathcal{L}((\mathcal{S}), (\mathcal{S}))$ and $\mathcal{L}((\mathcal{S})^*, (\mathcal{S})^*)$.

Lemma 6.1 ([23]) *It holds that for any* $\zeta \in \mathcal{S}$ *and* $\Phi \in (\mathcal{S})^*$,

$$[A(\zeta), M_\Phi] = M_{A(\zeta)\Phi}, \qquad [M_\Phi, A(\zeta)^*] = M_{A(\zeta)\Phi}.$$

Theorem 6.2 *Let* $\phi \in (\mathcal{S})$. *Then* $M_\phi \in \mathrm{Dom}\,(\Delta_{\mathrm{G}}^{\mathrm{Q}})$ *and*

$$\Delta_{\mathrm{G}}^{\mathrm{Q}-} M_\phi = \Delta_{\mathrm{G}}^{\mathrm{Q}+} M_\phi = M_{\Delta_{\mathrm{G}}\phi}. \tag{6.1}$$

In particular,

$$\frac{1}{2} \Delta_{\mathrm{G}}^{\mathrm{Q}} M_\phi = M_{\Delta_{\mathrm{G}}\phi}.$$

PROOF. Since $\phi_\xi \phi_\eta = \phi_{\xi+\eta} e^{\langle \xi,\, \eta \rangle}$ for any $\xi, \eta \in \mathcal{S}$,

$$w M_\phi(\xi, \eta) = \langle\!\langle \phi,\, \phi_\xi \phi_\eta \rangle\!\rangle \, e^{-\langle \xi,\, \eta \rangle} = S\phi(\xi + \eta). \tag{6.2}$$

Therefore, $M_\phi \in \mathrm{Dom}\,(\Delta_{\mathrm{G}}^{\mathrm{Q}})$ and by Lemma 6.1 we have

$$\Delta_{\mathrm{G}}^{\mathrm{Q}-} M_\phi = \sum_{k=1}^{\infty} D_{e_k}^{-} D_{e_k}^{-} M_\phi = \lim_{N\to\infty} M_{\sum_{k=1}^{N} A(e_k)A(e_k)\phi}.$$

On the other hand, $\Delta_{\mathrm{G}}\phi = \sum_{k=1}^{\infty} A(e_k)A(e_k)\phi$ and the map $\phi \mapsto M_\phi$ is a continuous injection from $(\mathcal{S})$ into $\mathcal{L}((\mathcal{S}),(\mathcal{S}))$. Therefore, we have

$$\Delta_{\mathrm{G}}^{\mathrm{Q}-} M_\phi = \lim_{N\to\infty} M_{\sum_{k=1}^{N} A(e_k)A(e_k)\phi} = M_{\Delta_{\mathrm{G}}\phi}.$$

Similarly, we prove that

$$\Delta_{\mathrm{G}}^{\mathrm{Q}+} M_\phi = M_{\Delta_{\mathrm{G}}\phi}.$$

The last assertion is obvious from (6.1). ∎

Theorem 6.3 *Let* $\Xi \in \mathrm{Dom}\,(\Delta_{\mathrm{G}}^{\mathrm{Q}+})$. *Then we have*

$$\left(\Delta_{\mathrm{G}}^{\mathrm{Q}+}\Xi\right)\phi_0 = \Delta_{\mathrm{G}}(\Xi\phi_0).$$

Proof. Since $\Xi \in \mathrm{Dom}\,(\Delta_{\mathrm{G}}^{\mathrm{Q}+})$,

$$\begin{aligned}\left(\Delta_{\mathrm{G}}^{\mathrm{Q}+}\Xi\right)\phi_0 &= \left(\sum_{k=1}^{\infty} D_{e_k}^{+} D_{e_k}^{+}\Xi\right)\phi_0 \\ &= \sum_{k=1}^{\infty}\left(A(e_k)^2\Xi - 2A(e_k)\Xi A(e_k) + \Xi A(e_k)^2\right)\phi_0 \\ &= \sum_{k=1}^{\infty} A(e_k)^2\Xi\phi_0\end{aligned}$$

which proves the assertion from (3.1). ∎

Remark 6.4 ([23]) Let $\Phi \in \mathrm{Dom}\,(\Delta_{\mathrm{L}})$. Then $M_\Phi \in \mathrm{Dom}\,(\Delta_{\mathrm{L}}^{\mathrm{Q}})$ and

$$\Delta_{\mathrm{L}}^{\mathrm{Q}-} M_\Phi = \Delta_{\mathrm{L}}^{\mathrm{Q}+} M_\Phi = M_{\Delta_{\mathrm{L}}\Phi}.$$

In particular,

$$\frac{1}{2}\Delta_{\mathrm{L}}^{\mathrm{Q}} M_\Phi = M_{\Delta_{\mathrm{L}}\Phi}.$$

Let $\mathrm{Dom}\,(\Delta_{\mathrm{L}}^{\mathrm{Q}+})$ be the set of all $\Xi \in \mathcal{L}((\mathcal{S}),(\mathcal{S})^*)$ such that the limit

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} D_{\ell_k}^{+} D_{\ell_k}^{+}\Xi$$

exists in $\mathcal{L}((\mathcal{S}),(\mathcal{S})^*)$. Then, by Theorem 5.9, $\mathbf{L}_{\mathrm{Q}} \subset \mathrm{Dom}\,(\Delta_{\mathrm{L}}^{\mathrm{Q}+})$ and for each $\Xi \in \mathrm{Dom}\,(\Delta_{\mathrm{L}}^{\mathrm{Q}+})$ we have

$$\Delta_{\mathrm{L}}^{\mathrm{Q}+}\Xi = \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} D_{\ell_k}^{+} D_{\ell_k}^{+}\Xi.$$

Then by similar argument used in the proof of Theorem 6.3, we prove that

$$\left(\Delta_{\mathrm{L}}^{\mathrm{Q}+}\Xi\right)\phi_0 = \Delta_{\mathrm{L}}(\Xi\phi_0). \tag{6.3}$$

Proposition 6.5 *For each* $\Phi \in (\mathcal{S})^*$ *with* $M_\Phi \in \mathrm{Dom}\,(\Delta_{\mathrm{L}}^{\mathrm{Q}+})$ *we have*

$$\Delta_{\mathrm{L}}\Phi = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} A(e_n)^2\Phi.$$

PROOF. By (6.3), we prove that

$$\Delta_{\mathrm{L}}\Phi = \Big(\Delta_{\mathrm{L}}^{\mathrm{Q}+} M_\Phi\Big)\phi_0 = \Big(\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} D_{\ell_k}^{+} D_{\ell_k}^{+} M_\Phi\Big)\phi_0 = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} A(e_n)^2\Phi$$

which proves the assertion. ∎

7 Heat Equations

Now, we consider the following normal-ordered white noise equation:

$$\frac{d\Xi_t}{dt} = \Big(A(x_t) + A^*(y_t)\Big) \diamond \Xi_t, \qquad \Xi_0 = I, \tag{7.1}$$

where the maps $t \mapsto x_t \in \mathcal{S}^*$ and $t \mapsto y_t \in \mathcal{S}^*$ are continuous. It is known that (7.1) has a unique solution in $\mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$. In fact, the solution of (7.1) is given by the Wick-exponential:

$$\Xi_t = \sum_{n=0}^{\infty} \frac{1}{n!} \Big\{ \int_0^t \Big(A(x_s) + A^*(y_s)\Big) ds \Big\}^{\diamond n} \tag{7.2}$$

of which the Wick symbol is given by

$$w\Xi_t(\xi, \eta) = \exp\Big\{ \int_0^t \Big(\langle x_s,\, \xi\rangle + \langle y_s,\, \eta\rangle \Big) ds \Big\}, \quad \xi, \eta \in \mathcal{S}, \tag{7.3}$$

see [11].

7.1 Heat Equation Associated with the Quantum Gross Laplacian

Let $\mathfrak{G}$ be the set of all $\mathcal{S}^*$-valued continuous maps on $[0, T]$ such that the series

$$G(x, t) \equiv \sum_{n=1}^{\infty} \left(\int_0^t \langle x_s,\, e_n\rangle\, ds \right)^2, \qquad t \in [0, T]$$

converges and the map $t \mapsto G(x, t)$ is bounded. The map $\delta : [0, T] \ni t \mapsto \delta_t \in \mathcal{S}^*$ is continuous and

$$G(\delta, t) = t, \qquad t \in [0, T]$$

which implies that $\delta \in \mathfrak{G}$. Let $f \in C([0, T])$ and $g \in H$. We consider the H-valued continuous function $x(t) = f(t)g \in H$, $t \in [0, T]$. Then we can easily show that

$$G(x, t) = \left(\int_0^t f(s) ds \right)^2 \langle g,\, g\rangle\,, \qquad t \in [0, T].$$

Therefore, $x \in \mathfrak{G}$ and the algebraic tensor product $C([0, T]) \otimes_{\mathrm{alg}} H$ of $C([0, T])$ and H belongs to $\mathfrak{G}$.

Theorem 7.1 *Let $x, y \in \mathfrak{G}$ and let Ξ_t $(t \in [0,T])$ be the solution of* (7.1). *Then*

$$\Delta_{\rm G}^{\rm Q-}\Xi_t = G(x,t)\Xi_t, \qquad \Delta_{\rm G}^{\rm Q+}\Xi_t = G(y,t)\Xi_t, \qquad t \in [0,T]. \tag{7.4}$$

Moreover, for any $t \in [0,T]$

$$\Delta_{\rm G}^{\rm Q}\Xi_t = (G(x,t) + G(y,t))\,\Xi_t.$$

Proof. By (4.1) and (7.3), for any $t \in [0,T]$ and $\xi, \eta \in \mathcal{S}$ we have

$$\begin{aligned} w(D_{e_n}^+ D_{e_n}^+ \Xi_t)(\xi,\eta) &= \left.\frac{d^2}{dz^2}\right|_{z=0} w\Xi_t(\xi, \eta + z e_n) \\ &= \left(\int_0^t \langle y_s,\, e_n\rangle\, ds\right)^2 \exp\left\{\int_0^t (\langle x_s,\, \xi\rangle + \langle y_s,\, \eta\rangle)\, ds\right\}. \end{aligned}$$

Similarly, for any $\xi, \eta \in \mathcal{S}$

$$w(D_{e_n}^- D_{e_n}^- \Xi_t)(\xi,\eta) = \left(\int_0^t \langle x_s,\, e_n\rangle\, ds\right)^2 \exp\left\{\int_0^t (\langle x_s,\, \xi\rangle + \langle y_s,\, \eta\rangle)\, ds\right\}.$$

Therefore, by (5.5), for any $t \in [0,T]$

$$\Delta_{\rm G}^{\rm Q+}\Xi_t = \sum_{n=1}^{\infty}\left(\int_0^t \langle y_s,\, e_n\rangle\, ds\right)^2 \Xi_t, \qquad \Delta_{\rm G}^{\rm Q-}\Xi_t = \sum_{n=1}^{\infty}\left(\int_0^t \langle x_s,\, e_n\rangle\, ds\right)^2 \Xi_t$$

which proves (7.4). The last assertion is immediate from (7.4). ∎

The above theorem proves that, for any $x, y \in \mathfrak{G}$, the solution Ξ_t of the normal-ordered white noise differential equation (7.1) is an eigenvector of the quantum Gross Laplacian with eigenvalue $G(x,t) + G(y,t)$. The following result is immediate.

Theorem 7.2 *Let $x, y \in \mathfrak{G}$ and Ξ_t be the solution of* (7.1). *Let μ be a finite measure on $[0,1]$ and $\alpha \in \mathbf{C}$. Define for any $t \in \mathbf{R}$*

$$Y_t^+ = \int_0^1 e^{\alpha t G(y,s)}\Xi_s\,\mu(ds), \qquad Y_t^- = \int_0^1 e^{\alpha t G(x,s)}\Xi_s\,\mu(ds),$$

and

$$Y_t = \int_0^1 e^{\alpha t (G(x,s)+G(y,s))}\Xi_s\,\mu(ds).$$

Then $Y_t^\epsilon \in \mathcal{L}((\mathcal{S}),(\mathcal{S})^*)$ is a solution to the following Cauchy problem:

$$\frac{\partial Y_t^\epsilon}{\partial t} = \alpha\Delta_{\rm G}^{{\rm Q}\epsilon} Y_t^\epsilon, \qquad Y_0^\epsilon = Y_0 = \int_0^1 \Xi_s \mu(ds),$$

where $\epsilon = +, -,$ or empty.

From Theorems 7.2 and 6.3, the following result is immediate.

Corollary 7.3 *Let Y_t^+ be as in Theorem 7.2 and set $\Phi_t = Y_t\phi_0$. Then $\Phi_t \in (\mathcal{S})^*$ is a solution to the following Cauchy problem:*

$$\frac{\partial \Phi_t}{\partial t} = \alpha\Delta_{\rm G}\Phi_t, \qquad \Phi_0 = \int_0^1 \Xi_s\phi_0\mu(ds) \in (\mathcal{S})^*.$$

7.2 Heat Equation Associated with the Quantum Lévy Laplacian

Recall that the Lévy Laplacian depends on the choice of an infinite sequence $\{\ell_k\}_{k=1}^{\infty} \subset \mathcal{E}$. Let $\mathfrak{L}$ be the set of all $\mathcal{S}^*$-valued continuous maps on $[0,T]$ such that the limit

$$L(x,t) \equiv \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \left(\int_0^t \langle x_s,\, \ell_k \rangle \, ds \right)^2, \qquad t \in [0,T]$$

exists and the map $t \mapsto L(x,t)$ is bounded. For the given infinite sequence $\{\ell_k\}_{k=1}^{\infty} \subset \mathcal{E}$, if we assume that the following limit

$$\langle \ell \rangle_{\mathrm{L}}(t) \equiv \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \left(\ell_k(t) - \ell_k(0) \right)^2, \qquad t \in [0,T]$$

exists and the map $t \mapsto \langle \ell \rangle_{\mathrm{L}}(t)$ is bounded, then $\delta' \in \mathfrak{L}$. In fact, the map $\delta' : [0,T] \ni t \mapsto \delta'_t \in \mathcal{S}^*$ is continuous and for any $t \in [0,T]$

$$\begin{aligned} L(\delta',t) &= \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \left(\int_0^t \langle \delta'(s),\, \ell_k \rangle \, ds \right)^2 \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \left(-\int_0^t \ell'_k(s) ds \right)^2 \\ &= \langle \ell \rangle_{\mathrm{L}}(t). \end{aligned}$$

Let $f \in C([0,T])$ and $x \in \mathbf{L} \subset \mathcal{S}^*$. We consider the $\mathcal{S}^*$-valued continuous function $z(t) = f(t)x \in \mathcal{S}^*$, $t \in [0,T]$. Then we can easily show that

$$L(z,t) = \left(\int_0^t f(s) ds \right)^2 \langle x \otimes x \rangle_{\mathrm{L}}, \qquad t \in [0,T].$$

Therefore, $z \in \mathfrak{L}$ and the algebraic tensor product $C([0,T]) \otimes_{\mathrm{alg}} \mathbf{L}$ of $C([0,T])$ and $\mathbf{L}$ belongs to $\mathfrak{L}$.

Theorem 7.4 *Let $x, y \in \mathfrak{L}$ and let Ξ_t $(t \in [0,T])$ be the solution of* (7.1). *Then*

$$\Delta_{\mathrm{L}}^{\mathrm{Q}-} \Xi_t = L(x,t)\Xi_t, \qquad \Delta_{\mathrm{L}}^{\mathrm{Q}+} \Xi_t = L(y,t)\Xi_t, \qquad t \in [0,T].$$

Moreover, for any $t \in [0,T]$

$$\Delta_{\mathrm{L}}^{\mathrm{Q}} \Xi_t = \left(L(x,t) + L(y,t) \right) \Xi_t.$$

PROOF. The proof is a simple modification of the proof of Theorem 7.1. ∎

The above theorem proves that, for any $x, y \in \mathfrak{L}$, the solution Ξ_t of the normal-ordered white noise differential equation (7.1) is an eigenvector of the quantum Lévy Laplacian with eigenvalue $L(x,t) + L(y,t)$. The following result is immediate.

Theorem 7.5 *Let $x, y \in \mathfrak{L}$ and Ξ_t be the solution of* (7.1)*. Let ν be a finite measure on $[0,1]$ and $\alpha \in \mathbf{C}$. Define for any $t \in \mathbf{R}$*

$$Z_t^+ = \int_0^1 e^{\alpha t L(y,s)} \Xi_s \, \nu(ds), \qquad Z_t^- = \int_0^1 e^{\alpha t L(x,s)} \Xi_s \, \nu(ds),$$

and

$$Z_t = \int_0^1 e^{\alpha t (L(x,s)+L(y,s))} \Xi_s \, \nu(ds).$$

Then $Z_t^\epsilon \in \mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$ is a solution to the following Cauchy problem:

$$\frac{\partial Z_t^\epsilon}{\partial t} = \alpha \Delta_{\mathrm{L}}^{\mathrm{Q}\epsilon} Z_t^\epsilon, \qquad Z_0^\epsilon = Z_0 = \int_0^1 \Xi_s \nu(ds),$$

where $\epsilon = +, -,$ or empty.

From Theorem 7.5 and (6.3), the following result is immediate.

Corollary 7.6 *Let Z_t^+ be as in Theorem 7.5 and set $\Psi_t = Z_t \phi_0$. Then $\Psi_t \in (\mathcal{S})^*$ is a solution to the following Cauchy problem:*

$$\frac{\partial \Psi_t}{\partial t} = \alpha \Delta_{\mathrm{L}} \Psi_t, \qquad \Psi_0 = \int_0^1 \Xi_s \phi_0 \nu(ds) \in (\mathcal{S})^*.$$

Remark 7.7 A relation between heat equation associated with the quantum Lévy Laplacian and quadratic quantum white noises $\{a_t^2, a_t^{*2}; t \in \mathbf{R}\}$ has been discussed in [23]. In fact, a solution to the heat equation associated with the quantum Lévy Laplacian can be obtained from a normal-ordered white noise differential equation involving the quadratic quantum white noise.

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