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Capacity per Unit-Energy of Gaussian Many-Access Channels

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***Abstract*—We consider a Gaussian multiple-access channel where the number of transmitters grows with the blocklength n. For this setup, the maximum number of bits that can be transmitted reliably per unit-energy is analyzed. We show that if the number of users is of an order strictly above $n/\log n$, then the users cannot achieve any positive rate per unit-energy. In contrast, if the number of users is of order strictly below $n/\log n$, then each user can achieve the single-user capacity per unit-energy $(\log e)/N_0$ (where $N_0/2$ is the noise power) by using an orthogonal access scheme such as time division multiple access. We further demonstrate that orthogonal codebooks, which achieve the capacity per unit-energy when the number of users is bounded, can be strictly suboptimal.**

I. Introduction

The capacity per unit-energy $\dot{C}$ is defined as the largest number of bits per unit-energy that can be transmitted reliably over a channel. Verdú [1] showed that $\dot{C}$ can be obtained from the capacity-cost function $C(P)$, defined as the largest number of bits per channel use that can be transmitted reliably with average power per symbol not exceeding P, as $\dot{C} = \sup_{P>0} C(P)/P$. For the Gaussian channel with noise power $N_0/2$, this is equal to $\frac{\log e}{N_0}$. Verdú further showed that the capacity per unit-energy can be achieved by a codebook that is orthogonal in the sense that the nonzero components of different codewords do not overlap. In general, we shall say that a codebook is orthogonal if the inner product between different codewords is zero. The two-user Gaussian multiple access channel (MAC) was also studied in [1], and it was demonstrated that both users can achieve the single-user capacity per unit-energy by timesharing the channel between the users, i.e., while one user transmits the other user remains silent. This is an orthogonal access scheme in the sense that the nonzero components of codewords of different users do not overlap. In general, we shall say that an access scheme is orthogonal if the inner product between codewords of different users is zero.[1] To summarize, in a two-user Gaussian MAC both users can achieve the rate per unit-energy $\frac{\log e}{N_0}$ by combining an orthogonal access scheme with orthogonal codebooks. This result can be directly generalized to any finite number of users.

The picture changes when the number of users grows without bound with the blocklength n. This scenario was investigated recently by Chen *et al.* [2], who referred to such a channel model as a many-access channel (MnAC). Specifically, the MnAC was introduced to model systems consisting of a single receiver and many transmitters, the number of which is comparable to or even larger than the blocklength. This situation could, *e.g.*, occur in a machine-to-machine communication system with many thousands of devices in a given cell. In [2], Chen *et al.* considered a Gaussian MnAC with k_n users and determined the number of messages M_n each user can transmit reliably with a codebook of average power not exceeding P. In particular, they showed that the largest sequence $\{M_n\}$ such that the error probability vanishes as n tends to infinity satisfies $\log M_n = \frac{n}{2k_n}\log(1+k_nP) + o(n\log(1+k_nP)/k_n)$. This implies that the per-user rate $(\log M_n)/n$ vanishes as $n \to \infty$ unless k_n is bounded in n.

In this paper, we study the capacity per unit-energy of the Gaussian MnAC. We show that, in contrast to the per-user rate, the per-user rate per unit-energy can converge to a positive value as $n \to \infty$ even if k_n grows without bound. Specifically, we demonstrate that, if the order of growth of k_n is strictly below $n/\log n$, then each user can achieve the capacity per unit-energy $\frac{\log e}{N_0}$ by an orthogonal access scheme. Conversely, if the order of growth of k_n is strictly above $n/\log n$, then the capacity per unit-energy is zero. Thus, there is a sharp transition between orders of growth where interference-free communication is feasible and orders of growth where reliable communication at any positive rate per unit-energy is infeasible. We further characterize the largest rate per unit-energy that can be achieved with an orthogonal access scheme and orthogonal codebooks. Our characterization shows that orthogonal codebooks are only optimal if k_n grows more slowly than any positive power of n.

The paper is organized as follows. In Section II, we define the problem and introduce some preliminary notations. In Section III, we present the converse result when the order of k_n is strictly above $n/\log n$. In Section IV, we present the achievability result when the order of k_n is strictly below $n/\log n$. In Section V, we analyze the performance of orthogonal codebooks.

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[1]Note, however, that in an orthogonal access scheme the codebooks are not required to be orthogonal. That is, codewords of different codebooks are orthogonal to each other, but codewords of the same codebook need not be.

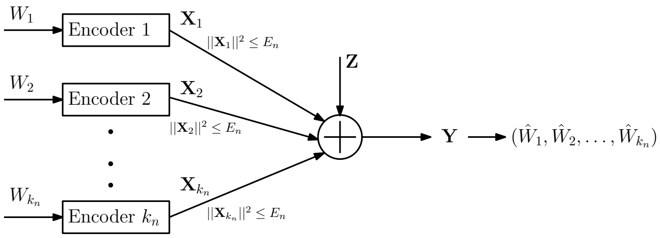

Fig. 1. Many-access channel with k_n users at blocklength n

II. Problem Formulation and Preliminaries

A. Model and Definitions

Suppose there are k users that wish to transmit their messages $W_i, i = 1, \ldots, k$, which are assumed to be independent and uniformly distributed on $\{1, \ldots, M_n^{(i)}\}$, to one common receiver; see Fig. 1. To achieve this, they send a codeword of n symbols over the channel. We refer to n as the blocklength. We consider a many-access scenario where the number of users k grows with n, hence, we denote it as k_n.

We further consider a Gaussian channel model where, for k_n users and blocklength n, the received vector $\mathbf{Y}$ is given by

$$\mathbf{Y} = \sum_{i=1}^{k_n} \mathbf{X}_i(W_i) + \mathbf{Z}.$$

Here $\mathbf{X}_i(W_i)$ is the n-length transmitted codeword from user i for message W_i and $\mathbf{Z}$ is a vector of n i.i.d. Gaussian components $Z_j \sim \mathcal{N}(0, N_0/2)$ independent of $\mathbf{X}_i$. We denote the vector of all transmitted codewords by $\mathbf{X} := (\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{k_n})$.

Definition 1: For $0 \le \epsilon < 1$, an $\left(n, \{M_n^{(\cdot)}\}, \{E_n^{(\cdot)}\}, \epsilon\right)$-code for the Gaussian many-access channel consists of:

1) Encoding functions $f_i : \{1, \ldots, M_n^{(i)}\} \to \mathcal{X}^n, i = 1, \ldots, k_n$, which map user i's message W_i to the codeword $\mathbf{X}_i(W_i)$, satisfying the energy constraint
$$\sum_{j=1}^{n} X_{ij}^2(W_i) \le E_n^{(i)}, \tag{1}$$
where X_{ij} is the jth symbol of the transmitted codeword.
2) Decoding function $g : \mathcal{Y}^n \to \{M_n^{(\cdot)}\}$ which maps the received vector $\mathbf{Y}$ to the messages of all users and whose average probability of error satisfies
$$P_e^{(n)} := P\{g(\mathbf{Y}) \neq (W_1, \ldots, W_{k_n})\} \le \epsilon.$$

We shall say that the $\left(n, \{M_n^{(\cdot)}\}, \{E_n^{(\cdot)}\}, \epsilon\right)$-code is symmetric if $M_n^{(i)} = M_n$ and $E_n^{(i)} = E_n$ for all $i = 1, \ldots, k_n$. For compactness, we denote a symmetric code by (n, M_n, E_n, ϵ). In this paper, we restrict ourselves to symmetric codes.

Definition 2: For a symmetric code, the rate per unit-energy $\dot{R}$ is said to be ϵ-achievable if for every $\alpha > 0$, there exists an n_0 such that if $n \ge n_0$, then an (n, M_n, E_n, ϵ)-code can be found whose rate per unit-energy satisfies $\frac{\log M_n}{E_n} > \dot{R} - \alpha$. Furthermore, $\dot{R}$ is said to be achievable if it is ϵ-achievable for all $0 < \epsilon < 1$. The capacity per unit-energy $\dot{C}$ is the supremum of all achievable rates per unit-energy.

B. Order Notations

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers. We write $a_n = O(b_n)$ if there exists an n_0 and a positive real number S such that, for all $n \ge n_0$, $a_n \le S b_n$. We write $a_n = o(b_n)$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, and $a_n = \Omega(b_n)$ if $\liminf_{n\to\infty} \frac{a_n}{b_n} > 0$. Similarly, $a_n = \Theta(b_n)$ indicates that there exist $0 < l_1 < l_2$ and n_0 such that $l_1 b_n \le a_n \le l_2 b_n$ for all $n \ge n_0$. Finally, we write $a_n = \omega(b_n)$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$.

III. Infeasible Order of Growth

We shall refer to orders of k_n for which no positive rate per unit-energy is achievable as infeasible orders of growth. In the next theorem, we show that any order of growth which is strictly above $n/\log n$ is infeasible.

Theorem 1: If $k_n = \omega(n/\log n)$, then $\dot{C} = 0$. In words, if the order of k_n is strictly above $n/\log n$, then no coding scheme achieves a positive rate per unit-energy.

Proof: Let $\mathbf{W}$ and $\hat{\mathbf{W}}$ denote the vectors $(W_1, \ldots, W_{k_n})$ and $(\hat{W}_1, \ldots, \hat{W}_{k_n})$, respectively. Then

$$\begin{aligned} k_n \log M_n &= H(\mathbf{W}) \\ &= H(\mathbf{W}|\hat{\mathbf{W}}) + I(\mathbf{W}; \hat{\mathbf{W}}) \\ &\le 1 + P_e^{(n)} k_n \log M_n + I(\mathbf{X}; \mathbf{Y}), \end{aligned}$$

by Fano's inequality and the data processing inequality. By following [3, Section 9.2], it can be shown that for the Gaussian channel $I(\mathbf{X}; \mathbf{Y}) \le \frac{n}{2} \log\left(1 + \frac{2k_n E_n}{n N_0}\right)$. Consequently,

$$\frac{\log M_n}{E_n} \le \frac{1}{k_n E_n} + \frac{P_e^{(n)} \log M_n}{E_n} + \frac{n}{2k_n E_n} \log\left(1 + \frac{2k_n E_n}{n N_0}\right).$$

This implies that the rate per unit-energy $\dot{R} = (\log M_n)/E_n$ is upper-bounded by

$$\dot{R} \le \frac{\frac{1}{k_n E_n} + \frac{n}{2k_n E_n} \log(1 + \frac{2k_n E_n}{n N_0})}{1 - P_e^{(n)}}. \tag{2}$$

We next show by contradiction that if $k_n = \omega(n/\log n)$, then $P_e^{(n)} \to 0$ as $n \to \infty$ only if $\dot{C} = 0$. Thus, assume that $k_n = \omega(n/\log n)$ and that there exists a code with rate per unit-energy $\dot{R} > 0$ such that $P_e^{(n)} \to 0$ as $n \to \infty$. To prove that there is a contradiction we need the following lemma.

Lemma 1: If $M_n \ge 2$, then $P_e^{(n)} \to 0$ only if $E_n \to \infty$.

Proof: See [4]. ■

By the assumption $\dot{R} > 0$, we have that $M_n \ge 2$. Since we further assumed that $P_e^{(n)} \to 0$, Lemma 1 implies that $E_n \to \infty$. Together with (2), this in turn implies that $\dot{R} > 0$ is only possible if $k_n E_n / n$ is bounded in n. Thus,

$$E_n = O(n/k_n). \tag{3}$$

The next lemma presents another condition on the order of E_n which contradicts (3).

Lemma 2: If $\dot{R} > 0$, then $P_e^{(n)} \to 0$ only if $E_n = \Omega(\log k_n)$.

Proof: See appendix. ■

We finish the proof by showing that, if $k_n = \omega(n/\log n)$, then there exists no sequence $\{E_n\}$ of order $\Omega(\log k_n)$ that satisfies (3). Indeed, $E_n = \Omega(\log k_n)$ and $k_n = \omega(n/\log n)$ imply that

$$E_n = \Omega(\log n), \tag{4}$$

because the order of E_n is lower-bounded by the order of $\log n - \log\log n$, and $\log n - \log\log n = \Theta(\log n)$. Furthermore, $E_n = O(n/k_n)$ and $k_n = \omega(n/\log n)$ imply that

$$E_n = o(\log n). \tag{5}$$

Since no sequence $\{E_n\}$ can simultaneously satisfy (4) and (5), it follows that, if $k_n = \omega(n/\log n)$, then no positive rate per unit-energy is achievable. ■

IV. Feasible Order of Growth

In this section, we show that if the order of the growth of k_n is strictly below $n/\log n$, then each user can achieve the single-user capacity per unit-energy $\frac{\log e}{N_0}$. Hence, in this case, the users can communicate as if free of interference. The achievability uses an orthogonal access scheme, where only one user transmits, all other users remain silent. For further reference, the probability of correct decoding of any orthogonal access scheme is given by

$$P_c^{(n)} = \prod_{i=1}^{k_n} \left(1 - P(\mathcal{E}_i)\right),$$

where $P(\mathcal{E}_i)$ denotes the probability of error in decoding user i's message. In addition, if each user follows the same coding scheme, then the probability of correct decoding is given by

$$P_c^{(n)} = \left(1 - P(\mathcal{E}_1)\right)^{k_n}. \tag{6}$$

We have the following theorem.

Theorem 2: If $k_n = o(n/\log n)$, then any rate per unit-energy satisfying $\dot{R} < \frac{\log e}{N_0}$ is achievable. Hence, $\dot{C} = \frac{\log e}{N_0}$.

Proof: For a Gaussian point-to-point channel with power constraint P, there exists an encoding and decoding scheme whose average probability of error is upper-bounded by

$$P(\mathcal{E}) \leq M_n^{\rho} \exp[-nE_0(\rho, P)], \quad \text{for every } 0 < \rho \leq 1, \tag{7}$$

where

$$E_0(\rho, P) := \frac{\rho}{2} \ln\left(1 + \frac{2P}{(1+\rho)N_0}\right).$$

This bound is due to Gallager and can be found in [5, Section 7.4].

Now let us consider an orthogonal access scheme in which each user gets n/k_n channel uses and we timeshare between users. Each user follows the coding scheme which achieves (7) with power constraint $P_n = \frac{E_n}{n/k_n}$. Note that this satisfies the energy constraint (1). Then by substituting n with n/k_n and P with $P_n = \frac{E_n}{n/k_n}$ in (7), we get the following bound on $P(\mathcal{E}_1)$ as a function of the rate per unit-energy $\dot{R} = \frac{\log M_n}{E_n}$:

$$\begin{aligned} P(\mathcal{E}_1) &\leq M_n^{\rho} \exp\left[-\frac{n}{k_n} E_0(\rho, P_n)\right] \\ &= \exp\left[\rho \ln M_n - \frac{n}{k_n}\frac{\rho}{2} \ln\left(1 + \frac{2E_n k_n/n}{(1+\rho)N_0}\right)\right] \\ &= \exp\left[-E_n \rho \left(\frac{\ln(1 + \frac{2E_n k_n/n}{(1+\rho)N_0})}{2E_n k_n/n} - \frac{\dot{R}}{\log e}\right)\right]. \end{aligned} \tag{8}$$

Combining (8) with (6), we obtain that the probability of correct decoding can be lower-bounded as

$$\begin{aligned} &1 - P_e^{(n)} \\ &\geq \left(1 - \exp\left[-E_n \rho \left(\frac{\ln(1 + \frac{2E_n k_n/n}{(1+\rho)N_0})}{2E_n k_n/n} - \frac{\dot{R}}{\log e}\right)\right]\right)^{k_n}. \end{aligned} \tag{9}$$

We next choose $E_n = c_n \ln n$ with $c_n := \ln\left(\frac{n}{k_n \ln n}\right)$. Since, by assumption, $k_n = o(n/\log n)$, this implies that $\frac{k_n E_n}{n} \to 0$ as $n \to \infty$. Consequently, the first term in the inner most bracket in (9) tends to $1/((1+\rho)N_0)$ as $n \to \infty$. It follows that for $\dot{R} < \frac{\log e}{N_0}$, there exists a sufficiently large n_0, a $\rho > 0$, and a $\delta > 0$ such that, for $n \geq n_0$, the RHS of (9) is lower-bounded by $(1 - \exp[-E_n \rho\delta])^{k_n}$. Since $c_n \delta\rho \to \infty$ as $n \to \infty$, we have

$$\begin{aligned} (1 - \exp[-E_n \rho\delta])^{k_n} &\geq \left(1 - \frac{1}{n^2}\right)^{k_n} \\ &\geq \left(1 - \frac{1}{n^2}\right)^{\frac{n}{\log n}} \\ &= \left[\left(1 - \frac{1}{n^2}\right)^{n^2}\right]^{\frac{1}{n \log n}}, \end{aligned} \tag{10}$$

for sufficiently large $n \geq n_0$ such that $c_n \delta\rho \geq 2$ and $k_n \leq \frac{n}{\log n}$. Noting that $(1 - \frac{1}{n^2})^{n^2} \to 1/e$ and $\frac{1}{n \log n} \to 0$ as $n \to \infty$, we obtain that the RHS of (10) goes to one as $n \to \infty$. This implies that, if $k_n = o(n/\log n)$, then any rate per unit-energy $\dot{R} < \frac{\log e}{N_0}$ is achievable. ■

V. Performance of Orthogonal Codebooks

As mentioned in the introduction, when the number of users is bounded, the capacity per unit-energy $\dot{C} = \frac{\log e}{N_0}$ can be achieved with orthogonal codebooks. In the following theorem, we characterize the largest rate per unit-energy achievable with orthogonal codebooks, denoted by $\dot{C}_\perp$, when the number of users grows with the blocklength.

Theorem 3: Suppose the users apply an orthogonal access scheme and each user uses orthogonal codebooks. Then:

1) If $k_n = o(n^c)$ for every $c > 0$, then $\dot{C}_\perp = \frac{\log e}{N_0}$.
2) If $k_n = \Theta(n^c)$, then

$$\dot{C}_\perp = \begin{cases} \frac{\log e}{N_0} \frac{1}{\left(1+\sqrt{\frac{c}{1-c}}\right)^2}, & \text{if } 0 < c \leq 1/2 \\ \frac{\log e}{2N_0}(1-c), & \text{if } 1/2 < c < 1. \end{cases}$$

Proof: To prove Theorem 3, we shall first present in the following lemma bounds on the probability of error achievable over a Gaussian point-to-point channel with an orthogonal codebook. The bounds are obtained using similar methods given in [6, Section 2.5].

Lemma 3: For an orthogonal codebook with M codewords of energy less than or equal to E, the probability of error satisfies the following bounds:

1) For $0 < \dot{R} \leq \frac{1}{4}\frac{\log e}{N_0}$,

$$\exp\left[-\frac{\ln M}{\dot{R}}\left(\frac{\log e}{2N_0} - \dot{R} + o(1)\right)\right] \leq P_e \leq \exp\left[-\frac{\ln M}{\dot{R}}\left(\frac{\log e}{2N_0} - \dot{R}\right)\right]. \quad (11)$$

2) For $\frac{1}{4}\frac{\log e}{N_0} \leq \dot{R} \leq \frac{\log e}{N_0}$,

$$\exp\left[-\frac{\ln M}{\dot{R}}\left(\left(\sqrt{\frac{\log e}{N_0}} - \sqrt{\dot{R}}\right)^2 + o(1)\right)\right] \leq P_e \leq \exp\left[-\frac{\ln M}{\dot{R}}\left(\sqrt{\frac{\log e}{N_0}} - \sqrt{\dot{R}}\right)^2\right]. \quad (12)$$

In (11) and (12), $o(1) \to 0$ as $E \to \infty$.

Proof: See [4]. ■

Next, we define

$$a := \begin{cases} \frac{\left(\frac{\log e}{2N_0} - \dot{R}\right)}{\dot{R}}, & \text{if } 0 < \dot{R} \leq \frac{1}{4}\frac{\log e}{N_0} \\ \frac{\left(\sqrt{\frac{\log e}{N_0}} - \sqrt{\dot{R}}\right)^2}{\dot{R}}, & \text{if } \frac{1}{4}\frac{\log e}{N_0} \leq \dot{R} \leq \frac{\log e}{N_0} \end{cases} \quad (13)$$

and let $a_E := a + o(1)$. Then the bounds in Lemma 3 can be written as

$$1/M^{a_E} \leq P_e \leq 1/M^{a}. \quad (14)$$

Now let us consider the case where the users apply an orthogonal access scheme and each user uses an orthogonal codebook. For an orthogonal access scheme with orthogonal codebooks, the collection of codewords from all users is orthogonal, hence there are at most n codewords of length n. Since with a symmetric code each user transmits the same number of messages, it follows that each user transmits $M_n = n/k_n$ messages with codewords of energy less than or equal to E_n. In this case, we obtain from (6) and (14) that

$$\left(1 - \left(\frac{k_n}{n}\right)^{a}\right)^{k_n} \leq (1 - P(\mathcal{E}_1))^{k_n} \leq \left(1 - \left(\frac{k_n}{n}\right)^{a_{E_n}}\right)^{k_n},$$

which, denoting $a_n := a_{E_n}$, can be written as

$$\left[\left(1 - \left(\frac{k_n}{n}\right)^{a}\right)^{(\frac{n}{k_n})^a}\right]^{\frac{k_n^{1+a}}{n^a}} \leq (1 - P(\mathcal{E}_1))^{k_n} \leq \left[\left(1 - \left(\frac{k_n}{n}\right)^{a_n}\right)^{(\frac{n}{k_n})^{a_n}}\right]^{\frac{k_n^{1+a_n}}{n^{a_n}}}. \quad (15)$$

Since Theorem 3 only concerns a sublinear number of users, we have

$$\lim_{n\to\infty}\left(1 - \left(\frac{k_n}{n}\right)^{a}\right)^{(\frac{n}{k_n})^a} = \frac{1}{e}.$$

Furthermore, if $P_e^{(n)} \to 0$ then, by Lemma 1, $E_n \to \infty$ as $n \to \infty$, in which case a_n converges to the finite value a as $n \to \infty$. It follows that

$$\lim_{n\to\infty}\left(1 - \left(\frac{k_n}{n}\right)^{a_n}\right)^{(\frac{n}{k_n})^{a_n}} = \frac{1}{e}.$$

Thus, (15) implies that $P_e^{(n)} \to 0$ as $n \to \infty$ if

$$\lim_{n\to\infty}\frac{k_n^{1+a}}{n^a} = 0, \quad (16)$$

and only if

$$\lim_{n\to\infty}\frac{k_n^{1+a_n}}{n^{a_n}} = 0. \quad (17)$$

We next use these observation to prove Parts 1) and 2) of Theorem 3. We begin with Part 1). Let $\dot{R} < \frac{\log e}{N_0}$. Thus, we have $a > 0$ which implies that we can find a constant $\eta < a/(1+a)$ such that $n^{\eta(1+a)}/n^a \to 0$ as $n \to \infty$. Since, by assumption, $k_n = o(n^c)$ for every $c > 0$, it follows that there exists an n_0 such that, for all $n \geq n_0$, we have $k_n \leq n^{\eta(1+a)}$. This implies that (16) is satisfied, from which Part 1) follows.

We next prove Part 2) of Theorem 3. Indeed, if $k_n = \Theta(n^c)$, $0 < c < 1$, then there exist $0 < l_1 < l_2$ and n_0 such that, for all $n \geq n_0$, we have $(l_1 n)^c \leq k_n \leq (l_2 n)^c$. Consequently,

$$\frac{(l_1 n)^{c(1+a_n)}}{n^{a_n}} \leq \frac{k_n^{1+a_n}}{n^{a_n}} \leq \frac{(l_2 n)^{c(1+a_n)}}{n^{a_n}}. \quad (18)$$

If $P_e^{(n)} \to 0$, then from (17) we have $\frac{k_n^{1+a_n}}{n^{a_n}} \to 0$. Thus, (18) implies that $c(1+a_n) - a_n$ converges to a negative value. Since $c(1+a_n) - a_n$ tends to $c(1+a) - a$ as $n \to \infty$, it follows that $P_e^{(n)} \to 0$ only if $c(1+a) - a < 0$, which is the same as $a > c/(1-c)$. Using similar arguments, it follows from (16) that if $a > c/(1-c)$, then $P_e^{(n)} \to 0$. Hence, $P_e^{(n)} \to 0$ if, and only if, $a > c/(1-c)$. It can be observed from (13) that a is a monotonically decreasing function of $\dot{R}$. So for $k_n = \Theta(n^c), 0 < c < 1$, the capacity per unit-energy $\dot{C}_\perp$ is given by

$$\dot{C}_\perp = \sup\{\dot{R} \geq 0 : a(\dot{R}) > c/(1-c)\},$$

where we write $a(\dot{R})$ to make it clear that a as defined in (13) is a function of $\dot{R}$. This supremum can be computed as

$$\dot{C}_\perp = \begin{cases} \frac{\log e}{N_0}\left(\frac{1}{1+\sqrt{\frac{c}{1-c}}}\right)^2, & \text{if } 0 < c \leq 1/2 \\ \frac{\log e}{2N_0}(1-c), & \text{if } 1/2 < c < 1, \end{cases}$$

which proves Part 2) of Theorem 3. ■

Appendix
Proof of Lemma 2

Let $\mathcal{W}$ denote the set of $M_n^{k_n}$ messages of all users at blocklength n. To prove the lemma, we first show that

$$\frac{1}{M_n^{k_n}} \sum_{\mathbf{w}\in\mathcal{W}} P_e(\mathbf{w}) \geq 1 - \frac{64E_n/N_0 + \log 2}{\log\left(k_n(M_n-1)\right)},$$

where $P_e(\mathbf{w})$ denotes the probability of error in decoding the set of messages $\mathbf{w} = (w_1, \ldots, w_{k_n})$. To this end, we show that there exists a partition $\mathcal{S}_d$, $d = 1, \ldots, D$ of $\mathcal{W}$ such that for every d we have

$$\frac{1}{|\mathcal{S}_d|} \sum_{\mathbf{w}\in\mathcal{S}_d} P_e(\mathbf{w}) \geq 1 - \frac{64E_n/N_0 + \log 2}{\log\left(k_n(M_n-1)\right)}, \qquad (19)$$

where we use $|\cdot|$ to denote the cardinality of a set. This implies that the RHS of (19) is also a lower bound on

$$\frac{1}{M_n^{k_n}} \sum_{\mathbf{w}\in\mathcal{W}} P_e(\mathbf{w}) = \frac{1}{M_n^{k_n}} \sum_{d=1}^{D} \sum_{\mathbf{w}\in\mathcal{S}_d} P_e(\mathbf{w}).$$

To describe the partition, we use the following representation for $\mathbf{w} \in \mathcal{W}$: Each $\mathbf{w} \in \mathcal{W}$ is denoted using a k_n-length vector such that the i^{th} position of the vector is set to j if user i has message j, where $1 \leq j \leq M_n$. The Hamming distance d_H between two messages $\mathbf{w} = (w_1, \ldots, w_{k_n})$ and $\mathbf{w}' = (w'_1, \ldots, w'_{k_n})$ is defined as the number of positions at which $\mathbf{w}$ differs from $\mathbf{w}'$, i.e., $d_H(\mathbf{w}, \mathbf{w}') := |\{i : w_i \neq w'_i\}|$.

We next show that one can find a partition $\mathcal{S}_d$, $d = 1, \ldots, D$ such that for each set $\mathcal{S}_d$, there exists a $\tilde{\mathbf{w}} \in \mathcal{S}_d$ such that $d_H(\tilde{\mathbf{w}}, \mathbf{w}) \leq 2$ for all $\mathbf{w} \in \mathcal{S}_d$. Let $\mathcal{C}$ be a code in $\mathcal{W}$ with minimum Hamming distance 3, such that for any $\mathbf{w} \in \mathcal{W}$ there exists at least one codeword in $\mathcal{C}$ which is at most at a distance 2 from it. Such a code exists because if for some $\mathbf{w} \in \mathcal{W}$ all codewords were at a distance 3 or more, then we could add $\mathbf{w}$ to $\mathcal{C}$ without affecting its minimum distance. Let $\mathbf{c}(1), \ldots, \mathbf{c}(|\mathcal{C}|)$ denote the codewords of code $\mathcal{C}$. Next we partition the set $\mathcal{W}$ into the $D = |\mathcal{C}|$ sets $\mathcal{S}_d$, $d = 1, \ldots, D$ as follows:

For a given $d = 1, \ldots, D$, we assign $\mathbf{c}(d)$ to $\mathcal{S}_d$ as well as all $\mathbf{w} \in \mathcal{W}$ that satisfy $d_H(\mathbf{w}, \mathbf{c}(d)) = 1$. These assignments are unique since the code $\mathcal{C}$ has minimum Hamming distance 3. We next consider all $\mathbf{w} \in \mathcal{W}$ for which there is no codeword $\mathbf{c}(d)$ satisfying $d_H(\mathbf{w}, \mathbf{c}(d)) = 1$ and assign it to the set with index $d = \min\{i = 1, \ldots, D : d_H(\mathbf{w}, \mathbf{c}(i)) = 2\}$. Like this, we obtain a partition of $\mathcal{W}$, and since for every codeword there are $k_n(M_n - 1)$ sequences at Hamming distance one from it, this partition satisfies $|\mathcal{S}_d| \geq 1 + k_n(M_n - 1), d = 1, \ldots, D$.

We next derive the lower bound (19) using a stronger form of Fano's inequality known as Birgé's inequality.

Lemma 4 (Birgé's inequality): Let $(\mathcal{Y}, \mathcal{B})$ be a measurable space with a σ-field and $P_1, \ldots, P_N$ be probability measures defined on $\mathcal{B}$. Further let $\mathcal{A}_i$, $i = 1, \ldots, N$ denote N events defined on $\mathcal{Y}$, where $N \geq 2$. Then

$$\frac{1}{N} \sum_{i=1}^{N} P_i(\mathcal{A}_i) \leq \frac{\frac{1}{N^2}\sum_{i,j} D(P_i \| P_j) + \log 2}{\log(N-1)}.$$

Proof: See [7] and references therein. ■

To apply Lemma 4 to the problem at hand, we set $N = |\mathcal{S}_d|$ and $P_i = P_{Y|\mathbf{X}}(\cdot|\mathbf{x}(i))$, where $\mathbf{x}(i)$ denotes the set of codewords transmitted to convey the set of messages $i \in \mathcal{S}_d$. We further let $\mathcal{A}_i$ denote the subset of $\mathcal{Y}^n$ for which the decoder declares the set of messages $i \in \mathcal{S}_d$. Then, the probability of error in decoding messages $i \in \mathcal{S}_d$ is given by $P_e(i) = 1 - P_i(\mathcal{A}_i)$, and $\frac{1}{|\mathcal{S}_d|}\sum_{i\in\mathcal{S}_d} P_i(\mathcal{A}_i)$ denotes the average probability of correctly decoding a message in $\mathcal{S}_d$.

For two multivariate Gaussian distributions $\mathbf{Z}_1 \sim \mathcal{N}(\boldsymbol{\mu_1}, \frac{N_0}{2}I)$ and $\mathbf{Z}_2 \sim \mathcal{N}(\boldsymbol{\mu_2}, \frac{N_0}{2}I)$ (where I denotes the identity matrix), the relative entropy $D(\mathbf{Z}_1\|\mathbf{Z}_2)$ is given by $\frac{\|\boldsymbol{\mu_1}-\boldsymbol{\mu_2}\|^2}{N_0}$. We next note that $P_{\mathbf{w}} = \mathcal{N}(\overline{\mathbf{x}}(\mathbf{w}), \frac{N_0}{2}I)$ and $P_{\mathbf{w}'} = \mathcal{N}(\overline{\mathbf{x}}(\mathbf{w}'), \frac{N_0}{2}I)$, where $\overline{\mathbf{x}}(i)$ denotes the sum of codewords contained in $\mathbf{x}(i)$. Since the energy of a codeword for any user is upper-bounded by E_n, and since for any two $\mathbf{w}, \mathbf{w}' \in \mathcal{S}_d$, $d_H(\mathbf{w}, \mathbf{w}') \leq 4$, we get that $\|\overline{\mathbf{x}}(\mathbf{w}) - \overline{\mathbf{x}}(\mathbf{w}')\|^2 \leq 64E_n$ (see [4] for details). Consequently,

$$D(P_{\mathbf{w}}\|P_{\mathbf{w}'}) \leq 64E_n/N_0, \quad \mathbf{w}, \mathbf{w}' \in \mathcal{S}_d.$$

It thus follows from Birgé's inequality that

$$\frac{1}{|\mathcal{S}_d|} \sum_{\mathbf{w}\in\mathcal{S}_d} P_e(\mathbf{w}) \geq 1 - \frac{64E_n/N_0 + \log 2}{\log(|\mathcal{S}_d| - 1)}$$

which yields (19) by noting that $|\mathcal{S}_d| - 1 \geq k_n(M_n - 1)$. Note that (19) holds for all $d = 1, \ldots, D$, so

$$P_e^{(n)} \geq 1 - \frac{64E_n/N_0 + \log 2}{\log\left(k_n(M_n-1)\right)}.$$

This shows that $P_e^{(n)}$ goes to zero only if

$$\begin{aligned} E_n &= \Omega\left(\log(k_n(M_n-1))\right) \\ &= \Omega\left(\log M_n + \log k_n\right), \end{aligned} \qquad (20)$$

where (20) follows because, by Lemma 1, $P_e^{(n)} \to 0$ as $n \to \infty$ only if $E_n \to \infty$, which by the assumption $\dot{R} > 0$ implies that $M_n \to \infty$. Using that $\log M_n = E_n\dot{R}$, (20) can be written as $E_n = \Omega(E_n\dot{R} + \log k_n)$, which is equivalent to $E_n = \Theta(E_n\dot{R} + \log k_n)$. However, this holds only if $\log k_n = O(E_n)$ or, equivalently, $E_n = \Omega(\log k_n)$. This proves Lemma 2.

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