Universidad Carlos III de Madrid

# On several extremal problems in graph theory involving Gromov Hyperbolicity Constant

**Author:** Verónica Hernández Martínez
Departamento de Matemáticas

**Advisor:** José Manuel Rodríguez García
Departamento de Matemáticas

**Co-Advisor:** Domingo de Guzmán Pestana Galván
Departamento de Matemáticas

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**Author:** Verónica Hernández Martínez
Department of Mathematics

**Advisor:** José Manuel Rodríguez García
Department of Mathematics

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Departamento de Matemáticas


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*to my family*

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# Contents

**Introduction** **6**

**1 A brief introduction to graph theory** **12**
1.1 Basic notions . . . 14
1.2 Subgraphs . . . 16
1.3 Connectivity of graphs . . . 17
1.4 Some special graphs . . . 19

**2 An introduction to Gromov hyperbolicity** **23**
2.1 Definition of Gromov Hyperbolicy and examples . . . 25
2.2 Gromov hyperbolicity, Mathematical Analysis and Geometry . . . 28
2.3 Main results on hyperbolic spaces . . . 30
2.4 Previous results on hyperbolic graphs . . . 32

**3 On the order, size, and hyperbolicity constant** **37**
3.1 Bounds for $B(n, m)$ . . . 38
3.2 Difference of the bounds of $B(n, m)$ . . . 46
3.3 An inequality involving the diameter of a graph . . . 51
3.4 Computation of $A(n, m)$ . . . 52
3.5 Non-simple and weighted graphs . . . 56
3.6 Random graphs . . . 59

**4 Diameter, minimum and maximum degree** **61**
4.1 On a classical Theorem on the diameter and minimum degree . . . 62
4.2 Upper bounds for $b(n, \delta_0)$ . . . 71
4.3 Lower bounds and some precise values for $b(n, \delta_0)$ . . . 77
4.4 Computation of $a(n, \delta_0)$ . . . 83
4.5 Computation of $\beta(n, \Delta)$ . . . 85
4.6 Computation of $\alpha(n, \Delta)$ . . . 88

**5 Circumference, girth, and hyperbolicity constant** **90**
5.1 Technical results . . . 91
5.2 Bounds for $\mathcal{A}(g, c, n)$ . . . 98
5.3 Computation of $\mathcal{B}(g, c, n)$ . . . 106
5.4 Bounds for $\mathfrak{A}(g, c, m)$ . . . 108
5.5 Computation of $\mathfrak{B}(g, c, m)$ . . . 111

**Conclusions** **113**

**Open problems** **116**

**Bibliography** **118**

# List of Figures

1.1 The graph of the bridges of Königsberg. . . . . . . . . . . . . . . . . . . . . . 12
1.2 Simple graph $G_1$ and non-simple graph $G_2$. . . . . . . . . . . . . . . . . . . . . 14
1.3 A subgraph $G_1$ and an induced subgraph $G_2$ of the graph $G$. . . . . . . . . . 16
1.4 Representation of a connected graph $G_1$ and a disconnected graph $G_2$. . . . . 17
1.5 Path graphs. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 19
1.6 Cycle graphs. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 19
1.7 Complete graphs. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 19
1.8 Complete bipartite graphs. . . . . . . . . . . . . . . . . . . . . . . . . . . . 20
1.9 Star graphs. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 20
1.10 Wheel graphs. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 21

2.1 $\mathbb{R}$ and $\mathbb{R}^2$ as examples of hyperbolic and non-hyperbolic spaces. . . . . . . . 26
2.2 Any metric tree $T$ verifies $\delta(T) = 0$. . . . . . . . . . . . . . . . . . . . . . . . 26
2.3 First steps in order to compute the hyperbolicity constant of $X$. . . . . . . . 27
2.4 Calculating the supremum over all geodesic triangles. . . . . . . . . . . . . . . 27

# Introduction

Gromov hyperbolicity was introduced by the Russian mathematician Mikhail Leonidovich Gromov in the setting of geometric group theory [59], [57], [41], but has played an increasing role in analysis on general metric spaces [19], [20], [8], with applications to the Martin boundary, invariant metrics in several complex variables [7] and extendability of Lipschitz mappings [79]. The concept of hyperbolicity appears also in discrete mathematics, algorithms and networking. For example, it has been shown empirically in [105] that the internet topology embeds with better accuracy into a hyperbolic space than into a Euclidean space of comparable dimension (formal proofs that the distortion is related to the hyperbolicity can be found in [109]); furthermore, it is evidenced that many real networks are hyperbolic (see, e.g., [1, 2, 40, 85]). Another important application of these spaces is the study of the spread of viruses through the internet (see [67, 68]). Furthermore, hyperbolic spaces are useful in secure transmission of information on the network (see [67, 68]). In [72] the authors study hyperbolicity in large scale networks (such as communication, citation, collaboration, peer-to-peer, friendship and other social networks) and propose that hyperbolicity, in conjunction with other local characteristics of networks, such as the degree distribution and clustering coefficients, provide a more complete unifying picture of networks, and helps classify in a parsimonious way what is otherwise a bewildering and complex array of features and characteristics specific to each natural and man-made network. The hyperbolicity has also been used extensively in the context of random graphs (see, e.g., [102, 103, 104]).

In a geodesic metric space $X$, a geodesic triangle is the union of three geodesic in $X$. A geodesic metric space $X$ is called hyperbolic (in the Gromov sense) if it satisfies the Rips condition, i.e. there is an upper bound of the distance of every point in a side of any geodesic triangle in $X$ to the union of the two other sides (see Definition 2.1.1). The sharp upper bound in the previous definition is called the hyperbolicity constant of $X$ and is denoted by $\delta(X)$. It is important to point out that there are several definitions of Gromov Hyperbolicity (see Chapter 2), although in this Thesis we work with Definition 2.1.1. For detailed expositions about Gromov hyperbolicity, see e.g. [3], [41], [57] or [108] (in this work we use the notations of [57]).

In particular, graphs can be seen as geodesic metric spaces, considering that the points in a graph $G$ are the vertices and, also, the points in the interior of any edge of $G$. The study of Gromov hyperbolic graphs is a subject of increasing interest in graph theory; see, e.g., [6, 16, 23, 28, 35, 37, 40, 42, 43, 54, 55, 67, 68, 75, 77, 81, 82, 85, 94, 107, 111, 113] and the references therein.

Last years several researchers have been interested in showing that metrics used in geometric function theory are Gromov hyperbolic. In particular, the equivalence of the hyperbolicity of Riemannian manifolds and the hyperbolicity of a very simple graph was proved in [92, 94, 107], hence, it is useful to know hyperbolicity criteria for graphs.

The main examples of hyperbolic graphs are trees. In fact, the hyperbolicity constant of a geodesic metric space can be viewed as a measure of how "tree-like" the space is, since those spaces $X$ with $\delta(X) = 0$ are precisely the metric trees. This is an interesting subject since, in many applications, one finds that the borderline between tractable and intractable cases may be the tree-like degree of the structure to be dealt with (see, e.g., [36]).

For a finite graph with $n$ vertices it is possible to compute $\delta(G)$ in time $O(n^{3.69})$ [54] (this is improved in [40, 42]). Given a Cayley graph (of a presentation with solvable word problem) there is an algorithm which allows to decide if it is hyperbolic [89]. However, deciding whether or not a general infinite graph is hyperbolic is usually very difficult. Therefore, it is interesting to relate hyperbolicity with other properties of graphs. The papers [23, 113, 11, 30] prove, respectively, that chordal, $k$-chordal, edge-chordal and join graphs are hyperbolic. Moreover, in [11] it is shown that hyperbolic graphs are path-chordal graphs. These results relating chordality and hyperbolicity are improved in [81]. Some other authors have obtained results on hyperbolicity for particular classes of graphs: vertex-symmetric graphs, bipartite and intersection graphs, bridged graphs, expanders and median graphs [25, 43, 75, 80, 106].

The three main problems on Gromov hyperbolic graphs are the following:

**I.** To obtain inequalities relating the hyperbolicity constant and other parameters of graphs.

**II.** To study the hyperbolicity for important classes of graphs.

**III.** To study the invariance of the hyperbolicity of graphs under appropriate transformations.

In this Thesis we study the extremal problems of maximazing and minimazing the hyperbolicity constant on several families of graphs (problem II in the above list). In order to properly raise our research problem, we need to introduce some important definitions and make some remarks on the graphs we study.

We consider simple (without loops or multiple edges) and connected graphs such that every edge has length 1. Note that to exclude multiple edges and loops is not an important loss of generality, since [16, Theorems 8 and 10] reduce the problem of computing the hyperbolicity constant of graphs with multiple edges and/or loops to the study of simple graphs. Also, if we consider a graph $G$ whose edges have length equal to one and a graph $G_k$ obtained from $G$ stretching out their edges until length $k$, then $\delta(G_k) = k\delta(G)$. Therefore, all the results in this work can be generalized when the edges of the graph have length equal to $k$.

The vertex set of a graph $G$ is denoted by $V(G)$, and the *order* $n$ of a graph is the number of its vertices ($n = |V(G)|$). The edge set of a graph $G$ is denoted by $E(G)$, and the *size* $m$ of a graph is the number of its edges ($m = |E(G)|$).

The *degree* of $v \in V(G)$ is the number of edges incident to the vertex and is denoted $\deg(v)$. The *minimum degree* is defined as $\delta_0 := \min\{\deg(v) \mid v \in V(G)\}$, whereas the *maximum degree* of a graph $G$ is defined as $\Delta := \max\{\deg(v) \mid v \in V(G)\}$.

Throughout this work, by *cycle* in a graph we mean a simple closed curve, i.e., a path with different vertices, except for the last one, which is equal to the first vertex. The *girth* of a graph (denoted by $g(G)$) is the length of any shortest cycle contained in the graph, whereas the *circumference* of a graph (denoted by $c(G)$) is the length of any longest cycle in a graph.

Let $\mathcal{G}(n, m)$ be the family of graphs $G$ of order $n$ and size $m$. Let us define

$$A(n, m) := \min\{\delta(G) \mid G \in \mathcal{G}(n, m)\},$$

$$B(n, m) := \max\{\delta(G) \mid G \in \mathcal{G}(n, m)\}.$$

Let $\mathcal{H}(n, \delta_0)$ be the family of graphs $G$ of order $n$ and minimum degree $\delta_0$. Let us define

$$a(n, \delta_0) := \min\{\delta(G) \mid G \in \mathcal{H}(n, \delta_0)\},$$

$$b(n, \delta_0) := \max\{\delta(G) \mid G \in \mathcal{H}(n, \delta_0)\}.$$

Let $\mathcal{J}(n, \Delta)$ be the family of graphs $G$ of order $n$ and maximum degree $\Delta$. Let us define

$$\alpha(n, \Delta) := \min\{\delta(G) \mid G \in \mathcal{J}(n, \Delta)\},$$

$$\beta(n, \Delta) := \max\{\delta(G) \mid G \in \mathcal{J}(n, \Delta)\}.$$

Let $\mathcal{M}(g, c, n)$ be the family of graphs $G$ of girth $g$, circumference $c$, and order $n$. Let us define

$$\mathcal{A}(g, c, n) := \min\{\delta(G) \mid G \in \mathcal{M}(g, c, n)\},$$

$$\mathcal{B}(g, c, n) := \max\{\delta(G) \mid G \in \mathcal{M}(g, c, n)\}.$$

Let $\mathcal{N}(g, c, m)$ be the family of graphs $G$ of girth $g$, circumference $c$, and size $m$. Let us define

$$\mathfrak{A}(g, c, m) := \min\{\delta(G) \mid G \in \mathcal{N}(g, c, m)\},$$

$$\mathfrak{B}(g, c, m) := \max\{\delta(G) \mid G \in \mathcal{N}(g, c, m)\}.$$

Our aim in this work is to estimate $A(n, m)$, $B(n, m)$, $a(n, \delta_0)$, $b(n, \delta_0)$, $\alpha(n, \Delta)$, $\beta(n, \Delta)$, $\mathcal{A}(g, c, n)$, $\mathcal{B}(g, c, n)$, $\mathfrak{A}(g, c, m)$ and $\mathfrak{B}(g, c, m)$, i.e., to study the extremal problems of maximazing and minimazing $\delta(G)$ on the sets $\mathcal{G}(n, m)$, $\mathcal{H}(n, \delta_0)$, $\mathcal{J}(n, \Delta)$, $\mathcal{M}(g, c, n)$ and $\mathcal{N}(g, c, m)$. In this way, we find bounds for $\delta(G)$ in terms of important parameters of the graph.

The outline of this PhD Thesis is as follows.

Chapter 1 is an introduction to graph theory. In Chapter 2 we give a brief introduction to hyperbolic spaces in the Gromov sense and present key definitions, as well as previous results regarding hyperbolicity, which are used in the Thesis.

In Chapter 3 we focus on estimating $A(n, m)$ and $B(n, m)$. In section 3.1 we can find two of our main results in this chapter: Theorems 3.1.11 and 3.1.13, which give upper and lower bounds for $B(n, m)$, respectively. In Section 3.2 we prove that the difference between the upper and lower bounds of $B(n, m)$ is $O(\sqrt{n})$, which means it is a good estimate. In Section 3.3, Theorem 3.4.10 gives the precise value of $A(n, m)$. In Section 3.4 we deal with non-simple and weighted graphs. The last section in this chapter gives a brief introduction to random graphs, while explaining that our results can be applied to a modified Erdös-Rényi random graph. As a consequence of our results, we also obtain an inequality involving the diameter, the order and the size of any graph (see Theorem 3.3.2).

In Chapter 4, Section 4.1 we obtain good upper bounds for the diameter of any graph in terms of its minimum degree and its order. One of our key results is Theorem 4.1.2, which improves a classical theorem due to Erdös, Pach, Pollack and Tuza. Also, Theorems 4.1.5, 4.1.7, 4.1.9, 4.1.11, 4.1.13, 4.1.15 and 4.1.17 provide better estimations of $\operatorname{diam} V(G)$ for some values of $\delta_0$. We later use these bounds in order to study hyperbolic graphs in the Gromov sense. In Section 4.2, Theorem 4.2.11 gives upper bounds for $b(n, \delta_0)$. Moreover, in Section 4.3, Theorem 4.3.11 gives the precise value of $b(n, \delta_0)$ for many values of $n$ and $\delta_0$. In Section 4.4, 4.5 and 4.6 we compute the precise values of $a(n, \delta_0)$, $\beta(n, \Delta)$ and $\alpha(n, \Delta)$, respectively (see Theorems 4.4.1, 4.5.2 and 4.6.1).

Finally, in Chapter 5 our aim is to study the extremal problems of maximazing and minimazing $\delta(G)$ on the sets $\mathcal{M}(g, c, n)$ and $\mathcal{N}(g, c, m)$. In Section 5.1 we present some useful definitions and previous results. Sections 5.2 and 5.4 contain good bounds for $\mathcal{A}(g, c, n)$ and $\mathfrak{A}(g, c, m)$, respectively. In Sections 5.3 and 5.5, Theorems 5.3.3 and 5.5.1 give the precise value of $\mathcal{B}(g, c, n)$ and $\mathfrak{B}(g, c, m)$, respectively.

The results in this work appear in [61, 62, 63]; these papers have been published or submitted to international mathematical journals which appear in the Journal Citation Reports.

These results were presented in the following international and national conferences:

- X Enuentro Andaluz de Matemática Discreta, July 2017, Escuela Politécnica Superior de Algeciras, Spain.

- XVII Encuentros de Análisis Real y Complejo, May 2017, Universidad de La Laguna, Spain.

- 7th European Congress of Mathematics, July 2016, Technische Universitat Berlin, Germany.

- Cologne - Twente Workshop, June 2016, Universita degli Studi di Milano, Italy.
- IX Enuentro Andaluz de Matemática Discreta, October 2015, Universidad de Almería, Spain.
- 9th Workshop of Young Researchers in Mathematics, September 2015, Universidad Complutense de Madrid, Spain.
- Congreso de Jóvenes Investigadores RSME, September 2015, Universidad de Murcia, Spain.
- XVI Encuentros de Análisis Real y Complejo, May 2015, Universidad de Sevilla, Spain.
- 8th Workshop of Young Researchers in Mathematics, September 2014, Universidad Complutense de Madrid, Spain.
- IX Conference on Discrete Mathematics and Algorithms, July 2014, Universitat Roviera I Virgili, Spain.

They were presented also in the following Seminars:

- Seminar of Discrete Mathematics of IMFM and FNM, October 2016, Maribor, Slovenia.
- Leoben - Ljubljana Graph Theory Seminar, September 2016, Judenburg, Austria.
- Seminar on Orthogonality, Approximation Theory and Applications. Group of Applied Mathematical Analysis (GAMA), June 2014, Universidad Carlos III de Madrid, Spain

# Chapter 1

# A brief introduction to graph theory

In this chapter, we lay the foundations for a proper study of graph theory. For further discussion, see [18] and [44].

We want to start by introducing the The Königsberg bridge problem, which is often said to have been the birth of graph theory. The city of Königsberg was located on the Pregel river in Prussia. The river divided the city into four separate landmasses. These four regions were linked by seven bridges. The Königsberg bridge problem asks if the seven bridges can all be traversed in a single trip without doubling back, with the additional requirement that the trip ends in the same place it began. The Swiss mathematician Leonhard Euler solved this problem in 1736, by proving that it is in fact impossible to cross each of the seven bridges of Königsberg exactly once. The key to Euler's solution was in a very simple abstraction of the puzzle: redraw the standard diagram of the city by representing each of the land masses as a point and representing each bridge as a line connecting the points corresponding to the land masses (see Figure 1.1 below).

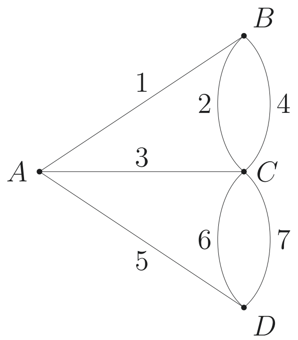



Figure 1.1: The graph of the bridges of Königsberg.

This model makes it easy to argue that the desired travesal does not exist. Each time we enter and leave a point (or land mass) we use two bridges ending at it. Thus, existence of the desired traversal requires an even number of bridges for each land mass. This neccesary condition did not hold in Königsberg.

Since the 18th century graph theory has developed into an extensive and popular branch of mathematics, which has been applied to many problems in computer science, and other scientific areas. As in the Königsberg bridge problem, many real-world situations can conveniently be described by means of a diagram consisting of a set of points together with lines joining certain pairs of these points. For example, computers, roads, railways or electric networks. Note that in this type of diagrams we are interested mainly if two given points are connected by a line, the way they come together is immaterial. The mathematical abstraction of situations of this type gives rise to the concept of graphs.

## 1.1 Basic notions

Graphs are so named because they can be represented graphically, and it is this graphical representation which helps us to understand many of their properties. Each vertex is indicated by a point, and each edge by a line joining the points representing its ends. More precisely, a *graph*, usually denoted $G(V(G), E(G))$ or $G = (V, E)$, consists of a set of vertices $V(G)$ together with a set $E(G)$ of unordered pairs of vertices called edges. The number of vertices in a graph is usually denoted $n = |V(G)|$ while the number of edges is usually denoted $m = |E(G)|$, these two basic parameters are called the *order* and *size* of $G$, respectively. We say that a graph $G$ is *finite* if and only if $n < \infty$ and $m < \infty$. Otherwise we say that the graph is *infinite*. For our purposes all graphs will be finite graphs, unless specifically stated otherwise.

An *edge* joining the vertices $u \in V(G)$ and $v \in V(G)$ on many occasions is denoted by $[uv]$, but we will use the notation $[u, v]$ to denote it, since the notation $[uv]$ will be used in this work for geodesics, which will be discussed in Chapter 2.

Any graph with just one vertex is referred to as *trivial graph*. All other graphs are *non-trivial*.

We say that two vertices $u \in V(G), v \in V(G)$ are *adjacent* or *neighbours* if $[u, v] \in E(G)$ and we also denote it by $u \sim v$; likewise, two edges are adjacent if they have one vertex in common; similarly, if $e = [u, v]$ we say that the edge $e \in E(G)$ is *incident* to the vertices $u$ and $v$. The set of neighbours of a vertex $v$ in a graph $G$ is denoted by $N_G(v)$, i.e., $N_G(v) := \{u \in V(G) : [u, v] \in E(G)\}$.

The *degree* of a vertex is the number of neighbors it has in the graph. The degree of $v \in V(G)$ is denoted by $\deg(v) := |N_G(v)|$.

The number $\delta_0(G) := \min\{\deg(v) : v \in V(G)\}$ is the *minimum degree* of $G$ and the number $\Delta(G) := \max\{\deg(v) : v \in V(G)\}$ is its *maximum degree*. In Figure 1.2, $\delta_0(G_1) = 0$ and $\Delta(G_2) = 4$.

If the degree of a vertex is 0, we say that is an *isolated vertex*. In Figure 1.2, the vertex $D$ in the graph $G_1$ is an isolated vertex.


Figure 1.2: Simple graph $G_1$ and non-simple graph $G_2$.

An edge with identical ends is called a *loop*, and an edge with distinct ends a *link*. Two or more links with the same pair of ends are said to be *multiple edges*.

In the graph $G_2$ of Figure 1.2, the edge $c$ is a loop, and all other edges are links; the edges $e$ and $d$ are multiple edges.

A *simple graph* is one that has a single edge joining any two adjacent vertices, i.e., a graph without loops and multiple edges (see the graph $G_1$ in Figure 1.2). Although some authors consider non-simple graphs (allowing loops and multiple edges), unless otherwise stated, by graph we mean a simple graph.

## 1.2 Subgraphs

Apart from the study of the characteristics or properties of a graph in its entirety, one can also consider only a region or a part thereof. For example, we can study arbitrary sets of vertices and edges of any graph. Moreover, in many cases, it is appropriate to consider graphs that are included "within" other. We will call them *subgraphs*.

**Definition 1.2.1.** (***Subgraph***)
*If $G = (V, E)$ is a graph then $G_1 = (V_1, E_1)$ is a* subgraph *of $G$ if $\emptyset \neq V_1 \subseteq V$ and $E_1 \subseteq E$ where each edge in $E_1$ is incident to vertices of $V_1$.*

See in Figure 1.3 the subgraphs $G_1$ and $G_2$ of the graph $G$. Particular types of subgraphs are obtained by removing in some vertex in a graph. We have formalized this idea in the following definitions. Let $v$ be a vertex of a graph $G = (V(G), E(G))$. The subgraph $G - v$ of $G$ is that graph whose vertex set is $V(G) - \{v\}$ and edge set is $E(G - v)$ (all edges of the graph $G$ except the incident edges to $v$).

A relevant class of subgraphs are the *induced subgraphs*.

**Definition 1.2.2.** (***Induced subgraph***)
*A subgraph obtained by vertex deletions only is called an* induced subgraph. *If $X$ is the set of vertices deleted, the resulting subgraph is denoted by $G - X$. Frequently, it is the set $Y := V \setminus X$ of vertices which remain that is the focus of interest.*

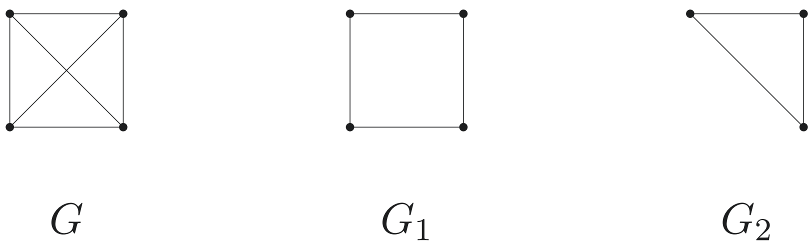



Figure 1.3: A subgraph $G_1$ and an induced subgraph $G_2$ of the graph $G$.

In Figure 1.3, $G_2$ is an induced subgraph of $G$. We can see graphically that it is the result of removing a vertex in the graph $G$.

## 1.3 Connectivity of graphs

One of the most significant properties that a graph may have is its connectivity. To understand this concept, it is necessary to give some definitions that describe what it means to go from one vertex to another.

**Definition 1.3.1.** (***Path***)
*A* path *of a graph $G = (V, E)$ is a sequence of vertices $P = \{v_0, v_1, v_2, \dots, v_n\}$ such that $v_{i-1}$ is adjacent to $v_i$, for $i = 1, 2, \dots, n$; a* simple path *is a path in which all vertices are different.*

**Definition 1.3.2.** (***Cycle***)
*By* cycle *we mean a simple closed curve, i.e., a path with different vertices, unless the last one, which is equal to the first vertex.*

The *length* of a path or a cycle is the number of its edges. We denote by $L(g)$ the length of the path $g$.

**Definition 1.3.3.** (***Connectivity***)
*A graph is* connected *if, for every partition of its vertex set into two nonempty sets $X$ and $Y$, there is an edge with one end in $X$ and one end in $Y$; otherwise, the graph is* disconnected *or* non-connected.

Given a connected graph $G = (V, A)$ and any two distinct vertices $u, v \in V$, we can find a path that connects them. Examples of connected and disconnected graphs are displayed in Figure 1.4.


Figure 1.4: Representation of a connected graph $G_1$ and a disconnected graph $G_2$.

**Definition 1.3.4.** (***Connected component***)
*A* connected component *of a graph $G$ is a connected subgraph of $G$ which is not properly contained on any other connected subgraph of $G$, that is, a connected component of $G$ is a subgraph that is maximal with respect to the property of being connected.*

**Definition 1.3.5.** (***Distance between two vertices***) *In a graph $G$ we define the* distance *between two vertices $u, v$ denoted by $d_G(u, v)$ or $d(u, v)$ as*

$$d_G(u, v) := \inf\{L(g) \mid g \textit{ is a path joining } u \textit{ and } v\}.$$

If there is not a path joining $u$ and $v$, we set $d(u, v) := \infty$. In a connected graph $G$, for every $u, v \in V(G)$ we have $d_G(u, v) < \infty$.

**Definition 1.3.6.** (***Diameter***) *Given a graph $G$, we define the diameter as*

$$\operatorname{diam} V(G) := \sup\big\{d(u, v) \,|\, u, v \in V(G)\big\}, \qquad \operatorname{diam} G := \sup\big\{d(x, y) \,|\, x, y \in G\big\}.$$

## 1.4 Some special graphs

Some graphs appear frequently in many applications and, hence, they have standard names.

**Definition 1.4.1.** (***Path graph***)
*A* path graph *is a graph* $P = (V, E)$ *with* $V = \{v_1, v_2, \ldots, v_n\}$, $n \geq 2$ *and* $E = \{[v_1, v_2], [v_2, v_3], \ldots, [v_{n-1}, v_n]\}$. *The path graph with* $n$ *vertices is denoted by* $P_n$. *The vertices* $v_1$ *and* $v_n$ *are called its ends; the vertices* $v_2, \ldots, v_{n-1}$ *are the inner vertices of* $P_n$.

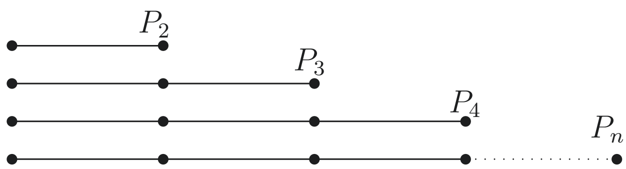



Figure 1.5: Path graphs.

**Definition 1.4.2.** (***Cycle graph***)
*A* cycle graph *of* $n$ *vertices is a graph* $G = (V, E)$ *with* $V = \{v_1, v_2, \ldots, v_n\}$, $n \geq 3$ *and* $E = \{[v_1, v_2], [v_2, v_3], \ldots, [v_{n-1}, v_n], [v_n, v_1]\}$. *It is denoted by* $C_n$.

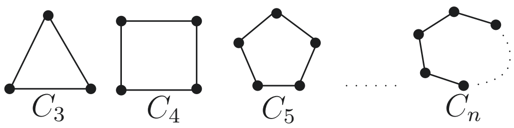



Figure 1.6: Cycle graphs.

**Definition 1.4.3.** (***Complete graph***)
*A* complete graph *is a graph in which every pair of vertices are joined by exactly one edge, i.e., all pairs of vertices of* $G$ *are adjacent. The complete graph with* $n$ *vertices is denoted by* $K_n$. *At each vertex* $v \in V(G)$ *we have* $\deg_G(v) = n - 1$.

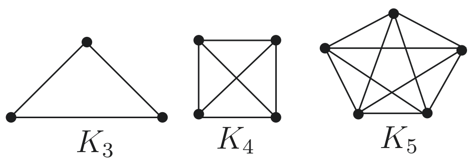



Figure 1.7: Complete graphs.

Note that $K_n$ has $n(n-1)/2$ edges.

**Definition 1.4.4.** (***Empty graph***)

*An* empty graph *is a graph whose edge set is empty. We denote by $E_n$ the empty graph with $n$ vertices. In an empty graph all vertices have degree* 0.

**Definition 1.4.5.** (***Bipartite graph***)

*A graph is* bipartite *if its vertex set can be partitioned into two nonempty subsets $V_1$ and $V_2$ so that no edge has both ends in $V_1$ or both ends $V_2$.*

**Definition 1.4.6.** (***Complete bipartite graph***)

*A bipartite graph is said to be a* complete bipartite graph *if each vertex of $V_1$ is adjacent with each vertex of $V_2$. If $|V_1| = m$ and $|V_2| = n$, then this graph is denoted by $K_{m,n}$.*

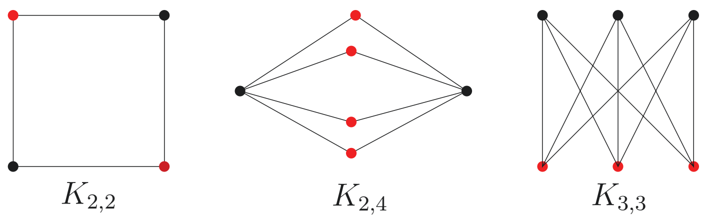



Figure 1.8: Complete bipartite graphs.

**Definition 1.4.7.** (***Star graph***)

*The complete bipartite graph $K_{n-1,1}$ is called an $n$* star graph *and it is denote by $S_n$.*

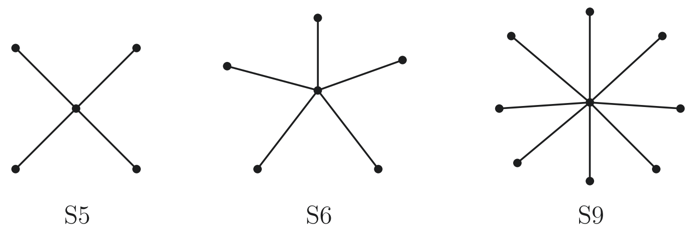



Figure 1.9: Star graphs.

**Definition 1.4.8.** (***Tree***)

*A* tree *is an acyclic and connected graph, i.e., a connected graph without cycles.*

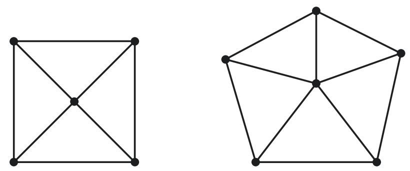
**Remark 1.4.9.** *In many ways a tree is the simplest non-trivial type of graph. It has several nice properties, such as the fact that any two vertices are connected by a unique path. Note that the star graph and the path graph defined before, are both particular cases of trees. A disjoint union of trees is called a forest.*

**Definition 1.4.10.** (***Regular graph***)

*A graph $G = (V, E)$ is* regular *if all vertices have the same degree $k$, and we say that it is $k$-regular. Every regular graph $G$ satisfies the equality $\delta_0(G) = \Delta(G)$. In fact, a graph $G$ is regular if and only if $\delta_0(G) = \Delta(G)$*

**Definition 1.4.11.** (***Wheel graph***)

*The* wheel graph *$W_n$ is a graph with $n$ vertices formed by connecting a single vertex to each vertex of a cycle $C_{n-1}$.*



$W_5$ $W_6$

Figure 1.10: Wheel graphs.

# Chapter 2

# An introduction to Gromov hyperbolicity

The concept of hyperbolicity offers a global approach to spaces like the hyperbolic plane, simply-connected Riemannian manifolds with negative sectional curvature, metric trees and other classical hyperbolic spaces. Several of their properties were introduced by Mikhael Leonidovich Gromov in the context of finitely generated groups but it generaly reached new horizons.

Next, we provide some basic concepts which are needed when studying Gromov Hyperbolicity.

If $\gamma : [a, b] \longrightarrow X$ is a continuous curve in a metric space $(X, d)$, we can define the *length* of $\gamma$ as $L(\gamma) := \sup \left\{ \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) : \, a = t_0 < t_1 < \cdots < t_n = b \right\}$. If $X$ is a metric space, we say that the curve $\gamma : [a, b] \longrightarrow X$ is a *geodesic* if it is an isometry, i.e., if we have $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$(then $\gamma$ is equipped with an arc-length parametrization). The metric space $X$ is said to be *geodesic* if for every couple of points in $X$ there exists a geodesic joining them; we denote by $[xy]$ any geodesic joining $x$ and $y$; this notation is ambiguous, since in general we do not have uniqueness of geodesics, but it is very convenient. Consequently, any geodesic metric space is connected. If the metric space $X$ is a graph, then the edge joining the vertices $u$ and $v$ will be denoted by $[u, v]$.

Recall that we consider only simple (without loops or multiple edges) and connected graphs such that every edge has length 1. In order to consider a graph $G$ as a geodesic metric space, we identify (by an isometry) any edge $[u, v] \in E(G)$, where $E(G)$ denotes the edge set of $G$, with the interval $[0, 1]$ in the real line; then the edge $[u, v]$ (considered as a graph with just one edge) is isometric to the interval $[0, 1]$. Thus, the points in $G$ are the vertices and, also, the points in the interior of any edge of $G$. In this way, any graph $G$ has a natural distance defined on its points, induced by taking the shortest paths in $G$, and we can see $G$ as a metric graph.

In this Chapter we introduce the concept of Gromov hyperbolicity and the main results concerning this theory. In the first section we formally define the concept of hyperbolic spaces in the Gromov sense. There are several equivalent definitions of Gromov Hyperbolicity, which are introduced in Section 2.2. Nevertheless, in this work we use the definition described in Section 2.1 (given by the Rips condition for geodesic triangles) because of its deep geometric meaning [57]. Finally, the last two sections provide important previous results which will be useful in this Thesis.

## 2.1 Definition of Gromov Hyperbolicy and examples

If $X$ is a geodesic metric space and $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $T_1 := [x_1x_2]$, $T_2 := [x_2x_3]$ and $T_3 := [x_3x_1]$. Sometimes we write the geodesic triangle $T$ as $T = \{[x_1x_2], [x_2x_3], [x_3x_1]\}$.

If $X$ is a geodesic metric space and $T = \{T_1, T_2, T_3\}$ is geodesic triangle with sides $T_i \subseteq X$, we say that $T$ is *$\delta$-thin* if each of its sides is contained in the $\delta$-neighborhood of the union of the other sides, i.e., $d(x, \cup_{j\neq i}T_j) \leq \delta$ for every $x \in T_i$. We denote by $\delta(T)$ the sharp thin constant of $T$, i.e., $\delta(T) = \inf\{\delta \geq 0 | \, T \text{ is } \delta\text{-thin}\}$.

**Definition 2.1.1.** *The space $X$ is $\delta$-*hyperbolic *(or satiesfies the Rips condition with constant $\delta$) if every geodesic triangle in $X$ is $\delta$-thin.*

We denote by $\delta(X)$ the sharp hyperbolicity constant of $X$, i.e.,

$$\delta(X) := \sup\{\delta(T) | \, T \text{ is a geodesic triangle in } X\}.$$

We say that $X$ is *hyperbolic* if $X$ is $\delta$-hyperbolic for some $\delta \geq 0$.

**Remark 2.1.2.** *There are several definitions of Gromov hyperbolicity. These different definitions are equivalent in the sense that if $X$ is $\delta$-hyperbolic with respect to the definition A, then it is $\delta'$-hyperbolic with respect to the definition B for some $\delta'$ (see, e.g., [22, 57]). We have chosen Definition 2.1.1 since it has a deep geometric meaning (see, e.g., [57]).*

Next, we collect some interesting examples of hyperbolic spaces in the Gromov sense.

**Example 2.1.3.** *Every bounded metric space $X$ is $(\frac{1}{2}\operatorname{diam} X)$-hyperbolic.*

**Example 2.1.4.** *The real line $\mathbb{R}$ is 0-hyperbolic: In fact, any point of a geodesic triangle in the real line belongs to two sides of the triangle simultaneously, and therefore any geodesic triangle in $\mathbb{R}$ is 0-thin.*

**Example 2.1.5.** *The Euclidean plane $\mathbb{R}^2$ is not hyperbolic, since the midpoint of a side on a large equilateral triangle is far from all points on the other two sides.*

These arguments can be applied to higher dimensions:

**Example 2.1.6.** *A normed real vector space is hyperbolic if and only if it has dimension* 1.

**Example 2.1.7.** *Every metric tree with arbitrary edge lengths is 0-hyperbolic, by the same reason that the real line.*

**Example 2.1.8.** *The unit disk $\mathbb{D}$ (with its Poincaré metric) is $\log(1+\sqrt{2}\,)$-thin: Consider any geodesic triangle $T$ in $\mathbb{D}$. It is clear that $T$ is contained in an ideal triangle $T'$, all of whose sides are of infinite length, with $\delta(T) \leq \delta(T')$. Since all ideal triangles are isometric, we can consider just one fixed $T'$. Then, a computation gives $\delta(T') = \log(1+\sqrt{2}\,)$.*

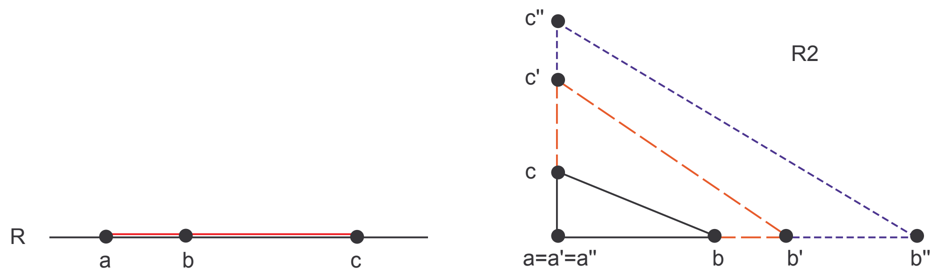



Figure 2.1: $\mathbb{R}$ and $\mathbb{R}^2$ as examples of hyperbolic and non-hyperbolic spaces.

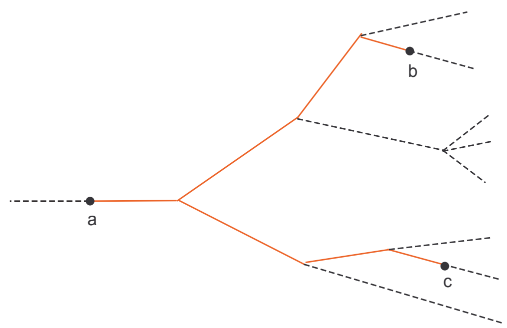



Figure 2.2: Any metric tree $T$ verifies $\delta(T) = 0$.

**Example 2.1.9.** *Every simply connected complete Riemannian manifold with sectional curvatures verifying $K \leq -c^2 < 0$, for some constant $c$, is hyperbolic (see, e.g., [57, p.52]).*

**Example 2.1.10.** *The graph $\Gamma$ of the routing infrastructure of the Internet is also empirically shown to be hyperbolic (see [9]). One can think that this is a trivial (and then a non-useful) fact, since every bounded metric space $X$ is $(\frac{1}{2}\operatorname{diam} X)$-hyperbolic. The point is that the quotient*

$$\frac{\delta(\Gamma)}{\operatorname{diam}\Gamma}$$

*is very small, and this makes the tools of hyperbolic spaces applicable to $\Gamma$ (see, e.g., [35]).*

We would like to point out that deciding whether or not a space is hyperbolic is usually very difficult. Notice that, first of all, we have to consider an arbitrary geodesic triangle $T$, and calculate the minimum distance from an arbitrary point $P$ of $T$ to the union of the other two sides of the triangle to which $P$ does not belong to. Thereafter, we have to take the supremum over all the possible choices for $P$ and then over all the possible choices for $T$. This means that if our space is, for instance, an $n$-dimensional manifold and we select two points $P$ and $Q$ on different sides of a triangle $T$, then the function $F$ that measures the distance between $P$ and $Q$ is a $(3n + 2)$-variable function ($3n$ variables describe the three

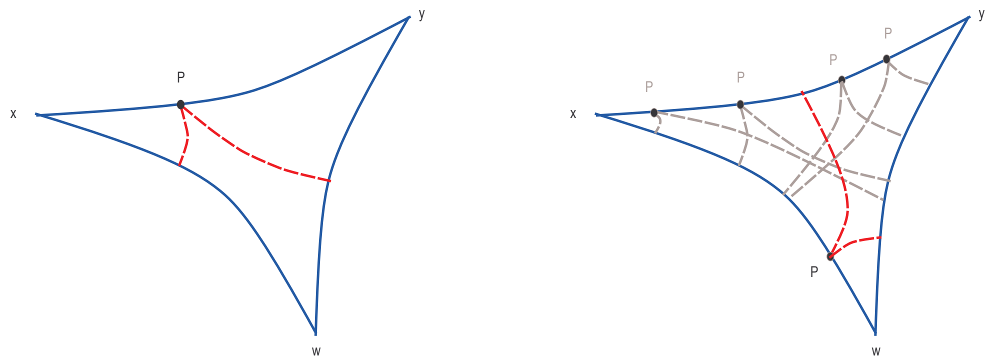



Figure 2.3: First steps in order to compute the hyperbolicity constant of $X$.

vertices of $T$ and two variables describe the points $P$ and $Q$ in the closed curve given by $T$). In order to prove that our space is hyperbolic we would have to take the minimum of $F$ over the variable that describes $Q$, and then the supremum over the remaining $3n+1$ variables, or at least to prove that it is finite.

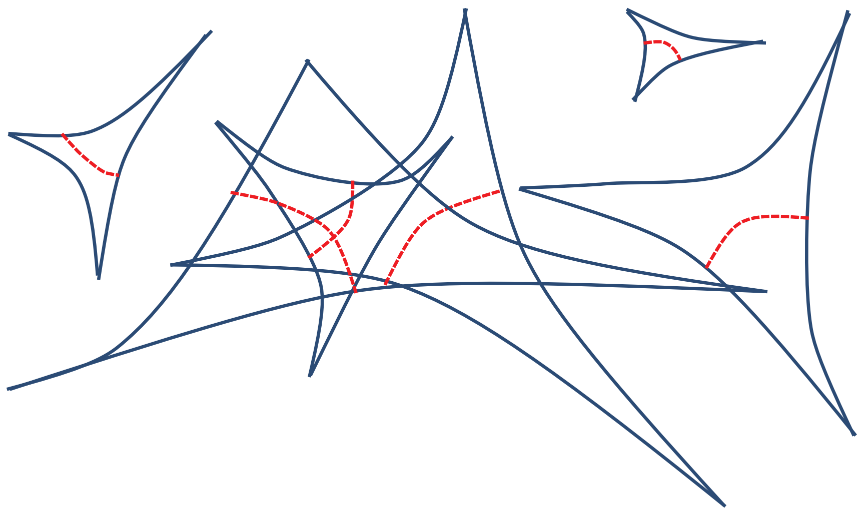


Figure 2.4: Calculating the supremum over all geodesic triangles.

Without disregarding the difficulty of solving a $(3n+2)$-variable minimax problem, notice that in general the main obstacle is that we do not know the location of geodesics in the space. Therefore, it is interesting to obtain inequalities involving the hyperbolicity constant and other parameters of graphs.

## 2.2 Gromov hyperbolicity, Mathematical Analysis and Geometry

The ideal boundary of a metric space is a type of boundary at infinity which is a very useful concept when dealing with negatively curved spaces. We want to talk about some subjects in which this boundary is useful.

A main problem in the study of Partial Differential Equations on Riemannian manifolds is whether or not there exist nonconstant bounded harmonic functions. A way to approach this problem is to study whether the so-called Dirichlet problem at infinity (or the asymptotic Dirichlet problem) is solvable on a complete Riemannian manifold $M$. That is to say, raising the question as to whether every continuous function on the boundary $\partial M$ has a (unique) harmonic extension to $M$. Of course, the answer, in general, is no, since the simplest manifold $\mathbb{R}^n$ admits no positive harmonic functions other than constants. However, the answer is positive for the unit disk $\mathbb{D}$.

In [4] Ancona studied the asymptotic Dirichlet problem on Gromov hyperbolic graphs and in [5] on Gromov hyperbolic Riemannian manifolds with bounded geometry and a positive lower bound $\lambda_1(M) > 0$ for Dirichlet eigenvalues. In the papers [24] and [73] conditions on Gromov hyperbolic manifolds $M$ that imply the positivity of $\lambda_1(M)$ are given and, consequently, the Dirichlet problem is solvable for many Gromov hyperbolic manifolds.

One of the most important features of the transition from a Gromov hyperbolic space to its Gromov boundary is that it is functorial. If $f : X \longrightarrow Y$ is in a certain class of maps between two Gromov hyperbolic spaces $X$ and $Y$, then there is a boundary map $\partial f : \partial X \longrightarrow \partial Y$ which is in some other class of maps. In particular, if $f$ is a quasi-isometry, then $\partial f$ is a bihölder map (with respect to the Gromov metric on the boundary).

It is well known that biholomorphic maps between domains (with smooth boundaries) in $\mathbb{C}$ can be extended as a homeomorphism between their boundaries. If we consider domains in $\mathbb{C}^n$ ($n > 1$) instead in $\mathbb{C}$, then the problem is very difficult. C. Fefferman (Fields medallist) showed in Inventiones Mathematicae (see [52]), with a very long and technical proof, that biholomorphic maps between bounded strictly pseudoconvex domains with smooth boundaries can be extended as a homeomorphism between their boundaries. It is possible to give a “more elementary” proof of this extension result using the functoriality of Gromov hyperbolic spaces: If we consider the Carathéodory metric on a bounded smooth strictly pseudoconvex domain in $\mathbb{C}^n$, then it is Gromov hyperbolic, and the Gromov boundary is homeomorphic to the topological boundary (see [7]). Since any biholomorphic map $f$ between such two domains is an isometry for the Carathéodory metrics, the boundary map $\partial f$ is essentially a boundary extension of $f$ that is a homeomorphism between the boundaries (in fact, it is bihölder with respect to the Carnot-Carathéodory metrics in the boundaries). Fefferman's result gives much more precise information, but this last proof is simpler and gives information about a class of maps that is much more general than biholomorphic maps:

the quasi-isometries for the Carathéodory metrics.

In applications to various areas of mathematics, the Gromov boundary can be similarly be proved (under appropriate conditions) to coincide with other "finite" boundaries, such as the Euclidean or inner Euclidean boundary, or the Martin boundary, so we obtain a variety of boundary extension results as above.

Isometries (and quasi-isometries) in a hyperbolic space $X$ can be extended (as an homeomorphism) to the Gromov boundary $\partial X$ of the space. This fact allows to classify the isometries as *hyperbolic*, *parabolic* and *elliptic*, like the Möbius maps in $\mathbb{D}$, in terms of their fixed points in $X \cup \partial X$.

There are just three possibilities:

- There are exactly two fixed points in $X \cup \partial X$ and both are in $\partial X$ (hyperbolic isometry).
- There is a single fixed point in $X \cup \partial X$ and it is in $\partial X$ (parabolic isometry).
- There is a single fixed point in $X \cup \partial X$ and it is in $X$ (elliptic isometry).

A main ingredient in the proof of this result in the unit disk $\mathbb{D}$ is that the isometries are holomorphic functions. Surprisingly, the tools in hyperbolic spaces provide a new and general proof just in terms of distances!

## 2.3 Main results on hyperbolic spaces

We state now some of the main facts about hyperbolic spaces.

**Definition 2.3.1.** *Let* $(X, d_X)$ *and* $(Y, d_Y)$ *be two metric spaces. A map* $f : X \longrightarrow Y$ *is said to be an* $(\alpha, \beta)$-quasi-isometric embedding*, with constants* $\alpha \geq 1$, $\beta \geq 0$ *if for every* $x, y \in X$*:*

$$\alpha^{-1} d_X(x, y) - \beta \leq d_Y(f(x), f(y)) \leq \alpha d_X(x, y) + \beta.$$

*The function* $f$ *is* $\varepsilon$-full *if for each* $y \in Y$ *there exists* $x \in X$ *with* $d_Y(f(x), y) \leq \varepsilon$*.*

**Definition 2.3.2.** *A map* $f : X \longrightarrow Y$ *is said to be a* quasi-isometry*, if there exist constants* $\alpha \geq 1$, $\beta, \varepsilon \geq 0$ *such that* $f$ *is an* $\varepsilon$*-full* $(\alpha, \beta)$*-quasi-isometric embedding.*

**Definition 2.3.3.** *An* $(\alpha, \beta)$-quasigeodesic *in* $X$ *is an* $(\alpha, \beta)$*-quasi-isometric embedding between an interval of* $\mathbb{R}$ *and* $X$*.*

In the study of any mathematical property, the class of maps which preserve that property plays a central role in the theory. The following result shows that quasi-isometries preserve hyperbolicity.

**Theorem 2.3.4** (Invariance of hyperbolicity)**.** *Let* $f : X \longrightarrow Y$ *be an* $(\alpha, \beta)$*-quasi-isometric embedding between the geodesic metric spaces* $X$ *and* $Y$*. If* $Y$ *is hyperbolic, then* $X$ *is hyperbolic.*

*Besides, if* $f$ *is* $\varepsilon$*-full for some* $\varepsilon \geq 0$ *(a quasi-isometry), then* $X$ *is hyperbolic if and only if* $Y$ *is hyperbolic.*

We next discuss the connection between hyperbolicity and *geodesic stability.* In the complex plane (with its Euclidean distance), there is only one optimal way of joining two points: a straight line segment. However if we allow “limited suboptimality”, the set of “reasonably efficient paths” (quasigeodesics) are well spread. For instance, if we split the circle $\partial D(0, R) \subset \mathbb{C}$ into its two semicircles between the points $R$ and $-R$, then we have two reasonably efficient paths (two $(\pi/2, 0)$-quasigeodesics) between these endpoints such that the point $Ri$ on one of the semicircles is far from all points on the other semicircle provided that $R$ is large. Even an additive suboptimality can lead to paths that fail to stay close together. For instance, the union of the two line segments in $\mathbb{C}$ given by $[0, R + i\sqrt{R}\,]$ and $[R + i\sqrt{R}\,, 2R]$ gives a path of length less than $2R + 1$ (since $2\sqrt{R^2 + R} \leq 2R + 1$), and so is “additively inefficient” by less than 1 (it is a $(1, 1)$-quasigeodesic). However, its corner point is very far from all points on the line segment $[0, 2R]$ when $R$ is very large.

The situation in Gromov hyperbolic spaces is very different, since all such reasonably efficient paths ($(\alpha, \beta)$-quasigeodesics for fixed $\alpha, \beta$) stay within a bounded distance of each other.

**Definition 2.3.5.** *Let $X$ be a metric space, $Y$ a non-empty subset of $X$ and $\varepsilon$ a positive number. We call* $\varepsilon$-neighborhood *of $Y$ in $X$, denoted by $V_\varepsilon(Y)$, to the set $\{x \in X : d_X(x, Y) \le \varepsilon\}$. The* Hausdorff distance *between two subsets $Y$ and $Z$ of $X$, denoted by $\mathcal{H}(Y, Z)$, is the number defined by:*

$$\inf\{\varepsilon > 0 : Y \subset V_\varepsilon(Z) \text{ and } Z \subset V_\varepsilon(Y)\}.$$

**Theorem 2.3.6** (Geodesic stability)**.** *For any constants $\alpha \ge 1$ and $\beta, \delta \ge 0$ there exists a constant $H = H(\delta, \alpha, \beta)$ such that for every $\delta$-hyperbolic geodesic metric space and for every pair of $(\alpha, \beta)$-quasigeodesics $g, h$ with the same endpoints, $\mathcal{H}(g, h) \le H$.*

The geodesic stability is not just a useful property of hyperbolic spaces; in fact, M. Bonk proves in [21] that the geodesic stability is equivalent to the hyperbolicity:

**Theorem 2.3.7.** *([21, p.286]) Let $X$ be a geodesic metric space with the following property: For each $a \ge 1$ there exists a constant $H$ such that for every $x, y \in X$ and any $(a, 0)$-quasigeodesic $g$ in $X$ starting in $x$ and finishing in $y$ there exists a geodesic $\gamma$ joining $x$ and $y$ satisfy $\mathcal{H}(g, \gamma) \le H$. Then $X$ is hyperbolic.*

Theorem 2.3.6 allows to prove Theorem 2.3.4:

*Proof of Theorem 2.3.4.* By hypothesis there exists $\delta \ge 0$ such that $Y$ is $\delta$-hyperbolic.

Let $T$ be a geodesic triangle in $X$ with sides $g_1, g_2$ y $g_3$, and $T_Y$ the triangle with $(\alpha, \beta)$-quasigeodesic sides $f(g_1)$, $f(g_2)$ y $f(g_3)$ in $Y$. Let $\gamma_j$ be a geodesic joining the endpoints of $f(g_j)$, for $j = 1, 2, 3$, and $T'$ the geodesic triangle in $Y$ with sides $\gamma_1, \gamma_2, \gamma_3$.

Let $p$ be any point in $f(g_1)$. We are going to prove that there exists a point $q \in f(g_2) \cup f(g_3)$ with $d_Y(p, q) \le K$, where $K := \delta + 2H(\delta, \alpha, \beta)$. By Theorem 2.3.6, there exists a point $p' \in \gamma_1$ with $d_Y(p, p') \le H(\delta, \alpha, \beta)$. Since $T'$ is a geodesic triangle, it is $\delta$-thin and there exists $q' \in \gamma_2 \cup \gamma_3$ with $d_Y(p', q') \le \delta$. Using again Theorem 2.3.6, there exists a point $q \in f(g_2) \cup f(g_3)$ con $d_Y(q, q') \le H(\delta, \alpha, \beta)$. Therefore,

$$\begin{aligned} d_Y(p, f(g_2) \cup f(g_3)) &\le d_Y(p, q) \le d_Y(p, p') + d_Y(p', q') + d_Y(q', q) \\ &\le H(\delta, \alpha, \beta) + \delta + H(\delta, \alpha, \beta). \end{aligned}$$

Let $z \in T$; without loss of generality we can assume that $z \in g_1$. We have seen that there exists a point $q \in f(g_2) \cup f(g_3)$ with $d_Y(f(z), q) \le K$. If $w \in g_2 \cup g_3$ satisfies $f(w) = q$, then

$$d_X(z, g_2 \cup g_3) \le d_X(z, w) \le \alpha d_Y(f(z), q) + \alpha\beta \le \alpha K + \alpha\beta.$$

Hence, $T$ is $(\alpha K + \alpha\beta)$-thin. Since $T$ is an arbitrary geodesic triangle, $X$ is $(\alpha\delta + 2\alpha H(\delta, \alpha, \beta) + \alpha\beta)$-hyperbolic.

Assume now that $f$ is $\varepsilon$-full. One can check that an "inverse" quasi-isometry $f^- : Y \longrightarrow X$ can be constructed as follows: for $y \in Y$ choose $x \in X$ with $d_Y(f(x), y) \le \varepsilon$ and define $f^-(y) := x$. Then the first part of the Theorem gives the result. $\square$

## 2.4 Previous results on hyperbolic graphs

Let us return to our framework: graphs as geodesic metric spaces. In this section we collect some important results which will be useful for the development of our work.

It is known (see [100, Lemma 2.1]) that, for every graph $G$, it is satisfied

$$\delta(G) = \sup\{\delta(T) : T \text{ is a geodesic triangle in } G \text{ that is a cycle}\}.$$

Given a graph $G$ and $[v, w] \in E(G)$, we say that $p$ is the *midpoint* of $[v, w]$ if $d_G(p, v) = d_G(p, w) = 1/2$; let us denote by $J(G)$ the union of the set $V(G)$ and the midpoints of the edges of $G$. Consider the set $\mathbb{T}_1$ of geodesic triangles $T$ in $G$ that are cycles and such that the three vertices of the triangle $T$ belong to $J(G)$.

The following result states that in the hyperbolic graphs there always exists a geodesic triangle $T$ for which the hyperbolicity constant is attained and, furthermore, $T \in \mathbb{T}_1$. It appears in [15, Theorem 2.7].

**Theorem 2.4.1.** *For any hyperbolic graph $G$ there exists a geodesic triangle $T \in \mathbb{T}_1$ such that $\delta(T) = \delta(G)$.*

Recall that the diameter of $G$ and $V(G)$ are defined as:

$$\operatorname{diam}(G) = \max\{d(x, y) : \ x, y \in G\}, \qquad \operatorname{diam} V(G) = \max\{d(u, v) : \ u, v \in V(G)\}.$$

It is easy to see that

$$\operatorname{diam} V(G) \leq \operatorname{diam}(G) \leq \operatorname{diam} V(G) + 1 \tag{2.1}$$

and both inequalities are sharp, since $\operatorname{diam} V(G) = \operatorname{diam}(G)$ if $G$ is a tree or a cycle with even length, and $2 = \operatorname{diam}(K_n) = \operatorname{diam} V(K_n) + 1$ if $K_n$ is a complete graph with $n \geq 4$.

The next theorem is a well-known fact (see, e.g., [99, Theorem 8] for a proof).

**Theorem 2.4.2.** *Let $G$ be any graph. Then*

$$\delta(G) \leq \frac{1}{2} \operatorname{diam}(G).$$

The following theorem gives the precise value of the hyperbolicity constant for several famous graphs (see [99, Theorem 11]).

**Theorem 2.4.3.** *The following graphs with edges of length 1 have the following hyperbolicity constants:*

- *The path graphs verify $\delta(P_n) = 0$ for every $n \geq 1$.*
- *The cycle graphs verify $\delta(C_n) = n/4$ for every $n \geq 3$.*

- *The complete graphs verify* $\delta(K_1) = \delta(K_2) = 0$, $\delta(K_3) = 3/4$, $\delta(K_n) = 1$ *for every* $n \geq 4$.
- *The complete bipartite graphs verify* $\delta(K_{1,1}) = \delta(K_{1,2}) = \delta(K_{2,1}) = 0$, $\delta(K_{m,n}) = 1$ *for every* $m, n \geq 2$.
- *The wheel graph with* $n$ *vertices* $W_n$ *verifies* $\delta(W_4) = \delta(W_5) = 1$, $\delta(W_n) = 3/2$ *for every* $7 \leq n \leq 10$, *and* $\delta(W_n) = 5/4$ *for* $n = 6$ *and for every* $n \geq 11$.

The following theorem appears in [15, Theorem 2.6].

**Theorem 2.4.4.** *For every graph* $G$, $\delta(G)$ *is a multiple of* $1/4$.

The next result appears in [82, Theorem 11].

**Theorem 2.4.5.** *Let* $G$ *be any graph.*

- $\delta(G) < 1/4$ *if and only if* $G$ *is a tree.*
- $\delta(G) < 1$ *if and only if every cycle* $g$ *in* $G$ *has length* $L(g) \leq 3$.

*Furthermore, if* $\delta(G) < 1$, *then* $\delta(G) \in \{0, 3/4\}$.

The following family of graphs allows to characterize the extremal graphs in Theorem 2.4.7 below. Let $\mathcal{F}_n$ be the set of Hamiltonian graphs $G$ of order $n$ with every edge of length 1 and such that there exists a Hamiltonian cycle $G_0$ which is the union of two geodesics $\Gamma_1, \Gamma_2$ in $G$ with length $n/2$ such that the midpoint $x_0$ of $\Gamma_1$ satisfies $d_G(x_0, \Gamma_2) = n/4$.

In [82, Proposition 29] the authors give the following precise description of $\mathcal{F}_n$.

**Proposition 2.4.6.** *For* $n \geq 3$, *let us consider the cycle graph* $C_n$ *with edges of length* $k$. *Fix a vertex* $z \in V(C_n)$ *and the geodesics (in* $C_n$*)* $\Gamma_1^*, \Gamma_2^*$ *with lengths* $nk/2$ *joining the vertex* $z$ *and the point* $w$ *and* $C_n = \Gamma_1^* \cup \Gamma_2^*$. *Denote by* $w_i^j$ *the vertex in* $\Gamma_i^*$ *with* $d(w_i^j, z) = jk$, *for* $i = 1, 2$ *and* $j \geq 1$ *(with* $w_i^j \neq w$*).*

- *If* $n$ *is even, we have* $1 \leq j \leq n/2 - 1$. *Then a graph belongs to* $\mathcal{F}_n$ *if and only if it is isomorphic (and hence, isometric) to a graph obtained by adding to* $C_n$ *any amount of multiple edges and/or loops and a subset (proper or not) of either*

$$\left\{ [w_1^1, w_2^1], [w_1^1, w_2^2], [w_1^{n/2-1}, w_2^{n/2-1}], [w_1^{n/2-1}, w_2^{n/2-2}] \right\}$$

*or*

$$\left\{ [z, w_2^2], [w_1^{n/2-1}, w_2^{n/2-1}] \right\}.$$

- *If $n$ is odd, we have $1 \leq j \leq (n-1)/2$. Then a graph belongs to $\mathcal{F}_n$ if and only if it is isomorphic (and hence, isometric) to a graph obtained by adding to $C_n$ any amount of multiple edges and/or loops and a subset (proper or not) of*

$$\left\{ [w_1^1, w_2^1], [w_1^1, w_2^2], [w_1^{(n-1)/2}, w_2^{(n-1)/2-1}] \right\}.$$

In [82, Theorem 30] we find the following result, which gives an optimal inequality between the order of a graph and its hyperbolicity constant.

**Theorem 2.4.7.** *Let $G$ be any graph with $n$ vertices. If every edge has length 1, then*

$$\delta(G) \leq n/4.$$

*Moreover, if $n \geq 3$ we have $\delta(G) = n/4$ if and only if $G \in \mathcal{F}_n$;*

**Definition 2.4.8.** (***Cut-vertex***) *We say that a vertex $v$ in a graph $G$ is a* cut-vertex *if $G \setminus \{v\}$ is not connected.*

**Definition 2.4.9.** (***Biconnected graph***) *A graph is* biconnected *if it does not contain cut-vertices.*

**Definition 2.4.10.** (***T-decomposition***) *Given a graph $G$, we say that a family of subgraphs $\{G_s\}_s$ of $G$ is a* T-decomposition *of $G$ if $\cup_s G_s = G$ and $G_s \cap G_r$ is either a cut-vertex or the empty set for each $s \neq r$.*

Every graph has a T-decomposition, as the following example shows. Given any edge in $G$, let us consider the maximal two-connected subgraph containing it: this is the well-known *biconnected decomposition* of $G$. Note that every $G_s$ in the *biconnected decomposition* of $G$ is an isometric subgraph of $G$.

The following result appears in [16, Theorem 3].

**Theorem 2.4.11.** *Let $G$ be a graph and $\{G_s\}$ any T-decomposition of $G$. Then, $\delta(G) = \sup_s \delta(G_s)$.*

It is known that a graph with small hyperbolicity constant can have an arbitrarily large diameter (consider, for example, the path graph $P_n$). However, there is a concept related with the diameter, the *effective diameter*, which is small when the hyperbolicity constant is small. This concept plays a main role in the theory of hyperbolic graphs (see [12]).

**Definition 2.4.12.** (***Effective Diameter***) *Given a graph $G$, let $\{G_s\}$ be the biconnected decomposition of $G$. We define the effective diameter as*

$$\operatorname{diameff} V(G) := \sup_s \, diam\, V(G_s), \qquad \operatorname{diameff} G := \sup_s \, diam\, G_s.$$

It is clear that if $\{G_s\}$ is any T-decomposition of $G$, then

$$\operatorname{diameff} V(G) \le \sup_s \operatorname{diam} V(G_s), \qquad \operatorname{diameff} G \le \sup_s \operatorname{diam} G_s.$$

Theorems 2.4.11 and 2.4.2 have the following consequence.

**Corollary 2.4.13.** *Let $G$ be any graph. Then*

$$\delta(G) \le \frac{1}{2} \operatorname{diameff}(G).$$

# Chapter 3

# On the order, size, and hyperbolicity constant

Let $\mathcal{G}(n,m)$ be the family of graphs $G$ of order $n$ and size $m$.

If $m = n-1$, then every $G \in \mathcal{G}(n,m)$ is a tree and $\delta(G) = 0$. On the other hand, the complete graph $K_n$ belongs to $\mathcal{G}(n,m)$ with $m = \binom{n}{2}$. Thus we consider $n-1 \le m \le \binom{n}{2}$.

Recall that $A(n,m)$ and $B(n,m)$ are defined as

$$A(n,m) := \min\{\delta(G) \mid G \in \mathcal{G}(n,m)\},$$

$$B(n,m) := \max\{\delta(G) \mid G \in \mathcal{G}(n,m)\},$$

respectively.

Our ultimate goal in this Chapter is to estimate $A(n,m)$ and $B(n,m)$.

## 3.1 Bounds for $B(n, m)$

First, our purpose is to find an upper bound for $B(n, m)$. In order to simplify this proof, we prove some technical lemmas.

We begin by proving Lemma 3.1.3. One of the most important tools used in the proof are Karush-Kuhn-Tucker necessary conditions for nonlinear optimization problems with inequality constraints.

Let $X$ be a non-empty open set of $\mathbb{R}^n$ and $f, g_j$ $(j = 1, \dots, k)$ functions of $X \subseteq \mathbb{R}^n$ in $\mathbb{R}$. Consider the problem:

$$\text{P:} \quad \min_{x \in V} f(x),$$

with $\quad V := \{x \in X \mid g_j(x) \leq 0, \ \ j = 1, \dots, k\}$.

Given $x^* \in V$, let $I(x^*)$ be the set of subscripts $j$ for which $g_j(x^*) = 0$.

**Definition 3.1.1.** *We say that a point $x^* \in V$ is* regular *if the vectors $\nabla g_j(x^*)$ $(j \in I(x^*))$ are linearly independent.*

**Theorem 3.1.2.** *Let $x^*$ be a point in $V$. Suppose that $f, g_j$ $(j \in I(x^*))$ are continuously differentiable functions and $g_j$ $(j \notin I(x^*))$ are continuous functions at $x^*$. If $x^*$ is a regular point and a local minimum of $f$ in $V$, then there exist unique scalars $\mu_j$ $(j \in I(x^*))$ such that:*

$$\nabla f(x^*) + \sum_{j \in I(x^*)} \mu_j \nabla g_j(x^*) = 0, \qquad \mu_j \geq 0, \quad j \in I(x^*).$$

The above conditions can be written as:

$$\nabla f(x^*) + \sum_{j=1}^{k} \mu_j \nabla g_j(x^*) = 0, \qquad \mu_j g_j(x^*) = 0, \quad j = 1, \dots, k, \qquad \mu_j \geq 0, \quad j = 1, \dots, k.$$

Consider $G \in \mathcal{G}(n, m)$. Fix $\operatorname{diam} V(G) = r$ and choose $u, v \in V(G)$ such that $d(u, v) = r$. Let $k_j = \# S_j$ where $S_j := \{w \in V(G) : \ d(w, u) = j\}$ $(0 \leq j \leq r)$. The number of edges that we must eliminate from the complete graph of $n$ vertices in order to obtain $G$ is at least

$$f_r(k_1, k_2, \dots, k_r) := \sum_{t=2}^{r} k_t \sum_{s=0}^{t-2} k_s,$$

since the vertices in the sphere $S_t$ $(2 \leq t \leq r)$ can not be connected by an edge with the vertices of the spheres $S_0, S_1, \cdots, S_{t-2}$, and therefore the corresponding edges in the complete graph must be deleted to obtain $G$.

In the next result we compute the minimum value of $f_r$ with $k_j \in \mathbb{R}$, $k_j \geq 2$ $(1 \leq j \leq r - 1)$. Observe that

$$\min_{k_j \in \mathbb{N}, k_j \geq 2} f_r(k_1, k_2, \dots, k_r) \geq \min_{k_j \in \mathbb{R}, k_j \geq 2} f_r(k_1, k_2, \dots, k_r)$$

and therefore, in order to obtain a lower bound of the minimum when $k_j \in \mathbb{N}$, it suffices to compute the minimum when $k_j \in \mathbb{R}$. We are not loosing nothing at all by using this inequality since it is, in fact, an equality (the proof of Lemma 3.1.3 gives that the minimum in the real case is attained at points with integer coordinates). Hence, this lower bound of the number of deleted edges is sharp.

The upper bound of $B(n, m)$ obtained in Theorem 3.1.11 by using this minimum is good, since, as we prove in Section 3.2, the difference between our upper and lower bounds of $B(n, m)$ is at most of order $\sqrt{n}$, while $\delta(G)$ is always less or equal than $n/4$ (and this estimation is sharp).

**Lemma 3.1.3.** *Consider the following optimization problem:*

$$\Delta_r := \min_{x \in W} f_r, \quad \textit{with} \quad f_r(k_1, k_2, \dots, k_r) := \sum_{t=2}^{r} k_t \sum_{s=0}^{t-2} k_s, \quad 2 \le r \le n/2,$$

*and* $\quad W := \{k_0 = 1, \quad k_j \ge 2, \quad \textit{if} \quad 1 \le j \le r-1, \quad k_r \ge 1, \quad 1 + k_1 + k_2 + \dots + k_r = n\}.$

*Then* $\Delta_2 = 1$, $\Delta_3 = n - 1$ *and* $\Delta_r = 2n(r-3) - 2r^2 + 6r + 5$ *for* $r \ge 4$.

**Remark 3.1.4.** *Note that if* $W \ne \emptyset$, *then* $n = 1 + \sum_{t=1}^{r} k_t \ge 1 + 2(r-1) + 1$ *and* $2r \le n$. *Conversely, if* $2r \le n$, *then* $W \ne \emptyset$. *Hence, we are assuming* $2r \le n$.

*Proof.* If $r = 2$, then $f_2(k_1, k_2) = k_2$, with $k_2 \ge 1$. Hence $\Delta_2 = 1$.

Consider now the case $r \ge 3$. Let us define the real-valued functions

$$g_j(\vec{k}) := -k_j + 2\,, \quad \text{for } 1 \le j \le r-1\,, \qquad g_r(\vec{k}) := -k_r + 1\,, \qquad h(\vec{k}) := 1 + k_1 + k_2 + \dots + k_r - n\,.$$

Then, the set $W$ can be written as:

$$W = \{\vec{k} \mid \quad g_j \le 0 \quad \text{if} \quad 1 \le j \le r-1, \; g_r \le 0, \; h = 0\}\,.$$

We eliminate a variable of our problem by solving $k_r$ in the equality restriction. Substituting the expression obtained in $f_r$, the original problem is reduced to the following one in $\mathbb{R}^{r-1}$:

$$\Delta_r = \min_{\vec{k^1} \in W^1} f_r^1(\vec{k^1}), \qquad \text{with} \quad f_r^1(\vec{k^1}) = f_r^1(k_1, k_2, \dots, k_{r-1}) := \sum_{t=2}^{r-1} k_t \sum_{s=0}^{t-2} k_s + \Big(n - \sum_{s=0}^{r-1} k_s\Big) \sum_{s=0}^{r-2} k_s,$$

and

$$W^1 := \{\vec{k^1} = (k_1, \cdots, k_{r-1}) \in \mathbb{R}^{r-1} \mid \quad g_j \le 0 \quad \text{if} \quad 1 \le j \le r-1, \quad g_r \le 0\}\,,$$

where as before

$$g_j(\vec{k^1}) := -k_j + 2\,, \quad \text{for } 1 \le j \le r-1\,, \qquad g_r(\vec{k^1}) := -k_r + 1 = 2 - n + \sum_{s=1}^{r-1} k_s\,.$$

Let us consider first the case in which there is a point where the minimum is attained which is not a regular point (in this case $g_j(\vec{k^1}) = 0$ for every $1 \le j \le r$). Hence:

$$h = 1 + 2(r-1) + 1 - n = 0 \quad \Rightarrow \quad 2r = n\,.$$

Therefore, the minimum is attained at $\vec{k^1}^* = (2,\dots,2)$, $W^1 = \{\vec{k^1}^*\}$ and evaluating $f_r$ at $\vec{k}^* = (\vec{k^1}^*, 1) = (2,\dots,2,1)$ we get:

$$f_r(\vec{k}^*) = \sum_{t=2}^{r-1} 2\Big(1 + \sum_{s=1}^{t-2} 2\Big) + \Big(1 + \sum_{s=1}^{r-2} 2\Big) = 2\sum_{t=2}^{r-1}(2t-3) + 2r - 3 = (1 + 2r - 5)(r-2) + 2r - 3 = 2r^2 - 6r + 5,$$

and then $\Delta_r = 2r^2 - 6r + 5$.

Now let us assume that the points where the minimum is attained are all regular points. Note that the vectors $\{\nabla g_j(\vec{k^1}),\ j = 1,\dots,r\}$ are linearly dependent in $\mathbb{R}^{r-1}$ but in order to get from them a linearly independent set it is necessary and sufficient to remove at least one of them. Therefore, it suffices to consider that at least one of the coefficients $\mu_j$ is zero. Also, me must have $g_j(\vec{k^1}) < 0$ for some $1 \le j \le r$. Since:

$$\frac{\partial f_r^1}{\partial k_{r-1}} = \sum_{s=0}^{r-3} k_s - \sum_{s=0}^{r-2} k_s = -k_{r-2},$$

by applying Theorem 3.1.2 we conclude that the following equality in $\mathbb{R}^{r-1}$ must be satisfied at a regular point where the minimum is attained:

$$\begin{pmatrix} * \\ \vdots \\ * \\ -k_{r-2} \end{pmatrix} + \mu_1 \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \mu_{r-1} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ -1 \end{pmatrix} + \mu_r \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

with $\mu_j \ge 0$ for $j = 1,\dots,r$.

Assuming that $\mu_r = 0$, from the previous expression we obtain that $-k_{r-2} = \mu_{r-1}$. The restriction $g_{r-2} \le 0$ of the problem and the positivity of the coefficient $\mu_{r-1}$ implies that $-2 \ge -k_{r-2} = \mu_{r-1} \ge 0$ and this is a contradiction, therefore $\mu_r > 0$.

Considering the condition $\mu_r g_r(\vec{k^1}^*) = 0$ we deduce that $g_r(\vec{k^1}^*) = -k_r + 1 = 0$ and $k_r = 1$.

We now write again our optimization problem in $\mathbb{R}^{r-1}$, with $k_r = 1$:

$$\Delta_r = \min_{\vec{k^2} \in W^2} f_r^2(\vec{k^2}), \quad \text{with} \quad f_r^2(\vec{k^2}) = f_r^2(k_1, k_2, \dots, k_{r-1}) := \sum_{t=2}^{r-1} k_t \sum_{s=0}^{t-2} k_s + \sum_{s=0}^{r-2} k_s,$$

and $W^2 := \{\vec{k^2} = (k_1, k_2, \dots, k_{r-1}) \in \mathbb{R}^{r-1} \mid \ k_j \ge 2 \ \text{ for } \ 1 \le j \le r-1, \ \ k_1 + k_2 + ... + k_{r-1} = n - 2\}$.

If $r = 3$, then $f_3^2(k_1, k_2) = k_2 + 1 + k_1$, with $k_1, k_2 \geq 2$ and $k_1 + k_2 = n - 2$. Hence, $\Delta_3 = n - 1$.

Let us suppose now that $r \geq 4$. Note that:

$$\begin{aligned} f_r^2 &= k_2 + \sum_{t=3}^{r-1} k_t \Big(1 + \sum_{s=1}^{t-2} k_s\Big) + 1 + \sum_{s=1}^{r-2} k_s = 1 - k_1 - k_{r-1} + 2\sum_{t=1}^{r-1} k_t + \sum_{t=3}^{r-1}\sum_{s=1}^{t-2} k_t k_s \\ &= 2n - 3 - k_1 - k_{r-1} + \sum_{t-2\geq s} k_t k_s, \end{aligned}$$

with $\sum_{t=1}^{r-1} k_t = n - 2$.

Consider now the expression $\Big(\sum_{t=1}^{r-1} k_t\Big)^2$:

$$\Big(\sum_t k_t\Big)^2 = \sum_t k_t^2 + 2\sum_{t-1\geq s} k_t k_s = \sum_t k_t^2 + 2\sum_{t-1=s} k_t k_s + 2\sum_{t-2\geq s} k_t k_s.$$

Moreover, we can write:

$$\sum_{t-2\geq s} k_t k_s = \frac{1}{2}\Big(\sum_{t=1}^{r-1} k_t\Big)^2 - \frac{1}{2}\sum_{t=1}^{r-1} k_t^2 - \sum_{t=2}^{r-1} k_t k_{t-1} = \frac{1}{2}(n-2)^2 - \frac{1}{2}\sum_{t=1}^{r-1} k_t^2 - \sum_{t=2}^{r-1} k_t k_{t-1}.$$

Thus, we have deduced that $\Delta_r = \min_{\vec{k^3}\in W^3} f_r^3(\vec{k^3})$, with:

$$f_r^3(\vec{k^3}) = f_r^3(k_1, k_2, \ldots, k_{r-1}) := \frac{1}{2}n^2 - 1 - k_1 - k_{r-1} - \frac{1}{2}\sum_{t=1}^{r-1} k_t^2 - \sum_{t=2}^{r-1} k_t k_{t-1},$$

and $W^3 := \{\vec{k^3} = (k_1, k_2, \ldots, k_{r-1}) \in \mathbb{R}^{r-1} \mid \quad k_j \geq 2 \quad \text{if} \quad 1 \leq j \leq r-1, \;\; k_1 + k_2 + \cdots + k_{r-1} = n - 2\}$.

This formulation allows us to see that the problem is symmetric in the variables $k_t$ and $k_{r-t}$ for every $1 \leq t \leq r - 1$.

Substituting $k_r = 1$ and $k_{r-1} = n - 2 - \sum_{t=1}^{r-2} k_t$ in $f_r$ we obtain that we must solve the following optimization problem in $\mathbb{R}^{r-2}$: $\quad \Delta_r = \min_{\vec{k^4}\in W^4} f_r^4(\vec{k^4})$, with:

$$f_r^4(\vec{k^4}) = f_r^4(k_1, k_2, \ldots, k_{r-2}) := \Big(n - 2 - \sum_{t=1}^{r-2} k_t\Big)\sum_{s=0}^{r-3} k_s + \sum_{t=2}^{r-2} k_t \sum_{s=0}^{t-2} k_s + \sum_{s=0}^{r-2} k_s,$$

and $W^4 := \{\vec{k^4} = (k_1, k_2, \ldots, k_{r-2}) \in \mathbb{R}^{r-2} \mid \quad k_j \geq 2 \quad \text{if} \quad 1 \leq j \leq r-2, \;\; k_{r-1} := n - 2 - \sum_{t=1}^{r-2} k_t \geq 2\}$.

Then $k_1 \in [2, n-4-\sum_{t=2}^{r-2} k_t]$. Computing the second derivative of $f_r^4$ with respect to $k_1$ we get:

$$\frac{\partial^2 f_r^4}{\partial k_1^2} = -2 < 0.$$

That is, the function is convex and the minimum is reached at the endpoints of the interval, $k_1 = 2$ or $k_1 = n-4-\sum_{t=2}^{r-2} k_t$, i.e., $k_1 = 2$ or $k_{r-1} = 2$.

By iterating this argument one can check that if $\vec{k3}^* = (k_1, k_2, \dots, k_{r-1})$ satisfies $f_r^3(\vec{k3}^*) = \Delta_r$, then $k_j = 2$ except for one $j_0$ with $1 \le j_0 \le r-1$, and $k_{j_0} = n-2r+2$. By symmetry, the cases $j_0 = 1$ and $j_0 = r-1$ provide the same value; furthermore, the cases $1 < j_0 < r-1$ provide the same value.

If $j_0 = 1$ or $j_0 = r-1$, then

$$\begin{aligned} f_r^3(\vec{k3}^*) &= \frac{1}{2}n^2 - 1 - n + 2r - 2 - 2 - \frac{1}{2}(n-2r+2)^2 - \frac{1}{2}4(r-2) - 2(n-2r+2) - 4(r-3) \\ &= n(2r-5) - 2r^2 + 4r + 5. \end{aligned}$$

If $1 < j_0 < r-1$, substituting $\vec{k3}^* = (2, \dots, 2, n-2r+2, 2, \dots, 2)$ in $f_r^3$ we get

$$\begin{aligned} f_r^3(\vec{k3}^*) &= \frac{1}{2}n^2 - 5 - \frac{1}{2}(n-2r+2)^2 - \frac{1}{2}4(r-2) - 4(n-2r+2) - 4(r-4) \\ &= 2n(r-3) - 2r^2 + 6r + 5. \end{aligned}$$

Then $\Delta_r = 2n(r-3) - 2r^2 + 6r + 5$ for $r \ge 4$, since $n \ge 2r$.

Note that if $n = 2r$, then $\Delta_r = 2r^2 - 6r + 5$, for every $r \ge 2$. □

We define $M(n,r) := \binom{n}{2} - \Delta_r$, for $2 \le r \le n/2$.

We have the following expression for $M(n,r)$:

$$M(n,2) = \frac{1}{2}[n^2 - n - 2], \quad M(n,3) = \frac{1}{2}[n^2 - 3n + 2]$$

and

$$M(n,r) = \frac{1}{2}[(n-2r+3)^2 + 5n - 19], \quad \text{if} \quad r \ge 4.$$

**Lemma 3.1.5.** *If $G \in \mathcal{G}(n,m)$ and* $\operatorname{diameff} V(G) = \operatorname{diam} V(G) = r$, *then $m \le M(n,r)$.*

*Proof.* Let us consider $u, v \in V(G)$ such that $d(u,v) = \operatorname{diam} V(G) = r$. Denote by $k_j$ the cardinal of $S_j := \{w \in V(G) \mid d(w,u) = j\}$ for $0 \le j \le r$. Since $\operatorname{diameff} V(G) = \operatorname{diam} V(G) = r$, we have $k_0 = 1, k_j \ge 2$ for $1 \le j \le r-1$, and $k_r \ge 1$.

Note that a vertex of $S_j$ and a vertex of $S_0 \cup S_1 \cup \dots \cup S_{j-2}$ can not be neighbors for $2 \le j \le r$. Denote by $x$ the minimum number of edges that can be removed from the complete graph with $n$ vertices in order to obtain $G$. Since the diameter of $V(G)$ is $r$, we have obtained the following lower bound for $x$:

$$x \geq k_2+k_3(1+k_1)+k_4(1+k_1+k_2)+\ldots+k_{r-1}(1+k_1+\ldots+k_{r-3})+k_r(1+k_1+\ldots+k_{r-2}) = f_r.$$

Then $x \geq \Delta_r$ by Lemma 3.1.3 and $m = \binom{n}{2} - x \leq \binom{n}{2} - \Delta_r = M(n,r)$. $\square$

**Lemma 3.1.6.** *The inequality*

$$\binom{n-n_0+1}{2} \leq M(n,r) - M(n_0,r)$$

*holds for* $2 \leq r \leq n_0/2$ *and* $n > n_0$.

*Proof.* If $r \geq 4$, then the inequality holds if and only if

$$\frac{1}{2}(n-n_0+1)(n-n_0) \leq \frac{1}{2}n(n-1) - \frac{1}{2}n_0(n_0-1) - 2(n-n_0)(r-3)$$

$$\Leftrightarrow \quad (n-n_0+1)(n-n_0) \leq n^2 - n_0^2 - (n-n_0) - 4(n-n_0)(r-3)$$

$$\Leftrightarrow \quad n-n_0+1 \leq n+n_0-1-4(r-3) \quad \Leftrightarrow \quad 2r \leq n_0+5,$$

and this holds since $2r \leq n_0$.

If $r = 3$, then

$$\frac{1}{2}(n-n_0+1)(n-n_0) \leq \frac{1}{2}n(n-1) - \frac{1}{2}n_0(n_0-1) - (n-1) - (n_0-1)$$

$$\Leftrightarrow \quad (n-n_0+1)(n-n_0) \leq n^2 - n_0^2 - (n-n_0) - 2(n-n_0)$$

$$\Leftrightarrow \quad n-n_0+1 \leq n+n_0-3 \quad \Leftrightarrow \quad n_0 \geq 2,$$

and this holds since $n_0 \geq 2r = 6$.

If $r = 2$, then

$$\frac{1}{2}(n-n_0+1)(n-n_0) \leq \frac{1}{2}n(n-1)-\frac{1}{2}n_0(n_0-1) \quad \Leftrightarrow \quad n-n_0+1 \leq n+n_0-1 \quad \Leftrightarrow \quad n_0 \geq 1.$$

$\square$

**Lemma 3.1.7.** *If* $G \in \mathcal{G}(n,m)$ *and* diameff $V(G) = r$, *then* $m \leq M(n,r)$.

*Proof.* Given a graph $G \in \mathcal{G}(n,m)$ with biconnected decomposition $\{G_s\}$, let $G_k$ be a subgraph with diameff $V(G_k) =$ diameff $V(G) = r$. If $G_k$ has $n_0$ vertices and $m_0$ edges, then $m_0 \leq M(n_0,r)$ by Lemma 3.1.5. Note that $2r \leq n_0$.

Completing $G_k$ with the complete graph of $n-n_0+1$ vertices (one of the vertices belongs to $G_k$) we get that $m \leq m_0 + \binom{n-n_0+1}{2}$.

By Lemma 3.1.6 we have $m \leq m_0 + M(n,r) - M(n_0,r)$ and, since $m_0 \leq M(n_0,r)$, we conclude $m \leq M(n,r)$. $\square$

**Corollary 3.1.8.** *If $G \in \mathcal{G}(n,m)$, $2 \le r \le n/2$ and $m > M(n,r)$, then* diameff $V(G) \neq r$.

We will show now that, in fact, this result can be improved.

**Theorem 3.1.9.** *If $G \in \mathcal{G}(n,m)$, $2 \le r \le n/2$ and $m > M(n,r)$, then* diameff $V(G) < r$.

*Proof.* By Corollary 3.1.8, it suffices to prove that $M(n,r)$ is a decreasing function of $r$. We have $\Delta_2 \le \Delta_3 \le \Delta_4$, since $1 \le n-1 \le 2n-3$. Thus, $M(n,2) \ge M(n,3) \ge M(n,4)$. If $r \ge 4$, then $M(n,r)$ decreases as a function of $r$ since $2r \le n$ gives $\frac{\partial M(n,r)}{\partial r} = -2(n-2r+3) \le 0$. $\square$

Since diameff $V(G) < r$ implies diameff $G \le r$, Lemma 2.4.13 and Theorem 3.1.9 imply the following theorems.

**Theorem 3.1.10.** *If $G \in \mathcal{G}(n,m)$, $2 \le r \le n/2$ and $m > M(n,r)$, then $\delta(G) \le r/2$.*

We define $M(n,1) := n(n-1)/2$.

**Theorem 3.1.11.** *If $n \ge 1$ and $m = n-1$, then $B(n,m) = 0$. If $n \ge 3$ and $n \le m \le n+3$, then $B(n,m) = n/4$. If $M(n,r) < m \le M(n,r-1)$ for some $2 \le r \le n/2$, then $B(n,m) \le r/2$. Otherwise, $B(n,m) \le n/4$.*

*Proof.* If $n \ge 1$ and $m = n-1$, then every $G \in \mathcal{G}(n,m)$ is a tree and $\delta(G) = 0$; consequently, $B(n,m) = 0$. We have $\delta(G) \le n/4$ for every graph $G$ by Theorem 2.4.7. If $n \ge 3$ and $n \le m \le n+3$, then Theorem 2.4.7 gives that there exists $G_0 \in \mathcal{G}(n,m)$ with $\delta(G_0) = n/4$. Hence, $B(n,m) = n/4$ for $3 \le n \le m \le n+3$. Finally, the third part of the statement is a consequence of Theorem 3.1.10. $\square$

Now, let us prove a lower bound for $B(n,m)$.

**Theorem 3.1.12.** *If $3 \le n_0 \le n$ and $n < m \le n + \binom{n_0-1}{2}$, then $B(n,m) \ge (n-n_0+3)/4$.*

*Proof.* Let us consider a cycle graph with $n$ vertices $C_n$. Given $n_0 \ge 3$, choose a path $\{v_1, ..., v_{n_0}\}$ in $C_n$ and add $\binom{n_0}{2} - (n_0-1) = \binom{n_0-1}{2}$ edges to $C_n$ if $n_0 < n$, or $\binom{n}{2} - n$ if $n_0 = n$, obtaining a graph $G_{n,n_0}$ such that the induced subgraph by $\{v_1, ..., v_{n_0}\}$ in $G_{n,n_0}$ is isomorphic to the complete graph with $n_0$ vertices.

Choose a path $\{v_1, ..., v_{n_0}\}$ in $C_n$ and add $m-n$ edges to $C_n$, obtaining a subgraph $G$ of $G_{n,n_0}$ with at least some $v_i$ verifying $[v_i, v_1], [v_i, v_{n_0}] \in E(G)$. If $n_0 = 3$, then $n < m \le n+1$ and $m = n+1$.

Note that $G \in \mathcal{G}(n,m)$. Let $\eta$ be the path in $C_n$ joining $v_1$ and $v_{n_0}$ with $v_2, ..., v_{n_0-1} \notin \eta$ and let $y$ be the midpoint of $\eta$. Define $x := v_i$, $\gamma_1 = [x, v_1] \cup [v_1 y]$ and $\gamma_2 = [x, v_{n_0}] \cup [v_{n_0} y]$. Then $\gamma_1$ and $\gamma_2$ are geodesics from $x$ to $y$ and

$$d_G(x,y) = 1 + \frac{n-(n_0-1)}{2} = \frac{n-n_0+3}{2}.$$

Consider the geodesic bigon $T = \{\gamma_1, \gamma_2\}$ and the midpoint $p$ of $\gamma_1$. Then

$$B(n,m) \geq \delta(G) \geq d_G(p, \gamma_2) = \frac{1}{2} L(\gamma_1) = \frac{n - n_0 + 3}{4}.$$

$\square$

Theorems 3.1.11 and 3.1.12 have the following direct consequence.

**Theorem 3.1.13.** *If $n \geq 1$ and $m = n-1$, then $B(n,m) = 0$. If $n \geq 3$ and $n \leq m \leq n+3$, then $B(n,m) = n/4$. If $5 \leq n_0 \leq n$ and $n + \binom{n_0-2}{2} < m \leq n + \binom{n_0-1}{2}$, then $B(n,m) \geq (n - n_0 + 3)/4$.*

## 3.2 Difference of the bounds of $B(n, m)$

Let $b_1(n, m)$ and $b_2(n, m)$ be the lower and upper bounds of $B(n, m)$ obtained in Theorems 3.1.13 and 3.1.11, respectively. In this section we prove that the difference between $b_2$ and $b_1$ is $O(\sqrt{n}\,)$. This is a good estimate, since the sharp upper bound for graphs with $n$ vertices is $n/4$ (see Theorem 2.4.7).

**Lemma 3.2.1.** *Given integers $n$ and $r$ with $2 \le r \le n/2$, let $n_0$ be the smallest natural number such that $3 \le n_0 \le n$ and $M(n, r) < n + \binom{n_0-1}{2}$. The following holds for $M(n, r) < m \le n + \binom{n_0-1}{2}$.*

- *If $r = 2$, then $b_2(n, m) = b_1(n, m)$.*
- *If $r = 3$, then $b_2(n, m) - b_1(n, m) < 3/4$.*
- *If $4 \le r \le n/2$, then $b_2(n, m) - b_1(n, m) < \sqrt{3n}/4$.*

**Remark 3.2.2.** *Note that we always have $M(n, r) \le n(n-1)/2 < n + \binom{n-1}{2}$, and this implies the existence of $n_0$.*

*Proof.* If $r = 2$, then $M(n, r) = \binom{n}{2} - 1$ and $M(n, 2) < m$ implies $m = \binom{n}{2}$. Hence, every graph $G \in \mathcal{G}(n, m)$ is isomorphic to the complete graph with $n$ vertices, and $\delta(G) = 1$ since $n \ge 4$. Thus, $A(n, m) = B(n, m) = 1$ and $b_1(n, m) = b_2(n, m) = 1$.

If $r = 3$, then

$$M(n, 3) < n + \binom{n_0 - 1}{2} \quad \Leftrightarrow \quad \binom{n}{2} - (n-1) < n + \binom{n_0 - 1}{2} \quad \Leftrightarrow \quad n^2 - 5n < n_0^2 - 3n_0.$$

Let us define $\lambda_3 := n^2 - 5n$. Since $n \ge 4$, the smallest $n_0$ verifying the previous inequality is the smallest $n_0$ satisfying $n_0 > (3 + \sqrt{9 + 4\lambda_3}\,)/2$. Thus $n_0 \le (5 + \sqrt{9 + 4\lambda_3}\,)/2 =: n_0'$.

Then, the following holds

$$\frac{r}{2} - \frac{n - n_0 + 3}{4} \le \frac{3}{2} - \frac{n - n_0' + 3}{4} = \frac{11 - 2n + \sqrt{9 + 4\lambda_3}}{8} = \frac{11 - 2n + \sqrt{4(n - 5/2)^2 - 16}}{8}.$$

Note that

$$\frac{11 - 2n + \sqrt{4(n - 5/2)^2 - 16}}{8} < \frac{11 - 2n + \sqrt{4(n - 5/2)^2}}{8} = \frac{11 - 2n + 2(n - 5/2)}{8} = \frac{3}{4}.$$

Therefore, for $r = 3$, we obtain

$$b_2(n, m) - b_1(n, m) = \frac{r}{2} - \frac{n - n_0 + 3}{4} < \frac{3}{4}.$$

Note that if $r \geq 4$, then

$$M(n,r) < n + \binom{n_0-1}{2} \quad \Leftrightarrow \quad \binom{n}{2} - (2n(r-3) - 2r^2 + 6r + 5) < n + \binom{n_0-1}{2}$$
$$\Leftrightarrow \quad n^2 + 9n - 4nr + 4r^2 - 12r - 12 < n_0^2 - 3n_0.$$

Let us define $\lambda_r := n^2 + 9n - 4nr + 4r^2 - 12r - 12$. Then, the smallest $n_0$ verifying the previous inequality is the smallest $n_0$ satisfying $n_0 > (3 + \sqrt{9 + 4\lambda_r}\,)/2$. Thus $n_0 \leq (5 + \sqrt{9 + 4\lambda_r}\,)/2 =: n_0'$.

Note that

$$\frac{r}{2} - \frac{n - n_0 + 3}{4} \leq \frac{r}{2} - \frac{n - n_0' + 3}{4} = \frac{4r + \sqrt{9 + 4\lambda_r} - 2n - 1}{8}.$$

Let us fix $n$ and consider the function $F(r) = 4r + \sqrt{9 + 4\lambda_r}$. It can be easily checked that the following holds for all $r \in [4, n/2]$

$$F'(r) = 4 + \frac{2(-4n + 8r - 12)}{\sqrt{9 + 4\lambda_r}} > 0 \quad \Leftrightarrow \quad n > 6.$$

Since $r \geq 4$, we have $n \geq 8$, $F(r)$ is an increasing function and $F(n/2) = 2n + \sqrt{9 + 4(3n - 12)}$ is the maximum value of $F(r)$.

Then, the following inequalities hold

$$b_2(n,m) - b_1(n,m) = \frac{r}{2} - \frac{n - n_0 + 3}{4} \leq \frac{F(n/2) - 2n - 1}{8} < \frac{\sqrt{9 + 4(3n - 12)}}{8} < \frac{2\sqrt{3n}}{8} = \frac{\sqrt{3n}}{4}.$$

□

**Lemma 3.2.3.** *Given integers $n$ and $r$ with $3 \leq r \leq n/2$, let $n_1$ be the smallest natural number such that $3 \leq n_1 \leq n$ and $M(n, r-1) < n + \binom{n_1-1}{2}$. Consider $n_0$ defined as in Lemma 3.2.1. The following holds.*

- *If $r = 3$, $r = 4$ or $r = n/2$, then $n_1 - n_0 \leq 2$.*
- *If $5 \leq r < n/2$, then $n_1 - n_0 \leq 4$.*

*Proof.* If $r = 3$, then

$$M(n, r-1) < n + \binom{n_1 - 1}{2} \quad \Leftrightarrow \quad n^2 - 3n - 4 < n_1^2 - 3n_1.$$

Using the definition of $\lambda_r$ in the proof of Lemma 3.2.1, we deduce that the smallest natural number $n_1$ verifying the previous inequality satisfies $n_1 \le (5+\sqrt{9+4\lambda_2}\,)/2 =: n_1'$.

If $r=4$, then

$$M(n,r-1) < n + \binom{n_1-1}{2} \quad \Leftrightarrow \quad n^2-5n < n_1^2-3n_1.$$

Therefore, the smallest $n_1$ verifying the previous inequality satisfies $n_1 \le (5+\sqrt{9+4\lambda_3}\,)/2 =: n_1'$.

Note that if $r \ge 5$, then

$$M(n,r-1) < n + \binom{n_1-1}{2}$$

$$\Leftrightarrow \quad n^2+13n-4nr+4r^2-20r+4 = n^2+9n-4n(r-1)+4(r-1)^2-12(r-1)-12 < n_0^2-3n_0.$$

Thus, the smallest $n_1$ verifying the previous inequality satisfies $n_1 \le (5+\sqrt{9+4\lambda_{r-1}}\,)/2 =: n_1'$.

Now we estimate the difference between $n_1$ and $n_0$.

$$n_1-n_0 < n_1'-(n_0'-1) = \frac{\sqrt{9+4\lambda_{r-1}}-\sqrt{9+4\lambda_r}}{2}+1 = \frac{2(\lambda_{r-1}-\lambda_r)}{\sqrt{9+4\lambda_{r-1}}+\sqrt{9+4\lambda_r}}+1 \le \frac{\lambda_{r-1}-\lambda_r}{\sqrt{9+4\lambda_r}}+1.$$

If $r=3$, then $n \ge 6$ and

$$n_1-n_0 < \frac{\lambda_2-\lambda_3}{\sqrt{9+4\lambda_3}}+1 = \frac{2(n-2)}{\sqrt{9+4\lambda_3}}+1 < \frac{n-2}{\sqrt{\lambda_3}}+1.$$

The following holds

$$\frac{n^2-4n+4}{n^2-5n} = \frac{n^2-5n+n+4}{n^2-5n} < 3 \quad \Rightarrow \quad \frac{n-2}{\sqrt{\lambda_3}} < \sqrt{3}.$$

Therefore,

$$n_1-n_0 < \sqrt{3}+1 \quad \Rightarrow \quad n_1-n_0 \le 2.$$

If $r=4$, then $n \ge 8$ and

$$n_1-n_0 < \frac{\lambda_3-\lambda_4}{\sqrt{9+4\lambda_4}}+1 = \frac{2(n-2)}{\sqrt{9+4\lambda_4}}+1 < \frac{n-2}{\sqrt{\lambda_4}}+1.$$

The following holds

$$\frac{n^2-4n+4}{n^2-7n+4} = \frac{n^2-7n+4+3n}{n^2-7n+4} \leq 3 \quad \Rightarrow \quad \frac{n-2}{\sqrt{\lambda_4}} \leq \sqrt{3}.$$

Therefore,

$$n_1 - n_0 < \sqrt{3}+1 \quad \Rightarrow \quad n_1 - n_0 \leq 2.$$

If $r \geq 5$, then

$$n_1 - n_0 < \frac{\lambda_{r-1}-\lambda_r}{\sqrt{9+4\lambda_r}} + 1 = \frac{4(n-2r+4)}{\sqrt{9+4\lambda_r}} + 1.$$

Note that

$$\lambda_r = (n-2r)^2 + 9(n-2r) + 6r - 12 \geq (n-2r)^2 + 6r - 12 \geq (n-2r)^2 + 18.$$

If $r < n/2$, then

$$\frac{4(n-2r+4)}{\sqrt{9+4\lambda_r}} \leq \frac{4(n-2r+4)}{\sqrt{81+4(n-2r)^2}} < 2\frac{n-2r}{n-2r} + \frac{16}{9} < 4.$$

Thus, $n_1 - n_0 < 5$ and $n_1 - n_0 \leq 4$.If $r = n/2$, then

$$\frac{4(n-2r+4)}{\sqrt{9+4\lambda_r}} \leq \frac{4(n-2r+4)}{\sqrt{81+4(n-2r)^2}} = \frac{16}{9} < 2.$$

Therefore $n_1 - n_0 < 3$ and $n_1 - n_0 \leq 2$. □

The following result is a consequence of the two previous lemmas.

**Lemma 3.2.4.** *Given integers $n$ and $r$ with $3 \leq r \leq n/2$, let $n_0$ be defined as in Lemma 3.2.1. Assume $M(n,r-1) > n + \binom{n_0-1}{2}$. The following holds for $n + \binom{n_0-1}{2} < m \leq M(n,r-1)$.*

- *If $r = 3$, then $b_2(n,m) - b_1(n,m) < 5/4$.*
- *If $r = 4$ or $r = n/2$, then $b_2(n,m) - b_1(n,m) < \sqrt{3n}/4 + 1/2$.*
- *If $5 \leq r < n/2$, then $b_2(n,m) - b_1(n,m) < \sqrt{3n}/4 + 1$.*

*Proof.* Let $n_1$ be defined as in Lemma 3.2.3.

On the one hand, $m \leq M(n,r-1) < n + \binom{n_1-1}{2}$ and Theorem 3.1.12 gives $b_1(n,m) \geq (n-n_1+3)/4$.

On the other hand, $M(n,r) < n + \binom{n_0-1}{2} < m \leq M(n,r-1)$ and Theorem 3.1.11 gives $b_2(n,m) = r/2$.

The following holds

$$b_2(n,m) - b_1(n,m) = b_2(n,m) - \frac{n-n_0+3}{4} + \frac{n-n_0+3}{4} - b_1(n,m).$$

Notice that

$$\frac{n-n_0+3}{4} - b_1(n,m) \le \frac{n-n_0+3}{4} - \frac{n-n_1+3}{4} = \frac{n_1-n_0}{4}.$$

Then, applying Lemmas 3.2.1 and 3.2.3, in order to bound $b_2(n,m)-(n-n_0+3)/4$ and $n_1-n_0$, respectively, we obtain the desired upper bounds. □

Lemmas 3.2.1 and 3.2.4 have the following consequence.

**Theorem 3.2.5.** *The following holds for all $n \ge 3$.*

$$b_2(n,m) - b_1(n,m) < \frac{\sqrt{3n}}{4} + 1. \tag{3.1}$$

*Proof.* If $m > M(n,3)$, then $b_2(n,m) \le 3/2$ by Theorem 3.1.10, and

$$b_2(n,m) - b_1(n,m) \le b_2(n,m) \le \frac{3}{2} < \frac{3}{4} + 1 \le \frac{\sqrt{3n}}{4} + 1.$$

Consider now $r \ge 3$ and $n_0$ defined as in Lemma 3.2.1. If $M(n,r) < m \le n + \binom{n_0-1}{2}$ or $M(n,r-1) < m \le n + \binom{n_1-1}{2}$, then Lemma 3.2.1 gives

$$b_2(n,m) - b_1(n,m) < \frac{\sqrt{3n}}{4}. \tag{3.2}$$

If $M(n,r-1) \le n + \binom{n_0-1}{2}$, then equation (3.2) holds for $M(n,r) < m \le M(n,r-1)$.

If $n + \binom{n_0-1}{2} < M(n,r-1)$ and $n + \binom{n_0-1}{2} < m \le M(n,r-1)$, then Lemma 3.2.4 implies (3.1). Thus, (3.1) holds for $M(n,r) < m \le M(n,r-1)$. Hence, (3.1) holds for every $m > M(n,\lfloor n/2 \rfloor)$.

If $m \le n+3$, then $b_2(n,m) = b_1(n,m)$.

Finally, assume that $n+3 < m \le M(n,\lfloor n/2 \rfloor)$. First, note that if $M(n,\lfloor n/2 \rfloor) < m \le \min\left\{n + \binom{n_0-1}{2}, M(n,\lfloor n/2 \rfloor - 1)\right\}$, then Lemma 3.2.1 implies

$$b_2(n,m) - b_1(n,m) = \frac{\lfloor n/2 \rfloor}{2} - \frac{n-n_0+3}{4} < \frac{\sqrt{3n}}{4}.$$

Consider now $m \le M(n,\lfloor n/2 \rfloor)$, then

$$b_2(n,m) - b_1(n,m) \le \frac{n}{4} - \frac{n-n_0+3}{4} < \frac{2(\lfloor n/2 \rfloor + 1)}{4} - \frac{n-n_0+3}{4} = \frac{\lfloor n/2 \rfloor}{2} - \frac{n-n_0+3}{4} + \frac{1}{2} < \frac{\sqrt{3n}}{4} + \frac{1}{2}.$$

Hence, (3.1) holds for every $m \le M(n,\lfloor n/2 \rfloor)$. □

## 3.3 An inequality involving the diameter of a graph

We consider a similar optimization problem to the one in Lemma 3.1.3.

**Lemma 3.3.1.** *Consider the following optimization problem:*

$$\Lambda_r' = \min_{x \in W_0} f_r, \quad \textit{with} \quad f_r(k_1, k_2, \dots, k_r) := \Sigma_{t=2}^r k_t \Sigma_{s=0}^{t-2} k_s, \quad 2 \le r \le n-1,$$

$$\textit{and} \quad W_0 := \ \{k_0 = 1, \quad k_j \ge 1 \quad \textit{if} \quad 1 \le j \le r,$$
$$1 + k_1 + k_2 + \dots + k_r = n\}.$$

*Then* $\Lambda_2' = 1$ *and* $\Lambda_r' = n(r-2) - \frac{1}{2}r^2 + \frac{1}{2}r + 2$ *for* $r \ge 3$.

The same arguments in the proof of Lemma 3.1.3 allow to prove Lemma 3.3.1; in fact, the proof of Lemma 3.3.1 is simpler, since $W_0$ is more symmetric than $W$.

To find inequalities relating the diameter, the order and another parameter of any graph is an important problem in graph theory (see, e.g., [39] and [84]). If the third parameter is a bound of the degree, we have the well-known Moore bounds (see, e.g., [84]).

We present here an upper bound of the size of the graph in terms of its diameter and its order, that is a consequence of Lemma 3.3.1.

**Theorem 3.3.2.** *Let $G$ be any graph with $n$ vertices and $m$ edges.*

- *If* $\operatorname{diam} V(G) = 2$, *then*
$$m \le \binom{n}{2} - 1.$$

- *If* $\operatorname{diam} V(G) \ge 3$, *then*
$$m \le \binom{n}{2} + 2(n-1) + \frac{1}{2}\big(\operatorname{diam} V(G)\big)^2 - \Big(n + \frac{1}{2}\Big) \operatorname{diam} V(G).$$

## 3.4 Computation of $A(n, m)$

Denote by $\Gamma_3$ the set of graphs such that every cycle has length 3 and every edge belongs to some cycle.

**Proposition 3.4.1.** *Consider a graph $G \in \mathcal{G}(n, m) \cap \Gamma_3$. If $k$ denotes the number of cycles of $G$, then $n = 2k + 1$ and $m = 3k$.*

*Proof.* Let us prove the result by induction on $k$. If $k = 1$, then $G$ is isomorphic to $C_3$ and $n = m = 3$. Assume that the statement holds for every graph $G_0$ with $k - 1$ cycles. Then $G_0$ has $n_0 = 2(k - 1) + 1$ vertices and $m_0 = 3(k - 1)$ edges. Any graph $G$ with $k$ cycles can be obtained by adding 2 vertices and 3 edges to some graph $G_0$ with $k - 1$ cycles, that is, $n = n_0 + 2 = 2k + 1$ and $m = m_0 + 3 = 3k$. □

We say that an edge $g$ of a graph $G$ is a *cut-edge* if $G \setminus \{g\}$ is not connected. Given a graph $G$, the *T-edge-decomposition* of $G$ is a *T-decomposition* such that each component $G_s$ is either a *cut-edge* or it does not contain cut-edges.

**Proposition 3.4.2.** *Let $G \in \mathcal{G}(n, m)$ be a graph such that every cycle has length* 3*. Then $2m \leq 3n - 3$.*

*Proof.* The canonical T-edge-decomposition of $G$ has $r \geq 1$ graphs $\{G_1, ..., G_r\}$ in $\Gamma_3$ and $s \geq 0$ edges $\{G_{r+1}, ..., G_{r+s}\}$. For each component $G_i \in \Gamma_3$ we have, by 3.4.1,

$$n_i = 2k_i + 1, \;\; m_i = 3k_i, \;\; 1 \leq i \leq r,$$

where $n_i$, $m_i$ and $k_i$ denote the number of vertices, edges and cycles in $G_i$, respectively.

Let us denote by $k = \sum_{i=1}^{r} k_i$ the number of cycles of $G$. Let $n_0$ and $m_0$ be the number of vertices and edges we add to complete $G$, i.e, $n_0 = n - \sum_{i=1}^{r} n_i$, $m_0 = m - \sum_{i=1}^{r} m_i$. Then we have

$$n = \sum_{i=0}^{r} n_i = n_0 + \sum_{i=1}^{r} (2k_i + 1) = n_0 + 2k + r,$$

$$m = \sum_{i=0}^{r} m_i = m_0 + \sum_{i=1}^{r} (3k_i) = m_0 + 3k.$$

Hence,

$$n = n_0 + 2\,\frac{m - m_0}{3} + r.$$

One can check that if $n_0 = 0$, then $m_0 = r - 1$ and if $n_0 \geq 1$, then $m_0 = n_0 + r - 1$. Therefore,

$$n = n_0 + 2\,\frac{m - (n_0 + r - 1)}{3} + r \;\; \Rightarrow \;\; 2m = 3n - n_0 - r - 2 \;\; \Rightarrow \;\; 2m = 3n - m_0 - 3.$$

Then $2m \leq 3n - 3$. □

**Proposition 3.4.3.** *If $m \geq n$ and $2m \leq 3n-3$, then $A(n,m) = 3/4$.*

**Remark 3.4.4.** *Note that $n \leq m \leq (3n-3)/2$ implies $n \geq 3$.*

*Proof.* Since $m \geq n \geq 3$, if $G \in \mathcal{G}(n,m)$, then $G$ is not a tree. Hence Theorem 2.4.5 gives $\delta(G) \geq 3/4$ and $A(n,m) \geq 3/4$.

Fix $n, m$ verifying the hypotheses. Define $n_0 := m_0 := 3n-3-2m$ and $k := m+1-n$. Then

$$n = 2k + 1 + n_0, \qquad m = 3k + n_0.$$

Let us consider $k$ graphs $G_1, \dots, G_k$ isomorphic to $C_3$ and $n_0$ graphs $\Gamma_1, \dots, \Gamma_{n_0}$ isomorphic to $P_2$. Fix vertices $v_1 \in V(G_1), \dots, v_k \in V(G_k), w_1 \in V(\Gamma_1), \dots, w_{n_0} \in V(\Gamma_{n_0})$ and consider the graph $G$ obtained from $G_1, \dots, G_k, \Gamma_1, \dots, \Gamma_{n_0}$ by identifying $v_1, \dots, v_k, w_1, \dots, w_{n_0}$ in a single vertex. Then $G \in \mathcal{G}(n,m)$ and $\delta(G) = 3/4$. Therefore, $A(n,m) \leq 3/4$ and we conclude $A(n,m) = 3/4$. $\square$

**Definition 3.4.5.** *Let $K_n$ be the complete graph with $n$ vertices and consider the numbers $N_i$, $i = 1, \dots, s$, $(s \geq 1)$ such that $2 \leq N_1, \dots, N_s < n$, $N_1 + \cdots + N_s \leq n$. Choose sets of vertices $V_1, \dots, V_s \subset V(K_n)$ with $V_i \cap V_j = \emptyset$ if $i \neq j$ and $\#V_i = N_i$ for $i = 1, \dots, s$. Let $K_n^{N_1,\dots,N_s}$ be the graph obtained from $K_n$ by removing the edges joining any two vertices in $V_i$ for every $i = 1, \dots, s$.*

**Lemma 3.4.6.** *We always have $\delta(K_n^{N_1,\dots,N_s}) \leq 1$.*

*Proof.* Fist of all, note that $\operatorname{diam} V(K_n^{N_1,\dots,N_s}) = 2$. Hence, in order to prove $\operatorname{diam}(K_n^{N_1,\dots,N_s}) = 2$, it suffices to check that $d(x,y) \leq 2$ for every midpoint $x$ of any edge in $E(K_n^{N_1,\dots,N_s})$ and every $y \in K_n^{N_1,\dots,N_s}$.

Fix $i \in \{1, \dots, s\}$ and $u \in V_i$. Then, $d(u,v) = 1$ for every $v \in V(K_n^{N_1,\dots,N_s}) \setminus V_i$ and $d(u,v) = 2$ for every $v \in V_i \setminus \{u\}$.

Given a fixed vertex $u \in V_i$, let $x$ be the midpoint of the edge $[u,v]$ (then $v \notin V_i$). If $w \in V_i$, then there exists an edge joining $v$ with $w$. Therefore, we have $d(x,w) \leq d(x,v) + d(v,w) = 3/2$. If $w \notin V_i$, then $[u,w] \in E(K_n^{N_1,\dots,N_s})$ and $d(x,w) \leq d(x,u) + d(u,w) = 3/2$. Hence, $d(x,v) \leq 3/2$ for every $v \in V(K_n^{N_1,\dots,N_s})$; thus, $d(x,y) \leq 2$ for every $y \in K_n^{N_1,\dots,N_s}$.

If $N_1 + \cdots + N_s \leq n-2$, let $x$ be the midpoint of $[v^1, v^2]$, where $v^1, v^2 \notin \cup_i V_i$. If $v \in V(K_n^{N_1,\dots,N_s})$, then there exists an edge joining $v$ with $v^1$. Thus, we have $d(x,v) \leq d(x,v^1) + d(v^1,v) = 3/2$ for every $v \in V(K_n^{N_1,\dots,N_s})$. Hence, $d(x,y) \leq 2$ for every $y \in K_n^{N_1,\dots,N_s}$.

Therefore $\operatorname{diam}(K_n^{N_1,\dots,N_s}) = 2$ and $\delta(K_n^{N_1,\dots,N_s}) \leq 1$ by Theorem 2.4.2. $\square$

In order to prove our next result we need the following combinatorial lemma.

**Lemma 3.4.7.** *For all $t \geq 3$, $(t \neq 4, 5)$, there exist numbers $t_i \geq 2$, $i = 1, \ldots, s$ $(s \geq 1)$, such that*

$$\sum_i t_i \leq t \quad \text{and} \quad \sum_i \binom{t_i}{2} = t.$$

*Proof.* If $t = 3$, then choose $t_1 = 3$, $3 \leq 3$ and $\binom{3}{2} = 3$. If $t = 6$, then choose $t_1 = 4$, $4 \leq 6$ and $\binom{4}{2} = 6$. If $t = 7$, then choose $t_1 = 4$, $t_2 = 2$, $4 + 2 \leq 7$ and $\binom{4}{2} + \binom{2}{2} = 7$. If $t = 8$, then choose $t_1 = 4$, $t_2 = 2$, $t_3 = 2$, $4 + 2 + 2 \leq 8$ and $\binom{4}{2} + \binom{2}{2} + \binom{2}{2} = 8$. If $t = 9$, then choose $t_1 = 4$, $t_2 = 3$, $4 + 3 \leq 9$ and $\binom{4}{2} + \binom{3}{2} = 9$.

Let us prove the result by induction on $t$. We have seen that

$$\sum_i t_i \leq t, \quad \sum_i \binom{t_i}{2} = t$$

holds for $6 \leq t \leq 9$. Assume now that it holds for every value $3, 6, 7, \ldots, t-1$, with $t > 9$. Then it holds for $t-3 \geq 6$ and there exist numbers $t_i \geq 2$, $i = 1, \ldots, s$, such that $\sum_i t_i \leq t-3$ and $\sum_i \binom{t_i}{2} = t - 3$.

Therefore, there exist numbers $t_i' \geq 2$, $t_i' = t_i$ for $i = 1, \ldots, s$, $t_{s+1}' = 3$ such that

$$\sum_i t_i' = \sum_i t_i + 3 \leq t$$

and

$$\sum_i \binom{t_i'}{2} = \sum_i \binom{t_i}{2} + \binom{3}{2} = t.$$

So we have shown that the statement holds at $t$ when it is assumed to be true for $3, 6, 7, \ldots, t-1$. □

**Corollary 3.4.8.** *For all $t \geq 1$, there exist numbers $t_i \geq 2$, $i = 1, \ldots, s$, $(s \geq 1)$ such that $\sum_i t_i \leq t + 2$ and $\sum_i \binom{t_i}{2} = t$.*

*Proof.* If $t \neq 1, 2, 4, 5$, then Lemma 3.4.7 gives the result. If $t = 1$, then choose $t_1 = 2$, $2 \leq 3$ and $\binom{2}{2} = 1$. If $t = 2$, then choose $t_1 = 2$, $t_2 = 2$, $2 + 2 \leq 4$ and $\binom{2}{2} + \binom{2}{2} = 2$. If $t = 4$, then choose $t_1 = 3$, $t_2 = 2$, $3 + 2 \leq 6$ and $\binom{3}{2} + \binom{2}{2} = 4$. If $t = 5$, then choose $t_1 = 3$, $t_2 = 2$, $t_3 = 2$, $3 + 2 + 2 \leq 7$ and $\binom{3}{2} + \binom{2}{2} + \binom{2}{2} = 5$. □

**Proposition 3.4.9.** *If $m \geq n$ and $2m > 3n - 3$, then $A(n, m) = 1$.*

*Proof.* Consider any $G \in \mathcal{G}(n, m)$. Proposition 3.4.2 gives that there exists at least one cycle in $G$ with length greater or equal than 4. Then Theorem 2.4.5 gives $\delta(G) \geq 1$ for every $G \in \mathcal{G}(n, m)$ and, consequently, $A(n, m) \geq 1$.

In order to finish the proof it suffices to find a graph $G \in \mathcal{G}(n, m)$ with $\delta(G) \leq 1$. Note that $n \geq 4$ since $2m > 3n - 3$.

If $m = n+1$, then consider a graph $G_1$ with 4 vertices and 5 edges and a path graph $G_2$ with $n-3$ vertices and $n-4$ edges. Fix vertices $v_1 \in G_1$ and $v_2 \in G_2$. Let $G$ be the graph obtained by identifying $v_1$ and $v_2$ in a single vertex, then $G$ has $n$ vertices and $m = n+1$ edges, and $\delta(G) = \delta(G_1) = 1$. Therefore $A(n,m) \le \delta(G) \le 1$ and we conclude $A(n,m) = 1$.

If $m = \binom{n}{2}$ and $G \in \mathcal{G}(n,m)$, the $G$ is isomorphic to $K_n$ and $\delta(G) = 1$. Therefore $A(n,m) = 1$.

Assume now that $n+2 \le m < \binom{n}{2}$. Then $m-6 \ge n-4$ and we can define

$$n_0 - 1 := \max\Big\{4 \le j \le n-1 \mid m - \binom{j}{2} \ge n-j\Big\}.$$

Then $3 \le n_0 \le n$ and we have

$$\binom{n_0-1}{2} + n - n_0 + 1 \le m < \binom{n_0}{2} + n - n_0.$$

Define $T := \binom{n_0}{2} + n - n_0 - m$. Notice that

$$1 \le T \le \binom{n_0}{2} + n - n_0 - \binom{n_0-1}{2} - n + n_0 - 1 = n_0 - 2.$$

It follows from Corollary 3.4.8 that there exist numbers $t_i \ge 2$, $i = 1, \dots, s$, such that $\sum_i t_i \le T + 2 \le n_0$ and $\sum_i \binom{t_i}{2} = T$.

Choose sets of vertices $V_1, \dots, V_s \subset V(K_{n_0})$ with $V_i \cap V_j = \emptyset$ if $i \ne j$ and $\#V_i = t_i$ for $i = 1, \dots, s$. Let us denote by $G_1$ the graph obtained from $K_{n_0}$ by removing the $T = \sum_i \binom{t_i}{2}$ edges joining any two vertices in $V_i$ for every $i = 1, \dots, s$. Then $G_1 \in \mathcal{G}(n_0, m-n+n_0)$ and Lemma 3.4.6 implies $\delta(G_1) = \delta(K_{n_0}^{t_1,\dots,t_s}) \le 1$.

Let us define $G_2$ as a path graph with $n - n_0 + 1$ vertices and $n - n_0$ edges. Fix vertices $v_1 \in G_1$ and $v_2 \in G_2$. Let $G$ be the graph obtained from $G_1$ and $G_2$ by identifying $v_1$ and $v_2$ in a single vertex, then $G \in \mathcal{G}(n,m)$ and $\delta(G) = \delta(G_1) = 1$. Therefore $A(n,m) \le \delta(G) = 1$ and we conclude $A(n,m) = 1$. $\square$

The previous results have the following consequence.

**Theorem 3.4.10.** *If $m = n-1$, then $A(n,m) = 0$.*
*If $m \ge n$ and $2m \le 3n-3$, then $A(n,m) = 3/4$.*
*If $m \ge n$ and $2m > 3n-3$, then $A(n,m) = 1$.*

## 3.5 Non-simple and weighted graphs

In this section, we deal first with non-simple graphs (we allow loops and/or multiple edges). Let $\mathcal{G}'(n,m)$ be the set of non-simple graphs $G$ with $n$ vertices and $m$ edges, and such that every edge has length equal to 1 (we consider $\mathcal{G}(n,m) \subseteq \mathcal{G}'(n,m)$). If $m = n-1$, then every $G \in \mathcal{G}'(n,m)$ is a tree and $\delta(G) = 0$. Thus, for $m \geq n-1$, let us define

$$A'(n,m) := \min\{\delta(G) \mid G \in \mathcal{G}'(n,m)\},$$

$$B'(n,m) := \max\{\delta(G) \mid G \in \mathcal{G}'(n,m)\}.$$

The following result provides the precise value of $A'(n,m)$ and $B'(n,m)$ for all values of $n$ and $m$.

**Proposition 3.5.1.** *If $m = n-1$, then $A'(n,m) = B'(n,m) = 0$. If $m > n-1$, then $A'(n,m) = 1/4$ and $B'(n,m) = n/4$.*

*Proof.* If $m = n-1$, then every $G \in \mathcal{G}'(n,m)$ is a tree and $A'(n,m) = B'(n,m) = 0$. Assume now $m > n-1$. Given any $G \in \mathcal{G}'(n,m)$ with loops, let $G^*$ be the subgraph of $G$ obtained by deleting the set of loops $\{G_j\}$ of $G$. Since $\{G^*, \{G_j\}\}$ is a T-decomposition of $G$, Lemma 2.4.11 gives

$$\delta(G) = \max\left\{\delta(G^*),\, \sup_j \delta(G_j)\right\} = \max\left\{\delta(G^*),\, 1/4\right\}. \tag{3.3}$$

If $G^*$ is any tree with $n$ vertices, then consider any $G \in \mathcal{G}'(n,m)$ obtained from $G^*$ by adding $m-(n-1)$ loops. By (3.3), $\delta(G) = 1/4$ and thus $A'(n,m) \leq 1/4$. Since any non-simple graph $G$ which is not a tree contains a cycle with length at least 1, we have $\delta(G) \geq 1/4$, and we conclude $A'(n,m) = 1/4$.

Since any non-simple graph with $n$ vertices verifies $\delta(G) \leq n/4$, we have $B'(n,m) \leq n/4$. If $G^*$ is a cycle graph with $n$ vertices, then consider any $G \in \mathcal{G}'(n,m)$ obtained from $G^*$ by adding $m-n$ loops. By (3.3), $\delta(G) = \delta(G^*) = n/4$ and thus $B'(n,m) = n/4$. $\square$

We deal now with (edge) weighted (simple) graphs, since this kind of graphs is interesting in the field of applied graph theory. Given $w = (w_1, \dots, w_m) \in \mathbb{R}_+^m$, let $\mathcal{G}(n,m;w)$ be the set of weighted graphs $G$ with $n$ vertices and $m$ edges, and such that the lengths of their edges are precisely $\{w_1, \dots, w_m\}$. Note that if $w_1 < \cdots < w_m$, then there exist $m!$ isomorphic graphs in $\mathcal{G}(n,m;w)$ which are not isometric. Define

$$A(n,m;w) := \min\{\delta(G) \mid G \in \mathcal{G}(n,m;w)\},$$
$$B(n,m;w) := \max\{\delta(G) \mid G \in \mathcal{G}(n,m;w)\}.$$

To obtain bounds of $A(n,m;w)$ and $B(n,m;w)$ is a more complicated task. We present here some results which can be deduced from our previous work.

Let $b_1(n,m)$ and $b_2(n,m)$ be the lower and upper bounds of $B(n,m)$ obtained in Theorems 3.1.13 and 3.1.11, respectively. Since we have proved Theorems 3.1.13 and 3.1.11 with combinatorial arguments and by bounding the diameter of the graphs in $\mathcal{G}(n,m)$, respectively, we obtain the following consequence for weighted graphs.

**Proposition 3.5.2.** *We have* $b_1(n,m)\min_j w_j \le B(n,m;w) \le b_2(n,m)\max_j w_j$.

Fix $n$ and $m$ with $m \ge n$ and $2m \le 3n-3$. We can assume that $w = (w_1,\dots,w_m)$ verifies the condition $w_1 \le \cdots \le w_m$. As in the proof of Proposition 3.4.3, define $k := m+1-n$. For any permutation $\sigma$ of the set $\{1,\dots,3k\}$ we define

$$S(\sigma,w) := \max_{1\le j\le k} \frac{1}{4}\big(w_{\sigma(3j-2)} + w_{\sigma(3j-1)} + w_{\sigma(3j)}\big).$$

**Proposition 3.5.3.** *If* $m = n-1$, *then* $A(n,m;w) = B(n,m;w) = 0$.
*If* $m \ge n$ *and* $2m \le 3n-3$, *then* $A(n,m;w) = \min_\sigma S(\sigma,w)$.
*If* $m \ge n$ *and* $2m > 3n-3$, *then* $\min_j w_j \le A(n,m;w) \le \max_j w_j$.

*Proof.* The case $m = n-1$ is direct.

Assume now $m \ge n$ and $2m \le 3n-3$. As in the proof of Proposition 3.4.3, we know that there are graphs $G \in \mathcal{G}(n,m;w)$ such every cycle in $G$ has three edges; furthermore, $G$ has $k = m+1-n$ cycles, and we denote them by $C_1,\dots,C_k$. Since

$$\delta(G) = \max_{1\le j\le k} \delta(C_j) = \max_{1\le j\le k} \frac{1}{4} L(C_j) = S(\sigma,w),$$

for some permutation $\sigma$, we have $A(n,m;w) = \min_\sigma S(\sigma,w)$.

Finally, assume that $m \ge n$ and $2m > 3n-3$.

Since the proof of Proposition 3.4.9 involves mainly combinatorial arguments and estimates of diameter of graphs, we obtain $A(n,m;w) \le \max_j w_j$.

In order to prove the lower bound of $A(n,m;w)$, define $s := \min_j w_j$. For each $G \in \mathcal{G}(n,m;w)$, define $\mathfrak{C}(G) := \big\{G \text{ cycle in } G \,|\, L(C) \ge 4s\big\}$. Let $C_0 \in \mathfrak{C}(G)$ such that $L(C_0) \le L(C)$ for every $C \in \mathfrak{C}(G)$. Note that if $C_0$ is an isometric subgraph of $G$, then $\delta(G) \ge \delta(C_0) = L(C_0)/4 \ge s$.

By a *shortcut* of $C_0$ we mean a geodesic $\gamma$ in $G$ joining two vertices $a,b \in C_0 \cap V(G)$ such that $\gamma \cap C_0 = \{a,b\}$. Given a shortcut $\gamma = [ab]$ of $C_0$, we denote by $C_0^1, C_0^2$ the two curves joining $a,b \in C_0 \cap V(G)$ with $C_0^1 \cup C_0^2 = C_0$ and $C_0^1 \cap C_0^2 = \{a,b\}$. By symmetry, we can assume that the number of edges in $C_0^1$ is greater or equal than the number of edges in $C_0^2$.

It is clear that $C_0$ is an isometric subgraph of $G$ if and only if $C_0$ does not have shortcuts.

We prove now that any shortcut of $C_0$ is an edge. Seeking for a contradiction assume that there is a shortcut $\gamma$ containing more than one edge. Since $C_0^1$ contains at least two edges, $C_0' := C_0^1 \cup \gamma$ has at least four edges; hence, it verifies $L(C_0') \ge 4s$ and $L(C_0') < L(C_0)$, which is a contradiction. Therefore, any shortcut of $C_0$ is an edge.

Case (1): $C_0$ contains at least five edges. Seeking for a contradiction assume that $C_0$ is not an isometric subgraph. Thus there exists a shortcut $\gamma$ of $C_0$ joining $a,b \in C_0 \cap V(G)$. Since $C_0$ contains at least five edges, $C_0' := C_0^1 \cup \gamma$ has at least four edges; hence, it verifies $L(C_0') \ge 4s$ and $L(C_0') < L(C_0)$, which is a contradiction. Therefore, $C_0$ is an isometric subgraph and $\delta(G) \ge \delta(C_0) = L(C_0)/4 \ge s$.

Case (2): $C_0$ contains exactly three edges. Since any shortcut of $C_0$ is an edge and $G$ is a simple graph, $C_0$ does not have shortcuts. Then $C_0$ is an isometric subgraph of $G$ and $\delta(G) \ge \delta(C_0) = L(C_0)/4 \ge s$.

Case (3): $C_0$ contains exactly four edges.

Case (3.1): $C_0$ is an isometric subgraph. Thus $\delta(G) \ge \delta(C_0) = L(C_0)/4 \ge s$.

Case (3.2): $C_0$ has just a shortcut $\gamma = [a, b]$. Then $C_1 := C_0 \cup \gamma$ is an isometric subgraph of $G$ and $\delta(G) \ge \delta(C_1)$. Let $x, y$ be the midpoints of $C_0^1, C_0^2$, respectively. Let $\gamma^1, \gamma^2$ be the two geodesics joining $x, y$ with $\gamma^1 \cup \gamma^2 = C_0$ and $\gamma^1 \cap \gamma^2 = \{x, y\}$. Let $p$ be the midpoint of $\gamma^1$. Since $L(\gamma^1) = L(C_0)/2 \ge 2s$ and $L([a, b]) \ge s$, we have $\delta(G) \ge \delta(C_1) \ge d(p, \gamma^2) \ge s$.

Case (3.3): $C_0$ has at least two shortcuts. Since any shortcut of $C_0$ is an edge and $G$ is a simple graph, there exist exactly two shortcuts $\gamma_1, \gamma_2$ of $C_0$, and $C_2 := C_0 \cup \gamma_1 \cup \gamma_2$ is isomorphic to the complete graph $K_4$. Then $C_2$ is an isometric subgraph of $G$ and $\delta(G) \ge \delta(C_2)$. Let $e_0 = [u, v]$ be a shortest edge in $C_2$ and $p$ be the midpoint of $e_0$. Let $e_u, e_v$ be two edges in $C_2$ incident to $u, v$, respectively, with $e_u \cap e_v = \emptyset$, and $x, y$ the midpoints of $e_u, e_v$, respectively. Let $e'$ be the unique edge in $C_2$ such that $e_0 \cup e_u \cup e_v \cup e'$ is a cycle, and $z$ the midpoint of $e'$. Thus $e_0 \subset [xy]$ and $[xy] \cup [yz] \cup [zx] = e_0 \cup e_u \cup e_v \cup e'$. Since $L([xy]) \ge 2s$ and the length of every edge is at least $s$, we have $\delta(G) \ge \delta(C_2) \ge d(p, [yz] \cup [zx]) \ge s$. $\square$

## 3.6 Random graphs

The field of random graphs was started in the late fifties and early sixties of the last century by Erdös and Rényi, see [49, 50, 65, 51]. At first, the study of random graphs was used to prove deterministic properties of graphs. For example, if we can show that a random graph has a certain property with a positive probability, then a graph must exist with this property. Lately there has been a great amount of work on the field. The practical applications of random graphs are found, for instance, in areas in which complex networks need to be modeled. See the standard reference on the subject [65] for the state of the art.

Erdös and Rényi studied in [50] the simplest imaginable random graph, which is now named after them. Given $n$ fixed vertices, the Erdös-Rényi random graph $R(n,m)$ is characterized by $m$ edges distributed uniformly at random among all possible $\binom{n}{2}$ edges. However, in order to avoid disconnected graphs, which are not geodesic metric spaces, a random tree of order $n$ is first generated and then the remaining $m-(n-1)$ edges are distributed uniformly at random over the remaining $\binom{n}{2}-n+1$ possible edges. Call this new model $R'(n,m)$. This modified Erdös-Rényi random graph $R'(n,m)$ has a number of desirable properties as a model of a network, see [69].

We can apply the results obtained in this work to $R'(n,m)$:

For all $G \in R'(n,m)$ we have $A(n,m) \le \delta(G) \le B(n,m)$, and Theorems 3.4.10 and 3.1.11 give the precise value for $A(n,m)$ and an upper bound of $B(n,m)$.

# Chapter 4

# Diameter, minimum and maximum degree and hyperbolicity constant

Let $\mathcal{H}(n, \delta_0)$ be the family of graphs $G$ of order $n$ and minimum degree $\delta_0$. Similarly, let $\mathcal{J}(n, \Delta)$ be the family of graphs $G$ of order $n$ and maximum degree $\Delta$.

Our goal in this Chapter is to estimate

$$a(n, \delta_0) := \min\{\delta(G) \mid G \in \mathcal{H}(n, \delta_0)\},$$

$$b(n, \delta_0) := \max\{\delta(G) \mid G \in \mathcal{H}(n, \delta_0)\},$$

$$\alpha(n, \Delta) := \min\{\delta(G) \mid G \in \mathcal{J}(n, \Delta)\},$$

$$\beta(n, \Delta) := \max\{\delta(G) \mid G \in \mathcal{J}(n, \Delta)\}.$$

## 4.1 On a classical Theorem on the diameter and minimum degree

In the design of communication networks, it is common to take into account limitations on the vertex degrees and the diameter. In this context, the degree of a vertex represents the number of the connections attached to a node, while the diameter of the vertices of the graph indicates the largest number of links that must be traversed in order to transmit a message between any two nodes. That is, the maximal message delay index, in the network, is expressed in terms of the diameter of the vertices of the graph.

Diameter-related problems arise in network optimization as early as in 1962, when Erdös and Rényi asked the problem of scheduling airplanes flights between $n$ cities so that it is possible to fly from any one city to another with only a few intermediate stopovers along the way, subject to capacity constraints on the airports (see [48]).

The so called degree/diameter problem is to determine the largest graphs of given maximum degree and given diameter, that is, given natural numbers $\Delta$ and $D$, to find the largest possible number of vertices $n$ in a graph of maximum degree $\Delta$ and diameter $D$.

Throughout the years these kind of problems have attracted the attention of many researchers. There are several different versions of the degree/diameter problem and they have numerous applications; for instance, diameter-related problems often arise in connection with analyzing the computational complexity of routing, distributing and scheduling algorithms. Mirka Miller and Jozef Sirán [84] give an overview on results related to this topic.

A particularly important parameter in networks is the reliability of the network: it is desirable that if some stations (respectively, branches) are unable to work, the message can still be always transmitted. This corresponds to the connectivity (respectively, edge-connectivity) of the associated graph. It is well-known that the connectivity is less than or equal to the edge-connectivity, which is less than or equal to the minimum degree of the graph.

Thus, it is interesting to obtain inequalities relating the minimum degree with other parameters of the graph, such as the order and the diameter. Seidman [101] gives an upper bound for the diameter of a connected graph in terms of its number of vertices, minimum degree and connectivity. Earlier results in this direction were also obtained by Watkins [112] and Kramer [74]. Fiol [53] considers the relation between connectivity and other parameters of a graph G, namely, its order, minimum degree, maximum degree, diameter, and girth. See [84] for more examples of papers on this subject.

In this section, we focus on obtaining good bounds for the diameter in terms of the order and minimum degree of a graph.

The following result gives an asymptotically sharp upper bound for the diameter of a connected graph (see [47, Theorem 1]).

**Theorem 4.1.1** (Erdös, Pach, Pollack and Tuza). *Let $G \in \mathcal{H}(n, \delta_0)$ with $\delta_0 \geq 2$. Then*

$$\operatorname{diam} V(G) \leq \left\lfloor \frac{3n}{\delta_0 + 1} \right\rfloor - 1.$$

The next results provide better estimations of $\operatorname{diam} V(G)$.

**Theorem 4.1.2.** *If $G \in \mathcal{H}(n, \delta_0)$, then $\operatorname{diam} V(G) \leq n - 1$ if $\delta_0 = 1$, and*

$$\operatorname{diam} V(G) \leq \max \left\{ 2, \left\lfloor \frac{3n - 4}{\delta_0 + 1} \right\rfloor - 1 \right\}$$

*for every $\delta_0 \geq 2$.*

*Proof.* The inequality for $\delta_0 = 1$ is direct. Assume now that $\delta_0 \geq 2$. Consider a graph $G \in \mathcal{H}(n, \delta_0)$ such that $\operatorname{diam} V(G) = r$ and choose $x, y \in V(G)$ with $d(x, y) = r$. We can assume $r \geq 2$, since otherwise the inequality holds. Denote by $k_j$ the cardinal of $S_j := \{w \in V(G) : \ d(w, x) = j\}$ $(0 \leq j \leq r)$. Note that a vertex of $S_j$ and a vertex of $S_0 \cup S_1 \cup \cdots \cup S_{j-2}$ can not be neighbors for $2 \leq j \leq r$. Clearly, $\sum_{j=0}^{r} k_j = n$. Since the minimum degree of $G$ is $\delta_0$, we have $k_0 = 1$, $k_1 \geq \delta_0$ and $k_j \geq 1$ $(2 \leq j \leq r)$.

Let us define $k_{r+1} := 0$ and

$$\Lambda_r := \sum_{j=3}^{r} \big( k_{j-1} + k_j + k_{j+1} \big) = \begin{cases} k_2 + 2k_3 + 3 \sum_{j=4}^{r-1} k_j + 2k_r, & \text{if } r > 4, \\ k_2 + 2k_3 + 2k_4, & \text{if } r = 4. \end{cases}$$

Note that $k_{j-1} + k_j + k_{j+1} \geq \delta_0 + 1$ $(3 \leq j \leq r)$. Therefore, summing up these inequalities for $3 \leq j \leq r$, we obtain $\Lambda_r \geq (r - 2)(\delta_0 + 1)$. Note that

$$3n = 3 + 3k_1 + 2k_2 + k_3 + k_r + \Lambda_r \geq 3 + 3\delta_0 + 2 + 1 + 1 + (r - 2)(\delta_0 + 1) = (r + 1)(\delta_0 + 1) + 4.$$

If $r = 3$, then $n = 1 + k_1 + k_2 + k_3 \geq 2(\delta_0 + 1)$, since $k_1 \geq \delta_0$ and $k_2 + k_3 \geq \delta_0 + 1$, and $3n \geq 6(1 + \delta_0) = 4(\delta_0 + 1) + 2(\delta_0 + 1) \geq 4(\delta_0 + 1) + 4$.

We conclude that if $r \geq 3$, then $(3n - 4)/(\delta_0 + 1) \geq r + 1$ and $\operatorname{diam} V(G) \leq \lfloor \frac{3n-4}{\delta_0+1} \rfloor - 1$. Hence,

$$\operatorname{diam} V(G) \leq \max \left\{ 2, \left\lfloor \frac{3n - 4}{\delta_0 + 1} \right\rfloor - 1 \right\}.$$

□

**Remark 4.1.3.** *Since*

$$\left\lfloor \frac{3n}{\delta_0 + 1} \right\rfloor - 1 \geq \left\lfloor \frac{3n}{n - 1 + 1} \right\rfloor - 1 = 2 \quad \text{and} \quad \left\lfloor \frac{3n}{\delta_0 + 1} \right\rfloor - 1 \geq \left\lfloor \frac{3n - 4}{\delta_0 + 1} \right\rfloor - 1,$$

*Theorem 4.1.2 improves the bound in Theorem 4.1.1.*

The argument in the proof of Theorem 4.1.2 allows to prove also the following result.

**Theorem 4.1.4.** *Consider $G \in \mathcal{H}(n, \delta_0)$.*

(1) *If $G$ does not have cut-vertices, then*

$$\operatorname{diam} V(G) = 1 \quad \text{if } \delta_0 = 1, \qquad \text{and} \qquad \operatorname{diam} V(G) \le \max\left\{3, \left\lfloor \frac{3n-7}{\delta_0+1} \right\rfloor - 1\right\} \quad \text{if } \delta_0 \ge 2.$$

(2) *If $G$ has some cut-vertex, then*

$$\operatorname{diameff} V(G) \le \left\lfloor \frac{n-1}{2} \right\rfloor \quad \text{if } \delta_0 = 1,$$

$$\operatorname{diameff} V(G) \le \max\left\{3, \left\lfloor \frac{3n-9}{\delta_0+1} \right\rfloor - 2\right\} \quad \text{if } \delta_0 \ge 2.$$

(3) *We have in any case*

$$\operatorname{diameff} V(G) \le \max\left\{1, \left\lfloor \frac{n-1}{2} \right\rfloor\right\} \quad \text{if } \delta_0 = 1,$$

$$\operatorname{diameff} V(G) \le \max\left\{3, \left\lfloor \frac{3n-7}{\delta_0+1} \right\rfloor - 1\right\} \quad \text{if } \delta_0 \ge 2.$$

*Proof.* The inequalities for $\delta_0 = 1$ are not difficult. Let us consider $G \in \mathcal{H}(n, 1)$. If $G$ does not have cut-vertices, then $G$ has just one edge and so $\operatorname{diam} V(G) = 1$. If $G$ has some cut-vertex and $n < 4$, then $G$ is a tree and $\operatorname{diameff} V(G) = 1$. If $G$ has some cut-vertex and $n \ge 4$, then one check that the extremal case for $G$ is any graph obtained by attaching an edge to some vertex of a cycle graph $C_{n-1}$, and so $\operatorname{diameff} V(G) \le \operatorname{diam} V(C_{n-1}) = \lfloor (n-1)/2 \rfloor$. The third statement follows from the previous inequality and the first equality.

Consider now a graph $G \in \mathcal{H}(n, \delta_0)$ with $\delta_0 \ge 2$.

Assume first that $G$ does not have cut-vertices. Thus, $\operatorname{diam} V(G) = \operatorname{diameff} V(G)$. Define $r := \operatorname{diam} V(G)$ and choose $x, y \in V(G)$ with $d(x, y) = r$. We can assume $r \ge 4$, since otherwise the equalities hold. Using the same notation as in the proof of Theorem 4.1.2, we have $k_0 = 1, k_1 \ge \delta_0, k_r \ge 1$ and $k_j \ge 2$ for $1 < j < r$. Note that

$$3n = 3 + 3k_1 + 2k_2 + k_3 + k_r + \Lambda_r \ge 3 + 3\delta_0 + 4 + 2 + 1 + (r-2)(\delta_0+1) = (r+1)(\delta_0+1) + 7,$$

and we have the first inequality.

Now, assume that $G$ has some cut-vertex and consider the biconnected decomposition of $G$. Then there exists a subgraph $G_s$ in this biconnected decomposition with $\operatorname{diam} V(G_s) = \operatorname{diameff} V(G)$. Define $r := \operatorname{diameff} V(G)$ and consider $x, y \in V(G_s)$ with $d_{G_s}(x, y) = r$. Without loss of generality, assume $r \ge 4$. Let $v_0$ be any cut-vertex of $G$ contained in $G_s$. Consider $z \in N(v_0) \setminus V(G_s)$. Since $v_0$ is a cut-vertex, $N(z) \cap V(G_s) = \{v_0\}$. Denote by

$G^{v_0}$ the connected component of $(G \setminus G_s) \cup \{v_0\}$ containing $v_0$; thus, $|V(G^{v_0}) \setminus V(G_s)| \geq |\{z\} \cup N(z) \setminus \{v_0\}| \geq \delta_0$. If $v_1$ is another cut-vertex of $G$ contained in $G_s$, then the same argument gives $|V(G^{v_1}) \setminus V(G_s)| \geq \delta_0$; we also have $V(G^{v_0}) \cap V(G^{v_1}) = \emptyset$ since $G_s$ is a subgraph in the biconnected decomposition.

Let us denote by $k_{j,s}$ the cardinal of $S_{j,s} := \{w \in V(G_s) : \ d_{G_s}(w,x) = j\}$; obviously, $k_{-1,s} = k_{r+1,s} = 0$. Denote by $k'_{j,s}$ the sum of $k_{j,s}$ and the cardinal of the vertices in the subgraphs $G^v \setminus \{v\}$ with $v$ a cut-vertex of $G$ and $v \in S_{j,s}$; thus, $k'_{-1,s} = k'_{r+1,s} = 0$. Since $G_s$ does not have cut-vertices, we have $k'_{j,s} \geq k_{j,s} \geq 2$ for $0 < j < r$. Since the minimum degree of $G$ is $\delta_0$, we obtain $k'_{j-1,s} + k'_{j,s} + k'_{j+1,s} \geq k_{j-1,s} + k'_{j,s} + k_{j+1,s} \geq \delta_0 + 1$ for every $0 \leq j \leq r$. Since we consider the biconnected decomposition of $G$, we have $n = \sum_{j=0}^{r} k'_{j,s}$.

Case (A). Assume that $x$ and some vertex in $S_{r,s}$ are both cut-vertices of $G$. Thus, $k'_{0,s}, k'_{r,s} \geq \delta_0 + 1$ and

$$\begin{aligned} 3n = \sum_{j=0}^{r} 3k'_{j,s} &= 3k'_{0,s} + 2k'_{1,s} + k'_{2,s} + \sum_{j=2}^{r-2} \left(k'_{j-1,s} + k'_{j,s} + k'_{j+1,s}\right) + k'_{r-2,s} + 2k'_{r-1,s} + 3k'_{r,s} \\ &\geq 3(\delta_0+1) + 4 + 2 + (r-3)(\delta_0+1) + 2 + 4 + 3(\delta_0+1) = (r+3)(\delta_0+1) + 12, \end{aligned}$$

and we conclude

$$\left\lfloor \frac{3n-9}{\delta_0+1} \right\rfloor - 2 \geq \left\lfloor \frac{3n-12}{\delta_0+1} \right\rfloor - 3 \geq r = \operatorname{diameff} V(G).$$

Case (B). Assume that $x$ is not a cut-vertex of $G$ and some vertex in $S_{r,s}$ is a cut-vertex of $G$. Thus, $k'_{0,s} + k'_{1,s} = 1 + k'_{1,s} \geq \delta_0 + 1$ and $k'_{r,s} \geq \delta_0 + 1$

$$\begin{aligned} 3n = \sum_{j=0}^{r} 3k'_{j,s} &= k'_{0,s} + 2(k'_{0,s} + k'_{1,s}) + k'_{2,s} + \sum_{j=2}^{r-2} \left(k'_{j-1,s} + k'_{j,s} + k'_{j+1,s}\right) + k'_{r-2,s} + 2k'_{r-1,s} + 3k'_{r,s} \\ &\geq 1 + 2(\delta_0+1) + 2 + (r-3)(\delta_0+1) + 2 + 4 + 3(\delta_0+1) = (r+2)(\delta_0+1) + 9, \end{aligned}$$

and we conclude

$$\left\lfloor \frac{3n-9}{\delta_0+1} \right\rfloor - 2 \geq r = \operatorname{diameff} V(G).$$

Case (C). If $x$ is a cut-vertex of $G$ and the vertices in $S_{r,s}$ are not cut-vertices of $G$, then a symmetric argument to the one in Case (B) gives the same inequality.

Case (D). Assume that $x$ and the vertices in $S_{r,s}$ are not cut-vertices of $G$.

Case (D.1). Assume that there exist $0 < j < r$ and $w_1, w_2 \in S_{j,s}$ such that $w_1$ is not a cut-vertex of $G$ and $w_2$ is a cut-vertex of $G$. Since $w_1$ is not a cut-vertex, $k_{j-1,s} + k_{j,s} + k_{j+1,s} \geq \delta_0 + 1$. Since $w_2$ is a cut-vertex, $k'_{j,s} - k_{j+1,s} \geq \delta_0$. Then the argument in the proof of (1) gives $3n \geq (r+1)(\delta_0+1) + 7 + 3(k'_{j,s} - k_{j+1,s}) \geq (r+1)(\delta_0+1) + 7 + 3\delta_0 = (r+4)(\delta_0+1) + 4$ and, since $\delta_0 \geq 2$, we have

$$\left\lfloor \frac{3n-9}{\delta_0+1} \right\rfloor - 2 = \left\lfloor \frac{3n-9}{\delta_0+1} - 2 \right\rfloor \geq \left\lfloor \frac{3n-4}{\delta_0+1} - 4 \right\rfloor = \left\lfloor \frac{3n-4}{\delta_0+1} \right\rfloor - 4 \geq r = \operatorname{diameff} V(G).$$

Case (D.2). If the hypothesis in (D.1) does not hold, then every vertex in $\cup_{0<j<r} S_{j,s}$ is a cut-vertex of $G$. Therefore, $k'_{j,s} \geq k_{j,s} + k_{j,s}\delta_0 \geq 2(\delta_0 + 1)$ for every $0 < j < r$, and so $n = \sum_{j=0}^{r} k'_{j,s} \geq 1 + 2(r-1)(\delta_0 + 1) + 1$. Since $r \geq 4$, we have $n \geq 2 + 6(\delta_0 + 1) = 6\delta_0 + 8$, $5n \geq 6\delta_0 + 22$, $6n - 18 \geq n - 2 + 6(\delta_0 + 1)$ and

$$\left\lfloor \frac{3n-9}{\delta_0+1} \right\rfloor - 2 = \left\lfloor \frac{3n-9}{\delta_0+1} - 2 \right\rfloor \geq \left\lfloor \frac{n-2}{2(\delta_0+1)} + 1 \right\rfloor = \left\lfloor \frac{n-2}{2(\delta_0+1)} \right\rfloor + 1 \geq r = \operatorname{diameff} V(G).$$

Hence, if $G$ has some cut-vertex and $\delta_0 \geq 2$, we have

$$\operatorname{diameff} V(G) \leq \max\left\{3, \left\lfloor \frac{3n-9}{\delta_0+1} \right\rfloor - 2\right\}.$$

The third statement follows from the first two inequalities, since

$$\left\lfloor \frac{3n-7}{\delta_0+1} \right\rfloor - 1 \geq \left\lfloor \frac{3n-9}{\delta_0+1} \right\rfloor - 2.$$

□

Given any graph $G$ and any $v \in V(G)$, let us denote by $N(v)$ the set of neighbors of $v$.

The next results provide better estimations of $\operatorname{diam} V(G)$ for some values of $\delta_0$. They will be useful in the next sections.

**Theorem 4.1.5.** *Let $G \in \mathcal{H}(n, \delta_0)$. If $\delta_0 \geq (n-1)/2$, then* $\operatorname{diam} V(G) \leq 2$.

*Proof.* Let $x$ and $y$ be two vertices in $V(G)$ with $d(x,y) = \operatorname{diam} V(G)$. If $d(x,y) = 1$, then $\operatorname{diam} V(G) = 1$. Assume that $d(x,y) \geq 2$. Let $X := N(x)$ and $Y := N(y)$. If $X$ and $Y$ are disjoint, then

$$n \geq 2 + |X| + |Y| \geq 2 + 2\delta_0 \geq 2 + n - 1 = n + 1,$$

which is a contradiction. Therefore, there exists $v \in X \cap Y$ and so $2 = d(x,y) = \operatorname{diam} V(G)$. □

**Remark 4.1.6.** *If $\delta_0 = (n-1)/2$, then $\left\lfloor \frac{3n-4}{\delta_0+1} \right\rfloor - 1 = \left\lfloor \frac{6n-8}{n+1} \right\rfloor - 1$. Note that $2 \leq \frac{6n-8}{n+1} - 2 < \left\lfloor \frac{6n-8}{n+1} \right\rfloor - 1$ for all $n \geq 6$. Therefore, Theorem 4.1.5 improves the upper bound of* $\operatorname{diam} V(G)$ *in Theorem 4.1.2 for this $\delta_0$ and infinitely many values of $n$.*

**Theorem 4.1.7.** *Let $G \in \mathcal{H}(n, \delta_0)$. If $\delta_0 \geq (n-2)/2$, then* $\operatorname{diam} V(G) \leq 3$.

*Proof.* Let $x$ and $y$ be two vertices in $V(G)$ with $d(x,y) = \operatorname{diam} V(G)$. We can assume that $d(x,y) \geq 3$. Let $X := N(x)$, $Y := N(y)$ and $B := V(G) \setminus (\{x\} \cup X \cup Y \cup \{y\})$.

Since $d(x,y) \geq 3$, we have that $X$ and $Y$ are disjoint. Thus,

$$n = 2 + |X| + |B| + |Y| \geq 2 + 2\delta_0 + |B| \geq 2 + n - 2 + |B| = n + |B|.$$

Therefore, $|B| = 0$ and $B = \emptyset$. Since $G$ is connected, $d(X,Y) = 1$ and $d(x,y) = 3$. □

**Remark 4.1.8.** *If $\delta_0 = (n-2)/2$, then $\lfloor \frac{3n-4}{\delta_0+1} \rfloor - 1 = \lfloor \frac{6n-8}{n} \rfloor - 1$. Note that $3 \le \frac{6n-8}{n} - 2 < \lfloor \frac{6n-8}{n} \rfloor - 1$ for all $n \ge 8$. Therefore, Theorem 4.1.7 improves the upper bound of* $\operatorname{diam} V(G)$ *in Theorem 4.1.2 for this $\delta_0$ and infinitely many values of $n$.*

**Theorem 4.1.9.** *Let $G \in \mathcal{H}(n, \delta_0)$. If $\delta_0 \ge (n-3)/2$, then*

$$\operatorname{diam} V(G) \le 4 \quad \textit{and} \quad \operatorname{diameff} V(G) \le 3.$$

*Furthermore, if* $\operatorname{diam} V(G) = 4$, *then* $\operatorname{diameff}(G) \le 3$.

*Proof.* Let $x$ and $y$ be two vertices in $V(G)$ with $d(x,y) = \operatorname{diam} V(G)$. We can assume that $d(x,y) \ge 3$. Let $X := N(x)$, $Y := N(y)$ and $B := V(G) \setminus (\{x\} \cup X \cup Y \cup \{y\})$.

Since $d(x,y) \ge 3$, we have that $X$ and $Y$ are disjoint. Thus,

$$n = 2 + |X| + |B| + |Y| \ge 2 + 2\delta_0 + |B| \ge 2 + n - 3 + |B| = n - 1 + |B|. \tag{4.1}$$

Therefore, $|B| \le 1$.

If $d(X,Y) = 1$, then $d(x,y) = 3$ and $\operatorname{diameff} V(G) \le \operatorname{diam} V(G) = 3$.

Since $|B| \le 1$, if $d(X,Y) \ge 2$, then $d(X,Y) = 2$ and $|B| = 1$, and so $d(x,y) = 4$. Let $B = \{p\}$. We just need to prove that $\operatorname{diameff}(G) \le 3$ in this case. Let us start by proving $\operatorname{diameff} V(G) \le 3$. Note that $p \in B$ is a cut-vertex and (4.1) gives $|X| = |Y| = \delta_0 = (n-3)/2$. Consider the connected subgraphs $G_1, G_2$ of $G$ such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = \{p\}$ (thus, $\{G_1, G_2\}$ is a T-decomposition of $G$). Note that $d(x,p) = 2$ and $d(y,p) = 2$. By symmetry, we can assume that $x \in G_1$, and thus $V(G_1) = \{x\} \cup X \cup \{p\}$. If $\alpha, \beta \in X$, then

$$\begin{aligned} d(\alpha,\beta) &= d_{G_1}(\alpha,\beta) \le d_{G_1}(\alpha,x) + d_{G_1}(x,\beta) = 2, \\ d(\alpha,p) &= d_{G_1}(\alpha,p) \le d_{G_1}(\alpha,x) + d_{G_1}(x,p) = 3. \end{aligned}$$

Therefore, $\operatorname{diam} V(G_1) \le 3$. If $\alpha \in X \cap N(p)$, then $d_{G_1}(\alpha, w) \le 2$ for every $w \in V(G_1)$. If $\alpha \in X \setminus N(p)$, then $N(\alpha) = \{x\} \cup X \setminus \{\alpha\}$, since $|X| = \delta_0$, and we also have $d_{G_1}(\alpha, w) \le 2$ for every $w \in V(G_1)$. Since every edge of $G_1$ is adjacent to at least a vertex in $X$, if $q$ is the midpoint of some edge of $G_1$, then $d_{G_1}(q, w) \le 5/2$ for every $w \in V(G_1)$; therefore, $d_{G_1}(q, z) \le 3$ for every $z \in G_1$, and we conclude $\operatorname{diam}(G_1) \le 3$.

We can prove in a similar way that $\operatorname{diam}(G_2) \le 3$, and thus $\operatorname{diameff}(G) \le 3$. □

**Remark 4.1.10.** *If $\delta_0 = (n-3)/2$, then $\lfloor \frac{3n-4}{\delta_0+1} \rfloor - 1 = \lfloor \frac{6n-8}{n-1} \rfloor - 1$. Note that $\lfloor \frac{6n-8}{n-1} \rfloor - 1 \ge \lfloor \frac{6n-8}{n} \rfloor - 1 \ge 4$ for $n \ge 8$.*

**Theorem 4.1.11.** *Let $G \in \mathcal{H}(n, \delta_0)$. If $\delta_0 \ge (n-4)/2$, then*

$$\operatorname{diam} V(G) \le 5 \quad \textit{and} \quad \operatorname{diameff} V(G) \le 4.$$

*Furthermore, if* $\operatorname{diam} V(G) = 5$, *then* $\operatorname{diameff}(G) \le 3$.

*Proof.* Let $x$ and $y$ be two vertices in $V(G)$ with $d(x,y) = \operatorname{diam} V(G)$. We can assume that $d(x,y) \geq 3$. Let $X := N(x)$, $Y := N(y)$ and $B := V(G) \setminus (\{x\} \cup X \cup Y \cup \{y\})$.

Since $d(x,y) \geq 3$, we have that $X$ and $Y$ are disjoint. Thus,

$$n = 2 + |X| + |B| + |Y| \geq 2 + 2\delta_0 + |B| \geq 2 + n - 4 + |B| = n - 2 + |B|. \tag{4.2}$$

Therefore, $|B| \leq 2$.

Case (A): First, assume that $d(X,Y) \leq 2$. Then, $d(x,y) \leq 4$ and $\operatorname{diameff} V(G) \leq \operatorname{diam} V(G) \leq 4$.

Case (B): Assume now that $d(X,Y) \geq 3$. Since $|B| \leq 2$, we have $d(X,Y) = 3$ and $|B| = 2$. Let $B = \{s,t\}$. Since the graph is connected, $[s,t] \in E(G)$, $s$ and $t$ are cut-vertices and $d(x,y) = 5$. Note that (4.2) gives $|X| = |Y| = \delta_0 = (n-4)/2$. Consider the family of connected subgraphs $\{G_1, G_2, G_3\}$ of $G$ such that $G_2 = \{[s,t]\}$, $G_1 \cup G_2 \cup G_3 = G$, $G_1 \cap G_2 = \{s\}$, $G_2 \cap G_3 = \{t\}$ and $G_1 \cap G_3 = \emptyset$ (thus, $\{G_1, G_2, G_3\}$ is a T-decomposition of $G$).

Using a similar argument to the one in the proof of Theorem 4.1.9, we obtain that $\operatorname{diam}(G_1) \leq 3$ and $\operatorname{diam}(G_3) \leq 3$. Since $\operatorname{diam}(G_2) = 1$, we have $\operatorname{diameff}(G) \leq 3$. □

**Remark 4.1.12.** *If $\delta_0 = (n-4)/2$, then $\left\lfloor \frac{3n-4}{\delta_0+1} \right\rfloor - 1 = \left\lfloor \frac{6n-8}{n-2} \right\rfloor - 1$. Note that $\left\lfloor \frac{6n-8}{n-2} \right\rfloor - 1 \geq 5$ for $n \geq 8$.*

**Theorem 4.1.13.** *Let $G \in \mathcal{H}(n, \delta_0)$. If $\delta_0 \geq (n-5)/2$, then*

$$\operatorname{diam} V(G) \leq 6 \quad \textit{and} \quad \operatorname{diameff} V(G) \leq 4.$$

*Furthermore, if* $\operatorname{diam} V(G) = 6$*, then* $\operatorname{diameff} V(G) \leq 3$.

*Proof.* Let $x$ and $y$ be two vertices in $V(G)$ with $d(x,y) = \operatorname{diam} V(G)$. We can assume that $d(x,y) \geq 3$. Let $X := N(x)$, $Y := N(y)$ and $B := V(G) \setminus (\{x\} \cup X \cup Y \cup \{y\})$.

Since $d(x,y) \geq 3$, we have that $X$ and $Y$ are disjoint. Thus,

$$n = 2 + |X| + |B| + |Y| \geq 2 + n - 5 + |B| = n - 3 + |B|.$$

Therefore, $|B| \leq 3$.

Case (A): First, assume that $d(X,Y) \leq 2$. Then, $d(x,y) \leq 4$ and $\operatorname{diameff} V(G) \leq \operatorname{diam} V(G) \leq 4$.

Case (B): Now, assume that $d(X,Y) \geq 4$. Then, $|B| = 3$, $d(X,Y) = 4$ and $d(x,y) = 6$. Let $B = \{r,s,t\}$. By symmetry, we can assume that $d(X,r) = 1 = d(Y,t)$ and so $[r,t] \notin E(G)$. Then, since the graph is connected, $[s,t], [s,r] \in E(G)$.

In this case, $r$, $s$ and $t$ are cut-vertices. Consider the family of connected subgraphs $\{G_1, G_2, G_3, G_4\}$ of $G$ such that $G_2 = \{[r,s]\}$, $G_3 = \{[s,t]\}$, $\cup_i G_i = G$, $G_1 \cap G_2 = \{r\}$, $G_3 \cap G_4 = \{t\}$ and $G_1 \cap G_3 = G_1 \cap G_4 = G_2 \cap G_4 = \emptyset$ (thus, $\{G_1, G_2, G_3, G_4\}$ is a T-decomposition of $G$). We can assume $x \in V(G_1)$ and $y \in V(G_4)$. If $\alpha, \beta \in X$, then

$$d(\alpha, \beta) = d_{G_1}(\alpha, \beta) \leq d_{G_1}(\alpha, x) + d_{G_1}(x, \beta) = 2,$$

$$d(\alpha, r) = d_{G_1}(\alpha, r) \le d_{G_1}(\alpha, x) + d_{G_1}(x, r) = 3.$$

Therefore, $\operatorname{diam} V(G_1) \le 3$. We can prove in a similar way that $\operatorname{diam} V(G_4) \le 3$. Since $\operatorname{diam} V(G_2) = \operatorname{diam} V(G_3) = 1$, we have $\operatorname{diameff} V(G) \le 3$.

Case (C): Finally, assume that $d(X, Y) = 3$. Then, $|B| \ge 2$ and $d(x, y) = 5$. Assume that $|B| = 3$ and let $B = \{r, s, t\}$ (if $|B| = 2$, then the argument is simpler).

If $\delta_0 \le 2$, then $2 \ge \delta_0 \ge (n-5)/2$ and $n \le 9$; hence, $\operatorname{diameff} V(G) \le 4$. Therefore, we can assume $\delta_0 \ge 3$. Since $\delta_0 \ge 3$, by symmetry, we can assume that $x \in V(G_1)$, $y \in V(G_2)$, both $r$, $s$ have neighbors in $X$, $t$ has neighbors in $Y$ and $[r, t] \in E(G_1)$.

In this case, $t$ is a cut-vertex and $|X| = |Y| = \delta_0 = (n-5)/2$. Consider the family of connected subgraphs $\{G_1, G_2\}$ of $G$ such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = \{t\}$ (thus, $\{G_1, G_2\}$ is a T-decomposition of $G$).

If $\alpha, \beta \in X$ and $\omega \in \{r, s\}$, then

$$\begin{aligned} d(\alpha, \beta) &= d_{G_1}(\alpha, \beta) \le d_{G_1}(\alpha, x) + d_{G_1}(x, \beta) = 2, \\ d(\alpha, t) &= d_{G_1}(\alpha, t) \le d_{G_1}(\alpha, x) + d_{G_1}(x, t) = 4, \\ d(\alpha, \omega) &= d_{G_1}(\alpha, \omega) \le d_{G_1}(\alpha, x) + d_{G_1}(x, \omega) = 3, \\ d(r, s) &= d_{G_1}(r, s) \le d_{G_1}(r, x) + d_{G_1}(x, s) = 4. \end{aligned}$$

Thus $\deg(s) \ge \delta_0 = |X|$ and we conclude $s$ is either a neighbor of all vertices in $X$ or a neighbor of $r$ or $t$. Thus, $d(s, t) \le 3$ and $\operatorname{diam} V(G_1) \le 4$. We can prove in a similar way that $\operatorname{diam} V(G_2) \le 3$. Hence, we have $\operatorname{diameff} V(G) \le 4$. □

**Remark 4.1.14.** *If $\delta_0 = (n-5)/2$, then $\left\lfloor \frac{3n-4}{\delta_0+1} \right\rfloor - 1 = \left\lfloor \frac{6n-8}{n-3} \right\rfloor - 1$. Note that $\frac{6n-8}{n-3} \ge 7$ if $n \le 13$. Hence, $\left\lfloor \frac{6n-8}{n-3} \right\rfloor - 1 \ge 6$ for every $n \le 13$.*

**Theorem 4.1.15.** *Let $G \in \mathcal{H}(n, \delta_0)$. If $\delta_0 \ge (n-2)/3$, then* $\operatorname{diam} V(G) \le 5$.

*Proof.* Let $x$ and $y$ be two vertices in $V(G)$ with $d(x, y) = \operatorname{diam} V(G)$. We can assume that $d(x, y) \ge 5$. Let $X := N(x)$ and $Y := N(y)$. Let $A$ and $C$ be the sets of vertices at distance 2 from $x$ and $y$, respectively, and $B$ the set $B := V(G) \setminus (\{x\} \cup X \cup A \cup C \cup Y \cup \{y\})$.

Since $d(x, y) \ge 5$, we have that $X, A, C, Y$ are pairwise disjoint. Thus,

$$n = 2 + |X| + |A| + |B| + |C| + |Y| \ge 2 + 2\delta_0 + |A| + |B| + |C|.$$

If $|A| + |B| + |C| > \delta_0$, then the following holds

$$n - 2 - 2\delta_0 \ge |A| + |B| + |C| > \delta_0 \quad \Rightarrow \quad \frac{n-2}{3} > \delta_0,$$

which is a contradiction. Therefore, $|A| + |B| + |C| \le \delta_0$.

Seeking for a contradiction, assume that $B \ne \emptyset$ and consider $p$ in $B$. Since $p$ has no neighbors in $\{x\} \cup X \cup Y \cup \{y\}$, we have

$$\delta_0 \le \deg(p) \le |B| - 1 + |A| + |C| \le \delta_0 - 1 < \delta_0.$$

This contradiction implies, $B = \emptyset$. Since $G$ is connected, $d(A, C) = 1$ and $d(x, y) = 5$. □

**Remark 4.1.16.** *If* $\delta_0 = (n-2)/3$, *then* $\lfloor \frac{3n-4}{\delta_0+1} \rfloor - 1 = \lfloor \frac{9n-12}{n+1} \rfloor - 1$. *Note that* $5 \leq \lfloor \frac{9n-12}{n+1} \rfloor - 1$ *for all* $n \geq 6$.

**Theorem 4.1.17.** *Let* $G \in \mathcal{H}(n, \delta_0)$. *If* $\delta_0 \geq (n-3)/3$, *then* $\operatorname{diam} V(G) \leq 6$.

*Proof.* Let $x$ and $y$ be two vertices in $V(G)$ with $d(x,y) = \operatorname{diam} V(G)$. We can assume that $d(x,y) \geq 5$. Let $X := N(x)$ and $Y := N(y)$. Let $A$ and $C$ be the sets of vertices at distance 2 from $x$ and $y$, respectively, and $B$ the set $B := V(G) \setminus (\{x\} \cup X \cup A \cup C \cup Y \cup \{y\})$.

Since $d(x,y) \geq 5$, we have that $X, A, C, Y$ are pairwise disjoint. Thus,

$$n = 2 + |X| + |A| + |B| + |C| + |Y| \geq 2 + 2\delta_0 + |A| + |B| + |C|.$$

Therefore, $|A| + |B| + |C| \leq n - 2 - 2\delta_0$.

Since $G$ is connected, if $B = \emptyset$, then $d(A, C) = 1$ and $d(x,y) = 5$.

Now, assume that $B \neq \emptyset$ and consider $p$ in $B$. Since $p$ has no neighbors in $\{x\} \cup X \cup Y \cup \{y\}$, we have

$$\delta_0 \leq \deg(p) \leq |B| - 1 + |A| + |C| \leq n - 3 - 2\delta_0 \quad \Rightarrow \quad \delta_0 \leq \frac{n-3}{3},$$

which implies $\delta_0 = (n-3)/3$. Therefore, $\deg(p) = |B| - 1 + |A| + |C|$ and for any $p \in B$, $v \in A$ and $w \in C$ we have $[v,p] \in E(G)$ and $[w,p] \in E(G)$. Thus, $d(A,C) = 2$ and $d(x,y) = 6$. $\square$

**Remark 4.1.18.** *If* $\delta_0 = (n-3)/3$, *then* $\lfloor \frac{3n-4}{\delta_0+1} \rfloor - 1 = \lfloor \frac{9n-12}{n} \rfloor - 1$. *Note that* $6 \leq \lfloor \frac{9n-12}{n} \rfloor - 1$ *for all* $n \geq 6$.

## 4.2 Upper bounds for $b(n, \delta_0)$

Theorems 4.1.4 and 2.4.2, provide the following upper bound for $b(n, \delta_0)$.

**Theorem 4.2.1.** *For every $n$ and $\delta_0 \geq 2$ we have*

$$b(n, \delta_0) \leq \max \left\{ 2, \frac{1}{2} \left\lfloor \frac{3n-7}{\delta_0+1} \right\rfloor \right\}.$$

In this section we obatin further upper bounds for $b(n, \delta_0)$. Since the proofs of these bounds are long, in order to make the arguments more transparent, we collect some results in technical lemmas and propositions.

**Lemma 4.2.2.** *Let $G$ be a graph with minimum degree $\delta_0$, $T = \{x, y, z\}$ a geodesic triangle that is a cycle in $G$ and $p \in [xy]$ such that $L([xy]) \geq 3$, $d(p, [xz] \cup [zy]) = \delta(T) = \delta(G)$ and $x, y, z \in J(G)$. Assume that $x$ and $y$ are the midpoints of $[x_1, x_2]$ and $[y_1, y_2]$, respectively, with $x_1, x_2, y_1, y_2 \in V(G)$.*

- *If $\delta(G) = L([xy])/2$, then $\left|N(x_1) \cup N(x_2) \setminus \{x_1, x_2\}\right| \geq \delta_0$ and $\left|N(y_1) \cup N(y_2) \setminus \{y_1, y_2\}\right| \geq \delta_0$.*
- *If $\delta(G) = L([xy])/2 - 1/4$, then $\left|N(x_1) \cup N(x_2) \setminus \{x_1, x_2\}\right| \geq \delta_0$ or $\left|N(y_1) \cup N(y_2) \setminus \{y_1, y_2\}\right| \geq \delta_0$.*
- *If $\delta(G) \geq L([xy])/2 - 1/4$, then $\left|N(x_1) \cup N(x_2) \setminus \{x_1, x_2\}\right| + \left|N(y_1) \cup N(y_2) \setminus \{y_1, y_2\}\right| \geq 2\delta_0 - 1$.*

*Proof.* Define $p_0$ as the midpoint of the geodesic $[xy]$.

Assume first that $\delta(G) = L([xy])/2$. Thus $\delta(G) = L([xy])/2 \geq d(p, \{x, y\}) \geq d(p, [xz] \cup [zy]) = \delta(G)$, and we conclude $L([xy])/2 = d(p, x) = d(p, y) = d(p, [xz] \cup [zy]) = \delta(T) = \delta(G)$ and $p = p_0$.

We have for every $w \in [xp]$ the inequality $d(w, [xz] \cup [zy]) \leq d(w, x)$, and

$$d(x, w) + d(w, p) = d(x, p) = d(p, [xz] \cup [zy]) \leq d(p, w) + d(w, [xz] \cup [zy]). \tag{4.3}$$

Thus $d(x, w) \leq d(w, [xz] \cup [zy])$, and we conclude $d(w, [xz] \cup [zy]) = d(w, x)$ for every $w \in [xp]$.

Define $X_0 := N(x_1) \cup N(x_2) \setminus \{x_1, x_2\}$. Seeking for a contradiction, assume $|X_0| < \delta_0$. Since $\delta_0$ is the minimum degree of $G$, thus $|X_0| \geq |N(x_1) \setminus \{x_2\}| \geq \delta_0 - 1$, and we conclude $|X_0| = \delta_0 - 1$. Note that $N(x_1) \subseteq \{x_2\} \cup X_0$ and $N(x_2) \subseteq \{x_1\} \cup X_0$. Hence,

$$\left|\{x_2\} \cup X_0\right| = \left|\{x_1\} \cup X_0\right| = 1 + \delta_0 - 1 = \delta_0 \quad \Rightarrow \quad N(x_1) = \{x_2\} \cup X_0, \quad N(x_2) = \{x_1\} \cup X_0.$$

Therefore, $X_0 = N(x_1) \setminus \{x_2\} = N(x_2) \setminus \{x_1\}$.

Since $T$ is a cycle, without loss of generality, assume $x_1 \in [xy]$ and $x_2 \in [xz] \cup [yz]$. Let $p_1$ be the point in $[xy]$ with $d(p_1, x) = 3/2$. Note that $p_1 \in [xp]$, since $L([xy]) \geq 3$. We have $d(p_1, x_1) = 1$, and then $p_1 \in N(x_1) \setminus \{x_2\} = N(x_2) \setminus \{x_1\}$. This fact and (4.3) give $3/2 = d(p_1, x) = d(p_1, [xz] \cup [yz]) \leq d(p_1, x_2) = 1$, which is a contradiction. Thus, $|X_0| \geq \delta_0$.

By symmetry, we have $\big|N(y_1) \cup N(y_2) \setminus \{y_1, y_2\}\big| \geq \delta_0$.

Assume now that $\delta(G) = L([xy])/2 - 1/4$. Since

$$d(p, [xz] \cup [zy]) = \min \big\{ d\big(p, \{x, y\}\big), \, d\big(p, ([xz] \cup [zy]) \cap V(G)\big) \big\},$$

$2L([xy]) - 1$ is odd and $d(p, [xz] \cup [zy]) = (2L([xy]) - 1)/4$, we have $d(p, p_0) = 1/4$.

If $p \in [xp_0]$, then the previous argument gives $|N(x_1) \cup N(x_2) \setminus \{x_1, x_2\}| \geq \delta_0$. By symmetry, if $p \in [p_0 y]$, then $\big|N(y_1) \cup N(y_2) \setminus \{y_1, y_2\}\big| \geq \delta_0$.

Finally, assume that $\delta(G) \geq L([xy])/2 - 1/4$. Since $\delta(G) = d(p, [xz] \cup [zy]) \leq d(p, \{x, y\}) \leq L([xy])/2$, by Theorem 2.4.4 we have either $\delta(G) = L([xy])/2$ or $\delta(G) = L([xy])/2 - 1/4$, and the first two items of this lemma give $\big|N(x_1) \cup N(x_2) \setminus \{x_1, x_2\}\big| \geq \delta_0$ or $\big|N(y_1) \cup N(y_2) \setminus \{y_1, y_2\}\big| \geq \delta_0$. Since $\big|N(x_1) \cup N(x_2) \setminus \{x_1, x_2\}\big| \geq \delta_0 - 1$ and $\big|N(y_1) \cup N(y_2) \setminus \{y_1, y_2\}\big| \geq \delta_0 - 1$, we conclude $\big|N(x_1) \cup N(x_2) \setminus \{x_1, x_2\}\big| + \big|N(y_1) \cup N(y_2) \setminus \{y_1, y_2\}\big| \geq 2\delta_0 - 1$. □

**Lemma 4.2.3.** *Let $G$ be a graph with minimum degree $\delta_0$, $\operatorname{diam} V(G) \geq 2$ and $\delta(G) = (\operatorname{diam} V(G) + 1)/2$. Consider any geodesic triangle $T = \{x, y, z\}$ that is a cycle in $G$ and $p \in [xy]$ such that $d(p, [xz] \cup [zy]) = \delta(T) = \delta(G)$ and $x, y, z \in J(G)$. Then $\operatorname{diam}(G) = \operatorname{diam} V(G) + 1 = L([xy])$, $d(p, \{x, y\}) = (\operatorname{diam} V(G) + 1)/2$, $x$ and $y$ are the midpoints of $[x_1, x_2]$ and $[y_1, y_2]$, respectively, with $x_1, x_2, y_1, y_2 \in V(G)$, $d([x_1, x_2], [y_1, y_2]) = \operatorname{diam} V(G)$, $\big|N(x_1) \cup N(x_2) \setminus \{x_1, x_2\}\big| \geq \delta_0$ and $\big|N(y_1) \cup N(y_2) \setminus \{y_1, y_2\}\big| \geq \delta_0$. The same results hold if we replace $\operatorname{diam} V(G)$ and $\operatorname{diam}(G)$ by $\operatorname{diameff} V(G)$ and $\operatorname{diameff}(G)$, respectively.*

*Proof.* Fix any geodesic triangle $T = \{x, y, z\}$ that is a cycle in $G$ and $p \in [xy]$ such that $d(p, [xz] \cup [zy]) = \delta(T) = \delta(G)$ and $x, y, z \in J(G)$. By Theorem 2.4.1, we know that there exists at least a triangle verifying these properties. Since $d(p, \{x, y\}) \geq d(p, [xz] \cup [zy]) = (\operatorname{diam} V(G) + 1)/2$, we have $\operatorname{diam}(G) \geq L([xy]) \geq \operatorname{diam} V(G) + 1$. Since

$$\operatorname{diam}(G) \leq \operatorname{diam} V(G) + 1 = 2\delta(G) \leq \operatorname{diam}(G),$$

we conclude $\operatorname{diam}(G) = \operatorname{diam} V(G) + 1$, $L([xy]) = \operatorname{diam} V(G) + 1$ and $d(p, \{x, y\}) = (\operatorname{diam} V(G) + 1)/2$. Furthermore, we have $x, y \in J(G) \setminus V(G)$ since $d(x, y) = \operatorname{diam} V(G) + 1$. If $x$ and $y$ are the midpoints of $[x_1, x_2]$ and $[y_1, y_2]$, respectively, then $d([x_1, x_2], [y_1, y_2]) = \operatorname{diam} V(G)$. Since $2\delta(G) = L([xy]) = \operatorname{diam} V(G) + 1 \geq 3$, Lemma 4.2.2 gives $\big|N(x_1) \cup N(x_2) \setminus \{x_1, x_2\}\big| \geq \delta_0$ and $\big|N(y_1) \cup N(y_2) \setminus \{y_1, y_2\}\big| \geq \delta_0$.

Applying this first part of the lemma to a subgraph $G_s$ in the biconnected decomposition of $G$ with $\delta(G) = \delta(G_s) = (\operatorname{diam} V(G_s) + 1)/2$, we obtain that the same result holds if we replace $\operatorname{diam} V(G)$ and $\operatorname{diam}(G)$ by $\operatorname{diameff} V(G)$ and $\operatorname{diameff}(G)$, respectively. Note that the argument works considering the sets $N(x_1)$ and $N(x_2)$ (it is not necessary to consider in this case the set of neighbors of $x_1$ and $x_2$ in $G_s$). □

The argument in the proof of Lemma 4.2.2 also gives the following result.

**Lemma 4.2.4.** *Let $G$ be a graph with minimum degree $\delta_0$, $T = \{x, y, z\}$ a geodesic triangle that is a cycle in $G$ and $p \in [xy]$ such that $L([xy]) \geq 7/2$, $d(p, [xz] \cup [zy]) = \delta(T) = \delta(G)$ and $x, y, z \in J(G)$. Assume that $x$ is the midpoint of $[x_1, x_2]$ with $x_1, x_2 \in V(G)$ and $y \in V(G)$. If $\delta(G) = L([xy])/2$, then $\left|N(x_1) \cup N(x_2) \setminus \{x_1, x_2\}\right| \geq \delta_0$.*

Proposition 4.2.5 below improves the estimation of $\delta(G)$ obtained by Theorems 4.1.7 and 2.4.2.

**Proposition 4.2.5.** *Let $G \in \mathcal{H}(n, \delta_0)$. If $\delta_0 \geq (n-2)/2$, then $\delta(G) \leq 3/2$.*

*Proof.* If $\delta_0 \geq (n-2)/2$, then Theorems 4.1.7 and 2.4.2 imply $\operatorname{diam} V(G) \leq 3$ and $\delta(G) \leq 2$.

Theorem 2.4.1 gives that there exist a geodesic triangle $T = \{x, y, z\}$ that is a cycle in $G$ and $p \in [xy]$ such that $d(p, [xz] \cup [zy]) = \delta(T) = \delta(G)$ and $x, y, z \in J(G)$. Since $\operatorname{diam} V(G) \leq 3$, we have $L([xy]) \leq 4$.

Seeking for a contradiction, assume that $\delta(G) \geq 7/4$. Theorem 2.4.4 gives $\delta(G) \in \{7/4, 2\}$.

Assume that $L([xy]) = 4$. Since $\operatorname{diam} V(G) \leq 3$, we have $\operatorname{diam}(G) = 4$ and $\operatorname{diam} V(G) = 3$. Hence, $x$ and $y$ are the midpoints of $[x_1, x_2]$ and $[y_1, y_2]$, respectively, with $x_1, x_2, y_1, y_2 \in V(G)$, and $d([x_1, x_2], [y_1, y_2]) = 3$. Let $X_0 := N(x_1) \cup N(x_2) \setminus \{x_1, x_2\}$ and $Y_0 := N(y_1) \cup N(y_2) \setminus \{y_1, y_2\}$. Since $\delta(G) \geq 7/4 = L([xy])/2 - 1/4$, Lemma 4.2.2 gives $|X_0| + |Y_0| \geq 2\delta_0 - 1$. Since $d([x_1, x_2], [y_1, y_2]) = 3$, we have $X_0 \cap Y_0 = \emptyset$ and so

$$n \geq 2 + |X_0| + |Y_0| + 2 \geq 3 + 2\delta_0 \geq n + 1,$$

which is a contradiction. Hence, $L([xy]) \leq 7/2$ and, since $\delta(G) \geq 7/4$, we conclude $L([xy]) = 7/2$ and $\delta(G) = 7/4 = L([xy])/2$. Thus we have either $x \in J(G) \setminus V(G)$ and $y \in V(G)$, or $x \in V(G)$ and $y \in J(G) \setminus V(G)$. By symmetry, we can assume that $x$ is the midpoint of $[x_1, x_2]$, with $x_1, x_2 \in V(G)$, and $y \in V(G)$. Let us define $X_0$ as before. Lemma 4.2.4 gives $|X_0| \geq \delta_0$. Since $d([x_1, x_2], y) = 3$, we have $X_0 \cap N(y) = \emptyset$ and so

$$n \geq 2 + |X_0| + |N(y)| + 1 \geq 3 + 2\delta_0 \geq n + 1,$$

which is a contradiction. Thus, $\delta(G) < 7/4$ and Theorem 2.4.4 gives $\delta(G) \leq 3/2$. $\square$

The next result improves the bound of $\delta(G)$ obtained by Theorem 4.1.9 and Corollary 2.4.13.

**Proposition 4.2.6.** *Let $G \in \mathcal{H}(n, \delta_0)$. If $\delta_0 \geq (n-3)/2$, then $\delta(G) \leq 7/4$.*

*Proof.* If $\delta_0 \geq (n-3)/2$, then Theorem 4.1.9 and Corollary 2.4.13 imply $\operatorname{diameff} V(G) \leq 3$ and $\delta(G) \leq 2$.

Seeking for a contradiction, assume $\delta(G) = 2$. Thus $\operatorname{diameff}(G) \geq 4$ by Corollary 2.4.13, and we conclude $\operatorname{diameff} V(G) = 3$ and $\operatorname{diameff}(G) = 4$. Then $\delta(G) = (\operatorname{diameff} V(G) + 1)/2$.

Theorem 2.4.1 gives that there exist a geodesic triangle $T = \{x, y, z\}$ that is a cycle in $G$ and $p \in [xy]$ such that $d(p, [xz] \cup [zy]) = \delta(T) = 2$ and $x, y, z \in J(G)$. Since $T$ is a cycle, $T$ is contained in some subgraph $G_s$ in the biconnected decomposition of $G$.

By Lemma 4.2.3, $L([xy]) = 4$, $d(p, \{x, y\}) = 2$, $x$ and $y$ are the midpoints of $[x_1, x_2]$ and $[y_1, y_2]$, respectively, with $x_1, x_2, y_1, y_2 \in V(G)$, $d([x_1, x_2], [y_1, y_2]) = 3$, $\big|N(x_1) \cup N(x_2) \setminus \{x_1, x_2\}\big| \geq \delta_0$ and $\big|N(y_1) \cup N(y_2) \setminus \{y_1, y_2\}\big| \geq \delta_0$.

Let $X_0 := N(x_1) \cup N(x_2) \setminus \{x_1, x_2\}$ and $Y_0 := N(y_1) \cup N(y_2) \setminus \{y_1, y_2\}$.

Since $d([x_1, x_2], [y_1, y_2]) = 3$, we have $X_0 \cap Y_0 = \emptyset$.

Since $|X_0|, |Y_0| \geq \delta_0$, we deduce

$$n \geq 2 + |X_0| + |Y_0| + 2 \geq 4 + 2\delta_0 \geq n + 1,$$

which is a contradiction. Thus, $\delta(G) < 2$ and Theorem 2.4.4 gives $\delta(G) \leq 7/4$. □

Proposition 4.2.7 below improves the estimation of $\delta(G)$ obtained by Theorem 4.1.11 and Corollary 2.4.13.

**Proposition 4.2.7.** *Let $G \in \mathcal{H}(n, \delta_0)$. If $\delta_0 \geq (n-4)/2$, then $\delta(G) \leq 2$.*

*Proof.* If $\delta_0 \geq (n-4)/2$, then Theorem 4.1.11 and Corollary 2.4.13 imply $\operatorname{diameff} V(G) \leq 4$ and $\delta(G) \leq 5/2$.

Theorem 2.4.1 gives that there exist a geodesic triangle $T = \{x, y, z\}$ that is a cycle in $G$ and $p \in [xy]$ such that $d(p, [xz] \cup [zy]) = \delta(T) = \delta(G)$ and $x, y, z \in J(G)$. Note that since $T$ is a cycle, $T$ is contained in some subgraph $G_s$ in the biconnected decomposition of $G$. Since $\operatorname{diameff} V(G) \leq 4$, we have $L([xy]) \leq 5$.

Seeking for a contradiction, assume that $\delta(G) \geq 9/4$. Theorem 2.4.4 gives $\delta(G) \in \{9/4, 5/2\}$.

Assume that $L([xy]) = 5$. Since $\operatorname{diameff} V(G) \leq 4$, we have $\operatorname{diameff}(G) = 5$ and $\operatorname{diameff} V(G) = 4$. Hence, $x$ and $y$ are the midpoints of $[x_1, x_2]$ and $[y_1, y_2]$, respectively, with $x_1, x_2, y_1, y_2 \in V(G)$, and $d([x_1, x_2], [y_1, y_2]) = 4$. Let $X_0 := N(x_1) \cup N(x_2) \setminus \{x_1, x_2\}$ and $Y_0 := N(y_1) \cup N(y_2) \setminus \{y_1, y_2\}$. Since $\delta(G) \geq 9/4 = L([xy])/2 - 1/4$, Lemma 4.2.2 gives $|X_0| + |Y_0| \geq 2\delta_0 - 1$. Define $B_0 := V(G) \setminus (\{x_1, x_2\} \cup X_0 \cup Y_0 \cup \{y_1, y_2\})$. Since $d([x_1, x_2], [y_1, y_2]) = 4$, we have $X_0 \cap Y_0 = \emptyset$ and so

$$n \geq 2 + |X_0| + |B_0| + |Y_0| + 2 \geq 3 + 2\delta_0 + |B_0| \geq n - 1 + |B_0|,$$

and $|B_0| \leq 1$. If $|B_0| = 0$, then $d(X_0, Y_0) = 1$ and $d([x_1, x_2], [y_1, y_2]) = 3$, which is a contradiction. If $|B_0| = 1$, then $B_0 = \{w\}$ and $w$ belongs to both $[xy]$ and $[yz] \cup [zx]$. Since $w \neq x, y$ and $T$ is a cycle, this is a contradiction. Hence, $L([xy]) \leq 9/2$ and, since $\delta(G) \geq 9/4$, we conclude $L([xy]) = 9/2$ and $\delta(G) = 9/4 = L([xy])/2$. Thus we have either $x \in J(G) \setminus V(G)$ and $y \in V(G)$, or $x \in V(G)$ and $y \in J(G) \setminus V(G)$. By symmetry, we can assume that $x$ is the midpoint of $[x_1, x_2]$, with $x_1, x_2 \in V(G)$, and $y \in V(G)$. Let us define

$X_0$ as before, and $B := V(G) \setminus (\{x_1, x_2\} \cup X_0 \cup Y_0 \cup \{y\})$. Lemma 4.2.4 gives $|X_0| \geq \delta_0$. Since $d([x_1, x_2], y) = 3$, we have $X_0 \cap N(y) = \emptyset$ and so

$$n \geq 2 + |X_0| + |B| + |N(y)| + 1 \geq 3 + 2\delta_0 + |B| \geq n - 1 + |B|,$$

and $|B| \leq 1$, which we have seen that is a contradiction. Thus, $\delta(G) < 9/4$ and Theorem 2.4.4 gives $\delta(G) \leq 2$. $\square$

The next result improves the upper bound of $\delta(G)$ obtained by Theorem 4.1.13 and Corollary 2.4.13.

**Proposition 4.2.8.** *Let $G \in \mathcal{H}(n, \delta_0)$. If $\delta_0 \geq (n-5)/2$, then $\delta(G) \leq 9/4$.*

*Proof.* If $\delta_0 \geq (n-5)/2$, then Theorem 4.1.13 and Corollary 2.4.13 imply $\operatorname{diameff} V(G) \leq 4$ and $\delta(G) \leq 5/2$.

Seeking for a contradiction, assume $\delta(G) = 5/2$. Thus $\operatorname{diameff}(G) \geq 5$ by Corollary 2.4.13, and we conclude $\operatorname{diameff} V(G) = 4$ and $\operatorname{diameff}(G) = 5$. Then $\delta(G) = (\operatorname{diameff} V(G) + 1)/2$.

Theorem 2.4.1 gives that there exist a geodesic triangle $T = \{x, y, z\}$ that is a cycle in $G$ and $p \in [xy]$ such that $d(p, [xz] \cup [zy]) = \delta(T) = 5/2$ and $x, y, z \in J(G)$. Since $T$ is a cycle, $T$ is contained in some subgraph $G_s$ in the biconnected decomposition of $G$.

By Lemma 4.2.3, $L([xy]) = 5$, $d(p, \{x, y\}) = 5/2$, $x$ and $y$ are the midpoints of $[x_1, x_2]$ and $[y_1, y_2]$, respectively, with $x_1, x_2, y_1, y_2 \in V(G)$, $d([x_1, x_2], [y_1, y_2]) = 4$, $\big|N(x_1) \cup N(x_2) \setminus \{x_1, x_2\}\big| \geq \delta_0$ and $\big|N(y_1) \cup N(y_2) \setminus \{y_1, y_2\}\big| \geq \delta_0$.

Let $X_0 := N(x_1) \cup N(x_2) \setminus \{x_1, x_2\}$ and $Y_0 := N(y_1) \cup N(y_2) \setminus \{y_1, y_2\}$.

Since $d([x_1, x_2], [y_1, y_2]) = 4$, we have $X_0 \cap Y_0 = \emptyset$. Define $B_0 := V(G) \setminus (\{x_1, x_2\} \cup X_0 \cup Y_0 \cup \{y_1, y_2\})$. Since $|X_0|, |Y_0| \geq \delta_0$, we deduce

$$n \geq 2 + |X_0| + |B_0| + |Y_0| + 2 \geq 4 + 2\delta_0 + |B_0| \geq n - 1 + |B_0|,$$

and $|B_0| \leq 1$. The argument in the proof of Proposition 4.2.7 shows that this is a contradiction. Thus, $\delta(G) < 5/2$ and Theorem 2.4.4 gives $\delta(G) \leq 9/4$. $\square$

The following result improves the bound of $\delta(G)$ obtained by Theorems 4.1.17 and 2.4.2.

**Proposition 4.2.9.** *Let $G \in \mathcal{H}(n, \delta_0)$. If $\delta_0 \geq (n-3)/3$, then $\delta(G) \leq 3$.*

*Proof.* If $\delta_0 \geq (n-3)/3$, then Theorems 4.1.17 and 2.4.2 imply $\operatorname{diam} V(G) \leq 6$ and $\delta(G) \leq 7/2$.

Theorem 2.4.1 gives that there exist a geodesic triangle $T = \{x, y, z\}$ that is a cycle in $G$ and $p \in [xy]$ such that $d(p, [xz] \cup [zy]) = \delta(T) = \delta(G)$ and $x, y, z \in J(G)$.

Seeking for a contradiction, assume that $\delta(G) \geq 13/4$. Theorem 2.4.4 gives $\delta(G) \in \{13/4, 7/2\}$.

Assume that $L([xy]) = 7$. Since $\operatorname{diam} V(G) \leq 6$, we have $\operatorname{diam}(G) = 7$ and $\operatorname{diam} V(G) = 6$. Hence, $x$ and $y$ are the midpoints of $[x_1, x_2]$ and $[y_1, y_2]$, respectively, with $x_1, x_2, y_1, y_2 \in$

$V(G)$, and $d([x_1, x_2], [y_1, y_2]) = 6$. Let $X_0 := N(x_1) \cup N(x_2) \setminus \{x_1, x_2\}$ and $Y_0 := N(y_1) \cup N(y_2) \setminus \{y_1, y_2\}$. Let $A_0$ and $C_0$ be the sets of vertices at distance 2 from $\{x_1, x_2\}$ and $\{y_1, y_2\}$, respectively, and $B_0$ the set $B_0 := V(G) \setminus (\{x_1, x_2\} \cup X_0 \cup A_0 \cup C_0 \cup Y_0 \cup \{y_1, y_2\})$. Since $d([x_1, x_2], [y_1, y_2]) = 6$, we have that $X_0, A_0, C_0, Y_0$ are pairwise disjoint and $B_0 \neq \emptyset$. Since $\delta(G) \geq 13/4 = L([xy])/2 - 1/4$, Lemma 4.2.2 gives $|X_0| + |Y_0| \geq 2\delta_0 - 1$. Thus,

$$n = 4 + |X_0| + |A_0| + |B_0| + |C_0| + |Y_0| \geq 3 + 2\delta_0 + |A_0| + |B_0| + |C_0|.$$

Therefore, $|A_0| + |B_0| + |C_0| \leq n - 3 - 2\delta_0$.

Consider $p$ in $B_0$. Since $p$ has no neighbors in $\{x_1, x_2\} \cup X_0 \cup Y_0 \cup \{y_1, y_2\}$, we have

$$\delta_0 \leq \deg(p) \leq |B_0| - 1 + |A_0| + |C_0| \leq n - 4 - 2\delta_0 \quad \Rightarrow \quad \delta_0 \leq \frac{n-4}{3},$$

which contradicts $\delta_0 \geq (n-3)/3$. Hence, $L([xy]) \leq 13/2$ and, since $\delta(G) \geq 13/4$, we conclude $L([xy]) = 13/2$ and $\delta(G) = 13/4 = L([xy])/2$. Thus we have either $x \in J(G) \setminus V(G)$ and $y \in V(G)$, or $x \in V(G)$ and $y \in J(G) \setminus V(G)$. By symmetry, we can assume that $x$ is the midpoint of $[x_1, x_2]$, with $x_1, x_2 \in V(G)$, and $y \in V(G)$.

Let us define $X_0$ and $A_0$ as before, let $C$ be the set of vertices at distance 2 from $y$, and $B := V(G) \setminus (\{x_1, x_2\} \cup X_0 \cup A_0 \cup C \cup N(y) \cup \{y\})$. Since $d([x_1, x_2], y) = 6$, we have that $X_0, A_0, C, N(y)$ are pairwise disjoint and $B \neq \emptyset$. Lemma 4.2.4 gives $|X_0| \geq \delta_0$ and thus

$$n = 3 + |X_0| + |A_0| + |B| + |C| + |N(y)| \geq 3 + 2\delta_0 + |A_0| + |B| + |C|.$$

Therefore, $|A_0| + |B| + |C| \leq n - 3 - 2\delta_0$, and the previous argument gives a contradiction. Thus, $\delta(G) < 13/4$ and Theorem 2.4.4 gives $\delta(G) \leq 3$. □

The argument in the proof of Proposition 4.2.9 also gives the following result.

**Proposition 4.2.10.** *Let $G \in \mathcal{H}(n, \delta_0)$. If $\delta_0 \geq (n-2)/3$, then $\delta(G) \leq 5/2$.*

The next result is a direct consequence of Propositions 4.2.5, 4.2.6, 4.2.7, 4.2.8, 4.2.9, 4.2.10, Theorem 2.4.2 and Corollary 2.4.13.

**Theorem 4.2.11.** *Consider any $1 \leq \delta_0 \leq n - 1$.*

- *If $\delta_0 \geq (n-2)/2$, then $b(n, \delta_0) \leq 3/2$.*
- *If $\delta_0 \geq (n-3)/2$, then $b(n, \delta_0) \leq 7/4$.*
- *If $\delta_0 \geq (n-4)/2$, then $b(n, \delta_0) \leq 2$.*
- *If $\delta_0 \geq (n-5)/2$, then $b(n, \delta_0) \leq 9/4$.*
- *If $\delta_0 \geq (n-2)/3$, then $b(n, \delta_0) \leq 5/2$.*
- *If $\delta_0 \geq (n-3)/3$, then $b(n, \delta_0) \leq 3$.*

## 4.3 Lower bounds and some precise values for $b(n, \delta_0)$

In order to obtain a lower bound for $b(n, \delta_0)$, we need some previous results.

A subgraph $H$ of $G$ is said *isometric* if $d_H(x, y) = d_G(x, y)$ for every $x, y \in H$. Note that this condition is equivalent to $d_H(u, v) = d_G(u, v)$ for every vertices $u, v \in V(H)$.

The following results appear in [16, Lemma 9] and [99, Theorem 11].

**Lemma 4.3.1.** *If $H$ is an isometric subgraph of $G$, then $\delta(H) \le \delta(G)$.*

**Lemma 4.3.2.** *If $C_n$ is the cycle graph with $n$ vertices, then $\delta(C_n) = L(C_n)/4 = n/4$.*

**Corollary 4.3.3.** *If a graph $G$ contains an isometric cycle with length $r$, then $\delta(G) \ge r/4$.*

**Theorem 4.3.4.** *Let $2 \le \delta_0 \le n-1$ and $r := \left\lfloor \frac{3n}{\delta_0+1} \right\rfloor$. Then $b(n, \delta_0) \ge \frac{1}{4}(r-1)$.*

*Furthermore, this inequality can be improved in many cases: if we have either*

- $r \equiv 0 \mod 3$,
- $\delta_0 + 1 \equiv 0 \mod 3$,
- $r \equiv 1 \mod 3$, $\delta_0 + 1 \equiv 1 \mod 3$, *and* $3n \not\equiv 0, 1 \mod \delta_0 + 1$,
- $r \equiv 1 \mod 3$, $\delta_0 + 1 \equiv 2 \mod 3$, *and* $3n \not\equiv 0 \mod \delta_0 + 1$,
- $r \equiv 2 \mod 3$, $\delta_0 + 1 \equiv 1 \mod 3$, *and* $3n \not\equiv 0 \mod \delta_0 + 1$, *or*
- $r \equiv 2 \mod 3$, $\delta_0 + 1 \equiv 2 \mod 3$,

*then $b(n, \delta_0) \ge \frac{1}{4} r$.*

**Remark 4.3.5.** *This lower bound is sharp: Proposition 4.3.8 will show that the equality $b(n, \delta_0) = \frac{1}{4} r$ holds for $\delta_0 = 2$ and every $n \ge 3$.*

*Note that the upper and lower bounds on $b(n, \delta_0)$ in Theorems 4.2.1 and 4.3.4 are asymptotically $3n/(2\delta_0)$ and $3n/(4\delta_0)$, respectively. Thus, both bounds have the same order of magnitude.*

*Proof.* We prove first the second statement. We are going to construct a graph $G \in \mathcal{H}(n, \delta_0)$ containing an isometric cycle with length $r$. Thus, Corollary 4.3.3 will give the inequality.

We are going to prove that the hypotheses allow to choose positive integers $k_1, k_2, \dots, k_r$ with $k_1 + k_2 + \cdots + k_r = n$ and $k_{j-1} + k_j + k_{j+1} \ge \delta_0 + 1$ for every integer $j$ if we define $k_{j'} = k_j$ when $j' \equiv j \mod r$. Note that it suffices to find positive integers $k_1^*, k_2^*, \dots, k_r^*$ with $k_1^* + k_2^* + \cdots + k_r^* \le n$ and $k_{j-1}^* + k_j^* + k_{j+1}^* \ge \delta_0 + 1$ for every integer $j$ if we define $k_{j'}^* := k_j^*$ when $j' \equiv j \mod r$.

Define $k_1^* = \left\lceil \frac{\delta_0+1}{3} \right\rceil$, $k_2^* = k_3^* = \left\lfloor \frac{\delta_0+1}{3} \right\rfloor$ and $k_{j'}^* = k_j^*$ for $1 \le j, j' \le r$ and $j' \equiv j \mod 3$. Since $\left\lceil \frac{\delta_0+1}{3} \right\rceil + \left\lfloor \frac{\delta_0+1}{3} \right\rfloor + \left\lfloor \frac{\delta_0+1}{3} \right\rfloor = \delta_0 + 1$, we have $k_{j-1}^* + k_j^* + k_{j+1}^* \ge \delta_0 + 1$ for every integer $j$.

Case (a). If $r \equiv 0 \mod 3$, then

$$k_1^* + k_2^* + \cdots + k_r^* = (\delta_0 + 1)\frac{r}{3} \le \frac{\delta_0 + 1}{3}\frac{3n}{\delta_0 + 1} = n.$$

Case (b). If $r \equiv 1 \mod 3$, then

$$\begin{aligned} k_1^* + k_2^* + \cdots + k_r^* &= (\delta_0 + 1)\left\lfloor \frac{r}{3} \right\rfloor + \left\lceil \frac{\delta_0 + 1}{3} \right\rceil = (\delta_0 + 1)\frac{r-1}{3} + \left\lceil \frac{\delta_0 + 1}{3} \right\rceil \\ &= (\delta_0 + 1)\frac{r}{3} + \left\lceil \frac{\delta_0 + 1}{3} \right\rceil - \frac{\delta_0 + 1}{3}. \end{aligned} \tag{4.4}$$

Case (b.1). If $\delta_0 + 1 \equiv 0 \mod 3$, then $k_1^* + \cdots + k_r^* = (\delta_0 + 1)\frac{r}{3} \le n$.

Case (b.2). Assume $\delta_0 + 1 \equiv 1 \mod 3$, and $3n \not\equiv 0, 1 \mod \delta_0 + 1$. Note that $3n \not\equiv 0, 1 \mod \delta_0 + 1$ is equivalent to

$$\frac{3n}{\delta_0 + 1} - \left\lfloor \frac{3n}{\delta_0 + 1} \right\rfloor \ge \frac{2}{\delta_0 + 1} \quad \Leftrightarrow \quad (\delta_0 + 1)\frac{r}{3} \le n - \frac{2}{3}. \tag{4.5}$$

Since $\delta_0 + 1 \equiv 1 \mod 3$, we have

$$\left\lceil \frac{\delta_0 + 1}{3} \right\rceil - \frac{\delta_0 + 1}{3} = \frac{\delta_0 + 3}{3} - \frac{\delta_0 + 1}{3} = \frac{2}{3}.$$

This last equality, (4.4) and (4.5) give $k_1^* + \cdots + k_r^* \le n$.

Case (b.3). Assume $\delta_0 + 1 \equiv 2 \mod 3$, and $3n \not\equiv 0 \mod \delta_0 + 1$. Note that $3n \not\equiv 0 \mod \delta_0 + 1$ is equivalent to

$$\frac{3n}{\delta_0 + 1} - \left\lfloor \frac{3n}{\delta_0 + 1} \right\rfloor \ge \frac{1}{\delta_0 + 1} \quad \Leftrightarrow \quad (\delta_0 + 1)\frac{r}{3} \le n - \frac{1}{3}. \tag{4.6}$$

Since $\delta_0 + 1 \equiv 2 \mod 3$, we have

$$\left\lceil \frac{\delta_0 + 1}{3} \right\rceil - \frac{\delta_0 + 1}{3} = \frac{\delta_0 + 2}{3} - \frac{\delta_0 + 1}{3} = \frac{1}{3}.$$

This last equality, (4.4) and (4.6) give $k_1^* + \cdots + k_r^* \le n$.

Case (c). If $r \equiv 2 \mod 3$, then

$$\begin{aligned} k_1^* + k_2^* + \cdots + k_r^* &= (\delta_0 + 1)\left\lfloor \frac{r}{3} \right\rfloor + \left\lceil \frac{\delta_0 + 1}{3} \right\rceil + \left\lfloor \frac{\delta_0 + 1}{3} \right\rfloor \\ &= (\delta_0 + 1)\frac{r-2}{3} + \left\lceil \frac{\delta_0 + 1}{3} \right\rceil + \left\lfloor \frac{\delta_0 + 1}{3} \right\rfloor \\ &= (\delta_0 + 1)\frac{r}{3} + \left\lceil \frac{\delta_0 + 1}{3} \right\rceil + \left\lfloor \frac{\delta_0 + 1}{3} \right\rfloor - 2\frac{\delta_0 + 1}{3}. \end{aligned} \tag{4.7}$$

Case (c.1). If $\delta_0 + 1 \equiv 0 \mod 3$, then $k_1^* + \cdots + k_r^* = (\delta_0 + 1)\frac{r}{3} \le n$.

Case (c.2). Assume $\delta_0+1 \equiv 1 \mod 3$, and $3n \not\equiv 0 \mod \delta_0+1$. Since $\delta_0+1 \equiv 1 \mod 3$, we have

$$\left\lceil \frac{\delta_0+1}{3} \right\rceil + \left\lfloor \frac{\delta_0+1}{3} \right\rfloor - 2\frac{\delta_0+1}{3} = \frac{\delta_0+3}{3} + \frac{\delta_0}{3} - 2\frac{\delta_0+1}{3} = \frac{1}{3}\,.$$

This last equality, (4.7) and (4.6) give $k_1^* + \cdots + k_r^* \le n$.

Case (c.3). Assume $\delta_0+1 \equiv 2 \mod 3$. Thus we have

$$\left\lceil \frac{\delta_0+1}{3} \right\rceil + \left\lfloor \frac{\delta_0+1}{3} \right\rfloor - 2\frac{\delta_0+1}{3} = \frac{\delta_0+2}{3} + \frac{\delta_0-1}{3} - 2\frac{\delta_0+1}{3} = \frac{-1}{3}\,.$$

This last equality and (4.7) give $k_1^* + \cdots + k_r^* < (\delta_0+1)\frac{r}{3} \le n$.

Hence, $k_1^* + \cdots + k_r^* \le n$ holds in any case.

Let $K_1, K_2, \dots, K_r$ be pairwise disjoint sets of points such that $K_j$ has cardinal $k_j$ for every $1 \le j \le r$. In order to obtain $G$, define $V(G) := K_1 \cup K_2 \cup \cdots \cup K_r$. Define also $K_{j'} := K_j$ when $j' \equiv j \mod r$.

For each vertex $v \in K_j$ with $1 \le j \le k$ we can choose at least $\delta_0$ edges joining $v$ with other vertices in $K_{j-1} \cup K_j \cup K_{j+1} \setminus \{v\}$, since $k_{j-1}+k_j+k_{j+1} \ge \delta_0+1$. Furthermore, we can choose the edges with the additional property $\deg(v_0) = \delta_0$ for some $v_0 \in V(G)$. If $G$ is any graph obtained in this way, then $G \in \mathcal{H}(n,\delta_0)$. Since $N(v) \subset K_{j-1} \cup K_j \cup K_{j+1}$ for every $v \in K_j$ and every $j$, the graph $G$ contains an isometric cycle with length $r$ and Corollary 4.3.3 gives the inequality.

Let us prove now the first statement. Let us define $r' := \left\lfloor \frac{3n}{\delta_0+1} \right\rfloor - 1 = r-1$. Hence, every upper bound of $k_1^* + \cdots + k_r^*$ obtained in the previous argument holds if we replace $(\delta_0+1)\frac{r}{3}$ by

$$(\delta_0+1)\frac{r-1}{3} \le \frac{\delta_0+1}{3}\Big(\frac{3n}{\delta_0+1} - 1\Big) = n - \frac{\delta_0+1}{3} \le n-1.$$

Since we have in every case

$$\left\lceil \frac{\delta_0+1}{3} \right\rceil - \frac{\delta_0+1}{3} \le 1, \qquad \left\lceil \frac{\delta_0+1}{3} \right\rceil + \left\lfloor \frac{\delta_0+1}{3} \right\rfloor - 2\frac{\delta_0+1}{3} \le 1,$$

the argument in the proof of the second statement gives $b(n,\delta_0) \ge \frac{1}{4}r'$. □

We prove now another lower bound which will be useful.

**Lemma 4.3.6.** *Consider positive integers $6 \le k \le 10$ and $n \ge k+2$ such that $3 \le \delta_0 \le \left\lfloor \frac{n-k+4}{2} \right\rfloor$. Then $b(n,\delta_0) \ge k/4$.*

*Proof.* Define $t := \left\lceil \frac{n-k}{2} \right\rceil$ and $t' := n-k-t = \left\lfloor \frac{n-k}{2} \right\rfloor$.

Assume first that $k$ is even. Consider a cycle graph $C_k$ with (ordered) vertices $v_1, v_2, \dots, v_k$, and two complete graphs $G_1, G_2$ with $t, t'$ vertices, respectively, and join $v_1$ with $\delta_0-2$ vertices in $G_1$, $v_2, v_3, v_{k-1}, v_k$ with every vertex in $G_1$ and $v_{k/2-1}, v_{k/2}, v_{k/2+1}, v_{k/2+2}, v_{k/2+3}$ with every vertex in $G_2$. Denote by $G$ the graph obtained in this way. Thus, $\deg(v_1) = \delta_0$ and $\deg(v_j) \ge$

$2+\lfloor\frac{n-k}{2}\rfloor = \lfloor\frac{n-k+4}{2}\rfloor \geq \delta_0$ for every $1 < j \leq k$. Since $\deg(v) \geq 4+\lfloor\frac{n-k}{2}\rfloor - 1 = \lfloor\frac{n-k+6}{2}\rfloor > \delta_0$ for every $v \in V(G_1) \cup V(G_2)$, we have $G \in \mathcal{H}(n,\delta_0)$. Define $x = v_1$ and $y = v_{k/2+1}$, and consider two geodesics $g_1, g_2$ in $C_k$ joining $x, y$ with $g_1 \cup g_2 = C_k$ and $g_1 \cap g_2 = \{x, y\}$ (thus $d_G(x, y) = k/2$ and $g_1, g_2$ are geodesics in $G$). If $T$ is the geodesic bigon $T = \{g_1, g_2\}$ and $p$ is the midpoint of $g_1$, then $d(p, g_2) = d(p, \{x, y\}) = k/4$. Hence, $b(n,\delta_0) \geq \delta(G) \geq \delta(T) \geq k/4$.

Assume now that $k$ is odd. Consider a cycle graph $C_k$ with (ordered) vertices $v_1, v_2, \dots, v_k$, and two complete graphs $G_1, G_2$ with $t, t'$ vertices, respectively, and join $v_1$ with $\delta_0 - 2$ vertices in $G_1$, $v_2, v_3, v_{k-1}, v_k$ with every vertex in $G_1$ and $v_{(k-1)/2}, v_{(k+1)/2}, v_{(k+3)/2}, v_{(k+5)/2}$ with every vertex in $G_2$. Denote by $G$ the graph obtained in this way. As before, $G \in \mathcal{H}(n,\delta_0)$. Define $x = v_1$ and $y$ as the midpoint of the edge $[v_{(k+1)/2}, v_{(k+3)/2}]$, and consider two geodesics $g_1, g_2$ in $C_k$ joining $x, y$ with $g_1 \cup g_2 = C_k$ and $g_1 \cap g_2 = \{x, y\}$ (thus $d_G(x, y) = k/2$ and $g_1, g_2$ are geodesics in $G$). If $T$ is the geodesic bigon $T = \{g_1, g_2\}$ and $p$ is the midpoint of $g_1$, then $b(n,\delta_0) \geq \delta(G) \geq \delta(T) \geq d(p, g_2) = d(p, \{x, y\}) = k/4$. □

Theorem 4.3.11 below gives good bounds for $b(n,\delta_0)$ when $\delta_0$ is big enough, and even it provides the precise value of $b(n,\delta_0)$ in many cases.

In order to prove it, we need the following result in [82, Proposition 29 and Theorem 30].

**Theorem 4.3.7.** *If $G$ is a graph with $n$ vertices, then $\delta(G) \leq n/4$. Furthermore, if $\delta(G) = n/4$, then the minimum degree of $G$ is at least 2.*

**Proposition 4.3.8.** *Consider $n \geq 2$.*

- $b(n, 1) = 0$ *for $n < 4$, and* $b(n, 1) = (n-1)/4$ *for every $n \geq 4$.*
- $b(n, 2) = n/4$ *for every $n \geq 3$.*

*Proof.* If $G \in \mathcal{H}(2, 1)$, then $G$ is isomorphic to the path graph $P_2$ and $b(2, 1) = \delta(P_2) = 0$. If $G \in \mathcal{H}(3, 1)$, then $G$ is isomorphic to the path graph $P_3$ and $b(3, 1) = \delta(P_3) = 0$.

Consider $n \geq 4$ and $G \in \mathcal{H}(n, 1)$. Proposition 2.4.6 and Theorem 4.3.7 imply that $\delta(G) \leq n/4$; furthermore, if $\delta(G) = n/4$, then the minimum degree of $G$ is at least 2. Thus, $b(n, 1) < n/4$ and Theorem 2.4.4 implies $b(n, 1) \leq (n-1)/4$. If $G$ is any graph obtained by attaching an edge to some vertex of a cycle graph $C_{n-1}$, then $G \in \mathcal{H}(n, 1)$ and $b(n, 1) \geq \delta(G) = \delta(C_{n-1}) = (n-1)/4$. Hence, $b(n, 1) = (n-1)/4$.

Theorem Theorem 2.4.7 gives $b(n, 2) \leq n/4$. Theorem 4.3.4 (with $\delta_0 = 2$ and $r = n$) implies $b(n, 2) \geq n/4$. Hence, $b(n, 2) = n/4$. □

Note that by Proposition 4.3.8, in order to estimate $b(n,\delta_0)$, it suffices to consider $3 \leq \delta_0 \leq n-1$ and $n \geq 4$.

In order to prove Theorem 4.3.11 we need the following results, which appear in [82, Proposition 27] and [82, Theorem 28], respectively.

**Proposition 4.3.9.** *Let $G$ be a graph with $n \geq 4$ vertices. If $\deg(v) \geq n-2$ for every vertex $v \in V(G)$, then $\delta(G) = 1$ and* $\operatorname{diam} G = 2$.

**Theorem 4.3.10.** *For each $n \geq 5$ and $1 \leq m \leq n-2$, let $G_{n,m}$ be the graph obtained by removing $m$ edges starting in the same vertex from the complete graph $K_n$. Then $\delta(G_{n,m}) = 1$ if $m = 1$ or $m = n-2$, and $\delta(G_{n,m}) = 5/4$ if $1 < m < n-2$.*

**Theorem 4.3.11.** *Consider any $n \geq 4$ and $3 \leq \delta_0 \leq n-1$.*

- *If $\delta_0 \geq n-2$, then $b(n, \delta_0) = 1$.*
- *If $\delta_0 = n-3$, then $b(n, \delta_0) = 5/4$.*
- *If $(n-2)/2 \leq \delta_0 \leq n-4$, then $b(n, \delta_0) = 3/2$.*
- *If $\delta_0 = (n-3)/2$, then $b(n, \delta_0) = 7/4$.*
- *If $\delta_0 = (n-4)/2$, then $b(n, \delta_0) = 2$.*
- *If $\delta_0 = (n-5)/2$, then $b(n, \delta_0) = 9/4$.*
- *If $(n-2)/3 \leq \delta_0 < (n-5)/2$, then $b(n, \delta_0) = 5/2$.*
- *If $\delta_0 = (n-3)/3$, then $5/2 \leq b(n, \delta_0) \leq 3$.*

*Proof.* If $\delta_0 \geq n-2$, then Proposition 4.3.9 implies $b(n, \delta_0) = 1$.

If $\delta_0 = n-3$, then $n \geq 6$ and Theorem 4.3.10 implies $b(n, \delta_0) \geq 5/4$. Since $n \geq 5$, we have $\delta_0 = n-3 \geq (n-1)/2$. Theorems 4.1.5 and 4.2.11 give $\operatorname{diam} V(G) \leq 2$ for every $G \in \mathcal{H}(n, \delta_0)$ and $b(n, \delta_0) \leq 3/2$. Seeking for a contradiction assume that $b(n, \delta_0) > 5/4$. Then Theorem 2.4.4 gives $b(n, \delta_0) = 3/2$. Thus there exists $G \in \mathcal{H}(n, \delta_0)$ with $\delta(G) = 3/2$. By Theorem 2.4.1 there exist a geodesic triangle $T = \{x, y, z\}$ that is a cycle in $G$ and $p \in [xy]$ such that $d(p, [xz] \cup [zy]) = \delta(T) = 3/2$ and $x, y, z \in J(G)$. As in the proof of Lemma 4.2.3, one can check that $\operatorname{diam} V(G) = 2$, $\operatorname{diam}(G) = \operatorname{diam} V(G) + 1 = 3$, $L([xy]) = 3$ and $d(p, \{x, y\}) = 3/2$. Hence, we have $x, y \in J(G) \setminus V(G)$ and $p \in V(G)$. Since $L([xz] \cup [zy]) \geq L([xy]) = 3$ and $x, y \in J(G) \setminus V(G)$, there are at least three points in $V(G) \cap ([xz] \cup [zy])$. Since $d(p, [xz] \cup [zy]) = 3/2$, the vertex $p$ is not adjacent to the points in $V(G) \cap ([xz] \cup [zy])$, and we conclude that $n-3 = \delta_0 \leq \deg(p) \leq n-4$, which is a contradiction. Hence, $b(n, \delta_0) = 5/4$.

If $(n-2)/2 \leq \delta_0 \leq n-4$, then Theorem 4.2.11 gives $b(n, \delta_0) \leq 3/2$. Note first that $3 \leq \delta_0 \leq n-4$ gives $n \geq 7$. We are going to construct a graph $G \in \mathcal{H}(n, \delta_0)$ in the following way. Given a cycle graph $C_6$ and a complete graph $K_{n-6}$, we join a fixed vertex $v_0 \in C_6$ with $\delta_0 - 2$ vertices in $K_{n-6}$ and each vertex in $V(C_6) \setminus \{v_0\}$ with every vertex in $K_{n-6}$ (we can do that since $\delta_0 - 2 \leq n-6$). Since $\delta_0 - 2 \leq n-6$, we have $\deg(v) \geq \deg(v_0) = \delta_0$ for every $v \in V(G)$ and so $G \in \mathcal{H}(n, \delta_0)$. Consider $x, y \in J(C_6) \setminus V(C_6)$ with $d_{C_6}(x, y) = 3$ and two geodesics $g_1, g_2$ joining $x, y$ in $C_6$ with $g_1 \cup g_2 = C_6$ and $g_1 \cap g_2 = \{x, y\}$ (thus $d_G(x, y) = 3$ and $g_1, g_2$ are geodesics in $G$). If $T$ is the geodesic bigon $T = \{g_1, g_2\}$ and $p$ is the midpoint of $g_1$, then one can check that $d(p, g_2) = d(p, \{x, y\}) = 3/2$. Hence, $b(n, \delta_0) \geq \delta(G) \geq \delta(T) \geq 3/2$ and we conclude $b(n, \delta_0) = 3/2$.

If $\delta_0 = (n-3)/2$, then $n$ is odd and Theorem 4.2.11 gives $b(n, \delta_0) \leq 7/4$. Note that $3 \leq \delta_0 < (n-1)/2$ gives $n \geq 9$. Since $k = 7$ is odd and $\delta_0 = \frac{n-k+4}{2} = \left\lfloor \frac{n-k+4}{2} \right\rfloor$, Lemma 4.3.6 gives $b(n, \delta_0) \geq 7/4$ and we have $b(n, \delta_0) = 7/4$.

If $\delta_0 = (n-4)/2$, then $n$ is even, $n \geq 10$ and Theorem 4.2.11 gives $b(n, \delta_0) \leq 2$. Since $k = 8$ is even and $\delta_0 = (n-k+4)/2$, Lemma 4.3.6 gives $b(n, \delta_0) \geq 2$, and thus $b(n, \delta_0) = 2$.

If $\delta_0 = (n-5)/2$, then $n$ is odd, $n \geq 11$ and Theorem 4.2.11 gives $b(n, \delta_0) \leq 9/4$. Since $k = 9$ is odd and $\delta_0 = (n-k+4)/2$, Lemma 4.3.6 gives $b(n, \delta_0) \geq 9/4$, and so $b(n, \delta_0) = 9/4$.

If $(n-2)/3 \leq \delta_0 < (n-5)/2$, then $n \geq 12$ and Theorem 4.2.11 gives $b(n, \delta_0) \leq 5/2$. Consider $k = 10$. If $n$ is even, then $\delta_0 \leq \frac{n-6}{2} = \left\lfloor \frac{n-6}{2} \right\rfloor = \left\lfloor \frac{n-k+4}{2} \right\rfloor$; if $n$ is odd, then $\delta_0 \leq \frac{n-7}{2} = \left\lfloor \frac{n-6}{2} \right\rfloor = \left\lfloor \frac{n-k+4}{2} \right\rfloor$. Thus, in both cases Lemma 4.3.6 implies $b(n, \delta_0) \geq 5/2$, and thus $b(n, \delta_0) = 5/2$.

If $\delta_0 = (n-3)/3$, then $n \geq 12$ and Theorem 4.2.11 gives $b(n, \delta_0) \leq 3$. Consider $k = 10$. Since $\frac{n-3}{3} \leq \left\lfloor \frac{n-6}{2} \right\rfloor = \left\lfloor \frac{n-k+4}{2} \right\rfloor$, Lemma 4.3.6 gives $b(n, \delta_0) \geq 5/2$. $\square$

## 4.4 Computation of $a(n, \delta_0)$

**Theorem 4.4.1.** *Consider $G \in \mathcal{H}(n, \delta_0)$ with $1 \le \delta_0 \le n-1$.*

- *If $\delta_0 = 1$, then $a(n, \delta_0) = 0$ for all values of $n$.*
- *If $\delta_0 = 2$, then $a(n, \delta_0) = 1$ if $n = 4$ and $a(n, \delta_0) = 3/4$ if $n \neq 4$.*
- *If $\delta_0 \ge 3$, then $a(n, \delta_0) = 1$.*

*Proof.* For every $n$ there is some tree in $\mathcal{H}(n, 1)$, and so $a(n, 1) = 0$.

Consider $\delta_0 = 2$ and $G \in \mathcal{H}(n, 2)$.

If $n = 3$, then $G$ is isomorphic to $C_3$ and $a(3, 2) = 3/4$.

If $n = 4$, then $G$ is isomorphic to either a cycle graph $C_4$ or a complete graph $K_4$ without an edge, and both of them have hyperbolicity constant equal to 1. Thus, $a(4, 2) = 1$.

Now consider $n > 4$. Let $\{v_1, \dots, v_n\} = V(G)$. If $m = |E(G)|$, then we have

$$m = \frac{1}{2}\sum_{i=1}^{n} \deg(v_i) \ge \frac{1}{2} n\delta_0 = n > n - 1,$$

and so $G$ is not a tree. Therefore, Theorem 2.4.5 gives $\delta(G) \ge 3/4$ and $a(n, 2) \ge 3/4$.

If $n$ is odd, let us consider $k = (n-1)/2$ graphs $G_1, \dots, G_k$ isomorphic to $C_3$. Fix vertices $v_1 \in V(G_1), \dots, v_k \in V(G_k)$ and consider the graph $G$ obtained from $G_1, \dots, G_k$ by identifying $v_1, \dots, v_k$ in a single vertex. Note that $n = 2k+1$ and $G \in \mathcal{H}(n, 2)$. Furthermore, $\{G_1, \dots, G_k\}$ is the biconnected decomposition of $G$ and Theorem 2.4.11 and Lemma 4.3.2 give $\delta(G) = \delta(C_3) = 3/4$. Therefore, $a(n, 2) \le 3/4$ and we conclude $a(n, 2) = 3/4$.

If $n$ is even, let us consider $k = (n-2)/2$ graphs $G_1, \dots, G_k$ isomorphic to $C_3$ and a graph $\Gamma$ isomorphic to $P_2$. Fix vertices $v_1 \in V(G_1), \dots, v_k \in V(G_k)$ and let $\{w_1, w_2\} = V(\Gamma)$. Identify $v_1 \in V(G_1)$ and $w_1 \in V(\Gamma)$ in a single vertex $v^*$ and $v_2 \in V(G_2), v_3 \in V(G_3), \dots, v_k \in V(G_k), w_2 \in V(\Gamma)$ in a single vertex $w^*$. We obtain in this way a graph $G \in \mathcal{H}(n, 2)$ from $G_1, \dots, G_k, \Gamma$, since $n = 2k + 2$. Furthermore, $\{G_1, \dots, G_k, \Gamma\}$ is the biconnected decomposition of $G$ and Theorem 2.4.11 and Lemma 4.3.2 give $\delta(G) = \delta(C_3) = 3/4$. Therefore, $a(n, 2) \le 3/4$ and we conclude $a(n, 2) = 3/4$.

Consider $G \in \mathcal{H}(n, \delta_0)$ with $\delta_0 \ge 3$, then

$$2m = \sum_{i=1}^{n} \deg(v_i) \ge n\delta_0 \ge 3n > 3n - 3,$$

and Theorem 3.4.10 gives $\delta(G) \ge 1$. Hence, $a(n, \delta_0) \ge 1$.

Let $K_n$ be the complete graph with $n$ vertices and consider $1 < N_1 < n$. Choose a set of vertices $V_1 \subset V(K_n)$ with $|V_1| = N_1$. Let $K_n^{N_1}$ be the graph obtained from $K_n$ by removing the edges joining any two vertices in $V_1$. We have $\delta(K_n^{N_1}) \le 1$ by Lemma 3.4.6. Since

$K_n^{n-\delta_0} \in \mathcal{H}(n, \delta_0)$ if $1 < n - \delta_0 < n$ (i.e., $1 \leq \delta_0 < n-1$), we have $a(n, \delta_0) \leq \delta(K_n^{N_1}) \leq 1$ and we conclude $a(n, \delta_0) = 1$. Finally, consider the case $\delta_0 = n-1$. Note that $K_n \in \mathcal{H}(n, n-1)$. Since $\delta_0 \geq 3$, we have $n \geq 4$ and thus $\delta(K_n) = 1$. Therefore, $a(n, \delta_0) \leq \delta(K_n) = 1$ and so $a(n, \delta_0) = 1$. □

## 4.5 Computation of $\beta(n,\Delta)$

**Lemma 4.5.1.** *For all $n$ and $\Delta$ we have $\beta(n,\Delta) \le (n-\Delta+5)/4$.*

*Proof.* If $\Delta \le 5$, then Theorem 2.4.7 gives $\delta(G) \le n/4 \le (n-\Delta+5)/4$ and therefore $\beta(n,\Delta) \le (n-\Delta+5)/4$.

Let us assume now that $\Delta \ge 6$. Fix $G \in \mathcal{J}(n,\Delta)$ and consider $v \in V(G)$ such that $\deg(v) = \Delta$. By Theorem 2.4.1 there exist a geodesic triangle $T = \{x,y,z\}$ with $x,y,z \in J(G)$ and $p \in [xy]$ with $\delta(G) = d(p,[xz]\cup[yz])$. If $L([xy]) < 3/2$, then

$$\delta(G) = d(p,[xz]\cup[yz]) \le d(p,\{x,y\}) \le \frac{1}{2}\,L([xy]) < \frac{3}{4} < \frac{6}{4} \le \frac{n-\Delta+5}{4}\,.$$

Hence, we can assume that $L([xy]) \ge 3/2$. We can assume also that $x,y,z$ are three different vertices, since if $T$ is a geodesic bigon, then the argument in the proof is similar and simpler.

We consider now several cases:

Case (A): Assume first that $v \notin T$.

Case (A.1): If $|N(v)\cap T| \le 6$, then $V(G)\cap T \subseteq V(G) \setminus \{v\} \setminus (N(v)\setminus T)$ and $L(T) = |V(G)\cap T| \le n-1-(\Delta-6) = n-\Delta+5$. Since $[xy]$ is a geodesic, we have $L([xy]) \le L(T)/2 \le (n-\Delta+5)/2$.

Case (A.2): Assume $|N(v)\cap T| \ge 7$. Since $[xy]$ is a geodesic, we have $|N(v)\cap[xy]| \le 3$. Thus, $|N(v)\cap(T\setminus[xy])| \ge 4$, which implies $N(v)\cap[xz] \ne \emptyset$ and $N(v)\cap[yz] \ne \emptyset$. Let us denote by $v_1$ the closest vertex to $x$ from $N(v)\cap[xz]$ and by $v_2$ the closest vertex to $y$ from $N(v)\cap[yz]$. Let us denote by $\gamma_1$ the path in $[xz]$ joining $x$ and $v_1$, and denote by $\gamma_2$ the path in $[yz]$ joining $y$ and $v_2$ (if $x \in N(v)$, then $v_1 = x$ and so $\gamma_1 = \{x\}$; if $y \in N(v)$, then $v_2 = y$ and $\gamma_2 = \{y\}$). We have $N(v)\cap\gamma_1 = \{v_1\}$ and $N(v)\cap\gamma_2 = \{v_2\}$. Consider the cycle $C := [xy]\cup\gamma$, where $\gamma := \gamma_2\cup[v_2,v]\cup[v,v_1]\cup\gamma_1$. Since $|N(v)\cap C| = |(N(v)\cap[xy])\cup\{v_1\}\cup\{v_2\}| \le |N(v)\cap[xy]|+2 \le 5$, we have $L(C) = |V(G)\cap C| \le |V(G)| - |N(v)\setminus C| \le |V(G)| - (\Delta-5) = n-\Delta+5$. Since $[xy]$ is a geodesic, we have $L([xy]) \le L(C)/2 \le (n-\Delta+5)/2$.

Hence, in both cases we have

$$\delta(G) = d(p,[xz]\cup[yz]) \le d(p,\{x,y\}) \le \frac{1}{2}L([xy]) \le \frac{n-\Delta+5}{4}.$$

Case (B): Now, assume that $v \in T$.

Case (B.1): If $|N(v)\cap T| \le 5$, then $V(G)\cap T \subseteq V(G)\setminus(N(v)\setminus T)$ and $L(T) = |V(G)\cap T| \le n-(\Delta-5)$. Thus, $\delta(G) \le (n-\Delta+5)/4$.

Case (B.2): Assume $|N(v)\cap T| \ge 6$. Let $\sigma$ be a geodesic side of $T$ with $v \in \sigma$. Thus $|N(v)\cap\sigma| \le 2$. Seeking for a contradiction assume that $N(v)\cap\sigma = \emptyset$. This hypothesis and $x,y,z \in J(G)$ give $L(\sigma) \le 1$. Since $L([xy]) \ge 3/2$, we have $\sigma \ne [xy]$. By symmetry, we can assume that $\sigma = [xz]$. The inequality $|N(v)\cap T| \ge 6$ and $N(v)\cap[xz] = \emptyset$ give

$|N(v) \cap [xy]| = 3 = |N(v) \cap [yz]|$. Since $x, z \in J(G)$, $v \in [xz]$ and $N(v) \cap [xz] = \emptyset$, we have either $L([xz]) = 1/2$ or $L([xz]) = 1$ and $v$ is the midpoint of $[xz]$.

Case (B.2.1): Assume that $L([xz]) = 1/2$. Since $V(G) \cap [xz] = \{v\}$ and $v \notin [xy]$, we deduce $v = z$ and $|N(v) \cap [xz]| \leq 2$, which is a contradiction.

Case (B.2.2): Assume that $L([xz]) = 1$ and $v$ is the midpoint of $[xz]$. Since $|N(v) \cap [xy]| = 3$, we have $N(v) \cap [xy] = \{v_1, v_2, v_3\}$, with $d(x, v_j) = j - 1/2$ for $j = 1, 2, 3$. Therefore, $5/2 = d(x, v_3) \leq d(x, v) + d(v, v_3) = 1/2 + 1 = 3/2$, which is a contradiction.

Hence, we conclude in both cases $|N(v) \cap \sigma| \geq 1$. Since $1 \leq |N(v) \cap \sigma| \leq 2$ and $|N(v) \cap T| \geq 6$, we have $N(v) \cap [xz] \neq \emptyset$ and $N(v) \cap [yz] \neq \emptyset$. Using the previous argument in Case (A.2) we obtain $\delta(G) \leq (n - \Delta + 5)/4$.

Hence, $\beta(n, \Delta) \leq (n - \Delta + 5)/4$. □

**Theorem 4.5.2.** *Consider any* $1 \leq \Delta \leq n - 1$.

- *If* $\Delta = 1$*, then* $n = 2$ *and* $\beta(2, 1) = 0$.
- *If* $2 \leq \Delta \leq 4$*, then* $\beta(n, \Delta) = n/4$.
- *If* $\Delta = 5$*, then* $\beta(n, 5) = (n - 1)/4$.
- *If* $\Delta \geq 6$*, then* $\beta(n, \Delta) = (n - \Delta + 5)/4$.

*Proof.* For every $n$ and $\Delta$, Theorem 4.3.7 gives $\beta(n, \Delta) \leq n/4$.

If $\Delta = 1$ and $G \in \mathcal{J}(n, 1)$, then $G$ is isomorphic to the path graph $P_2$. Thus, $n = 2$ and $\beta(2, 1) = 0$.

If $\Delta = 2$, then every graph $G \in \mathcal{J}(n, 2)$ is isomorphic to either the path graph $P_n$ (if $\delta = 1$) or the cycle graph $C_n$ (if $\delta = 2$). Since $\delta(C_n) = n/4$, we conclude $\beta(n, \Delta) = n/4$.

If $\Delta = 3$ or $\Delta = 4$, then Proposition 2.4.6 and Theorem 2.4.7 provide graphs $G_{n,\Delta} \in \mathcal{J}(n, \Delta)$ with $\delta(G_{n,\Delta}) = n/4$, which implies $\beta(n, \Delta) = n/4$.

Assume $\Delta = 5$ (thus $n \geq 6$). Note that Proposition 2.4.6 and Theorem 2.4.7 give $\beta(n, 5) < n/4$, and Theorem 2.4.4 gives $\beta(n, 5) \leq (n - 1)/4$. Since $\beta(n, 4) = n/4$ for every $n \geq 5$, there exists a graph $F_n \in \mathcal{J}(n - 1, 4)$ with $\delta(F_n) = (n - 1)/4$ and $w \in V(F_n)$ such that $\deg w = 4$ for each $n \geq 6$. Consider a graph $\Gamma$ isomorphic to $P_2$ and fix a vertex $v \in V(\Gamma)$. Identify $v$ and $w$ in a single vertex $v^*$. We obtain in this way a graph $G_n \in \mathcal{J}(n, 5)$ from $F_n$ and $\Gamma$, since $\Delta = \deg v^* = 4 + 1$. Furthermore, $\{F_n, \Gamma\}$ is the biconnected decomposition of $G_n$ and Theorem 2.4.11 gives $\delta(G_n) = \delta(F_n) = (n - 1)/4$. Therefore, $\beta(n, 5) \geq \delta(G_n) = (n - 1)/4$, and we conclude $\beta(n, 5) = (n - 1)/4$.

Assume now $\Delta \geq 6$. Since $n - \Delta \geq 1$ we can consider a graph $G_1$ isomorphic to the cycle graph $C_{n-\Delta+5}$. Consider two antipodal points $x, y \in G_1$, with $x \in V(G_1)$. Denote by $\Gamma_1, \Gamma_2$ the geodesics in $G_1$ joining $x$ and $y$ with $G_1 = \Gamma_1 \cup \Gamma_2$. Denote by $v_i^j$ the vertex in $\Gamma_i$ with $d(v_i^j, x) = j$, for $i = 1, 2$ and $1 \leq j \leq (n - \Delta + 5)/2$. Note that $(n - \Delta + 5)/2 \geq 3$.

Consider a graph $G_2$ isomorphic to the star graph with $\Delta+1$ vertices $S_\Delta \in \mathcal{J}(\Delta+1,\Delta)$. Denote by $v^* \in V(G_2)$ the vertex of maximum degree in $G_2$, that is, $\deg v^* = \Delta$. Since $\Delta \geq 6$, we can choose vertices $w_j \in V(G_2) \setminus \{v^*\}$ $(j = 1,\dots,6)$.

Identify $x$ and $w_6$ in a single vertex $w^*$. For each $1 \leq j \leq 2$, identify $v_1^j \in V(\Gamma_1)$ and $w_j \in V(G_2)$ in a single vertex $v_j^*$, and for each $1 \leq j \leq 3$, identify $v_2^j \in V(\Gamma_2)$ and $w_{j+2} \in V(G_2)$ in a single vertex $v_{j+2}^*$. We obtain in this way a graph $G \in \mathcal{J}(n,\Delta)$ from $G_1$ and $G_2$, since $|V(G)| = n-\Delta+5+\Delta+1-6 = n$ and $\Delta = \deg v^*$.

Consider the geodesic triangle $T = \{x,y,z\}$ in $G$ with $z = v_4^*$, $\Gamma_1 = [xy]$ and $\Gamma_2 = [xz] \cup [zy]$. If we consider the midpoint $p$ of $\Gamma_1$, then

$$\beta(n,\Delta) \geq \delta(G) \geq d_G(p,\Gamma_2) = d_G(p,\{x,y\}) = \frac{1}{2}L(\Gamma_1) = \frac{n-\Delta+5}{4}.$$

Since Lemma 4.5.1 implies $\beta(n,\Delta) \leq (n-\Delta+5)/4$, we conclude $\beta(n,\Delta) = (n-\Delta+5)/4$. $\square$

## 4.6 Computation of $\alpha(n, \Delta)$

**Theorem 4.6.1.** *For all $n$ and $\Delta$ we have $\alpha(n, \Delta) = 0$.*

*Proof.* Consider any $n$, $\Delta$ and $G \in \mathcal{J}(n, \Delta)$. We have $\delta(G) \geq 0$ and, consequently, $\alpha(n, \Delta) \geq 0$.

Let us consider $\Delta - 1$ graphs $G_1, \ldots, G_{\Delta-1}$ isomorphic to $P_2$ and a graph $G_\Delta$ isomorphic to $P_{n-\Delta+1}$. Fix vertices $v_1 \in V(G_1), \ldots, v_\Delta \in V(G_\Delta)$ with degree 1 and consider the graph $G$ obtained from $G_1, \ldots, G_\Delta$ by identifying $v_1, \ldots, v_\Delta$ in a single vertex. Then $G \in \mathcal{J}(n, \Delta)$. Since $G$ is a tree, we have $\delta(G) = 0$. Therefore, $\alpha(n, \Delta) \leq 0$ and we conclude $\alpha(n, \Delta) = 0$. $\square$

# Chapter 5

# Circumference, girth, and hyperbolicity constant

In this Chapter we study two families of graphs: $\mathcal{M}(g,c,n)$ and $\mathcal{N}(g,c,m)$, where $\mathcal{M}(g,c,n)$ denotes the graphs $G$ of girth $g$, circumference $c$, and order $n$, whilst $\mathcal{N}(g,c,m)$ denotes the graphs $G$ of girth $g$, circumference $c$, and size $m$.

Our goal in this Chapter is to estimate $\mathcal{A}(g,c,n)$, $\mathcal{B}(g,c,n)$, $\mathfrak{A}(g,c,m)$ and $\mathfrak{B}(g,c,m)$, which are defined as follows

$$\mathcal{A}(g,c,n) := \min\{\delta(G) \mid G \in \mathcal{M}(g,c,n)\},$$

$$\mathcal{B}(g,c,n) := \max\{\delta(G) \mid G \in \mathcal{M}(g,c,n)\},$$
$$\mathfrak{A}(g,c,m) := \min\{\delta(G) \mid G \in \mathcal{N}(g,c,m)\},$$
$$\mathfrak{B}(g,c,m) := \max\{\delta(G) \mid G \in \mathcal{N}(g,c,m)\}.$$

## 5.1 Technical results

In this section we deal with several technical results, which are needed in order to estimate $\mathcal{A}(g,c,n)$, $\mathcal{B}(g,c,n)$, $\mathfrak{A}(g,c,m)$ and $\mathfrak{B}(g,c,m)$.

The following theorem gives lower and upper bounds for the hyperolicity constant of any graph in terms of its circumference and girth. It is a direct consequence of [82, Theorem 17] and [33, Lemma 2.11].

**Theorem 5.1.1.** *For every graph $G$ with $g(G) = g$ and $c(G) = c$*

$$\frac{g}{4} \le \delta(G) \le \frac{c}{4},$$

*and both inequalities are sharp.*

**Corollary 5.1.2.** *We always have*

$$\frac{g}{4} \le \mathcal{A}(g,c,n) \le \mathcal{B}(g,c,n) \le \frac{c}{4},$$

$$\frac{g}{4} \le \mathfrak{A}(g,c,m) \le \mathfrak{B}(g,c,m) \le \frac{c}{4}.$$

Now we define a family of graphs which will be useful.

**Definition 5.1.3.** *Consider non-negative integers $k$, $\beta_j$, $\beta'_j$ $(0 \le j \le k)$, and $\alpha_j$ $(0 \le j \le k+1)$, with $\beta'_0 = \beta'_k = \alpha_0 = \alpha_{k+1} = 0$, and such that*

$$\alpha_j < \beta_j + \alpha_{j+1} + \beta'_j, \tag{5.1}$$

$$\alpha_j < \beta_{j-1} + \alpha_{j-1} + \beta'_{j-1}, \tag{5.2}$$

*for $1 \le j \le k$.*

*Let $B_j$ (respectively, $B'_j$) be a path graph with endpoints $u_j$ and $v_j$ (respectively, $u'_j$ and $v'_j$) and length $\beta_j$ (respectively, $\beta'_j$) for $0 \le j \le k$ (respectively, $1 \le j \le k-1$). Let $A_j$ be a path graph with endpoints $a_j$ and $a'_j$ and length $\alpha_j$ for $1 \le j \le k$.*

*If $A = (\alpha_1, \dots, \alpha_k)$, $B = (\beta_0, \dots, \beta_k)$, $B' = (\beta'_1, \dots, \beta'_{k-1})$, then we define $G_{A,B,B'}$ as the graph obtained from $A_1, \dots, A_k$, $B_0, \dots, B_k$, $B'_1, \dots, B'_{k-1}$ by identifying the vertices $v_{j-1}$, $u_j$, and $a_j$ in a single vertex $p_j$ for each $1 \le j \le k$, the vertices $v'_{j-1}$, $u'_j$, and $a'_j$ in a single vertex $p'_j$ for each $1 < j < k$, the vertices $u_0$, $u'_1$,$a'_1$ in a single vertex $p'_1$ and the vertices $v'_{k-1}$, $v_k$, $a'_k$ in a single vertex $p'_k$.*

Denote by $C_j$ the cycle in $G_{A,B,B'}$ induced by $V(B_j) \cup V(B'_j) \cup V(A_j) \cup V(A_{j+1})$, for $0 \le j \le k$, where $V(B'_0) = V(B'_k) = V(A_0) = V(A_{k+1}) = \emptyset$. Note that $C_j \cap C_{j+1} = A_{j+1}$ for every $0 \le j \le k-1$ and $C_j \cap C_i = \emptyset$ if $|i-j| > 1$.

The following result is a direct consequence of inequalities (5.1) and (5.2).

**Lemma 5.1.4.** *If $x, y \in C_j$ with $0 \le j \le k$ and $\gamma$ is a geodesic in $G_{A,B,B'}$ joining $x$ and $y$, then $\gamma$ is contained in $C_j$.*

The following proposition gives an upper bound for the hyperbolicity constant of the graphs in this family.

**Proposition 5.1.5.**

$$\delta(G_{A,B,B'}) \le \max\left\{ \max_{0\le j\le k} \frac{\beta_j + \beta_j' + \max\{3\alpha_j + \alpha_{j+1}, \alpha_j + 3\alpha_{j+1}\}}{4}, \max_{0<j<k} \frac{\alpha_j + \alpha_{j+1} + \max\{\beta_j, \beta_j'\}}{2} \right\}.$$

*Proof.* In order to simplify notation, we shall write $G = G_{A,B,B'}$.

Theorem 2.4.1 gives that there exists some geodesic triangle $T = \{x, y, z\} \in \mathbb{T}_1$ and $p \in [xy]$ such that $\delta(G) = \delta(T) = d_G(p, [xz] \cup [yz])$.

Case (1). Assume first that $T = C_j$ for some $0 \le j \le k$. Thus, $\delta(G) = \delta(T) = L(C_j)/4 = (\beta_j + \beta_j' + \alpha_j + \alpha_{j+1})/4$.

Case (2). Assume now that $T$ is the closure of $(C_i \cup C_{i+1} \cup \cdots \cup C_{i+r}) \setminus (A_{i+1} \cup A_{i+2} \cup \cdots \cup A_{i+r})$, for some $0 \le i < i + r \le k$.

Case (2.1). Assume $p \in C_j$, with $i < j < i + r$. Then, $p \in B_j \cup B_j'$.

Assume $p \in B_j$. Since $i < j < i + r$, Lemma 5.2.6 gives that $B_j' \subseteq [xz] \cup [yz]$. Thus, we conclude

$$\delta(G) = d_G(p, [xz] \cup [yz]) \le d_G(p, \{p_j', p_{j+1}'\}) \le \frac{1}{2}(\beta_j + \alpha_j + \alpha_{j+1}).$$

Similarly, if $p \in B_j'$, we conclude $\delta(G) \le (\beta_j' + \alpha_j + \alpha_{j+1})/2$.

Case (2.2). Assume $p \in C_i$.

If $[xy] \cap \{p_{i+1}, p_{i+1}'\} = \emptyset$, then $[xy] \subset C_i$ and $L([xy]) \le L(C_i)/2$. Thus,

$$\delta(G) = d_G(p, [xz] \cup [yz]) \le d_G(p, \{x, y\}) \le \frac{1}{2}L([xy]) \le \frac{1}{4}L(C_i) \le \frac{1}{4}(\beta_i + \beta_i' + \alpha_i + \alpha_{i+1}).$$

If $p_{i+1} \in [xy]$, then by inequality (5.2) we have $p_{i+1}' \in [xz] \cup [yz]$ and $L([xy] \cap C_i) \le L(C_i)/2$.

Note that $p \in [xy] \cap C_i \subset ([xy] \cap C_i) \cup A_{i+1}$, $L(([xy] \cap C_i) \cup A_{i+1}) \le L(C_i)/2 + L(A_{i+1})$ and the endpoints of $([xy] \cap C_i) \cup A_{i+1}$ belong to $[xz] \cup [yz]$. Thus,

$$\delta(G) \le \frac{L(C_i)}{4} + \frac{\alpha_{i+1}}{2} \le \frac{\beta_i + \beta_i' + \alpha_i + \alpha_{i+1}}{4} + \frac{\alpha_{i+1}}{2} = \frac{\beta_i + \beta_i' + \alpha_i + 3\alpha_{i+1}}{4}.$$

Analogously, if $p_{i+1}' \in [xy]$, we obtain the same result.

Case (2.3). Finally, if $p \in C_{i+r}$, then a similar argument to the one in (2.2) gives

$$\delta(G) \le \frac{\beta_{i+r} + \beta_{i+r}' + 3\alpha_{i+r} + \alpha_{i+r+1}}{4}.$$

Since

$$\beta_0 + \beta_0' + \alpha_0 + 3\alpha_1 = \beta_0 + 3\alpha_1 = \beta_0 + \beta_0' + \max\{3\alpha_0 + \alpha_1, \alpha_0 + 3\alpha_1\},$$

$$\beta_k + \beta_k' + 3\alpha_k + \alpha_{k+1} = \beta_k + 3\alpha_k = \beta_k + \beta_k' + \max\{3\alpha_k + \alpha_{k+1}, \alpha_k + 3\alpha_{k+1}\},$$

we conclude

$$\delta(G_{A,B,B'}) \le \max\left\{ \max_{0\le j\le k} \frac{\beta_j + \beta_j' + \max\{3\alpha_j + \alpha_{j+1}, \alpha_j + 3\alpha_{j+1}\}}{4}, \max_{0<j<k} \frac{\alpha_j + \alpha_{j+1} + \max\{\beta_j, \beta_j'\}}{2} \right\}$$

in any case. □

Proposition 5.1.5 has the following consequence.

**Corollary 5.1.6.** *If $\alpha_j = 1$ for $0 < j < k$, then*

$$\delta(G_{A,B,B'}) \le \max\left\{ \frac{1}{2} + \frac{1}{4}L(C_0), \frac{1}{2} + \frac{1}{4}L(C_k), \max_{0<j<k} \frac{2 + \max\{\beta_j, \beta_j'\}}{2} \right\}.$$

*Proof.* The following equalities hold for $0 \le j \le k$

$$\beta_j + \beta_j' + \max\{3\alpha_j + \alpha_{j+1}, \alpha_j + 3\alpha_{j+1}\} = \beta_j + \beta_j' + \alpha_j + \alpha_{j+1} + \max\{2\alpha_j, 2\alpha_{j+1}\} = L(C_j) + 2.$$

Furthermore, if $0 < j < k$, then

$$\beta_j + \beta_j' + \max\{3\alpha_j + \alpha_{j+1}, \alpha_j + 3\alpha_{j+1}\} = \beta_j + \beta_j' + 4 \le 2(\max\{\beta_j, \beta_j'\} + 2).$$

Since $2\max\{\beta_j, \beta_j'\} \le \beta_j + \beta_j' + 1$, we deduce

$$\frac{\alpha_j + \alpha_{j+1} + \max\{\beta_j, \beta_j'\}}{2} \le \frac{2 + (\beta_j + \beta_j' + 1)/2}{2} = \frac{3/2 + (\beta_j + \beta_j' + 2)/2}{2} = \frac{3}{4} + \frac{1}{4}L(C_j).$$

□

In what follows, we denote by $C_{a_1,a_2,a_3}$ the graph with three paths with lengths $a_1 \le a_2 \le a_3$ joining two fixed vertices.

The next corollary was proved in [99, Theorem 12]. We provide here a simpler proof following a different approach.

**Corollary 5.1.7.** $\delta(C_{a_1,a_2,a_3}) = (a_3 + \min\{a_2, 3a_1\})/4.$

*Proof.* Consider a graph $G_{A,B,B'}$ as in Definition 5.1.3, with $k = 1$, $\alpha_1 = a_1$, $\beta_0 = a_3$ and $\beta_1 = a_2$. Thus, $G_{A,B,B'}$ is the union of three paths $A_0, B_0, B_1$ joining $p_1$ and $p_1'$.

Since $\beta_0' = \beta_1' = \alpha_0 = \alpha_2 = 0$ and $a_1 < a_2 \le a_3$, we have $\alpha_1 < \min\{\beta_0, \beta_1\}$, and equations (5.1) and (5.2) hold. Thus, we can write $G_{A,B,B'} = C_{a_1,a_2,a_3}$.

Assume first that $3a_1 \le a_2$.

Let $T = \{x, y, z\}$ be the geodesic triangle which is the closure of $(C_0 \cup C_1) \setminus A_1$, with $x \in B_0$, $y, z \in B_1$, $d(x, p_1) = (a_3 + a_1)/2$, $d(y, p_1) = a_1$ and $d(z, p_1) = 3a_1$. Let $p$ be the midpoint of $[xy]$. Then,

$$\delta(T) = d_G(p, [xz] \cup [yz]) = d_G(p, \{x, y\}) = \frac{L([xy])}{2} = \frac{1}{2}\Big(\frac{a_3 + a_1}{2} + a_1\Big) = \frac{a_3 + 3a_1}{4}.$$

Therefore, $\delta(C_{a_1,a_2,a_3}) \geq (a_3 + 3a_1)/4$.

If $a_2 < 3a_1$, then let $T = \{x, y, z\}$ be the geodesic triangle which is the closure of $(C_0 \cup C_1) \setminus A_1$, with $x \in B_0$, $y \in B_1$, $d(x, p_1) = (a_3 + a_1)/2$, $d(y, p_1) = (a_2 - a_1)/2 < a_1$ and $z = p_1'$. Let $p$ be the midpoint of $[xy]$. Then,

$$\delta(T) = d_G(p, [xz] \cup [yz]) = d_G(p, \{x, y\}) = \frac{L([xy])}{2} = \frac{1}{2}\Big(\frac{a_3 + a_1}{2} + \frac{a_2 - a_1}{2}\Big) = \frac{a_3 + a_2}{4}.$$

Thus, $\delta(C_{a_1,a_2,a_3}) \geq (a_3 + \min\{a_2, 3a_1\})/4$ in both cases.

Let us prove the converse inequality. Assume first that $a_1 < a_2$. Proposition 5.1.5 gives $\delta(C_{a_1,a_2,a_3}) \leq \max\{a_3 + 3a_1, a_2 + 3a_1\}/4 = (a_3 + 3a_1)/4$. On the other hand, Theorem 5.1.1 gives $\delta(C_{a_1,a_2,a_3}) \leq c(C_{a_1,a_2,a_3})/4 = (a_3 + a_2)/4$. Thus, we conclude $\delta(C_{a_1,a_2,a_3}) \leq (a_3 + \min\{a_2, 3a_1\})/4$.

Finally, assume that $a_1 = a_2$. Thus, $\delta(C_{a_1,a_2,a_3}) \leq c(C_{a_1,a_2,a_3})/4 = (a_3 + a_2)/4 = (a_3 + \min\{a_2, 3a_1\})/4$.

Thus, we conclude $\delta(C_{a_1,a_2,a_3}) = (a_3 + \min\{a_2, 3a_1\})/4$. □

**Corollary 5.1.8.** *$\delta(C_{a_1,a_2,a_3}) \leq (a_3 + 3a_1)/4$.*

**Lemma 5.1.9.** *For every graph $G$, $\operatorname{diam}(G) \leq 2$ if and only if $d(v, e) \leq 1$ for every $v \in V(G)$ and $e \in E(G)$.*

*Proof.* Assume that $\operatorname{diam}(G) \leq 2$. Given $v \in V(G)$ and $e \in E(G)$, if $p$ is the midpoint of $e$, then $d(v, p) \leq 3/2$, since $d(v, p)$ is an odd multiple of $1/2$ less than 2. Hence, $d(v, e) = d(v, p) - 1/2 \leq 1$.

Assume now that $d(v, e) \leq 1$ for $v \in V(G)$ and $e \in E(G)$. Given $v, w \in V(G)$, choose $e \in E(G)$ with $w \in e$; thus, $d(v, w) \leq d(v, e) + 1 \leq 2$. If $v \in V(G)$ and $p$ is the midpoint of $e \in E(G)$, then $d(p, v) \leq d(v, e) + 1/2 \leq 3/2$. Finally, consider $p, q$ midpoints of $e_p, e_q \in E(G)$, respectively; if $v$ is a vertex of $e_q$, then $d(v, p) \leq 3/2$ and $d(p, q) \leq d(p, v) + d(v, q) \leq 3/2 + 1/2 = 2$. Hence, $\operatorname{diam}(G) \leq 2$. □

**Lemma 5.1.10.** *The integers $a_1 := n - c + 1$, $a_2 := g + c - n - 1$ and $a_3 := n - g + 1$ are the only real numbers satisfying*

(1) *the following equations:*

$$a_1 + a_2 = g, \tag{5.3}$$

$$a_2 + a_3 = c, \tag{5.4}$$

$$a_1 + a_2 + a_3 = n + 1. \tag{5.5}$$

(2) $a_1 \le a_2 \Leftrightarrow n \le c - 1 + g/2$.
(3) $a_2 \le a_3 \Leftrightarrow n \ge g - 1 + c/2$.
(4) $a_2 \le 3a_1 \Leftrightarrow n \ge c - 1 + g/4$.
(5) $a_1 \ge 1$ *and* $a_2 \ge 2$ *if* $a_1 = 1$.

*Proof.* Equations (5.6), (5.7), (5.8), and (5) follow directly.

Consider the system of linear equations in (1). Since the coefficient matrix is non-sigular, $a_1$, $a_2$ and $a_3$ are the only real numbers satisfying (5.6), (5.7), and (5.8).

The condition $n \le c + g/2 - 1$ is equivalent to

$$2n + 2 \le g + 2c \qquad \Leftrightarrow \qquad n - c + 1 \le g + c - n - 1 \qquad \Leftrightarrow \qquad a_1 \le a_2.$$

On the other hand, the condition $n \ge g - 1 + c/2$ is equivalent to

$$c + 2g - 2 \le 2n \qquad \Leftrightarrow \qquad g + c - n - 1 \le n - g + 1 \qquad \Leftrightarrow \qquad a_2 \le a_3.$$

Finally, the condition $n \ge c - 1 + g/4$ is equivalent to

$$a_1 + a_2 + a_3 - 1 \ge a_2 + a_3 + (a_1 + a_2)/4 - 1 \qquad \Leftrightarrow \qquad a_2 \le 3a_1.$$

□

We say that the triplet $(g, c, n)$ is *v-admissible* if $\mathcal{G}(c, g, n)$ is not the empty set.

**Lemma 5.1.11.** *The triplet $(g, c, n)$ is v-admissible if and only if we have either $g = c \le n$ or $g < c$ and $n \ge g - 1 + c/2$.*

*Proof.* Assume that $(g, c, n)$ is v-admissible. If $g = c$, then there is nothing to prove.

Assume $g < c$ and consider any graph $G \in \mathcal{M}(c, g, n)$. Denote by $C_g$ and $C_c$ two cycles in $G$ with lenghts $g$ and $c$, respectively. If there is no path $\eta$ joining two different vertices of $C_c$ with $\eta \not\subset C_c$, then $C_c \cap C_g$ contains at most a vertex and we conclude $n \ge g-1+c > g-1+c/2$.

Assume now that such path $\eta$ exists. Without loss of generality we can assume that $\eta \cap C_c$ is exactly two vertices. Let $\{u, v\} = \eta \cap C_c$ and consider the two different paths $\eta_1, \eta_2$ contained in $C_c$ and joining $u$ and $v$.

Define $b_0 = L(\eta)$, $b_1 = L(\eta_1)$ and $b_2 = L(\eta_2)$. Thus, $b_1 + b_2 = c$, $b_0 + b_1 \ge g$ and $b_0 + b_2 \ge g$. Then $b_0 \ge g - b_1$, $b_0 \ge g - b_2$, $b_0 \ge (2g - b_1 - b_2)/2 = g - c/2$ and $n + 1 \ge b_0 + c \ge g - c/2 + c = g + c/2$.

Consider now positive integers $g, c, n$ with either $g = c$ or $g < c$ and $n \ge g - 1 + c/2$.

If $g = c$, then let us define define $k := n - g \ge 0$. Consider a graph $G_0$ isomorphic to the cycle graph $C_g$, and $k$ graphs $G_i$, $1 \le i \le k$ isomorphic to the path graph $P_2$. Fix a vertex $v_i \in V(G_i)$ for each $0 \le i \le k$. Let $G$ be the graph obtained from $G_0, G_1, \dots, G_k$ by identifying $v_0, v_1, \dots, v_k$ in a single vertex. It is clear that $|V(G)| = g + k = n$, $g(G) = c(G) = c(G_0) = g$. Thus, $G \in \mathcal{M}(g, g, n)$.

Consider now the case where $g < c$ and $n \geq g - 1 + c/2$.

Assume first that $g - 1 + c/2 \leq n \leq c - 1 + g/2$.

Consider three natural numbers $a_1 = n - c + 1$, $a_2 = g + c - n - 1$ and $a_3 = n - g + 1$. Lemma 5.1.10 gives $a_1 + a_2 = g$, $a_2 + a_3 = c$, $a_1 + a_2 + a_3 = n + 1$, and $a_1 \leq a_2 \leq a_3$. Thus, $C_{a_1,a_2,a_3} \in \mathcal{M}(g, c, n)$ and $(g, c, n)$ is v-admissible.

Finally, assume that $n > c - 1 + g/2$. Let us define $a_1 = \lfloor g/2 \rfloor$, $a_2 = g - a_1$, $a_3 = c - a_2$, where $\lfloor t \rfloor$ denotes the lower integer part of $t$, i.e., the largest integer not greater than $t$. Since $2a_2 \leq g < c = a_2 + a_3$, we have $a_2 < a_3$; furthermore, $a_1 \leq a_2$, and we can define $G_0 := C_{a_1,a_2,a_3}$. Note that $g(G_0) = g$, $c(G_0) = c$, and $|V(G_0)| = c + a_1 - 1$. Let $k := n - (c - 1 + a_1) > 0$ and consider $k$ graphs $G_i$, $1 \leq i \leq k$ isomorphic to the path graph $P_2$. Fix a vertex $v_i \in V(G_i)$ for each $0 \leq i \leq k$. Let $G$ be the graph obtained from $G_0, G_1, \dots, G_k$ by identifying $v_0, v_1, \dots, v_k$ in a single vertex. It is clear that $|V(G)| = |V(G_0)| + k = c + a_1 - 1 + n - (c - a_1 - 1) = n$, $c(G) = c(G_0) = c$ and $g(G) = g(G_0) = g$. Thus, $G \in \mathcal{M}(g, c, n)$ and $(g, c, n)$ is v-admissible. □

**Lemma 5.1.12.** *If $(g, c, n)$ is a v-admissible triplet and $n'$ is an integer with $n' \geq n$, then $(g, c, n')$ is a v-admissible triplet and*

$$\mathcal{A}(g, c, n') \leq \mathcal{A}(g, c, n) \leq \mathcal{B}(g, c, n) \leq \mathcal{B}(g, c, n').$$

*Proof.* Lemma 5.1.11 gives that $(g, c, n')$ is a v-admissible triplet.

If $n' = n$, Corollary 5.1.2 gives the desired result. Thus, assume that $n' > n$.

It suffices to prove that for each $G_0 \in \mathcal{M}(g, c, n)$, there exists $G \in \mathcal{M}(g, c, n')$ with $\delta(G) = \delta(G_0)$.

If $g = c$, then Corollary 5.1.2 implies $\mathcal{A}(c, c, n') = \mathcal{A}(c, c, n) = \mathcal{B}(c, c, n) = \mathcal{B}(c, c, n') = c/4$.

Assume now that $g < c$ and consider a graph $G_0 \in \mathcal{M}(g, c, n)$ and graphs $G_i$ isomorphic to $P_2$, $1 \leq i \leq n' - n$. Fix vertices $u_i \in V(G_i)$, for $0 \leq i \leq n' - n$. Denote by $G$ the graph obtained from $G_0, G_1, \dots, G_{n'-n}$ by identifying $u_0, u_1, \dots, u_{n'-n}$ in a single vertex $v$. Since $v$ is a cut-vertex, the graphs $G_i$, $0 \leq i \leq n' - n$ are a T-decomposition of $G$ and Theorem 2.4.11 implies $\delta(G) = \delta(G_0)$. □

**Theorem 5.1.13.** *Let $(g, c, n)$ be a v-admissible triplet and $r$ a positive integer. Consider graphs $G_1, G_2 \in \mathcal{M}(g, c, n)$ with $m_1, m_2$ edges, respectively, and such that $\delta(G_1) = \mathcal{A}(g, c, n)$ and $\delta(G_2) = \mathcal{B}(g, c, n)$. Then*

$$\mathcal{A}(rg, rc, n_1) \leq r\mathcal{A}(g, c, n) \leq r\mathcal{B}(g, c, n) \leq \mathcal{B}(rg, rc, n_2),$$

*for every $n_1 \geq n + (r - 1)m_1$ and $n_2 \geq n + (r - 1)m_2$.*

*Proof.* Denote by $G_1^{(r)}$ the graph obtained from $G_1$ by replacing each original edge with a path of legth $r$. Thus, $|V(G_1^{(r)})| = n + (r - 1)m_1$, $g(G_1^{(r)}) = rg$ and $c(G_1^{(r)}) = rc$. It is clear that

$$\mathcal{A}(rg, rc, n + (r - 1)m_1) \leq \delta(G_1^{(r)}) = r\delta(G_1) = r\mathcal{A}(g, c, n).$$

If $n_1 \geq n + (r-1)m_1$, then Lemma 5.1.9 allows to conclude $\mathcal{A}(rg, rc, n_1) \leq \mathcal{A}(rg, rc, n + (r-1)m_1) \leq r\mathcal{A}(g, c, n)$.

Analogously, we have $\mathcal{B}(rg, rc, n_2) \geq r\mathcal{B}(g, c, n)$. □

**Corollary 5.1.14.** *Let $(g, c, n)$ be a v-admissible triplet and $r$ a positive integer. Consider graphs $G_1, G_2 \in \mathcal{M}(g, c, n)$ with $m_1, m_2$ edges, respectively, and such that $\delta(G_1) = \mathcal{A}(g, c, n) = g/4$ and $\delta(G_2) = \mathcal{B}(g, c, n) = c/4$. Then $\mathcal{A}(rg, rc, n_1) = rg/4$ for every $n_1 \geq n + (r-1)m_1$ and $\mathcal{B}(rg, rc, n_2) = rc/4$ for every $n_2 \geq n + (r-1)m_2$.*

The argument in the proof of Lemma 5.1.10 gives the following result.

**Lemma 5.1.15.** *The integers $a_1 := m - c$, $a_2 := g + c - m$ and $a_3 := m - g$ are the only real numbers satisfying*

(1) *the following equations:*

$$a_1 + a_2 = g, \tag{5.6}$$

$$a_2 + a_3 = c, \tag{5.7}$$

$$a_1 + a_2 + a_3 = m. \tag{5.8}$$

(2) $a_1 \leq a_2 \Leftrightarrow m \leq c + g/2$.
(3) $a_2 \leq a_3 \Leftrightarrow m \geq g + c/2$.
(4) $a_2 \leq 3a_1 \Leftrightarrow m \geq c + g/4$.
(5) $a_1 \geq 1$ *and* $a_2 \geq 2$ *if* $a_1 = 1$.

We say that the triplet $(g, c, m)$ is *e-admissible* if $\mathcal{N}(c, g, m)$ is not the empty set.

The argument in the proof of Lemma 5.1.11, using Lemma 5.1.15 instead of Lemma 5.1.10, gives the following result.

**Lemma 5.1.16.** *The triplet $(g, c, m)$ is e-admissible if and only if we have either $g = c \leq m$ or $g < c$ and $m \geq g + c/2$.*

The arguments in the proofs of Lemma 5.1.12 and Theorem 5.1.13, respectively, give the following results.

**Lemma 5.1.17.** *If $(g, c, m)$ is a e-admissible triplet and $m'$ is an integer with $m' \geq m$, then*

$$\mathfrak{A}(g, c, m') \leq \mathfrak{A}(g, c, m) \leq \mathfrak{B}(g, c, m) \leq \mathfrak{B}(g, c, m').$$

**Theorem 5.1.18.** *If $(g, c, m)$ is an e-admissible triplet and $r$ is a positive integer, then*

$$\mathfrak{A}(rg, rc, rm) \leq r\mathfrak{A}(g, c, m) \leq r\mathfrak{B}(g, c, m) \leq \mathfrak{B}(rg, rc, rm).$$

**Corollary 5.1.19.** *Let $(g, c, m)$ be an e-admissible triplet and $r$ a positive integer. Consider graphs $G_1, G_2 \in \mathcal{N}(g, c, m)$ such that $\delta(G_1) = \mathfrak{A}(g, c, m) = g/4$ and $\delta(G_2) = \mathfrak{B}(g, c, m) = c/4$. Then, $\mathfrak{A}(rg, rc, rm) = rg/4$ and $\mathfrak{B}(rg, rc, rm) = rc/4$.*

## 5.2 Bounds for $\mathcal{A}(g, c, n)$

The following result characterizes the graphs with hyperbolicity constant 1 (see [12, Theorem 3]).

**Theorem 5.2.1.** *Let $G$ be any graph. Then $\delta(G) = 1$ if and only if* $\operatorname{diameff}(G) = 2$.

The following theorem appears in [82, Theorem 7].

**Theorem 5.2.2.** *Let $G$ be any graph. If there exists a cycle $C$ in $G$ with length $L(C) \geq 4$, then*

$$\delta(G) \geq \frac{1}{4} \min \{\sigma \text{ is a cycle in } G \text{ with } L(\sigma) \geq 4\}.$$

Let us start by computing $\mathcal{A}(g, c, n)$ for $g = 3$ and $g = 4$.

**Theorem 5.2.3.** *For any integers $3 \leq c \leq n$ we have*

$$\mathcal{A}(3, c, n) = \begin{cases} 3/4, & \text{if } c = 3, \\ 1, & \text{if } c > 3. \end{cases}$$

*Proof.* If $g = c = 3$, Corollary 5.1.2 gives $A(3, 3, n) = 3/4$.

If $g = 3$, $c \geq 4$ and $G \in \mathcal{M}(3, c, n)$, then Theorem 5.2.2 gives $\delta(G) \geq 1$. Thus, $\mathcal{A}(3, c, n) \geq 1$.

Let us consider the complete graph with $c$ vertices $K_c$, and $n - c$ graphs $G_1, \dots, G_{n-c}$ isomorphic to the path graph $P_2$. Fix $v_0 \in V(K_c)$ and $v_j \in G_j$ for $1 \leq j \leq n - c$. Let $G_0$ be the graph obtained from $K_c, G_1, \dots, G_{n-c}$ by identifying the vertices $v_0, v_1, \dots, v_{n-c}$ in a single vertex $v$. Thus, $G_0 \in \mathcal{G}(3, c, n)$. Since $v$ is a cut vertex of $G_0$, $\{K_c, G_1, \dots, G_{n-c}\}$ is the biconnected decomposition of $G_0$. We have $\delta(G_1) = \cdots = \delta(G_{n-c}) = 0$, and Theorem 2.4.11 gives $\delta(G_0) = \delta(K_c) = 1$. Since $\mathcal{A}(3, c, n) \leq \delta(G_0) = 1$, we conclude $\mathcal{A}(3, c, n) = 1$. $\square$

**Theorem 5.2.4.** *For every v-admissible triplet $(4, c, n)$,*

$$\mathcal{A}(4, c, n) = \begin{cases} 1, & \text{if } c \text{ is even}, \\ 5/4, & \text{if } c \text{ is odd}. \end{cases}$$

*Proof.* Corollary 5.1.2 gives $\mathcal{A}(4, c, n) \geq 1$.

Assume first that $c$ is even. Let $\Gamma_c$ be the graph defined by $V(\Gamma_c) = \{v_1, \dots, v_c\}$ and $E(\Gamma_c) = \{[v_i, v_j] \mid 1 \leq i, j \leq c,\ i + j \text{ is odd}\}$. In particular, $[v_1, v_2], \dots, [v_{c-1}, v_c], [v_c, v_1] \in E(\Gamma_c)$, $g(\Gamma_c) = 4$ and $c(\Gamma_c) = c$. That is, we have $\Gamma_c \in \mathcal{M}(4, c, c)$.

Let us prove that given two edges $e_1, e_2 \in E(\Gamma_c)$, there exists a cycle $\sigma$ with $e_1, e_2 \subset \sigma$ and $L(\sigma) = 4$. If $e_1 = [v_{i_1}, v_{i_2}]$ and $e_2 = [v_{i_2}, v_{i_3}]$, then $i_1 + i_2$ and $i_2 + i_3$ are odd, and thus, $i_1 + i_3$ is even. Since $c \geq 4$, there exists $i_4 \notin \{i_1, i_2, i_3\}$ such that $i_2 + i_4$ is even. Hence, $i_1 + i_4$ and $i_3 + i_4$ are even, and the cycle $[v_{i_1}, v_{i_2}] \cup [v_{i_2}, v_{i_3}] \cup [v_{i_3}, v_{i_4}] \cup [v_{i_4}, v_{i_1}]$ contains $e_1$ and $e_2$.

Assume that $e_1 = [v_{j_1}, v_{j_2}]$ and $e_2 = [v_{j_3}, v_{j_4}]$, with $e_1 \cap e_2 = \emptyset$. Since $j_3 + j_4$ is odd, we have that either $j_1 + j_3$ or $j_1 + j_4$ is odd. By symmetry, we can assume that $j_1 + j_3$ is odd. Thus, $j_1 + j_4$ is even and $j_2 + j_4$ is odd. Hence, $[v_{j_1}, v_{j_2}] \cup [v_{j_2}, v_{j_4}] \cup [v_{j_4}, v_{j_3}] \cup [v_{j_3}, v_{j_1}]$ is the required cycle.

Therefore, we conclude that $\operatorname{diam} V(\Gamma_c) \leq 2$, since every two points in $\Gamma_c$ are contained in a cycle with length 4. Finally, Theorem 2.4.2 gives $\delta(\Gamma_c) \leq 1$.

Thus, Lemma 5.1.12 gives $1 \leq \mathcal{A}(4, c, n) \leq \mathcal{A}(4, c, c) \leq \delta(\Gamma_c) \leq 1$, and we deduce $A(4, c, n) = 1$.

Assume that $c$ is odd. Seeking for a contradiction, assume that $\mathcal{A}(4, c, n) = 1$, i.e., there exists $G \in \mathcal{M}(4, c, n)$ with $\delta(G) = 1$. Let $C_c$ be a cycle in $G$ with $L(C_c) = c$ and $G_0$ be the two-connected component of $G$ containing $C_c$. By Theorem 5.2.1, $\operatorname{diam}(G_0) \leq 2$. Fix $v \in V(C_c)$. By Lemma 5.1.9, we have $d_{G_0}(v, e) \leq 1$ for every $e \in E(G_0)$.

Denote by $v, v_2, \dots, v_c$ the vertices in $C_c$ such that $[v, v_2], [v_2, v_3], \dots, [v_c, v] \subset C_c$. Since $d_{G_0}(v, e) \leq 1$ for every $e \in E(G_0)$ and $g(G_0) \geq g(G) = 4$, we can prove inductively that $[v, v_{2j}] \in E(G_0)$ for every $1 \leq j \leq (c-1)/2$ and $[v, v_{2j+1}] \notin E(G_0)$ for every $1 \leq j \leq (c-1)/2$. In particular, if $j = (c-1)/2$, then $[v, v_c] \notin E(G_0)$, which is a contradiction. Hence, $\mathcal{A}(4, c, n) > 1$. By Theorem Theorem 2.4.4, we have $\mathcal{A}(4, c, n) \geq 5/4$.

Assume first that $c = 5$. Then, Lemma 5.1.11 gives that the tripet $(4, 5, n)$ is v-admissible if and only if $n \geq 6$. Corollary 5.1.2 and Lemma 5.1.12 give $5/4 \leq \mathcal{A}(4, 5, n) \leq \mathcal{A}(4, 5, 6) \leq 5/4$. Thus, $\mathcal{A}(4, 5, n) = 5/4$ for every v-admissible triplet $(4, 5, n)$.

Assume $c > 5$. Consider the graph $\Gamma_{c-1}$ defined as before. Denote by $\Lambda_c$ the graph obtained from $\Gamma_{c-1}$ by replacing a fixed edge $e_0 \in E(C_{c-1})$ by a path $\eta$ of length 2. Since $c \neq 5$, $\Lambda_c \in \mathcal{M}(4, c, c)$ and $(4, c, c)$ is v-admissible. The previous argument gives that any two points in $\Lambda_c$ are contained in a cycle with length at most 5, and therefore, $\operatorname{diam}(\Lambda_c) \leq 5/2$. Thus, Lemma 2.4.2 implies $\delta(\Lambda_c) \leq 5/4$. Lemma 5.1.12 gives $5/4 \leq \mathcal{A}(4, c, n) \leq \mathcal{A}(4, c, c) \leq \delta(\Lambda_c) \leq 5/4$, and we conclude $\mathcal{A}(4, c, n) = 5/4$. ☐

The next result provides good bounds for $\mathcal{A}(g, c, n)$.

**Theorem 5.2.5.** *Let $(g, c, n)$ be a v-admissible triplet with $g \geq 5$.*

- *If $2g - 2 \leq c < 3g - 4$ with $c = 2g - 2 + s$ $(0 \leq s \leq g - 3)$, then*
$$\frac{g}{4} \leq \mathcal{A}(g, c, n) \leq \frac{g + 2 + s}{4}.$$
- *If $r$ is a positive integer, $g$ is even and $c = 2g - 2 + r(g - 2)$, then*
$$\frac{g}{4} \leq \mathcal{A}(g, c, n) \leq \frac{g + 2}{4}.$$
- *If $r$ is a positive integer, $g$ is odd and $2g - 2 + r(g - 2) \leq c \leq 2g - 1 + r(g - 2)$, then*
$$\frac{g}{4} \leq \mathcal{A}(g, c, n) \leq \frac{g + 3}{4}.$$

- *If $r$ and $s$ are integers with $r \geq 1$, $s \geq 0$, $g$ is even and $2g-2+r(g-2) < c \leq 2g-2+r(g-2)+2(r+1)(s+1)$, then*

$$\frac{g}{4} \leq \mathcal{A}(g,c,n) \leq \frac{g+4+2s}{4}.$$

- *If $r$ and $s$ are integers with $r \geq 1$, $s \geq 0$, $g$ is odd and $2g-1+r(g-2) < c \leq 2g-2+r(g-2)+2(r+1)(s+1)$, then*

$$\frac{g}{4} \leq \mathcal{A}(g,c,n) \leq \frac{g+5+2s}{4}.$$

*Proof.* Case 1.

Assume that $2g-2 \leq c < 3g-4$ with $c = 2g-2+s$ $(0 \leq s \leq g-3)$.

Consider the graph $C_{a_1,a_2,a_3}$ with $a_1 = 1$, $a_2 = g-1$ and $a_3 = g-1+s$. Note that $g(C_{a_1,a_2,a_3}) = a_1 + a_2 = g$, $c(C_{a_1,a_2,a_3}) = a_2 + a_3 = 2g-2+s = c$ and thus, $C_{a_1,a_2,a_3} \in \mathcal{M}(g,c,c)$. Since $g \geq 5$, Corollary 5.1.7 gives $\delta(C_{a_1,a_2,a_3}) = (a_3 + \min\{a_2, 3a_1\})/4 = (g-1+s+\min\{g-1,3\})/4 = (g+2+s)/4$. Thus, Corollary 5.1.2 and Lemma 5.1.12 imply $g/4 \leq \mathcal{A}(g,c,n) \leq \mathcal{A}(g,c,c) \leq \delta(C_{a_1,a_2,a_3}) \leq (g+2+s)/4$.

Case 2.

Assume that $2g-2+r(g-2) \leq c \leq 2g-1+r(g-2)$, with $r \geq 1$.

Since $r \geq 1$, it follows that $c \geq 2g-2+r(g-2) \geq 3g-4$.

Consider a graph $G_{A,B,B'}$ as in Definition 5.1.3, with $k = r+1$, $\beta_0 = g-1$ and $\alpha_j = 1$ for $1 \leq j \leq k$.

Case 2.1.

If $g$ is even and $c = 2g-2+r(g-2)$, then let $\beta_k = g-1$ and $\beta_j = \beta'_j = g/2-1$ for $1 \leq j \leq k-1$ in the previous graph $G_{A,B,B'}$. Thus,

$$\sum_{j=0}^{k} (\beta_j + \beta'_j) = g-1+2r(g/2-1)+g-1 = 2g-2+r(g-2) = c.$$

Note that $g(G_{A,B,B'}) = L(C_0) = \alpha_1 + \beta_0 = g$, $c(G_{A,B,B'}) = \sum_{j=0}^{k}(\beta_j + \beta'_j) = c$ and thus, $G_{A,B,B'} \in \mathcal{G}(g,c,c)$.

Since $L(C_j) = g$ for $0 \leq j \leq k$ and $2 + \max\{\beta_j, \beta'_j\} = 1 + g/2$ for $0 < j < k$, Corollaries 5.1.2 and 5.1.6 and Lemma 5.1.12 give

$$\frac{g}{4} \leq \mathcal{A}(g,c,n) \leq \mathcal{A}(g,c,c) \leq \delta(G_{A,B,B'}) \leq \max\left\{\frac{2+g}{4}, \frac{2+g}{4}\right\} = \frac{2+g}{4}.$$

Case 2.2.

Similarly, if $g$ is odd, consider a graph $G_{A,B,B'}$ as before with $g-1 \le \beta_k \le g$, $\beta_j = (g-1)/2$ and $\beta'_j = (g-3)/2$ for $1 \le j \le k-1$. Since

$$g-1+r((g-1)/2+(g-3)/2)+g-1 \le \sum_{j=0}^{k}(\beta_j+\beta'_j) \le g-1+r((g-1)/2+(g-3)/2)+g,$$

$$2g-2+r(g-2) \le \sum_{j=0}^{k}(\beta_j+\beta'_j) \le 2g-1+r(g-2),$$

we can choose $\beta_k$ with the additional property $\sum_{j=0}^{k}(\beta_j+\beta'_j) = c$.

Note that $g(G_{A,B,B'}) = L(C_0) = \alpha_1 + \beta_0 = g$, $c(G_{A,B,B'}) = \sum_{j=0}^{k}(\beta_j+\beta'_j) = c$ and thus, $G_{A,B,B'} \in \mathcal{M}(g,c,c)$.

Since $L(C_j) \le g+1$ for $0 \le j \le k$ and $2+\max\{\beta_j,\beta'_j\} = (g+3)/2$ for $0 < j < k$, Corollaries 5.1.2 and 5.1.6 and Lemma 5.1.12 give

$$\frac{g}{4} \le \mathcal{A}(g,c,n) \le \mathcal{A}(g,c,c) \le \delta(G_{A,B,B'}) \le \max\left\{\frac{2+g+1}{4}, \frac{3+g}{4}\right\} = \frac{3+g}{4}.$$

Case 3.

Assume now that $r$ and $s$ are integers with $r \ge 1$, $s \ge 0$, $g$ is even and $2g-2+r(g-2) < c \le 2g-2+r(g-2)+2(r+1)(s+1)$.

Consider a graph $G_{A,B,B'}$ as in Definition 5.1.3, with $k = r+1$, $\beta_0 = g-1$, $g-1+2s < \beta_k \le g-1+2(s+1)$ and $\alpha_j = 1$ for $1 \le j \le k$.

If $g$ is even, let $g/2-1+s \le \beta_j, \beta'_j \le g/2+s$ for $0 < j < k$. Since

$$g-1+2r(g/2-1+s)+g-1+2s < \sum_{j=0}^{k}(\beta_j+\beta'_j) \le g-1+2r(g/2+s)+g-1+2(s+1),$$

$$2g-2+r(g-2) \le 2g-2+r(g-2)+2(r+1)s < \sum_{j=0}^{k}(\beta_j+\beta'_j) \le 2g-2+r(g-2)+2(r+1)(s+1),$$

we can choose $\beta_j, \beta'_j$ with the additional property $\sum_{j=0}^{k}(\beta_j+\beta'_j) = c$.

Note that $g(G_{A,B,B'}) = L(C_0) = \alpha_1 + \beta_0 = g$, $c(G_{A,B,B'}) = \sum_{j=0}^{k}(\beta_j+\beta'_j) = c$ and thus, $G_{A,B,B'} \in \mathcal{M}(g,c,c)$. Since $L(C_j) \le g+2s+2$ for $0 \le j \le k$ and $2+\max\{\beta_j,\beta'_j\} \le (g+4)/2+s$ for $0 < j < k$, Corollaries 5.1.2 and 5.1.6 and Lemma 5.1.12 give

$$\frac{g}{4} \le \mathcal{A}(g,c,n) \le \mathcal{A}(g,c,c) \le \delta(G_{A,B,B'}) \le \max\left\{\frac{4+g+2s}{4}, \frac{4+g+2s}{4}\right\} = \frac{g+4+2s}{4}.$$

Case 4.

Assume now that $r$ and $s$ are integers with $r \geq 1$, $s \geq 0$, $g$ is odd and $2g-1+r(g-2) < c \leq 2g-2+r(g-2)+2(r+1)(s+1)$.

Consider a graph $G_{A,B,B'}$ as in Definition 5.1.3, with $k = r+1$, $\beta_0 = g-1$, $g+2s < \beta_k \leq g-1+2(s+1)$, $\alpha_j = 1$ for $1 \leq j \leq k$ and $(g-1)/2+s \leq \beta_j \leq (g+1)/2+s$, $(g-3)/2+s \leq \beta'_j \leq (g-1)/2+s$ for $0 < j < k$. Since

$$g-1+r((g-1)/2+s+(g-3)/2+s)+g+2s < \sum_{j=0}^{k}(\beta_j+\beta'_j)$$

$$\leq g-1+r((g+1)/2+s+(g-1)/2+s)+g-1+2(s+1),$$

$$2g-1+r(g-2) < \sum_{j=0}^{k}(\beta_j+\beta'_j) \leq 2g-2+r(g-2)+2(r+1)(s+1),$$

we can choose $\beta_j, \beta'_j$ with the additional property $\sum_{j=0}^{k}(\beta_j+\beta'_j) = c$.

Note that $g(G_{A,B,B'}) = L(C_0) = \alpha_1+\beta_0 = g$, $c(G_{A,B,B'}) = \sum_{j=0}^{k}(\beta_j+\beta'_j) = c$ and thus, $G_{A,B,B'} \in \mathcal{M}(g,c,c)$. Since $L(C_j) \leq g+2s+2$ for $0 \leq j \leq k$ and $2+\max\{\beta_j,\beta'_j\} \leq (g+5)/2+s$ for $0 < j < k$, Corollaries 5.1.2 and 5.1.6 and Lemma 5.1.12 give

$$\frac{g}{4} \leq \mathcal{A}(g,c,n) \leq \mathcal{A}(g,c,c) \leq \delta(G_{A,B,B'}) \leq \max\left\{\frac{4+g+2s}{4}, \frac{5+g+2s}{4}\right\} = \frac{g+5+2s}{4}.$$

□

The following result shows that the lower bound $g/4 \leq \mathcal{A}(g,c,n)$ is attained for infinitely many v-admissible triplets.

**Proposition 5.2.6.** *For any positive integer $u$, we have $\mathcal{A}(4u,6u,n) = g/4$ for every $n \geq 9u-3$.*

*Proof.* Consider a cycle graph $C_6$ with vertices $v_1,\dots,v_6$ and the graph $G$ with $V(G) = V(C_6)$ and $E(G) = E(C_6) \cup \{[v_1,v_4],[v_2,v_5],[v_3,v_6]\}$. Thus, $G \in \mathcal{M}(4,6,6)$. One can check that $\operatorname{diam}(G) = 1$, and Theorem 2.4.2 gives $\delta(G) \leq 1$. Hence, Corollary 5.1.2 implies $1 \leq \mathcal{A}(4,6,6) \leq \delta(G) \leq 1$ and $\mathcal{A}(4,6,6) = 1$. Since $G$ has 9 edges, Corollary 5.1.13 gives $\mathcal{A}(4u,6u,n) = g/4$ for every $n \geq 6+(u-1)9 = 9u-3$. □

We give now some bounds for $A(g,c,m)$ which do not depend on $r$ and $s$.

**Theorem 5.2.7.** *Let $(g,c,n)$ be a v-admissible triplet with $g \geq 5$.*

- *If $c < 3g-4$, then*

$$\frac{g}{4} \leq \mathcal{A}(g,c,n) \leq \frac{2g-1}{4}.$$

- *If $c = 3g - 4$, then*

$$\frac{g}{4} \le \mathcal{A}(g,c,n) \le \frac{g+2}{4} \qquad \textit{if } g \textit{ is even},$$

$$\frac{g}{4} \le \mathcal{A}(g,c,n) \le \frac{g+3}{4} \qquad \textit{if } g \textit{ is odd}.$$

- *If $c > 3g - 4$, then*

$$\frac{g}{4} \le \mathcal{A}(g,c,n) \le \frac{3g+5}{8} \qquad \textit{if } g \textit{ is even},$$

$$\frac{g}{4} \le \mathcal{A}(g,c,n) \le \frac{3g+7}{8} \qquad \textit{if } g \textit{ is odd}.$$

*Proof.* By Corollary 5.1.2, it suffices to prove the upper bounds.

Case 1.

If $c < 2g-2$, then Corollary 5.1.2 gives the result. If $2g-2 \le c < 3g-4$, then Theorem 5.2.5 gives $\mathcal{A}(g,c,n) \le (g+2+s)/4 \le (2g-1)/4$.

Case 2.

If $c = 3g-4$, then Theorem 5.2.5 with $r = 1$ gives the inequalities.

Case 3.

Consider the case $c > 3g-4$.

Let us define $r := \lceil \frac{c-2g+2}{g-2} \rceil - 1$. Therefore,

$$\frac{c-2g+2}{g-2} - 1 \le r < \frac{c-2g+2}{g-2},$$
$$2g-2+r(g-2) < c \le 2g-2+(r+1)(g-2).$$

Since $c > 3g-4$, we have $c-2g+2 > g-2$ and $r \ge 1$.

Define now $s := \lceil \frac{c-2g+2-r(g-2)}{2(r+1)} \rceil - 1$. Thus,

$$\frac{c-2g+2-r(g-2)}{2(r+1)} - 1 \le s < \frac{c-2g+2-r(g-2)}{2(r+1)},$$
$$2g-2+r(g-2)+2(r+1)s < c \le 2g-2+r(g-2)+2(r+1)(s+1).$$

Since $c > 2g-2+r(g-2)$, we have $s \ge 0$.

Since $2g-2+r(g-2)+2(r+1)s < c \le 2g-2+(r+1)(g-2)$, we have $2(r+1)s \le g-3$ and thus, $2s \le (g-3)/(r+1) \le (g-3)/2$.

If $g$ is even, then Theorem 5.2.5 gives

$$\mathcal{A}(g,c,n) \le \frac{g+4+2s}{4} \le \frac{g+4+\frac{g-3}{2}}{4} \le \frac{3g+5}{8}.$$

If $g$ is odd, then Theorem 5.2.5 gives

$$\mathcal{A}(g,c,n) \le \frac{g+5+2s}{4} \le \frac{g+5+\frac{g-3}{2}}{4} \le \frac{3g+7}{8}.$$

□

We can improve the bounds in Theorem 5.2.7 when $c$ is large enough.

**Theorem 5.2.8.** *Let $(g,c,n)$ be a v-admissible triplet with $g \ge 5$ and $2c \ge g^2 - 2g + 4$.*

- *If $g$ is even, then*
$$\frac{g}{4} \le \mathcal{A}(g,c,n) \le \frac{g+4}{4}.$$
- *If $g$ is odd, then*
$$\frac{g}{4} \le \mathcal{A}(g,c,n) \le \frac{g+5}{4}.$$

*Proof.* The lower bounds are consequence of Corollary 5.1.2. Let us prove the upper bounds.

Let us define $r := \left\lfloor \frac{c-2g+2}{g-2} \right\rfloor$. We have

$$\frac{c-2g+2}{g-2} - 1 < r \le \frac{c-2g+2}{g-2},$$
$$2g-2+r(g-2) \le c < 2g-2+(r+1)(g-2),$$
$$2g-2+r(g-2) \le c \le 2g-3+(r+1)(g-2).$$

Assume first that $g = 5$. Thus, $8 + 3r \le c \le 10 + 3r$. Inequality $2c \ge g^2 - 2g + 4$ gives $c \ge 10$. Thus, it suffices to consider the case $10 < g \le 10 + 3r$.

If $c = 10$, we have $c = 10 < 11 = 3g - 4$, and Theorem 5.2.7 gives $\mathcal{A}(g,c,n) \le 9/4 = (g+4)/4 < (g+5)/4$.

Assume that $g \ge 6$.

If $c = 2g-2+r(g-2)$, Theorem 5.2.5 gives $\mathcal{A}(g,c,n) \le (g+2)/4$ if $g$ is even and $\mathcal{A}(g,c,n) \le (g+3)/4$ if $g$ is odd.

If $c = 2g-1+r(g-2)$, Theorem 5.2.5 gives $\mathcal{A}(g,c,n) \le (g+3)/4$ if $g$ is odd.

Thus, we consider $2g-2+r(g-2) < c \le 2g-3+(r+1)(g-2)$ if $g$ is even and $2g-1+r(g-2) < c \le 2g-3+(r+1)(g-2)$ if $g$ is odd.

Since $g \ge 6$, we have $(g^2-2g+4)/2 \ge 3g-4$. Thus, $c \ge 3g-4$, which implies $c-2g+2 \ge g-2$ and $r \ge 1$. Hence, $r \ge 1$ for every $g \ge 5$.

On the other hand, note that for any value of $g$ we have

$$2c \ge g^2 - 2g + 4 \quad\Leftrightarrow\quad 2c - 4g + 4 \ge g^2 - 6g + 8 \quad\Leftrightarrow\quad \frac{c-2g+2}{g-2} \ge \frac{g-4}{2}.$$

Thus, $(g-4)/2 < r+1$ and we obtain $2r > g-6$. Therefore, $2r \geq g-5$. Note that

$$2r \geq g-5 \quad \Leftrightarrow \quad 2(r+1) \geq g-2-1 \quad \Leftrightarrow \quad 2g-2+r(g-2)+2(r+1) \geq 2g-2+(r+1)(g-2)-1.$$

Thus,

$$c \leq 2g-3+(r+1)(g-2) \leq 2g-2+r(g-2)+2(r+1)$$

and Theorem 5.2.5 gives

$$\mathcal{A}(g,c,n) \leq \frac{g+4}{4} \qquad \text{if } g \text{ is even,}$$

$$\mathcal{A}(g,c,n) \leq \frac{g+5}{4} \qquad \text{if } g \text{ is odd.}$$

$\square$

## 5.3 Computation of $\mathcal{B}(g, c, n)$

We compute in this Section the exact value of $\mathcal{B}(g, c, n)$ for every v-admissible triplet. Let us tart with two lemmas.

**Lemma 5.3.1.** *If $(g, c, n)$ is v-admissible and we have either $g = c$ or $g < c$ and $n \geq c - 1 + g/4$, then $\mathcal{B}(g, c, n) = c/4$.*

*Proof.* If $g = c$, then Corollary 5.1.2 implies $\mathcal{B}(g, c, n) = c/4$. Assume now $g < c$ and $n \geq c - 1 + g/4$.

If $n \leq c - 1 + g/2$, then $\max\{g - 1 + c/2, c - 1 + g/4\} \leq n \leq c - 1 + g/2$. Consider three natural numbers $a_1 := n - c + 1$, $a_2 := g + c - n - 1$, and $a_3 := n - g + 1$. Lemma 5.1.10 gives $C_{a_1,a_2,a_3} \in \mathcal{M}(g, c, n)$, with $a_2 \leq 3a_1$. Corollary 5.1.7 gives $\delta(C_{a_1,a_2,a_3}) = (a_3 + \min\{a_2, 3a_1\})/4 = (a_3 + a_2)/4 = c/4$. Thus, $\mathcal{B}(g, c, n) \geq c/4$, and Corollary 5.1.2 implies $\mathcal{B}(g, c, n) = c/4$.

If $n > c - 1 + g/2$, then Corollary 5.1.2 and Lemma 5.1.12 give the result (since $\lceil g/4 \rceil \leq g/2$, where $\lceil t \rceil$ denotes the upper integer part of $t$, there exists an integer $n_0$ such that $\max\{g - 1 + c/2, c - 1 + g/4\} \leq n_0 \leq c - 1 + g/2$). Thus, we conclude $\mathcal{B}(g, c, n) = c/4$ in any case. $\square$

**Lemma 5.3.2.** *If $(g, c, n)$ is v-admissible, $g < c$ and $n < c - 1 + g/4$, then $\mathcal{B}(g, c, n) = n + 1 - (g + 3c)/4$.*

*Proof.* First, let us prove that $\mathcal{B}(g, c, n) \geq n + 1 - (g + 3c)/4$.

Consider three natural numbers $a_1 := n - c + 1$, $a_2 := g + c - n - 1$ and $a_3 := n - g + 1$. Since $g < c$, we have $g - 1 + c/2 \leq n$ by Lemma 5.1.11. Since $g - 1 + c/2 \leq n < c - 1 + g/4 < c - 1 + g/2$, Lemma 5.1.10 gives $a_1 \leq a_2 \leq a_3$ and $C_{a_1,a_2,a_3} \in \mathcal{M}(g, c, n)$. By Lemma 5.1.10, $n < c - 1 + g/4$ is equivalent to $3a_1 < a_2$. Corollary 5.1.7 gives $\delta(C_{a_1,a_2,a_3}) = (a_3 + \min\{a_2, 3a_1\})/4 = (a_3 + 3a_1)/4 = n + 1 - (g + 3c)/4$. Thus, $\mathcal{B}(g, c, n) \geq n + 1 - (g + 3c)/4$.

Now, let us prove that $\mathcal{B}(g, c, n) \leq n + 1 - (g + 3c)/4$.

Consider any graph $G \in \mathcal{M}(g, c, n)$.

Let $T$ be any fixed geodesic triangle in $G$. Note that $g \leq L(T) \leq c$.

Assume first that $L(T) = c$.

Let us denote by $C_g$ a cycle in $G$ with length $g$. Since $n < c - 1 + g/4$, $E(T) \cap E(C_g) \neq \emptyset$. Let $\eta$ be a fixed connected component of $C_g \setminus T$.

Let us denote by $G_0$ the subgraph of $G$ such that $V(G_0) = V(T) \cup V(\eta)$ and $E(G_0) = E(T) \cup E(\eta)$. Since $L(T) = c$, we have $G_0 \in \mathcal{M}(g_0, c, n_0)$, with $n_0 := |V(G_0)| \leq n$ and $g_0 := g(G_0) \geq g$. Let us define $a_1 := L(\eta) = n_0 - c + 1$ and consider the two curves $\eta_2, \eta_3$ contained in $T$ joining the endpoints of $\eta$. By symmetry we can assume that $a_2 := L(\eta_2) \leq a_3 := L(\eta_3)$. Since $\eta \subset C_g$, we have $a_1 = L(\eta) \leq a_2$ and the following equations hold: $a_1 + a_2 = g_0$, $a_2 + a_3 = c$, $a_1 + a_2 + a_3 = n_0 + 1$. Thus, Lemma 5.1.10 gives $a_2 = g_0 + c - n_0 - 1$ and $a_3 = n_0 - g_0 + 1$. Note that $G_0 = C_{a_1,a_2,a_3}$. Corollary 5.1.8 gives

$$\delta(G_0) \leq \frac{a_3 + 3a_1}{4} = \frac{n_0 - g_0 + 1 + 3(n_0 - c + 1)}{4} = n_0 + 1 - \frac{g_0 + 3c}{4} \leq n + 1 - \frac{g + 3c}{4}.$$

Since $d_G(u,v) \le d_{G_0}(u,v)$ for every $u,v \in G_0$, we have that any geodesic $\gamma$ in $G$ contained in $G_0$ is a geodesic in $G_0$, $T$ is also a geodesic triangle in $G_0$, and

$$\delta_G(T) \le \delta_{G_0}(T) \le \delta(G_0) \le n+1-\frac{g+3c}{4}.$$

Assume now that $L(T) < c$.

Let us denote by $C_c$ a cycle in $G$ with length $c$. Since $n < c-1+g/4 \le c-1+L(T)/4$, we have $E(T) \cap E(C_c) \neq \emptyset$.

Denote by $k \ge 1$ the cardinality of the connected components of $T \setminus C_c$.

Let us denote by $L_1,\dots,L_k$ the lengths of the connected components $\eta_1,\eta_2,\dots,\eta_k$ of $T \setminus C_c$, respectively. Denote by $\eta_1',\eta_2',\dots,\eta_k'$ the connected components of $C_c \setminus T$ such that $\eta_j$ and $\eta_j'$ have the same endpoints for $1 \le j \le k$. Let us denote by $L_1',\dots,L_k'$ the lengths of $\eta_1',\eta_2',\dots,\eta_k'$, respectively. Since the length of the smallest cycle in $G$ is $g$, we have $L_j' \ge g - L_j$, for $1 \le j \le k$.

Since $c \le n$, inequality $n < c-1+g/4$ implies $g > 4$. Note that

$$g>4 \text{ and } k\ge 1 \quad\Rightarrow\quad g-4 \le k(g-4) \quad\Rightarrow\quad 0 < 4\sum_{j=1}^{k} L_j + \sum_{j=1}^{k}(g-2L_j) + 4 - 4k - g$$

$$\Rightarrow\quad 0 < 4\sum_{j=1}^{k} L_j + \sum_{j=1}^{k}(L_j'-L_j) + 4 - 4\sum_{j=1}^{k} 1 - g$$

$$\Leftrightarrow\quad c - \sum_{j=1}^{k} L_j' + \sum_{j=1}^{k} L_j < 4\Big(c + \sum_{j=1}^{k}(L_j-1)\Big) + 4 - g - 3c$$

$$\Rightarrow\quad L(T) < 4n+4-g-3c \quad\Rightarrow\quad \delta(T) < n+1-\frac{g+3c}{4}.$$

Hence, $\delta(G) \le n+1-(g+3c)/4$. Thus, $\mathcal{B}(g,c,n) \le n+1-(g+3c)/4$ and we conclude $\mathcal{B}(g,c,n) = n+1-(g+3c)/4$. □

We can summarize Lemmas 5.1.11, 4.2.11 and 5.3.2 in the following result.

**Theorem 5.3.3.** *For any v-admissible triplet $(g,c,n)$, the value of $\mathcal{B}(g,c,n)$ is as follows.*

*(1) If we have either $g=c$ or $g<c$ and $n \ge c-1+g/4$, then $\mathcal{B}(g,c,n)=c/4$.*

*(2) If $g<c$ and $n<c-1+g/4$, then $\mathcal{B}(g,c,n)=n+1-(g+3c)/4$.*

## 5.4 Bounds for $\mathfrak{A}(g, c, m)$

Let us start with some bounds for $\mathfrak{A}(g, c, m)$ similar to the ones in Theorem 5.2.5.

**Theorem 5.4.1.** *Let $(g, c, m)$ be an e-admissible triplet.*

- *If $2g - 2 \le c < 3g - 4$ with $c = 2g - 2 + s$ $(0 \le s \le g - 3)$, then*
$$\frac{g}{4} \le \mathfrak{A}(g, c, m) \le \frac{g + 2 + s}{4}.$$
- *If $m \ge c + r$, $r$ is a positive integer, $g$ is even and $c = 2g - 2 + r(g - 2)$, then*
$$\frac{g}{4} \le \mathfrak{A}(g, c, m) \le \frac{g + 2}{4}.$$
- *If $m \ge c+r$, $r$ is a positive integer, $g$ is odd and $2g-2+r(g-2) \le c \le 2g-1+r(g-2)$, then*
$$\frac{g}{4} \le \mathfrak{A}(g, c, m) \le \frac{g + 3}{4}.$$
- *If $m \ge c+r$, $r$ and $s$ are integers with $r \ge 1$, $s \ge 0$, $g$ is even and $2g-2+r(g-2) < c \le 2g - 2 + r(g - 2) + 2(r + 1)(s + 1)$, then*
$$\frac{g}{4} \le \mathfrak{A}(g, c, m) \le \frac{g + 4 + 2s}{4}.$$
- *If $m \ge c+r$, $r$ and $s$ are integers with $r \ge 1$, $s \ge 0$, $g$ is odd and $2g-1+r(g-2) < c \le 2g - 2 + r(g - 2) + 2(r + 1)(s + 1)$, then*
$$\frac{g}{4} \le \mathfrak{A}(g, c, m) \le \frac{g + 5 + 2s}{4}.$$

*Proof.* Corollary 5.1.2 gives the lower bound. Let us prove the upper bounds.

Assume that $2g-2 \le c < 3g-4$ with $c = 2g-2+s$ $(0 \le s \le g-3)$. Since $g < 2g-2 \le c$, we have $m \ge c + 1$.

Assume that $2g-2 \le c < 3g-4$. Consider the graph $C_{a_1,a_2,a_3}$ with $a_1 = 1$, $a_2 = g-1$ and $a_3 = g - 1 + s$. Note that $g(C_{a_1,a_2,a_3}) = a_1 + a_2 = g$, $c(C_{a_1,a_2,a_3}) = a_2 + a_3 = 2g - 2 + s = c$, $m = c + 1$ and thus, $C_{a_1,a_2,a_3} \in \mathcal{N}(g, c, c + 1)$. Corollary 5.1.7 gives $\delta(C_{a_1,a_2,a_3}) = (a_3 + \min\{a_2, 3a_1\})/4 = (g - 1 + s + \min\{g - 1, 3\})/4 \le (g + 2 + s)/4$.

Thus, Lemma 5.1.17 implies $\mathfrak{A}(g, c, m) \le \mathfrak{A}(g, c, c + 1) \le \delta(C_{a_1,a_2,a_3}) \le (g + 2 + s)/4$.

The proof in the other cases follows the argument in the proof of Theorem 5.2.5, since $m \ge c + r$. ∎

The following result provides bounds for $\mathfrak{A}(g,c,m)$ when $m < c+r$ and we can not apply Theorem 5.4.1.

**Theorem 5.4.2.** *Let $(g,c,m)$ be an e-admissible triplet.*

- *If $c = g$, then*
$$\mathfrak{A}(g,g,m) = \mathfrak{B}(g,g,m) = \frac{g}{4}.$$
- *If $m = c+1$, then*
$$\mathfrak{A}(g,c,c+1) = \mathfrak{B}(g,g,c+1) = \frac{c-g+1+\min\{3,g-1\}}{4}.$$
- *If $m \ge c+u$, with $2 \le u \le \frac{c-g+1}{g-1}$, then*
$$\frac{g}{4} \le \mathfrak{A}(g,c,m) \le \frac{5}{4} + \frac{1}{4}\left\lceil \frac{c-g+1}{u} \right\rceil$$

*Proof.* Corollary 5.1.2 gives the lower bound. Let us prove the upper bounds.

Case 1.

If $c = g$, then Corollary 5.1.2 gives $\mathfrak{A}(g,g,m) = g/4$.

If $m = c$, then $g = c = m$ and we have proved $\mathfrak{A}(g,g,g) = g/4$. Thus, we can assume $c > g$ and $m \ge c+1$. Furthermore, Lemma 5.1.16 gives $m \ge g + c/2$.

Case 2.

Let $m = c+1$. As in Section 2, denote by $C_{a_1,a_2,a_3}$ the graph with three paths with lenghts $a_1 \le a_2 \le a_3$ joining two fixed vertices. Let $a_1 = 1$, $a_2 = g-1$, $a_3 = c-g+1$. Since $m \ge g + c/2$ and $m = c+1$, we have $g-1 \le c-g+1$. Then, every graph $G \in \mathcal{N}(g,c,c+1)$ is isomorphic to $C_{1,g-1,c-g+1}$. Corollary 5.1.7 gives $\delta(G) = (c-g+1+\min\{3,g-1\})/4$ and thus, $\mathfrak{A}(g,c,c+1) = \beta(g,c,c+1) = (c-g+1+\min\{3,g-1\})/4$.

Case 3.

Let $m = c+u$, with $2 \le u \le (c-g+1)/(g-1)$.

Consider a graph $G_{A,B,B'}$ as in Definition 5.1.3, with $k = u$, $\alpha_j = 1$ for $1 \le j \le u$ and $\beta_0 = g-1$. For each $1 \le j \le u-1$, choose $\beta_j, \beta'_j$ such that we either have $\beta_j + \beta'_j = \lfloor \frac{c-g+1}{u} \rfloor$ or $\beta_j + \beta'_j = \lceil \frac{c-g+1}{u} \rceil$ , and $\sum_{j=1}^{u}(\beta_j + \beta'_j) = c-g+1$. Hence, $\sum_{j=0}^{u}(\beta_j + \beta'_j) = c$.

Note that $L(C_0) = g$,

$$u \le \frac{c-g+1}{g-1} \qquad \Rightarrow \qquad g \le \frac{c-g+1}{u} + 1 \qquad \Rightarrow \qquad g \le \left\lfloor \frac{c-g+1}{u} \right\rfloor + 1$$
$$\Rightarrow \qquad g \le \beta_j + \beta'_j + 1$$

and thus, $L(C_j) \ge g$ for every $0 \le j \le u$. Thus, $G_{A,B,B'} \in \mathcal{N}(g,c,c+u)$.

Since $\beta_j + \beta'_j \leq \lceil \frac{c-g+1}{u} \rceil$ for each $1 \leq j \leq u-1$, it is possible to choose $\beta_j, \beta'_j$ with the additional condition $\max\{\beta_j, \beta'_j\} \leq \frac{1}{2}(\lceil \frac{c-g+1}{u} \rceil + 1)$.

Corollary 5.1.6 gives $\delta(G_{A,B,B'}) \leq \max\{\frac{1}{2} + \frac{g}{4}, \frac{3}{4} + \frac{1}{4}\lceil \frac{c-g+1}{u} \rceil, \frac{5}{4} + \frac{1}{4}\lceil \frac{c-g+1}{u} \rceil\} = \frac{5}{4} + \frac{1}{4}\lceil \frac{c-g+1}{u} \rceil$. Thus, $\alpha(g,c,c+u) \leq \delta(G_{A,B,B'}) \leq \frac{5}{4} + \frac{1}{4}\lceil \frac{c-g+1}{u} \rceil$.

Finally, Lemma 5.1.17 gives the desired result. □

We improve now Theorem 5.4.2 for the case $g = 3$.

**Theorem 5.4.3.** *Let $(3, c, m)$ be an e-admissible triplet.*

- *If $c = 3$, then*
$$\mathfrak{A}(3,3,m) = \mathfrak{B}(3,3,m) = \frac{3}{4}.$$
- *If $c \geq 4$ and $m = c+1$, then*
$$\mathfrak{A}(3,c,c+1) = \mathfrak{B}(3,c,c+1) = \frac{c}{4}.$$
- *If $c \geq 4$ and $m = c+u$, with $2 \leq u < \lfloor \frac{c-2}{2} \rfloor$, then*
$$1 \leq \mathfrak{A}(3,c,c+u) \leq \frac{5}{4} + \frac{1}{4}\left\lceil \frac{c-2}{u} \right\rceil.$$
- *If $c \geq 4$ and $c + \lfloor \frac{c-2}{2} \rfloor \leq m < \binom{c}{2}$, then*
$$1 \leq \mathfrak{A}(3,c,m) \leq \frac{3}{2}.$$
- *If $c \geq 4$ and $m \geq \binom{c}{2}$, then*
$$\mathfrak{A}(3,c,m) = 1.$$

*Proof.* Theorem 5.4.2 gives the first three items.

If $c \geq 4$, Theorem 5.2.2 gives $\mathfrak{A}(3,c,m) \geq 1$.

Assume that $m = c + \lfloor \frac{c-2}{2} \rfloor$.

If $c$ is even, then $m = c + (c-2)/2$. Consider a graph $G_{A,B,B'}$ as in Definition 5.1.3, with $k = (c-2)/2$, $\alpha_j = 1$ for $1 \leq j \leq (c-2)/2$, $\beta_0 = \beta_{(c-2)/2} = 2$ and $\beta_j = \beta'_j = 1$ for $1 \leq j \leq (c-2)/2 - 1$. Corollary 5.1.6 gives $\delta(G_{A,B,B'}) \leq \max\{5/4, 3/2\} = 3/2$. Note that $G_{A,B,B'} \in \mathcal{N}(3,c,c+(c-2)/2)$. If $m \geq c + (c-2)/2$, then Lemma 5.1.17 implies $\mathfrak{A}(3,c,m) \leq \mathfrak{A}(3,c,c+(c-2)/2) \leq \delta(G_{A,B,B'}) \leq 3/2$.

If $c$ is odd, then $m = c + (c-3)/2$. Consider a graph $G_{A,B,B'}$ as in Definition 5.1.3, with $k = (c-3)/2$, $\alpha_j = 1$ for $1 \leq j \leq (c-3)/2$, $\beta_j = \beta'_j = 1$ for $1 \leq j \leq (c-3)/2 - 1$, $\beta_0 = 3$ and $\beta_{(c-3)/2} = 2$. Corollary 5.1.6 gives $\delta(G_{A,B,B'}) \leq \max\{5/4, 3/2\} = 3/2$. Note that $G_{A,B,B'} \in \mathcal{N}(3,c,c+(c-3)/2)$. If $m \geq c + (c-3)/2$, then Lemma 5.1.17 implies $\mathfrak{A}(3,c,m) \leq \mathfrak{A}(3,c,c+(c-3)/2) \leq \delta(G_{A,B,B'}) \leq 3/2$.

Assume now that $m = \binom{c}{2}$. Let $G$ be the complte graph with $c$ vertices. Thus, $G \in \mathcal{N}(3,c,\binom{c}{2})$, $\delta(G) = 1$ and $\mathfrak{A}(3,c,\binom{c}{2}) = 1$. Finally, Lemma 5.1.12 gives the desired result. □

## 5.5 Computation of $\mathfrak{B}(g, c, m)$

The arguments in the proofs of Lemmas 5.3.1 and 5.3.2, but using Lemmas 5.1.15 and 5.1.16 instead of Lemmas 5.1.10 and 5.1.11, respectively, give the following result.

**Theorem 5.5.1.** *For any e-admissible triplet $(g, c, m)$, the value of $\mathfrak{B}(g, c, m)$ is as follows.*

*(1) If we have either $g = c$ or $g < c$ and $m \geq c + g/4$, then $\mathfrak{B}(g, c, m) = c/4$.*

*(2) If $g < c$ and $m < c + g/4$, then $\mathfrak{B}(g, c, m) = m - (g + 3c)/4$.*

# Conclusions

For a general graph deciding whether or not the space is Gromov hyperbolic seems an intractable problem. Therefore, it is interesting to study the hyperbolicity of particular classes of graphs.

In this Thesis we consider simple graphs with every edge of length 1. We study the hyperbolicity constant of several classes of graphs, obtaining good bounds for it in terms of important parameters of the graph.

We start with the class of graphs with $n$ vertices and $m$ edges and denote by $A(n,m)$ and $B(n,m)$ the minimum and the maximum, respectively, of the hyperbolicity constants of the graphs in this class. First we estimate $A(n,m)$ and $B(n,m)$: Theorems 3.1.11 and 3.1.13 give upper and lower bounds for $B(n,m)$, respectively, while Theorem 3.4.10 gives the precise value of $A(n,m)$ for all values of $n$ and $m$. Furthermore, we compute these minimum and maximum values in the case of non-simple graphs, and we give bounds for them in the case of weighted graphs. Besides, these results can be applied to Erdös-Rényi random graphs. As a consequence of the results obtained in this Chapter, we also prove an inequality involving the diameter, the order and the size of any graph (see Theorem 3.3.2).

In this work, we study also a natural problem: Obtaining upper bounds for the diameter of a graph in terms of its minimum degree and its order, improving a classical theorem due to Erdös, Pach, Pollack and Tuza (see Theorem 4.1.2). We use these bounds in order to solve four interesting extremal problems involving the hyperbolicity constant, the order, and the maximum and minimum degree. Three of these extremal problems are completely solved; more precisely, we compute the precise values of $a(n,\delta_0)$, $\beta(n,\Delta)$ and $\alpha(n,\Delta)$, respectively (see Theorems 4.4.1, 4.5.2 and 4.6.1). The fourth extremal problem, regarding $b(n,\delta_0)$ (the hardest one), is completely solved in most cases (see Theorem 4.3.11).

Finally, we study the extremal problems of maximazing and minimazing $\delta(G)$ on the sets $\mathcal{M}(g,c,n)$ and $\mathcal{N}(g,c,m)$. Sections 5.2 and 5.4 contain good bounds for $\mathcal{A}(g,c,n)$ and $\mathfrak{A}(g,c,m)$, respectively. Theorems 5.3.3 and 5.5.1 give the precise value of $\mathcal{B}(g,c,n)$ and $\mathfrak{B}(g,c,m)$, respectively.

Summing up, along this Thesis we focus on estimating $A(n,m)$, $B(n,m)$, $a(n,\delta_0)$, $b(n,\delta_0)$, $\alpha(n,\Delta)$, $\beta(n,\Delta)$, $\mathcal{A}(g,c,n)$, $\mathcal{B}(g,c,n)$, $\mathfrak{A}(g,c,m)$ and $\mathfrak{B}(g,c,m)$, i.e., we study the extremal problems of maximazing and minimazing $\delta(G)$ on the sets $\mathcal{G}(n,m)$, $\mathcal{H}(n,\delta_0)$, $\mathcal{J}(n,\Delta)$, $\mathcal{M}(g,c,n)$ and $\mathcal{N}(g,c,m)$.

# Open problems

As a continuation of this work, we would like to deal with some interesting problems. Before stating them, we need to introduce some new definitions.

In a connected graph $G$, a *Steiner tree* of a (multi)set $W \subseteq V(G)$, is a minimum order tree in $G$ that contains all vertices of $W$ (see [64] for an introduction to Steiner trees).

The notion of a Steiner tree of a multiset of vertices $W$ can be considered as a generalization of the geodesic because when $W$ consists of two different vertices, then a Steiner tree of $W$ is a geodesic between the vertices. Therefore, it seems natural to try to generalize the concept of hyperbolicity constant in terms of Steiner trees.

We can consider Steiner trees of a set $W$ of points in $G$, where the elements in $W$ are not necessarily vertices. The set of Steiner trees of $W$ is denoted by $T_W$, whilst a given Steiner tree in $T_W$ is denoted by $t_W$. We define the following parameter, called Gromov-Steiner constant.

**Definition 5.5.2** (Gromov-Steiner constant)**.** *For fixed natural numbers $t$ and $k$, $1 < t < k$, consider a set of points $S' = \{u_1, \dots, u_k\}$ in a given graph $G$ and any $t-$ subset $S$ of $S'$ where $S = \{u_{i_1}, \dots, u_{i_t}\}$. We denote by $T$ a $(t+1)$-tuplet of Steiner trees $(t_S, t_{S_1}, \dots, t_{S_t})$ where $S_j = S' - \{u_{i_j}\}$, $1 \le j \le t$. We say that $T$ is $\delta_{t,k}$-thin if $d\left(p, \cup_{j=1}^{t} t_{S_j}\right) \le \delta_{t,k}$ for any point $p \in t_S$. Let us denote by $\delta_{t,k}(T)$ the sharp $\delta_{t,k}$-thin constant of $T$, i.e., $\delta_{t,k}(T) :=$ $\inf\{\delta_{t,k} \ge 0 : T \text{ is } \delta_{t,k}-\ thin\}$. The Gromov-Steiner constant for the graph $G$ is $\delta_{t,k}(G) :=$ $\sup\{\delta_{t,k}(T) : T$ is a (t+1)$-$tuplet$\}$.*

Im other words, for any fixed natural numbers $t$ and $k$, and sets $S$ and $S'$ defined as in the previous definition, any point of a Steiner tree of $S$ must be at distance at most $\delta_{t,k}$ from the union of Steiner trees of $S_j = S' - \{u_{i_j}\}$, $1 \le j \le t$.

For future works, it would be interesting to address the following problems.

- To study the Gromov-Steiner constant and document the similarities and differences obtained when dealing with similar problems to the ones contained in this work.
- To obtain other bounds for the hyperbolicity constant of weighted graphs (as a complement of the results found in Chapter 3)
- To prove some results about the average hyperbolicity constant of Erdös-Rényi graphs.

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