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OPTIMAL PORTFOLIO WITH INSIDER INFORMATION ON THE STOCHASTIC INTEREST RATE

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Abstract

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Keywords: Optimal portfolio, Enlargement of filtrations, Vasicek interest rate model, Value of the information

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Optimal portfolio with insider information on the stochastic interest rate

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Abstract

We consider the optimal portfolio problem where the interest rate is stochastic and the agent has insider information on its value at a finite terminal time. The agent's objective is to optimize the terminal value of her portfolio under a logarithmic utility function. Using techniques of initial enlargement of filtration, we identify the optimal strategy and compute the value of the information. The interest rate is first assumed to be an affine diffusion, then more explicit formulas are computed for the Vasicek interest rate model where the interest rate moves according to an Ornstein-Uhlenbeck process. We show that when the interest rate process is correlated with the price process of the risky asset, the value of the information is infinite, as is usually the case for initial-enlargement-type problems. However since the agent does not know exactly the correlation factor, this may induce an infinite loss instead of an infinite gain. Finally weakening the information own by the agent, and assuming that she only knows a lower-bound for the terminal value of the interest rate process, we show that the value of the information is finite.

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1 Introduction

The mathematical models used to construct optimal portfolio strategies usually assume that investors, or traders, rationally use at each time all the information at their disposal in order to optimize their future utility. In the standard setting, the accessible information is given by the historical prices at which the assets that make up the portfolio have been traded in the past.

However, the information own by an investor could actually be larger than the standard one. For example, the agent may get additional information on the business underlying the asset, as well as she may include public information generally accessed by financial publications.

In some specific situations, the trader may access and take advantage of private or privileged information, even if this last option is usually considered not legal according to the rules governing the public stock exchanges. This type of information is of different nature with respect of the one mentioned above as, in this case, it anticipates the future trend of the risky asset and may generate arbitrage opportunities.

It is therefore of interest to understand how to introduce anticipative information in the stochastic models used to construct the optimization strategies and to value the performances of these strategies with respect to the ones that do not make use of the privileged information. This approach, besides their intrinsic mathematical interest may furnish a tool to detect irregularities in the markets as well as to estimate the real value of the favored position of the insider traders.

Continuous-time portfolio selection problems were introduced in Merton [1969], Pliska [1986] an Karatzas et al. [1987], these treated single-agent consumption/investment problems and constructed an optimal portfolio for power and logarithm utilities. The portfolio optimization problem in complete market was originally introduced by Karatzas et al. [1987], who introduced the use of the martingale methodology.

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Then, the portfolio optimization with insider information was first studied by Pikovsky and Karatzas [1996]. There, the technique of enlargement of filtrations was used to compute the martingale representation of the price process with respect to the enlarged filtration of the insider trader and, by a standard application of Itô calculus, the utility maximization problem was solved by getting the insider trader optimal strategy.

Ever since, many authors have used this approach to study the insider trader problem, such as in Amendinger et al. [2003], where the financial value of initial information is calculated by modeling the market as semi-martingales.

Another approach to the problem, differing from the enlargement of filtration, was considered by Baudoin [2002]. The author introduces the conditioned stochastic differential equations and defines the concept of weak information, that is, the insider has knowledge of the initial filtration and the law of some functional of the paste and future price history. Among the latest techniques used in this context we mention the (generalized) Hida-Malliavin calculus and white noise theory that uses forward stochastic integration, see for example Kohatsu-Higa and Sulem [2006] and Biagini and Øksendal [2005].

The original contribution of our work is to analyze the case when the additional information is not directly related with the price process of the risky asset but it concerns the future value of the interest rate process. In this respect we show that when the driving processes of the price process of the risky asset and the interest rate process are correlated, the informed agent may take advantage of an arbitrage.

Depending on the type of information the arbitrage may be unbounded. However we show that if the insider trader uses an inaccurate model, meaning that her estimation of the correlation coefficient is not precise, her opportunity can turn into an infinite loss.

Finally we show that when the privileged information is not exact, but it only gives a lower bound for the future value of the interest process, the value of the information is actually finite.

It is worth to point out that the assumption of an affine diffusion structure for the interest rate process, leads to explicit and almost handful expressions. This class of models includes, as a special case, the Ornstein-Uhlenbeck process and therefore it allows to analyze in very detail the well known Vasicek model, introduced in Vasicek [1977].

The optimal strategy with insider information strongly depends on the privileged information, and it largely differs form the Merton optimal strategy. It seems reasonable to think that it may lead to the design of effective tests for fraud detection, similarly to as suggested in Bernard and Vanduffel [2014] and in Grorud and Pontier [1998]. However this line of research is not addressed in this paper.

The analysis done in this paper follows and extends the stochastic models for insider trading strategies studied by Pikovsky and Karatzas [1996], Amendinger et al. [1998] and Baudoin [2002]. Our approach therefore differs from the one followed in Guasoni [2006], Buckley et al. [2012] and Buckley et al. [2014]. As explicitly mentioned in Buckley et al. [2016], the approach there is the opposite one, starting from a larger filtration, the one known by the institutional investors, that is then restricted to take into account the smaller information own by the retail investors. It is remarkable to note that the mean-reverting process used in Buckley et al. [2016] to model the mispricing and the asymmetric information is again an Ornstein-Uhlenbeck process.

As explained more in details in the following sections, we look for an optimal strategy that maximizes the expected logarithmic utility of the terminal wealth of a portfolio made of two assets, one risky and one riskless:

$$\sup_{\pi\in\mathcal{A}_{\mathbb{H}}} \boldsymbol{E}[U(X_T^{\pi})] =: \boldsymbol{E}[U(X_T^{\pi^*})] \ , \tag{1a}$$

$$dX_t^{\pi} = (1-\pi_t)X_t^{\pi}R_t\,dt + \pi_t X_t^{\pi}(\eta_t\,dt + \xi_t\,dB_t) \ . \tag{1b}$$

In (1b), π_t is the agent's strategy, X_t represents the wealth at time t of her portfolio, R_t is the stochastic interest rate and B_t is a standard Brownian motion. We assume that the agent owns insider information, and we model this assumption by specializing the filtration $\mathbb{H}$, that determines the class of adapted policies, $\mathcal{A}_{\mathbb{H}}$, among which the optimal strategy must be chosen. In particular we focus on different kinds of information that the agent may own about the terminal value of the interest rate process, and for these cases we obtain the optimal portfolio together with a quantitative estimation for the value of her insider information.

We start by assuming that the process $R = (R_t,\, t \geq 0)$ belongs to the class of affine diffusion. Then to make the results more explicit we specialize the computation for the case R is an Ornstein-Uhlenbeck process, that we denote by Y.

In Section 2 we introduce in more details the general model with the interest rate process modeled as an affine diffusion. The end of the section contains a brief summary of the used mathematical notation.

We analyze the general model under the insider information assumption in Section 3 where we compute explicitly the optimal strategy. In Section 4 we specialize the model by assuming that the interest rates follow an Ornstein-Uhlenbeck process. This corresponds to the popular Vasicek model. For this model we analyze a strong type of insider information, by assuming known the final value of the interest rate process, and for it we compute the optimal portfolio. Then, in Section 5, we show that this type of information carries an infinite value, that may be positive or negative according to the accuracy of the estimated correlation coefficient. In Section 6 we introduce a weaker type of information assuming that only a lower bound for the final value of the interest rate process is known and in this case we compute again the optimal portfolio and the value of the information that in this case turns out to be finite. We conclude in Section 7 with some concluding remarks.

2 Model and Notation

As a general setup we assume to work in a probability space $(\Omega, \mathcal{F}, \mathbb{F}, \boldsymbol{P})$ where $\mathcal{F}$ is the event sigma-algebra, and $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ is an augmented filtration that is generated by (or at least contains) the natural filtration of a bi-dimensional Brownian motion $(B^R, B^S) = ((B^R_t, B^S_t), t \geq 0)$, whose components have constant correlation ρ.

We assume that the portfolio is made of only two assets, one risky, that we call $S = (S_t,\, t \geq 0)$ and the other risk-less $D = (D_t,\, t \geq 0)$, and both processes are adapted semi-martingales in the defined probability space. In particular their dynamics are defined by the following SDEs,

$$dD_t = D_t\, R_t\, dt, \tag{2a}$$

$$dS_t = S_t\, \left(\eta_t dt + \xi_t dB^S_t\right) \tag{2b}$$

where $R = (R_t,\, t \geq 0)$ is the *instantaneous risk-free interest rate*, sometimes also called the *short term rate* [Gibson et al., 2010]. The drift and the volatility of the risky asset are given by the processes $\eta = (\eta_t \in \mathbb{R},\, t \geq 0)$ and $\xi = (\xi_t \in \mathbb{R}_+,\, t \geq 0)$, respectively. They are assumed to be adapted to the natural filtration of the process B^S with η and $1/\xi$ bounded. The interest rate process R is assumed to be an affine diffusion, satisfying the following SDE

$$dR_t = [a_1(t)R_t + a_2(t)]\, dt + b_2(t) dB^R_t\ , \tag{3}$$

where the deterministic functions a_1, a_2, b_2 are sufficiently smooth functions. This class of processes includes as a particular case the Ornstein-Uhlenbeck process, in this paper denoted by $Y = (Y_t,\, t \geq 0)$, satisfying the well known SDE

$$dY_t = k(\mu - Y_t)dt + \sigma dB^R_t \tag{4}$$

where k, μ, σ are given parameters. This process was proposed for modeling the interest rate in Vasicek [1977].

Using the above set-up, we are going to assume that an investor can control her portfolio by a given self-financial strategy $\pi = (\pi_t,\, 0 \leq t \leq T)$, with the aim to optimize her utility function at a finite terminal time $T > 0$.

If we denote by $X^\pi = (X^\pi_t,\, 0 \leq t \leq T)$ the wealth of the portfolio of the investor under her strategy π, its dynamics are given by the following stochastic differential equation, for $0 \leq t \leq T$,

$$\frac{dX^\pi_t}{X^\pi_t} = (1 - \pi_t)\frac{dD_t}{D_t} + \pi_t \frac{dS_t}{S_t}\ , \quad X_0 = x_0\ , \tag{5}$$

that can be reduced, using the expressions in (2), to the following form

$$dX^\pi_t = (1 - \pi_t) X^\pi_t R_t\, dt + \pi_t X^\pi_t \left(\eta_t\, dt + \xi_t\, dB^S_t\right)\ , \quad X_0 = x_0\ . \tag{6}$$

Usually it is assumed that the strategy π makes optimal use of all information at disposal of the agent at each instant, and in general we are going to assume that the agent's flow of information, modeled by the filtration $\mathbb{H} = (\mathcal{H}_t,\, 0 \leq t \leq T)$, is possibly larger than filtration $\mathbb{F}$, that is $\mathbb{F} \subset \mathbb{H}$.

Defining by $\mathcal{A}_\mathbb{H}$ all the admissible $\mathbb{H}$ adapted processes, we define the optimal portfolio $\pi^* = (\pi^*_t,\, 0 \leq t \leq T)$, as the solution of the following optimization problem,

$$\mathbb{V}^\mathbb{H}_T \coloneqq \sup_{\pi \in \mathcal{A}_\mathbb{H}} \boldsymbol{E}[U(X^\pi_T)] = \boldsymbol{E}[U(X^{\pi^*}_T)]\ , \tag{7}$$

where $\mathbb{V}_T^{\mathbb{H}}$ is defined as the optimal value of the portfolio at time T given the information $\mathbb{H}$, and the function $U : \mathbb{R}^+ \to \mathbb{R}$ denotes the utility function of the investor. This function is assumed to be continuous, increasing and concave. For sake of simplicity, following the main trend in the literature — as it allows to determine the solution in an explicit form —, we assume a logarithmic function for the utility, that is $U(x) = \ln(x)$.

In the following sections we consider two kinds of initial enlargements; a first one, *stronger*, under which the investor is assumed to know exactly the future value of the interest process, R_T, and a second one, *weaker*, where the investor knows only the value of a lower/upper bound, $\mathbb{1}\{R_T \geq c\}$. The filtration $\mathbb{H}$ in the former case will be denoted by $\mathbb{G} = (\mathcal{G}_t,\, 0 \leq t \leq T)$, with

$$\mathcal{G}_t = \mathcal{F}_t \bigvee \sigma(R_T) \;, \tag{8}$$

and we denote the corresponding strategies in $\mathcal{A}_{\mathbb{G}}$, with $\hat{\pi}$. In particular we will use the $\hat{}$ decoration to denote all the functions that will make use of the additional information in $\mathbb{G}$. In the latter case the filtration is denoted by $\tilde{\mathbb{G}} = (\tilde{\mathcal{G}}_t,\, 0 \leq t \leq T)$ where

$$\tilde{\mathcal{G}}_t = \mathcal{F}_t \bigvee \sigma(\mathbb{1}\{R_T \geq c\}) \;, \tag{9}$$

and we will use the corresponding decoration $\tilde{}$, such as in $\tilde{\pi} \in \mathcal{A}_{\tilde{\mathbb{G}}}$. It is immediate to see that the following inclusions hold

$$\mathbb{F} \subset \tilde{\mathbb{G}} \subset \mathbb{G} \;. \tag{10}$$

2.1 Additional Notation

Given two random variables X and Y, we write $X \approx Y$ to indicate that they have the same distribution. The notation $\mathcal{N}(\mu, \sigma^2)$ denotes a normal random variable with mean μ and variance σ^2. $\Phi(z)$ denotes the cumulative ditribution of a standard Normal random variable. With $f_X(x)$, we denote the density function of X evaluated at x and by $f_{X|Y}(x|y)$ the value of the conditioned density function at x given $\{Y = y\}$. $\boldsymbol{P}$, $\boldsymbol{E}$ and $\boldsymbol{V}$ denote the probability, the expectation and the variance operators. $(fg)(x)$ is sometimes used to denote the product $f(x)g(x)$ and $f \overset{\delta}{\sim} g$ means that $\lim_{\delta\to 0} f(\delta)/g(\delta) = 1$. We may omit to explicitly indicate δ when it is clear form the context.

3 General Interest Rate

Combining equations (3) and (6) we get the following system of stochastic differential equations,

$$\begin{cases} dX_t^\pi = (1 - \pi_t)X_t^\pi R_t\, dt + \pi_t X_t^\pi \left(\eta_t\, dt + \xi_t\, dB_t^S\right) \\ dR_t = [a_1(t)R_t + a_2(t)]\, dt + b_2(t) dB_t^R \end{cases} \tag{11}$$

that solved with respect to the filtration $\mathbb{F}$ gives the evolution of the interest rate process and the portfolio wealth, as seen by a non-informed investor.

In order to analyze the same processes adapted to the enlarged filtration $\mathbb{G}$ defined in (8), following standard techniques of enlargement of filtrations, we look for the semi-martingale decomposition of the pair (B^R, B^S) with respect to a new bi-dimensional $\mathbb{G}$-Brownian motion (W^R, W^S), whose coordinates will be shown to share the same correlation ρ.

We achieve this new representation by expressing the process R, now seen as adapted to the filtration $\mathbb{G}$, in the following way

$$dR_t = \hat{u}(R_t, R_T, t, T)dt + \hat{v}(R_t, R_T, t, T)dW_t^R \;, \tag{12}$$

where we compute the functions u and v as the infinitesimal $\mathcal{G}_t$-conditional mean and variance of the increment process of R_t. Similar results could have been achieved by applying Jacod's theorem [Amendinger et al., 1998, Jacod, 1985] as shown in Jeanblanc et al. [2009, Theorem 5.9.3.1], however we prefer to go for a more direct approach that gives, as by-product, the complete distribution of $R_{t+\delta}|\mathcal{G}_t$, with $0 \leq \delta \leq T - t$.

3.1 Analysis of the R process

Assuming integrability for the functions a_1, a_2, and b_2 in (3) it is easy to see [Jeanblanc et al., 2009, Example 1.5.4.8] by applying Itô's lemma, that the process R admits the following explicit solution

$$R_t = \Psi_t \left[R_0 + \int_0^t (a_2 \Psi^{-1})(x)\, dx + \int_0^t (b_2 \Psi^{-1})(x)\, dB_x^R \right] \,, \tag{13}$$

where the function $\Psi_{s,t} = \exp\left(\int_s^t a_1(x)dx\right)$, and we used the simplified notation $\Psi(t) = \Psi_t = \Psi_{0,t}$. The process R is Markov and it is Gaussian when R_0 is normal distributed.

To compute the distribution of $R_{t+\delta}|\mathcal{G}_t$ for $0 < \delta < T - t$, we need the following lemma, that besides its simplicity we state here as reference for the following calculations.

Lemma 1. *Let Z_t be a Markov process adapted to a filtration $\mathbb{F}$, then under the filtration $\mathbb{G}$ it holds*

$$f_{Z_{t+\delta}|Z_t,Z_T}(b\,|\,a,c) = f_{Z_{t+\delta}|Z_t}(b\,|\,a)\, f_{Z_T|Z_{t+\delta}}(c\,|\,b)/f_{Z_T|Z_t}(c\,|\,a) \tag{14}$$

$$f_{Z_{t+\delta}|\mathcal{G}_t}(b) = f_{Z_{t+\delta}|Z_t,Z_T}(b\,|\,Z_t, Z_T) \,, \tag{15}$$

and in particular, $\boldsymbol{E}_{\mathcal{G}_t}[Z_t] = \boldsymbol{E}_{\mathcal{F}_t}[Z_t|Z_T]$.

Proof. Equation (14) follows by a direct application of twice the definition of conditional density function

$$\begin{aligned} f_{Z_{t+\delta}|Z_t,Z_T}(b\,|\,a,c) &= \frac{f_{Z_t}(a)\, f_{Z_{t+\delta}|Z_t}(b\,|\,a)\, f_{Z_T|Z_t,Z_{t+\delta}}(c\,|\,a,b)}{f_{Z_t}(a)\, f_{Z_T|Z_t}(c\,|\,a)} \\ &= \frac{f_{Z_{t+\delta}|Z_t}(b\,|\,a)\, f_{Z_T|Z_{t+\delta}}(c\,|\,b)}{f_{Z_T|Z_t}(c\,|\,a)}, \end{aligned}$$

where in the second equality, we used the Markov property. Also (15) follows by the Markov property, i.e. $\boldsymbol{E}[f_{Z_{t+\delta}}(b)|\mathcal{G}_t] = \boldsymbol{E}[f_{Z_{t+\delta}}(b)|\mathcal{F}_t, Z_T] = \boldsymbol{E}[f_{Z_{t+\delta}}(b)|Z_t, Z_T]$. □

Given that R is a Markov process, by (14) – (15) we reduce our problem to the study of the distribution of $(R_s|R_u)$ for $0 \le u \le s$. We can calculate it by conveniently handling the explicit expression in (13) as we show in the following lemma.

Lemma 2. *Let R_t be the process defined by* (3)*. For $0 \le u \le s$, the conditioned random variable $(R_s|R_u)$ has the following distribution*

$$(R_s|R_u) \approx \mathcal{N}\left(\Psi_{u,s} R_u + \Psi_s \int_u^s \left(a_2\Psi^{-1}\right)(x)dx,\ \Psi_s^2 \int_u^s \left(b_2\Psi^{-1}\right)^2(x)dx \right) \tag{16}$$

Proof. Using equation (13), we can express the value of R_s in terms of its value at time u in the following way,

$$\begin{aligned} R_s = \Psi_u \Psi_{u,s} \Big[R_0 &+ \int_0^u \left(a_2\Psi^{-1}\right)(x)dx + \int_u^s \left(a_2\Psi^{-1}\right)(x)dx \\ &+ \int_0^u \left(b_2\Psi^{-1}\right)(x)dB_x^R + \int_u^s \left(b_2\Psi^{-1}\right)(x)dB_x^R \Big] \\ = \Psi_{u,s} R_u &+ \Psi_s \left[\int_u^s \left(a_2\Psi^{-1}\right)(x)dx + \int_u^s \left(b_2\Psi^{-1}\right)(x)dB_x^R \right] \,. \end{aligned} \tag{17}$$

The result then follows by identifying the deterministic and stochastic part of formula (17), the latter gives the variance by applying the Itô isometry. □

Using the above expression we finally can compute the complete distribution of $R_{t+\delta}|\mathcal{G}_t$ as shown in the following theorem.

Theorem 3. *The conditioned random variable $(R_{t+\delta}|R_t, R_T) \approx (R_{t+\delta}|\mathcal{G}_t)$ is Gaussian, whose parameters are given by*

$$\boldsymbol{E}[R_{t+\delta}|R_t, R_T] = \frac{\int_{t+\delta}^{T} \left(b_2\Psi^{-1}\right)^2(x)dx}{\int_t^T \left(b_2\Psi^{-1}\right)^2(x)dx} \left[\Psi_{t,t+\delta}R_t + \Psi_{t+\delta}\int_t^{t+\delta}(a_2\Psi^{-1})(x)dx\right] + \frac{\int_t^{t+\delta}(b_2\Psi^{-1})^2(x)dx}{\int_t^T(b_2\Psi^{-1})^2(x)dx}\frac{\left[R_T - \Psi_T\int_{t+\delta}^T(a_2\Psi^{-1})(x)dx\right]}{\Psi_{t+\delta,T}}\,, \tag{18a}$$

$$\boldsymbol{V}[R_{t+\delta}|R_t, R_T] = \frac{\int_{t+\delta}^{T} \left(b_2\Psi^{-1}\right)^2(x)dx}{\int_t^T \left(b_2\Psi^{-1}\right)^2(x)dx} \left[\Psi_{t+\delta}^2\int_t^{t+\delta}\left(b_2\Psi^{-1}\right)^2(x)dx\right]\,. \tag{18b}$$

Proof. The proof follows by applying Lemma 1 and Lemma 2 and by rearranging terms in such a way to eventually identify the density function of a Normal distribution and its parameters. Since the steps are quite technical we defer the details to the appendix. □

3.2 Differential Coefficients

Known the density of the variable $(R_{t+\delta}\,|\,R_t, R_T)$, we can compute the first term of the Taylor expansion in δ of its parameters, that allows to compute the functions $\hat{u}$ and $\hat{v}$ in (12) by the formulas

$$\boldsymbol{E}[R_{t+\delta} - R_t|R_t, R_T] = \hat{u}(R_t, R_T, t, T)\,\delta + o(\delta)\,, \tag{19a}$$

$$\boldsymbol{E}[(R_{t+\delta} - R_t)^2|R_t, R_T] = \hat{v}^2(R_t, R_T, t, T)\,\delta + o(\delta)\,. \tag{19b}$$

Since the quadratic variation does not depend on the filtration, it will follow that $\hat{v}(R_t, R_T, t, T) = b_2(t)$.

Lemma 4. *The $\hat{u}$ and $\hat{v}$ terms in the expansions* (19) *are given by*

$$\hat{u}(R_t, R_T, t, T) = a_1(t)R_t + a_2(t) + \hat{g}_{t,T}(R_t, R_T)\,, \tag{20a}$$

$$\hat{v}(R_t, R_T, t, T) = b_2(t) \tag{20b}$$

where

$$\hat{g}_{t,T}(R_t, R_T) \coloneqq \frac{(b_2\Psi^{-1})^2(t)\left[R_T - \Psi_{t,T}\,R_t - \Psi_T\int_t^T(a_2\Psi^{-1})(x)dx\right]}{\Psi_{t,T}\int_t^T(b_2\Psi^{-1})^2(x)dx}\,. \tag{21}$$

Proof. The term in (20a) follows by computing in the expression (18a) the $\lim_{\delta\to 0}\boldsymbol{E}[R_{t+\delta} - R_t|R_t, R_T]/\delta$. The result follows by applying the following estimates

$$\frac{\Psi_{t+\delta}}{\delta}\int_t^{t+\delta}(a_2\Psi^{-1})(x)dx \overset{\delta}{\sim} a_2(t)\,;$$

$$\frac{1}{\delta}\int_t^{t+\delta}(b_2\Psi^{-1})^2dx \overset{\delta}{\sim} (b_2\Psi^{-1})^2(t)\,;$$

$$\frac{1}{\delta}\left(\Psi_{t,t+\delta}\frac{\int_{t+\delta}^T(b_2\Psi^{-1})^2\,dx}{\int_t^T(b_2\Psi^{-1})^2\,dx} - 1\right) \overset{\delta}{\sim} a_1(t) - \frac{(b_2\Psi^{-1})^2(t)}{\int_t^T(b_2\Psi^{-1})^2\,dx}\,.$$

In the same way, using that $\boldsymbol{E}[(R_{t+\delta} - R_t)^2|R_t, R_T] \overset{\delta}{\sim} \boldsymbol{V}[R_{t+\delta} - R_t|R_t, R_T]$ we get (20b) from (18b) and the estimate $\Psi_t^2(b_2\Psi^{-1})^2(t) \overset{\delta}{\sim} [b_2(t)]^2$. □

3.3 Optimal Portfolio

The analysis above allows to rewrite the SDE (11) expressed in the filtration $\mathbb{F}$ under the filtration $\mathbb{G}$ as shown by the following proposition.

Proposition 5. *Under the filtration $\mathbb{G}$ the processes $X^{\hat{\pi}} = (X_t^{\hat{\pi}},\, 0 \le t \le T)$ and $R = (R_t,\, 0 \le t \le T)$ satisfy the following SDE:*

$$\begin{cases} dX_t^{\hat{\pi}} = (1 - \hat{\pi}_t)X_t^{\hat{\pi}}R_t\,dt + \hat{\pi}_tX_t^{\hat{\pi}}(\eta_t dt + \xi_t dB_t^S) \\ dR_t = \hat{g}_{t,T}(R_t, R_T)\,dt + [a_1(t)R_t + a_2(t)]\,dt + b_2(t)dW_t^R \\ dB_t^S = \dfrac{\rho}{b_2(t)}\hat{g}_{t,T}(R_t, R_T)\,dt + dW_t^S \end{cases} \tag{22}$$

where $(W^R, W^S) = ((W_t^R, W_t^S), 0 \leq t \leq T)$ *is a bi-dimensional* $\mathbb{G}$*-Brownian motion with constant correlation* ρ*.*

Proof. Since B^R and B^S have constant correlation ρ, we can write

$$B_t^S = \rho B_t^R + \sqrt{1-\rho^2}\, W_t \tag{23}$$

where $W = (W_t,\, 0 \leq t \leq T)$ is an $\mathbb{F}$-Brownian motion independent of B^R. By Lemma 4, the semi-martingale representation of R under $\mathbb{H}$ is given by

$$dR_t = \hat{g}_{t,T}(R_t, R_T)\, dt + [a_1(t)R_t + a_2(t)]\, dt + b_2(t) dW_t^R \tag{24}$$

with $W^R = (W_t^R,\, 0 \leq t \leq T)$ being a $\mathbb{G}$-Brownian motion. Expressing B^R in terms of R and W^R we get

$$dB_t^R = \frac{dR_t - [a_1(t)R_t + a_2(t)]dt}{b_2(t)} = \frac{\hat{g}_{t,T}(R_t, R_T)}{b_2(t)} dt + dW_t^R\ , \tag{25}$$

and using (23) we get the $\mathbb{H}$ semi-martingale representation of B^S as

$$\begin{aligned} dB_t^S &= \rho\, dB_t^R + \sqrt{1-\rho^2} dW_t \\ &= \frac{\rho}{b_2(t)} \hat{g}_{t,T}(R_t, R_T)\, dt + \rho\, dW_t^R + \sqrt{1-\rho^2}\, dW_t\ . \end{aligned} \tag{26}$$

To complete the proof, we define the process $W^S = (W_t^S,\, 0 \leq t \leq T)$ by setting $W_t^S = \rho\, W_t^R + \sqrt{1-\rho^2}\, W_t$, that satisfies $\boldsymbol{E}[W_t^S W_t^R] = \rho$. □

The semi-martingale representation of the wealth process in $\mathbb{G}$ allows to solve for the optimal strategy $\hat{\pi}$ that maximizes $\boldsymbol{E}[\ln(X_T^\pi)]$ along the lines of Karatzas et al. [1987] and Merton [1969]. This is summarized by the following main result.

Theorem 6. *The solution of the optimal portfolio problem*

$$\sup_{\hat{\pi} \in \mathcal{A}_{\mathbb{G}}} \boldsymbol{E}[\ln(X_T^{\hat{\pi}})]\ ; \qquad \textit{with } \mathbb{G} = \mathbb{F} \bigvee \sigma(R_T) \tag{27}$$

where $(X_t^{\hat{\pi}},\, 0 \leq t \leq T)$ *satisfies* (22)*, is given by*

$$\hat{\pi}_t^* = \frac{\eta_t - R_t}{\xi_t^2} + \frac{\rho}{b_2(t)\, \xi_t} \frac{(b_2 \Psi^{-1})^2(t) \left(R_T - \Psi_{t,T} R_t - \Psi_T \int_t^T a_2 \Psi^{-1}(x) dx \right)}{\Psi_{t,T} \int_t^T (b_2 \Psi^{-1})^2(x) dx}\ . \tag{28}$$

Proof. Using the expression (22) and applying Ito's lemma we compute the expected value of $\ln X_T^{\hat{\pi}}$ as

$$\boldsymbol{E}\left[\ln \frac{X_t^{\hat{\pi}}}{X_0} \right] = \int_0^T \boldsymbol{E}\left[\xi_t^2\, I_{t,T}(\hat{\pi}_t) \right] dt \tag{29}$$

where

$$I_{t,T}(x) = \frac{R_t}{\xi_t^2} + \left(\frac{\eta_t - R_t}{\xi_t^2} + \frac{\rho}{b_2(t)\, \xi_t} \hat{g}_{t,T}(R_t, R_T) \right) x - \frac{\xi_t^2}{2} x^2\ . \tag{30}$$

We immediately get that

$$\sup_{\hat{\pi} \in \mathcal{A}_{\mathbb{G}}} \boldsymbol{E}\left[\ln \frac{X_t^{\hat{\pi}}}{X_0} \right] \leq \int_0^T \boldsymbol{E}[\sup_{\hat{\pi} \in \mathcal{A}_{\mathbb{G}}} I_{t,T}(\hat{\pi}_t)]\, dt\ , \tag{31}$$

and the equality follows from the fact that the optimal strategy that optimizes the right-hand-side of (31) belongs to $\mathcal{A}_{\mathbb{G}}$. To compute it we equate to 0 the first derivative of $I_{t,T}$, obtaining (28). Since $I''_{t,T}(\hat{\pi}_t^*) = -\xi_t^2 < 0$, the solution indeed identifies a maximum. □

4 Ornstein-Uhlenbeck Model

In this section, we consider the case in which the functions a_1, a_2, b_2 are constant. Doing so, the model of the interest rate is assumed to be an Ornstein-Uhlenbeck process and satisfies the following SDE

$$dY_t = k(\mu - Y_t)dt + \sigma dB_t^Y, t \geq 0, \tag{32}$$

where $\mu \in \mathbb{R}$ and $\sigma, k \in \mathbb{R}^+$. This model was introduced in the financial setting in Vasicek [1977], and in the previous context, it consists in setting

$$a_1(t) = -k\,,\ a_2(t) = k\mu\,,\ b_2(t) = \sigma\ . \tag{33}$$

By Ito's lemma, it is easy to verify that (32) admits the following explicit solution,

$$Y_t = \mu + (Y_0 - \mu)\, e^{-kt} + \mu\,(1 - e^{-kt}) + \sigma \int_0^t e^{-k(t-s)} dB_s^Y, \tag{34}$$

and it is a Markov process, Gaussian if Y_0 is Normal distributed. Given the starting value, its marginal distribution at time t is given by,

$$(Y_t|Y_0) \approx \mathcal{N}\left(\mu(t, Y_0), \sigma^2(t)\right)\ , \tag{35}$$

where $\mu(t,y) \coloneqq \mu + (y-\mu)e^{-kt}$ and $\sigma^2(t) \coloneqq \sigma^2\,(1-e^{-2kt})/2k = \sigma^2 \sinh(kt)e^{-kt}/k$.

Proceeding along the lines of Section 3, we rewrite the semi-martingale representation of Y under the filtration $\mathbb{G}$, that in this case allows for explicit expressions.

4.1 Analysis of the Y process

The Ornstein-Uhlenbeck process has the characteristic property to be the unique Gaussian Markov process being stationary, as it was firstly shown in Doob [1942]. Stationarity means that $(Y_{t+\delta}|\mathcal{F}_t) \approx (Y_\delta|\mathcal{F}_0)$, for $t, \delta > 0$, and it leads to a strong simplification of formulas as shown by the following lemma that details the results of Theorem 3 in this specific case.

Lemma 7. *The conditioned random variable $(Y_{t+\delta}|Y_t, Y_T) \approx (Y_{t+\delta}|\mathcal{G}_t)$ is Gaussian, whose parameters are given by*

$$\boldsymbol{E}[Y_{t+\delta}|Y_t, Y_T] = \frac{\sigma^2(\delta)e^{-2k(T-t-\delta)}}{\sigma^2(T-t)}\mu(t+\delta-T, Y_T) + \frac{\sigma^2(T-t-\delta)}{\sigma^2(T-t)}\mu(\delta, Y_t), \tag{36a}$$

$$\boldsymbol{V}[Y_{t+\delta}|Y_t, Y_T] = \frac{\sigma^2(T-t-\delta)\sigma^2(\delta)}{\sigma^2(T-t)}\ . \tag{36b}$$

Looking at the first order Taylor expansions of the above expressions we get

Proposition 8. *The variable $(Y_{t+\delta} - Y_t|\mathcal{G}_t)$ has the following differential mean and variance*

$$\boldsymbol{E}[Y_{t+\delta} - Y_t|Y_t, Y_T] \overset{\delta}{\sim} k(\mu - Y_t) + \hat{f}_{t,T}(Y_t, Y_T) \tag{37}$$

$$\boldsymbol{V}[(Y_{t+\delta} - Y_t)^2|Y_t, Y_T] \overset{\delta}{\sim} \sigma^2 \tag{38}$$

where

$$\hat{f}_{t,T}(Y_t, Y_T) \coloneqq \frac{k}{\sinh(k(T-t))}\Big((\mu - Y_t)\ e^{-k(T-t)} - (\mu - Y_T)\Big)\ . \tag{39}$$

4.2 Optimal Portfolio

We are now ready to formulate and solve the optimal portfolio problem for the insider trader under the assumption that the interest rate follows the model given in (32). We repeat here the dynamics of the portfolio of the investor given in (11),

$$\begin{cases} dX_t^{\hat{\pi}} = (1-\hat{\pi}_t)X_t^{\hat{\pi}}Y_t\,dt + \hat{\pi}_t X_t^{\hat{\pi}}(\eta_t\,dt + \xi_t\,dB_t^S) \\ dY_t = k(\mu - Y_t)dt + \sigma dB_t^Y \end{cases} \tag{40}$$

and we remind that the strategy $\hat{\pi}$ is looked for in the set $\mathcal{A}_{\mathbb{G}}$ of $\mathbb{G}$-adapted functions with the aim to optimize the terminal expected value of the wealth, $\boldsymbol{E}[\ln(X_T^{\hat{\pi}})]$. Since $\mathbb{G} = \mathbb{F} \bigvee \sigma(Y_T)$, the investor is informed, since the beginning, about the final value of the interest process, Y_T.

To solve the optimization problem it is useful to rewrite (40) in an equivalent form, because, in the filtration $\mathbb{G}$, the bi-dimensional process $(B^R, B^S) = ((B_t^R, B_t^S), 0 \le t \le T)$ is not anymore a Brownian motion, but just a semi-martingale. We use the expressions (37) given in Proposition (8) to get its martingale decomposition in the filtration $\mathbb{G}$, as shown in the following proposition.

Proposition 9. *The dynamics of the wealth of the $\mathbb{G}$-adapted portfolio solve the following system of SDEs*

$$\begin{cases} dX_t^{\hat{\pi}} = (1-\hat{\pi}_t) X_t^{\hat{\pi}} Y_t \, dt + \hat{\pi}_t X_t^{\hat{\pi}} (\eta_t dt + \xi_t dB_t^S) \\ \;\; dY_t = \hat{f}_{t,T}(Y_t, Y_T)\, dt + k(\mu - Y_t)\, dt + \sigma \, dW_t^Y \\ dB_t^S = \dfrac{\rho}{\sigma} \hat{f}_{t,T}(Y_t, Y_T)\, dt + dW_t^S \end{cases} \tag{41}$$

where $(W^Y, W^S) = ((W_t^Y, W_t^S), 0 \le t \le T)$ is a bi-dimensional $\mathbb{G}$-Brownian motion with constant correlation ρ.

Proof. The proof follows along the lines of the proof of Proposition 5. □

Using the above representation and by applying standard optimization techniques we are finally able to find the optimal strategy.

Theorem 10. *The solution of the optimal portfolio problem*

$$\sup_{\hat{\pi} \in \mathcal{A}_{\mathbb{G}}} \boldsymbol{E}[\ln(X_T^{\hat{\pi}})] \; ; \qquad \textit{with } \mathbb{G} = \mathbb{F} \bigvee \sigma(R_T) \tag{42}$$

where $(X_t^{\hat{\pi}}, 0 \le t \le T)$ satisfies (41), is given by

$$\hat{\pi}_t^* = \frac{\eta_t - Y_t}{\xi_t^2} + \frac{\rho}{\sigma\, \xi_t} \frac{k}{\sinh(k(T-t))} \Big((\mu - Y_t)\, e^{-k(T-t)} - (\mu - Y_T) \Big) \; . \tag{43}$$

The optimal value of the portfolio is given by

$$\mathbb{V}_T^{\mathbb{G}} = \int_0^T \boldsymbol{E}\left[Y_t + \frac{(\xi_t\, \hat{\pi}_t^*)^2}{2} \right] dt \; . \tag{44}$$

Proof. The proof follows along the lines of the proof of Theorem 6, by expressing the expected value of the utility of the terminal wealth in the following form

$$\boldsymbol{E}\left[\ln \frac{X_t^{\hat{\pi}}}{X_0} \right] = \int_0^T \boldsymbol{E}\left[\xi_t^2 \, I_{t,T}(\hat{\pi}_t) \right] dt \tag{45}$$

where

$$I_{t,T}(x) = \frac{Y_t}{\xi_t^2} + \left(\frac{\eta_t - Y_t}{\xi_t^2} + \frac{\rho}{\sigma\, \xi_t} \hat{f}_{t,T}(Y_t, Y_T) \right) x - \frac{x^2}{2} \; . \tag{46}$$

Equation (44) follows by substituting the optimal value $\hat{\pi}_t^*$ in (45). □

5 The Price of Information

In this section, we calculate the benefit that an insider trader would obtain from the additional information on the future value of the interest. We recall formula (7) that given a filtration $\mathbb{H}$ containing the basic or natural information flow $\mathbb{F}$, i.e. $\mathbb{F} \subset \mathbb{H}$, defines the optimal value of the portfolio as

$$\mathbb{V}_T^{\mathbb{H}} := \sup_{\pi \in \mathcal{A}_{\mathbb{H}}} \boldsymbol{E}[\ln(X_T^{\pi})] = \boldsymbol{E}[\ln(X_T^{\pi^*})] \; . \tag{47}$$

This allows to define the advantage of the additional information carried by $\mathbb{H}$ as the increment in the expected value of the optimal portfolio with respect to the one constructed by using only the accessible information in $\mathbb{F}$.

Definition 1. The price of the information of a filtration $\mathbb{H} \supset \mathbb{F}$, is given by

$$\Delta \mathbb{V}_T^{\mathbb{H}} = \mathbb{V}_T^{\mathbb{H}} - \mathbb{V}_T^{\mathbb{F}} \tag{48}$$

Where the quantities on the right-hand-side are defined in (47).

In the following we continue to work with the Vasicek model in (32). By Merton [1969], it is known that the optimal portfolio in the absence of insider information is given by the strategy

$$\pi_t^* = \frac{\eta_t - Y_t}{\xi_t^2} \ , \tag{49}$$

while, according to the results of Theorem 10, using the insider information, modeled by the enlarged filtration $\mathbb{G}$, we have that the optimal strategy is given by

$$\hat{\pi}_t^* = \pi_t^* + \frac{\rho}{\sigma\, \xi_t} \frac{k}{\sinh(k(T-t))} \left((Y_T - \mu) - (Y_t - \mu) e^{-k(T-t)} \right) \ . \tag{50}$$

The results of this section surprisingly show that the information carried by $\mathbb{G}$, even if it refers to the only interest-rate process, is so strong that implies an infinite value. However since the sign of this value is shown to strictly depend on the accuracy of the model used by the trader in following her strategy, it implies that she can incur in an infinite loss when the accuracy is low.

To simplify calculations, we are going to make the following standing assumption

Assumption 1. *The processes η and ξ are deterministic.*

Hiwever this assumption can be easily relaxed at the price of having more complicated formulas.

Lemma 11. *Characterization of the moments of the process π^*:*

$$\lim_{t \to T} \boldsymbol{E}[\hat{\pi}_t^*] < + \ , \infty \tag{51}$$

$$\int_0^T \boldsymbol{V}[\hat{\pi}_t^*] dt = +\infty \ . \tag{52}$$

Proof. To get (51), by (50) and using the expansion $1/\sinh(x) = 1/x + o(1)$, it is enough to prove that

$$Y_T - \mu - (\boldsymbol{E}[Y_t | Y_0, Y_T] - \mu) e^{-k(T-t)} = O(T-t) \ .$$

Moreover, expanding $\exp(-k(T-t))$, this is equivalent to show that

$$Y_T - \boldsymbol{E}[Y_t | Y_0, Y_T] = O(T-t).$$

Using (18a), after some algebraic manipulations, we get

$$\begin{aligned} \boldsymbol{E}[Y_t | Y_0, Y_T] &= \frac{\sigma^2(t)}{\sigma^2(T)} e^{-2k(T-t)} \mu(t-T, Y_T) + \frac{\sigma^2(T-t)}{\sigma^2(T)} \mu(t, Y_0) \\ &\overset{T-t}{\sim} Y_T + (T-t) \left(k \frac{Y_T\, e^{-kT}}{\sinh(kT)} - k(Y_T - \mu) + \frac{\sigma^2}{\sigma^2(T)} \mu(t, Y_0) \right) \ , \end{aligned}$$

that implies the result. To prove (52) we assume $\rho > 0$, the case $\rho < 0$ follows along similar arguments by using (51).

$$\begin{aligned} \int_0^T \boldsymbol{V}[\hat{\pi}_t^*] dt &\geq \int_0^T \boldsymbol{V}[Y_t | Y_0, Y_T] \left(\rho \frac{k}{\sigma \xi_t} \frac{e^{-k(T-t)}}{\sinh(k(T-t))} \right)^2 dt \\ &= \int_0^T \frac{\sigma^2}{2k} \frac{\left(1 - e^{-2k(T-t)}\right)\left(1 - e^{-2kt}\right)}{1 - e^{-2kT}} \left(\rho \frac{2k}{\sigma \xi_t} \frac{e^{-2k(T-t)}}{1 - e^{-2k(T-t)}} \right)^2 dt \\ &= \rho^2 \frac{4k}{1 - e^{-2kT}} \int_0^T \frac{\left(1 - e^{-2kt}\right)}{\xi_t^2} \left(\frac{e^{-4k(T-t)}}{1 - e^{-2k(T-t)}} \right) dt = +\infty \ , \end{aligned}$$

where in the first step we used the fact that the strategy $\hat{\pi}^*$ can be written as

$$\hat{\pi}_t^* = \hat{\pi}_t^*(Y_t, Y_T) = -Y_t \left(\frac{1}{\xi_t^2} + \frac{k\rho}{\sigma \xi_t} \frac{e^{-k(T-t)}}{\sinh(k(T-t))} \right) + h_{t,T}(Y_T) \tag{53}$$

for some deterministic function h. $\square$

Proposition 12. *The value of the information, $\Delta\mathbb{V}_T^{\mathbb{G}}$, of the insider trader is infinite.*

Proof. To see this, just use $\boldsymbol{E}[(\pi_t^*)^2] = \boldsymbol{E}[\pi_t^*]^2 + \boldsymbol{V}[\pi_t^*]$ and apply Lemma 11. $\square$

Now, we assume that the insider trader does not know the exact value of the correlation ρ. We denote her estimation of the correlation by $\bar{\rho}$.

Proposition 13. *Assuming that the insider trader uses an estimation of the correlacion factor ρ, say $\bar{\rho}$, with $\bar{\rho} \neq \rho$ the value of her information is still unbounded but it is equal to*

$$\Delta\mathbb{V}_T = \begin{cases} +\infty & \text{if } \rho/\bar{\rho} > 1/2 \\ -\infty & \text{otherwise} \end{cases} . \tag{54}$$

Proof. We begin by assuming that the insider trader follows the following strategy, with $\bar{\rho} \neq \rho$.

$$\bar{\pi}_t = \pi_t^* + \frac{\bar{\rho}}{\sigma\,\xi_t}\hat{f}_{t,T}(Y_t, Y_T) \tag{55}$$

therefore the expected terminal utility of her portfolio will be equal to

$$\begin{aligned} \boldsymbol{E}[\ln X_t^{\bar{\pi}}] &= \int_0^T \boldsymbol{E}\left[(1-\bar{\pi}_t)Y_t + \frac{\bar{\rho}}{\sigma\,\xi_t}\hat{f}_{t,T}(Y_t,Y_T)\bar{\pi}_t - \frac{\bar{\pi}_t^2\xi_t^2}{2}\right] dt \\ &= \int_0^T \boldsymbol{E}\Big[Y_t + \frac{Y_t^2}{2\xi_t^2} - \frac{\eta_t^2}{2\xi_t^2} + (\rho-\bar{\rho})\eta_t\frac{\hat{f}_{t,T}(Y_t,Y_T)}{\sigma\xi_t} - \rho Y_t\frac{\hat{f}_{t,T}(Y_t,Y_T)}{\sigma\xi_t} \\ &\quad + \frac{1}{\sigma}\left(\rho\bar{\rho} - \frac{1}{2}\bar{\rho}^2\right)\left(\hat{f}_{t,T}(Y_t,Y_T)\right)^2\Big] dt\ . \end{aligned}$$

Using Lemma 11, all terms are integrable but $\hat{f}^2_{t,T}(Y_t, Y_T)$. Therefore we are left with studying the sign of $\rho\bar{\rho} - \bar{\rho}^2/2 = \bar{\rho}^2(\rho/\bar{\rho} - 1/2)$, from which the result follows. $\square$

As foretold, Proposition 13 shows that the insider trader is required to estimate with some accuracy the actual value of the correlation coefficient ρ in order to gain from her privileged knowledge and not to incur in an infinite loss.

6 Interval-type information

In this section, we assume the insider trader doesn't know the final value of the interest rate, Y_T, but she knows if it will be greater than a given value p. To this aim we introduce the random variable $A = \mathbb{1}\{Y_T \geq p\}$, together with the following filtration

$$\widetilde{\mathbb{G}} = \mathbb{F} \bigvee \sigma(A). \tag{56}$$

It is obvious that $\mathbb{F} \subset \widetilde{\mathbb{G}} \subset \mathbb{G}$. To calculate the optimal portfolio $\tilde{\pi}$, we will compute the new drift of the conditioned process adapted to $\widetilde{\mathbb{G}}$. By mimicking the calculations done in Sections 3 and 4, we introduce the following correction function for the drift of the interest rate

$$\begin{aligned} \tilde{f}_{t,T}(Y_t, A) &= \lim_{\delta\to 0}\frac{1}{\delta}\boldsymbol{E}\left[Y_{t+\delta} - Y_t|\widetilde{\mathcal{G}}_t\right] - (k(\mu - Y_t)) = \lim_{\delta\to 0}\frac{1}{\delta}\boldsymbol{E}\left[\boldsymbol{E}[Y_{t+\delta} - Y_t|\mathcal{G}_t]|\widetilde{\mathcal{G}}_t\right] - (k(\mu - Y_t)) \\ &= \boldsymbol{E}\left[\hat{f}_{t,T}(Y_t, Y_T)|\widetilde{\mathcal{G}}_t\right]\ . \end{aligned} \tag{57}$$

The next proposition gives the probabilistic interpretation of the above function and a simple expression to compute it.

Proposition 14. *The variable $(Y_{t+\delta} - Y_t|\tilde{\mathcal{G}}_t)$ is Gaussian and it has the following differential mean and variance*

$$\boldsymbol{E}[Y_{t+\delta} - Y_t|Y_t, A] \overset{\delta}{\sim} k(\mu - Y_t) + \tilde{f}_{t,T}(Y_t, A) \tag{58}$$

$$\boldsymbol{V}[(Y_{t+\delta} - Y_t)^2|Y_t, A] \overset{\delta}{\sim} \sigma^2 \tag{59}$$

where

$$\tilde{f}_{t,T}(y, a) = \sigma^2 e^{-k(T-t)}\frac{f_{Y_T}(p\,|\,Y_t = y)}{(1-a) + (2a-1)\boldsymbol{P}(Y_T \geq p|Y_t = y)}, \tag{60}$$

with $a \in \{0, 1\}$ and $y \in \mathbb{R}$.

Proof. Applying the definition of the conditioned expectation to (57), we can compute the drift of the interest rate process, for example when the condition $\{Y_T \geq p\}$ is satisfied, as follows

$$\boldsymbol{E}[\mathbb{1}\{Y_T \geq p\}\tilde{f}_{t,T}(Y_t, 1)] = \boldsymbol{E}\left[\mathbb{1}\{Y_T \geq p\}\hat{f}_{t,T}(Y_t, Y_T)\right] .$$

This formula allows to compute the value of $\tilde{f}_{t,T}(Y_t, 1)$ as in equation (61a) below. Equation (61b) follows along similar arguments.

$$\tilde{f}_{t,T}(y, 1) = \frac{\int_{\{Y_T \geq p\}} \hat{f}_{t,T}(y, u)\, dP_{T-t}(u|y)}{\boldsymbol{P}(Y_T \geq p\,|Y_t = y)} \tag{61a}$$

$$\tilde{f}_{t,T}(y, 0) = \frac{\int_{\{Y_T < p\}} \hat{f}_{t,T}(y, u)\, dP_{T-t}(u|y)}{\boldsymbol{P}(Y_T \geq p\,|Y_t = y)} \;, \tag{61b}$$

where $P_{T-t}(\cdot\,|\,y)$ is the distribution of $(Y_{T-t}|Y_0 = y)$. Substituting in (61a) the explicit expression of $\hat{f}$, given in (39), the numerator can be written in the following form

$$\begin{aligned}
\int_p^\infty \hat{f}_{t,T} dP_{T-t} &= \int_p^\infty \frac{k}{\sinh(k(T-t))}\,(u - \mu(T-t, y))\, dP_{T-t}(u|y) \\
&= \frac{k}{\sinh(k(T-t))}\boldsymbol{E}[\mathbb{1}\{Y_T \geq p\}(Y_T - \mu(T-t, y)) \mid Y_t = y] \\
&= \frac{k}{\sinh(k(T-t))}\sigma^2(T-t) f_{Y_T}(p \mid Y_t = y) \;,
\end{aligned}$$

with $\mu(t, y)$ and $\sigma^2(t)$ defined below equation (35), and where, in the last step, we used the obvious fact that if $Z \sim_d N(\mu, \sigma^2)$, then $\boldsymbol{E}[\mathbb{1}\{Z \geq c\}(Z - \mu)] = \sigma^2 f_Z(c)$.

Substituting back in (61a) we finally get,

$$\tilde{f}_{t,T}(y, 1) = \sigma^2 e^{-k(T-t)} \frac{f_{Y_T}(p\,|\,Y_t = y)}{\boldsymbol{P}(Y_T \geq p|Y_t = y)} \;,$$

and repeating the same procedure for (61b) we get the result. □

Since $\widetilde{\mathbb{G}} \subset \mathbb{G}$, the $\mathbb{G}$-Brownian motion (W^R, W^S) is also a $\widetilde{\mathbb{G}}$-Brownian motion, and we can write the process Y_t as the solution of the following SDE

$$dY_t = \tilde{f}_{t,T}(Y_t, A)dt + k(\mu - Y_t)dt + \sigma dW_t^Y \;.$$

The above expression together with the arguments of Subsection 3.3, allows to write the dynamics of the portfolio under the strategy of the insider trader under the information flow $\widetilde{\mathbb{G}}$.

Proposition 15. *The dynamics of the wealth of the $\widetilde{\mathbb{G}}$-adapted portfolio solve the following system of SDEs*

$$\begin{cases} dX_t^{\tilde{\pi}} = (1 - \tilde{\pi}_t)X_t^{\tilde{\pi}} Y_t\, dt + \tilde{\pi}_t X_t^{\tilde{\pi}}(\eta_t\, dt + \xi_t\, dB_t^S) \\ dY_t = \tilde{f}_{t,T}(Y_t, A)dt + k(\mu - Y_t)dt + \sigma dW_t^Y \\ dB_t^S = \dfrac{\rho}{\sigma}\tilde{f}_{t,T}(Y_t, A)dt + dW_t^S \end{cases} \tag{62}$$

where $(W^Y, W^S) = ((W_t^Y, W_t^S), 0 \leq t \leq T)$ is a bi-dimensional $\widetilde{\mathbb{G}}$-Brownian motion with constant correlation ρ. In particular, the optimal portfolio in the market is given by:

$$\tilde{\pi}_t^* = \tilde{\pi}_t^*(Y_t, A) = \frac{\eta_t - Y_t}{\xi_t^2} + \frac{\rho}{\sigma\,\xi_t}\tilde{f}_{t,T}(Y_t, A) \;. \tag{63}$$

We are now ready to state the main result of this section.

Theorem 16. *The value of the information, $\Delta\mathbb{V}_T^{\tilde{\mathbb{G}}}$, of the insider trader is finite.*

Proof. Proceeding as in formula (44) and using the expression in (63) we can compute the value of the information as

$$
\begin{aligned}
\mathbb{V}_T^{\tilde{\mathbb{G}}} &= \int_0^T \boldsymbol{E}\left[Y_t + \frac{(\xi_t \tilde{\pi}_t^*)^2}{2}\right] dt \\
&= \int_0^T \boldsymbol{E}\left[Y_t + \frac{1}{2}\left(\frac{\eta_t - Y_t}{\xi_t^2} + \frac{\rho}{\sigma\,\xi_t}\tilde{f}_{t,T}(Y_t, A)\right)^2\right] dt\ . \qquad (64)
\end{aligned}
$$

The mean and variance of Y_t are integrable,

$$
\begin{aligned}
\int_0^T \boldsymbol{E}[Y_t]dt &= \int_0^T \mu + (Y_0 - \mu)e^{-kt}dt = \mu T + \frac{(Y_0 - \mu)}{k}(1 - e^{-kT}) < +\infty\ , \\
\int_0^T \boldsymbol{V}[Y_t]dt &= \int_0^T \frac{\sigma^2}{2k}(1 - e^{-2kt})dt = \frac{\sigma^2}{2k}T - \frac{\sigma^2}{4k^2}(1 - e^{-2kT}) < +\infty\ .
\end{aligned}
$$

A repeated application of the inequality $(a+b)^2 \leq 2(a^2 + b^2)$ implies that to have a finite value of the information we are left with proving that $\boldsymbol{E}[(\tilde{f}_{t,T}(Y_t, A))^2]$ is integrable in $[0, T]$. By rewriting

$$
\boldsymbol{E}\left[\left(\tilde{f}_{t,T}(Y_t, A)\right)^2\right] = \boldsymbol{E}\left[\left(\tilde{f}_{t,T}(Y_t, 0)\right)^2 \boldsymbol{P}(A = 0|Y_t) + \left(\tilde{f}_{t,T}(Y_t, 1)\right)^2 \boldsymbol{P}(A = 1|Y_t)\right]
$$

and using the following definitions

$$
\begin{aligned}
u(z,t) &= \frac{\sigma^4 e^{-2k(T-t)}}{\sigma^2(T-t)}\left(\frac{1}{\bar{\Phi}(-z)} + \frac{1}{\bar{\Phi}(z)}\right)\left(\Phi'(-z)\right)^2 \qquad (65) \\
z(y) &= (\mu(T-t, y) - p)/\sigma(T-t) \qquad (66)
\end{aligned}
$$

we have

$$
\boldsymbol{E}\left[\left(\tilde{f}_{t,T}(Y_t, A)\right)^2\right] = \boldsymbol{E}[u(z(Y_t), t)] = \frac{1}{\sigma(t)}\int_{-\infty}^{\infty} u(z(y), t)\Phi'\left(\frac{y - \mu(t, Y_0)}{\sigma(t)}\right) dy\ .
$$

Applying the change of variable in (66) with

$$
\begin{aligned}
y(z) &= \mu + (p - \mu)e^{k(T-t)} + z\,\sigma(T-t)\,e^{k(T-t)} \\
a(z) &= \frac{y(z) - \mu(t, Y_0)}{\sigma(t)} = \frac{e^{k(T-t)}}{\sigma(t)}\left(z\,\sigma(T-t) + (p-\mu) - (Y_0 - \mu)e^{-kT}\right)
\end{aligned}
$$

we have

$$
\begin{aligned}
\boldsymbol{E}[u(z(Y_t), t)] &= \frac{\sigma^4 e^{-k(T-t)}}{\sigma(t)\sigma(T-t)}\int_{-\infty}^{\infty}\left(\frac{1}{\bar{\Phi}(z)} + \frac{1}{\bar{\Phi}(-z)}\right)\left(\Phi'(-z)\right)^2 \Phi'(a(z))\,dz \\
&\leq \frac{1}{\sqrt{2\pi}}\frac{\sigma^4 e^{-k(T-t)}}{\sigma(t)\sigma(T-t)}\int_{-\infty}^{\infty}\left(\frac{1}{\bar{\Phi}(z)} + \frac{1}{\bar{\Phi}(-z)}\right)\left(\Phi'(-z)\right)^2 dz \\
&= \frac{1}{\sqrt{2\pi}}\,\psi(t)\,I
\end{aligned}
$$

where we used that $\Phi'(z) \leq 1/\sqrt{2\pi}$ and we made the following definition:

$$
\begin{aligned}
\psi(t) &\coloneqq \frac{\sigma^4 e^{-k(T-t)}}{\sigma(t)\sigma(T-t)} \qquad (67) \\
I &\coloneqq \int_{-\infty}^{\infty}\left(\frac{1}{\bar{\Phi}(z)} + \frac{1}{\bar{\Phi}(-z)}\right)\left(\Phi'(-z)\right)^2 dz\ . \qquad (68)
\end{aligned}
$$

By Lemma 17 and Lemma 18 in the Appendix, I is bounded by a constant, and the function $\psi(t)$ is integrable in $[0, T]$. Therefore

$$
\int_0^T \boldsymbol{E}\left[\left(\tilde{f}_{t,T}(Y_t, A)\right)^2\right] dt = \int_0^T \boldsymbol{E}[u(z(Y_t), t]\,dt < \infty \qquad (69)
$$

and the value of the information is finite. □

7 Conclusions

In this paper, we showed how it is possible to include privileged information about the interest rate process in a portfolio, and how to determine the modified optimal strategy. If the information about the future is very precise, giving the value of the interest rate process at a terminal time T, we showed that an arbitrage opportunity of infinite value is created.

At first sight, it may sound counter intuitive the fact that an insider information on the future value of the interest rate has an infinite value. However to understand why this is true, it is important to remember that the driving processes for the interest rate and the stock asset are correlated. When the time approaches the terminal epoch T, the interest rate process behaves like a Gaussian bridge and therefore exhibits an unbounded drift that, by correlation, it is passed to the risky asset. It is by taking advantage of this drift explosion that the insider trader is able to generate infinite profits.

We also showed that the arbitrage opportunity may lead to an unbounded gain as well as an infinite loss depending on the accuracy of the correlation coefficient of the model with respect to the *true value*.

Finally we showed that if the privileged information is not very accurate, for example it gives only a lower bound for the terminal value of the interest rate process, then the value of such information is finite, and we give a methodology about how to compute it.

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A Appendix

Proof. Proof of Theorem 3 We need to plug into (15) of Lemma 1 the expression obtained from (16) in Lemma 2 and than to match the coefficients of the following Gaussian form,

$$f_{R_{t+\delta}|R_t,R_T}(b\,|\,R_t,R_T)=\frac{1}{\sqrt{2\pi\,\sigma^2}}\exp\left(-\frac{(b-\mu)^2}{2\sigma^2}\right)\ .$$

By combining the three factors in (15) and the one of the expression above it is easy to check that the only possible candidate for σ is given by the following expression

$$\sigma^2=\Psi^2_{t+\delta}\frac{\int_t^{t+\delta}(b_2\Psi^{-1})^2(x)dx\int_{t+\delta}^T(b_2\Psi^{-1})^2(x)dx}{\int_t^T(b_2\Psi^{-1})^2(x)dx}=\Psi^2_{t+\delta}\frac{\triangle_t^{t+\delta}\triangle_{t+\delta}^T}{\triangle_t^T}\ , \tag{70}$$

where in the second equality we used the notation $\triangle_u^s:=\int_u^s(b_2\Psi^{-1})^2(x)dx$. For the following computations it will be useful to introduce also the notation $\triangledown_u^s:=\int_u^s a_2\Psi^{-1}(x)dx$.

The candidate for μ has to satisfy the following equality

$$\frac{(b - R_t\,\Psi_{t,t+\delta} - \nabla_t^{t+\delta}\Psi_{t+\delta})^2}{\triangle_t^{t+\delta}\Psi_{t+\delta}^2} + \frac{(R_T - b\,\Psi_{t+\delta,T} - \nabla_{t+\delta}^{T}\Psi_T)^2}{\triangle_{t+\delta}^{T}\Psi_T^2} - \frac{(R_T - R_t\,\Psi_{t,T} - \nabla_t^T)^2\Psi_T}{\triangle_t^T\Psi_T^2} = \frac{(b-\mu)^2}{\sigma^2}$$

Multiplying the expression for σ^2 given in (70) to both members of the equality above we get

$$\frac{\triangle_{t+\delta}^T}{\triangle_t^T}\left(b - R_t\,\Psi_{t,t+\delta} - \nabla_t^{t+\delta}\Psi_{t+\delta}\right)^2 + \frac{\triangle_t^{t+\delta}}{\triangle_t^T}\frac{\Psi_{t+\delta}^2}{\Psi_T^2}\left(R_T - b\,\Psi_{t+\delta,T} - \nabla_{t+\delta}^T\Psi_T\right)^2 - \frac{\triangle_t^{t+\delta}\triangle_{t+\delta}^T}{(\triangle_t^T)^2}\frac{\Psi_{t+\delta}^2}{\Psi_T^2}\left(R_T - R_t\,\Psi_{t,T} - \nabla_t^T\Psi_T\right)^2 = (b-\mu)^2 \tag{71}$$

It is left to equate the terms in $b^2, -2b^1$ and b^0 of both sides of (71) are the same.

Quadratic Term:

$$\frac{\triangle_{t+\delta}^T}{\triangle_t^T} + \frac{\triangle_t^{t+\delta}}{\triangle_t^T}\frac{\Psi_{t+\delta}^2\Psi_{t+\delta,T}^2}{\Psi_T^2} = \frac{\triangle_{t+\delta}^T}{\triangle_t^T} + \frac{\triangle_t^{t+\delta}}{\triangle_t^T}\frac{\Psi_T^2}{\Psi_T^2} = \frac{\triangle_{t+\delta}^T + \triangle_t^{t+\delta}}{\triangle_t^T} = \frac{\triangle_t^T}{\triangle_t^T} = 1 \tag{72}$$

Linear Term: (divided by -2).

$$\frac{\triangle_{t+\delta}^T}{\triangle_t^T}[R_t\Psi_{t,t+\delta} + \nabla_t^{t+\delta}\Psi_{t+\delta}] + \frac{\triangle_t^{t+\delta}}{\triangle_t^T}\frac{\Psi_{t+\delta}}{\Psi_T}[R_T - \nabla_{t+\delta}^T\Psi_T] = \mu \tag{73}$$

that gives a candidate for μ. In the LHS of (73) we used that $\Psi_{t+\delta}\Psi_{t+\delta,T} = \Psi_T$.

Constant Term:
We are finally left with checking that the constant term in the LHS of (71), and shown below, is equal to μ^2 with μ given in (73).

$$\frac{\triangle_{t+\delta}^T}{\triangle_t^T}[R_t\Psi_{t,t+\delta} + \nabla_t^{t+\delta}\Psi_{t+\delta}]^2 + \frac{\triangle_t^{t+\delta}}{\triangle_t^T}\frac{\Psi_{t+\delta}^2}{\Psi_T^2}[R_T - \nabla_{t+\delta}^T\Psi_T]^2 - \frac{\triangle_t^{t+\delta}\triangle_{t+\delta}^T}{(\triangle_t^T)^2}\frac{\Psi_{t+\delta}^2}{\Psi_T^2}[R_T - R_t\Psi_{t,T} - \nabla_t^T\Psi_T]^2 \tag{74}$$

By writing $\nabla_t^T = \nabla_t^{t+\delta} + \nabla_{t+\delta}^T$, and $\Psi_T = \Psi_{t+\delta}\Psi_{t+\delta,T}$, we can expand last term in (74) in the following way

$$\begin{aligned}
[R_T - R_t\Psi_{t,T} - \nabla_t^T\Psi_T]^2 &= \left(R_T - \nabla_{t+\delta}^T\Psi_T - \Psi_{t+\delta,T}(R_t\Psi_{t,t+\delta} + \nabla_t^{t+\delta}\Psi_{t+\delta})\right)^2 \\
&= [R_T - \nabla_{t+\delta}^T\Psi_T]^2 - 2\Psi_{t+\delta,T}[R_T - \nabla_{t+\delta}^T\Psi_T][R_t\Psi_{t,t+\delta} + \nabla_t^{t+\delta}\Psi_{t+\delta}] \\
&\quad + \Psi_{t+\delta,T}^2[R_t\Psi_{t,t+\delta} + \nabla_t^{t+\delta}\Psi_{t+\delta}]^2
\end{aligned} \tag{75}$$

Finally we substitute (75) in (74) and compare the factors of $(R_t\Psi_{t,t+\delta} + \nabla_t^{t+\delta}\Psi_{t+\delta})^2$ in (74) and the square of (73)

$$\frac{\triangle_{t+\delta}^T}{\triangle_t^T} - \frac{\triangle_t^{t+\delta}\triangle_{t+\delta}^T}{(\triangle_t^T)^2}\frac{\Psi_{t+\delta}^2\Psi_{t+\delta,T}}{\Psi_T^2} = \frac{\triangle_{t+\delta}^T(\triangle_t^T - \triangle_t^{t+\delta})}{(\triangle_t^T)^2} = \left(\frac{\triangle_{t+\delta}^T}{\triangle_t^T}\right)^2 .$$

We do the same with the factors of $(R_T - \nabla_{t+\delta}^T\Psi_T)^2$,

$$\frac{\triangle_t^{t+\delta}}{\triangle_t^T}\frac{\Psi_{t+\delta}^2}{\Psi_T^2} - \frac{\triangle_t^{t+\delta}\triangle_{t+\delta}^T}{(\triangle_t^T)^2}\frac{\Psi_{t+\delta}^2}{\Psi_T^2} = \frac{(\triangle_t^{t+\delta})^2}{(\triangle_t^T)^2}\frac{\Psi_{t+\delta}^2}{\Psi_T^2} ,$$

and finally with the factors of the double product $2(R_t\Psi_{t,t+\delta} + \nabla_t^{t+\delta}\Psi_{t+\delta})(R_T - \nabla_{t+\delta}^T\Psi_T)$

$$\frac{\triangle_t^{t+\delta}\triangle_{t+\delta}^T}{(\triangle_t^T)^2}\frac{\Psi_{t+\delta}^2}{\Psi_T^2}\Psi_{t+\delta,T} = \frac{\triangle_t^{t+\delta}\triangle_{t+\delta}^T}{(\triangle_t^T)^2}\frac{\Psi_{t+\delta}}{\Psi_T} .$$

Since they all coincide, the expressions (73) and (70) are the right parameters given in (18a) and (18b).
□

Lemma 17.

$$\int_{-\infty}^{\infty}\left(\frac{1}{\bar{\Phi}(z)}+\frac{1}{\bar{\Phi}(-z)}\right)\left(\Phi'(-z)\right)^2\Phi'(a_t(z))dz<\infty \tag{76}$$

Proof. Using the definition of I in (68) we have

$$\begin{aligned} I \le \int_{-\infty}^{\infty}\left(\frac{1}{\bar{\Phi}(z)}+\frac{1}{\bar{\Phi}(-z)}\right)\left(\Phi'(-z)\right)^2 dz &= \int_{-\infty}^{\infty}\frac{\left(\Phi'(-z)\right)^2}{\bar{\Phi}(z)}dz+\int_{-\infty}^{\infty}\frac{\left(\Phi'(-z)\right)^2}{\bar{\Phi}(-z)}dz \\ &= 2\left(\int_{-\infty}^{0}+\int_{0}^{\infty}\right)\frac{\Phi'(z)}{\bar{\Phi}(z)}d\Phi(z) \end{aligned}$$

We have

$$\int_{-\infty}^{0}\frac{\Phi'(z)}{\bar{\Phi}(z)}d\Phi(z)\le 2\int_{-\infty}^{0}\Phi'(z)d\Phi(z)\le\sqrt{\frac{2}{\pi}}\int_{-\infty}^{0}d\Phi(z)=\sqrt{\frac{2}{\pi}}\Phi(0)=\frac{1}{\sqrt{2\pi}}\ .$$

As for the other term, since $1/\bar{\Phi}(z)=e^{z^2/2}O(z)$ and $\Phi'(z)=e^{-z^2/2}O(1)$ we have

$$\frac{\Phi'(z)}{\bar{\Phi}(z)}=O(z) \tag{77}$$

and the result follows because $|z|$ is integrable with respect to $d\Phi(z)$. □

Lemma 18.

$$\int_{0}^{T}\frac{\sigma^4e^{-k(T-t)}}{\sigma(t)\sigma(T-t)}\le\infty \tag{78}$$

Proof. Having

$$\begin{aligned} \sigma^2(t) &= \frac{\sigma^2}{2k}(1-e^{-2kt})=\frac{\sigma^2}{2k}\sinh(kt)e^{-kt} \\ \sigma^2(T-t) &= \frac{\sigma^2}{2k}(1-e^{-2k(T-t)})=\frac{\sigma^2}{2k}\sinh(k(T-t))e^{-k(T-t)} \end{aligned}$$

it follows that

$$\begin{aligned} 1/\sigma(t) &= O(1/\sqrt{t}) \\ 1/\sigma(T-t) &= O(1/\sqrt{T-t}) \end{aligned}$$

and the result follows. □