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Ph.D. Thesis/Tesis Doctoral

Iterated Integrals of Orthogonal Polynomials and Applications

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A mi familia

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Resumen y aportaciones

La presente tesis doctoral tiene por objeto el estudio de familias de polinomios que son soluciones del siguiente problema con valores iniciales

$$\begin{cases} \mathcal{L}[f(z)] &= \lambda\, g(z), \\ \frac{d^k f}{dz^k}(\omega_k) &= 0, \qquad \omega_k \in \mathbb{C}, \;\; k = 0, \dots, m \in \mathbb{Z}_+, \end{cases} \tag{1}$$

donde tanto f como g son polinomios y $\mathcal{L}$ en los capítulos 2 y 3 es el operador derivada m-ésima de f. La diferencia en los dos casos anteriores es que mientras en el capítulo 2 se considera que g es un polinomio ortogonal clásico sobre la recta real, en el capítulo 3 denotamos por g al polinomio ortogonal con respecto a una cierta medida soportada sobre un arco de la circunferencia unidad y ω_k es constante para cada $k = 0, \dots, m$. El capítulo 4, se dedica a las aplicaciones al procesamiento digital de imágenes, de las soluciones del problema (1) cuando $f = g$, $\lambda = \lambda_n = \frac{n}{\alpha - 1}$ ($n \in \mathbb{Z}_+$, $0 < \alpha < 1$) y $\mathcal{L}$ es el operador en diferencia

$$\mathcal{L}[f(z)] = 2z\triangle f(z) + \frac{\alpha N - z}{1 - \alpha}\triangle_+ f(z) - 2z\triangle_- f(z),$$

donde $N \in \mathbb{Z}_+$, $\triangle f(z) = 2^{-1}(f(z+1) - f(z-1))$, $\triangle_+ f(z) = f(z+1) - f(z)$ y $\triangle_- f(z) = f(z) - f(z-1)$. Como puede apreciarse más adelante, este último caso corresponde a los conocidos polinomios de Krawtchouk.

Acerca de la localización de los puntos críticos de polinomios en términos de sus ceros existe una teoría amplia (vea [72, Part I] y [81]), cuyas bases fundamentales son los teoremas de Rolle, Gauss-Lucas y sus refinamientos. Sin embargo, no existen recíprocos generales de estos resultados. Es obvio, que dado un cero de un polinomio y sus puntos críticos, los restantes ceros están unívocamente determinados. No obstante, solo existen unos pocos resultados sobre localización de ceros en función de sus puntos críticos y uno de sus ceros, la mayoría de los cuales se pueden ver en [72, §4.5]. En general, estos resultados son corolarios del Teorema de Composición de Schur-Szegő (vea [72, Th.

3.4.1d]. Quizás, los resultados más significativos en este sentido sean los Teoremas de Walsh [72, Th. 4.5.1] y Biernacki [72, Th. 4.5.2]. Hasta donde conocemos, sobre la localización de ceros de integrales iteradas de polinomios, normalizados con la condición de anularse en el origen, solo existe el trabajo [16]. El mencionado artículo estudia varios casos particulares de familias de polinomios, entre ellos los polinomios de Legendre, y plantea una serie de conjeturas, algunas de las cuales se responden en los Capítulos 2 y 3 de esta memoria.

El Capítulo 2 de esta memoria está dedicado a las integrales iteradas de polinomios ortogonales clásicos sobre la recta real, con énfasis en el caso Jacobi. Los trabajos [9, 10] muestran el interés de este tipo de polinomios para las aplicaciones a los métodos numéricos de elementos finitos. Es bien conocido que el polinomio mónico de Hermite H_{n+m} de grado $(n+m) \in \mathbb{Z}_+$, donde tanto n como m son enteros no negativos, cumple la relación $\frac{d^m}{dz^m}(H_{n+m}) = \frac{(n+m)!}{n!} H_n$. Luego, las integrales iteradas de orden m tiene todos sus ceros reales y los ceros de las integrales iteradas consecutivas se entrelazan.

Como se mencionó anteriormente, el tercer capítulo se dedica al estudio del comportamientos asintótico los polinomios obtenidos mediante la integración iterada de los polinomios ortogonales con respecto a medias soportadas en un arco de la circunferencia unidad y el conjunto de acumulación de sus ceros. Se encuentra el comportamiento asintótico relativo entre las familias de polinomios ortogonales y sus respectivas familias de polinomios obtenidos por integración iterada. Se muestra la representación gráfica de regiones cerradas que contienen los ceros de las nuevas familias de polinomios y de curvas donde se acumulan los mismos en varios casos particulares.

El tema central del Capítulo 4 es la implementación de un algoritmo eficiente para la detección de bordes de imágenes digitales basado en las propiedades de los polinomios ortogonales de Krawtchouk en dos variables. La primera parte del capítulo se dedica a estudiar las propiedades de esta familia de polinomios ortogonales en dos variables, que son de interés para el algoritmo propuesto. Las novedades de este algoritmo que fundamentan la calificación de eficiente son las siguientes:

- La aproximación de las diferencias parciales (derivadas parciales discretas) se realiza mediante una combinación lineal de polinomios de Krawtchouk en dos variables, los cuales son ortogonales con respecto a un producto interior discreto que involucra a la distribución binomial. En consecuencia, ya no es necesario suavizar la imagen mediante un filtro Gaussiano en dos dimensiones antes de realizar la diferenciación numérica, con el fin de regularizar la naturaleza mal condicionada de la diferenciación (ver [91]) y por lo tanto mejorar la localización de los bordes.

- En [11, 36] los autores describen un procedimiento para la detección de bordes utilizando los polinomios discretos de Chebyshev y un único umbral de discriminación de bordes para toda la imagen. Aquí, el algoritmo propuesto utiliza dos niveles de umbrales adaptativos, lo que reduce la presencia de falsos positivos o negativos en la selección de pixels-bordes.

- El operador gradiente para submatrices bloques de 5×5, en lugar del tradicional 3×3, proporciona una mejor localización de los pixels-bordes, ya que los bordes tienden a ser más gruesos cuando el tamaño del bloque incrementa [36, 69].

- Para evitar el efecto de bordes gruesos y mejorar el resultado final en el algoritmo se aplican operaciones morfológicas (estrechar, erosionar y adelgazar) a la imagen de borde obtenida después del segundo paso de procesamiento del algoritmo.

Para demostrar la efectividad del algoritmo propuesto se utilizaron imágenes tomadas de dos campos de aplicación muy diferentes: imágenes naturales utilizadas para la detección de objetos, vigilancia, etc; así como mapas de profundidad utilizados actualmente en aplicaciones y servicios multimedia de video 3D. Los contornos de objetos superpuestos, como la identificación de objetos de primer plano en mapas de profundidad, se obtienen con bastante buena precisión.

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Chapter

1 Introduction

1.1 Orthogonal Polynomials

Let μ be a positive Borel measure on the complex plane $\mathbb{C}$, with an infinite number of points in its support. We denote by $L^2(\mu)$ the space of square-integrable functions with respect to μ. It is well known that $L^2(\mu)$ is an Hilbert space with inner product and norm given by

$$\langle f, g\rangle_\mu = \int_{\mathrm{supp}(\mu)} f(z)\overline{g(z)}d\mu(z), \qquad \|f\|_\mu = \sqrt{\langle f, f\rangle_\mu}, \quad f, g \in L^2(\mu),$$

where the support of the measure μ is

$$\mathrm{supp}\,(\mu) = \{z \in \mathbb{C} : \mu(B_{z,\epsilon}) > 0 \text{ for every } \epsilon > 0\},$$

with $B_{z,\epsilon}$ the open disk with center z and radius ϵ.

We say that the functions $f, g \in L^2(\mu)$ are orthogonal if $\langle f, g\rangle_\mu = 0$.

Since the support of the measure μ is an infinite set, the functions $1, z, z^2, \ldots$ are linearly independent in $L^2(\mu)$, so we can use the Gram-Schmidt procedure to obtain unique polynomials

$$q_n(z) = q_n(\mu, z) = l_n z^n + \cdots, \quad l_n > 0, \; n = 0, 1, \ldots$$

that form an orthonormal system in $L^2(\mu)$, i.e.

$$\langle q_m, q_n\rangle_\mu = \begin{cases} 0 & \text{if} \quad m \neq n, \\ 1 & \text{if} \quad m = n. \end{cases} \tag{1.1}$$

The polynomials that satisfy (1.1) are called *orthonormal polynomials* with respect to μ, and l_n is the leading coefficient. In what follows $q_n(z)/l_n = z^n + \cdots$ is called the *monic* orthogonal polynomial.

When $d\mu(x) = w(x)dx$ on some interval, then we talk about orthogonal polynomials with respect to the weight function w.

The polynomials (1.1) can be generated using

$$\int \frac{q_n(z)}{l_n}\overline{z^k}d\mu(z) = 0, \quad k = 0, 1, \ldots, n-1,$$

which is an $n \times n$ system of equations for the coefficients of $q_n(z)/l_n$ with matrix $(\sigma_{i,j})_{i,j}^{n-1}$ where

$$\sigma_{i,j} = \int z^i\,\overline{z^j}d\mu(z),$$

are the complex moments of μ. This matrix is nonsingular, so the system has a unique solution, and finally l_n comes from normalization. The complex moments determine explicitly the polynomials by the determinant formulae:

$$q_n(z) = \frac{1}{\sqrt{D_{n-1}D_n}}\begin{vmatrix} \sigma_{0,0} & \sigma_{0,1} & \cdots & \sigma_{0,n-1} & 1 \\ \sigma_{1,0} & \sigma_{1,1} & \cdots & \sigma_{1,n-1} & z \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \sigma_{n-1,0} & \sigma_{n-1,1} & \cdots & \sigma_{n-1,n-1} & z^{n-1} \\ \sigma_{n,0} & \sigma_{n,1} & \cdots & \sigma_{n,n-1} & z^n \end{vmatrix}, \quad (1.2)$$

where $D_n = |\sigma_{i,j}|_{i,j}^n$ is the so called Gram determinants.

In the case that μ is supported on the real line then

$$\sigma_{i,j} = \int x^{i+j}d\mu(x) := h_{i+j},$$

so $D_n = |h_{i+j}|_{i,j}^n$ is a Hankel determinant, while if μ is supported on the unit circle then

$$\sigma_{i,j} = \int z^{i-j}d\mu(z) := t_{i-j}, \quad (1.3)$$

where $D_n = |t_{i-j}|_{i,j}^n$ is a Toeplitz determinant. In these two important cases the orthogonal polynomials have many special properties that are missing in the general theory.

1.1.1 General properties of zeros

One of the most important properties of orthogonal polynomials, is the location of the zeros. We denote by $C_o(A)$ the convex hull of the set $A \subset \mathbb{C}$, i.e., the smallest convex set containing A, and a set $G \subset \mathbb{C}$ is convex if for each pair of points $x, y \in G$ the line connecting x and y is a subset of G. Obviously

$$C_o(A) = \bigcap_{A \subset G \subset \mathbb{C}} G, \qquad G \text{ convex},$$

and if A is closed then $C_o(A)$ is the intersection of all closed half-planes containing A.

Theorem 1.1.1. *(Fejér) If q_n is the orthonormal polynomial of degree n with respect to the measure μ, then all its zeros lie in $C_o(\operatorname{supp}(\mu))$.*

Proof. See [88, Th. 2.1.1]. □

Theorem 1.1.2. *If $C_o(\operatorname{supp}(\mu))$ is not a line segment and $\operatorname{supp}(\mu)$ is an infinite set, then the zeros of the orthogonal polynomial for the measure μ can not be on the boundary of the convex hull of $\operatorname{supp}(\mu)$.*

Proof. See [79, Th. 2.2]. □

By the above theorem, the set $C_o(\operatorname{supp}(\mu))$ gives a good idea of where the zeros of the orthogonal polynomials are located. Furthermore, one might expect the zeros to lie on $\operatorname{supp}(\mu)$, but this is not true, since there may be zeros on the set $C_o(E) \setminus E$, where $E = \operatorname{supp}(\mu)$. Indeed, if the set E has "holes", the zeros may be located there. In fact, all the zeros may be in the holes, as in the case of the orthogonal polynomials on the unit circle.

If Ω_∞ denotes the connected component of the complement of E that contains ∞, then Ω_∞ is open and $P_C(E) = \mathbb{C} \setminus \Omega_\infty$ is the *polynomial convex hull* of E. Clearly $P_C(E) \subset C_o(E)$. For a set $A \subset \mathbb{C}^n$ the polynomial convex hull is the set of all $x \in \mathbb{C}^n$ for which $|f(x)| \leq \sup_{z \in A} |f(z)|$ for all polynomials f. If we restrict f to polynomials of degree 1, then gives the convex hull. For one complex variable the polynomial convex hull of $A \subset \mathbb{C}$ coincides with the complement of the outer component of A. The next theorem says that most of the zeros of orthogonal polynomials are in the polynomial convex hull of the support of μ.

Theorem 1.1.3. *(Widom) Suppose K is a compact set in Ω_∞. Then the number of zeros of q_n in K is bounded by a constant independent of n.*

Proof. See [96, Lemma 4]. □

As far as the general properties of zeros and asymptotic of orthogonal polynomials is concerned, we suggest the references [52, 58, 79, 88, 89, 93].

1.1.2 Orthogonal polynomials on the real line

Let μ a measure supported on the set of real numbers $\mathbb{R}$. The Gram-Schmidt method may be used to orthogonalize $1, x, \ldots, x^n, \ldots$, and omitting the step where the new orthogonal element is normalized, we arrive at a uniquely determined sequence of polynomials

$$p_0(x), p_1(x), \ldots, p_n(x), \ldots, \tag{1.4}$$

with the following properties:

i) each p_n is a monic polynomial of degree n with real coefficients,

ii) $\langle p_m, p_n\rangle = \displaystyle\int p_m(x)p_n(x)d\mu(x) = 0$, if $m \neq n$.

The sequence (1.4) is the so called monic orthogonal polynomial sequence with respect to the measure μ. If we denote the positive numbers $\langle p_n, p_n\rangle$ by $\tilde{\gamma}_n$, $\quad (n = 0, 1, \ldots)$, the corresponding orthonormal polynomials are given by

$$q_n(x) = \tilde{\gamma}_n^{-1/2} p_n(x) \quad (n \in \mathbb{Z}_+), \tag{1.5}$$

where $\mathbb{Z}_+$ is the set of nonnegative integer numbers. As in general case (1.1) they satisfy $\langle q_m, q_n\rangle = \delta_{m,n}$, where $\delta_{m,n}$ is Kronecker's delta. Clearly, the polynomials (1.4) constitute a basis for the linear space of all polynomials, and so do the polynomials (1.5). The most important property that satisfy the inner products with respect to measures supported in the real line, is that $\langle xp_m, p_n\rangle_\mu = \langle p_m, xp_n\rangle_\mu$, which has the consequence that the p_n's (and so q_n's) obey a three-term recurrence formula.

Proposition 1.1.1. *(Recurrence formula) Let μ a measure supported on $\mathbb{R}$. Then the associated sequence $\{p_k\}_{k\in\mathbb{Z}_+}$ of monic orthogonal polynomials satisfies the recurrence formula*

$$p_{n+1}(x) = (x - \tilde{\alpha}_{n+1})p_n(x) - \tilde{\beta}_n p_{n-1}(x) \quad (n = 0, 1, \ldots), \tag{1.6}$$

where $p_{-1}(x) \equiv 0$*,* $p_0(x) \equiv 1$*,* $\tilde{\beta}_0 = 1$ *and*

$$\tilde{\alpha}_n = \frac{1}{\tilde{\gamma}_{n-1}} \int x p_{n-1}^2(x) d\mu(x), \quad \tilde{\beta}_n = \frac{\tilde{\gamma}_n}{\tilde{\gamma}_{n-1}}, \quad (n = 1, 2, \ldots),$$

with $\tilde{\gamma}_k := \langle p_k, p_k \rangle$ *for* $k \in \mathbb{Z}_+$*.*

It is important to mention that the three term recurrence relation has a converse. Given a sequence $(\tilde{\alpha}_n)_{n\in\mathbb{N}}$ of real numbers and $(\tilde{\beta}_n)_{n\in\mathbb{Z}_+}$ a sequence of positive numbers, the recurrence formula (1.6) produces a sequence $(p_n)_{n\in\mathbb{N}}$ of monic polynomials. Then, by a result of Favard, there exits a measure μ so that $p_0(x), p_1(x), \ldots$ is the system of monic orthogonal polynomials associated with μ.

The three term recurrence implies for the so called reproducing kernel the Christoffel-Darboux formula

$$\sum_{k=0}^{n} \frac{1}{\tilde{\gamma}_k} p_k(x) p_k(t) = \frac{1}{\tilde{\gamma}_k} \begin{cases} \dfrac{p_{n+1}(x)p_n(t) - p_n(x)p_{n+1}(t)}{x-t} & \text{if } x \neq t, \\ p'_{n+1}(x)p_n(x) - p_{n+1}(x)p'_n(x) & \text{if } x = t. \end{cases}$$

In the real case the zeros of p_n are real and simple and the zeros of p_n and p_{n+1} interlace, i.e., in between any zeros of p_{n+1} there is a zero of p_n. In fact, p_n must have n sign changes, for if it had only $m < n$, say at the points $y_1, \ldots, y_m \in \mathbb{R}$, then the polynomial $r(x) = \prod_{j=1}^{m}(x - y_j)$ of degree $m < n$, for then $r(x)p_n(x)$ would be of constant sign. Let now $x_1 < x_2 < \cdots < x_{n-1} < x_n$ be the zeros of p_n, and suppose that we already know that the zeros p_n and p_{n-1} interlace, which implies $sign(p_{n-1}(x_k)) = (-1)^{k-1}$. If we substitute x_k into the recurrence formula (1.6) then we have that $p_{n+1}(x_k)$ and $p_{n-1}(x_k)$ are opposite signs at x_k, i.e., $sign(p_{n+1}(x_k)) = (-1)^k$, and this gives that the zeros of p_n and p_{n+1} also interlace. Thus, the interlacing property follows by induction.

1.1.2.1 Classical Orthogonal Polynomials on the real line

The classical orthogonal polynomials are composed of the three families:

- the *Jacobi* polynomials $P_n^{(\alpha,\beta)}$, with $\alpha, \beta > -1$, are orthogonal with respect to the measure $d\mu^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta dx$ on $[-1, 1]$, and are defined by

$$P_n^{(\alpha,\beta)}(x) := \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta} \frac{d^n}{dx^n}\left((1-x)^{n+\alpha}(1+x)^{n+\beta}\right),$$

with the normalization

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}.$$

The most important specials cases are the *Legendre* polynomials ($\alpha = \beta = 0$), the *Ultraspherical* or *Gegenbauer* polynomials ($\alpha = \beta$), and the *Chebyshev* polynomials of the first and second kind, which are obtained when $\alpha = \beta = -1/2$ and $\alpha = \beta = 1/2$ respectively.

- the *Laguerre* polynomials $L^{(\gamma)}$, with $\gamma > -1$, are orthogonal with respect to $d\mu^{(\gamma)}(x) = x^\gamma e^{-x} dx$ on $[0, \infty)$, and are defined by

$$L_n^{(\gamma)}(x) := \frac{1}{n!} e^x x^{-\gamma} \frac{d^n}{dx^n} \left(x^{n+\gamma} e^{-x} \right),$$

with the normalization

$$L_n^{(\gamma)}(0) = \binom{n+\gamma}{n}.$$

- the *Hermite* polynomials H_n are orthogonal with respect to the measure $d\mu_\hbar(x) = e^{-x^2} dx$ on the real line $(-\infty, \infty)$, and are defined by

$$H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

These families of orthogonal polynomials are very special, for they posses many properties that no other orthogonal polynomial system does. In particular:

a) They satisfy a second order differential equation of the form

$$\sigma(x) y''(x) + \tau(x) y'(x) + \lambda_n y(x) = 0,$$

where σ is a polynomial of degree at most two and τ a polynomial of degree one, both independent of n. In order to obtain a solution y which is a polynomial of degree n, then comparing the leading coefficients shows that $\lambda_n = -n(n-1)\sigma''/2 - n\tau'$. For these specific values of λ_n, the polynomial solution y_n will consists of orthogonal polynomials.

b) They have derivatives which form again an orthonormal polynomial system.

c) All classical orthogonal polynomial satisfy a Rodrigues type formula

$$y_n(x) = \frac{1}{C_n w(x)} \frac{d^n}{dx^n} \{w(x)\sigma(x)^n\},$$

where C_n is independent of x and w is non-negative on a certain real interval, σ is a polynomial of degree at most two, independent of n and $w'/w = [\tau - \sigma']/\sigma$ is a rational function, with τ a polynomial of degree at most one. With these conditions one obtains polynomials y_n of degree n which are orthogonal.

Every one of these properties has a converse, namely if a system of orthogonal polynomials possesses any of these properties, then it is (up to a change of variables) one of the classical systems.

The Legendre polynomials were introduced in 1785 by the French mathematician A.M. Legendre in [47]. In the paper [20] of 1854 "*Théorie des mécanismes connus sous le nom de parallèlogrammes*", the Russian mathematician P.L. Chebyshev, who may be considered as the father of approximation theory, found the Chebyshev polynomials as solutions of extremal problems. The Jacobi polynomials were introduced by C.G.J. Jacobi [43] in 1859. The special case of Laguerre polynomials ($\alpha = 0$) appears in earlier works of N.H. Abel, J.L. Lagrange and Chebyshev in [21] and finally in the work [46] of E.N. Laguerre. The general case ($\alpha > -1$) is due to N.J. Sonin [87]. The Hermite polynomials were considered by P.S. Laplace in his most important work *"Traité de Mécanique Céleste"*, published in five volumes in the period $(1799 - 1825)$, then they were studied by Chebyshev in [21] and by C. Hermite in [37].

1.1.3 Discrete orthogonal polynomials

Let $N \in \mathbb{N}$, $\Lambda := \{x_0, x_1, \ldots, x_N\} \subset \mathbb{R}$, where $x_0 < x_1 < \ldots < x_N$, $\mathbf{F}(\Lambda)$ be the set of all real functions on Λ, $\mathbb{P}$ be the set of all real coefficient polynomials and $\mathbb{P}_N \subset \mathbb{P}$ be the set of polynomials of degree at most N. Note that any real function of a discrete variable $f \in \mathbf{F}(\Lambda)$ can be seen as the restriction on Λ of a number of functions of real variable, in particular the Lagrange interpolation polynomial $P \in \mathbb{R}_N$ such that $P(x_i) = f(x_i)$ for $i = 0, 1, \ldots, N$. Then we have a natural identification between the sets $\mathbf{F}(\Lambda)$ and $\mathbb{P}_N$.

We call *weight function* (or simply *weight*) any positive function μ on Λ and we say that it is *normalized* when $\sum_{k=0}^{N} \mu(x_k) = 1$.

Let us consider the pair (Λ, μ), where μ is a weight defined on Λ. The inner product on $\mathbb{P}_N$ associated to (Λ, μ) is defined by:

$$\langle f, g\rangle_{\Lambda,\mu} = \sum_{k=0}^{N} f(x_k)g(x_k)\mu(x_k), \quad f, g \in \mathbb{P}_N. \tag{1.7}$$

with a corresponding norm $\|f\|_{\Lambda,\mu} = \sqrt{\langle f, f\rangle_{\Lambda,\mu}}$.

A family of polynomials $\{p_{N,k}\}_{k=0}^{m}$ with $m \leq N$ is *orthogonal* with respect to the inner product (1.7) if p_k is a polynomial of degree k with positive leading coefficient and

$$\langle p_{N,n}, p_{N,m}\rangle_{\Lambda,\mu} = \begin{cases} \neq 0 & \text{if} \quad n = m, \\ = 0 & \text{if} \quad n \neq m. \end{cases} \tag{1.8}$$

If $\|p_{N,k}\|_{\Lambda,\mu} = 1$ for all $0 \leq k \leq N$, the family $\{p_{N,k}(x)\}_{k=0}^{m}$ is called *orthonormal* with respect to (4.1). Finally if for all $0 \leq k \leq N$ the leading coefficient of $p_{N,k}(x)$ is equal to one, then $\{p_{N,k}(x)\}_{k=0}^{m}$ is called a family of *monic* orthogonal polynomials with respect to (4.1). Analogously to the "continuous" case, the polynomials $p_{N,k}(z)$ may be built from the monomials $1, z, z^2, \ldots, z^{N-1}$ using the Gram-Schmidt process. The general elementary properties of the discrete orthogonal polynomials are:

- Each discrete polynomial $p_k(z)$ has k simple zeros in the interval $[x_0, x_N]$.
- No more than one zero lies in the closed interval $[x_n, x_{n+1}]$ between any two consecutive nodes.
- The discrete orthogonal polynomials satisfy a three term recurrence relation.

1.1.3.1 Classical discrete orthogonal polynomials on the real line

If f is a function on the set $\mathbf{F}(\Lambda)$, the forward and backward difference operators are defined by

$$\triangle_+ f(x) := f(x+1) - f(x), \qquad \triangle_- f(x) := f(x) - f(x-1), \tag{1.9}$$

with $\triangle_+^n f(x) = \triangle_+ \left[\triangle_+^{n-1} f(x)\right]$ for $n = 1, 2, \ldots$. These operators are the analogous of the derivative operator in the continuous case. The classical discrete orthogonal polynomials are composed by the four families:

- The *Hahn polynomials* $p_n^{(\alpha,\beta)}(x,N)$, with α, $\beta > -1$, are orthogonal on the interval $[0, N-1]$ are defined by the following Rodrigues type formula

$$p_n^{(\alpha,\beta)}(x,N) = \frac{(-1)^n}{n!\mu(x)} \triangle_+^n \left[\mu(x) \prod_{k=0}^{n-1} \sigma(x-k) \right],$$

  where $\sigma(x) = x(N+\alpha+x)$,

$$\mu(x) = \frac{\Gamma(N+\alpha-x)\Gamma(1+\beta+x)}{\Gamma(1+x)\Gamma(N-x)},$$

  and as usual $\Gamma(\cdot)$ denote the Gamma function. The special case of the Hahn polynomials, called the *Chebyshev polynomials of a discrete variable*, denoted by $t_n(x) = p_n^{(0,0)}(x,N)$ arises when $\mu(x) \equiv 1$.

- The *Charlier* polynomials $C_n^{(a)}$, with parameter $a > 0$, are orthogonal on the interval $[0,\infty)$ with respect to the weight

$$\mu(x) = \frac{e^{-a}a^x}{\Gamma(1+x)},$$

  and are defined by the following Rodrigues type formula

$$C_n^{(a)}(x) = \frac{1}{a^n \mu(x)} \triangle_+^n \left[\mu(x) \prod_{k=0}^{n-1} \sigma(x-k) \right],$$

  with $\sigma(x) = x$.

- The *Meixner* polynomials $M_n(\cdot) = M_n(\cdot; b, c)$, with parameters $b > 0$ and $0 < c < 1$, are orthogonal on the interval $[0,\infty)$ with respect to the weight

$$\mu(x) = \frac{c^x\Gamma(b+x)}{\Gamma(1+x)\Gamma(b)},$$

  and are defined by the following Rodrigues type formula

$$M_n(x) = \frac{1}{c^n \mu(x)} \triangle_+^n \left[\mu(x) \prod_{k=0}^{n-1} \sigma(x-k) \right],$$

  with $\sigma(x) = x$.

- Let $N \in \mathbb{N}$, $\Lambda_N = \{0, 1, 2, \ldots, N\}$, $\alpha \in]0, 1[$ and $w_{N,\alpha}(x)$ the weight function
$$w_{N,\alpha}(x) = \binom{N}{x} \alpha^x (1-\alpha)^{N-x}, \quad \text{for all } x \in \Lambda_N.$$
The monic *Krawtchouk* polynomials can be generated by the formula (c.f. [5, (5.4.3)])
$$\kappa_n^{\alpha}(x, N) = \sum_{j=0}^{n} \binom{n}{j} \alpha^{n-j} (1-\alpha)^j (x-N)_{n-j} (x-j+1)_j,$$
where $(a)_j$ denotes the Pochhammer symbol or shifted factorial as in [5, (1.1.8)].

The *Hahn* polynomials were introduced in 1949 by Wolfgang Hahn in [35]. The *Charlier* polynomials also known as "Poisson-Charlier" polynomials, were investigated by C.V.L. Charlier in [19]. The *Meixner* polynomials were considered by Joseph Meixner in [57]. The *Krawtchouk* polynomials [44] were introduced in 1929 by M. Krawtchouk. These are the orthogonal polynomials associated with the binomial distribution in probability theory.

1.2 An application of discrete orthogonal polynomials: Edge detection

Digital image processing is a discipline in mathematics and electrical engineering which is included within the more general field of *signal processing*, which deals with the analysis and processing of analog and digital signals, and with storing, filtering and many others operations on signals. There is a considerable variety of possible signals, among which it is worth mentioning sound and voice signals, transmission signals and image signals. Then as the name suggests, digital image processing is the set of techniques and procedures (algorithms) for the transformation of digital images.

An image is a two-dimensional signal or function $f(x, y)$, where x and y are the spatial coordinates, and the amplitude of f at any pair of coordinates (x, y) is called intensity of the image at that level. Digital images $\mathcal{I}(x, y)$ are obtained from "continuous" images $f(x, y)$, by means of a sampling procedure. Then, digital images are two-dimensional arrays which are composed by a finite numbers of points, each of them called **pixel** or **pel** (from "*pictured element*").

1.2.0.1 Image type data

In a two-dimensional (2D) image the *spatial resolution*, also called *digital resolution*, is the number of pixels used to cover the visual space captured by the image, and it is denoted by $C \times R$ (column by row), for example: images of resolution 640×480, 1024×768 etc. Among the most used formats of representation of 2D images are:

1. **Binary images** In this type of images each pixel only take the values zero (for black color), or the value one (for white color), though any two color can be used.

2. **Grey-scale images** Let L_g the number of discrete gray levels allowed in each pixel. Then due to processing, storage, and sampling considerations, we have [34, Ch. 2] that the number L_g is an integer power of 2: $L_g = 2^k$, and we usually refer to image as a "k-bit image". It is commonly assumed that the discrete gray levels are equally spaced and that they are integers in the interval $[0, L_g - 1]$. Then in the gray-scaled images, the values of pixels goes from 0 (black color) to $L_g - 1$ (white color). One of the most used gray-scale images is when $k = 8$ (or 8-bit image), which provides of $L_g = 2^8 = 256$ levels of luminance per pixel.

3. **RGB images** This type of images are represented by three dimensional arrays that assign three numerical values to each pixel, each value corresponding to the so called primary colors red (R), green (G) and blue (B), which can be combined in various proportions to obtain any color in the visible spectrum.

1.2.1 Edge detection

Edge detection can be considered as part of the most general image processing discipline called *Segmentation*, which roughly speaking consist in the separation or subdivision of the image into regions or objects to represent meaningful areas. The main goal of image segmentation is obtain a representation of the image which is easier to analyze. There is a great variety of applications to image segmentation, for example: *Machine vision*, *Medical imaging* in order to locate tumors, measure tissue volumes, surgery planning; *Object detection* (face detection, brake light detection, locate objects in satellite images, such as, roads, crops, forest); *Recognition tasks* such as face, fingerprint and iris recognition;

Traffic control systems, etc. There are several possible approaches to image segmentation:

i) *Edge/boundary* These methods consist in finding discontinuities or abrupt changes in intensity on the image in order to identify boundaries between regions.

ii) *Region-based* This approach is based on partitioning an image into regions that are similar according to a set of predefined criteria.

Basically there are three types of gray level discontinuities on digital images: points, lines and edges. An edge is a set of connected pixels that lie on the boundary between two regions. *Edge detection* can be defined as the set of procedures to find edge points on an image. In the same manner that the first derivative is used in one variable functions to detect critical points, Edge detection procedures makes use of discrete differential operators to detect changes in the gradients of the gray (or color) levels image. Edge detection is divided into two major categories: *first order* and *second order* Edge detection. In this work we will mainly deal with first order methods. It is well known that in the study of such methods, an efficient approach to the first discrete derivative is needed. For this purpose, it is necessary to introduce the notion of mask or kernels and discrete convolution.

Gradient

The gradient is a measure of change in a function, and a digital image $\mathcal{I}(x, y)$ can be considered to be an array of samples of some continuous function $f(x, y)$ of image intensity. Analogously to the one variable case, significant changes in the gray level values in a image can be detected by using a discrete approximation to the gradient, which is defined by the vector

$$\nabla f(x, y) = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right].$$

As is known, the gradient represent the direction of the greatest rate of increase of the function $f(x, y)$. Taking into account that the partial derivatives of $f(x, y)$ are dependent on the direction, i.e., is not isotropic (invariant), but the magnitude $|\nabla f(x, y)|$ of the gradient vector is, then a common approximation into image processing field is as follows

$$|\nabla f(x,y)| = \sqrt{\left[\frac{\partial f}{\partial x}\right]^2 + \left[\frac{\partial f}{\partial y}\right]^2} \approx \left|\frac{\partial f}{\partial x}\right| + \left|\frac{\partial f}{\partial y}\right|.$$

The first order partial discrete derivatives or partial differences of the image function $f(x,y)$ can be computed [78, Ch. 6, Sect. 6.3.1] by

$$\triangle_x f(x,y) = f(x+1,y) - f(x,y), \ \triangle_y f(x,y) = f(x,y+1) - f(x,y).$$

In order to obtain an efficient method to compute the discrete derivative in each point (pixel) of the image $\mathcal{I}(x,y)$, one of the most used procedures is the *Spatial linear filtering*, which can be explained as follows: If the image $\mathcal{I}(x,y)$ is represented by the 6×6 matrix

$$\mathcal{I} = \begin{pmatrix} * & * & * & * & * & * \\ * & I(x-1,y-1) & I(x-1,y) & I(x-1,y+1) & * & * \\ * & I(x,y-1) & I(x,y) & I(x,y+1) & * & * \\ * & I(x+1,y-1) & I(x+1,y) & I(x+1,y+1) & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{pmatrix}, \tag{1.10}$$

and

$$\mathcal{M} = \begin{pmatrix} m(-1,-1) & m(-1,0) & m(-1,1) \\ m(0,-1) & m(0,0) & m(0,1) \\ m(1,-1) & m(1,0) & m(1,1) \end{pmatrix}, \tag{1.11}$$

denotes the *filter* or *mask* (also called: template, kernel or window). The 3×3 section of the matrix image (1.10) centered at (x,y) (also called neighborhood of (x,y)) is denoted by $\mathcal{I}_{3\times 3}$. The response of applying $\mathcal{M}$ to the matrix section $\mathcal{I}_{3\times 3}$, is given by linear combination

$$\begin{aligned} R = {} & m(-1,-1)I(x-1,y-1) + m(-1,0)I(x-1,y) \\ & + m(-1,1)I(x-1,y+1) + m(0,-1)I(x,y-1) \\ & + m(0,0)I(x,y) + m(0,1)I(x,y+1) + m(1,-1)I(x+1,y-1) \\ & + m(1,0)I(x+1,y) + m(1,1)I(x+1,y+1). \end{aligned} \tag{1.12}$$

The response (1.12) of the mask at the point $I(x,y)$, can also be written in the form $R = \langle \mathrm{vec}\,(\mathcal{M}), \mathrm{vec}\,(\mathcal{I}_{3\times 3})\rangle_2$ where $\langle \cdot, \cdot \rangle_2$ denotes the usual Euclidean inner product, and $\mathrm{vec}\,(\mathcal{M})$ is the column vector

$$(m(-1,-1), m(0,-1), m(1,-1), m(-1,0), m(0,0), m(1,0), m(-1,1),$$
$$m(0,1), m(1,1))^t\,,$$

and similarly for $\operatorname{vec}(\mathcal{I}_{3\times3})$. For example if we applied the mask

$$\mathcal{M}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{1.13}$$

to the section $\mathcal{I}_{3\times3}$, the result is precisely the partial discrete derivative with respect to x at the point (x,y):

$$\triangle_x \mathcal{I}_{3\times3}(x,y) = I(x+1,y) - I(x,y).$$

Usually masks are matrices of order $n \times m$, where $n = 2k_1 + 1$ and $m = 2k_2 + 1$, and k_1, k_2 are nonnegative integers, i.e., the filter mask has an odd numbers of rows and columns. This will ensure that the effect of filtering is over the center pixel (x,y) of the section matrix $\mathcal{I}_{n\times m}$. In general the *Spatial linear filtering* of an image $\mathcal{I}$ of size $M \times N$ with a filter mask of size $n \times m$ at the point (x,y) is given by

$$\mathcal{I}_{\mathcal{M}}(x,y) = \sum_{s=-k_1}^{k_1} \sum_{t=-k_2}^{k_2} m(s,t)\mathcal{I}(x+s,y+t). \tag{1.14}$$

In order to process all the pixels of an image, it is necessary to apply (1.14) varying $x = 0,1,\ldots,M-1$, $y = 0,1,\ldots,N-1$, thus obtaining the transformed image $\mathcal{I}_{\mathcal{M}}$, which can be written in compact form as follows

$$\mathcal{I}_{\mathcal{M}} = \mathcal{M} * \mathcal{I}, \tag{1.15}$$

where the symbol $*$ denotes the discrete convolution of the mask $\mathcal{M}$ with the image $\mathcal{I}$, see [69, Ch. 15], [34, Ch. 3 and 10], and [38, Ch. 15 Sect. 15.1.4].

1.2.2 Classical first-order methods of Edge detection

The most used mask of size 3×3 for approximate first order the discrete derivatives are:

Roberts This method is one of the earliest procedures for detecting edges, and was proposed in 1965 by L.G. Roberts [77]. The masks are given by:

$$\mathcal{M}_x = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{M}_y = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Prewitt The Prewitt operator was developed in 1970 by J.M.S. Prewitt in [71]. The filter mask are:

$$\mathcal{M}_x = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix}, \qquad \mathcal{M}_y = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}.$$

Sobel In his PhD dissertation [84] (see also [23, p. 271]), I.E. Sobel proposed the following masks for approximating first order derivatives of images:

$$\mathcal{M}_x = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{pmatrix}, \qquad \mathcal{M}_y = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{pmatrix}.$$

The edge detector using gradient magnitude has the following basic steps:

- Compute gradient vector at each pixel by convolving the image with horizontal and vertical derivative filters.
- Compute the gradient magnitude at each pixel.
- If the magnitude at a pixel exceeds a threshold, report a possible edge point.

The edge set obtained by an edge detector can be divided into three subsets: *correct edges or true edges*, corresponding to edges in the scene, *false edges* (also called false positive) which do not correspond to edges in the scene, and the *missing edges* (or false negative edges) defined as those edges in the scene that should have been detected.

Previous to the above steps for edge detection (using the magnitude of gradient), smoothing filters are commonly used to reduce noise, thus reducing the probability of obtaining false positive edges. Among the most widely used smoothing filter are: 1) *average (or lowpass) filters*, which consist in replacing the value of every pixel in a image by the average of the gray levels in the neighborhood defined by the filter mask. 2) *The median filter* and 3) Gaussian filters, see [69, Ch. 7] and [34, Ch. 3 and 4]. Another reason for applying smoothing filters, is to regularize the ill-posed nature of differentiation and therefore improve edge localization, as pointed out by Torre and Poggio in [91]. Other methods for edge detection were put forth proposed by, Hueckel [39] and Frei and Chen [26] in the 1970s. Many statistical [98] and several filtering approaches have also been used for edge detection. Algorithms based on the Laplacian of Gaussian [55] and based on the gradient of Gaussian [18] were very popular in the 1980s. Haralick [36] presented an edge detection scheme based on the second directional derivative. His procedure incorporated a form of image smoothing based on approximating the image with local surface patches.

One of the most used and complete method for edge detection since 1980s was proposed by J.F. Canny in [18]. This briefly consists in the following: 1) The image is first smoothed using Gaussian kernel. 2) Find the edge strength taking gradient of the image with the Sobel operator. 3) Calculate the edge direction. 4) Digitize the edge direction. 5) Non-maximum suppression. 6) Hysteresis.

1.2.3 Edge detection using two variables Krawtchouk polynomials

A gray-scale image with resolution $(N_1+1)\times(N_2+1)$ pixels ($N_1, N_2 \in \mathbb{N}$) can be considered as a function of two variables $\mathbf{I}(x,y)$ defined on the set $\Lambda_{N_1} \times \Lambda_{N_2}$, where $\Lambda_{N_1} = \{0,1,\ldots,N_1\}$ and $\Lambda_{N_2} = \{0,1,\ldots,N_2\}$, i.e.

$$\begin{array}{rccc} \mathbf{I}: & \Lambda_{N_1} \times \Lambda_{N_2} & \longrightarrow & [0,1], \\ & (x,y) & \longrightarrow & \mathbf{I}(x,y). \end{array}$$

Hence, the values of $\mathbf{I}$ on $\Lambda_{N_1} \times \Lambda_{N_2}$ can be represented by a matrix $\mathcal{I}$ of order $(N_1+1)\times(N_2+1)$. Let $\mathbb{P}_{N_1,N_2}$ be the linear space of polynomials in the variables x and y, of degree at most N_1 and N_2 respectively. To study an image as a polynomial in two variables, we use the Krawtchouk polynomials in two variables or bivariate Krawtchouk polynomials [75, Ch. 12], [25, Ch. 2].

Let $N_1, N_2 \in \mathbb{N}$, $\alpha_1, \alpha_2 \in]0,1[$, $\Lambda_{N_1} = \{0,\ldots,N_1\}$ and $\Lambda_{N_2} = \{0,\ldots,N_2\}$. We call *Two-dimensional Krawtchouk polynomials or 2D monic Krawtchouk polynomials* the polynomial of two variables

$$\mathbf{K}_{n,m}^{\alpha_1,\alpha_2}(x,y) = \kappa_n^{\alpha_1}(x,N_1)\,\kappa_m^{\alpha_2}(y,N_2),$$

where $(x,y) \in \Lambda_{N_1} \times \Lambda_{N_2}$.

The 2D monic Krawtchouk polynomials are orthogonal with respect to the next inner product on $\mathbb{P}_{N_1,N_2}$

$$\langle \mathbf{f}, \mathbf{g} \rangle_{2D} = \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} \mathbf{f}(x_i,x_j)\mathbf{g}(x_i,x_j) w_{N_1,\alpha_1}(x_i)\, w_{N_2,\alpha_2}(x_j),$$

(see [75, Lemma 12-1]). Furthermore

$$\begin{aligned}\langle \mathbf{K}_{n,m}^{\alpha_1,\alpha_2}, \mathbf{K}_{r,s}^{\alpha_1,\alpha_2}\rangle_{2D} &= \langle \kappa_n^{\alpha_1}, \kappa_r^{\alpha_1}\rangle_{N_1,\alpha_1} \langle \kappa_m^{\alpha_2}, \kappa_s^{\alpha_2}\rangle_{N_2,\alpha_2} \\ &= \begin{cases} 0, & |n-r|+|m-s|>0, \\ \|\mathbf{K}_{n,m}^{\alpha_1,\alpha_2}\|_{2D}^2 > 0, & |n-r|+|m-s|=0, \end{cases}\end{aligned}$$

where $\|\mathbf{f}\|_{2D} = \sqrt{\langle \mathbf{f}, \mathbf{f}\rangle_{2D}}$.

From the standard theory of approximation of functions (c.f. [75, Ch. 12]), for $M_1 \in \Lambda_{N_1} \setminus \{0\}$ and $M_2 \in \Lambda_{N_2} \setminus \{0\}$, the polynomial of total degree $(M_1-1) \times (M_2-1)$

$$\mathbf{P}_{M_1,M_2}(x,y) = \sum_{n=0}^{M_1-1} \sum_{m=0}^{M_2-1} \beta_{n,m} \mathbf{K}_{n,m}^{\alpha_1,\alpha_2}(x,y),$$

where $\beta_{n,m} = \dfrac{\langle \mathbf{I}, \mathbf{K}_{n,m}^{\alpha_1,\alpha_2}\rangle_{2D}}{\langle \mathbf{K}_{n,m}^{\alpha_1,\alpha_2}, \mathbf{K}_{n,m}^{\alpha_1,\alpha_2}\rangle_{2D}}$ is such that

$$\min_{\mathbf{Q}\in\mathbb{P}_{M_1,M_2}} \|\mathbf{I}-\mathbf{Q}\|_{2D} = \|\mathbf{I}-\mathbf{P}_{M_1,M_2}\|_{2D},$$

i.e. $\mathbf{P}_{M_1,M_2}$ is the polynomial of least square approximation of $\mathbf{I}$ in $\mathbb{P}_{M_1,M_2}$ and we write $\mathbf{I}(x,y) \approx \mathbf{P}_{M_1,M_2}(x,y)$. Furthermore, if $M_1 = N_1+1$ and $M_2 = N_2+1$, then $\mathbf{I} = \mathbf{P}_{N_1,N_2}$.

In chapter 4 we will propose an alternative method for edge detection on gray-scale images, that extends beyond classic first-order differential operators by using the differential properties of two variable Krawtchouk orthogonal polynomials, to obtain two matrices $\mathcal{P}_x$ and $\mathcal{P}_y$ of the same order of the original image $\mathcal{I}$, where each entry (i,j) is the partial derivative with respect to x or y, and hence achieve an approximation of the modulus of gradient on each point (i,j) of $\mathcal{I}$ which is the basis of the proposed algorithm.

1.3 Non-standard Orthogonal Polynomials

Let $\mathbb{P}$ the vector space of all polynomials with complex coefficients, $\mathbb{P}_n$ the vector subspace of all polynomials of degree at most n. We denote by $T_x : \mathbb{P} \to \mathbb{P}$ the multiplication operator defined as

$$\forall p \in \mathbb{P}, \qquad T_x(p) = xp. \tag{1.16}$$

Definition 1.3.1. *Let $\langle \cdot, \cdot \rangle : \mathbb{P}\times\mathbb{P} \to \mathbb{C}$ an inner product over $\mathbb{P}$. We will say that $\{p_n\}_{n=0}^{\infty}$ is a standard sequence of monic orthogonal polynomials with respect to $\langle \cdot, \cdot \rangle$ if*

$$\forall p, q \in \mathbb{P}, \qquad \langle T_x(p), q\rangle = \langle p, T_x(q)\rangle \,.$$

The monic orthogonal polynomial sequence with respect to the measure μ (1.4) is an example of standard sequence of monic orthogonal polynomials. Also, the discrete inner product (1.8) produces a set $\{p_{N,k}(x)\}_{k=0}^{m}$ of standard monic orthogonal polynomials in the discrete setting.

1.3.1 Orthogonal polynomials on the unit circle

Let μ be a nontrivial probability measure on the unit circle $\mathbb{T} = \{z : |z| = 1\}$, and suppose that $d\mu(\theta) = w(\theta)d\theta$ is the Radon-Nikodym derivative of μ with respect to the Lebesgue measure. Then

$$\int_{\mathbb{T}} f(z)d\mu(z) = \int_0^{2\pi} f(e^{i\,\theta})w(\theta)d\theta.$$

According to (1.3) the elements of Gram matrix (in this case called Toeplitz matrix) are given by

$$\langle z^n, z^m\rangle = \int_0^{2\pi} e^{-i(m-n)}w(\theta)d\theta := t_{m-n},$$

with explicit expression

$$T_n = \begin{pmatrix} t_0 & t_1 & \cdots & t_n \\ t_{-1} & t_0 & \cdots & t_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ t_{-n} & t_{-n+1} & \cdots & t_0 \end{pmatrix}.$$

If φ_n denotes the orthonormal polynomial of degree n with respect to μ, by (1.2) we have

$$\varphi_n(z) = \frac{1}{\sqrt{\widetilde{D}_{n-1}\widetilde{D}_n}} \begin{vmatrix} t_0 & t_1 & \cdots & t_n \\ t_{-1} & t_0 & \cdots & t_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ t_{-n+1} & t_{-n+2} & \cdots & t_{-1} \\ 1 & z & \cdots & z^n \end{vmatrix} = \kappa_n z^n + \cdots\,, \quad (1.17)$$

where $\widetilde{D}_n$ is the determinant of T_n and $\kappa_n = \sqrt{\frac{\widetilde{D}_{n-1}}{\widetilde{D}_n}}$. The polynomials (1.17) are also known as Szegő polynomials, and can be written in the form $\varphi_n(z) = \frac{\Phi_n(z)}{\|\Phi_n(z)\|}$, where $\Phi_n(z)$ is the corresponding monic polynomial and the leading coefficients are given by $\kappa_n = \|\Phi_n\|^{-1}$. If $p_n(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n, the *reversed polynomial* is defined by

$$p_n^*(z) = \sum_{j=0}^{n} \bar{a}_j z^{n-j} = z^n \overline{p_n(1/\bar{z})}. \tag{1.18}$$

The polynomials Φ_n and Φ_n^* satisfy the following orthogonality conditions

$$\left\langle \Phi_n, z^k \right\rangle = 0, \quad 0 \le k \le n-1,$$
$$\left\langle \Phi_n^*, z^k \right\rangle = 0, \quad 1 \le k \le n.$$

As a consequence of Fejér Theorem 1.1.1 we have the following

Theorem 1.3.1. *([82, Part 1, Ch. 1, Sect 1.7]) For each n, all the zeros of φ_n lie in the unit disc $\mathbb{D} = \{z : |z| < 1\}$.*

Theorem 1.3.2. *([82, Part 1, Ch. 1]) If $K_n(z,\xi) = \sum_{k=0}^{n} \varphi_k(z)\overline{\varphi_k(\xi)}$ is the reproducing kernel for the orthogonal polynomials φ_n on the unit circle, then*

$$K_n(z,0) = \sum_{k=0}^{n} \varphi_k(z)\overline{\varphi_k(0)} = \kappa_n \varphi_n^*(z), \tag{1.19}$$

and

$$K_n(0,0) = \sum_{k=0}^{n} |\varphi_k(0)|^2 = \kappa_n^2. \tag{1.20}$$

From (1.3) it follows that $\{\Phi_n\}_{n=0}^{\infty}$ is an non standard orthogonal polynomials sequence, for this reason these polynomials do not satisfy a three term recurrence relation, nonetheless they satisfy the following

Theorem 1.3.3. *(Szegő recurrence, [82, Part 1, Ch. 1]) For the monic orthogonal polynomials Φ_n one has*

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n \Phi_n^*(z), \tag{1.21}$$
$$\Phi_{n+1}^*(z) = \Phi_n^*(z) - \alpha_n z \Phi_n(z). \tag{1.22}$$

Moreover,

$$\begin{aligned}\|\Phi_{n+1}\|^2 &= \left(1-|\alpha_n|^2\right)\|\Phi_n\|^2 \\ &= \prod_{j=0}^{n}\left(1-|\alpha_j|^2\right). \end{aligned} \tag{1.23}$$

Using (1.21) and $\Phi_{n+1}^*(0)=1$, we have

$$\alpha_n = -\overline{\Phi_{n+1}(0)}. \tag{1.24}$$

The parameters α_n are called *Verblunsky coefficients* and satisfy $|\alpha_j|<1$, i.e., are all in $\mathbb{D}$. We define

$$\rho_j := \sqrt{1-|\alpha_j|^2},$$

so (1.23) and $\kappa_n = \|\Phi_n\|^{-1}$ imply

$$\kappa_n = \prod_{j=0}^{n-1}\rho^{-1} = \prod_{j=0}^{n-1}\left(1-|\alpha_j|^2\right)^{-1/2}.$$

Since $\varphi_n = \kappa_n\Phi_n$ and $\kappa_{n+1}\rho_n = \kappa_n$, the monic Szegő recursions (1.21) and (1.22) can be written as follows

$$\begin{aligned}\varphi_{n+1}(z) &= \rho_n^{-1}\left(z\varphi_n(z) - \bar{\alpha}_n\varphi_n^*(z)\right), \\ \varphi_{n+1}^*(z) &= \rho_n^{-1}\left(\varphi_n^*(z) - \alpha_n z\varphi_n(z)\right).\end{aligned}$$

Also, the monic recurrence formulas (1.21) and (1.22) can be written in matrix form as follows

$$\begin{pmatrix}\Phi_{n+1}\\ \Phi_{n+1}^*\end{pmatrix} = \begin{pmatrix} z & -\bar{\alpha}_n \\ -\alpha z & 1\end{pmatrix}\begin{pmatrix}\Phi_n \\ \Phi_n^*\end{pmatrix},$$

and analogously, for the orthonormal recurrence we have

$$\begin{pmatrix}\varphi_{n+1}\\ \varphi_{n+1}^*\end{pmatrix} = \rho_n^{-1}\begin{pmatrix} z & -\bar{\alpha}_n \\ -\alpha z & 1\end{pmatrix}\begin{pmatrix}\varphi_n \\ \varphi_n^*\end{pmatrix}.$$

As a consequence of recurrence formulas (1.21) and (1.22) we have the following expression for the reproducing kernel, similar to the Christoffel-Darboux formula on the real line.

Theorem 1.3.4. *([82, Part 1, Ch. 1]) If φ_n are the orthonormal polynomials on the unit circle, then*

$$K_n(z,\xi) = \sum_{k=0}^{n-1} \varphi_k(z)\overline{\varphi_k(\xi)} = \frac{\varphi_n^*(z)\overline{\varphi_k^*(\xi)} - \varphi_n(z)\overline{\varphi_k(\xi)}}{1 - z\xi}.$$

The most simple example of orthogonal polynomial on the unit circle "OPUC" is obtained taking the weight function $w(\theta) = 1$, which produces $\Phi_n(z) = z^n$ and $\Phi_n^*(z) = 1$, with Verblunsky coefficients $\alpha_n \equiv 0$. This example illustrate that unlike the orthogonal polynomials on the real line, the zeros of OPUC may not be simple, indeed in this case the unique zero is $z = 0$. Other examples of OPUC can be found on [82, §1.6]. Regarding the classical references of OPUC, we suggest the monographs [27, 31, 89], and the more recent [82].

1.3.2 Orthogonality with respect to a Differential Operator

Definition 1.3.2. *Assume that μ is a positive Borel measure on the real line and let $\{\rho_k\}_{k=0}^M$ be a set of functions such that,*

$$\int |x^j \rho_k(x)| d\mu(x) < \infty, \quad 0 \le j < \infty,$$

for all $k = 0, \dots, M$. Denote by

$$\mathcal{L}^{(M)} = \sum_{k=0}^{M} \rho_k(x) \frac{d^k}{dx^k}, \tag{1.25}$$

an operator acting over the space of polynomials $\mathbb{P}$. We say that $\{Q_n\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials with respect to the pair $(\mathcal{L}^{(M)}, \mu)$ if deg$[Q_n] \le n$ and

$$\int \mathcal{L}^{(M)}[Q_n](x) P(x) d\mu(x) = 0, \tag{1.26}$$

for any polynomial P such that deg$[P] \le n-1$.

1. In the case $M = 0$, i.e. $\mathcal{L}^{(M)}[f](x) = \rho_0(x) f(x)$, we obtain the classical construction of orthogonal polynomials with respect to a standard inner product

$$\int Q_n(x) P(x) \rho_0(x) d\mu(x) = 0, \quad deg[P] \le n-1.$$

2. Let $\zeta \in \mathbb{C}$ be fixed and consider the differential operator $\mathcal{L}_\zeta : W^{1,2}(\mu) \to L^2(\mu)$

$$\mathcal{L}_\zeta[f(x)] = f(x) + (x-\zeta)f'(x),$$

where $W^{1,2}(\mu) = \{f \in L^2(\mu) : f' \in L^2(\mu)\}$ is the Sobolev space of index 1. Let us consider a positive measure μ supported on a subset $\Delta \subset \mathbb{R}$. The polar polynomial associated to μ, see [65], is defined as the polynomial Q_n of degree n orthogonal with respect to $(\mathcal{L}_\zeta, \mu)$. Let us consider

$$\begin{aligned} P_{0,\zeta} &= 1 \\ P_{n+1,\zeta}(z) &= (z-\zeta)Q_n(z), \quad n \geq 0. \end{aligned}$$

Then it is not difficult to see that the family $\{P_{n+1,\zeta}\}_{n=0}^{\infty}$ is orthogonal with respect to the Sobolev inner product

$$\langle f, g\rangle_\zeta = \eta f(\zeta)g(\zeta) + \int_\Delta f'(x)g'(x)d\mu(x),$$

for some $\eta > 0$. The case $\mu = \mu_\lambda,\ \lambda > -\frac{1}{2}$, corresponding to the Gegenbauer or ultraspherical measure, i.e. $d\mu_\lambda(x) = (1-x^2)^{\lambda-\frac{1}{2}}dx$, was studied in detail in [6].

1.3.3 Properties of uniqueness for the sequence of orthogonal polynomials with respect to differential operators in general.

Taking into account that the sequence of orthogonal polynomials with respect to differential operators is not unique, in [3] the authors find that the notions of $T-$ system results to be a sufficient condition for normality of the sequence for linear homogeneous differential operators.

Definition 1.3.3. *A set $\{u_k\}_{k=0}^n$ of continuous functions on Δ is called a Chebyshev system ($T-$ system) on Δ if any linear combination*

$$\sum_{k=0}^{n} \alpha_k u_k,$$

has at most n zeros on this interval. If for each $0 \leq m \leq n$, the set of functions $\{u_k\}_{k=0}^m$ forms a $T-$ system it is called a Markov system ($M-$ system).

A sufficient condition for uniqueness is that any polynomial satisfying (1.26) has exactly degree n. Sufficient conditions for the question of uniqueness of the sequence of orthogonal polynomials with respect to linear homogeneous differential operators in general are given by the next three theorems.

Theorem 1.3.5. *Given $\mathcal{L}^{(M)}$ as in* (1.25)*, let us assume that the sequence $\{\mathcal{L}^{(M)}[x^\nu]\}_{\nu=0}^{n}$ is an $M-$system on* $\operatorname{supp}(\mu)$*. Then $deg[Q_n] = n$.*

Basically the above theorem allows to establish a correspondence between $T-$systems and fundamental systems of solutions of linear differential equations because any fundamental system $(u_0, \ldots, u_{M-1})$ of $\mathcal{L}^{(M)}[u] = 0$ satisfies $W(u_0, \ldots, u_{M-1}) \neq 0$. Therefore, any such solution is a $T-$system.

The next theorem gives a condition of normality in terms of a fundamental system of solutions:

Theorem 1.3.6. *Let $\{u_0, \ldots, u_{M-1}\}$ be a fundamental system of solutions of $\mathcal{L}^{(M)}[u] = 0$. Let us assume that $n \in \mathbb{N}$ is given and that $\{(u_0^{(\nu)}, \ldots, u_{M-1}^{(\nu)})\}$, is a $T-$ system for $\nu = 1, 2, \ldots, n+1$. If Q_n is the nth orthogonal polynomial with respect to $(\mathcal{L}^{(M)}, \mu)$, then* $\deg[Q_n] = n$.

The condition for normality provided by Theorem 1.3.6 gives,

Theorem 1.3.7. *Assume that $\mathcal{L}^{(M)}$ has infinitely differentiable coefficients $\{\rho_k\}_{k=0}^{M}$ on* $\operatorname{supp}(\mu)$*. Define recurrently the system of functions $\{\rho_{k,m}\}_{k=0}^{M}$, $m = 1, 2, \ldots,$ as $\{\rho_{k,0} := \rho_k\}_{k=0}^{M}$, and*

$$
\begin{aligned}
\rho_{k,m+1} &= \rho_{k,m} + \rho_{0,m}\left(\frac{\rho_{k+1,m}}{\rho_{k+1,m}}\right) \\
k &= 0, \ldots, M-1, \quad m \in \mathbb{N}, \\
\rho_{M,m} &\equiv 1, \quad m \in \mathbb{N}.
\end{aligned}
$$

Then $deg[Q_n] = n$ if for all $m = 1, 2, \ldots, n$ we have $\rho_{0,m}(x) \neq 0$ for $x \in$ $\operatorname{supp}(\mu)$*.*

Using the previous results, the authors of [3] proved, for some cases of differential operators, the normality of the associated sequence of orthogonal polynomials.

Chapter

2 Iterated integrals of Jacobi Polynomials

2.1 Introduction

There is extensive literature about the location of the critical points of a polynomial in terms of its zeros ([72, Part I] and [81]), whose main pillars are Rolle's Theorem, Gauss-Lucas Theorem and their refinements. However, proper converses of these theorems have yet to be found. It is obvious that given one of the zeros of a polynomial and its critical points, the remaining zeros are uniquely determined. Nonetheless, there are only some results about zero location of polynomials in terms of its critical points and a given zero, most of them contained in [72, §4.5]. In general, these follow from the Schur-Szegő composition theorem [72, Th. 3.4.1d]. Perhaps, the most relevant results in this sense are the theorems of Walsh [72, Th. 4.5.1] and Biernacki [72, Th. 4.5.2].

Let $P_n^{(\alpha,\beta)}$ be the nth monic Jacobi polynomials with parameters $\alpha, \beta \in \mathbb{R}$

$$P_n^{(\alpha,\beta)}(z) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\binom{n+\beta}{k}\binom{2n+\alpha+\beta}{n}^{-1} \cdot (z-1)^k(z+1)^{n-k}, \tag{2.1}$$

where $2n+\alpha+\beta \neq 0, 1, \ldots, n-1$, $\binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(a-b+1)\Gamma(b+1)}$ and $\Gamma(\cdot)$ is the usual Gamma function (see [89, (4.21.6) and (4.3.2)] for more details). These classical polynomials have been used extensively in mathematical analysis and practical applications (cf. [73, 89, 90]). Nowadays, there has been renewed interest in using the Jacobi polynomials in the numerical solution of differential equations. Some of these methods require explicit expressions of the integral of such polynomials and the localization of their zeros (e.g. see [9, 10]). Another area that demand this knowledge is the study of families of polynomials orthogonal in a non-standard sense, particulary the Sobolev-type orthogonality and the orthogonality with respect to a differential operator (e.g. [6, 14, 65]).

For a fixed $m \in \mathbb{Z}_+$, let $\mathcal{P}_{n,m}^{(\alpha,\beta)}$ be the monic polynomial of degree $n+m$ given by $\mathcal{P}_{n,m}^{(\alpha,\beta)} = P_{n+m}^{(\alpha-m,\beta-m)}$. From [89, (4.21.6)-(4.21.7)] it is known that

$$\frac{d^k\, P_n^{(\alpha,\beta)}}{dz^k}(z) = \frac{n!}{(n-k)!}\, P_{n-k}^{(\alpha+k,\beta+k)}(z), \quad \text{where } 0 \le k \le n. \tag{2.2}$$

Hence, $\frac{d^m\, \mathcal{P}_{n,m}^{(\alpha,\beta)}}{dz^m}(z) = \frac{(n+m)!}{n!}\, P_n^{(\alpha,\beta)}(z)$, i.e. $\mathcal{P}_{n,m}^{(\alpha,\beta)}$ is an mth iterated integral (or a primitive of order m) of $\frac{n!}{(n+m)!}\, P_n^{(\alpha,\beta)}$. In what follows, we shall refer to $\mathcal{P}_{n,m}^{(\alpha,\beta)}$ as the *mth fundamental iterated integral of* $\frac{n!}{(n+m)!}\, P_n^{(\alpha,\beta)}$.

Given m complex numbers $\omega_1, \dots, \omega_m \in \mathbb{C}$, let $\mathbf{\Omega_k} = (\omega_1, \dots, \omega_k)$ for $1 \le k \le m$, and $\mathcal{P}_{n,m,\mathbf{\Omega_m}}^{(\alpha,\beta)}$ be the mth iterated integral of $\frac{(n+m)!}{n!}\, P_n^{(\alpha,\beta)}$ normalized by the conditions

$$\frac{d^k\, \mathcal{P}_{n,m,\mathbf{\Omega_m}}^{(\alpha,\beta)}}{dz^k}(\omega_{m-k}) = 0, \quad k = 0, 1, \dots, m-1. \tag{2.3}$$

Furthermore, there exists a unique polynomial

$$\mathcal{A}_{n,m}(z) = \mathcal{A}_{n,m}(z; \omega_1, \dots, \omega_m)$$

of degree at most equal to $m-1$, satisfying the equations

$$\frac{d^k\, \mathcal{A}_{n,m}}{dz^k}(\omega_{m-k}) = \frac{d^k\, \mathcal{P}_{n,m}^{(\alpha,\beta)}}{dz^k}(\omega_{m-k}), \quad k = 0, 1, \dots, m-1. \tag{2.4}$$

The polynomial $\mathcal{A}_{n,m}$ is named the *Abel-Goncharov interpolation polynomial*, associated to the conditions (2.4). The existence and uniqueness of $\mathcal{A}_{n,m}$ is obvious if we observe that (2.4) is a triangular system of m equations and m unknown (the coefficients of $\mathcal{A}_{n,m}$) whose determinant is equal to $\prod_{k=0}^{m-1} k!$. The Abel-Goncharov interpolation polynomial is a generalization of Taylor's polynomial, which corresponds to the case $\omega_m = \omega_{m-1} = \cdots = \omega_1$. In section 2.3, explicit expressions of Abel-Goncharov polynomials and some of their properties are given, for more details see [2, 24, 90].

Now, the polynomial $\mathcal{P}_{n,m,\mathbf{\Omega_m}}^{(\alpha,\beta)}$ can be written as

$$\mathcal{P}_{n,m,\mathbf{\Omega_m}}^{(\alpha,\beta)}(z) = \mathcal{P}_{n,m}^{(\alpha,\beta)}(z) - \mathcal{A}_{n,m}(z), \tag{2.5}$$

and we can interpret the polynomial $\mathcal{P}_{n,m,\mathbf{\Omega_m}}^{(\alpha,\beta)}$ as the polynomial solution of the next Abel-Goncharov boundary value problem (see [2, §3.5])

$$\begin{cases} \frac{d^m Y}{dz^m}(z) &= \frac{(n+m)!}{n!} P_n^{(\alpha,\beta)}(z), \quad n > m, \\ \frac{d^k Y}{dz^k}(\omega_{m-k}) &= 0, \qquad\qquad\qquad k = 0, 1, \ldots, m-1. \end{cases}$$

Moreover, if $\alpha, \beta > -1$ then $\mathcal{P}_{n,m,\mathbf{\Omega_m}}^{(\alpha,\beta)}$ is the $(n+m)$th monic orthogonal polynomial with respect to the discrete-continuous Sobolev inner product (see [1, 6]) given by

$$\begin{aligned} \langle f, g \rangle_S = & \sum_{k=0}^{m-1} \frac{d^k f}{dz^k}(\omega_{m-k}) \frac{d^k g}{dz^k}(\omega_{m-k}) \\ & + \int_{-1}^{1} \frac{d^m f}{dz^m}(x) \frac{d^m g}{dz^m}(x)(1-x)^\alpha (1+x)^\beta dx. \end{aligned}$$

Let $m \in \mathbb{Z}_+$, $\{\omega_0, \omega_1, \ldots, \omega_m\} \subset \mathbb{C} \setminus [-1, 1]$ and $\alpha, \beta > -1$, the aim of this work is the study of algebraic and asymptotic properties of the sequence of monic polynomials $\left\{\mathcal{P}_{n,m,\mathbf{\Omega_m}}^{(\alpha,\beta)}\right\}_n$. The case $\alpha = \beta = \omega_1 = \cdots = \omega_m = 0$ was early studied in [16], where the authors wrote *"It would be interesting to obtain results, analogous to Theorem [16, Th. 2], for these polynomials"* referring to the Gegenbauer (or ultraspherical) polynomials ($\alpha = \beta > -1$). Our Theorem 2.4.2 is an extension of [16, Th. 2] for Jacobi polynomials when all the constants of integration ω_i are outside of the interval $[-1, 1]$.

In the next section we review some of the standard facts on Jacobi polynomials and we give the proof of some auxiliary results. The third section is devoted to study the Abel-Goncharov interpolation polynomial $\mathcal{A}_{n,m}(z)$ of the mth fundamental iterated integral of Jacobi polynomials. In the section 2.4 our main results on asymptotic behavior of the sequence of polynomials $\left\{\mathcal{P}_{n,m,\mathbf{\Omega_m}}^{(\alpha,\beta)}\right\}_n$ and its zeros, are stated and proved. In the last section, we studied the results analogous to section 2.2, for the Laguerre polynomials.

2.2 Fundamental iterated integral of Jacobi polynomials

Recall that, for a fixed $m, n \in \mathbb{Z}_+$, we denote by $\mathcal{P}_{n,m}^{(\alpha,\beta)}$ the Jacobi monic polynomial of degree $n+m$ given by $P_{n+m}^{(\alpha-m,\beta-m)}$. From [73, §135 (12) and §138 (14)-(15)] we have the next lemma.

Lemma 2.2.1. *For a fixed $m \in \mathbb{Z}_+$, let $\mathcal{P}_{n,m}^{(\alpha,\beta)}$ be the $(n+m)$th fundamental primitive of nth monic Jacobi polynomials with parameters $\alpha, \beta \in \mathbb{R}$, as defined on the introduction of the chapter. Then*

$$\mathcal{P}_{n,m}^{(\alpha,\beta)}(z) = \mathcal{P}_{n+1,m-1}^{(\alpha,\beta)}(z) + a_{n,m}^{(\alpha,\beta)}\,\mathcal{P}_{n,m-1}^{(\alpha,\beta)}(z) + b_{n,m}^{(\alpha,\beta)}\,\mathcal{P}_{n-1,m-1}^{(\alpha,\beta)}(z); \tag{2.6}$$

$$\begin{aligned} \textit{where} \quad a_{n,m}^{(\alpha,\beta)} &= \frac{2(n+m)(\alpha-\beta)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta)}, \\ b_{n,m}^{(\alpha,\beta)} &= \frac{-4(n+m)(n+m-1)(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)^2((2n+\alpha+\beta)^2-1)} \quad \textit{and} \\ \mathcal{P}_{n,0}^{(\alpha,\beta)}(z) &= P_n^{(\alpha,\beta)}(z). \end{aligned}$$

The asymptotic behavior of the sequence of polynomials $\{\mathcal{P}_{n,m}^{(\alpha,\beta)}\}_n$, stated in the following lemma is a direct consequence of [89, Th. 8.21.7 & Eqn. (4.21.6)]).

Lemma 2.2.2. *If $\alpha, \beta \in \mathbb{R}$ and $m \in \mathbb{Z}_+$, then*

1. *(Outer strong asymptotic). Uniformly on compact subsets of $\overline{\mathbb{C}} \setminus [-1,1]$*

$$\lim_{n\to\infty} \frac{\mathcal{P}_{n,m}^{(\alpha,\beta)}(z)}{\varphi^n(z)} = \psi_{\alpha,\beta,m}(z)\,\sqrt{\varphi(z)}, \qquad \textit{where} \tag{2.7}$$

$$\varphi(z) = \frac{1}{2}\left(z + \sqrt{z^2-1}\right)$$

$$\textit{with } \sqrt{z^2-1} > 0 \ \textit{ when } z > 1 \ \textit{ and}$$

$$\psi_{\alpha,\beta,m}(z) = \frac{2^{2m-\alpha-\beta}\,\left(\sqrt{z-1}+\sqrt{z+1}\right)^{\alpha+\beta-2m}}{\sqrt[4]{(z-1)^{2(\alpha-m)+1}}\,\sqrt[4]{(z+1)^{2(\beta-m)+1}}}.$$

2. *(nth root asymptotic behavior). Uniformly on compact subsets of $\overline{\mathbb{C}} \setminus [-1,1]$*

$$\lim_{n\to\infty} \left|\mathcal{P}_{n,m}^{(\alpha,\beta)}(z)\right|^{\frac{1}{n}} = |\varphi(z)|\,. \tag{2.8}$$

3. *(Comparative asymptotic behavior). Uniformly on compact subsets of $\overline{\mathbb{C}} \setminus [-1,1]$*

$$\lim_{n\to\infty} \frac{\mathcal{P}_{n,m}^{(\alpha,\beta)}(z)}{P_n^{(\alpha,\beta)}(z)} = \left(\frac{1}{\varphi'(z)}\right)^m. \tag{2.9}$$

The two lemmas listed below are deduced from the Rouché's Theorem (cf. [70, Th. 1.1.1]) and the Biernacki's Theorem (cf. [72, Th. 4.5.2]), respectively.

Lemma 2.2.3. (Rouché's Theorem, [70, Th. 1.1.1]) *Let f and g be polynomials, and γ a closed curve without self-intersections in the complex plane. If $|f(z)| < |g(z)|$ for all $z \in \gamma$, then the polynomials $f + g$ and g have the same number of zeros in the interior of γ.*

Lemma 2.2.4. *Let f be a polynomial whose critical points are on a compact subset $K \subset \mathbb{C}$. If there exists $\zeta \in \mathbb{C}$ such that $f(\zeta) = 0$, then the zeros of f lie in the compact set $[K]_\zeta = \{z \in \mathbb{C} : \inf_{w\in K} |z - w| \leq \mathbf{d}_{K_\zeta}\}$, where $\mathbf{d}_{K_\zeta}$ is the diameter of the compact set $K_\zeta = K \cup \{\zeta\}$ (i.e. $\mathbf{d}_{K_\zeta} = \sup_{u,v\in K_\zeta} |u - v|$).*

Of course, for all $\zeta \in \mathbb{C}$ we get $K \subset K_\zeta \subset [K]_\zeta$.

In the classical Szegő's book [89, §6.72], the reader can find a full description of the distribution of the zeros of $P_{n+m}^{(\alpha-m,\beta-m)}$, i.e. $\mathcal{P}_{n,m}^{(\alpha,\beta)}$, when $\alpha, \beta \in \mathbb{R}$ and $n, m \in \mathbb{Z}_+$ are fixed. Additionally, in the next theorem we state some aspect of interest about their asymptotic behavior.

Theorem 2.2.1. *Let $\alpha, \beta > -1$, $m \in \mathbb{N}$ fixed, $I = (-1, 1)$ and $\mathbf{Z}_{n,m}^{(\alpha,\beta)}(A)$ be the set of zeros of $\mathcal{P}_{n,m}^{(\alpha,\beta)}$ on the set $A \subset \mathbb{C}$. Then*

1. *For each $n > 2m$, at least $(n - 2m)$ distinct zeros of $\mathcal{P}_{n,m}^{(\alpha,\beta)}$ lie in I.*

2. *There exists a compact subset $K \subset \mathbb{C}$, such that $(-1, 1) \subset K$ and*

$$\bigcup_{n\geq 1} \mathbf{Z}_{k,m}^{(\alpha,\beta)}(\mathbb{C}) \subset K.$$

3. *All the roots of $\mathcal{P}_{n,m}^{(\alpha,\beta)}$ accumulate at $[-1, 1]$, i.e.*

$$\bigcap_{n\geq 1} \bigcup_{k\geq n} \mathbf{Z}_{k,m}^{(\alpha,\beta)}(\mathbb{C}) = [-1, 1].$$

Proof.

(1) From (2.6) for consecutive values of m, we get that there exist $(2m+1)$ constants a_0, $a_1, \ldots,$ a_{2m} such that

$$\mathcal{P}_{n,m}^{(\alpha,\beta)}(z) = \sum_{k=0}^{2m} a_k \, P_{n-m+k}^{(\alpha,\beta)}(z).$$

Hence $\mathcal{P}_{n,m}^{(\alpha,\beta)}$ is a quasi-orthogonal polynomial of order $2m$ with respect to the measure $(1-x)^{\alpha}(1+x)^{\beta}dx$ on I. Hence from [17, Th. 2] we have the first assertion of the theorem.

(2) If $m = 1$, all the critical points of $\mathcal{P}_{n,1}^{(\alpha,\beta)}$ lie in $(-1,1)$ and by the first sentence of the theorem at least $n-2$ of its zeros are on $I = [-1,1]$. Let $x_0 \in I$ such that $\mathcal{P}_{n,1}^{(\alpha,\beta)}(x_0) = 0$, then according to the notations in Lemma 2.2.4 we get that $I_{x_0} = I$ and $\mathbf{d}_{I_{x_0}} = 2$. Hence, from Lemma 2.2.4 we get $\left(\bigcup_{n\geq 1} \mathbf{Z}_{k,1}^{(\alpha,\beta)}(\mathbb{C})\right) \subset [I]_{\mathbf{x_0}}$.

Suppose that for a fixed $m \in \mathbb{N}$, there exists a compact set $K_m^{(\alpha,\beta)}$ such that $\left(\bigcup_{n\geq 1} \mathbf{Z}_{k,m}^{(\alpha,\beta)}(\mathbb{C})\right) \subset K_m^{(\alpha,\beta)}$. As the zeros of $\mathcal{P}_{n,m}^{(\alpha,\beta)}$ are the critical points of $\mathcal{P}_{n,m+1}^{(\alpha,\beta)}$, from Theorem 2.2.1-(1) and Lemma 2.2.4 we get the desired statement.

(3) For a fixed $m \in \mathbb{N}$, from the Theorem 2.2.1-(2) we known that the set of all zeros of $\{\mathcal{P}_{n,m}^{(\alpha,\beta)}\}$ are uniformly bounded.

Note that for all $n \in \mathbb{Z}_+$ the functions $\frac{\mathcal{P}_{n,m}^{(\alpha,\beta)}(z)}{P_n^{(\alpha,\beta)}(z)}$ and $\left(\frac{1}{\varphi'(z)}\right)^m = \left(\frac{\sqrt{z^2-1}}{\varphi(z)}\right)^m$ are analytic on $\overline{\mathbb{C}} \setminus [-1,1]$. Furthermore, $\left(\frac{\sqrt{z^2-1}}{\varphi(z)}\right)^m \neq 0$ if $z \in \overline{\mathbb{C}} \setminus [-1,1]$, hence (3) is a consequence of (2.9).

□

2.3 The Abel-Goncharov interpolation polynomial

Given m complex numbers $\omega_1, \ldots, \omega_m \in \mathbb{C}$, let $\mathbf{\Omega_k}$ for $1 \leq k \leq m$, as in (2.3), as it is shown in section 2.1, there exists a unique polynomial $\mathcal{A}_{n,m}$ of degree at most $m-1$, such that the equations (2.4) are satisfied. The polynomial $\mathcal{A}_{n,m}$ is the mth Abel-Goncharov polynomial of interpolation relative to the conditions (2.4) and is given by the expression

$$\mathcal{A}_{n,m}(z) = \mathcal{P}_{n,m}^{(\alpha,\beta)}(\omega_m) + \sum_{k=1}^{m-1} \frac{1}{k!} \frac{d^k \mathcal{P}_{n,m}^{(\alpha,\beta)}}{dz^k}(\omega_{m-k})\, \mathcal{G}_{k,m}(z) \tag{2.10}$$

where $\mathcal{G}_{k,m}(z) = \mathcal{G}_{k,m}(z; \omega_m, \omega_{m-1}, \ldots, \omega_{m-k})$ is the monic polynomial of degree k , generate by the kth iterated integral

$$\mathcal{G}_{k,m}(z) = k! \int_{\omega_m}^{z} \int_{\omega_{m-1}}^{s_{m-1}} \cdots \int_{\omega_{m-(k-1)}}^{s_{m-(k-1)}} ds_{m-1}\, ds_{m-2} \cdots ds_{m-k}, \tag{2.11}$$

see [90, §4.1.4 (15)-(16)] for more details. The polynomial $\mathcal{G}_{k,m}$ is called the kth Goncharov's polynomial associated with $\{\omega_1, \ldots, \omega_m\}$.

Example 2.3.1 (Abel's polynomials)**.** *If $\omega_1, \ldots, \omega_m$ form an arithmetic progression, i.e. $\omega_{m-k} = \omega + k\vartheta$, where $\omega, \vartheta \in \mathbb{C}$ are fixed and $k = 0, 1, \ldots, m-1$, it is well known that in this case the kth Goncharov polynomials*

$$\mathcal{G}_{k,m}(z) = (z-\omega)(z-\omega-(m-k)\vartheta)^{k-1}, \tag{2.12}$$

is the so called kth Abel's polynomials.

If $\vartheta = 0$, we have the special case $\mathcal{G}_{k,m}(z) = (z-\omega)^k$ (Taylor's case) and then the mth Abel-Goncharov polynomial of interpolation (2.10) *becomes into the Taylor's expansion of $\mathcal{P}_{n,m}^{(\alpha,\beta)}$ in ω , as we mentioned in the introduction.*

According to (2.2) it follows that

$$\frac{1}{k!} \frac{d^k \mathcal{P}_{n,m}^{(\alpha,\beta)}}{dz^k}(\omega_{m-k}) = \binom{n+m}{k} \mathcal{P}_{n,m-k}^{(\alpha,\beta)}(\omega_{m-k})$$

and replacing this formula in (2.10) we thus get

$$\mathcal{A}_{n,m}(z) = \mathcal{P}_{n,m}^{(\alpha,\beta)}(\omega_m) + \sum_{k=1}^{m-1} \binom{n+m}{k} \mathcal{P}_{n,m-k}^{(\alpha,\beta)}(\omega_{m-k})\, \mathcal{G}_{k,m}(z). \tag{2.13}$$

Theorem 2.3.1. *Given $m > 0$ and $\omega_1, \ldots, \omega_m \in \mathbb{C} \setminus [-1,1]$ fixed, let $\mathcal{A}_{n,m}(z)$ be the Abel-Goncharov polynomial of interpolation associate to* *the conditions* (2.4), $\rho_m = \max_{0 \le k \le m-1} |\varphi(\omega_{m-k})|$, $U = \{k : |\varphi(\omega_{m-k})| = \rho_m\}$ *and* $\hat{k} = \max_{k \in U} |k|$. *Then uniformly on compact subsets of* $\overline{\mathbb{C}}$

$$\lim_{n\to\infty} \frac{\mathcal{A}_{n,m}(z)}{n^{\hat{k}}\, \mathcal{P}^{(\alpha,\beta)}_{n,m-\hat{k}}(\omega_{m-\hat{k}})} = \frac{\mathcal{G}_{\hat{k},m}(z)}{\hat{k}!}, \tag{2.14}$$

$$\lim_{n\to\infty} |\mathcal{A}_{n,m}(z)|^{\frac{1}{n}} = \rho_m. \tag{2.15}$$

The branch of the square root contained in the explicit formula of φ *is chosen so that* $|\varphi(\omega_{m-k})| > 1$, *for each* $0 \le k \le m-1$.

Proof. Let $V = \{k : |\varphi(\omega_{m-k})| < \rho_m\}$, obviously $U \cap V = \emptyset$ and $U \cup V = \{1, 2, \cdots, m\}$. From (2.13) we get

$$\left(\frac{(n+m-\hat{k})!}{(n+m)!}\right) \frac{\mathcal{A}_{n,m}(z)}{\mathcal{P}^{(\alpha,\beta)}_{n,m-\hat{k}}(\omega_{m-\hat{k}})} = \frac{\mathcal{G}_{\hat{k},m}(z)}{\hat{k}!} + \sum_{k \in U\setminus\{\hat{k}\}} A_{n,m,k} \frac{\mathcal{G}_{k,m}(z)}{k!} + \sum_{k\in V} A_{n,m,k} \frac{\mathcal{G}_{k,m}(z)}{k!}, \tag{2.16}$$

$$\text{where} \quad A_{n,m,k} = \frac{(n+m-\hat{k})!}{(n+m-k)!} \frac{\mathcal{P}^{(\alpha,\beta)}_{n,m-k}(\omega_{m-k})}{\mathcal{P}^{(\alpha,\beta)}_{n,m-\hat{k}}(\omega_{m-\hat{k}})}.$$

Firstly we will prove that for all $k \in (U \cup V) \setminus \{\hat{k}\}$

$$\lim_{n\to\infty} A_{n,m,k} = 0, . \tag{2.17}$$

If $k \in V$, then $|\varphi(\omega_{m-k})| < |\varphi(\omega_{m-\hat{k}})|$,

$$A_{n,m,k} = \frac{(n+m-\hat{k})!}{(n+m-k)!} \left(\frac{\varphi(\omega_{m-k})}{\varphi(\omega_{m-\hat{k}})}\right)^n \frac{\mathcal{P}^{(\alpha,\beta)}_{n,m-k}(\omega_{m-k})}{\varphi^n(\omega_{m-k})} \frac{\varphi^n(\omega_{m-\hat{k}})}{\mathcal{P}^{(\alpha,\beta)}_{n,m-\hat{k}}(\omega_{m-\hat{k}})}$$

and from (2.7) we can assert that for $k \in V$ we get (2.17).

If $k \in U \setminus \{\hat{k}\}$, therefore $k < \hat{k}$ and $|\varphi(\omega_{m-k})| = |\varphi(\omega_{m-\hat{k}})|$. Let us write $\varphi(\omega_{m-k}) = |\varphi(\omega_{m-\hat{k}})|e^{i\theta}$ and $\varphi(\omega_{m-\hat{k}}) = |\varphi(\omega_{m-\hat{k}})|e^{i\hat{\theta}}$, with $\theta, \hat{\theta} \in [0.2\pi)$. Then

$$A_{n,m,k} = \left(\frac{(n+m-\hat{k})!}{(n+m-k)!}\right) e^{n(\theta-\hat{\theta})\,i} \left(\frac{\mathcal{P}^{(\alpha,\beta)}_{n,m-k}(\omega_{m-k})}{\varphi^n(\omega_{m-k})}\right) \left(\frac{\varphi^n(\omega_{m-\hat{k}})}{\mathcal{P}^{(\alpha,\beta)}_{n,m-\hat{k}}(\omega_{m-\hat{k}})}\right)$$

and as in the previous reasoning, from (2.7) we can assert that for $k \in U \setminus \{\hat{k}\}$ we get (2.17).

Now, according to (2.16)-(2.17) we get (2.14). Finally, (2.15) is a consequence of (2.14) and (2.8). $\square$

In example 2.3.1, if for each $0 \leq k \leq m-1$ it holds that $(\omega + k\vartheta) \notin [-1,1]$, then all the zeros of the Abel's polynomials (2.12) are out to the interval $[-1,1]$.

2.4 General primitive of Jacobi polynomials and its zeros

$$
\begin{aligned}
\overline{\mathbf{E}}_\rho &= \left\{ z \in \mathbb{C} : |z-1| + |z+1| > \rho + \rho^{-1} \right\}, \\
\underline{\mathbf{E}}_\rho &= \left\{ z \in \mathbb{C} : |z-1| + |z+1| \leq \rho + \rho^{-1} \right\}.
\end{aligned}
$$

Analogously to the notations introduced in Theorem 2.2.1, we denote $\mathbf{Z}^{(\alpha,\beta)}_{n,m,\Omega_m} = \left\{ z \in \mathbb{C} : \mathcal{P}^{(\alpha,\beta)}_{n,m,\Omega_m}(z) = 0 \right\}$ (i.e. set of the $(n+m)$ zeros of $\mathcal{P}^{(\alpha,\beta)}_{n,m,\Omega_m}$, and $\mathbf{Z}^{(\alpha,\beta)}_{m,\Omega_m} = \bigcap_{n\geq 1} \bigcup_{k\geq n} \mathbf{Z}^{(\alpha,\beta)}_{k,m,\Omega_m}$ (i.e. set of accumulation points of zeros of $\{\mathcal{P}^{(\alpha,\beta)}_{n,m,\Omega}\}$).

Lemma 2.4.1. *Let $\alpha, \beta > -1$, $m \in \mathbb{N}$ and $\Omega_m = (\omega_1, \ldots, \omega_m) \in \mathbb{C}^m$ fixed. Then there exists a compact subset $K \subset \mathbb{C}$, such that $(-1,1) \subset K$ and $\mathbf{Z}^{(\alpha,\beta)}_{n,m,\Omega_m} \subset K$ for all n.*

Proof. This claim is proved analogously that the Theorem 2.2.1-(2). If $m = 1$, for all $n \geq 1$ the critical points of $\mathcal{P}^{(\alpha,\beta)}_{n,1,\omega_1}$ are on $I = [-1,1]$, then from Lemma 2.2.4 we get $\mathbf{Z}^{(\alpha,\beta)}_{n,1,\Omega_1}$ is a subset of the compact set $[I]_{\omega_1}$, defined in Lemma 2.2.4.

Suppose that for a fixed $m \in \mathbb{N}$, there exists a compact set K_{m-1} such that $\mathbf{Z}^{(\alpha,\beta)}_{n,m-1,\Omega_{m-1}} \subset K_{m-1}$. As the zeros of $\mathcal{P}^{(\alpha,\beta)}_{n,m-1,\Omega_{m-1}}$ are the critical points of $\mathcal{P}^{(\alpha,\beta)}_{n,m,\Omega_m}$, from Lemma 2.2.4 we get $\mathbf{Z}^{(\alpha,\beta)}_{n,m,\Omega_m} \subset [K_{m-1}]_{\omega_m}$. $\square$

Theorem 2.4.1. *Given $m > 0$ and $\omega_1, \ldots, \omega_m \in \mathbb{C} \setminus [-1,1]$ fixed, let ρ_m as in (2.15). Then uniformly on compact subsets of $\overline{\mathbf{E}}_{\rho_m}$*

$$\lim_{n\to\infty} \frac{\mathcal{P}^{(\alpha,\beta)}_{n,m,\Omega_m}(z)}{P^{(\alpha,\beta)}_n(z)} = \left(\frac{1}{\varphi'(z)}\right)^m. \tag{2.18}$$

Furthermore, $\mathbf{Z}^{(\alpha,\beta)}_{m,\Omega_m} \subset \underline{\mathbf{E}}_{\rho_m}$.

Proof. From (2.5) we known that

$$\frac{\mathcal{P}^{(\alpha,\beta)}_{n,m,\Omega_m}(z)}{P^{(\alpha,\beta)}_n(z)} = \frac{\mathcal{P}^{(\alpha,\beta)}_{n,m}(z)}{P^{(\alpha,\beta)}_n(z)} - \frac{\mathcal{A}_{n,m}(z)}{P^{(\alpha,\beta)}_n(z)}.$$

The uniform limit of the first quotient in the right side is given by (2.9). Hence to proof (2.18), it is sufficient to proof that

$$\lim_{n\to\infty} \frac{\mathcal{A}_{n,m}(z)}{P^{(\alpha,\beta)}_n(z)} = 0, \quad \text{uniformly on compact subsets of } \overline{\mathbf{E}}_{\rho_m}. \tag{2.19}$$

From (2.13) we have

$$\frac{\mathcal{A}_{n,m}(z)}{P^{(\alpha,\beta)}_n(z)} = \sum_{k=0}^{m-1} \frac{\mathcal{P}^{(\alpha,\beta)}_{n,m-k}(\omega_{m-k})}{P^{(\alpha,\beta)}_n(\omega_{m-k})} \left(\frac{(n+m)!}{(n+m-k)!} \frac{P^{(\alpha,\beta)}_n(\omega_{m-k})}{P^{(\alpha,\beta)}_n(z)}\right) \frac{\mathcal{G}_{k,m}(z)}{k!},$$

where $\mathcal{G}_m(z) \equiv 1$. For $k = 0, 1, \ldots, m-1$ we get

$$\begin{aligned} \frac{(n+m)!}{(n+m-k)!} \frac{P^{(\alpha,\beta)}_n(\omega_{m-k})}{P^{(\alpha,\beta)}_n(z)} &= \frac{(n+m)!}{(n+m-k)!} \left(\frac{\varphi(\omega_{m-k})}{\varphi(z)}\right)^n \\ &\quad \cdot \frac{P^{(\alpha,\beta)}_n(\omega_{m-k})}{\varphi^n(\omega_{m-k})} \frac{\varphi^n(z)}{P^{(\alpha,\beta)}_n(z)}. \end{aligned}$$

As $|\varphi(\omega_{m-k})| < |\varphi(z)|$ for all $z \in \overline{\mathbf{E}}_{\rho_*}$, from (2.9) it follows (2.19).

Finally, the assertion $\mathbf{Z}^{(\alpha,\beta)}_{m,\Omega_m} \subset \underline{\mathbf{E}}_{\rho_m}$ is a consequence of (2.18) and Lemma 2.4.1, using analogous argument as in the proof of Theorem 2.2.1-(3). □

Theorem 2.4.2. *Assume that $m > 0$ and $\omega_1, \ldots, \omega_m \in \mathbb{C} \setminus [-1,1]$,then the accumulation points of zeros of $\{\mathcal{P}^{(\alpha,\beta)}_{n,m,\Omega_m}\}$ are located on the union of the interval $[-1,1]$ and the ellipse*

$$\mathbf{E}_{\rho_m} = \left\{z \in \mathbb{C} : |z-1| + |z+1| = \rho_m + \rho_m^{-1}\right\}, \tag{2.20}$$

where $\rho_m = \max_{1\le k\le m} |\varphi(\omega_k)|$ *and the branch of the square root contained in the explicit formula of* φ *is chosen so that* $|\varphi(\omega_k)| > 1$, *for each* $1 \le k \le m$.

Proof. From (2.5) the zeros of the polynomial $\mathcal{P}^{(\alpha,\beta)}_{n,m,\Omega_m}$ satisfy the equation

$$\left|\mathcal{P}^{(\alpha,\beta)}_{n,m}(z)\right|^{\frac{1}{n}} = |\mathcal{A}_{n,m}(z)|^{\frac{1}{n}} . \tag{2.21}$$

Taking the limit as $n \to \infty$ of both sides of (2.21), from (2.15) and (2.8), we have that $\mathbf{Z}^{(\alpha,\beta)}_{m,\Omega_m} \subset \mathbf{E}_{\rho_m}$ where

$$\mathbf{E}_{\rho_m} = \left\{ z \in \mathbb{C} : |z + \sqrt{z^2-1}| = \rho_m \right\} .$$

Let $\tilde{k}$ be an index, $1 \le \tilde{k} \le m$, such that $\varphi(\omega_{\tilde{k}}) = \rho_m e^{i\tilde{\theta}}$, $0 \le \tilde{\theta} < 2\pi$. Hence, we have that $z + \sqrt{z^2-1} = \rho_m\, e^{i\tilde{\theta}}$, $z - \sqrt{z^2-1} = \rho_m^{-1}\, e^{-i\tilde{\theta}}$ and taking the difference between both we get $\sqrt{z^2-1} = (\rho_m e^{i\tilde{\theta}} + \rho_m^{-1} e^{-i\tilde{\theta}})/2$. Thus,

$$|z-1| + |z+1| = \frac{|\rho_m e^{i\tilde{\theta}} - 1|^2 + |\rho_m e^{i\tilde{\theta}} + 1|^2}{2\rho_m} ,$$

which is equivalent to the equation of the ellipse in (2.20). As the limit that we have taken is uniform on compact subsets $\mathbb{C}\setminus[-1,1]$ the theorem is proved. ☐

Corollary 2.4.2.1. *Under the assumptions of theorems 2.3.1 and 2.4.2, if the zeros of the Goncharov polynomial* $\mathcal{G}_{\hat{k},m}$ *are outside to the interval* $[-1,1]$ *then the accumulation points of zeros of* $\{\mathcal{P}^{(\alpha,\beta)}_{n,m,\Omega}\}$ *are located on the ellipse* $\mathbf{E}_{\rho_m}$.

Proof. Obviously, from Theorem 2.4.2 it is sufficient to prove that there does not exist an accumulation point of zeros of $\{\mathcal{P}^{(\alpha,\beta)}_{n,m,\Omega}\}$ are located on the interval $[-1,1]$.

Let $\varepsilon \in \mathbb{R}$ such that $\omega_1, \dots, \omega_m$ and the zeros of $\mathcal{G}_{\hat{k},m}$ are on the exterior of the ellipse $\mathbf{E}_{1+\varepsilon}$. Thus, if $w \in \mathbf{E}_{1+\varepsilon}$, from (2.7) and (2.14) we get for sufficiently large values of n

$$\mathcal{A}_{n,m}(w) \approx \binom{n+m}{\hat{k}} \psi_{\alpha,\beta,m-\hat{k}}(\omega_{m-\hat{k}})\, \varphi^{n+\frac{1}{2}}(\omega_{m-\hat{k}})\, \mathcal{G}_{\hat{k},m}(w), \tag{2.22}$$

$$\mathcal{P}^{(\alpha,\beta)}_{n,m}(w) \approx \psi_{\alpha,\beta,m}(w)\, \varphi^{n+\frac{1}{2}}(w). \tag{2.23}$$

As the zeros of the Goncharov polynomial $\mathcal{G}_{\hat{k},m}$ are on the exterior of the ellipse $\mathbf{E}_{1+\varepsilon}$, then from (2.22), there exists $N_1 \in \mathbb{Z}_+$ such that for $n > N_1$ the zeros of the polynomial $\mathcal{A}_{n,m}$ are on the exterior of the ellipse $\mathbf{E}_{1+\varepsilon}$ too. From (2.22)-(2.23)

$$\begin{aligned} |\mathcal{A}_{n,m}(w)| \approx & \binom{n+m}{\hat{k}} \left| \frac{\mathcal{G}_{\hat{k},m}(w)\ \psi_{\alpha,\beta,m-\hat{k}}(\omega_{m-\hat{k}})}{\psi_{\alpha,\beta,m}(w)} \right| \left| \frac{\varphi(\omega_{m-\hat{k}})}{\varphi(w)} \right|^{n+\frac{1}{2}} \\ & \cdot \left| \psi_{\alpha,\beta,m}(w)\ \varphi^{n+\frac{1}{2}}(w) \right| \\ \geq & \left| \frac{\mathcal{G}_{\hat{k},m}(w)\ \psi_{\alpha,\beta,m}(\omega_{m-\hat{k}})}{\psi_{\alpha,\beta,m}(w)} \right| \left| \frac{\varphi(\omega_{m-\hat{k}})}{\varphi(w)} \right|^{n+\frac{1}{2}} \\ & \cdot \left| \mathcal{P}_{n,m}^{(\alpha,\beta)}(w) \right| \end{aligned} \tag{2.24}$$

As is well known from classical complex analysis (cf. [54, §51]), $\varphi(z)$ is an inverse Joukowsky mapping that maps every confocal ellipse $|z-1| + |z+1| = r + \frac{1}{r}$, with $r > 0$, on to the circumference $|z| = r$. Hence, as each ω_k are on the exterior of the ellipse $\mathbf{E}_{1+\varepsilon}$ and $w \in \mathbf{E}_{1+\varepsilon}$, we get that $|\varphi(\omega_{m-\hat{k}})| > |\varphi(w)|$. Thus, from (2.24) there exists $N_2 \in \mathbb{Z}_+$ such that for $n > N_2$ the following inequality holds

$$|\mathcal{A}_{n,m}(w)| > \left| \mathcal{P}_{n,m}^{(\alpha,\beta)}(w) \right|.$$

Finally, from Lemma 2.2.3 and theorem 2.4.2 the corollary is proven. □

2.5 The Laguerre case

Recall that, for a $\gamma \in \mathbb{R}$, we denote by $L_n^{(\gamma)}$ the Laguerre monic polynomial of degree n, as in Section 1.1.2, The next lemma summarizes properties of monic Laguerre polynomials based on [89, (5.1.6),(5.1.8), (5.1.13)–(5.1.14) and (5.2.1)].

Lemma 2.5.1. *Let $L_n^{(\gamma)}$ be the nth monic Laguerre polynomials with* *parameter $\gamma \in \mathbb{R}$, as in Section 1.1.2, then*

$$L_n^{(\gamma)}(z) = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \frac{\Gamma(n+\gamma+1)}{\Gamma(k+\gamma+1)} z^k, \quad (2.25)$$

$$\begin{aligned}(n+1)(n+\gamma+1)L_n^{(\gamma)}(z) = &- L_{n+2}^{(\gamma)}(z) \\ &+ (z-(2n+\gamma+3))L_{n+1}^{(\gamma)}(z), \quad (2.26)\\ &L_{-1}^{(\gamma)}(z) = 0, \quad L_0^{(\gamma)}(z) = 1.\end{aligned}$$

$$L_n^{(\gamma)}(z) = z^{-\gamma} L_{n+\gamma}^{(-\gamma)}(z), \quad \textit{where } \gamma = 1, \cdots, n. \quad (2.27)$$

$$\frac{d^k}{dz^k}\left(L_n^{(\gamma)}(z)\right) = \frac{n!}{(n-k)!} L_{n-k}^{(\gamma+k)}(z), \quad (2.28)$$

$$\textit{where } 0 \le k \le n.$$

$$L_{n+1}^{(\gamma-1)}(z) = L_{n+1}^{(\gamma)}(z) + (n+1)L_n^{(\gamma)}(z). \quad (2.29)$$

Let us define

$$\mathcal{L}_{n,1}^{(\gamma)}(z) = L_{n+1}^{(\gamma-1)}(z) = L_{n+1}^{(\gamma)}(z) + (n+1)L_n^{(\gamma)}(z), \quad (2.30)$$

hence from (2.28) we get

$$\frac{d}{dz}\left(\mathcal{L}_{n,1}^{(\gamma)}(z)\right) = (n+1)L_n^{(\gamma)}(z), \quad (2.31)$$

i.e. $(n+1)^{-1}\mathcal{L}_{n,1}^{(\gamma)}(z)$ is a primitive of $L_n^{(\gamma)}(z)$, that we call *fundamental primitive of order one for $L_n^{(\gamma)}(z)$.*

The following theorem describes the interlacing properties of zeros between Laguerre polynomials and its fundamental primitives, and between two consecutive fundamental primitives.

Theorem 2.5.1. *Let $x_{n,1} < x_{n,2} < \cdots < x_{n,n}$ be the n zeros of $L_n^{(\gamma)}$ and $\mathcal{L}_{n,1}^{(\gamma)}$ as* (2.30)*, where $\gamma > -1$. Then*

1.- $\mathcal{L}_{n,1}^{(\gamma)}$ has real and simple zeros and at least n of them lie on $(0, \infty)$.

2-. Let $z_{n,1} < z_{n,2} < \cdots < z_{n,n+1}$ be the $(n+1)$ *zeros of $\mathcal{L}_{n,1}^{(\gamma)}$. Then the zeros of $L_n^{(\gamma)}$ and $\mathcal{L}_{n,1}^{(\gamma)}$ are interlaced as $z_{n,1} < x_{n,1} < z_{n,2} < x_{n,2} < \cdots < z_{n,n} < x_{n,n} < z_{n+1,n+1}$.*

3-. For $z_{n,1}$, the smallest zero of $\mathcal{L}_{n,1}^{(\gamma)}$, we have:

3.1.- if $\gamma > 0$, then $z_{n,1} > 0$.

3.2.- if $\gamma = 0$, then $z_{n,1} = 0$.

3.3.- if $-1 < \gamma < 0$, then there exist a constant $c_\gamma > 0$ for all $n \geq 3$, such that $z_{n,1} \in (-\frac{c_\gamma}{n}, 0)$.

4.- Let $z_{n+1,1} < z_{n+1,2} < \cdots < z_{n+1,n+2}$ be the $n+2$ zeros of $\mathcal{L}_{n+1,1}^{(\gamma)}$. Then the zeros of $\mathcal{L}_{n,1}^{(\gamma)}$ and $\mathcal{L}_{n+1,1}^{(\gamma)}$ are interlaced as:

4.1.- if $\gamma \geq 0$, then $0 \leq z_{n+1,1} \leq z_{n,1} < z_{n+1,2} < z_{n,2} < \cdots < z_{n,n+1} < z_{n+1,n+2}$. The equalities occur only if $\gamma = 0$.

4.2.- if $-1 < \gamma < 0$, then $z_{n,1} < z_{n+1,1} < 0 < z_{n+1,2} < z_{n,2} < \cdots < z_{n+1,n+1} < z_{n,n+1} < z_{n+1,n+2}$.

Proof.

1.- From (2.30) $\mathcal{L}_{n,1}^{(\gamma)}$ is a quasi-orthogonal polynomial of order 1 with respect to the measure $d\mu^{(\gamma)}(x) = x^\gamma e^{-x}dx$ on $[0,\infty)$. Hence, $\mathcal{L}_{n,1}^{(\gamma)}$ has at least n zeros of odd multiplicity on $(0,\infty)$. Furthermore, as $\mathcal{L}_{n,1}^{(\gamma)}$ is a polynomial with real coefficients, the remaining zero must be a real number and all their zeros are simple.

2.- As the critical points of $\mathcal{L}_{n,1}^{(\gamma)}$ are the zeros of $L_n^{(\gamma)}$, from the Rolle's Theorem, the interlacing property between zeros of $L_n^{(\gamma)}$ and $\mathcal{L}_{n,1}^{(\gamma)}$ is straightforward.

3.- The statement *3.1.* is straightforward because from (2.30) if $\gamma > 0$ then $\mathcal{L}_{n,1}^{(\gamma)}$ is a Laguerre orthogonal polynomial.

From [89, (5.1.7), (5.1.8)] and (2.30)

$$\mathcal{L}_{n,1}^{(\gamma)}(0) = (-1)^{n+1}\,\gamma\,\frac{\Gamma(n+\gamma+1)}{\Gamma(\gamma+1)}, \tag{2.32}$$

where we have the assertion *3.2.*

Let sgn$\cdot$ be the signum function defined by sgn$0 = 0$ and sgn$x = x/|x|$ for all $x \in \mathbb{R} \setminus \{0\}$, then for $n \geq 2$

$$\operatorname{sgn}\Big(\mathcal{L}_{n,1}^{(\gamma)}(0)\Big) = \begin{cases} (-1)^{n+1}, & \text{if } \gamma > 0, \\ 0, & \text{if } \gamma = 0, \\ (-1)^n, & \text{if } -1 < \gamma < 0. \end{cases} \tag{2.33}$$

Hence, if $-1 < \gamma < 0$ we get that $\operatorname{sgn}\mathcal{L}_{n,1}^{(\gamma)}(0) \neq \operatorname{sgn}\mathcal{L}_{n,1}^{(\gamma)}(\infty)$ and there exist $z^* \in (-\infty, 0)$ such that $\mathcal{L}_{n,1}^{(\gamma)}(z^*) = 0$. From the above interlacing property *2*, $z_{n,2} \in (x_{n,1}, x_{n,2})$ and from [89, Th. 6.31.3] there exist a constant $c_\gamma \in \mathbb{R}_+$ such that $(x_{n,2} - x_{n,1}) < c_\gamma/n$. Now, taking into consideration the Bisector Theorem ([81, Th. 5.7.7]) is straightforward that $z^* \in (-c_\gamma/n, x_{n,1})$ and the assertion *3.3.* is proved.

4.- If $\gamma > 0$ the statement *4.1.* is the well known interlacing property between two consecutives Laguerre orthogonal polynomials .

If $\mathcal{L}_{n+1,1}^{(\gamma)}(z_{n+1,k}) = 0$ for $k = 1, 2, \ldots, n+2$, from (2.30) we get $L_{n+2}^{(\gamma)}(z_{n+1,k}) = -(n+2)L_{n+1}^{(\gamma)}(z_{n+1,k})$ and from (2.26)

$$(n+\gamma+1)\mathcal{L}_{n,1}^{(\gamma)}(z) = -L_{n+2}^{(\gamma)}(z) + (z-(n+2))L_{n+1}^{(\gamma)}(z), \quad (2.34)$$

Hence

$$\begin{aligned}(n+\gamma+1)\mathcal{L}_{n,1}^{(\gamma)}(z_{n+1,k}) =& z_{n+1,k}L_{n+1}^{(\gamma)}(z_{n+1,k}),\\ \mathcal{L}_{n,1}^{(\gamma)}(z_{n+1,k})\mathcal{L}_{n,1}^{(\gamma)}(z_{n+1,k+1}) =& \frac{z_{n+1,k}\, z_{n+1,k+1}}{(n+\gamma+1)^2}L_{n+1}^{(\gamma)}(z_{n+1,k})\\ &\cdot L_{n+1}^{(\gamma)}(z_{n+1,k+1}) \qquad (2.35)\end{aligned}$$

- If $\gamma = 0$ from *2* and *3.2* , we get $\mathcal{L}_{n,1}^{(\gamma)}(z_{n+1,k})\mathcal{L}_{n,1}^{(\gamma)}(z_{n+1,k+1}) < 0$ for $k = 2, \cdots, n+1$.
- If $-1 < \gamma < 0$, from (2.26) and (2.30):

$$\begin{aligned}\mathcal{L}_{n+1,1}^{(\gamma)}(z) =& -(n+1)(n+\gamma+1)L_n^{(\gamma)}(z)\\ &+(z-(n+\gamma+1))L_{n+1}^{(\gamma)}(z), \qquad (2.36)\\ \mathcal{L}_{n+1,1}^{(\gamma)}(0) =& -(n+\gamma+1)\mathcal{L}_{n,1}^{(\gamma)}(0).\end{aligned}$$

If $\mathcal{L}_{n,1}^{(\gamma)}(z_{n,1}) = 0$, hence $L_{n+1}^{(\gamma)}(z_{n,1}) = -(n+1)L_n^{(\gamma)}(z_{n,1})$ and from (2.36) $\mathcal{L}_{n+1,1}^{(\gamma)}(z_{n,1}) = x_k L_{n+1}^{(\gamma)}(z_{n,1})$. Hence, using [89, (5.1.7), (5.1.8)], *3.3* of Theorem 2.5.1 and (2.33), we obtain

$$\begin{aligned}\mathcal{L}_{n+1,1}^{(\gamma)}(z_{n,1})\mathcal{L}_{n+1,1}^{(\gamma)}(0) &= -(n+\gamma+1)z_{n,1}L_{n+1}^{(\gamma)}(z_{n,1})\mathcal{L}_{n,1}^{(\gamma)}(0)\\ \operatorname{sgn}\mathcal{L}_{n+1,1}^{(\gamma)}(z_{n,1})\mathcal{L}_{n+1,1}^{(\gamma)}(0) &= \operatorname{sgn}L_{n+1}^{(\gamma)}(z_{n,1})\mathcal{L}_{n,1}^{(\gamma)}(0)\\ &= \operatorname{sgn}L_{n+1}^{(\gamma)}(0)\mathcal{L}_{n,1}^{(\gamma)}(0)\\ &= (-1)^{n+1}(-1)^n = -1,\end{aligned}$$

i.e. $z_{n,1} < z_{n+1,1} < 0$. The interlacing between positive zeros of $\mathcal{L}_{n,1}^{(\gamma)}$ and $\mathcal{L}_{n+1,1}^{(\gamma)}$ is analogous to the previous case for $\gamma = 0$.

□

For all $m \in \mathbb{N}$, we define recursively the monic polynomial of degree $n+m$

$$\mathcal{L}_{n,m}^{(\gamma)}(z) = \mathcal{L}_{n+1,m-1}^{(\gamma)}(z) + (n+m)\,\mathcal{L}_{n,m-1}^{(\gamma)}(z); \qquad (2.37)$$

where $\mathcal{L}_{n,0}^{(\gamma)}(z) = L_n^{(\gamma)}(z)$. Reasoning by mathematical induction, we get

$$\frac{d^m}{dx^m}\left(\mathcal{L}_{n,m}^{(\gamma)}(z)\right) = \frac{(n+m)!}{n!}\,L_n^{(\gamma)}(z),\ m \in \mathbb{N}. \qquad (2.38)$$

Indeed, for $m = 1$ the equation (2.38) becomes (2.31), and from (2.37), (2.38) and (2.29)

$$\begin{aligned}
\frac{d^{m+1}}{dx^{m+1}}\left(\mathcal{L}_{n,m+1}^{(\gamma)}(z)\right) &= \frac{d}{dx}\left(\frac{d^m}{dx^m}\left(\mathcal{L}_{n+1,m}^{(\gamma)}(z)\right)\right. \\
&\quad \left. +(n+m+1)\frac{d^m}{dx^m}\left(\mathcal{L}_{n,m}^{(\gamma)}(z)\right)\right) \\
&= \frac{(n+m+1)!}{(n+1)!}\,\frac{d}{dx}\left(L_{n+1}^{(\gamma)}(z) + (n+1)\,L_n^{(\gamma)}(z)\right) \\
&= \frac{(n+m+1)!}{(n+1)!}\,\frac{d}{dx}\left(L_{n+1}^{(\gamma-1)}(z)\right) \\
&= \frac{(n+m+1)!}{n!}\,L_n^{(\gamma)}(z).
\end{aligned}$$

For $m \in \mathbb{N}$, $m \le n$, we call *mth fundamental iterate integral of* $L_n^{(\gamma)}$ to the polynomial $\dfrac{n!\,\mathcal{L}_{n,m}^{(\gamma)}}{(n+m)!}$, where $\mathcal{L}_{n,m}^{(\gamma)}$ is given by (2.37).

Theorem 2.5.2. *Let $\mathcal{L}_{n,m}^{(\gamma)}$ be the mth monic iterated integral defined by* (2.37)*, where $n, m \in \mathbb{N}$, then $\mathcal{L}_{n,m}^{(\gamma)}(z) = L_{n+m}^{(\gamma-m)}(z)$.*

Proof. For $m = 1$, from (2.29) we get

$$\mathcal{L}_{n,1}^{(\gamma)}(z) = L_{n+1}^{(\gamma)}(z) + (n+1)L_n^{(\gamma)}(z) = L_{n+1}^{(\gamma-1)}(z).$$

Assume that $\mathcal{L}_{n,m}^{(\gamma)}(z) = L_{n+m}^{(\gamma-m)}(z)$ for $m \in \mathbb{N}$, then

$$\begin{aligned}
\mathcal{L}_{n,m+1}^{(\gamma)}(z) &= \mathcal{L}_{n+1,m}^{(\gamma)}(z) + (n+m+1)\,\mathcal{L}_{n,m}^{(\gamma)}(z) \\
&= L_{n+m+1}^{(\gamma-m)}(z) + (n+m+1)\,L_{n+m}^{(\gamma-m)}(z) = L_{n+m+1}^{(\gamma-m-1)}(z).
\end{aligned}$$

□

The following corollary clarifies the location of the real zeros of $\mathcal{L}_{n,m}^{(\gamma)}$.

Corollary 2.5.2.1. *Let $\mathcal{L}_{n,m}^{(\gamma)}$ be the monic primitive of degree $(n+m)$ defined by* (2.37)*, where $n, m \in \mathbb{N}$.*

1.- If $\gamma - m \in (-1, +\infty)$, the $(n+m)$ zeros of $\mathcal{L}_{n,m}^{(\gamma)}$ lie on $(0, +\infty)$ and are distinct. Furthermore, two consecutive polynomials $\mathcal{L}_{n,m}^{(\gamma)}$ and $\mathcal{L}_{n+1,m}^{(\gamma)}$ have strictly interlacing zeros.

2.- If $\gamma - m = -\eta$, where $\eta = 1, 2, \cdots, n+m$, the polynomial $\mathcal{L}_{n,m}^{(\gamma)}$ has $(n+m-\eta)$ distinct zeros lie on $(0, +\infty)$ and a zero of multiplicity η at $z = 0$. Furthermore, the positive zeros of the consecutive polynomials $\mathcal{L}_{n,m}^{(\gamma)}$ and $\mathcal{L}_{n+1,m}^{(\gamma)}$ have strictly interlacing zeros.

3.- If $\gamma - m \in (-n-m, -1) \setminus \mathbb{Z}$ the polynomial $\mathcal{L}_{n,m}^{(\gamma)}$ has $(n + [\gamma - m] + 1)$ distinct zeros lie on $(0, +\infty)$ and

3.1- $\gamma - m \in (-\eta - 1, -\eta)$ with $\eta = 1, 3, \cdots, 2\left[\frac{n+m-1}{2}\right] + 1$, there is only a negative real zero.

3.2- $\gamma - m \in (-\eta - 1, -\eta)$ with $\eta = 2, 4, \cdots, 2\left[\frac{n+m-1}{2}\right]$, there are no negative zeros.

4.- If $\gamma - m \in (-\infty, -n)$, there are no positive zeros at all and there is only a negative real zero if n is odd.

The symbol $[x]$ denotes the greatest integer less than or equal to the real number x.

Proof. The first assertion of the theorem is straightforward form of the basic properties of the zeros of Laguerre polynomials $L_{n+m}^{(\gamma-m)}(z)$, where $\gamma > m - 1$.

If $\gamma = m - \eta$, where $\eta = 1, 2, \cdots, n$, from Theorem 2.5.2 and (2.27) we get $\mathcal{L}_{n,m}^{(\gamma)}(z) = L_{n+m}^{(-\eta)}(z) = z^{\eta}\, L_{n+m-\eta}^{(\eta)}(z)$. This means that $\mathcal{L}_{n,m}^{(\gamma)}$ has a zero of multiplicity η at $z = 0$ and there are $n + m - \eta$ distinct zeros on $(0, +\infty)$.

The cases $\gamma - m \in (-n - m, -1) \setminus \mathbb{Z}$ and $\gamma < -n$ follow from [89, §6.73]. □

Corollary 2.5.2.2 (Perron's type formula for fundamental primitive of Laguerre polynomials)**.** *Let* $\gamma \in \mathbb{R}$ *and* $m \in \mathbb{Z}_+$*. Then*

$$\begin{aligned} \mathcal{L}_{n,m}^{(\gamma)}(z) =& \frac{(-1)^{n+m}}{\sqrt{2}} \; (-z)^{-\frac{\gamma}{2}+\frac{2m-1}{4}} \; n^{\frac{\gamma}{2}+\frac{5}{4}} \\ & \cdot \left(\frac{n}{e}\right)^n \; e^{\frac{z}{2}+2\sqrt{-nz}} \left(1+\mathbf{O}(n^{-\frac{1}{2}})\right). \end{aligned} \tag{2.39}$$

This relation holds for z *in the complex plane cut along the positive real semiaxis; both* $(-z)^{-\frac{\gamma}{2}+\frac{2m-1}{4}}$ *and* $(-z)^{-\frac{1}{2}}$ *must be taken real and positive if* $z < 0$*. The bound of the remainder holds uniformly on every closed domain which does not overlap with the positive real semiaxis.*

Proof. From Theorem 2.5.2 and [89, Perron's formula (Th. 8.22.3)]. □

Corollary 2.5.2.3. *Let* $\gamma \in \mathbb{R}$ *and* $m \in \mathbb{Z}_+$ *fixed. Then all zeros* $\{\mathcal{L}_{n,m}^{(\gamma)}\}$ *accumulate at* $[0,\infty)$*. i.e.*

$$\bigcap_{n\geq 1} \bigcup_{k\geq n} \{z \in \mathbb{C} : \mathcal{L}_{k,m}^{(\gamma)}(z) = 0\} = [0,\infty)$$

Proof. From (2.39) and [89, Perron's formula (Th. 8.22.3)]

$$\lim_{n\to\infty} \frac{\mathcal{L}_{n,m}^{(\gamma)}(z)}{nL_n^{(\gamma)}(z)} = \frac{(-z)^{\frac{m}{2}}}{\sqrt{2}}, \quad \text{uniformly on compact subsets of } \mathbb{C} \setminus [0,\infty). \tag{2.40}$$

Observe that the functions $\frac{\mathcal{L}_{n,m}^{(\gamma)}(z)}{nL_n^{(\gamma)}(z)}$ with $n \in \mathbb{Z}_+$ and $z^{\frac{m}{2}}$ are analytic on $\overline{\mathbb{C}} \setminus [0,\infty)$, as $z^{\frac{m}{2}} \neq 0$ if $z \in \overline{\mathbb{C}} \setminus [0,\infty)$.

□

Chapter

3 Iterated integrals of Orthogonal Polynomials on an arc of the Unit Circle

3.1 Introduction

Let $\mathbb{A} = \mathbb{A}(\theta_0, \theta_1) = \{e^{it} : \theta_0 \leq t \leq \theta_1, 0 \leq \theta_0 < \theta_1 < 2\pi\}$ be a proper closed arc of the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ (i.e. $\mathbb{A} \neq \mathbb{T}$) and $\mathcal{M}(\mathbb{A})$ be the linear space of all probability measure on $\mathbb{A}$, whose support is an infinite set (i.e. a nontrivial probability measure on $\mathbb{A}$). We exclude the case $\theta_0 = \theta_1$ because it degenerates into a proper arc with one point, and it is not interesting because then we would get a discrete measure.

Let us denote by $\phi_n(z) = z^n + \cdots$ the nth monic orthogonal polynomial with respect to $\sigma \in \mathcal{M}(\mathbb{A})$, which is uniquely determined by the relations

$$\int_{\mathbb{A}} \phi_n(t)\, t^{-k}\, d\sigma(t) = 0\,, \quad k = 0, 1, 2, \ldots, n-1\,, \tag{3.1}$$

so in the Hilbert space $L^2_\sigma(\mathbb{A})$, if $n \neq m$ then

$$\langle \phi_n, \phi_m \rangle = \int_{\mathbb{A}} \phi_n(t)\, \overline{\phi_m(t)}\, d\sigma(t) = 0 \tag{3.2}$$
$$\text{and } \|\phi_n\| = \sqrt{\langle \phi_n, \phi_n \rangle} > 0.$$

As far as the zeros of ϕ_n are concerned, it is known that they lie in the interior of the convex hull of $\mathbb{A}$, and if K is a closed set such that $K \cap \mathbb{A} = \emptyset$, then the number of zeros of ϕ_n on K is uniformly bounded in n (cf. [79, Th. 2.2] and [96, Lemma 4]).

We say that a measure $\sigma \in \mathcal{M}(\mathbb{A})$ is *regular* if

$$\lim_{n\to\infty} \sqrt[n]{\|\phi_n\|} = \mathbf{cap}\,(\mathbb{A}) = \sin\left(\frac{\theta_1 - \theta_0}{4}\right), \tag{3.3}$$

where the constant $\mathbf{cap}\,(\mathbb{A})$ is the logarithmic capacity of $\mathbb{A}$ (cf. [92, Cor. III.39.2] or [74, Section 5.2, Table 5.1]). From the well known

Erdős-Turán criterion (cf. [88, Th. 4.1.1]), the condition $\sigma' > 0$ a.e. on $\mathbb{A}$ implies that σ is *regular*. We will write that $\sigma \in \mathcal{M}'(\mathbb{A})$ if $\sigma \in \mathcal{M}(\mathbb{A})$ and $\sigma' > 0$ a.e. on $\mathbb{A}$. An important consequence of the condition $\sigma' > 0$ a.e. on $\mathbb{A}$, is that the zeros of $\{\phi_n\}$ concentrate on $\mathbb{A}$ in the sense that for each compact subset $K \subset \Omega = \mathbb{C} \setminus \mathbb{A}$ there exists n' such that for $n \geq n'$, ϕ_n has no zeros lying on K (cf. [7, Lemma 3]).

For each $m \in \mathbb{Z}_+$ and $\lambda \in \mathbb{C}$, let $\Phi_{n,m,\lambda}$ be the monic polynomial of degree $n+m$, such that

$$\frac{d^m \,\Phi_{n,m,\lambda}(z)}{dz^m} = \frac{(n+m)!}{n!}\phi_n(z), \tag{3.4}$$

normalized by the conditions

$$\frac{d^k \,\Phi_{n,m,\lambda}}{dz^k}(\lambda) = 0, \quad k = 1, \ldots, m; \tag{3.5}$$

it is assumed that $\Phi_{n,0,\lambda} = \phi_n$. Clearly, $\Phi_{n,m,\lambda}$ may be written as the m-times iterated integrals

$$\begin{aligned} \Phi_{n,m,\lambda}(z) &= \frac{n!}{(n+m)!} \underbrace{\int_\lambda^z \int_\lambda^{s_m} \cdots \int_\lambda^{s_2}}_{m-\text{times}} \phi_n(s_1)\, ds_1\, ds_2 \cdots ds_m, \\ &= \frac{n!}{(n+m)!\,(m-1)!} \int_\lambda^z (z-s)^{m-1}\phi_n(s)\, ds, \end{aligned} \tag{3.6}$$

where the last equality is obtained from the Cauchy formula for iterated integrals [62, (2.7.2)]. We refer the reader to [16] for a discussion on zero location of basic iterated integrals of a polynomial, normalized so that the constants of integration are all zero.

The aim of this chapter is the study of some algebraic and asymptotic properties of the iterated integral of orthogonal polynomials with respect to a measure supported on an arc of the unit circle and its zeros. The next section contains two extensions of known results from the analytic theory of polynomial, about the location of the zeros of polynomials whose critical points lie on the unit disc. In Section 3.3 we recall some knowledge of the Logarithmic Potential Theory and prove others auxiliary results. The last two sections are devoted to the aforementioned goal, when $\lambda \in \mathbb{A}_\theta$ and $\lambda \in \mathbb{C} \setminus \mathbb{A}_\theta$ respectively.

3.2 Zeros of polynomials with critical points on a disc

The following lemma is a particular case of the theorem [81, Th. 5.2.7] or [72, Th. 3.4.1b].

Lemma 3.2.1. *(Walsh's coincidence Lemma) Let $F(z_1, z_2, \cdots, z_n)$ be a polynomial in $z_1, z_2, \cdots, z_n$ of total degree n, symmetric in its variables, and of degree at most one in each of them. Then every disk containing the points $w_1, w_2, \cdots, w_n$, also contains at least one point w_0, such that $F(w_1, w_2, \cdots, w_n) = F(w_0, w_0, \cdots, w_0)$.*

The next lemma is a straightforward consequence of the bisector theorem [81, Th. 5.7.7] or the Grace-Heawood theorem [72, Th. 4.3.1].

Lemma 3.2.2. *(The bisector lemma) Let P be a polynomial of degree greater or equal to two and let z_1 and z_2 be distinct complex numbers such that $P(z_1) = P(z_2) = 0$. Then P' has at least one zero in each of the closed half-planes whose boundary is the mid-perpendicular of the line segment joining z_1 and z_1.*

It is clear, that if we knew the critical points of a polynomial and one of its zeros, the remaining zeros would be uniquely determined. Nonetheless, there are only a few general results about zero location of polynomials in terms of its critical points and a given zero, most of them contained in [72, §4.5].

For $r \in \mathbb{R}_+$ we denote $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$, $\partial\mathbb{D}_r = \{z \in \mathbb{C} : |z| = r\}$ and $\overline{\mathbb{D}}_r = \mathbb{D}_r \cup \partial\mathbb{D}_r$. The next useful theorem is a natural extension of [81, Th. 5.7.8] and the proof is carried out with analogous arguments.

Theorem 3.2.1. *Let P be a polynomial of degree $n \geq 2$ with all its critical points in the closed disc $\overline{\mathbb{D}}_r$, where $r \in \mathbb{R}_+$ are fixed. If $P(\lambda) = P(z) = 0$, with $\lambda, z \in \mathbb{C}$, then*

1. *there exists $w \in \overline{\mathbb{U}}$ such that*
$$z = F_r(w) = 2rw - \overline{\lambda}w^2. \tag{3.7}$$

2. *$|z| \leq 2r + |\lambda|$.*

3. *F_r is univalent on $\overline{\mathbb{U}}$ if and only if $|\lambda| \leq r$.*

Proof. As z and λ are zeros of P, from Lemma 3.2.2, if we draw a straight line ℓ which cuts perpendicularly the segment joining the two zeros at its middle point, then P' has at least one zero in each of the closed half planes in which ℓ divides the complex plane. But, we have assumed that all the zeros of P' lie in $\overline{\mathbb{D}}_r$ and therefore ℓ must intersect $\partial\mathbb{D}_r$. Hence, there exists $u \in \mathbb{T}$ such that $|z - r\,u| = |r\,u - \lambda| = \left|r - \overline{\lambda}\,u\right|$. It follows that there exists $v \in \mathbb{T}$ such that $z - r\,u = v\,\left(r - \overline{\lambda}\,u\right)$, where we have $z = r(u+v) - \overline{\lambda}\,u\,v$. This expresses z as a value of a symmetric linear form in the variables u and v taking their values on $\mathbb{T}$, and therefore in $\overline{\mathbb{U}} = \mathbb{U} \cup \mathbb{T}$. It follows from Lemma 3.2.1 that z is a value of the polynomial obtained by putting $w = u = v$ with $w \in \overline{\mathbb{U}}$, which establishes (3.7) and the inequality in 2. as an immediate consequence.

If $\lambda = 0$ then obviously F_r is univalent. Assume that $\lambda \neq 0$, if there exist $w_1, w_2 \in \overline{\mathbb{U}}$ such that $w_1 \neq w_2$ and $F_r(w_1) = F_r(w_2)$, we get that $w_1 + w_2 = 2(r/\overline{\lambda})$. Therefore, F_r is univalent on $\overline{\mathbb{U}}$ if and only if $|\lambda| \leq r$ and we get the third statement. □

Remark 3.2.1. *Let us mention some important consequences of Theorem 3.2.1, that are of general interest. Under the above assumptions, the possible region of zeros of P is the set $F_r(\overline{\mathbb{U}})$ and if $|\lambda| \leq r$ then F_r maps $\mathbb{T}$ onto a Jordan curve (for $r = 1$ see Figure 3.1).*

Corollary 3.2.1.1. *Given two integers $n, m > 0$ and $\lambda \in \mathbb{C}$, let $\rho = 2^m(|\lambda| + 1) - |\lambda|$. Then all the zeros of the polynomials $\Phi_{n,m,\lambda}$ lie in the closed disc $\overline{\mathbb{D}}_\rho$.*

Proof. For $m = 1$, as all the zeros of ϕ_n lie in $\mathbb{U}$ (i.e. the critical points of $\Phi_{n,1,\lambda}$). Then the assertion follows from Theorem 3.2.1. The rest of the proof runs by induction. □

3.3 Asymptotic behavior preliminaries

For any polynomial q of degree exactly n, we consider $\nu[q] := \frac{1}{n}\sum_{j=1}^{n} \delta_{z_j}$, where $z_1, \ldots, z_n$ are the zeros of q repeated according to their multiplicity, and δ_{z_j} is the Dirac measure with mass one at the point z_j. This is the so called *normalized zero counting measure associated with q.*

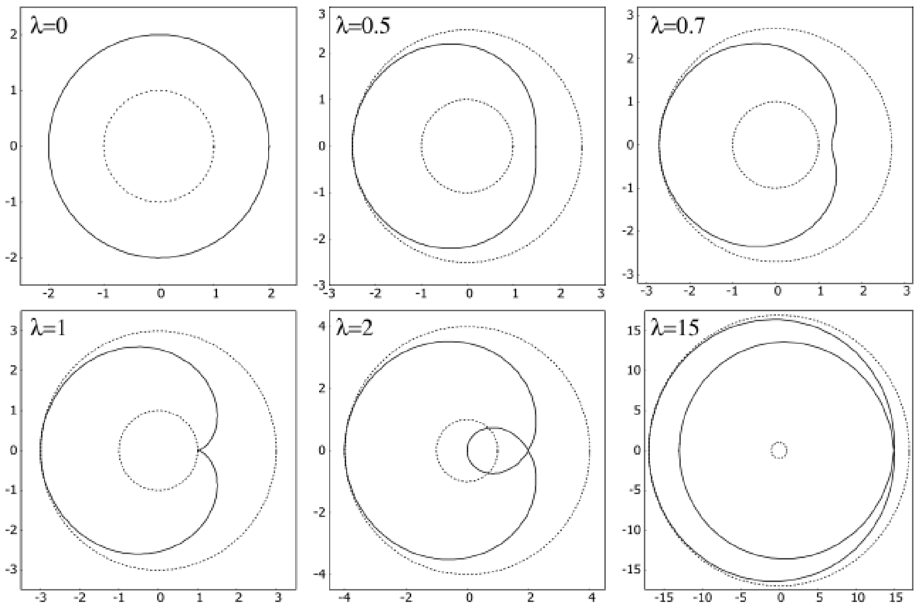


Figure 3.1: The cardioidal curve $F_r(\mathbb{T})$, for $r = 1$ and several values of λ. The interior circle is $\mathbb{T}$ and the exterior one is given by $|z| = 2 + |\lambda|$.

Let $\{\sigma_n\}$ be a sequence of measures in $\mathcal{M}(\mathbb{A})$. If there exits a measure $\sigma \in \mathcal{M}(\mathbb{A})$ such that

$$\lim_{n\to\infty} \int f d\sigma_n = \int f d\sigma,$$

for every continuous function on $\mathbb{A}$, we say that σ_n converges weakly to the measure σ as $n \to \infty$ and write $\text{w-}\lim_{n\to\infty} d\sigma_n = d\sigma$.

Without restriction of generality, from now on, we can assume $\mathbb{A}$ is an arc symmetric with respect to $\mathbb{R}$, i.e., $\mathbb{A} = \mathbb{A}_\theta = \{e^{it} : \theta \leq t \leq 2\pi - \theta, 0 < \theta < \pi\}$. Note that if $\mathbb{A}$ is not symmetric with respect to $\mathbb{R}$, the problem reduces to the symmetric case by a simple change of variables (rotation). From (3.3), it is straightforward that $C_\theta = \textbf{cap}\,(\mathbb{A}_\theta) = \cos(\theta/2)$.

The following lemma summarizes the most important consequence of the regularity (cf. (3.3)) for orthogonal polynomials with respect to a measure $\sigma \in \mathcal{M}'(\mathbb{A}_\theta)$.

Lemma 3.3.1. *Let $0 < \theta < \pi$ fixed, $\sigma \in \mathcal{M}'(\mathbb{A}_\theta)$ and ϕ_n be the nth monic orthogonal polynomial with respect to σ. Then*

1. $\operatorname*{w-lim}_{n\to\infty} d\nu[\phi_n] = d\mu_\theta$, *where*

$$d\mu_\theta(t) = \frac{1}{2\,\pi}\,\frac{\sin(t/2)\;dt}{\sqrt{\cos^2(\theta/2)-\cos^2(t/2)}},\qquad t\in[-\theta,\theta]. \tag{3.8}$$

2. $\lim_{n\to\infty}\sqrt[n]{|\phi_n(z)|} = e^{g_{\Omega_\theta}(z;\infty)}$ *uniformly on compact subset of* $\Omega_\theta = \mathbb{C}\setminus\mathbb{A}_\theta$, *where* $g_{\Omega_\theta}(z;\infty)$ *is the Green function on* Ω_θ *with pole at infinity and explicit formula*

$$\begin{aligned} g_{\Omega_\theta}(z;\infty) &= \log|\ell_\theta(z)|, \quad \textit{where} \qquad (3.9)\\ \ell_\theta(z) &= \frac{1}{2C_\theta}\left(z+1+\sqrt{(z-e^{i\theta})(z-e^{-i\theta})}\right), \end{aligned}$$

is the conformal mapping of $\overline{\mathbb{C}}\setminus\mathbb{A}_\theta$ *onto* $\overline{\mathbb{C}}\setminus\overline{\mathbb{U}}$ *such that* $\ell_\theta(\infty)=\infty$ *and* $\ell'_\theta(\infty)>0$.

Proof. From [83, Th. 3.4] we get the assertion *(1)* and formula (3.8) is given in [8, (3.1)]. The sentence *(2)* is obtained from [83, Th. 3.4] and formula 3.9 is computed by [68, Cor. 9.9] and [32, (14)-(15)]. □

Lemma 3.3.2. *Let* $0<\theta<\pi$, $\Omega_\theta = \mathbb{C}\setminus\mathbb{A}_\theta$, $g_{\Omega_\theta}(z;\infty)$ *be the Green function on* Ω_θ *with pole at infinity and* $d\mu_\theta$ *be the equilibrium measure of* $\mathbb{A}_\theta$. *Then*

$$\psi(z) = \int \frac{d\mu_\theta(w)}{z-w} = \frac{\ell'_\theta(z)}{\ell_\theta(z)}, \tag{3.10}$$

where ℓ_θ *is given by (3.10).*

Proof. Let us define $z=x+iy$ and $w=u+iv$. Obviously the functions $P(z,\mu_\theta)=\int\log(z-w)\,d\mu_\theta(w)$ and $T(z)=\log(\ell_\theta(z))+\log(C_\theta)$ are analytic functions on $\mathbb{C}\setminus\mathbb{A}_\theta$, with real part $\varrho_P(x,y)=\int\log|z-w|\,d\mu_\theta(w)$ and $\varrho_T(x,y)=\log|\ell_\theta(z)|+\log(C_\theta)$, respectively. From the Logarithmic Potential Theory (cf. [80], [88, Appendices A.II-A.V]) and (3.9), we get

$$\varrho_P(x,y) = g_{\Omega_\theta}(z;\infty)+\log(C_\theta) = \varrho_T(x,y).$$

Therefore,

$$\begin{aligned}\psi(z) = \int\frac{d\mu_\theta(w)}{z-w} &= \frac{dP(z,\mu_\theta)}{dz} = \frac{\partial\varrho_P}{\partial x} - i\,\frac{\partial\varrho_P}{\partial y}\\ &= \frac{\partial\varrho_T}{\partial x} - i\,\frac{\partial\varrho_T}{\partial y} = \frac{\ell'_\theta(z)}{\ell_\theta(z)}.\end{aligned}$$

□

Lemma 3.3.3. *For every sequence of polynomials $\{q_n\}$, $\deg q_n \leq n$ and $q_n \neq 0$ we have $\lim_{n\to\infty} \sqrt[n]{\frac{\|q_n\|_{\mathbb{A}_\theta}}{\|q_n\|}} = 1$ (cf. [88, Th. 3.2.3]). Here and in the following, $\|\cdot\|_K$ denotes the supremum norm on the compact set $K \subset \mathbb{C}$.*

We denote by T_n the monic Chebyshev polynomial of degree n for $\mathbb{A}_\theta$, i.e. the unique monic polynomial of degree n such that

$$\|T_n\|_{\mathbb{A}_\theta} = \inf_{Q(z)=z^n+\cdots} \|Q\|_{\mathbb{A}_\theta}. \tag{3.11}$$

Combining [74, Coro. 5.5.5] and [92, Coro. III.9.2], we get the next useful property of the Chebyshev polynomials on a circular arc.

Lemma 3.3.4. *Let $\mathbb{A}_\theta$ be a symmetric closet circular arc of the unit circle and T_n be the corresponding nth monic Chebyshev polynomial, then*

$$\lim_{n\to\infty} \sqrt[n]{\|T_n\|_{\mathbb{A}_\theta}} = C_\theta = \cos(\theta/2). \tag{3.12}$$

3.4 Asymptotic distribution of zeros for $\lambda \in \mathbb{A}_\theta$

Theorem 3.4.1. *If $0 < \theta < \pi$, $\sigma \in \mathcal{M}'(\mathbb{A}_\theta)$ and $\lambda \in \mathbb{A}_\theta$, then for all $m \in \mathbb{Z}_+$ fixed*

$$\lim_{n\to\infty} \sqrt[n]{\|\Phi_{n,m,\lambda}\|_{\mathbb{A}_\theta}} = C_\theta = \cos(\theta/2). \tag{3.13}$$

and

$$\operatorname*{w-lim}_{n\to\infty} d\nu[\Phi_{n,m,\lambda}] = d\mu_\theta, \tag{3.14}$$

where the measure $d\mu_\theta$ is given in (3.8).

Proof. From (3.11)-(3.12)

$$\cos(\theta/2) = \lim_{n\to\infty} \sqrt[n]{\|T_n\|_{\mathbb{A}_\theta}} \leq \lim_{n\to\infty} \sqrt[n]{\|\Phi_{n,m,\lambda}\|_{\mathbb{A}_\theta}}. \tag{3.15}$$

If $m = 0$, then $\|\Phi_{n,0,\lambda}\| = \|\phi_n\| \leq \|T_n\| \leq \|T_n\|_{\mathbb{A}_\theta}$. Hence, combining Lemma 3.3.3, (3.12) and (3.15), we get (3.13).

Assume that $m > 0$ and $z \in \mathbb{A}_\theta$

$$\begin{aligned}
|\Phi_{n,m,\lambda}(z)| &= \frac{n!}{(n+m)!\,(m-1)!}\left|\int_\lambda^z (z-s)^{m-1}\phi_n(s)\,ds\right| \\
&\leq \frac{n!}{(n+m)!\,(m-1)!}\int_\lambda^z |z-s|^{m-1}\,|\phi_n(s)|\ |ds| \\
&\leq \frac{n!\ |1+\lambda|^{m-1}}{(n+m)!\,(m-1)!}\int_\lambda^z |\phi_n(s)|\ |ds| \\
&\leq \frac{2^m\ n!}{(n+m)!\,(m-1)!}\,\|\phi_n\|_{\mathbb{A}_\theta}.
\end{aligned}$$

where the last inequality is given by the maximum modulus principle for holomorphic functions. Hence

$$\lim_{n\to\infty}\sqrt[n]{\|\Phi_{n,m,\lambda}\|_{\mathbb{A}_\theta}} \leq \lim_{n\to\infty}\sqrt[n]{\|\phi_n\|_{\mathbb{A}_\theta}} = \cos(\theta/2)\,. \tag{3.16}$$

Thus, (3.13) is established from (3.12), (3.15) and (3.16).

The arc $\mathbb{A}_\theta$ has empty interior and connected complement. It is well known (cf. [12]) that under such conditions (3.13) implies (3.14). □

Theorem 3.4.2. *Let $0 < \theta < \pi$ fixed, $\sigma \in \mathcal{M}'(\mathbb{A}_\theta)$, $\lambda \in \mathbb{A}_\theta$ and $m \in \mathbb{Z}_+$ fixed. Then*

1. *uniformly on compact subset of Ω_θ*

$$\lim_{n\to\infty}\frac{\phi_n(z)}{n^m\ \Phi_{n,m,\lambda}(z)} = \psi^m(z), \tag{3.17}$$

where the function $\psi(z)$ is given by (3.10).

2. *The zeros of $\{\Phi_{n,m,\lambda}\}$ concentrate on $\mathbb{A}_\theta$ in the sense that for each compact subset $K \subset \Omega_\theta = \mathbb{C}\setminus\mathbb{A}_\theta$ there exists n' such that for $n \geq n'$, $\Phi_{n,m,\lambda}$ has no zeros lying on K.*

Proof. Let $t_{n,k,\lambda}$, with $k = 1,\ldots,n+m-k$, the $n+m-k$ zeros of the polynomial $\Phi_{n,m-k,\lambda}$. From the well known Gauss-Lucas theorem all the critical points of the non–constant polynomials $\Phi_{n,m,\lambda}$ and their derivatives lie in the convex hull of their zeros. Therefore, from Corollary 3.2.1.1, $t_{n,k,\lambda} \in \overline{\mathbb{D}}_\rho$ for $k = 1,\ldots,n+m-j$, where $\rho = 2^{m+1}-1$. Using the partial fraction decomposition, we get

$$\begin{aligned}
\frac{n!\,\phi_n(z)}{(n+m)!\,\Phi_{n,m,\lambda}(z)} &= \left(\frac{n!}{(n+m)!}\right)^2 \prod_{j=0}^{m-1} \frac{\Phi^{(j+1)}_{n,m,\lambda}(z)}{\Phi^{(j)}_{n,m-j,\lambda}(z)} \\
&= \left(\frac{n!}{(n+m)!}\right) \prod_{j=0}^{m-1} \frac{\Phi_{n,m-j-1,\lambda}(z)}{\Phi_{n,m-j,\lambda}(z)} \\
&= \prod_{j=0}^{m-1} \left(\frac{1}{n+m-j} \sum_{k=1}^{n+m-j} \frac{1}{z-t_{n,k,\lambda}}\right) \\
&= \prod_{j=0}^{m-1} \int \frac{d\nu^{(j)}_{n,m,\lambda}(t)}{z-t}.
\end{aligned}$$

where $\nu^{(j)}_{n,m,\lambda} = \nu_n\left[\Phi^{(j)}_{n,m,\lambda}\right]$. Therefore, the family of functions

$$\left\{\frac{\Phi^{(j+1)}_{n,m,\lambda}}{n\,\Phi^{(j)}_{n,m,\lambda}}\right\}, \quad n \in \mathbb{Z}_+ , \tag{3.18}$$

is uniformly bounded on each compact subset of Ω_θ. Hence, from (3.10) and (3.14)

$$\lim_{n\to\infty} \frac{\phi_n(z)}{n^m\,\Phi_{n,m,\lambda}(z)} = \left(\int \frac{d\mu_\theta(w)}{z-w}\right)^m = \psi^m(z) = \left(\frac{\ell'_\theta(z)}{\ell_\theta(z)}\right)^m .$$

As $\sigma \in \mathcal{M}'(\mathbb{A}_\theta)$, the second assertion of the theorem is an immediate consequence of (3.17), taking into account that ψ^m is an analytic function without zeros on Ω_θ and it only has two poles located at the end points of the arc $e^{\pm i\theta}$. $\square$

Now, from Lemma 3.3.1, we obtain the next corollary.

Corollary 3.4.2.1. *Let $0 < \theta < \pi$ fixed, $\sigma \in \mathcal{M}'(\mathbb{A}_\theta)$, $\lambda \in \mathbb{A}_\theta$ and $m \in \mathbb{Z}_+$ fixed. Then uniformly on compact subset of Ω_θ*

$$\lim_{n\to\infty} \sqrt[n]{|\Phi_{n,m,\lambda}(z)|} = |\ell_\theta(z)|. \tag{3.19}$$

3.5 Asymptotic behavior of zeros of iterated integrals with $\lambda \in \Omega_\theta$

For a fixed $0 < \theta < \pi$ we define the arc of the unit circle $\mathbb{A}_\theta = \{e^{it} : \theta \leq t \leq 2\pi - \theta\}$. Let σ be a mesure in $\mathcal{M}'(\mathbb{A}_\theta)$ and $\{\phi_n\}$ be the sequence of monic orthogonal polynomials with respect to σ. For each $m \in \mathbb{Z}_+$ and $\lambda \in \mathbb{C}$, let $\Phi_{n,m,\lambda}$ be the monic polynomial of degree $n+m$, such that

$$\frac{d^m\, \Phi_{n,m,\lambda}(z)}{dz^m} = \frac{(n+m)!}{n!}\phi_n(z), \tag{3.20}$$

normalized by the conditions

$$\frac{d^k\, \Phi_{n,m,\lambda}}{dz^k}(\lambda) = 0, \quad k = 1, \ldots, m; \tag{3.21}$$

where we assume that $\Phi_{n,0,\lambda} = \phi_n$ and if $\lambda = -1$ we denote $\Phi_{n,m} = \Phi_{n,m,-1}$.

As we have already studied the case $\lambda \in \mathbb{A}_\theta$ in Section 3.4, here we assume that $\lambda \in \Omega_\theta = \mathbb{C} \setminus \mathbb{A}_\theta$. From (3.20)-(3.21), $\Phi_{n,m,\lambda}$ can be written in an alternative form of (3.6) as

$$\Phi_{n,m,\lambda}(z) = \Phi_{n,m}(z) - P_m(z), \tag{3.22}$$

where $P_m(z)$ is the Taylor polynomial of degree m of the function $\Phi_{n,m}$ in powers of $(z-\lambda)$. From (3.20) we have that $\frac{\Phi^{(k)}_{n,m}(\lambda)}{k!} = \binom{n+m}{k} \Phi_{n,m-k}(\lambda)$, therefore

$$\begin{aligned} P_m(z) &= P_m(z, \Phi_{n,m}, \lambda) = \sum_{j=0}^{m} \frac{\Phi^{(j)}_{n,m}(\lambda)}{j!} (z-\lambda)^j, \\ &= \sum_{j=0}^{m} \binom{n+m}{j} \Phi_{n,m-j}(\lambda)\,(z-\lambda)^j. \end{aligned} \tag{3.23}$$

Theorem 3.5.1. *If $m > 0$, $0 < \theta < \pi$ and $\lambda \in \Omega_\theta = \mathbb{C} \setminus \mathbb{A}_\theta$ fixed, let $\mathbf{Z}_{\theta,\lambda}$ be the set of all zeros of the polynomials $\{\Phi_{n,m,\lambda}\}$ and $\mathbf{Z}'_{\theta,\lambda}$ be the set of its accumulation points. Then*

1. *uniformly on compact subsets of Ω_θ*

$$\lim_{n\to\infty} \frac{P_m(z)}{n^{2m}\, \Phi_{n,m}(\lambda)} = \frac{(z-\lambda)^m}{m!}. \tag{3.24}$$

2. *uniformly on compact subsets of Ω_θ*

$$\lim_{n\to\infty} |P_m(z)|^{\frac{1}{n}} = |\ell_\theta(\lambda)|. \tag{3.25}$$

3. $\boldsymbol{Z}'_{\theta,\lambda} \subset (\mathbb{A}_\theta \cup \Upsilon(\lambda,\theta))$, *where*

$$\Upsilon(\lambda,\theta) = \left\{ z \in \mathbb{C} : \left| z + 1 + \sqrt{z^2 - 2\cos(\theta) z + 1} \right| = \rho \right\}, \tag{3.26}$$

and $\rho = \rho(\lambda,\theta) = \left| \lambda + 1 + \sqrt{\lambda^2 - 2\cos(\theta)\lambda + 1} \right|$.

The branch of the square root contained in the explicit formula of ℓ_θ is chosen so that $|\ell_\theta(\lambda)| > 1$, for each $0 \leq k \leq m-1$.

Proof. From (3.23) we get

$$\begin{aligned}
\frac{P_m(z)}{n^{2m}\,\Phi_{n,m}(\lambda)} &= \sum_{j=0}^{m} \frac{a_{n,m.j}}{j!}\,(z-\lambda)^j, \qquad \text{where} \\
a_{n,m.j} &= \frac{(n+m)!}{n^{2m}\,(n+m-j)!}\,\frac{\Phi_{n,m-j}(\lambda)}{\Phi_{n,m}(\lambda)}, \\
&= \frac{n^j\,(n+m)!}{n^{2m}\,(n+m-j)!}\,\frac{n^{m-j}\Phi_{n,m-j}(\lambda)}{\phi_n(\lambda)}\,\frac{\phi_n(\lambda)}{n^m\,\Phi_{n,m}(\lambda)}.
\end{aligned}$$

Taking limit as n tends to ∞, we get

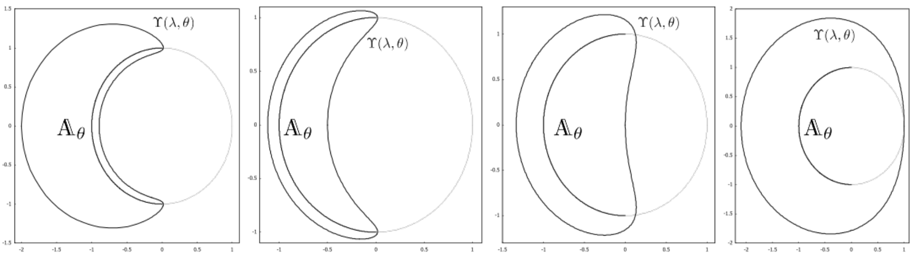
$$\lim_{n\to\infty} a_{n,m.j} = \begin{cases} 1, & \text{if } j = m, \\ 0, & \text{if } 0 \leq j < m, \end{cases} \tag{3.27}$$

which proves (3.24). Finally, we deduce (3.25) from Corollary 3.4.2.1 and (3.24).

As an immediate consequence of Theorem 3.2.1 and Corollary 3.2.1.1, we get that $\mathbf{Z}_{\theta,\lambda}$ is a bounded set. From (3.22) the zeros of the polynomial $\Phi_{n,m,\lambda}$ are on the polynomial lemniscate

$$|\Phi_{n,m}(z)| = |P_m(z)|\ .$$

Hence $|\Phi_{n,m}(z)|^{\frac{1}{n}} = |P_m(z)|^{\frac{1}{n}}$ and taking the limit as $n \to \infty$, from (3.19) and (3.25) we have that the set of accumulation points of zeros of $\{\Phi_{n,m,\lambda}\}$ are on the arc $\mathbb{A}_\theta$ or the curve $\Upsilon(\lambda,\theta)$. $\square$



$\theta = \dfrac{\pi}{2},\ \lambda = -2.$ $\theta = \dfrac{\pi}{2},\ \lambda = -\dfrac{1}{2}.$ $\theta = \dfrac{\pi}{2},\ \lambda = 0.$ $\theta = \dfrac{\pi}{2},\ \lambda = 1.$

Figure 3.2: The curve $\Upsilon(\lambda, \theta)$, for $\theta = \pi/2$ and several values of λ. The gray circle is $\mathbb{T}$ and the arc is $\mathbb{A}_\theta$.

Example 3.5.1. *By way of illustration, here is an example of the curve $\Upsilon(\lambda, \theta)$ defined in (3.26) when $\theta = \pi/2$. Obviously, for $\lambda \in \Omega_{\pi/2}$ fixed*

$$\Upsilon(\lambda, \pi/2) = \{z \in \mathbb{C} : \left|z + 1 + \sqrt{z^2+1}\right| = \rho_\lambda\},$$

where $\rho_\lambda = \left|\lambda + 1 + \sqrt{\lambda^2+1}\right|$. It is straightforward to see that there exists $t \in [0, 2\pi]$ such that

$$z + \sqrt{z^2+1} = \rho e^{it} - 1 = \frac{1}{\sqrt{z^2+1} - z},$$

and therefore $2z = \rho e^{it} - 1 - \dfrac{1}{\rho e^{it} - 1} = \dfrac{\rho^2 e^{i2t} - 2\rho e^{it}}{\rho e^{it} - 1} = \dfrac{\rho(\rho e^{it} - 2)}{\rho - e^{-it}}$. Hence, $\Upsilon(\lambda, \pi/2)$ is given by the parametric equation

$$z = \frac{\rho(\rho e^{it} - 2)(\rho - e^{it})}{2(\rho^2 - 2\cos(t) + 1)},$$

with $t \in [0, 2\pi]$ (for $\lambda = -2,\ -\frac{1}{2},\ 0$ or 1 see Figure 3.2).

CHAPTER

4 Edge Detection Based on Krawtchouk Polynomials

4.1 Introduction

Edge detection plays a relevant role in digital image processing algorithms for many different fields of application. From computer vision applications in the industry and medical fields [97] to 3D video and image coding [61], visual object contour detection is one of the functions based on edge detection, which is very often required as part of more complex operations. For instance, image segmentation for visual object identification and recognition, definition of regions of interest within a visual scene for selective coding, inspection or attention-based processing, all require fast and efficient edge detection algorithms [64].

There are various methods for edge detection in digital images based on a fairly consolidated theory (c.f. [69, Ch. 15] and [85, §4.5–§4.6]). Most of the edge extraction techniques operate a predefined format of digital image representation (e.g. RGB, gray-scale, etc) using differential operators to find relevant transitions in the image intensity. Such transitions represent edges, which are defined as the borders of either visual objects or image regions, also establishing the boundaries between overlapping objects or different regions.

In many applications the relevant region boundaries, i.e., edges, are found on the luminance component of images, which requires a gray-scale representation, even though other formats might by used (e.g. RGB). Since the majority of important low-level feature related information exists in gray-scale images, such as edges, smooth regions, textures and so on, in this work we only consider gray-scale representation images. The method for edge detection in gray-scale images proposed in this paper is based on approximating the derivatives of the function image using the Krawtchouk orthogonal polynomials properties.

The structure of this chapter is as follows. In the next section we present the theoretical framework and the basic properties of the Krawtchouk polynomials in one variable. In Section 4.3 we establish the Krawtchouk polynomials in two variables which are used to approximate the image in the sense of least squares. In Section 4.4 we explain how we analyze the image by blocks and for the numerical experiments we consider a particular cases in order to obtain close formulas. In Section 4.5, we describe our algorithm of edge detection based on the Krawtchouk polynomials. Finally in the last Section we present experimental results and concluding remarks of the proposed method.

4.2 Krawtchouk polynomials in one variable

This work relies on well known references often cited to establish the basic theory of orthogonal polynomials namely [22, 89]. However these monographs essentially deal with orthogonal polynomials with respect to a continuous inner product whilst for the purpose of our work we are focused on the discrete case as required by the type of data we are dealing with, i.e., digital images. As far as the theory of discrete orthogonal polynomials and applications is concerned, we suggest the references [4, 22, 30, 42, 59, 60] or [5, Ch. 5].

Although we already gave the general definitions of discrete inner product and discrete orthogonal polynomials in Section 1.1.2, it is convenient to state those definitions once again:

Let $N \in \mathbb{N}$, $\Lambda := \{x_0, x_1, \ldots, x_N\} \subset \mathbb{R}$, where $x_0 < x_1 < \ldots < x_N$, $\mathbf{F}(\Lambda)$ be the set of all real functions on Λ, $\mathbb{P}$ be the set of all real coefficient polynomials and $\mathbb{P}_N \subset \mathbb{P}$ be the set of polynomials of degree at most N. Note that any real function of a discrete variable $f \in \mathbf{F}(\Lambda)$ can be seen as the restriction on Λ of a number of functions of real variable, in particular the Lagrange interpolation polynomial $P \in \mathbb{R}_N$ such that $P(x_i) = f(x_i)$ for $i = 0, 1, \ldots, N$. Then we have a natural identification between the sets $\mathbf{F}(\Lambda)$ and $\mathbb{P}_N$.

We call *weight function* (or simply *weight*) to any positive function μ on Λ and we say that it is *normalized* when $\sum_{k=0}^{N} \mu(x_k) = 1$.

Let the pair (Λ, μ), where μ is a weight defined on Λ. The inner product on $\mathbb{P}_N$ associated to (Λ, μ) is defined by:

$$\langle f, g \rangle_{\Lambda,\mu} = \sum_{k=0}^{N} f(x_k) g(x_k) \mu(x_k), \quad f, g \in \mathbb{P}_N. \tag{4.1}$$

with a corresponding norm $\|f\|_{\Lambda,\mu} = \sqrt{\langle f, f \rangle_{\Lambda,\mu}}$.

A family of polynomials $\{p_{N,k}\}_{k=0}^{m}$ with $m \leq N$ is *orthogonal* with respect to the inner product (4.1) if p_k is a polynomial of degree k with positive leading coefficient and

$$\langle p_{N,n}, p_{N,m}\rangle_{\Lambda,\mu} = \begin{cases} \neq 0 & \text{if} \quad n = m, \\ = 0 & \text{if} \quad n \neq m. \end{cases} \tag{4.2}$$

If $\|p_{N,k}\|_{\Lambda,\mu} = 1$ for all $0 \leq k \leq N$, the family $\{p_{N,k}(x)\}_{k=0}^{m}$ is called *orthonormal* with respect to (4.1).

Note that (4.2) is equivalent to the condition

$$\left\langle p_n, x^j \right\rangle_{\Lambda,\mu} = \sum_{k=0}^{N} p_n(x_k) x_k^j \mu_k = 0, \quad \text{for} \quad j = 0, 1, \ldots n. \tag{4.3}$$

Then given a weight function μ, the relations (4.3) form a system of N equations with $N+1$ unknowns (the coefficients of polynomial), which determine the orthogonal polynomial $p_n(x)$ except for a multiplicative constant which may depend on n.

In this chapter we find a new form to obtain approximations to the derivatives in each point of the image making use of a family of discrete orthogonal polynomials called Krawtchouk polynomials, that are orthogonal with respect to the binomial distribution (c.f. [5, §5.4]). In the remainder of the section, we state some definitions and properties about the monic Krawtchouk polynomials in one variable.

Definition 4.2.1. *Let* $N \in \mathbb{N}$, $\Lambda_N = \{0, 1, 2, \ldots, N\}$, $\alpha \in]0, 1[$ *and* $w_{N,\alpha}(x)$ *the weight function*

$$w_{N,\alpha}(x) = \binom{N}{x} \alpha^x (1-\alpha)^{N-x}, \quad \textit{for all } x \in \Lambda_N. \tag{4.4}$$

We say that $\kappa_n^\alpha(x, N) = x^n + \ldots$, *with* $n \leq N$, *is the* nth monic Krawtchouk polynomial *with respect to the pair* $(\Lambda_N, w_{N,\alpha})$ *if*

$$\left\langle \kappa_n^\alpha(\cdot, N), x^j \right\rangle_{N,\alpha} = \sum_{i=0}^{N} \kappa_n^\alpha(i, N)\, x^j\, w_{N,\alpha}(i) = 0,$$

for all $j = 0, 1, \ldots, N$, *where* $\langle \cdot, \cdot \rangle_{N,\alpha} = \langle \cdot, \cdot \rangle_{\Lambda_N, w_{N,\alpha}}$.

Obviously, from the binomial theorem, the weight function (4.4) is normalized. The nth monic Krawtchouk polynomial in one variable can be generated by the formula (c.f. [5, (5.4.3)])

$$\kappa_n^\alpha(x, N) = \sum_{j=0}^{n} \binom{n}{j} \alpha^{n-j} (1-\alpha)^j (x-N)_{n-j} (x-j+1)_j, \tag{4.5}$$

where $(a)_j$ denotes the Pochhammer symbol or shifted factorial as in [5, (1.1.8)].

From [5, (5.2.8)], we obtain the norms of the nth polynomials (4.5)

$$\|\kappa_n^\alpha(\cdot, N)\|_{N,\alpha}^2 = \sum_{i=0}^{N} \left(\kappa_n^\alpha(i, N)\right)^2 \; w_{N,\alpha}(i) = \binom{N}{n} n!^2 (\alpha - \alpha^2)^n .$$

If f is a function of one variable, we define the differences of first order as $\triangle_+ f(x) = f(x+1) - f(x)$, $\triangle f(x) = \frac{1}{2}\left(f(x+1) - f(x-1)\right)$, where the differences $\triangle f$ of first order is the usual central-differences formulas on 2 nodes (c.f. [56, Table 6.3]). For a function of two variables $\mathbf{f}$ we define the partial differences of first order as:

$$\left.\begin{array}{rcl} \triangle_x \mathbf{f}(x, y) & = & \dfrac{\mathbf{f}(x+1, y) - \mathbf{f}(x-1, y)}{2}, \\ \triangle_y \mathbf{f}(x, y) & = & \dfrac{\mathbf{f}(x, y+1) - \mathbf{f}(x, y-1)}{2}. \end{array}\right\} \tag{4.6}$$

Most of the results contained in this chapter can be obtained analogously for other families of discrete orthogonal polynomials (see [5, Ch. 5]). However, we use the Krawtchouk polynomials because they allow to obtain closed expressions for the discrete derivatives (differences) of the polynomials in one and two variables, respectively. The next proposition is straightforward from the basic properties of Krawtchouk polynomials in [5, §5.4]. It includes some of the aforementioned closed expressions.

Proposition 4.2.1. *The monic Krawtchouk polynomial, with $\alpha \in]0, 1[$, satisfies the following relations:*

$$\triangle_+ \kappa_n^\alpha(x, N) = n\, \kappa_{n-1}^\alpha(x, N-1). \tag{4.7}$$

$$\triangle \kappa_n^\alpha(x, N) = \frac{n}{2}\left(\kappa_{n-1}^\alpha(x, N-1) + \kappa_{n-1}^\alpha(x-1, N-1)\right). \tag{4.8}$$

Proof. From [5, (5.4.4)], we get the relations (4.7) for the forward difference. The central difference $\triangle \kappa_n^\alpha(x, N)$ in (4.8), is a direct consequences of (4.7). □

4.3 Krawtchouk polynomials in two variables

A gray-scale image with resolution $(N_1+1)\times(N_2+1)$ pixels ($N_1, N_2 \in \mathbb{N}$) can be considered as a function of two variables $\mathbf{I}(x, y)$ defined on the set $\Lambda_{N_1} \times \Lambda_{N_2}$, where $\Lambda_{N_1} = \{0, 1, \ldots, N_1\}$ and $\Lambda_{N_2} = \{0, 1, \ldots, N_2\}$, i.e.

$$\begin{array}{rcl} \mathbf{I}: \; \Lambda_{N_1} \times \Lambda_{N_2} & \longrightarrow & [0,1], \\ (x,y) & \longrightarrow & \mathbf{I}(x,y). \end{array}$$

Hence, the values of $\mathbf{I}$ on $\Lambda_{N_1} \times \Lambda_{N_2}$ can be represented by the matrix $\mathcal{I}$ of order $(N_1+1) \times (N_2+1)$

$$\mathcal{I} = \begin{bmatrix} \mathbf{I}(0,0) & \mathbf{I}(0,1) & \cdots & \mathbf{I}(0,N_2) \\ \mathbf{I}(1,0) & \mathbf{I}(1,1) & \cdots & \mathbf{I}(1,N_2) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{I}(N_1,0) & \mathbf{I}(N_1,1) & \cdots & \mathbf{I}(N_1,N_2) \end{bmatrix}. \tag{4.9}$$

Let $\mathbb{P}_{N_1,N_2}$ be the linear space of polynomials in the variables x and y, of degree at most N_1 and N_2 respectively. To study an image as a polynomial in two variables, we need to introduce the Krawtchouk polynomials in two variables or bivariate Krawtchouk polynomials.

Definition 4.3.1. *Let* $N_1, N_2 \in \mathbb{N}$, $\alpha_1, \alpha_2 \in]0,1[$, $\Lambda_{N_1} = \{0, \ldots, N_1\}$ *and* $\Lambda_{N_2} = \{0, \ldots, N_2\}$. *We call* Two-dimensional Krawtchouk polynomials or 2D monic Krawtchouk polynomials *to the polynomial in two variables* $\mathbf{K}_{n,m}^{\alpha_1,\alpha_2}(x,y) = \kappa_n^{\alpha_1}(x,N_1)\,\kappa_m^{\alpha_2}(y,N_2)$, *where* $(x,y) \in \Lambda_{N_1} \times \Lambda_{N_2}$.

Note that the set of 2D monic Krawtchouk polynomials

$$\{\mathbf{K}_{n,m}^{\alpha_1,\alpha_2}\} = \{\kappa_n^{\alpha_1}(\cdot, N_1)\} \otimes \{\kappa_m^{\alpha_2}(\cdot, N_2)\},$$

where the symbol $\otimes$ denotes the tensor product of the set of polynomials $\{\kappa_n^{\alpha_1}(\cdot, N_1)\}$ and $\{\kappa_m^{\alpha_2}(\cdot, N_2)\}$ as in [75, §12-3]. The 2D monic Krawtchouk polynomials are orthogonal with respect to the following inner product on $\mathbb{P}_{N_1,N_2}$

$$\langle \mathbf{f}, \mathbf{g} \rangle_{2D} = \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} \mathbf{f}(x_i, x_j)\mathbf{g}(x_i, x_j) w_{N_1,\alpha_1}(x_i)\, w_{N_2,\alpha_2}(x_j). \tag{4.10}$$

(see [75, Lemma 12-1]). Furthermore

$$
\begin{aligned}
\langle \mathbf{K}_{n,m}^{\alpha_1,\alpha_2}, \mathbf{K}_{r,s}^{\alpha_1,\alpha_2} \rangle_{2D} &= \sum_{i=0}^{N_1}\sum_{j=0}^{N_2} \mathbf{K}_{n,m}^{\alpha_1,\alpha_2}(x_i,x_j)\mathbf{K}_{r,s}^{\alpha_1,\alpha_2}(x_i,x_j) \\
&\quad \cdot w_{N_1,\alpha_1}(x_i)\, w_{N_2,\alpha_2}(x_j) \\
&= \left(\sum_{i=0}^{N_1} \kappa_n^{\alpha_1}(x_i,N_1)\kappa_r^{\alpha_1}(x_i,N_1) w_{N_1,\alpha_1}(x_i) \right) \\
&\quad \left(\sum_{j=0}^{N_2} \kappa_m^{\alpha_2}(x_j,N_2)\kappa_s^{\alpha_2}(x_j,N_2) w_{N_2,\alpha_2}(x_j) \right) \\
&= \langle \kappa_n^{\alpha_1}, \kappa_r^{\alpha_1} \rangle_{N_1,\alpha_1} \langle \kappa_m^{\alpha_2}, \kappa_s^{\alpha_2} \rangle_{N_2,\alpha_2} \\
&= \begin{cases} 0, & |n-r|+|m-s|>0, \\ \|\mathbf{K}_{n,m}^{\alpha_1,\alpha_2}\|_{2D}^2 > 0, & |n-r|+|m-s|=0, \end{cases}
\end{aligned}
$$

where $\|\mathbf{f}\|_{2D} = \sqrt{\langle \mathbf{f}, \mathbf{f} \rangle_{2D}}$.

For the 2D monic Krawtchouk polynomials we have the following finite difference formulas:

$$
\begin{aligned}
\triangle_x \mathbf{K}_{n,m}^{\alpha_1,\alpha_2}(x,y) &= \left(\triangle_x \kappa_n^{\alpha_1}(x,N_1)\right) \kappa_m^{\alpha_2}(y,N_2). \\
\triangle_y \mathbf{K}_{n,m}^{\alpha_1,\alpha_2}(x,y) &= \left(\triangle_y \kappa_m^{\alpha_2}(y,N_2)\right) \kappa_n^{\alpha_1}(x,N_1).
\end{aligned}
\tag{4.11}
$$

From the standard theory of approximation of functions (c.f. [75, Ch. 12]), for $M_1 \in \Lambda_{N_1} \setminus \{0\}$ and $M_2 \in \Lambda_{N_2} \setminus \{0\}$, the polynomial of total degree $(M_1-1) \times (M_2-1)$

$$
\mathbf{P}_{M_1,M_2}(x,y) = \sum_{n=0}^{M_1-1} \sum_{m=0}^{M_2-1} \beta_{n,m} \mathbf{K}_{n,m}^{\alpha_1,\alpha_2}(x,y), \tag{4.12}
$$

with $\beta_{n,m} = \dfrac{\langle \mathbf{I}, \mathbf{K}_{n,m}^{\alpha_1,\alpha_2} \rangle_{2D}}{\langle \mathbf{K}_{n,m}^{\alpha_1,\alpha_2}, \mathbf{K}_{n,m}^{\alpha_1,\alpha_2} \rangle_{2D}}$, is such that

$$
\min_{\mathbf{Q} \in \mathbb{P}_{M_1,M_2}} \|\mathbf{I} - \mathbf{Q}\|_{2D} = \|\mathbf{I} - \mathbf{P}_{M_1,M_2}\|_{2D},
$$

i.e. $\mathbf{P}_{M_1,M_2}$ is the polynomial of least square approximation of $\mathbf{I}$ in $\mathbb{P}_{M_1,M_2}$ and we write $\mathbf{I}(x,y) \approx \mathbf{P}_{M_1,M_2}(x,y)$. Furthermore, if $M_1 = N_1 + 1$ and $M_2 = N_2 + 1$, then $\mathbf{I} = \mathbf{P}_{N_1,N_2}$.

4.4 Computation of the discrete derivative by blocks

In order to detect the edges points, we analyze the entire image $\mathcal{I}$, by blocks $\mathcal{I}_{i,j}$ of fixed-size $(n_1+1)\times(n_2+1)$, where $n_1 < N_1$ and $n_2 < N_2$. We recall that the blocks are all of the same size.

First, let us introduce some notations and definitions. If $\mathcal{A} = [a_{ij}]$ is a $u \times v$ real matrix, then $\mathcal{A}^T$ denotes the transpose matrix and $\operatorname{vec}(\mathcal{A})$ is the column vector of $(u\,v)$ entries defined as

$$\operatorname{vec}(\mathcal{A}) = (a_{11}, \ldots, a_{u1}, a_{12}, \ldots, a_{u2}, \ldots, a_{1v}, \ldots, a_{uv})^T.$$

Let $\overline{a}_n = (\mathbf{a}_n(0), \ldots, \mathbf{a}_n(n_1))$ and $\overline{b}_m = (\mathbf{b}_m(0), \ldots, \mathbf{b}_m(n_2))$ be two row vectors of order (n_1+1) and (n_2+1) respectively, where

$$\mathbf{a}_n(\nu) = \frac{\kappa_n^{\alpha_1}(\nu, n_1)\; w_{n_1,\alpha_1}(\nu)}{\|\kappa_n^{\alpha_1}\|^2_{n_1,\alpha_1}}, \quad \mathbf{b}_m(\nu) = \frac{\kappa_m^{\alpha_2}(\nu, n_2)\; w_{n_2,\alpha_2}(\nu)}{\|\kappa_m^{\alpha_2}\|^2_{n_2,\alpha_2}}.$$

The matrix $\mathcal{C}_{n_1,n_2}(n,m) = \overline{a}_n^T\,\overline{b}_m$ is of order $(n_1+1)\times(n_2+1)$ and only depends on the size of the blocks, not on its entries.

Let $\beta_{n,m}(i,j)$ be the coefficient given by (4.12), considering $\mathcal{I} = \mathcal{I}_{i,j}$. This coefficient can be computed as follows

$$\beta_{n,m}(i,j) = \langle \operatorname{vec}(\mathcal{I}_{i,j}), \operatorname{vec}(\mathcal{C}_{n_1,n_2}(n,m))\rangle_2\,, \tag{4.13}$$

where $\langle\cdot,\cdot\rangle_2$ is the usual Euclidean inner product on $\mathbb{R}^{(n_1+1)(n_2+1)}$.

Let $\mathcal{B}_{n,m}$ be the matrix of all coefficients $\beta_{n,m}(i,j)$, with $i = 0, \ldots, n_1$ and $j = 0, \ldots, n_2$. From (4.13), $\mathcal{B}_{n,m} = \mathcal{I} * \mathcal{C}_{n_1,n_2}(n,m)$, where the symbol $*$ indicates the 2-D discrete convolution of matrices (c.f. [38, §15.1.4]). Using the discrete Fourier transform, the convolution of these matrices can be optimized, improving significantly the CPU time (c.f. [38, Ch. 15]).

For each pixel (i,j) of the image $\mathcal{I}$, we compute the discrete partial derivative (4.6) of the approximation (4.12), only considering the information contained in the "neighborhood" $\mathcal{I}_{i,j}$. When this process is finished, we obtain two matrices $\mathcal{P}_x$ and $\mathcal{P}_y$ of the same size of $\mathcal{I}$, where each entry (i,j) is the partial derivatives with respect to x or y. In the next section we will see that, these matrices allow us to have a good estimate of the modulus of the gradient at each point (i,j) of $\mathcal{I}$.

In order to carry out numerical experiments we consider the parameters of Krawtchouk polynomials $\alpha_1 = \alpha_2 = \alpha = \frac{1}{2}$, and $n_1 = n_2 =$

$n_t = 4$, i.e. we approximate the discrete derivatives by considering the information of the block $\mathcal{I}_{i,j}$ of order 5×5, with center on the entry $\mathbf{I}(i,j)$:

$$\mathcal{I}_{i,j} = \begin{bmatrix} \mathbf{I}(i-2,j-2) & \mathbf{I}(i-2,j-1) & \mathbf{I}(i-2,j) & \mathbf{I}(i-2,j+1) & \mathbf{I}(i-2,j+2) \\ \mathbf{I}(i-1,j-2) & \mathbf{I}(i-1,j-1) & \mathbf{I}(i-1,j) & \mathbf{I}(i-1,j+1) & \mathbf{I}(i-1,j+2) \\ \mathbf{I}(i,j-2) & \mathbf{I}(i,j-1) & \mathbf{I}(i,j) & \mathbf{I}(i,j+1) & \mathbf{I}(i,j+2) \\ \mathbf{I}(i+1,j-2) & \mathbf{I}(i+1,j-1) & \mathbf{I}(i+1,j) & \mathbf{I}(i+1,j+1) & \mathbf{I}(i+1,j+2) \\ \mathbf{I}(i+2,j-2) & \mathbf{I}(i+2,j-1) & \mathbf{I}(i+2,j) & \mathbf{I}(i+2,j+1) & \mathbf{I}(i+2,j+2) \end{bmatrix},$$

From (4.12) with $M_1 = M_2 = M_t$, we have for $i = 0, \ldots, N_1$ and $j = 0, \ldots, N_2$ the polynomial block approximation:

$$\mathbf{P}^{(i,j)}_{M_t,M_t}(x,y) \approx \mathbf{I}(i+x-2, j+y-2), \; (x,y) \in \Lambda_{n_t} \times \Lambda_{n_t}, \tag{4.14}$$

where for each fixed point (i,j) the polynomial $\mathbf{P}^{(i,j)}_{M_t,M_t}(x,y)$ is given by (4.12), and the coefficients $\beta_{n,m}(i,j)$ by (4.13).

Now taking into account the fact that for $x = y = 2$ in the approximation (4.14) we stay right at the point $\mathbf{I}(i,j)$, the center of the block $\mathcal{I}_{i,j}$, then we can compute the first order partial differences of $\mathbf{P}^{(i,j)}_{M_t,M_t}(x,y)$ using the central-difference formula for Krawtchouk polynomials (4.8) together with (4.11). For example, for $M_t = 2$ we have

$$\left.\begin{array}{rcl} \triangle_x \mathbf{P}^{(i,j)}_{M_t,M_t}(2,2) & = & \beta_{1,0}(i,j) - \beta_{1,2}(i,j), \\ \triangle_y \mathbf{P}^{(i,j)}_{M_t,M_t}(2,2) & = & \beta_{0,1}(i,j) - \beta_{2,1}(i,j), \end{array}\right\} \tag{4.15}$$

and for $M_t = 4$

$$\begin{array}{rcl} \triangle_x \mathbf{P}^{(i,j)}_{M_t,M_t}(2,2) & = & \beta_{1,0}(i,j) - \beta_{1,2}(i,j) \\ & & +\frac{3}{2}\left(\beta_{3,2}(i,j) - \beta_{3,0}(i,j)\right), \\ \triangle_y \mathbf{P}^{(i,j)}_{M_t,M_t}(2,2) & = & \beta_{0,1}(i,j) - \beta_{2,1}(i,j) \\ & & +\frac{3}{2}\left(\beta_{2,3}(i,j) - \beta_{0,3}(i,j)\right). \end{array} \tag{4.16}$$

From (4.15) and (4.16), you can see that for the computation of $\mathcal{P}_x$ and $\mathcal{P}_y$ it is not necessary to compute all the matrices $\mathcal{B}_{n,m}$. In fact, by (4.16) we have that the matrices $\mathcal{P}_x$ and $\mathcal{P}_y$, of the partial derivatives with respect to x or y are respectively:

$$\mathcal{P}_x = \mathcal{B}_{1,0} - \mathcal{B}_{1,2} + \frac{3}{2}\left(\mathcal{B}_{3,2} - \mathcal{B}_{3,0}\right), \; \mathcal{P}_y = \mathcal{B}_{0,1} - \mathcal{B}_{2,1} + \frac{3}{2}\left(\mathcal{B}_{2,3} - \mathcal{B}_{0,3}\right).$$

In order to ensure that all the image boundary pixels are analyzed, we 'fill in' the missing pixels within the convolution operation, by mirroring the values that are inside the limits of the image $\mathcal{I}$ across the array border.

4.5 Edge Detection Based on Krawtchouk Polynomials

The proposed edge detection algorithm is based on the results of the previous sections and is described as follows:

1.- *Compute the matrix of gradient magnitude.*

Once we obtain the matrices $\mathcal{P}_x$ and $\mathcal{P}_y$, we compute the modulus of the gradient $\mathbf{G}(i,j)$ at each point (i,j) (edge strength).

$$\mathbf{G}(i,j) = \sqrt{\mathcal{P}_x^2(i,j) + \mathcal{P}_y^2(i,j)}.$$

2.- *Find the first threshold and the strong edge points.*

We compute the first level of adaptive threshold by:

$$\begin{aligned} \tau_{h_1} &= \text{mean}(\mathbf{G}(i,j)) + k \times \text{standard deviation}(\mathbf{G}(i,j)), \\ &\quad \text{where } k \in \mathbb{R}^+. \end{aligned}$$

If $\mathbf{G}(i,j) > \tau_{h_1}$, the point (i,j) is declared as a *strong edge point.*

3.- *Compute the second level of threshold and the weak edge points.*

Now we consider only the points (i_1, j_1) for which

$$\text{mean}(\mathbf{G}(i,j)) < \mathbf{G}(i_1,j_1) < \tau_{h_1},$$

and compute the second threshold $\tau_{h_2} < \tau_{h_1}$ by

$$\tau_{h_2} = \text{mean}(\mathbf{G}(i_1,j_1)) + k \times \text{standard deviation}(\mathbf{G}(i_1,j_1)).$$

If $\mathbf{G}(i_1,j_1) > \tau_{h_2}$, the point (i,j) is declared as a *weak edge point.*

4.- *Declaration of edge points.*

4.1.- Each strong edge point is considered an edge point.

4.1.- A weak edge point is considered an edge point if at least some of its eight neighboring pixels is a strong edge point.

5.- *Apply Morphological operations.*

The matrix E_{h_2} of edge points obtained by the proposed scheme, is a matrix in which in general the edge points tend to be thick and non-continuous. Then in order to avoid these effects, it is necessary to apply morphological operations [69, Ch. 14], which can be defined as combination of the two basics operations dilation and erosion [45]. Hence the final edge image E is obtained by performing first the thinning operation and then the linked operation.

4.6 Experimental results

4.6.1 Edge quality evaluation

In order to apply a quantitative approach to measure the effectiveness of the proposed edge detection method, we use the statistical error measures considered in [49], where the pixels in the candidate edge image are denoted by: True Positive (TP), False Positive (FP), True Negative (TN) and False Negative (FN). Using this classification, we define the measures of the quality of an edge image: $\phi(E, E_q)$, $\chi^2(E, E_q)$ and $F_\delta(E, E_q)$ as in [49, (11), (12) and (13)], respectively, where E_q is the true edge image. Finally to measure the errors in a edge image we use the following numbers:

$$\begin{aligned}
\phi^*(E, E_q) &= 1 - \phi(E, E_q), \\
\chi^{2*}(E, E_q) &= 1 - \chi^2(E, E_q), \\
F_\delta^*(E, E_q) &= 1 - F_\delta(E, E_q), \quad \text{where} \quad \delta \in [0, 1].
\end{aligned}$$

To compare, we take as a true edge image E_S and E_C which are the edge images given by the Sobel (see [38, Ex. 15.28]) and Canny methods (see [38, §16.4.3]). The results for the peppers image and depth map are displayed in Tables 4.1 and 4.2, where $EM(E, E_q)$ denotes the error measure between the image edge proposed E and $E_q = E_S$ (or $E_q = E_C$).

Images taken from two quite different fields of application were used to demonstrate the effectiveness of the proposed algorithm: (i) natural images used for object detection, surveillance, etc. (ii) depth maps

Measurement	ϕ^*	χ^{2*}	$F^*_{0.25}$	$F^*_{0.5}$	$F^*_{0.75}$
$EM(E, E_S)$	0.3639	0.5865	0.3536	0.3484	0.3432
$EM(E, E_C)$	0.4167	0.7317	0.4408	0.4714	0.4988

Table 4.1: Error measures for the peppers image.

Measurement	ϕ^*	χ^{2*}	$F^*_{0.25}$	$F^*_{0.5}$	$F^*_{0.75}$
$EM(E, E_S)$	0.1493	0.3616	0.1742	0.2002	0.2246
$EM(E, E_C)$	0.2046	0.3987	0.2124	0.2222	0.2318

Table 4.2: Error measures for the depth map image.

currently used in 3D video multimedia services and applications (e.g., depth-plus-video format [63]). The results show that the proposed algorithm is able to detect the edges of different types of images. Both the contours of overlapped objects and identification of foreground objects in depth maps are obtained with quite good accuracy, as shown in the Figures 4.1, 4.2 and Tables 4.1, 4.2.

Figure 4.1: Edge detection on peppers image.

Figure 4.2: Edge detection on depth map image.

CHAPTER

5 Conclusions and Future Research

5.1 Conclusions

In section *Resumen y aportaciones* we present an integrative approach that consider the functions to be studied in this work as solutions of the initial value problem (1). However, we can perfectly identify two parts of new knowledge generated that we explain below:

First Part.

This part is composed of the results contained in chapters 2 and 3, whose central objective is the study of algebraic and analytical properties of families of polynomials obtained by iterated integration until a fixed order $m \in \mathbb{Z}_+$, of families of orthogonal polynomials with respect to a measure supported on the real line or an arc of the unit circle. The results appear in [40, 66, 67].

Theorems 2.4.2 (for Jacobi polynomials) and 3.5.1 (for orthogonality on an arc) were first enunciated in the classical reference [16, Th. 2], when $\alpha = \beta = 0$ (Legendre polynomials) with $\omega_0 = \omega_1 = \cdots = \omega_m = 0$, i.e., $\mathcal{A}_{n,m}$ is a Maclaurin polynomial or equivalently a Taylor's polynomial about zero. The initial motivation of our work was the fourth remark in the last section of [16], where the author write *"It would be interesting to obtain results, analogous to Theorem [16, Th. 2], for these polynomials"* referring to the Gegenbauer (or ultraspherical) polynomials. Our result is an extension of [16, Th. 2] for iterated integral of Jacobi polynomials normalized by Abel-Goncharov conditions (see (2.4)) and orthogonal polynomials on an arc normalized by Taylor conditions (see (3.5)).

Another interesting result is the comparative asymptotic behavior between primitives and orthogonal polynomials proved in Theorems 2.4.1 (Jacobi case) and equation (3.24) of Theorem 3.5.1, for orthogonality on an arc. The proof of these theorems is based on the one hand in determining beforehand the asymptotic behavior of certain particular primitives as Jacobi fundamental primitives (Section 2.2, Lemma 2.2.2), Laguerre fundamental primitives (section 2.5, Theorem2.5.1) and the primitives of orthogonal polynomials on an arc studied in section 3.4 (Theorem 3.4.2). On the other hand, we find the asymptotic behavior of Abel-Goncharov interpolation polynomials (Section 2.3) or Taylor polynomials in the proof of Theorem 3.5.1.

Second Part.

The results achieved here appear in [76] and the proposed algorithm has the following characteristics.

- The approximation of the partial differences (derivatives) is carried out using a linear combination of bivariate Krawtchouk polynomials, which are orthogonal with respect to the inner product (4.10), which involves the product of binomial distributions (4.4). Therefore, it is not necessary to smooth the image with a 2-D Gaussian filter before numerical differentiation, in order to regularize the ill-posed nature of differentiation and therefore improve the edge localization. This is a well known procedure as pointed out in [91] and used in [48].

- In [11, 36] the authors describe edge detection procedures based on Chebyshev polynomials by using a unique threshold for the whole image. Here, we propose an algorithm that uses a two-level adaptive thresholds, that reduce the presence of false positive and false negative edge pixel.

- As consequence, a gradient operator of size 5×5 produces a better localized edge pixel, because the edges tend to be thicker as the size of the block $\mathcal{I}_{i,j}$ increases [36, 69].

- To avoid the thickness effect and improve the final result in our edge finder, we further apply morphological operations (close, erode and thin) to the edge image obtained after the second processing step of the proposed algorithm. As pointed out in recent work [45], this contributes to increasing the quality of the edges.

5.2 Some open problems

The open problems collected in this section, are in fact, future lines of research consequence of the work of this memory.

Once the problems dealt in the second chapter have been solved, it would seem natural to study the iterated integrals of families of orthogonal polynomials with respect to classes of general measures that are supported on an interval of the real line. Nevertheless, we have preferred to continue the study with iterated integrals of orthogonal polynomials with respect to measures supported on an arc of the unit circumference, motivated by the potentialities offered by some results of the analytical theory of polynomials and its consequences. The following two problems arise naturally:

Open Problem 5.1. *Study the analytic properties of iterated integral of families of orthogonal polynomials with respect to general measures supported on an interval of the real line (bounded or not). In particular, it could be of interest to determine the asymptotic behavior of their zeros.*

Open Problem 5.2. *The same aspect of the Open Problem 5.1, now for iterated integral of families of orthogonal polynomials with respect to general classes of measures whose support is contained on the unit circle.*

In [1, 6, 14, 28, 65] several applications to approximation of functions, electrostatics and hydrodynamics are described, such functions approximators can be considered as particular cases of the polynomials studied in the chapters 2 and 3, hence the following problem:

Open Problem 5.3. *Study the application of iterated integrals of families of orthogonal polynomials to physics problems and the approximation theory.*

From a more general point of view, the iterated primitives of orthogonal polynomials are related with orthogonality with respect to differential operators, as described in [3, 13, 14, 15]. Until now, most of the results obtained are essentially related to differential operators of first and second order. The results shown in the chapters 2 and 3 open the doors to the study of operators of higher order, hence we propose:

Open Problem 5.4. *Study the family of orthogonal polynomials with respect to higher order (> 2) operators, taking as starting point the families of primitives of orthogonal polynomials normalized with conditions of Abel-Goncharov type (described in Chapter 2), or other.*

In [6] the authors show that the primitives of orthogonal polynomials of first order can be interpreted as orthogonal polynomials with respect to a discrete-continuous Sobolev inner product, with derivatives of the same order of the iterated integral considered. The link between iterated integrals of orthogonal polynomials and Sobolev orthogonality could go deeper. For example, consider a product of the type:

$$\langle f, g\rangle_S = \sum_{k=0}^{m} \int_I f^{(k)} g^{(k)} d\mu_k, \tag{5.1}$$

where $m \in \mathbb{Z}_+$ is fixed, and we assume, without lost of generality, that μ_k for each $0 \leq k \leq m$ is a measure with "good properties", supported on a compact interval of the real line.

It is known (see e.g. [29, 50, 51, 53]) that the derivative of order m of orthogonal polynomials with respect to an inner product of the type (5.1) has (in some sense) similar asymptotic behavior that the orthogonal polynomials with respect to an standard inner product. Then the proper Sobolev orthogonal polynomials with respect to (5.1) are primitives of order m of their derivatives of order m.

Open Problem 5.5. *Study the implications of the results in [40, 66, 67] to describe the behavior of families of orthogonal polynomials with respect to* (5.1).

Open Problem 5.6. *Extend the results reached in [41] if in* (5.1) *we consider for* $1 \leq k \leq m$ *and that the measures* μ_k *are discrete and satisfying Abel-Goncharov conditions.*

As we have commented before chapter 4 offers a newfangled proposal of application of orthogonal polynomials on discrete sets to digital image processing. The computational advantages of the algorithm for edge detection has been recognized in an electronic private communication and in [86]. Notwithstanding the advantages offered by our procedure, it could be improved by solving the following problem:

Open Problem 5.7. *Determine the optimal form of choice of the thresholds, in order to minimize the probability of obtaining false positive or true negative edges.*

Another proposal that could be interesting in the field of image process is the following.

Open Problem 5.8. *Extend the main ideas of our algorithm to the approximation of the discrete operators that appear in other problems such as texture detection, face recognition and processing of medical images, as in the case of [86].*

In edge detection, the operators that we need approximate are difference operators, and as we have proved in [76], the Krawtchouk polynomials give optimal formulas. Perhaps the appropriate answer in this case would not come in terms of Krawtchouk polynomials, but rather in terms of other families of orthogonal polynomials on discrete sets.

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