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Bivariate generating functions for a class of linear recurrences: General structure

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ABSTRACT

We consider Problem 6.94 posed in the book *Concrete Mathematics* by Graham, Knuth, and Patashnik, and solve it by using bivariate exponential generating functions. The family of recurrence relations considered in the problem contains many cases of combinatorial interest for particular choices of the six parameters that define it. We give a complete classification of the partial differential equations satisfied by the exponential generating functions, and solve them in all cases. We also show that the recurrence relations defining the combinatorial numbers appearing in this problem display an interesting degeneracy that we study in detail. Finally, we obtain for all cases the corresponding univariate row generating polynomials.

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1. Introduction

Graham, Knuth and Patashnik (GKP), in their book *Concrete Mathematics* [11], posed the following "research problem" [11, Problem 6.94, pp. 319 and 564]:

Question 1.1. Develop a general theory of the solutions to the two-parameter recurrence

$$\left|{n \atop k}\right| = (\alpha n + \beta k + \gamma)\left|{n-1 \atop k}\right| + (\alpha' n + \beta' k + \gamma')\left|{n-1 \atop k-1}\right| + \delta_{n0}\delta_{k0} \tag{1.1}$$

for $n, k \in \mathbb{Z}$, assuming that $\left|{n \atop k}\right| = 0$ when $n < 0$ or $k < 0$. (Here and in the following δ_{ab} denotes the Kronecker delta.)

Many of the solutions to the recurrence (1.1) have been thoroughly studied in the literature [3–5,11,19]. They include classic examples such as the binomial coefficients, Stirling numbers of several kinds, Eulerian numbers and many others (see Table 1). Particular choices of the parameters defining the problem have been considered by Neuwirth in [16], where he found the solution of the recursion (1.1) for the particular case $\alpha' = 0$ by using Galton arrays. Also Spivey [20] has found explicit solutions (using finite differences) for the following three cases: (S1) $\alpha = -\beta$; (S2) $\beta = \beta' = 0$; and (S3) $\alpha/\beta = \alpha'/\beta' + 1$.

The previous studies focused on finding closed expressions for $\left|{n \atop k}\right|$ in terms of simpler combinatorial numbers but did not make significant use of generating functions. After completing the main computations of this paper, we learned[1] that those have been considered in the context of problem (1.1) by Théorêt [21–23] and Wilf [26]. In particular, Théorêt finds the exponential generating functions (EGF's) for the four particular cases explained above, and Wilf gives a general solution to the partial differential equations (PDE's) satisfied by the EGF's in terms of hypergeometric functions. However, in his own words [26]: "...we obtain a complete solution also, though its form is very unwieldy".

In this paper, we study in a systematic way the PDE's satisfied by the EGF's (in the next formula $P_n(x)$ are the so called *row generating polynomials*)

$$F(x, y) = \sum_{n,k \geqslant 0}\left|{n \atop k}\right|x^k\frac{y^n}{n!} = \sum_{n \geqslant 0}\frac{y^n}{n!}P_n(x), \tag{1.2}$$

defined by the sequences of numbers given by the recurrences (1.1). We propose a classification scheme that leads to a clean understanding of their solutions.

It is straightforward to show that the EGF (1.2) associated with the numbers $\left|{n \atop k}\right|$ satisfying the recurrence (1.1) is a solution to the PDE

$$-(\beta + \beta'x)xF_1 + (1 - \alpha y - \alpha'xy)F_2 = (\alpha + \gamma + (\alpha' + \beta' + \gamma')x)F, \tag{1.3}$$

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[1] We thank David Callan for calling our attention to Wilf's paper [26], which in turn refers to some earlier work by Théorêt [21–23].

Table 1

Some sequences of combinatorial interest satisfying (1.1). For each sequence, we give the type of the PDE for its EGF, the parameters defining its family $(\alpha, \beta; \alpha', \beta')$, the coefficients (γ, γ'), its description, and the corresponding entry in Ref. [17] (if any). More examples can be found in similar tables in Refs. [21,22].

Type	Family	(γ, γ')	Description	Entry
I	$(0,1;1,-1)$	$(1,0)$	Eulerian numbers $\left\langle {n \atop k} \right\rangle$ [11]	A173018
			[5,19]	A008292
	$(0,1;2,-1)$	$(1,-1)$	Second-order Eulerian numbers $\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle$ [10,11]	A008517
	$(0,1;3,-1)$	$(1,-2)$	Third-order Eulerian numbers [12,18]	A219512
	$(0,1;\nu,-1)$	$(1,1-\nu)$	ν-order Eulerian numbers $\left\langle {n \atop k} \right\rangle^{\nu}$ [2]	
	$(0,1;0,1)$	$(0,0)$	Surj(n,k) [3,9]	A019538
	$(0,1;1,1)$	$(0,-1)$	Ward numbers $\left\{\!\!\left\{ {n+k \atop k} \right\}\!\!\right\}$ [9]	A134991
	$(0,1;\nu,1)$	$(0,-\nu)$	ν-order Ward numbers [2]	
	$(1,1;1,1)$	$(-1,-1)$	$\left[\!\!\left[{n+k \atop k} \right]\!\!\right]$ [9]	
II	$(0,1;0,0)$	$(0,1)$	Stirling subset numbers $\left\{ {n \atop k} \right\}$ [11]	A008277
	$(-1,-1;0,0)$	$(1,-1)$	Lah numbers $L_{n,k}$ [5,19]	A008297
	$(1,1;0,0)$	$(-1,1)$	Unsigned Lah numbers $L(n,k)$ [24]	A105278
	$(2,1;0,0)$	$(-2,1)$	Generalization of $\left\{ {n \atop k} \right\}$ and $L(n,k)$ [14,15]	A035342
	$(1,-1;0,0)$	$(0,1)$	$[n \geqslant k]n!/k!$ [11,25]	A094587
	$(r-1,1;0,0)$	$(1-r,1)$	$S(r;n,k)$ [14,15]	
III	$(0,0;0,1)$	$(1,0)$	Inj(n,k) [9]	A008279
	$(0,0;-2,1)$	$(1,0)$	Coefficients of Laguerre polynomials in reverse order [1]	A021010
	$(0,0;-1,1)$	$(1,0)$	$(-1)^k \left\{ {n \atop n-k} \right\}$	A106800
	$(1,0;1,1)$	$(-1,-1)$	Ramanujan function $Q_{n+1,k}(-1)$ [8,27]	A075856
		$(0,-2)$	Ramanujan function $Q_{n,k}(1)$ [27]	A217922
		$(0,-1)$	Ramanujan function $Q_{n+1,k}(0)$ [27]	A054589
IV	$(0,0;0,0)$	$(1,1)$	Binomial coefficients $\binom{n}{k}$ [11]	A007318
	$(1,0;0,0)$	$(-1,1)$	Stirling cycle numbers $\left[{n \atop k} \right]$ [11]	A132393
	$(-1,0;0,0)$	$(1,1)$	Stirling numbers of the 1st kind $s(n,k)$ [3–5]	A008275

with the initial condition $F(x,0) = 1$. (Here and in the following F_i denotes the partial derivative of F with respect to its i-th variable.)

We classify now the PDE's satisfied by the EGF's solving Question 1.1 in terms of $(\alpha, \beta, \gamma; \alpha', \beta', \gamma')$. The dependence of the equations on the parameters $(\gamma; \gamma')$ is always fairly simple, so we will introduce *families* of equations characterized by the parameters $(\alpha, \beta; \alpha', \beta')$. In the paper we will sometimes refer to the full set of parameters defining a recurrence $(\alpha, \beta, \gamma; \alpha', \beta', \gamma')$, and sometimes just to the family $(\alpha, \beta; \alpha', \beta')$. A careful look at (1.3) reveals that the two most important parameters are β and β'. In fact, what really matters is whether these parameters are zero or non-zero. This leads us to introduce the following four different *types* of equations:

Definition 1.2. The PDE's for the EGF's relevant to solve Question 1.1 are classified in four different types: **Type I**: $\beta\beta' \neq 0$; **Type II**: $\beta \neq 0$ and $\beta' = 0$; **Type III**: $\beta = 0$ and $\beta' \neq 0$; and **Type IV**: $\beta = \beta' = 0$.

Remarks. 1. It is important to notice that although the parameters $(\alpha, \beta, \gamma; \alpha', \beta', \gamma')$ uniquely determine the numbers $\left|{n \atop k}\right|$, the converse is *not* true. For example, the *trivial sequence* $\left|{n \atop k}\right| = \delta_{n0}\delta_{k0}$ can be obtained through equations of any type by choosing the parameters to be $(\alpha, \beta, -\alpha; \alpha', \beta', -\alpha' - \beta')$, regardless of the specific values of α, β, α', and β', as can be easily seen by looking at Eq. (1.3). We will explore this phenomenon in the present paper and identify all the possible indeterminacies of this type. The reason why we classify equations instead of their solutions is a direct consequence of this fact.

2. If the numbers $\left|{n \atop k}\right|$ satisfy a recursion of the form (1.1) for certain parameters $(\alpha, \beta, \gamma; \alpha', \beta', \gamma')$, then the numbers

$$\left|{n \atop k}\right|^{\star} = \left|{n \atop n-k}\right| \tag{1.4}$$

also satisfy a recursion of the form (1.1) with parameters

$$\left(\alpha^{\star}, \beta^{\star}, \gamma^{\star}; \alpha'^{\star}, \beta'^{\star}, \gamma'^{\star}\right) = \left(\alpha' + \beta', -\beta', \gamma'; \alpha + \beta, -\beta, \gamma\right). \tag{1.5}$$

This involution was already introduced by Théorêt [22, Eq. (34)]. Hence, every Type II family is the $\star$-image of a Type III family and vice versa. On the other hand, the classes of Types I and IV are both closed under the $\star$-map, and it makes sense to talk about self-dual (symmetric) numbers as those satisfying

$$\left|{n \atop k}\right| = \left|{n \atop n-k}\right| = \left|{n \atop k}\right|^{\star}. \tag{1.6}$$

The families $(\alpha, \beta, \gamma; \alpha + \beta, -\beta, \gamma)$ are, for example, self-dual.

A more general involution can be defined as follows: let us denote by $\boldsymbol{z} = (x, y)$ the variables of the EGF (1.2), and by $\boldsymbol{\mu} = (\alpha, \beta, \gamma; \alpha', \beta', \gamma')$ the parameters of the corresponding recurrence (1.1), so the EGF can be compactly rewritten as $F(\boldsymbol{z}; \boldsymbol{\mu})$. We now define the $\star$-image of F as:

$$F^{\star}(\boldsymbol{z}; \boldsymbol{\mu}) = F\big(\mathcal{M}_1(\boldsymbol{z}, \boldsymbol{\mu}); \mathcal{M}_2(\boldsymbol{\mu})\big), \tag{1.7}$$

where $\mathcal{M}_2(\boldsymbol{\mu})$ is an involution (i.e., $\mathcal{M}_2(\mathcal{M}_2(\boldsymbol{\mu})) = \boldsymbol{\mu}$), and the function $\mathcal{M}_1$ satisfies

$$\mathcal{M}_1\big(\mathcal{M}_1(\boldsymbol{z}, \boldsymbol{\mu}), \mathcal{M}_2(\boldsymbol{\mu})\big) = \boldsymbol{z}, \quad \text{for all } \boldsymbol{z}. \tag{1.8}$$

Then, this $\star$-map is obviously an involution: i.e., $(F^{\star})^{\star}(\boldsymbol{z}; \boldsymbol{\mu}) = F(\boldsymbol{z}; \boldsymbol{\mu})$. The simplest of these involutions (i.e., those for which $\mathcal{M}_1$ is only a function of $\boldsymbol{z}$) are listed in Table 2. In Ref. [2] we will discuss in detail a more involved case.

The plan of the paper is the following. After this introduction, Section 2 is devoted to the derivation of the EGF's for the four distinct types of equations (cf. Definition 1.2).

Table 2
Involutions, their effects on the parameters $(\alpha, \beta, \gamma; \alpha', \beta', \gamma')$, and on the EGF's. They can be immediately checked by performing the change of variables defining the EGF transformation in the PDE (1.3).

Involution	Parameter transformation	EGF transformation
$\left\vert {n \atop k} \right\vert \to \left\vert {n \atop n-k} \right\vert$	$(\alpha, \beta, \gamma; \alpha', \beta', \gamma')$ $\to (\alpha' + \beta', -\beta', \gamma'; \alpha + \beta, -\beta, \gamma)$	$F^*(x, y) = F(1/x, xy)$
$\left\vert {n \atop k} \right\vert \to (-1)^k \left\vert {n \atop n-k} \right\vert$	$(\alpha, \beta, \gamma; \alpha', \beta', \gamma')$ $\to (\alpha' + \beta', -\beta', \gamma'; -\alpha - \beta, \beta, -\gamma)$	$F^*(x, y) = F(-1/x, -xy)$
$\left\vert {n \atop k} \right\vert \to (-1)^k \left\vert {n \atop k} \right\vert$	$(\alpha, \beta, \gamma; \alpha', \beta', \gamma')$ $\to (\alpha, \beta, \gamma; -\alpha', -\beta', -\gamma')$	$F^*(x, y) = F(-x, y)$
$\left\vert {n \atop k} \right\vert \to (-1)^{n-k} \left\vert {n \atop k} \right\vert$	$(\alpha, \beta, \gamma; \alpha', \beta', \gamma')$ $\to (-\alpha, -\beta, -\gamma; \alpha', \beta', \gamma')$	$F^*(x, y) = F(-x, -y)$
$\left\vert {n \atop k} \right\vert \to (-1)^{n} \left\vert {n \atop k} \right\vert$	$(\alpha, \beta, \gamma; \alpha', \beta', \gamma')$ $\to (-\alpha, -\beta, -\gamma; -\alpha', -\beta', -\gamma')$	$F^*(x, y) = F(x, -y)$

In Section 3, by using generating functions, we classify the parameter ambiguities in the problem; i.e., the possibility of obtaining the same solution to the recurrence with different sets of parameters. Finally, Section 4 is devoted to the study of polynomial generating functions in one variable, and Appendix A compiles some particular cases for which their EGF's can be computed in closed form.

2. Exponential generating functions

As we mentioned in the introduction, Wilf [26] has given general solutions for the EGF's solving (1.3) in terms of hypergeometric functions. This type of solution partially hides the structure of the EGF's, and it does not make it easy to see which parameter choices are the most relevant ones. To avoid these problems we have introduced a classification of the PDE's for the EGF's in four types that we discuss in the following subsections.

2.1. Type I equations

In this case both β and β' are non-zero. This fact can be used to rewrite the PDE (1.3) in a simpler way by performing the change of variables $(x, y) \mapsto (X, Y)$ defined by

$$X = \left\vert \frac{\beta'}{\beta} \right\vert x = \sigma \frac{\beta'}{\beta} x, \quad \text{where } \sigma = \text{sgn}(\beta\beta'), \tag{2.1a}$$

$$Y = \beta y. \tag{2.1b}$$

Notice that the signs of x and X are the same. This is not strictly necessary, but it is convenient to avoid absolute values in $\log x$ or $\log X$, as we will consider $x > 0$ and, hence, $X > 0$. The function $F(x, y)$ is then given by a function $\mathcal{F}(X, Y)$ via the relation

$$F(x, y) = \mathcal{F}(X, Y) = \mathcal{F}\left(\sigma \frac{\beta'}{\beta} x, \beta y\right), \tag{2.2}$$

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where $\mathcal{F}(X, Y)$ satisfies the PDE

$$-(1 + \sigma X)X\mathcal{F}_1 + \left(1 - rY - \sigma r'XY\right)\mathcal{F}_2 = \left(s - \sigma s'X\right)\mathcal{F}, \tag{2.3}$$

with initial condition

$$\mathcal{F}(X, 0) = 1. \tag{2.4}$$

The parameters r, r', s, s' appearing in (2.3) are defined as:

$$r = \frac{\alpha}{\beta}, \qquad r' = \frac{\alpha'}{\beta'}, \qquad s = \frac{\alpha + \gamma}{\beta}, \qquad s' = -1 - \frac{\alpha' + \gamma'}{\beta'}. \tag{2.5}$$

We have, hence, reduced the number of continuous parameters in two units: from $(\alpha, \beta, \gamma; \alpha', \beta', \gamma')$ to $(r, s; r', s')$, plus a discrete parameter $\sigma = \pm 1$. By doing this the expression for the PDE (2.3) becomes simpler than the original one (1.3). By using now the well-known method of characteristics, it is straightforward to prove the following

Theorem 2.1. *The solution $\mathcal{F}$ to (2.3) satisfying $\mathcal{F}(X, 0) = 1$ is given by*

$$\mathcal{F}(X, Y) = \left(\frac{G_{r,r',\sigma}(YX^{-r}(1 + \sigma X)^{r-r'} + G_{r,r',\sigma}^{-1}(X))}{X}\right)^s$$

$$\times \left(\frac{1 + \sigma X}{1 + \sigma G_{r,r',\sigma}(YX^{-r}(1 + \sigma X)^{r-r'} + G_{r,r',\sigma}^{-1}(X))}\right)^{s+s'}, \tag{2.6}$$

where

$$G_{r,r',\sigma}^{-1}(X) = \sum_{k \in \mathbb{Z}_0 \setminus \{r\}} \sigma^k \binom{-1 - r' + r}{k} \frac{X^{k-r}}{k-r} + \chi_{\mathbb{Z}_0}(r)\sigma^r \binom{-1 - r' + r}{r} \log X \tag{2.7}$$

for $0 < X < 1$, χ_A denotes the characteristic function of the set A, and $\mathbb{Z}_0 = \mathbb{N} \cup \{0\}$.

It is important to point out here that in many cases of combinatorial interest, the series defining $G_{r,r',\sigma}^{-1}$ can be summed in closed form to give simple functions (see Appendix A). Something similar happens for the other types of equations.

2.2. Type II equations

This type corresponds to $\beta \neq 0$ and $\beta' = 0$; then the PDE (1.3) simplifies to

$$-\beta x F_1 + \left(1 - (\alpha + \alpha'x)y\right)F_2 = \left(\alpha + \gamma + (\alpha' + \gamma')x\right)F. \tag{2.8}$$

This equation can be solved by using the method of characteristics. The solution is the content of the following theorem.

Theorem 2.2. *When $\beta \neq 0$ and $\beta' = 0$, the EGF is given by*

$$F(x,y) = \left(\frac{G_{\alpha,\beta,\alpha'}(yx^{-\alpha/\beta}e^{-\alpha'x/\beta} + G_{\alpha,\beta,\alpha'}^{-1}(x))}{x}\right)^{(\alpha+\gamma)/\beta}$$

$$\times \exp\left[-\frac{\alpha'+\gamma'}{\beta}\left(x - G_{\alpha,\beta,\alpha'}\left(yx^{-\alpha/\beta}e^{-\alpha'x/\beta} + G_{\alpha,\beta,\alpha'}^{-1}(x)\right)\right)\right], \quad (2.9)$$

where $G_{\alpha,\beta,\alpha'}^{-1}(x)$ is defined for any $x > 0$ as:

$$G_{\alpha,\beta,\alpha'}^{-1}(x) = \sum_{k\in\mathbb{Z}_0\backslash\{\alpha/\beta\}} \frac{(-\alpha'/\beta)^k}{k!\beta} \frac{x^{k-\alpha/\beta}}{k-\alpha/\beta} + \chi_{\mathbb{Z}_0}(\alpha/\beta)\frac{(-\alpha'/\beta)^{\alpha/\beta}}{(\alpha/\beta)!\beta}\log x. \quad (2.10)$$

2.3. Type III equations

This type corresponds to $\beta = 0$ and $\beta' \neq 0$. The PDE (1.3) reads now

$$-\beta'x^2 F_1 + \left(1 - (\alpha + \alpha'x)y\right)F_2 = \left(\alpha + \gamma + (\alpha' + \beta' + \gamma')x\right)F \quad (2.11)$$

and we have the following

Theorem 2.3. *When $\beta = 0$ and $\beta' \neq 0$, the EGF is given by*

$$F(x,y) = \left(\frac{G_{\alpha,\alpha',\beta'}(yx^{-\alpha'/\beta'}e^{\alpha/(\beta'x)} + G_{\alpha,\alpha',\beta'}^{-1}(x))}{x}\right)^{1+(\alpha'+\gamma')/\beta'}$$

$$\times \exp\left[\frac{\alpha+\gamma}{\beta'}\left[\frac{1}{x} - \frac{1}{G_{\alpha,\alpha',\beta'}(yx^{-\alpha'/\beta'}e^{\alpha/(\beta'x)} + G_{\alpha,\alpha',\beta'}^{-1}(x))}\right]\right], \quad (2.12)$$

where $G_{\alpha,\alpha',\beta'}^{-1}(x)$ is defined for $x > 0$ as:

$$G_{\alpha,\alpha',\beta'}^{-1}(x) = -\sum_{k\in\mathbb{Z}_0\backslash\{-1-\alpha'/\beta'\}} \frac{(\alpha/\beta')^k}{k!\beta'} \frac{1}{k+1+\alpha'/\beta'} \frac{1}{x^{k+1+\alpha'/\beta'}}$$

$$+ \chi_{\mathbb{N}}(-\alpha'/\beta')\frac{(\alpha/\beta')^{-1-\alpha'/\beta'}}{(-1-\alpha'/\beta')!\beta'}\log x. \quad (2.13)$$

It is interesting to note here that the involutions of the first two rows of Table 2 turn equations of Type II into equations of Type III (and vice versa) when the parameters and the arguments of the EGF's are transformed according to the rules given in any of those rows. Hence it would have sufficed, in principle, to discuss one of the two types of equations. We have considered both here for the sake of clarity.

This type is characterized by $\beta = \beta' = 0$, and corresponds to case (S2) of Spivey [20]. For the families $(\alpha, 0; \alpha', 0)$, Eq. (1.3) simplifies to the ordinary differential equation

$$\left(1 - (\alpha + \alpha' x)y\right)F_2 = \left(\alpha + \gamma + (\alpha' + \gamma')x\right)F. \tag{2.14}$$

Hence, in this case, it is trivial to obtain its closed form solutions satisfying the initial condition $F(x, 0) = 1$ (see also [22, Eq. (20)]):

Theorem 2.4. *When $\beta = \beta' = 0$ the EGF is*

$$F(x, y) = \begin{cases} \left(1 - (\alpha + \alpha' x)y\right)^{-\frac{\alpha + \gamma + (\alpha' + \gamma')x}{\alpha + \alpha' x}} & \text{if } (\alpha, \alpha') \neq (0, 0), \\ \exp((\gamma + \gamma' x)y) & \text{if } (\alpha, \alpha') = (0, 0). \end{cases} \tag{2.15}$$

In particular, the EGF's for the Type IV self-dual families $(\alpha, 0, \gamma; \alpha, 0, \gamma)$ are:

$$F(x, y) = \begin{cases} \left(1 - \alpha(1 + x)y\right)^{-\frac{\alpha + \gamma}{\alpha}} & \text{if } \alpha \neq 0, \\ \exp(\gamma(1 + x)y) & \text{if } \alpha = 0. \end{cases} \tag{2.16}$$

3. Parameter ambiguities

We discuss here the possibility of obtaining the same combinatorial numbers $\left|{n \atop k}\right|$ with different choices of parameters $(\alpha, \beta, \gamma; \alpha', \beta', \gamma')$. A straightforward way to do this is to consider Eq. (1.2) for the same EGF $F(x, y)$ and *two different sets of parameters* $(\alpha_1, \beta_1, \gamma_1; \alpha_1', \beta_1', \gamma_1')$ and $(\alpha_2, \beta_2, \gamma_2; \alpha_2', \beta_2', \gamma_2')$, i.e.,

$$-\left(\beta_1 + \beta_1' x\right)xF_1 + \left(1 - \alpha_1 y - \alpha_1' xy\right)F_2 = \left(\alpha_1 + \gamma_1 + (\alpha_1' + \beta_1' + \gamma_1')x\right)F, \quad \text{(3.1a)}$$

$$-\left(\beta_2 + \beta_2' x\right)xF_1 + \left(1 - \alpha_2 y - \alpha_2' xy\right)F_2 = \left(\alpha_2 + \gamma_2 + (\alpha_2' + \beta_2' + \gamma_2')x\right)F, \quad \text{(3.1b)}$$

with $F(x, 0) = 1$. By subtracting both equations, we see that a necessary condition that the EGF $F(x, y)$ must satisfy in order to give rise to the *same* family of combinatorial numbers is

$$-\left(\beta_{12} + \beta_{12}' x\right)xF_1 - \left(\alpha_{12} + \alpha_{12}' x\right)yF_2 = \left(\alpha_{12} + \gamma_{12} + (\alpha_{12}' + \beta_{12}' + \gamma_{12}')x\right)F, \quad \text{(3.2)}$$

where $\alpha_{12} = \alpha_1 - \alpha_2$, $\beta_{12} = \beta_1 - \beta_2$, $\gamma_{12} = \gamma_1 - \gamma_2$, $\alpha_{12}' = \alpha_1' - \alpha_2'$, $\beta_{12}' = \beta_1' - \beta_2'$, and $\gamma_{12}' = \gamma_1' - \gamma_2'$.

The simplest type of ambiguity occurs when $F(x, y) = F(x, 0) = 1$ (corresponding to the trivial case for which $\left|{n \atop k}\right| = \delta_{n0}\delta_{k0}$). In this case, Eqs. (3.1) imply that $\alpha_1 + \gamma_1 = \alpha_2 + \gamma_2 = 0$ and $\alpha_1' + \beta_1' + \gamma_1' = \alpha_2' + \beta_2' + \gamma_2' = 0$. This means that any choice of parameters $(\alpha, \beta, \gamma; \alpha', \beta', \gamma')$ such that $\alpha + \gamma = 0$ and $\alpha' + \beta' + \gamma' = 0$ defines the same (trivial) family of numbers $\left|{n \atop k}\right|$. In the following we will assume that $F(x, y)$ is not constant.

The solutions to Eq. (3.2) are remarkably simple, and can be written in terms of elementary functions. In order to solve it, one has to separately consider four cases again: (i) $\beta_{12}\beta'_{12} \neq 0$, (ii) $\beta_{12} = 0$, $\beta'_{12} \neq 0$, (iii) $\beta_{12} \neq 0$, $\beta'_{12} = 0$, and (iv) $\beta_{12} = 0$, $\beta'_{12} = 0$. The solutions for $F(x, y)$ are, respectively:

(i) $F(x,y) = x^{-\frac{\alpha_{12}+\gamma_{12}}{\beta_{12}}}(\beta_{12}+\beta'_{12}x)^{-1+\frac{\alpha_{12}+\gamma_{12}}{\beta_{12}}-\frac{\alpha'_{12}+\gamma'_{12}}{\beta'_{12}}}\Psi_1(yx^{-\frac{\alpha_{12}}{\beta_{12}}}(\beta_{12}+\beta'_{12}x)^{\frac{\alpha_{12}}{\beta_{12}}-\frac{\alpha'_{12}}{\beta'_{12}}})$.

(ii) $F(x,y) = \exp(\frac{\alpha_{12}+\gamma_{12}}{\beta'_{12}}\frac{1}{x})x^{-1-\frac{\alpha'_{12}+\gamma'_{12}}{\beta'_{12}}}\Psi_2(ye^{\frac{\alpha_{12}}{\beta'_{12}}\frac{1}{x}}x^{-\frac{\alpha'_{12}}{\beta'_{12}}})$.

(iii) $F(x,y) = \exp(-\frac{\alpha'_{12}+\gamma'_{12}}{\beta_{12}}x)x^{-\frac{\alpha_{12}+\gamma_{12}}{\beta_{12}}}\Psi_3(ye^{-\frac{\alpha'_{12}}{\beta_{12}}x}x^{-\frac{\alpha_{12}}{\beta_{12}}})$.

(iv) $F(x,y) = y^{-\frac{\alpha_{12}+\gamma_{12}+(\alpha'_{12}+\gamma'_{12})x}{\alpha_{12}+\alpha'_{12}x}}\Psi_4(x)$, if $\alpha_{12} \neq 0$ or $\alpha'_{12} \neq 0$. If $\alpha_{12} = \alpha'_{12} = 0$ then either $\gamma_{12} = \gamma'_{12} = 0$ – and $F(x,y)$ is arbitrary – or we must have $F(x,y) = 0$.

The functions Ψ_j, $j = 1, \ldots, 4$, appearing in the preceding expressions are arbitrary at this stage.

By demanding that $F(x, 0) = 1$ for all $x > 0$ in a neighborhood of 0, and requiring that Eq. (3.1a) (or equivalently, (3.1b)) is satisfied, we get the conditions that the parameters must satisfy in order to have non-trivial parameter indeterminacies, and also the functional forms of the functions Ψ_j. Once the EGF's are obtained, it is easy to derive closed formulas for the combinatorial numbers that they encode. All these steps are straightforward, so we just give here the final form of the degenerate families:

- $(\alpha, \beta, \gamma; \alpha', \beta', \gamma') = (\alpha, \alpha + \rho G, G - \alpha; -\rho H, \rho H + \alpha\frac{H}{G}, H - \alpha\frac{H}{G})$. Then

$$\left|{n \atop k}\right|_{\alpha, G, H} = \binom{n}{k}\left(\frac{H}{G}\right)^k \prod_{j=0}^{n-1}(G + \alpha j), \qquad (3.3)$$

  independent of ρ. Notice that $G = \alpha + \gamma$ and $H = \alpha' + \beta' + \gamma'$.
- $(\alpha, \beta, \gamma; \alpha', \beta', \gamma') = (\alpha, -\alpha, -\alpha; -\alpha', L - \alpha', \gamma')$. Then

$$\left|{n \atop k}\right|_{L, \gamma'} = \delta_{nk}\prod_{j=1}^{n}(\gamma' + Lj), \qquad (3.4)$$

  independent of α, α'. In this case $L = \alpha' + \beta'$.
- $(\alpha, \beta, \gamma; \alpha', \beta', \gamma') = (\alpha, \beta, M - \alpha; 0, \beta', -\beta')$. Then

$$\left|{n \atop k}\right|_{M, \alpha} = \delta_{k0}\prod_{j=0}^{n-1}(M + \alpha j), \qquad (3.5)$$

  independent of β, β'. In this case $M = \alpha + \gamma$.

- $(\alpha, \beta, \gamma; \alpha', \beta', \gamma') = (\alpha, \beta, -\alpha; \alpha', \beta', -\alpha' - \beta').$

$$\left| \begin{matrix} n \\ k \end{matrix} \right| = \delta_{k0}\delta_{n0}, \tag{3.6}$$

independent of α, β, α', β'. This is the trivial case.

As we can see, by adjusting the parameters within each of these families, it is possible to change the type of the PDE for their EGF. This is the reason why we introduced a classification for the equations instead of the combinatorial numbers themselves.

4. Polynomial generating functions in one variable

We study here the one-variable polynomials $P_n(x)$ defined in (1.2) for the four types of equations used in the solution to Question 1.1, as defined in Definition 1.2. We get these polynomials from the corresponding EGF's $F(x, y)$ by employing complex-variable methods. We will start with the most general case (Type I), and will work out the proof of the main theorem with some detail. For the other cases, the corresponding proofs are very similar, so we will just sketch them for the sake of brevity.

4.1. Type I case

When $\beta\beta' \neq 0$, it is convenient to work with the variables X, Y introduced in (2.1), and define the auxiliary polynomials

$$\mathcal{P}_n(X) = n! \big[Y^n \big] \mathcal{F}(X, Y), \tag{4.1}$$

so that

$$P_n(x) = \beta^n \mathcal{P}_n \left(\sigma \frac{\beta'}{\beta} x \right). \tag{4.2}$$

In this section we will get concrete expressions for $\mathcal{P}_n(X)$ by using Cauchy's theorem. The possibility of employing this procedure depends crucially on the analyticity properties of the generating functions $\mathcal{F}(X, Y)$ (cf. (2.2)) that, in turn, hinge upon those of the function $G_{r,r',\sigma}$ (cf. (2.7)). These can be studied by using the complex implicit-function theorem [13]. Our result can be summarized in the following

Theorem 4.1. *The polynomials* (4.1) *corresponding to EGF* (2.2) *satisfying Type-I equations are given by*

$$\mathcal{P}_n(X) = \frac{(1 + \sigma X)^{n(r - r') + s + s'}}{X^{s + rn}}$$

$$\times \lim_{Z \to X} \frac{\partial^n}{\partial Z^n} \left[\frac{Z^{s - r - 1}}{(1 + \sigma Z)^\eta} \left(\frac{Z - X}{G_{r,r',\sigma}^{-1}(Z) - G_{r,r',\sigma}^{-1}(X)} \right)^{n+1} \right], \tag{4.3}$$

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where $\eta = s + s' + 1 + r' - r$, or in the following alternative form if $r \in \mathbb{Z}_0$:

$$\mathcal{P}_n(X) = \frac{(1+\sigma X)^{n(r-r')+s+s'}}{X^{s+rn}} \sigma^{(n+1)r} \binom{-1-r'+r}{r}^{-n-1}$$

$$\times \lim_{Z \to X} \frac{\partial^n}{\partial Z^n} \left[\frac{Z^{s-r-1}(Z-X)^{n+1}}{(1+\sigma Z)^\eta} \left[\log \frac{Z\widehat{Q}^0_{r,r',\sigma}(Z)}{X\widehat{Q}^0_{r,r',\sigma}(X)} \right]^{-n-1} \right], \quad (4.4)$$

where

$$\widehat{Q}^0_{r,r',\sigma}(X) = \exp\left(\frac{Q^0_{r,r',\sigma}(X)}{\sigma^r \binom{-1-r'+r}{r}} \right), \quad (4.5a)$$

$$Q^0_{r,r',\sigma}(X) = \sum_{k \in \mathbb{Z}_0 \setminus \{r\}} \sigma^k \binom{-1-r'+r}{k} \frac{X^{k-r}}{k-r}. \quad (4.5b)$$

Proof. Let us pick $X \in \mathbb{C} \setminus \{0\}$ contained within the convergence disk of $Q^0_{r,r',\sigma}$ ($|X| < 1$; cf. (4.5b)). As $Q^0_{r,r',\sigma}$ contains a term of the form X^{-r}, the origin can be a singular point for specific choices of r (either a pole or a branch point). We consider the function

$$A \colon U \subset \mathbb{C}^3 \to \mathbb{C} \colon (X_1, X_2, X_3) \mapsto A(X_1, X_2, X_3) = \xi(X_1, X_2) - \xi(X_3, 0), \quad (4.6)$$

where $U \subset \mathbb{C}^3$ is an open neighborhood of $(X, 0, X)$ and

$$\xi(X_1, X_2) = X_2 X_1^{-r}(1+\sigma X_1)^{r-r'} + G^{-1}_{r,r',\sigma}(X_1). \quad (4.7)$$

Now, as $A(X, 0, X) = 0$ and $A_3(X, 0, X) \neq 0$ for all $X \in \mathbb{C}$ such that $0 < |X| < 1$, there exist open neighborhoods $U_1 \subset \mathbb{C}^2$ and $U_2 \subset \mathbb{C}$ of $(X, 0)$ and X, respectively, with $U_1 \times U_2 \subset U$, and a unique holomorphic function $\theta \colon U_1 \to U_2$ such that

$$A^{-1}(0) \cap (U_1 \times U_2) = \left\{ \big((X, Y), \theta(X, Y) \big) \colon (X, Y) \in U_1 \right\}. \quad (4.8)$$

An important consequence of this result and the definition of $G_{r,r',\sigma}$ (cf. (2.7)) is that

$$G_{r,r',\sigma}\big(\xi(X, Y)\big) = G_{r,r',\sigma}\big(\xi(\theta(X, Y), 0)\big) = \theta(X, Y). \quad (4.9)$$

The analyticity of θ implies that there exists an open neighborhood Ω of the origin of the complex Y-plane such that, for every $X \in \mathbb{C}$ satisfying $0 < |X| < 1$, the function $Y \mapsto \mathcal{F}(X, Y)$ given by (2.6) is analytic in Ω. By using Cauchy's theorem we can then write

$$[Y^n]\mathcal{F}(X, Y) = \frac{1}{2\pi i} \int_\Gamma \left[\frac{G_{r,r',\sigma}(\xi(X, Y))}{X} \right]^s \left[\frac{1+\sigma X}{1+\sigma G_{r,r',\sigma}(\xi(X, Y))} \right]^{s+s'} \frac{dY}{Y^{n+1}}, \quad (4.10)$$

where Γ is a simple closed curve of index $+1$, contained in Ω, surrounding the origin $Y = 0$ and no other singularity of the integrand. A natural change of variables, suggested by the form of (4.10), is to put (cf. (4.9))

$$Z = G_{r,r',\sigma}\big(\xi(X,Y)\big) = \theta(X,Y), \tag{4.11}$$

so let us consider the one-parameter family of holomorphic maps (π_2 denotes the projection onto the second argument)

$$Z_X : \pi_2\big((\{X\} \times \mathbb{C}) \cap U_1\big) \subset \mathbb{C} \to \mathbb{C} : Y \mapsto Z_X(Y) = G\big(\xi(X,Y)\big). \tag{4.12}$$

Notice that as $Z'_X(0) \neq 0$ we have that $Z_X(\Gamma)$, the image of the original integration contour Γ, will be also a closed, simple curve of index $+1$, contained in $Z_X(\Omega)$ and surrounding the point $Z = X$ in the complex Z-plane. Notice also that, given any open neighborhood V_X of $Z = X$, it is possible to choose the original integration contour Γ in such a way that $Z_X(\Gamma) \subset V_X$.

It is now straightforward to rewrite the integral in (4.10) as a contour integral on $Z_X(\Gamma)$ to obtain the following expression for the polynomials $\mathcal{P}_n(X)$:

$$\begin{aligned}
\mathcal{P}_n(X) = {}& n! \frac{(1 + \sigma X)^{n(r-r')+s+s'}}{X^{s+rn}} \\
& \times \frac{1}{2\pi i} \int_{Z_X(\Gamma)} \frac{Z^{s-r-1}}{(1+\sigma Z)^\eta} \frac{dZ}{(G_{r,r',\sigma}^{-1}(Z) - G_{r,r',\sigma}^{-1}(X))^{n+1}},
\end{aligned} \tag{4.13}$$

where $\eta = s + s' + 1 + r' - r$.

The analytic structure of the integrand of (4.13) shows that it is possible to choose Γ in such a way that (1) the singularity that may appear due to the term Z^{s-r-1} can be avoided, and (2) the only singularity surrounded by $Z_X(\Gamma)$ is $Z = X$. Hence, we can compute the integral by using the residue of the integrand at $Z = X$. This point can be immediately seen to be a pole of order $n + 1$ because $(G_{r,r',\sigma}^{-1})'(X) \neq 0$ if X satisfies $0 < |X| < 1$. By doing this, we immediately get (4.3).

When $r \in \mathbb{Z}_0$, it is convenient to explicitly take into account the logarithmic terms appearing in $G_{r,r',\sigma}^{-1}(Z)$ and $G_{r,r',\sigma}^{-1}(X)$. This gives (4.4). $\quad\square$

Remarks. 1. Notice that the procedure that we have followed above allows us to partially sidestep the difficulties associated with the impossibility to obtain closed form expressions for the function $G_{r,r',\sigma}(Z)$ from (2.7) in many cases; we only need the function $G_{r,r',\sigma}^{-1}(X)$ or $Q_{r,r',\sigma}^0(X)$.

2. As discussed above, the integrand of (4.13) has a pole of order $n + 1$ at $Z = X$. The expressions written above are based on the computation of the residue at this point and take advantage of the fact that the integrand is a meromorphic function in an open neighborhood of it. However, in many occasions it is possible to consider analytic

extensions of the integrand and move the integration contour to rewrite the integral in more convenient ways.

3. Formula (4.4) suggests the change of variables $e^U = Z\widehat{Q}^0(Z)$ and $e^V = X\widehat{Q}^0(X)$. This leads to simple Rodrigues-like formulas for the row polynomials.

4.2. Type II case

When $\beta \neq 0$ and $\beta' = 0$, the EGF $F(x, y)$ has the general form given by Theorem 2.2. In this case we can work with the original variables x, y. Our goal is to express the one-variable polynomials $P_n(x)$ as a contour integral by following the same steps that led to Theorem 4.1.

Using the complex implicit function theorem, we can show (as in the proof of Theorem 4.1) that there is an open neighborhood Ω of the origin of the complex y-plane where $F(x, y)$ is analytic as a function of y (for every $x \in \mathbb{C}$ satisfying $0 < |x| < 1$). Then, $P_n(x)$ can be expressed as a contour integral by using Cauchy's theorem:

$$P_n(x) = \frac{n!}{2\pi i} \int_{\Gamma_x} \left[\frac{G_{\alpha,\beta,\alpha'}(\xi(x,y))}{x} \right]^{\frac{\alpha+\gamma}{\beta}}$$
$$\times \exp\left[-\frac{\alpha'+\gamma'}{\beta}\left(x - G_{\alpha,\beta,\alpha'}\left(\xi(x,y)\right)\right) \right] \frac{dy}{y^{n+1}}, \tag{4.14}$$

where Γ_x is a closed, simple curve of index $+1$, that surrounds the origin $y = 0$ and no other singularity of the integrand, $\xi(x, y)$ is given by

$$\xi(x,y) = yx^{-\alpha/\beta}e^{-x\alpha'/\beta} + G_{\alpha,\beta,\alpha'}^{-1}(x),$$

and $G_{\alpha,\beta,\alpha'}$ is defined by (2.10).

As we did in the preceding section, it is convenient now to perform the change of variables $z = G_{\alpha,\beta,\alpha'}(\xi(x,y))$. The same steps followed in the proof of Theorem 4.1 lead to

$$P_n(x) = \frac{n!}{2\pi i \beta} \frac{e^{-((n+1)\alpha'+\gamma')x/\beta}}{x^{((n+1)\alpha+\gamma)/\beta}} \int_{\Gamma} \frac{z^{\gamma/\beta-1}e^{\gamma'z/\beta}}{(G_{\alpha,\beta,\alpha'}^{-1}(z) - G_{\alpha,\beta,\alpha'}^{-1}(x))^{n+1}} \, dz, \tag{4.15}$$

where the integration contour in (4.15) is a closed, simple curve of index $+1$, that surrounds the point $z = x$ and no other singularity of the integrand. The integrand in (4.15) has a pole of order $n + 1$ at $z = x$, so we can compute this integral using residues. The above discussion can be summarized in the following

Theorem 4.2. *The polynomials $P_n(x)$ corresponding to EGF satisfying Type-II equations are given by*

$$P_n(x) = \frac{e^{-((n+1)\alpha'+\gamma')x/\beta}}{\beta x^{((n+1)\alpha+\gamma)/\beta}} \lim_{z \to x} \frac{\partial^n}{\partial z^n} \frac{z^{\gamma/\beta-1}e^{\gamma'z/\beta}(z-x)^{n+1}}{(G_{\alpha,\beta,\alpha'}^{-1}(z) - G_{\alpha,\beta,\alpha'}^{-1}(x))^{n+1}}. \tag{4.16}$$

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4.3. Type III case

When $\beta = 0$ and $\beta' \neq 0$, the EGF $F(x, y)$ has the general form given by Theorem 2.3. Again, we can work with the original variables x, y and write the polynomials $P_n(x)$ as a contour integral.

Using the complex implicit function theorem, we can show (as in the proof of Theorem 4.1) that there is an open neighborhood Ω of the origin of the complex y-plane where $F(x, y)$ is analytic as a function of y (for every $x \in \mathbb{C}$ satisfying $0 < |x| < 1$). Then, $P_n(x)$ can be expressed as a contour integral by using Cauchy's theorem:

$$
P_n(x) = \frac{n!}{2\pi i} \int_{\Gamma_x} \left[\frac{G_{\alpha,\alpha',\beta'}(\xi(x,y))}{x} \right]^{1+\frac{\alpha'+\gamma'}{\beta'}}
$$
$$
\times \exp\left[\frac{\alpha+\gamma}{\beta'} \left[\frac{1}{x} - \frac{1}{G_{\alpha,\alpha',\beta'}(\xi(x,y))} \right] \right] \frac{dy}{y^{n+1}}, \tag{4.17}
$$

where Γ_x is a closed, simple curve of index $+1$, that surrounds the origin $y = 0$ and no other singularity of the integrand, $\xi(x, y)$ is given by

$$
\xi(x, y) = y x^{-\alpha'/\beta'} e^{\alpha/(\beta' x)} + G_{\alpha,\alpha',\beta'}^{-1}
$$

and $G_{\alpha,\alpha',\beta'}$ is defined by (2.13).

As we did in Section 4.1, we perform the change of variables $z = G_{\alpha,\alpha',\beta'}(\xi(x,y))$ to obtain

$$
P_n(x) = \frac{n!}{2\pi i \beta'} \frac{e^{((n+1)\alpha+\gamma)/(\beta' x)}}{x^{1+((n+1)\alpha'+\gamma')/\beta'}} \int_{\Gamma} \frac{z^{\gamma'/\beta'-1} e^{-\gamma/(\beta' z)}}{(G_{\alpha,\alpha',\beta'}^{-1}(z) - G_{\alpha,\alpha',\beta'}^{-1}(x))^{n+1}} \, dz, \tag{4.18}
$$

where the integration contour in (4.18) is a closed, simple curve of index $+1$, that surrounds the point $z = x$ and no other singularity of the integrand. The integrand in (4.18) has a pole of order $n + 1$ at $z = x$. Then, we can compute this integral using residues. The above discussion can be summarized in the following

Theorem 4.3. *The polynomials $P_n(x)$ corresponding to EGF satisfying Type-III equations are given by*

$$
P_n(x) = \frac{e^{((n+1)\alpha+\gamma)/(\beta' x)}}{\beta' x^{1+((n+1)\alpha'+\gamma')/\beta'}} \lim_{z \to x} \frac{\partial^n}{\partial z^n} \frac{z^{\gamma'/\beta'-1} e^{\gamma/(\beta' z)} (z-x)^{n+1}}{(G_{\alpha,\alpha',\beta'}^{-1}(z) - G_{\alpha,\alpha',\beta'}^{-1}(x))^{n+1}}. \tag{4.19}
$$

4.4. Type IV case

This corresponds to Spivey's case (S2) [20]. The EGF $F(x, y)$ for the families $(\alpha, 0; \alpha', 0)$ is given in closed form by (2.15), so it is not necessary to provide the integral representation used above. The result is easy to obtain, so we simply quote it here:

Theorem 4.4. *The polynomials $P_n(x)$ corresponding to EGF satisfying Type-IV equations are given by*

$$P_n(x) = \prod_{k=1}^{n} \big(k\alpha + \gamma + (k\alpha' + \gamma')x\big). \tag{4.20}$$

Actually, the form of the coefficients for this case is also easy to obtain:

Corollary 4.5. *The coefficients $\left|{n \atop k}\right|$ for $n \geqslant 0$ and $0 \leqslant k \leqslant n$ corresponding to solutions to* Question 1.1 *of Type IV are given by*

$$\left|{n \atop k}\right| = \sum_{t=0}^{n} \sum_{s=0}^{k} \left[{n \atop t}\right] \binom{t}{s} \binom{n-t}{k-s} (\alpha+\gamma)^{t-s} (\alpha'+\gamma')^{s} \alpha^{n-t+s-k} (\alpha')^{k-s}. \tag{4.21}$$

Acknowledgments

Appendix A. Some particular cases

In this appendix we will deal with particular cases of interest of the results obtained in Section 2. In particular, we will give closed formulas for the EGF's corresponding to cases where the functions G^{-1} of Theorems 2.1–2.3 can be written in closed form in terms of simple functions. Please, note that Type IV equations have been completely solved in Theorem 2.4.

A.1. Particular cases for Type I equations

In this subsection we will use the standard parameters $(\alpha, \beta, \gamma; \alpha', \beta', \gamma')$ (1.1) in the EGF, instead of the parameters (r, r', σ) (2.3)/(2.5) labeling $G_{r,r',\sigma}^{-1}$ (cf. (2.7)).

A.1.1. Solution for $1 + r' = r$
This is case (S3) of Spivey [20]. The function $G_{r,r-1,\sigma}^{-1}$ is given by:

$$G_{r,r-1,\sigma}^{-1}(X) = \begin{cases} -(rX^r)^{-1} & \text{if } r \neq 0, \\ \log X & \text{if } r = 0, \end{cases} \tag{A.1}$$

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and the EGF is

$$F(x,y) = \begin{cases} \dfrac{(\frac{\beta+\beta'x(1-\alpha y(\beta+\beta'x)/\beta)^{-\beta/\alpha}}{\beta+\beta'x})^{\gamma'/\beta'-\gamma/\beta}}{(1-\frac{\alpha y}{\beta}(\beta+\beta'x))^{(\alpha+\gamma)/\alpha}} & \text{if } \alpha \neq 0, \\[2ex] e^{y(\beta+\beta'x)\gamma/\beta}\big(\frac{\beta+\beta'xe^{y(\beta+\beta'x)}}{\beta+\beta'x}\big)^{\gamma'/\beta'-\gamma/\beta} & \text{if } \alpha = 0. \end{cases} \tag{A.2}$$

This EGF was also obtained by Théorêt [22, Proposition 3]. A particular case of this family corresponds to the Eulerian numbers defined by $(0,1,1;1,-1,0)$.

A.1.2. Solution for $r = -1$

This is case (S1) of Spivey [20]. The function $G_{-1,r',\sigma}^{-1}$ is given by:

$$G_{-1,r',\sigma}^{-1}(X) = \begin{cases} \frac{\sigma}{(1+r')}\big(1 - \frac{1}{(1+\sigma X)^{1+r'}}\big) & \text{if } r' \neq -1, \\[1ex] \sigma \log(1+\sigma X) & \text{if } r' = -1, \end{cases} \tag{A.3}$$

and this leads to

$$F(x,y) = \begin{cases} \big(\frac{1}{\beta'x}\frac{\beta+\beta'x}{(1-(\alpha'+\beta')xy)^{\beta'/(\alpha'+\beta')}} - \beta\big)^{\gamma/\beta-1} \\[1ex] \quad \times \big(\frac{1}{1-(\alpha'+\beta')xy}\big)^{\frac{2\beta\beta'+(\alpha'+\gamma')\beta-\gamma\beta'}{\beta(\alpha'+\beta')}} & \text{if } \alpha' \neq -\beta', \\[2ex] e^{xy\beta'(1+\gamma'/\beta'-\gamma/\beta)}\big(\frac{e^{xy\beta'}(\beta+\beta'x)-\beta}{\beta'x}\big)^{-1+\gamma/\beta} & \text{if } \alpha' = -\beta'. \end{cases} \tag{A.4}$$

This EGF can also be obtained starting from the EGF for $r' = 0$ (A.8), and using the involution (1.5) (cf. first row of Table 2), as suggested by Théorêt [22, p. 97].

A.1.3. Solution for $r = r' = 1$

In this case, the function to invert is

$$G_{1,1,\sigma}^{-1}(X) = -1/X + \sigma \log\big((1+\sigma X)/X\big). \tag{A.5}$$

This immediately gives

$$F(x,y) = \frac{1}{(\beta'x)^{1+\gamma/\beta}}\bigg(\frac{T(e^{\beta^2 y/(\beta'x)}T^{-1}(1+\beta/(\beta'x)))}{\beta+\beta'x}\bigg)^{1+\gamma'/\beta'-\gamma/\beta}$$

$$\times \bigg(\frac{\beta}{T(e^{\beta^2 y/(\beta'x)}T^{-1}(1+\beta/(\beta'x)))-1}\bigg)^{2+\gamma'/\beta'}, \tag{A.6}$$

in terms of the tree function T [6,7]. An interesting particular case corresponds to $(1,1,-1;1,1,-1)$ giving the numbers $\left[\!\!\left[\begin{smallmatrix} n+k \\ k \end{smallmatrix}\right]\!\!\right]$ introduced in [9].

A.1.4. Solution for $r' = 0$

This is the case studied by Neuwirth [16]. The function $G_{r,0,\sigma}^{-1}$ is given by:

$$G_{r,0,\sigma}^{-1}(X) = \begin{cases} -(1/r)((1+\sigma X)/X)^r & \text{if } r \neq 0, \\ \log(X/(1+\sigma X)) & \text{if } r = 0. \end{cases} \quad (A.7)$$

Then, we have

$$F(x,y) = \begin{cases} \left(\frac{\beta}{(\beta+\beta'x)(1-\alpha y)^{\beta/\alpha} - \beta'x}\right)^{1+\gamma'/\beta'} \\ \quad \times (1-\alpha y)^{(\beta/\alpha)(1+\gamma'/\beta' - (\alpha+\gamma)/\beta)} & \text{if } \alpha \neq 0, \\ e^{y\gamma}\left(\frac{\beta}{\beta+\beta'x(1-e^{y\beta})}\right)^{1+\gamma'/\beta'} & \text{if } \alpha = 0. \end{cases} \quad (A.8)$$

This EGF was also obtained by Théorêt [21,22].[2] An important particular case corresponds to the numbers $\text{Surj}(n,k)$ [9] defined by $(0,1,0;0,1,0)$.

A.1.5. Solution for $r = 0$ and $-r' \in \mathbb{N}$

As r' is a negative integer, it is convenient to define $\nu = -r' \in \mathbb{N}$. Then, the function $G_{0,-\nu,\sigma}^{-1}$ is given by

$$G_{0,-\nu,\sigma}^{-1}(X) = \log X + \sum_{k=1}^{\nu-1} \binom{\nu-1}{k} \frac{(\sigma X)^k}{k}. \quad (A.9)$$

It is convenient to define a new function T_ν as

$$T_\nu^{-1}(z) = z e^{Q_\nu(z)}, \quad \text{where } Q_\nu(z) = \sum_{k=1}^{\nu-1} \binom{\nu-1}{k} \frac{(-z)^k}{k}. \quad (A.10)$$

It is clear that T_1 is the identity, and T_2 is the tree function [6,7]. The EGF is

$$F(x,y) = \beta^{1-\nu+\gamma'/\beta'} \left(\frac{T_\nu(e^{y\beta^{1-\nu}(\beta+\beta'x)^\nu} T_\nu^{-1}(-\beta'x/\beta))}{(-\beta'x)}\right)^{\gamma/\beta}$$

$$\times \left(\frac{1 - T_\nu(e^{y\beta^{1-\nu}(\beta+\beta'x)^\nu} T_\nu^{-1}(-\beta'x/\beta))}{\beta+\beta'x}\right)^{1-\nu-\gamma/\beta+\gamma'/\beta'}. \quad (A.11)$$

It is possible to define ν-order Eulerian numbers as a generalization of ordinary and second order Eulerian numbers by the parameter choice $(0,1,1;\nu,-1,1-\nu)$ [2].

A.1.6. Solution for $r = 0$ and $r' \in \mathbb{N}$

It is convenient to redefine $r' = \nu \in \mathbb{N}$ in accordance with the previous section. The function $G_{0,\nu,\sigma}^{-1}$ is given by

---

[2] The expression for the case $\alpha = 0$ has a typo in [21, Eq. (4.66)]; but it is correct in [22, Eq. (16)].

$$G_{0,\nu,\sigma}^{-1}(X) = \log\left(\sigma T_{\nu+1}^{-1}\left(\frac{\sigma X}{1 + \sigma X}\right)\right), \tag{A.12}$$

where T_ν is given by (A.10). We have now that

$$F(x,y) = \frac{\beta^{1+\nu+\gamma'/\beta'}}{(\beta + \beta'x)^{1+\nu+\gamma'/\beta'-\gamma/\beta}}\left(\frac{T_{\nu+1}(e^{y\beta^{1+\nu}(\beta+\beta'x)^{-\nu}}T_{\nu+1}^{-1}(\beta'x/(\beta+\beta'x)))}{\beta'x}\right)^{\gamma/\beta}$$
$$\times \left(1 - T_{\nu+1}\left(e^{y\beta^{1+\nu}(\beta+\beta'x)^{-\nu}}T_{\nu+1}^{-1}\left(\beta'x/(\beta+\beta'x)\right)\right)\right)^{-1-\nu-\gamma'/\beta'}. \tag{A.13}$$

Interesting particular cases are the ν-order Ward numbers – a generalization of the ordinary Ward numbers – given by $(0,1,0;\nu,1,-\nu)$ [2].

A.2. Particular cases for Type II equations

In this subsection we will use the standard parameters $(\alpha,\beta,\gamma;\alpha',0,\gamma')$ in both $G_{\alpha,\beta,\alpha'}^{-1}$ (cf. (2.10)) and the EGF.

A.2.1. Solution for $(-\beta,\beta;\alpha',0)$
In this case $G_{-\beta,\beta,\alpha'}^{-1}$ can be computed in closed form to give

$$G_{-\beta,\beta,\alpha'}^{-1}(x) = \begin{cases} (1 - e^{-\alpha'x/\beta})/\alpha' & \text{if } \alpha' \neq 0, \\ x/\beta & \text{if } \alpha' = 0. \end{cases} \tag{A.14}$$

Hence

$$F(x,y) = \begin{cases} (1 - \frac{\beta\log(1-\alpha'xy)}{\alpha'x})^{\gamma/\beta-1}(1-\alpha'xy)^{-1-\gamma'/\alpha'} & \text{if } \alpha' \neq 0, \\ (1+\beta y)^{-1+\gamma/\beta}e^{\gamma'xy} & \text{if } \alpha' = 0. \end{cases} \tag{A.15}$$

A relevant particular case is defined by the parameters $(1,-1,0;0,0,1)$ and corresponds to the numbers $n^{\underline{n-k}}$.

A.2.2. Solution for $(\alpha,\beta;0,0)$
In this case $G_{\alpha,\beta,0}^{-1}$ can be easily summed to give

$$G_{\alpha,\beta,0}^{-1}(x) = \begin{cases} -x^{-\alpha/\beta}/\alpha & \text{if } \alpha \neq 0, \\ (1/\beta)\log x & \text{if } \alpha = 0. \end{cases} \tag{A.16}$$

Hence we have

$$F(x,y) = \begin{cases} (1-\alpha y)^{-(1+\gamma/\alpha)}\exp(-\frac{\gamma'x}{\beta}(1-(1-\alpha y)^{-\beta/\alpha})) & \text{if } \alpha \neq 0, \\ \exp(\gamma y - \gamma'(1-e^{\beta y})x/\beta) & \text{if } \alpha = 0. \end{cases} \tag{A.17}$$

The generalization of the Lah and Stirling subset numbers $S(r;n,k)$ [14,15] corresponding to $(r-1,1,1-r;0,0,1)$ is a particular interesting case.

In this subsection we will use the standard parameters $(\alpha, 0, \gamma; \alpha', \beta', \gamma')$ in both $G^{-1}_{\alpha,\alpha',\beta'}$ (cf. (2.13)) and the EGF.

A.3.1. Solution for $(\alpha, 0; \beta', \beta')$

The expression for $G^{-1}_{\alpha,\beta',\beta'}$ is given by

$$G^{-1}_{\alpha,\beta',\beta'}(x) = \begin{cases} e^{\alpha/(\beta'x)}(\beta'/\alpha^2 - 1/(\alpha x)) - \beta'/\alpha^2 & \text{if } \alpha \neq 0, \\ -1/(2\beta'x^2) & \text{if } \alpha = 0. \end{cases} \tag{A.18}$$

Therefore,

$$F(x,y) = \begin{cases} [\frac{\alpha}{x\beta'(1-T(\zeta))}]^{2+\gamma'/\beta'} \exp[\frac{\alpha+\gamma}{\beta'}(\frac{1}{x} - \frac{\beta'}{\alpha}(1 - T(\zeta)))] & \text{if } \alpha \neq 0, \\ (1 - 2\beta'xy)^{-(1+\gamma'/(2\beta'))} \exp[\frac{\gamma}{\beta'x}(1 - \sqrt{1 - 2\beta'xy})] & \text{if } \alpha = 0, \end{cases} \tag{A.19}$$

where

$$\zeta(x,y) = (\alpha^2 y + \beta'x - \alpha)\exp(\alpha/(\beta'x) - 1)/(\beta'x). \tag{A.20}$$

The particular family $(1, 0, \gamma; 1, 1, \gamma')$ contains numbers related to the Ramanujan functions $Q_{n,k}(x)$ [27] (see Table 1). These EGF's seem to be new.

A.3.2. Solution for $(0, 0; \alpha', \beta')$

A straightforward computation leads to the expression for $G^{-1}_{0,\alpha',\beta'}$:

$$G^{-1}_{0,\alpha',\beta'}(x) = \begin{cases} -x^{-1-\alpha'/\beta'}/(\alpha' + \beta') & \text{if } \alpha' \neq -\beta', \\ (\log x)/\beta' & \text{if } \alpha' = -\beta'. \end{cases} \tag{A.21}$$

Hence

$$F(x,y) = \begin{cases} \frac{\exp(\gamma(1-(1-xy(\alpha'+\beta'))^{\beta'/(\alpha'+\beta')})/(\beta'x))}{(1-xy(\alpha'+\beta'))^{1+\gamma'/(\alpha'+\beta')}} & \text{if } \alpha' \neq -\beta', \\ e^{xy\gamma'}\exp(\gamma(1 - e^{-xy\beta'})/(\beta'x)) & \text{if } \alpha' = -\beta'. \end{cases} \tag{A.22}$$

A.3.3. Solution for $(\alpha, 0; 0, \beta')$

The expression for $G^{-1}_{\alpha,0,\beta'}$ is given by:

$$G^{-1}_{\alpha,0,\beta'}(x) = \begin{cases} (1 - e^{\alpha/(\beta'x)})/\alpha & \text{if } \alpha \neq 0, \\ -1/(\beta'x) & \text{if } \alpha = 0. \end{cases} \tag{A.23}$$

The corresponding EGF is

$$F(x,y) = \begin{cases} \left(\frac{\alpha}{\alpha+\beta' x \log(1-\alpha y)}\right)^{1+\gamma'/\beta'}(1-\alpha y)^{-(1+\gamma/\alpha)} & \text{if } \alpha \neq 0, \\ e^{\gamma y}(1-xy\beta')^{-(1+\gamma'/\beta')} & \text{if } \alpha = 0. \end{cases} \tag{A.24}$$

A particular case corresponds to the injective numbers $\text{Inj}(n,k)$ [9] defined by the parameters $(0,0,1;0,1,0)$.

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