Universidad Carlos III de Madrid

Graphs with small hyperbolicity constant and hyperbolic minor graphs

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to my family

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Introduction

Hyperbolic spaces, defined by Gromov in [52], play an important role in geometric group theory and in the geometry of negatively curved spaces (see [3, 19, 50, 52]). The concept of Gromov hyperbolicity grasps the essence of negatively curved spaces like the classical hyperbolic space, Riemannian manifolds of negative sectional curvature bounded away from 0, and of discrete spaces like trees and the Cayley graphs of many finitely generated groups. It is remarkable that a simple concept leads to such a rich general theory (see [3, 19, 50, 52]).

The first works on Gromov hyperbolic spaces deal with finitely generated groups (see [52]). Initially, Gromov spaces were applied to the study of automatic groups in the science of computation (see, e.g., [72]); indeed, hyperbolic groups are strongly geodesically automatic, i.e., there is an automatic structure on the group [34].

The concept of hyperbolicity appears also in discrete mathematics, algorithms and networking. For example, it has been shown empirically in [88] that the internet topology embeds with better accuracy into a hyperbolic space than into an Euclidean space of comparable dimension (formal proofs that the distortion is related to the hyperbolicity can be found in [91]); furthermore, it is evidenced that many real networks are hyperbolic (see, e.g., [1, 2, 39, 63, 69]. A few algorithmic problems in hyperbolic spaces and hyperbolic graphs have been considered in recent papers (see [38, 45, 49, 62]). Another important application of these spaces is the study of the spread of viruses through the internet (see [54, 56]). Furthermore, hyperbolic spaces are useful in secure transmission of information on the network (see [55, 54, 56, 71]). The hyperbolicity has also been used extensively in the context of random graphs (see, e.g., [85, 86, 87]). For example, it was shown in [86, 87] that several types of small-world networks and networks with given expected degrees are not hyperbolic in some sense.

The study of Gromov hyperbolic graphs is a subject of increasing interest in graph theory; see, e.g., [6, 12, 14, 20, 28, 35, 37, 41, 44, 53, 55, 54, 56, 63, 66, 68, 69, 71, 78, 83, 89, 93, 94] and the references therein.

In our study on the hyperbolicity in graphs we use the notations of [50]. Now we present the basic facts about Gromov's spaces.

Definition 0.0.1. *If $\gamma : [a, b] \longrightarrow X$ is a continuous curve in a metric space (X, d), we can* *define the* length *of* γ *as*

$$L(\gamma) := \sup\Big\{ \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) : \, a = t_0 < t_1 < \cdots < t_n = b \Big\} .$$

We say that a curve $\gamma : [a, b] \to X$ in a metric space X is a *geodesic* if we have $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$, where L and d denote length and distance, respectively, and $\gamma|_{[t,s]}$ is the restriction of the curve γ to the interval $[t, s]$ (then γ is equipped with an arc-length parametrization). The metric space X is said *geodesic* if for every couple of points in X there exists a geodesic joining them; we denote by $[xy]$ any geodesic joining x and y; this notation is ambiguous, since in general we do not have uniqueness of geodesics, but it is very convenient. Consequently, any geodesic metric space is connected. If the metric space X is a graph, then the edge joining the vertices u and v will be denoted by $[u, v]$.

In order to consider a graph G as a geodesic metric space, we identify (by an isometry) any edge $[u, v] \in E(G)$ with the interval $[0, 1]$ in the real line; then the edge $[u, v]$ (considered as a graph with just one edge) is isometric to the interval $[0, 1]$. Thus, the points in G are the vertices and, also, the points in the interior of any edge of G. In this way, any connected graph G has a natural distance defined on its points, induced by taking shortest paths in G, and we can see G as a metric graph. Throughout this paper, $G = (V, E) = (V(G), E(G))$ denotes a connected graph such that every edge has length 1 and $V \neq \emptyset$. These properties guarantee that any connected graph is a geodesic metric space. We will work both with simple and non-simple graphs. The difference between them is that the first type does not contain either loops or multiple edges. Although the operation of contraction is naturally defined for non-simple graphs, simple graphs are a more usual context in the study of hyperbolicity.

If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, the union of three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ is a *geodesic triangle* that will be denoted by $T = \{x_1, x_2, x_3\}$ and we will say that x_1, x_2 and x_3 are the vertices of T; it is usual to write also $T = \{[x_1x_2], [x_2x_3], [x_3x_1]\}$. We say that T is δ-*thin* if any side of T is contained in the δ-neighborhood of the union of the two other sides. We denote by $\delta(T)$ the sharp thin constant of T, i.e., $\delta(T) := \inf\{\delta \geq 0 : \, T \text{ is } \delta\text{-thin}\,\}$. The space X is δ-*hyperbolic* (or satisfies the *Rips condition* with constant δ) if every geodesic triangle in X is δ-thin. We denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e., $\delta(X) := \sup\{\delta(T) : \, T \text{ is a geodesic triangle in } X\,\}$. We say that X is *hyperbolic* if X is δ-hyperbolic for some $\delta \geq 0$; then X is hyperbolic if and only if $\delta(X) < \infty$. If we have a triangle with two identical vertices, we call it a "bigon". Obviously, every bigon in a δ-hyperbolic space is δ-thin. If X has connected components $\{X_i\}_{i \in I}$, then we define $\delta(X) := \sup_{i \in I} \delta(X_i)$, and we say that X is hyperbolic if $\delta(X) < \infty$.

In the classical references on this subject (see, e.g., [19, 50]) appear several different definitions of Gromov hyperbolicity, which are equivalent in the sense that if X is δ-hyperbolic with respect to one definition, then it is δ'-hyperbolic with respect to another definition (for some δ' related to δ). We have chosen this definition by its deep geometric meaning [50].

Trivially, any bounded metric space X is $(\operatorname{diam} X)$-hyperbolic. A normed linear space is hyperbolic if and only if it has dimension one. If a complete Riemannian manifold is simply connected and its sectional curvatures satisfy $K \le c$ for some negative constant c, then it is hyperbolic. See the classical references [3, 19, 50] in order to find further results. We want to remark that the main examples of hyperbolic graphs are the trees. In fact, the hyperbolicity constant of a geodesic metric space can be viewed as a measure of how "tree-like" the space is, since those spaces X with $\delta(X) = 0$ are precisely the metric trees. This is an interesting subject since, in many applications, one finds that the borderline between tractable and intractable cases may be the tree-like degree of the structure to be dealt with (see, e.g., [36]).

Note that the hyperbolicity constant $\delta(X)$ of a geodesic metric space can be viewed as a measure of how "tree-like" the space is, since those spaces with $\delta(X) = 0$ are precisely the metric trees. This is an interesting subject since, in many applications, one finds that the borderline between tractable and intractable cases may be the tree-like degree of the structure to be dealt with (see, e.g., [36]).

For a finite graph with n vertices it is possible to compute $\delta(G)$ in time $O(n^{3.69})$ [47] (this is improved in [39, 41]). Given a Cayley graph (of a presentation with solvable word problem) there is an algorithm which allows to decide if it is hyperbolic [73]. However, deciding whether or not a general infinite graph is hyperbolic is usually very difficult. Therefore, it is interesting to study the invariance of the hyperbolicity of graphs under appropriate transformations and the hyperbolicity of particular classes of graphs. The invariance of the hyperbolicity under some natural transformations on graphs have been studied in previous papers, for instance, removing edges of a graph is studied in [14, 28]. Moreover, the hyperbolicity of some product graphs have been characterized: in [23, 24, 25, 26, 27, 30, 67] the authors characterize in a simple way the hyperbolicity of strong product of graphs, direct product of graphs, lexicographic product of graphs, Cartesian sum of graphs, graph join and corona, and Cartesian product of graphs. Some other authors have obtained results on hyperbolicity for particular classes of graphs: chordal graphs, vertex-symmetric graphs, bipartite and intersection graphs, bridged graphs and expanders [20, 94, 66, 22, 42, 61, 65].

To remove and to contract edges of a graph are also very natural transformations. In [28] the authors study the distortion of the hyperbolicity constant of the graph $G \setminus e$ obtained from a graph G by removing an edge e. These bounds allow to obtain the characterization, in a quantitative way, of the hyperbolicity of many graphs in terms of local hyperbolicity. A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from G by contracting some edges, deleting some edges, and deleting some isolated vertices. Minor graphs is an interesting class of graphs. This topic started with one well-known result on planar graph, independently proved by Kuratowski and Wagner, which says that a graph is planar if and only if it do not include as a minor neither the complete graph K_5 nor the complete bipartite graph $K_{3,3}$ (see [64, 92]). There are previous works relating minor graphs with tree-length and tree-width, which are parameters closely related to hyperbolicity (see

[16, 51, 79, 80]).

Three main problems on Gromov hyperbolic graphs are the following:

I. To obtain inequalities relating the hyperbolicity constant and other parameters of graphs.

II. To study the hyperbolicity for important classes of graphs.

III. To study the invariance of the hyperbolicity of graphs under appropriate transformations.

In this work, we study:

1. The graphs with small hyperbolicity constant.
2. The relationship of hyperbolicity constant and effective diameter.
3. The invariance of the hyperbolicity on many minor graphs.

Note that problem 1 is related to II, problem 2 to I and problem 3 to III.

The structure of this work is as follows.

Chapter 1 is an introduction to graph theory. In Chapter 2 we give a brief introduction to hyperbolic spaces in the Gromov sense and we consider some previous results regarding hyperbolicity.

In Chapter 3, Section 3.1 we study the properties of graphs with small hyperbolicity constants, i.e., the graphs which are like trees (in the Gromov sense). In Section 3.2 we give a partial answer to the question: What is the structure of graphs with small hyperbolicity constant? The answer relates the hyperbolicity constant to the effective diameter.

Finally, in Chapter 4 we obtain quantitative information about the distortion of the hyperbolicity constant of the graph $G \setminus e$ (respectively, G/e) obtained from the graph G by deleting (respectively, contracting) an arbitrary edge e from it.

Two of our main results in Chapter 3 are Theorems 3.1.8 and 3.2.14, which characterize in two simple ways the graphs G with $\delta(G) = 1$ (the case $\delta(G) < 1$ is known, see Theorem 3.1.1). We also characterize the graphs G with $\delta(G) = \frac{5}{4}$ in Theorem 3.2.21. Note that Theorems 3.1.2 and 3.2.9, Corollary 3.2.16 and Proposition 3.1.9 give necessary conditions and a sufficient condition in order to have $\delta(G) = \frac{5}{4}$. Proposition 3.1.10 gives a necessary condition in order to have $\delta(G) = \frac{3}{2}$. (Recall that Theorem 2.4.2 shows that $\delta(G)$ is a multiple of $\frac{1}{4}$.) Although it is not possible to obtain bounds for the diameter of graphs with small hyperbolicity constant, in Chapter 3 we obtain such bounds for the effective diameter if $\delta(G) < \frac{3}{2}$ (see Proposition 3.2.5 and Theorems 3.2.9 and 3.2.14). This is the only case where we can obtain them, since Remark 3.2.19 shows that it is not possible to obtain similar bounds if $\delta(G) \geq \frac{3}{2}$. Furthermore, Corollary 3.2.17 provides an explicit formula for the hyperbolicity constant of many graphs.

In Chapter 4 we obtain the invariance of the hyperbolicity under the contraction of a finite number of edges. Besides, we obtain quantitative information about the distortion of the hyperbolicity constant of the graph G/e obtained from the graph G by contracting an arbitrary edge e from it for simple and non-simple graphs, in Sections 4.1 and 4.3, respectively. In Sections 4.2 and 4.3 we obtain the invariance of the hyperbolicity on many minor graphs as a consequence of these results for simple and non-simple graphs, respectively.

This problem has attracted attention previously. There is an interesting work of Bandelt and Chepoi [6] characterizing 1-hyperbolic graphs. There are several definitions of Gromov hyperbolicity, these different definitions are equivalent in the sense that if X is δ-hyperbolic with respect to the definition A, then it is δ'-hyperbolic with respect to the definition B for some δ' (see, e.g., [19, 50]). Since [6] uses the so called 4-point definition for hyperbolicity instead of Rips condition, their result and ours are not equivalent. Furthermore, there is no relation between the set of 1-hyperbolic graphs with respect to the 4-point condition and the set of 1-hyperbolic graphs satisfying the Rips condition.

Note that, if we consider a graph G whose edges have length equal to one and a graph G_k obtained from G stretching out their edges until length k, then $\delta(G_k) = k\delta(G)$. Therefore, all the results in this work can be generalized when the edges of the graph have length equal to k.

Chapter 1

Introduction to graph theory

1.1 Graphs

The fundamental concept of graph theory is the graph, which (despite the name) is best thought of as a mathematical object rather than a diagram, even though graphs have a very natural graphical representation.

Many real-world situations can conveniently be described by means of a diagram consisting of a set of points together with lines joining certain pairs of these points. For example, computers, roads, railways or electric networks. Note that in this type of diagrams we are interested mainly if two given points are connected by a line, the way they come together is immaterial. The mathematical abstraction of situations of this type gives rise to the concept of graphs.

A *graph*, usually denoted $G(V(G), E(G))$ or $G = (V, E)$, consists of a set of vertices $V(G)$ together with a set $E(G)$ of unordered pairs of vertices called edges. The number of vertices in a graph is usually denoted $n = |V(G)|$ while the number of edges is usually denoted $m = |E(G)|$, these two basic parameters are called the *order* and *size* of G, respectively. We say that a graph G is *finite* if and only if $n < \infty$ and $m < \infty$. Otherwise we say that the graph is *infinite*. Since the edges are unordered pairs of vertices, we are always dealing with non-oriented graphs.

An *edge* joining the vertices $u \in V(G)$ and $v \in V(G)$ on many occasions is denoted by $[uv]$, but we will use the notation $[u, v]$ to denote this edge, since the notation $[uv]$ will be used in this work for geodesics, which will be discussed in Chapter 2.

Any graph with just one vertex is referred to as *trivial graph*. All other graphs are *non-trivial*.

Graphs are so named because they can be represented graphically, and it is this graphical representation which helps us to understand many of their properties. Each vertex is indicated by a point, and each edge by a line joining the points representing its ends. Most of the definitions and concepts in graph theory are suggested by its graphical representation as illustrated in Figure 1.1.

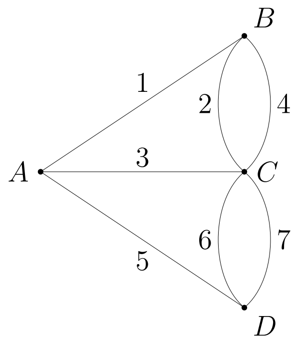

Figure 1.1: The graph of the bridges of Konigsberg.

We support us in this representation. We say that two vertices $u \in V(G), v \in V(G)$ are *adjacent* or *neighbours* if $[u, v] \in E(G)$ and we also denote it by $u \sim v$; likewise, two edges are adjacent if they have one vertex in common; similarly, if $e = [u, v]$ we say that the edge $e \in E(G)$ is *incident* to the vertices u and v. The set of neighbours of a vertex v in a graph G is denoted by $N_G(v)$, i.e., $N_G(v) := \{u \in V(G) : [u, v] \in E(G)\}$.

1.2 Degree of a vertex

The *degree* of a vertex is the number of neighbors it has in the graph. The degree of $v \in V(G)$ is denoted by $\deg(v) := |N_G(v)|$.

The number $\rho(G) := \min\{\deg(v) : v \in V(G)\}$ is the *minimum degree* of G and the number $\Delta(G) := \max\{\deg(v) : v \in V(G)\}$ is its *maximum degree*. In Figure 1.2, $\rho(G_1) = 0$ and $\Delta(G_2) = 4$.

If the degree of a vertex is 0, we say that is an *isolated vertex*. In Figure 1.2, the vertex D in the graph G_1 is an isolated vertex.

Figure 1.2: Simple graph G_1 and non-simple graph G_2.

Definition 1.2.1. (***Loop, Link***)
An edge with identical ends is called a loop*, and an edge with distinct ends a* link*. Two or more links with the same pair of ends are said to be* multiple edges*.*

In the graph G_2 of Figure 1.2, the edge c is a loop, and all other edges are links; the edges e and d are multiple edges.

A *simple graph* is one that has a single edge joining any two adjacent vertices, i.e., a graph without loops and multiple edges (see the graph G_1 in Figure 1.2).

Although some authors consider non-simple graphs (allowing loops and multiple edges), unless otherwise stated, we will work with simple graphs and then by graph we mean simple graph.

1.3 Subgraphs

Apart from the study of the characteristics or properties of a graph in its entirety, one can also consider only a region or a part thereof. For example, we can study arbitrary sets of vertices and edges of any graph. Moreover, in many cases, it is appropriate to consider graphs that are included "within" other. We will call them *subgraphs*.

Definition 1.3.1. (***Subgraph***)
If $G = (V, E)$ is a graph then $G_1 = (V_1, E_1)$ is a subgraph *of G if $\emptyset \neq V_1 \subseteq V$ and $E_1 \subseteq E$ where each edge in E_1 is incident to vertices of V_1.*

See in Figure 1.3 the subgraphs G_1 and G_2 of the graph G. Particular types of subgraphs are obtained by removing in some graph a vertex. We have formalized this idea in the following definitions. Let v be a vertex of a graph $G = (V(G), E(G))$. The subgraph $G - v$ of G is that graph whose vertex set is $V(G) - \{v\}$ and edge set is $E(G - v)$ (all edges of the graph G except the incident edges to v).

A relevant class of subgraphs are the *induced subgraphs*.

Definition 1.3.2. (***Induced subgraph***)
A subgraph obtained by vertex deletions only is called an induced subgraph. *If X is the set of vertices deleted, the resulting subgraph is denoted by $G - X$. Frequently, it is the set $Y := V \setminus X$ of vertices which remain that is the focus of interest.*

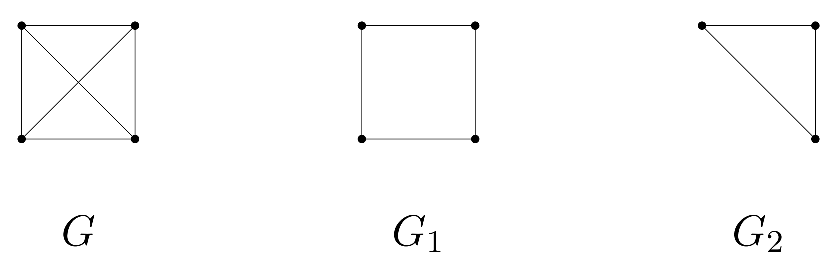

Figure 1.3: A subgraph G_1 and an induced subgraph G_2 of the graph G.

In Figure 1.3, G_2 is an induced subgraph of G. We can see graphically that it is the result of removing a vertex in the graph G.

1.4 Connectivity of graphs

One of the most significant properties that may have a graph, is connectivity. To understand this concept, it is necessary to give some definitions that describe us which means going from one vertex to another.

Definition 1.4.1. (***Path***)
A path *of a graph* $G = (V, E)$ *is a sequence of vertices* $P = \{v_0, v_1, v_2, \ldots, v_n\}$ *such that* v_{i-1} *is adjacent to* v_i*, for* $i = 1, 2, \ldots, n$*; a* simple path *is a path in which all vertices are different.*

Definition 1.4.2. (***Cycle***)
By cycle *we mean a simple closed curve, i.e., a path with different vertices, unless the last one, which is equal to the first vertex.*

The *length* of a path or a cycle is the number of its edges. We denote by $L(g)$ the length of the path g.

Definition 1.4.3. (***Connectivity***)
A graph is connected *if, for every partition of its vertex set into two nonempty sets* X *and* Y*, there is an edge with one end in* X *and one end in* Y*; otherwise, the graph is* disconnected *or* non-connected.

Given a connected graph $G = (V, A)$ and any two distinct vertices $u, v \in V$, we can find a path that connects them. Examples of connected and disconnected graphs are displayed in Figure 1.4.

Figure 1.4: Representation of a connected graph G_1 and a disconnected graph G_2.

A non-connected graph is formed by different "blocks" of vertices, each of which is a connected graph, what we call a *connected component*.

Definition 1.4.4. (***Connected component***)
A connected component *of a graph* G *is a connected subgraph of* G *which is not properly contained on any other connected subgraph of* G*, that is, a connected component of* G *is a subgraph that is maximal with respect to the property of being connected.*

In a graph G we define the *distance* of two vertices u, v denoted by $d_G(u, v)$ or $d(u, v)$ as

$$d_G(u, v) := \inf\{L(g) \mid g \text{ is a path joining } u \text{ and } v\}.$$

If there is not a path joining u and v, we set $d(u, v) := \infty$. In a connected graph G, for every $u, v \in V(G)$ we have $d_G(u, v) < \infty$. The greatest distance between any two vertices in G is the *diameter* of $V(G)$, denoted by $\operatorname{diam} V(G)$.

1.5 Some special graphs

Some graphs appear frequently in many applications and, hence, they have standard names.

Definition 1.5.1. (***Path graph***)
A path graph *is a graph* $P = (V, E)$ *with* $V = \{v_1, v_2, \ldots, v_n\}$, $n \geq 2$ *and* $E = \{[v_1, v_2], [v_2, v_3], \ldots, [v_{n-1}, v_n]\}$. *The path graph with* n *vertices is denoted by* P_n. *The vertices* v_1 *and* v_n *are called its ends; the vertices* $v_2, \ldots, v_{n-1}$ *are the inner vertices of* P_n.

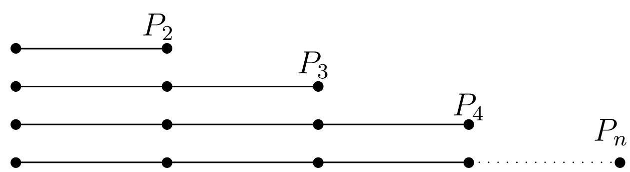

Figure 1.5: Path graphs.

Definition 1.5.2. (***Cycle graph***)
A cycle graph *of* n *vertices is a graph* $G = (V, E)$ *with* $V = \{v_1, v_2, \ldots, v_n\}$, $n \geq 3$ *and* $E = \{[v_1, v_2], [v_2, v_3], \ldots, [v_{n-1}, v_n], [v_n, v_1]\}$. *It is denoted by* C_n.

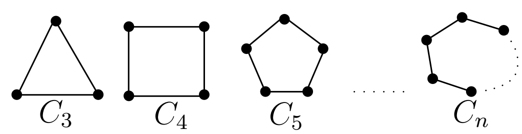

Figure 1.6: Cycle graphs.

Definition 1.5.3. (***Complete graph***)
A complete graph *is a graph in which every pair of vertices are joined by exactly one edge, i.e., all pairs of vertices of* G *are adjacent. The complete graph with* n *vertices is denoted by* K_n. *At each vertex* $v \in V(G)$ *we have* $\deg_G(v) = n - 1$.

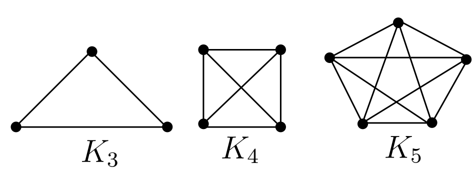

Figure 1.7: Complete graphs.

Definition 1.5.4. (***Empty graph***)
An empty graph *is a graph whose edge set is empty. We denote by E_n the empty graph with n vertices. In an empty graph all vertices have degree* 0.

Definition 1.5.5. (***Bipartite graph***)
A graph is bipartite *if its vertex set can be partitioned into two nonempty subsets V_1 and V_2 so that no edge has both ends in V_1 or both ends V_2.*

Definition 1.5.6. (***Complete bipartite graph***)
A bipartite graph is said to be a complete bipartite graph *if each vertex of V_1 is adjacent with each vertex of V_2. If $|V_1| = m$ and $|V_2| = n$, then this graph is denoted by $K_{m,n}$.*

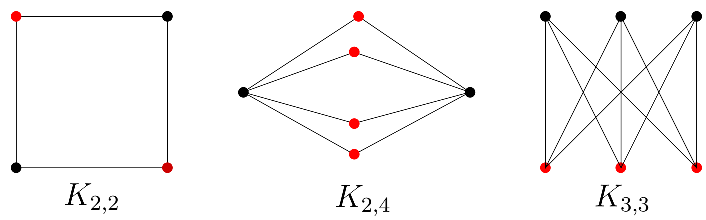

Figure 1.8: Complete bipartite graphs.

Definition 1.5.7. (***Star graph***)
The complete bipartite graph $K_{n-1,1}$ is called an n star graph *and it is denote by S_n.*

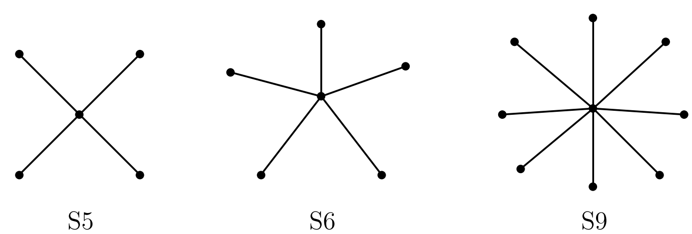

Figure 1.9: Star graphs.

Definition 1.5.8. (***Wheel graph***)
The wheel graph *W_n is a graph with n vertices formed by connecting a single vertex to each vertex of a cycle C_{n-1}.*

Definition 1.5.9. (***Regular graph***)
A graph $G = (V, E)$ is regular *if all vertices have the same degree k, and we say that it is k-regular. Every regular graph G satisfies the equality $\rho(G) = \Delta(G)$. In fact, a graph G is regular if and only if $\rho(G) = \Delta(G)$*

Definition 1.5.10. (***Tree***)
A tree *is an acyclic and connected graph, i.e., a connected graph without cycles.*

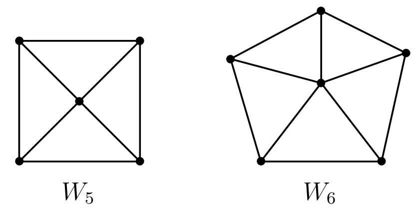

Figure 1.10: Wheel graphs.

1.6 Operations with graphs

In this section we define some of the most usual operations in graph theory and we will use them throughout the work. These operations produce new graphs from one or several graphs. We have unitary operations also called graph editing operations. They create a new graph from the original graph. Some examples of unitary operations are: adding or deleting a vertex or an edge, the contraction of an edge, line graph or graph complement. There are also binary operations that create a new graph from two initial graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$, such as: union of graphs or several kinds of products of graphs based on the Cartesian product of the set of vertices $V_1 \times V_2$.

1.6.1 Unitary operations

Most of the subgraphs worthwhile studying are those that differ minimally from the initial graphs, because they retain much of their properties and have small differences that show important details.

The operations of *deletion* and *contraction* of an edge are essential in the study of many properties of graphs.

The graph obtained by *deleting an edge* $e \in E$ of a graph $G = (E, V)$, is the subgraph of G denoted $G - e$ or $G \setminus e$ defined as $G \setminus e = (V, E \setminus e)$. We say that a subgraph is *expansive* when it contains all the vertices of the initial graph. Hence, every subgraph $G \setminus e$ is expansive.

The graph obtained by *contracting an edge* e in G, and denoted by G/e, results by identifying the endpoints of e followed by removing e. When e is a loop, G/e is the same as $G \setminus e$. It is not difficult to check that both deletion and contraction are commutative, and thus, for a subset of edges X, both $G \setminus X$ and G/X are well defined. Also, if $e \neq f$, then $(G \setminus e)/f$ and $(G/f) \setminus e$ are isomorphic; thus for disjoint subsets X, $X' \subseteq E(G)$, the graph $(G \setminus X)/X'$ is well-defined. A graph H isomorphic to $(G \setminus X)/X'$ for some choice of disjoint edge sets X and X' is called a *minor* of G.

Let us introduce another operation: *adding an edge* e of a graph G is the result of adding an edge to the set $E(G)$ connecting two vertices in $V(G)$; it is denoted by $G + e$.

Given a graph G with a finite number of connected components, an edge $e \in E(G)$ is a *bridge* or *cut edge* of G if the subgraph $G \setminus e$ has more connected components than G. We have that the edge $e \in E(G)$ is a bridge if and only if e does not belong to any cycle of G.

As we have seen, remove a vertex in a graph is not as simple as delete an edge, because when we remove a vertex all incident edges on it lose one end. Consequently, a good definition of this action is necessary: *Deleting a vertex* v of a graph G is to remove v from the set of vertices $V(G)$ and all the incident edges on v from the set of edges $E(G)$, obtaining a subgraph of G denoted by $G - v$ or $G \setminus v$.

Similarly, if G is a graph with a finite number of connected components, a vertex $v \in V(G)$ is a *cut vertex* of G if $G - v$ has more connected components than G.

We can obtain also the graph $G \cup \{v\}$ by adding to the graph G a single disjoint vertex v (i.e., $v \notin V(G)$). This operation is called *vertex addition*.

The *complement* $\overline{G}$ of the graph $G = (V, E)$ is the graph whose vertex set is V and whose edges are the pairs of non-adjacent vertices of G.

If $\mathtt{E} = \{[u, v] | u, v \in V, u \neq v\}$ is the set of all possible edges and $\overline{E} = \mathtt{E} \backslash E$ denotes the complement with respect to E, then $\overline{G} = (V, \overline{E})$.

Chapter 2

A brief introduction to Gromov hyperbolic graphs

There are several definitions of Gromov hyperbolicity. These different definitions are equivalent in the sense that if X is δ-hyperbolic with respect to the definition A, then it is δ'-hyperbolic with respect to the definition B for some δ' (see, e.g., [19, 50]). We have chosen the definition in the introduction since it has a deep geometric meaning (see, e.g., [50]).

First of all, we collect some basic facts about hyperbolic spaces.

The following are interesting examples of hyperbolic spaces.

Example 2.0.1. *Every bounded metric space X is $(\frac{1}{2}\operatorname{diam} X)$-hyperbolic.*

Example 2.0.2. *The real line $\mathbb{R}$ is 0-hyperbolic: In fact, any point of a geodesic triangle in the real line belongs to two sides of the triangle simultaneously, and therefore any geodesic triangle in $\mathbb{R}$ is 0-thin.*

Example 2.0.3. *The Euclidean plane $\mathbb{R}^2$ is not hyperbolic, since the midpoint of a side on a large equilateral triangle is far from all points on the other two sides.*

These arguments can be applied to higher dimensions:

Example 2.0.4. *A normed real vector space is hyperbolic if and only if it has dimension 1.*

Example 2.0.5. *Every metric tree with arbitrary edge lengths is 0-hyperbolic, by the same reason that the real line.*

Example 2.0.6. *The unit disk $\mathbb{D}$ (with its Poincaré metric) is $\log(1+\sqrt{2}\,)$-thin: Consider any geodesic triangle T in $\mathbb{D}$. It is clear that T is contained in an ideal triangle T', all of whose sides are of infinite length, with $\delta(T) \le \delta(T')$. Since all ideal triangles are isometric, we can consider just one fixed T'. Then, a computation gives $\delta(T') = \log(1+\sqrt{2}\,)$.*

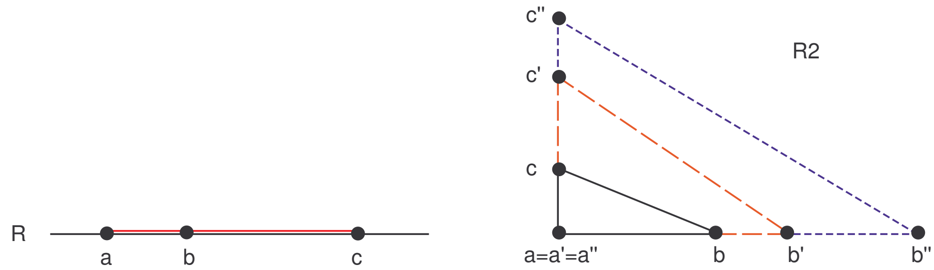

Figure 2.1: $\mathbb{R}$ and $\mathbb{R}^2$ as examples of hyperbolic and non-hyperbolic spaces.

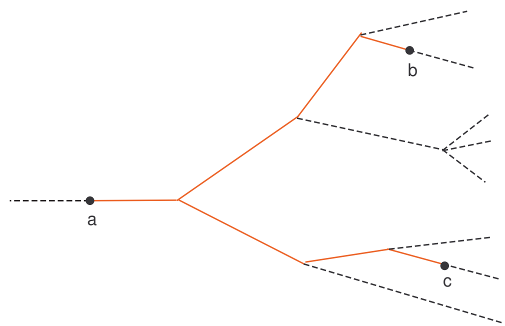

Figure 2.2: Any metric tree T verifies $\delta(T) = 0$.

Example 2.0.7. *Every simply connected complete Riemannian manifold with sectional curvatures verifying $K \le -c^2 < 0$, for some constant c, is hyperbolic (see, e.g., [50, p.52]).*

Example 2.0.8. *The graph Γ of the routing infrastructure of the Internet is also empirically shown to be hyperbolic (see [8]). One can think that this is a trivial (and then a non-useful) fact, since every bounded metric space X is $(\frac{1}{2}\operatorname{diam} X)$-hyperbolic. The point is that the quotient*

$$\frac{\delta(\Gamma)}{\operatorname{diam}\Gamma}$$

is very small, and this makes the tools of hyperbolic spaces applicable to Γ (see, e.g., [35]).

We would like to point out that deciding whether or not a space is hyperbolic is usually very difficult. Notice that, first of all, we have to consider an arbitrary geodesic triangle T, and calculate the minimum distance from an arbitrary point P of T to the union of the other two sides of the triangle to which P does not belong to. Thereafter, we have to take supremum over all the possible choices for P and then over all the possible choices for T. It means that if our space is, for instance, an n-dimensional manifold and we select two points P and Q on different sides of a triangle T, the function F that measures the distance between P and Q is a $(3n + 2)$-variable function ($3n$ variables describe the three vertices of T and

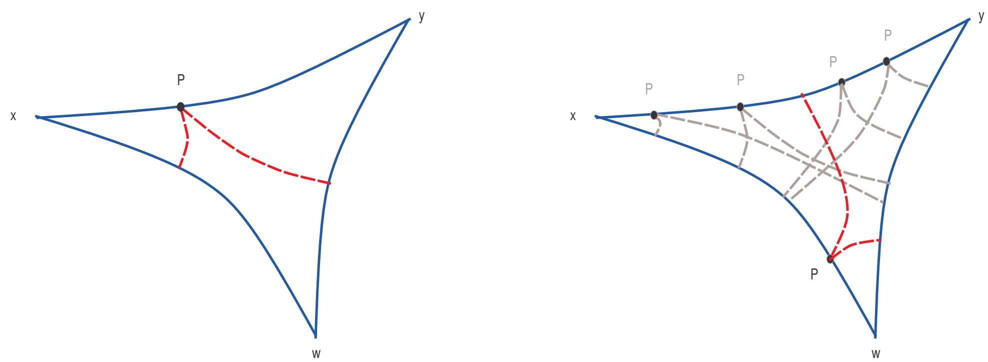

Figure 2.3: First steps in order to compute the hyperbolicity constant of X.

two variables describe the points P and Q in the closed curve given by T). In order to prove that our space is hyperbolic we would have to take the minimum of F over the variable that describes Q, and then the supremum over the remaining $3n+1$ variables, or at least to prove that it is finite. Without disregarding the difficulty of solving a $(3n+2)$-variable minimax problem, notice that the main obstacle is that we do not even know in an approximate way the location of geodesics in the space.

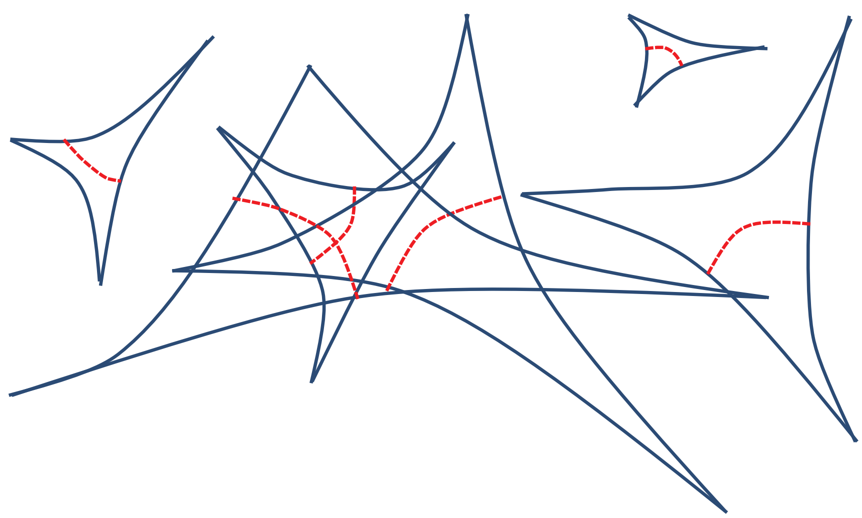

Figure 2.4: Calculating the supremum over all geodesic triangles.

Without disregarding the difficulty of solving this minimax problem, notice that in general the main obstacle is that we do not know the location of geodesics in the space. Therefore, it is interesting to obtain inequalities involving the hyperbolicity constant and other parameters of graphs. Another natural problem is to study the invariance of the hyperbolicity of graphs under appropriate transformations.

Since the hyperbolicity of many geodesic metric spaces is equivalent to the hyperbolicity of some graphs related to them (see, e.g., [19]), the study of hyperbolic graphs becomes an interesting topic.

2.1 Others definitions of Gromov hyperbolicity

2.1.1 Gromov product definition

Definition 2.1.1. *Given a metric space X, we define the* Gromov product *of $x, y \in X$ with base point $w \in X$ by*

$$(x|y)_w := \frac{1}{2}\big(d(x,w) + d(y,w) - d(x,y)\big). \tag{2.1}$$

We say that the Gromov product is δ-hyperbolic *if there is a constant $\delta \geq 0$ such that*

$$(x|z)_w \geq \min\big\{(x|y)_w, (y|z)_w\big\} - \delta$$

for every $x, y, z \in X$.

The following result (see [3, Proposition 2.2] and [52, Lemma 1.1A]) gives that the definition is independent of the base point.

Proposition 2.1.2. *Let X be a metric space and $w, w' \in X$. If the Gromov product based at w is δ-hyperbolic, then the Gromov product based at w' is 2δ-hyperbolic.*

We say that X is *δ-hyperbolic product* if its Gromov product is δ-hyperbolic for any base point, i.e.,

$$(x|z)_w \geq \min\big\{(x|y)_w, (y|z)_w\big\} - \delta \tag{2.2}$$

for every $x, y, z, w \in X$ (see, e.g., [50]).

It is well known that (2.2) is equivalent to our definition of Gromov hyperbolicity. Furthermore, we have the following quantitative result about this equivalence.

Theorem 2.1.3. *[50, Proposition 2.21, p.41] Let us consider a geodesic metric space X.*
(1) *If X is δ-hyperbolic, then it is 4δ-hyperbolic product.*
(2) *If X is δ-hyperbolic product, then it is 3δ-hyperbolic.*

2.1.2 Fine definition

Definition 2.1.4. *Given a geodesic triangle $T = \{x, y, z\}$ in a geodesic metric space X, let T_E be a Euclidean triangle with sides of the same length than T. Since there is no possible confusion, we will use the same notation for the corresponding points in T and T_E. The maximum inscribed circle in T_E meets the side $[xy]$ (respectively $[yz]$, $[zx]$) in a point z' (respectively x', y') such that $d(x,z') = d(x,y')$, $d(y,x') = d(y,z')$ and $d(z,x') = d(z,y')$. We call the points x', y', z', the* internal points *of $\{x, y, z\}$. There is a unique isometry f_{xyz} of $\{x, y, z\}$ onto a tripod (a star graph with one vertex w of degree 3, and three vertices x'', y'', z'' of degree one, such that $d(x'',w) = d(x,z') = d(x,y')$, $d(y'',w) = d(y,x') = d(y,z')$ and $d(z'',w) = d(z,x') = d(z,y')$), see Figure 2.5. The triangle $\{x, y, z\}$ is δ-*fine *if $f_{xyz}(p) = f_{xyz}(q)$ implies that $d(p,q) \leq \delta$. The space X is δ-*fine *if every geodesic triangle in X is δ-fine.*

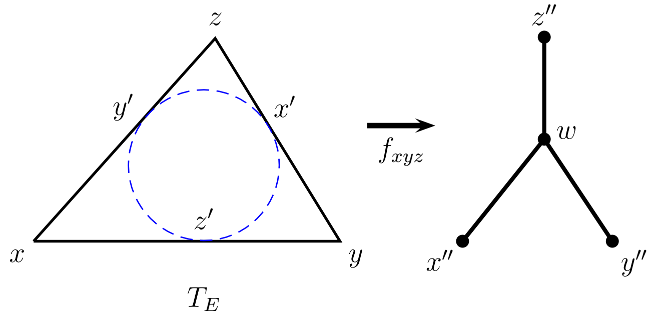

Figure 2.5: Isometry f_{xyz} of the triangle $T_E = \{x, y, z\}$ onto a tripod.

We also allow *degenerated tripods*, i.e., path graphs P_1, P_2 with one or two vertices, respectively. These situations correspond with triangles with several vertices repeated; in these cases the inscribed circle in T_E is a point.

It is known that this definition of fine is also equivalent to our definition of Gromov hyperbolicity. Furthermore, we have the following quantitative result.

Theorem 2.1.5. *[50, Proposition 2.21, p.41] Let us consider a geodesic metric space X.*
(1) *If X is δ-hyperbolic, then it is 4δ-fine.*
(2) *If X is δ-fine, then it is δ-hyperbolic.*

2.1.3 Insize definition

Definition 2.1.6. *Given a geodesic metric space X, let $T = \{x, y, z\}$ be a geodesic triangle in X and let x', y', z' be the internal points on T defined in Definition 2.1.4. We define the* insize *of the geodesic triangle T to be*

$$insize(T) := \operatorname{diam}\{x', y', z'\}. \tag{2.3}$$

*The space X is δ-*insize *if every geodesic triangle in X has insize at most δ.*

This definition of insize is also equivalent to our definition of Gromov hyperbolicity. Besides, we have the following quantitative result.

Theorem 2.1.7. *[50, Proposition 2.21, p.41] Let us consider a geodesic metric space X.*
(1) *If X is δ-hyperbolic, then it is 4δ-insize.*
(2) *If X is δ-insize, then it is 2δ-hyperbolic.*

2.1.4 Minsize definition

Definition 2.1.8. *Given a geodesic metric space X, let $T = \{x, y, z\}$ be a geodesic triangle in X and let $x' \in [yz]$, $y' \in [zx]$, $z' \in [xy]$. We define the* minsize *of the geodesic triangle T to be*

$$minsize(T) := \min_{x',y',z' \in T} \operatorname{diam}\{x', y', z'\}. \tag{2.4}$$

*The space X is δ-*minsize *if every geodesic triangle in X has minsize at most δ.*

It is known that this definition of minsize is also equivalent to our definition of Gromov hyperbolicity, in a quantitative way.

Theorem 2.1.9. *[50, Proposition 2.21, p.41] Let us consider a geodesic metric space X.*
(1) *If X is δ-hyperbolic, then it is 4δ-minsize.*
(2) *If X is δ-minsize, then it is 8δ-hyperbolic.*

2.1.5 Geodesics diverge

As usual, we denote by $B_k(x)$ the open ball in a metric space, i.e.,

$$B_k(x) := \{y \in X : d(x, y) < k\} \quad \text{for any } x \in X \text{ and } k > 0.$$

Definition 2.1.10. *Given a geodesic metric space X, we say that $e : [0, \infty) \to (0, \infty)$ is a* divergence function *for X, if for every point $x \in X$ and all geodesics $\gamma = [xy]$, $\gamma' = [xz]$, the function e satisfies the following condition:*

For every $R, r > 0$ such that $R + r \le \min\{L([xy]), L([xz])\}$, if $d(\gamma(R), \gamma'(R)) \ge e(0)$, and α is a path in $X \setminus B_{R+r}(x)$ from $\gamma(R + r)$ to $\gamma'(R + r)$, then we have $L(\alpha) > e(r)$ (see Figure 2.6).

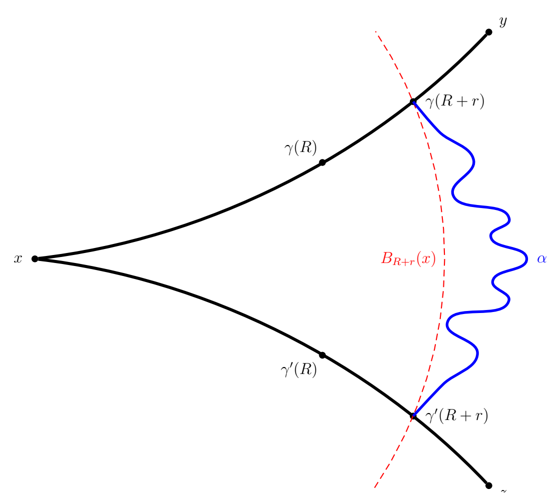

Figure 2.6: Geodesics diverge.

We say that *geodesics diverge* in X if there is a divergence function $e(r)$ such that

$$\lim_{r\to\infty} e(r) = \infty.$$

We say that *geodesics diverge exponentially* in X if there is an exponential divergence function. Theorem 1.1 in [74] shows that in a geodesic metric space X, geodesics diverge in X if and only if geodesics diverge exponentially in X.

It is known that Definition 2.1.10 is also equivalent to our definition of Gromov hyperbolicity (see [3, 74]). However, a quantitative result of this is not possible.

2.1.6 Gromov boundary

Let X be a metric space and we fix a base point $w \in X$. We say that a sequence $\widehat{x} = \{x_i\}_{i=1}^{\infty}$ in X is a *Gromov sequence* if $(x_i|x_j)_w \to \infty$ as $i, j \to \infty$.

Since, we have

$$\big|(x|y)_w - (x|y)_{w'}\big| = \frac{1}{2}\big|d(x,w) - d(x,w') + d(y,w) - d(y,w')\big| \le d(w,w'),$$

this concept is independent of the base point. The Gromov sequences are usually called sequences converging at infinity or tending to infinity (see, e.g., [3]). For the sake of brevity, we shall omit the base point w in the notation.

We say that two sequences $\widehat{x}$ and $\widehat{y}$ in X are *equivalent* and write $\widehat{x} \sim \widehat{y}$ if $(x_i|y_i) \to \infty$ as $i \to \infty$. This relation is always reflexive and symmetric, but it is transitive if X is hyperbolic (it is necessary to use (2.2) in order to prove it).

Definition 2.1.11. *Let X be a hyperbolic product metric space. Let $\overline{x}$ denote the equivalence class containing the Gromov sequence $\widehat{x}$ in X. The set of all equivalence classes*

$$\partial X := \{\overline{x} : \widehat{x} \textit{ is a Gromov sequence in } X\}$$

is the Gromov boundary *of X, and the set*

$$\overline{X} := X \cup \partial X$$

is the Gromov closure *of X.*

A *geodesic ray* in a space X is an isometric image of the half line $[0, \infty)$. In the case of geodesic metric spaces one can alternatively define a boundary point as an equivalence class of geodesic rays [50, p.119].

We want to define the Gromov product $(a|b)$ for all $a, b \in \overline{X}$. Assume first that $a, b \in \partial X$ and choose Gromov sequences $\widehat{x} \in a$, $\widehat{y} \in b$. The numbers $(x_i|y_j)$ need not converge to a limit but they converge to a rough limit in the following sense (see [90, Lemma 5.6]):

Lemma 2.1.12. *Let X be a δ-hyperbolic product metric space. Let $a, b \in \partial X$, $a \neq b$, and let $\widehat{x}, \widehat{x'} \in a$, $\widehat{y}, \widehat{y'} \in b$, $z \in X$. Then*

$$\limsup_{i,j\to\infty}(x'_i|y'_j) \leq \liminf_{i,j\to\infty}(x_i|y_j) + 2\delta < \infty,$$

$$\limsup_{i\to\infty}(x'_i|z) \leq \liminf_{i\to\infty}(x_i|z) + \delta < \infty.$$

Given $a, b \in \partial X$, there exist several definitions for $(a|b)$. We choose the following one, since it is very useful.

Definition 2.1.13. *Let X be a hyperbolic product metric space and $a, b \in \partial X$. We define*

$$(a|b) := \inf\{\liminf_{i,j\to\infty}(x_i|y_j) : \widehat{x} \in a, \widehat{y} \in b\}. \tag{2.5}$$

The same definition is used in [3, 40, 90], but [50] uses sup instead of inf.

In order to provide a topological structure to $\overline{X}$, we consider the set $\mathcal{B}$ consisting of all:

(1) open balls $B_r(x)$, for any $x \in X$ and $r > 0$,

(2) sets of the form $N_{x,k} := \{y \in \overline{X} : (x|y) > k\}$, for any $x \in \partial X$ and $k > 0$.

Proposition 4.8 in [3] shows that the set $\mathcal{B}$ is a basis for a topology of $\overline{X}$. Furthermore, we have the following result.

Proposition 2.1.14. *[3, Proposition 4.10] Let X be a locally compact hyperbolic product metric space. Then $\overline{X}$ is a Hausdorff compact metric space, and X is open and dense in $\overline{X}$.*

2.2 Gromov hyperbolicity, Mathematical Analysis and Geometry

The ideal boundary of a metric space is a type of boundary at infinity which is a very useful concept when dealing with negatively curved spaces. We want to talk about some subjects in which this boundary is useful.

A main problem in the study of Partial Differential Equations on Riemannian manifolds is whether or not there exist nonconstant bounded harmonic functions. A way to approach this problem is to study whether the so-called Dirichlet problem at infinity (or the asymptotic Dirichlet problem) is solvable on a complete Riemannian manifold M. That is to say, raising the question as to whether every continuous function on the boundary ∂M has a (unique) harmonic extension to M. Of course, the answer, in general, is no, since the simplest manifold $\mathbb{R}^n$ admits no positive harmonic functions other than constants. However, the answer is positive for the unit disk $\mathbb{D}$.

In [4] Ancona studied the asymptotic Dirichlet problem on Gromov hyperbolic graphs and in [5] on Gromov hyperbolic Riemannian manifolds with bounded geometry and a positive lower bound $\lambda_1(M) > 0$ for Dirichlet eigenvalues. In the papers [21] and [60] conditions on Gromov hyperbolic manifolds M that imply the positivity of $\lambda_1(M)$ are given and, consequently, the Dirichlet problem is solvable for many Gromov hyperbolic manifolds.

One of the most important features of the transition from a Gromov hyperbolic space to its Gromov boundary is that it is functorial. If $f : X \longrightarrow Y$ is in a certain class of maps between two Gromov hyperbolic spaces X and Y, then there is a boundary map $\partial f : \partial X \longrightarrow \partial Y$ which is in some other class of maps. In particular, if f is a quasi-isometry, then ∂f is a bihölder map (with respect to the Gromov metric on the boundary).

It is well known that biholomorphic maps between domains (with smooth boundaries) in $\mathbb{C}$ can be extended as a homeomorphism between their boundaries. If we consider domains in $\mathbb{C}^n$ $(n > 1)$ instead in $\mathbb{C}$, then the problem is very difficult. C. Fefferman (Fields medallist) showed in Inventiones Mathematicae (see [46]), with a very long and technical proof, that biholomorphic maps between bounded strictly pseudoconvex domains with smooth boundaries can be extended as a homeomorphism between their boundaries. It is possible to give a "more elementary" proof of this extension result using the functoriality of Gromov hyperbolic spaces: If we consider the Carathéodory metric on a bounded smooth strictly pseudoconvex domain in $\mathbb{C}^n$, then it is Gromov hyperbolic, and the Gromov boundary is homeomorphic to the topological boundary (see [7]). Since any biholomorphic map f between such two domains is an isometry for the Carathéodory metrics, the boundary map ∂f is essentially a boundary extension of f that is a homeomorphism between the boundaries (in fact, it is bihölder with respect to the Carnot-Carathéodory metrics in the boundaries). Fefferman's result gives much more precise information, but this last proof is simpler and gives information about a class of maps that is much more general than biholomorphic maps:

the quasi-isometries for the Carathéodory metrics.

In applications to various areas of mathematics, the Gromov boundary can be similarly be proved (under appropriate conditions) to coincide with other "finite" boundaries, such as the Euclidean or inner Euclidean boundary, or the Martin boundary, so we obtain a variety of boundary extension results as above.

Isometries (and quasi-isometries) in a hyperbolic space X can be extended (as an homeomorphism) to the Gromov boundary ∂X of the space. This fact allows to classify the isometries as *hyperbolic*, *parabolic* and *elliptic*, like the Möbius maps in $\mathbb{D}$, in terms of their fixed points in $X \cup \partial X$.

There are just three possibilities:

- There are exactly two fixed points in $X \cup \partial X$ and both are in ∂X (hyperbolic isometry).
- There is a single fixed point in $X \cup \partial X$ and it is in ∂X (parabolic isometry).
- There is a single fixed point in $X \cup \partial X$ and it is in X (elliptic isometry).

A main ingredient in the proof of this result in the unit disk $\mathbb{D}$ is that the isometries are holomorphic functions. Surprisingly, the tools in hyperbolic spaces provide a new and general proof just in terms of distances!

2.3 Main results on hyperbolic spaces

We state now some of the main facts about hyperbolic spaces.

Definition 2.3.1. *Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \longrightarrow Y$ is said to be an (α, β)-*quasi-isometric embedding*, with constants $\alpha \geq 1$, $\beta \geq 0$ if for every $x, y \in X$:*

$$\alpha^{-1} d_X(x, y) - \beta \leq d_Y(f(x), f(y)) \leq \alpha d_X(x, y) + \beta.$$

*The function f is ε-*full *if for each $y \in Y$ there exists $x \in X$ with $d_Y(f(x), y) \leq \varepsilon$.*

Definition 2.3.2. *A map $f : X \longrightarrow Y$ is said to be a* quasi-isometry*, if there exist constants $\alpha \geq 1$, $\beta, \varepsilon \geq 0$ such that f is an ε-full (α, β)-quasi-isometric embedding.*

Definition 2.3.3. *An (α, β)-*quasigeodesic *in X is an (α, β)-quasi-isometric embedding between an interval of $\mathbb{R}$ and X.*

In the study of any mathematical property, the class of maps which preserve that property plays a central role in the theory. The following result shows that quasi-isometries preserve hyperbolicity.

Theorem 2.3.4 (Invariance of hyperbolicity)**.** *Let $f : X \longrightarrow Y$ be an (α, β)-quasi-isometric embedding between the geodesic metric spaces X and Y. If Y is hyperbolic, then X is hyperbolic.*

Besides, if f is ε-full for some $\varepsilon \geq 0$ (a quasi-isometry), then X is hyperbolic if and only if Y is hyperbolic.

We next discuss the connection between hyperbolicity and *geodesic stability*. In the complex plane (with its Euclidean distance), there is only one optimal way of joining two points: a straight line segment. However if we allow “limited suboptimality”, the set of “reasonably efficient paths” (quasigeodesics) are well spread. For instance, if we split the circle $\partial D(0, R) \subset \mathbb{C}$ into its two semicircles between the points R and $-R$, then we have two reasonably efficient paths (two $(\pi/2, 0)$-quasigeodesics) between these endpoints such that the point Ri on one of the semicircles is far from all points on the other semicircle provided that R is large. Even an additive suboptimality can lead to paths that fail to stay close together. For instance, the union of the two line segments in $\mathbb{C}$ given by $[0, R + i\sqrt{R}]$ and $[R + i\sqrt{R}, 2R]$ gives a path of length less than $2R + 1$ (since $2\sqrt{R^2 + R} \leq 2R + 1$), and so is “additively inefficient” by less than 1 (it is a $(1, 1)$-quasigeodesic). However, its corner point is very far from all points on the line segment $[0, 2R]$ when R is very large.

The situation in Gromov hyperbolic spaces is very different, since all such reasonably efficient paths ((α, β)-quasigeodesics for fixed α, β) stay within a bounded distance of each other:

Definition 2.3.5. *Let X be a metric space, Y a non-empty subset of X and ε a positive number. We call ε*-neighborhood *of Y in X, denoted by $V_\varepsilon(Y)$, to the set $\{x \in X : d_X(x,Y) \le \varepsilon\}$. The* Hausdorff distance *between two subsets Y and Z of X, denoted by $\mathcal{H}(Y,Z)$, is the number defined by:*

$$\inf\{\varepsilon > 0 : Y \subset V_\varepsilon(Z) \text{ and } Z \subset V_\varepsilon(Y)\}.$$

Theorem 2.3.6 (Geodesic stability)**.** *For any constants $\alpha \ge 1$ and $\beta, \delta \ge 0$ there exists a constant $H = H(\delta,\alpha,\beta)$ such that for every δ-hyperbolic geodesic metric space and for every pair of (α,β)-quasigeodesics g, h with the same endpoints, $\mathcal{H}(g,h) \le H$.*

The geodesic stability is not just a useful property of hyperbolic spaces; in fact, M. Bonk proves in [18] that the geodesic stability is equivalent to the hyperbolicity:

Theorem 2.3.7. (*[18, p.286]*) *Let X be a geodesic metric space with the following property: For each $a \ge 1$ there exists a constant H such that for every $x, y \in X$ and any $(a,0)$-quasigeodesic g in X starting in x and finishing in y there exists a geodesic γ joining x and y satisfy $\mathcal{H}(g,\gamma) \le H$. Then X is hyperbolic.*

Theorem 2.3.6 allows to prove Theorem 4.1.4:

Proof of Theorem 4.1.4. By hypothesis there exists $\delta \ge 0$ such that Y is δ-hyperbolic.

Let T be a geodesic triangle in X with sides g_1, g_2 y g_3, and T_Y the triangle with (α,β)-quasigeodesic sides $f(g_1)$, $f(g_2)$ y $f(g_3)$ in Y. Let γ_j be a geodesic joining the endpoints of $f(g_j)$, for $j = 1,2,3$, and T' the geodesic triangle in Y with sides $\gamma_1, \gamma_2, \gamma_3$.

Let p be any point in $f(g_1)$. We are going to prove that there exists a point $q \in f(g_2) \cup f(g_3)$ with $d_Y(p,q) \le K$, where $K := \delta + 2H(\delta,\alpha,\beta)$. By Theorem 2.3.6, there exists a point $p' \in \gamma_1$ with $d_Y(p,p') \le H(\delta,\alpha,\beta)$. Since T' is a geodesic triangle, it is δ-thin and there exists $q' \in \gamma_2 \cup \gamma_3$ with $d_Y(p',q') \le \delta$. Using again Theorem 2.3.6, there exists a point $q \in f(g_2) \cup f(g_3)$ con $d_Y(q,q') \le H(\delta,\alpha,\beta)$. Therefore,

$$\begin{aligned} d_Y(p, f(g_2) \cup f(g_3)) &\le d_Y(p,q) \le d_Y(p,p') + d_Y(p',q') + d_Y(q',q) \\ &\le H(\delta,\alpha,\beta) + \delta + H(\delta,\alpha,\beta). \end{aligned}$$

Let $z \in T$; without loss of generality we can assume that $z \in g_1$. We have seen that there exists a point $q \in f(g_2) \cup f(g_3)$ with $d_Y(f(z),q) \le K$. If $w \in g_2 \cup g_3$ satisfies $f(w) = q$, then

$$d_X(z, g_2 \cup g_3) \le d_X(z,w) \le \alpha d_Y(f(z),q) + \alpha\beta \le \alpha K + \alpha\beta.$$

Hence, T is $(\alpha K + \alpha\beta)$-thin. Since T is an arbitrary geodesic triangle, X is $(\alpha\delta + 2\alpha H(\delta,\alpha,\beta) + \alpha\beta)$-hyperbolic.

Assume now that f is ε-full. One can check that an "inverse" quasi-isometry $f^- : Y \longrightarrow X$ can be constructed as follows: for $y \in Y$ choose $x \in X$ with $d_Y(f(x),y) \le \varepsilon$ and define $f^-(y) := x$. Then the first part of the Theorem gives the result. □

2.4 Previous results

We collect now some important results which would be useful for the development of our work.

As usual, by *cycle* we mean a simple closed curve, i.e., a path with different vertices, unless the last one, which is equal to the first vertex. It is known (see [84, Lemma 2.1]) that, for every graph G, it is satisfied

$$\delta(G) = \sup\{\delta(T) : T \text{ is a geodesic triangle in } G \text{ that is a cycle}\}.$$

We denote by $J(G)$ the union of the set $V(G)$ and the midpoints of the edges of G. Consider the set $\mathbb{T}_1$ of geodesic triangles T in G that are cycles and such that the three vertices of the triangle T belong to $J(G)$, and denote by $\delta_1(G)$ the infimum of the constants λ such that every triangle in $\mathbb{T}_1$ is λ-thin.

The following results, which appear in [12, Theorems 2.5, 2.6 and 2.7], will be used throughout the work.

Theorem 2.4.1. *For every graph G, we have $\delta_1(G) = \delta(G)$.*

Theorem 2.4.2. *For every hyperbolic graph G, $\delta(G)$ is a multiple of $\frac{1}{4}$.*

The following result is a direct consequence of Theorems 2.4.1 and 2.4.2; it states that in the hyperbolic graphs there always exists a geodesic triangle for which the hyperbolicity constant is attained.

Theorem 2.4.3. *For any hyperbolic graph G, there exists a geodesic triangle $T \in \mathbb{T}_1$ such that $\delta(T) = \delta(G)$.*

Given a graph G, we define

$$\operatorname{diam} V(G) := \sup\big\{d(u,v) \,|\, u,v \in V(G)\big\}, \qquad \operatorname{diam} G := \sup\big\{d(x,y) \,|\, x,y \in G\big\}.$$

It is clear that $\operatorname{diam} V(G) \le \operatorname{diam} G \le \operatorname{diam} V(G) + 1$.

We will need the following theorem (see [83, Theorem 8]).

Theorem 2.4.4. *In any graph G the inequality $\delta(G) \le \frac{1}{2} \operatorname{diam} G$ holds.*

If H is a subgraph of G and $w \in V(H)$, we denote by $\deg_H(w)$ the degree of the vertex w in the subgraph induced by $V(H)$. We always have $d_H(x,y) \ge d_G(x,y)$ for every $x,y \in H$. A subgraph H of G is said *isometric* if $d_H(x,y) = d_G(x,y)$ for every $x,y \in H$. Note that this condition is equivalent to $d_H(u,v) = d_G(u,v)$ for every vertices $u,v \in V(H)$.

The following results appear in [14, Lemma 9] and [83, Theorem 11].

Lemma 2.4.5. *If H is an isometric subgraph of G, then $\delta(H) \le \delta(G)$.*

Lemma 2.4.6. *If C_n is the cycle graph with n vertices, then $\delta(C_n) = \frac{1}{4} L(C_n) = \frac{n}{4}$.*

Corollary 2.4.7. *If G is a graph which contains an isometric subgraph which is isomorphic to C_n, then $\delta(G) \ge \frac{n}{4}$.*

Chapter 3

Small values of the hyperbolicity constant in graph

3.1 Graphs with small hyperbolicity constant

The results in this chapter show some characterizations for hyperbolic graphs with small hyperbolicity constant. In this sense, the following result in [68, Theorem 11] characterizes the graphs G with hyperbolicity constant $\delta(G) \leq \frac{3}{4}$.

Theorem 3.1.1. *Let G be any graph.*

(a) $\delta(G) = 0$ *if and only if G is a tree.*

(b) $\delta(G) = \frac{1}{4}, \frac{1}{2}$ *is not satisfied for any graph G.*

(c) $\delta(G) = \frac{3}{4}$ *if and only if G is not a tree and every cycle in G has length* 3.

In order to characterize the graphs whit hyperbolicity constant greater than $\frac{3}{4}$ it is necessary to obtain some previous results.

Theorem 3.1.2. *Let G be any graph. Then $\delta(G) \geq \frac{5}{4}$ if and only if there exists a cycle g in G with length $L(g) \geq 5$ and a vertex $w \in g$ such that* $\deg_g(w) = 2$.

Proof. Let us assume that there exists a cycle g in G with length $L(g) \geq 5$ and a vertex $w \in g$ such that $\deg_g(w) = 2$. If $L(g) = 5$, that is, $V(g) = \{u_1, u_2, u_3, u_4, u_5\}$ where $\deg_g(u_1) = 2$, taking $x = u_1$, y the midpoint in $[u_3, u_4]$, $z = u_4$ and p the midpoint in $[xy]$ (the geodesic containing u_2 and u_3), we obtain $d(p, [xz] \cup [zy]) = \frac{5}{4}$. If $L(g) = 6$, that is, $V(g) = \{u_1, \ldots, u_6\}$ where $\deg_g(u_1) = 2$, taking $x = u_1$, y the midpoint in $[u_3, u_4]$, z the midpoint in $[u_4, u_5]$ and p the midpoint in $[xy]$, we obtain $d(p, [xz] \cup [zy]) = \frac{5}{4}$. Now, we assume $L(g) = n$, where

$V(g) = \{u_1, \dots, u_n\}$, $\deg_g(u_1) = 2$ and $n \geq 7$, and we will prove the result by complete induction on n. Assume now that the conclusion of the theorem holds for cycles with length less or equal than $n-1$, that is, if there exist a cycle C such that $5 \leq L(C) \leq n-1$ and a vertex $w \in V(C)$ such that $\deg_C(w) = 2$, then $\delta(G) \geq \frac{5}{4}$. Let us consider the subgraph G_0 induced by $V(g)$. If $[u_i, u_j] \in E(G)$ for some $1 < i < j-1 \leq n-1$, then we obtain two cycles g_1 and g_2 such that $V(g_1) = \{u_1, \dots, u_i, u_j, \dots, u_n\}$ and $V(g_2) = \{u_i, \dots, u_j\}$. If $L(g_1) \geq 5$, the induction hypothesis gives the result, so we assume that $L(g_1) < 5$ and, therefore, $5 \leq L(g_2) \leq n-1$. If there exists $u_k \in \{u_i, \dots, u_j\}$ such that $\deg_{g_2}(u_k) = 2$, the induction hypothesis gives the result. Otherwise, we can choose two adjacent vertices $u_k, u_l \in \{u_i, \dots, u_j\}$, with $k \leq l-2$, such that $g_3 = \{u_1, \dots, u_k, u_l, \dots, u_n\}$ satisfies $5 \leq L(g_3) \leq n-1$ and $\deg_{g_3}(u_1) = 2$. Finally, we assume that $\deg_g(u_i) = 2$ for every $i \in \{1, 2, ..., n\}$, that is, $G_0 = g$. If G_0 is an isometric subgraph of G, then $\delta(G) \geq \delta(G_0) \geq \frac{n}{4} \geq \frac{7}{4}$. If G_0 is not an isometric subgraph of G, then let us consider the following family of paths in G:

$$P := \big\{\sigma : \sigma = [xy] \text{ is a geodesic in } G \text{ with } x, y \in V(g) \text{ and } L(\sigma) < d_g(x, y)\big\}.$$

Let $h = [uv] \in P$ with $L(h) = \min_{\sigma \in P} L(\sigma)$; note that $\deg_h(u) = 1$ since otherwise h is not a geodesic. Let h' be a geodesic in g joining u and v. Since $E(G_0) = E(g)$, we have that $L(h) \geq 2$; $L(h) < d_g(u, v) = L(h')$ gives that $L(h') \geq 3$. Therefore, $H := h \cup h'$ is a cycle in G with $5 \leq L(H) \leq n-1$ and $\deg_H(u) = 2$; hence, the induction hypothesis gives $\delta(G) \geq \frac{5}{4}$.

Now, we assume $\delta(G) \geq \frac{5}{4}$. By Theorem 2.4.3, there exist a simple geodesic triangle $T = \{x, y, z\}$ and $p \in [xy]$ such that $d(p, [xz] \cup [zy]) \geq \frac{5}{4}$. It is clear that $d(p, x), d(p, y) \geq \frac{5}{4}$; thus, $d(x, y) \geq \frac{5}{2}$ and $L(T) \geq 5$. If $\deg_T(u) = 2$ for some $u \in V(T)$, then we have finished the proof; hence, we can assume that $\deg_T(u) \geq 3$ for every $u \in V(T)$. If x (or y) is a vertex of the graph, since x has at least three adjacent vertices in T, then one of them has to belong to $[zy]$. Taking the vertex v in $[zy]$ adjacent to x and nearest to y, we obtain a cycle $g = [yx] \cup [x, v] \cup [vy]$ with length greater than or equal to 5 such that $\deg_g(x) = 2$. By Theorem 2.4.3, we can assume now that x and y are midpoints of different edges; we have $d(x, y) \geq 3$. Let $[u_x, v_x]$ be the edge which contains x and satisfies $v_x \in [xy]$, then $d(v_x, y) \geq \frac{5}{2}$; let v be the vertex in $[xz] \cup [zy]$ adjacent to v_x and nearest to y, and γ be the path in $[xz] \cup [zy]$ from v to y. Taking $g = [yv_x] \cup [v_x, v] \cup \gamma$, we obtain a cycle with length greater or equal to 5 such that $\deg_g(v_x) = 2$. □

For every $m \geq 4$, we say that a graph G with edges of length 1 is *m-chordal* (see [94]) if for any cycle C in G with length $L(C) \geq m$, there exists an edge joining two non-consecutive vertices x, y of C. Given a cycle C in G, we say that a geodesic $g = [uv]$ is a *shortcut* if $u, v \in V(C)$, $L(g) = d(u, v) < d_C(u, v)$ and $g \cap C = \{u, v\}$.

Corollary 3.1.3. *Let G be any graph with edges of length 1. If $\delta(G) \leq 1$, then G is 5-chordal.*

Proposition 3.1.4. *Let G be any graph. If $\delta(G) = 1$, then G has a cycle isomorphic to C_4.*

Proof. By Theorem 2.4.3, there exist a geodesic triangle $T = \{x, y, z\}$ that is a cycle and $p \in [xy]$ such that $d(p, [xz] \cup [zy]) = 1$; thus, $L(T) \geq 4$. Assume that $L(T) \geq 5$ and denote the vertices of T by $u_1, u_2, \ldots, u_n$ with $n \geq 5$. By Theorem 3.1.2 there exists a vertex $u_j \in \{u_4, \ldots, u_n\}$ adjacent to u_2. If none of the cycles $\{u_1, u_2, u_j, u_{j+1}, \ldots, u_n\}$ and $\{u_2, u_3, \ldots, u_j\}$ has length 4, at least one of them has length greater than or equal to 5. Iterating this process we get a cycle of length 4. □

Lemma 3.1.5. *Let G be any graph such that $\delta(G) = \frac{5}{4}$. If G contains a cycle C of length 6 and there exist $x, y \in C$ such that $d(x, y) = 3$, then G has a cycle isomorphic to C_5.*

Proof. Let C be a cycle whose vertices are $\{u_1, u_2, u_3, u_4, u_5, u_6\}$. One can check that x, y can be chosen such that we have either $x, y \in V(C)$ or x, y are midpoints of opposite edges. If x and y are two vertices in C such that $d(x, y) = 3$, then $\delta(G) \geq \frac{3}{2}$, a contradiction. Hence, x and y are midpoints of opposite edges, for instance $[u_1, u_2]$ and $[u_4, u_5]$, respectively. Since $1 < d(u_1, u_4) < 3$, then $d(u_1, u_4) = 2$. If there exists $v \in V(G) \setminus V(C)$ with $d(u_1, v) = d(u_4, v) = 1$, then there exists a cycle isomorphic to C_5. If there exists $v \in V(C)$ with $d(u_1, v) = d(u_4, v) = 1$, then $v \in \{u_3, u_6\}$ and in both cases there exists a cycle isomorphic to C_5. □

Proposition 3.1.6. *Let G be any graph. If $\delta(G) = \frac{5}{4}$, then G has a cycle isomorphic to C_5.*

Proof. Since $\delta(G) = \frac{5}{4}$, by Theorem 2.4.3, there exist a geodesic triangle $T = \{x, y, z\}$ that is a cycle whose vertices are in $J(G)$ and $p \in [xy]$ such that $d(p, [xz] \cup [yz]) = \frac{5}{4}$.

Case 1. Assume that x or y is a vertex. If $d(x, y) \geq 3$, we can take $p' \in [xy]$ such that p' is not a vertex and $d(p', \{x, y\}) = \frac{3}{2}$; thus $d(p', [xz] \cup [yz]) = \frac{3}{2}$, a contradiction. Therefore, $d(x, y) = \frac{5}{2}$, that is, x is a vertex, y is the midpoint of a edge $[u_y, v_y]$ (or viceversa), with $u_y \in [xy]$, and p is the midpoint of $[xy]$. We take $0 < \varepsilon < \frac{1}{4}$ and $y' \in [yz]$ such that $d(y, y') = \varepsilon$. If $p \in [xy']$, let us consider the geodesic triangle $T' = \{x, y', z\}$; taking $p' \in [xy']$ such that $d(x, p') = \frac{5}{4} + \frac{\varepsilon}{2}$, we obtain $d(p', [xz] \cup [y'z]) = \frac{5}{4} + \frac{\varepsilon}{2}$, a contradiction. If $p \notin [xy']$, then there exists a vertex w adjacent to x and v_y; thus, $[xu_y] \cup [u_y, v_y] \cup [v_y, w] \cup [w, x]$ is isomorphic to a cycle C_5, except for the case showed in Figure 3.1

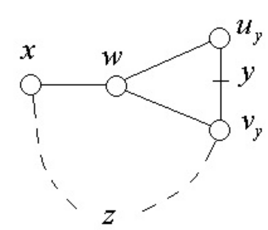

Figure 3.1: Example 1

In such a case, if $L([xz]\cup[zv_y]) = 2$ or 3, we have a cycle isomorphic to C_5, so we suppose that the vertices of $[xz]\cup[zv_y]$ are $\{x, w_1, ..., w_n, v_y\}$ where $x \sim w_1$, $w_i \sim w_{i+1}$, $w_n \sim v_y$ and $n \geq 3$. If $d(x, w_n) = 3$, we take the geodesic xwv_yw_n for $[xw_n]$ and $T' = \{x, w_n, z\}$, we obtain $\delta(T') \geq \frac{3}{2}$, a contradiction. If $d(x, w_n) = 1$ or $d(x, w_n) = 2$ and $w \notin [xw_n]$ then we have a cycle isomorphic to C_5. Finally, if $d(x, w_n) = 2$ and $w \in [xw_n]$, as it is showed in Figure 3.2, we can do the same with x and w_{n-1} in order to obtain a cycle isomorphic to C_5.

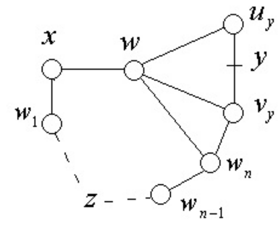

Figure 3.2:

Case 2. Assume that x and y are the midpoints of $[u_x, v_x]$ and $[u_y, v_y]$, respectively (with $v_x, u_y \in [xy]$). It is clear that $d(x, y) \geq 3$. If $d(x, y) \geq 4$, we can take $p' \in [xy]$ such that $d(p', \{x, y\}) = 2$ and, therefore, $d(p', [xz]\cup[yz]) \geq \frac{3}{2}$, a contradiction. Hence, $d(x, y) = 3$ and $d(v_x, u_y) = 2$. Since $\delta(G) = \frac{5}{4}$, $d(p, x) = \frac{5}{4}$ or $d(p, y) = \frac{5}{4}$. Assume, for instance, that $d(p, x) = \frac{5}{4}$, and we take $0 < \varepsilon < \frac{1}{4}$ and $x' \in [xu_x]$ such that $d(x, x') = \varepsilon$. If $p \in [x'y]$, let us consider the geodesic triangle $T' = \{x', y, z\}$; taking $p' \in [px']$ such that $d(p, p') = \frac{\varepsilon}{2}$, we conclude $d(p', [x'z]\cup[yz]) \geq \frac{5}{4} + \frac{\varepsilon}{2}$, a contradiction. In consequence, $p \notin [x'y]$, that is, $d(u_x, \{u_y, v_y\}) = 2$. Four cases could happen.

Case 2.1. If $\{u_x, w, v_y\}$ are the vertices of $[u_xv_y]$ and $w \notin [xy]$, then $\gamma = [u_xv_y]\cup[v_y, u_y]\cup[u_yv_x]\cup[v_x, u_x]$ is a cycle such that $L(\gamma) = 6$. Since $x, y \in \gamma$ and $d(x, y) = 3$, by Lemma 3.1.5, there exists a cycle isomorphic to C_5.

Case 2.2. If $\{u_x, w, v_y\}$ are the vertices of $[u_xv_y]$ and $w \in [xy]$, then we need to deal with several cases. If $L(T) = 6$, then $\gamma = [u_x, w]\cup[wy]\cup[yz]\cup[zu_x]$ is a cycle isomorphic to C_5. If $L(T) \geq 7$, then $L(\gamma) = p \geq 6$ and, since $\delta(G) = \frac{5}{4} < \frac{6}{4}$, Corollary 2.4.7 gives that γ is not an isometric subgraph, in consequence, there exists a shortcut $e_1 = [u_1v_1]$ in γ. Let γ_1, γ_1' be the two cycles in $\gamma\cup e_1$ containing e_1 such that γ_1 contains $[u_x, w]$ and γ_1' does not contain $[u_x, w]$. (a) If $L(\gamma_1) = 3$, then $e_1 \in E(G)$ and γ_1' is a cycle isomorphic to a cycle graph C_{p-1}. If $p - 1 = 5$, then there is nothing to prove. If $p - 1 > 5$, then $\delta(G) = \frac{5}{4} < \frac{p-1}{4}$; Corollary 2.4.7 gives that γ_1' is not an isometric subgraph, and therefore there exists a shortcut e_2 in γ_1'. Let γ_2, γ_2' be the two cycles in $\gamma_1'\cup e_2$ containing e_2 such that γ_2 contains e_1 and γ_2' does not contain e_1. If $L(\gamma_2) \in \{3, 4, 5\}$, then $\gamma_2\cup\gamma_1\cup[u_x, v_x]\cup[v_x, w]$ contains a cycle isomorphic to C_5. If $L(\gamma_2) \geq 6$, then $\delta(G) = \frac{5}{4} < \frac{6}{4}$; Corollary 2.4.7 gives that γ_2 is not an isometric subgraph, and therefore there exists a shortcut e_3 in γ_2. Let γ_3 be the cycle in $\gamma_2\cup e_3$ containing $e_1\cup e_3$; then $L(\gamma_3) < L(\gamma_2)$. If $L(\gamma_3) \in \{3, 4, 5\}$, then $\gamma_3\cup\gamma_1\cup[u_x, v_x]\cup[v_x, w]$

contains a cycle isomorphic to C_5. If $L(\gamma_3) \geq 6$, then we can obtain a cycle γ_4 containing e_1 with $L(\gamma_4) < L(\gamma_3)$. Iterating this process, finally we obtain a cycle isomorphic to C_5. (b) If $L(\gamma_1) > 3$, then a similar argument on γ_1 instead of γ_1' in (a) gives, in a step k, a cycle isomorphic to C_5 or a cycle γ_k such that $L(\gamma_k) = 3$, namely, $\gamma_k = [u_x, w] \cup [w, w'] \cup [w', u_x]$ where $w' \in V(\gamma)$. Finally, we can use γ_k instead of γ_1 in (a) to obtain the result.

Case 2.3. If $\{u_x, w, u_y\}$ are the vertices of $[u_x u_y]$ and $w \notin [xy]$, then $\gamma = [u_x u_y] \cup [u_y v_x] \cup [v_x, u_x]$ is a cycle such that $L(\gamma) = 5$.

Case 2.4. If $\{u_x, w, u_y\}$ are the vertices of $[u_x u_y]$ and $w \in [xy]$ $(w \neq v_x, u_y)$, then an argument similar to the one in Case 2.2 gives the result. □

Looking at Propositions 3.1.4 and 3.1.6 it seems logical to think that, if $\delta(G) = \frac{6}{4}$, then G has a cycle isomorphic to C_6 or, more generally, if $\delta(G) = \frac{n}{4}$, then G has a cycle isomorphic to C_n for every $n > 5$. But this is not true, as the following result shows.

Proposition 3.1.7. *For each $n > 5$ there exists a graph G_n such that $\delta(G_n) = \frac{n}{4}$ and G_n does not have any cycle isomorphic to C_n.*

Proof. Given $0 < a \leq b \leq c$, denote by $C_{a,b,c}$ the graph with three paths of lengths a, b, c joining two vertices. We know that $\delta(C_{a,b,c}) = (c + \min\{b, 3a\})/4$ by [83, Theorem 12].

Given $n > 6$, consider the graph $G_n = C_{1,(n-3),(n-3)}$. We have

$$\delta(G_n) = \frac{(n-3) + \min\{(n-3), 1\}}{4} = \frac{n}{4};$$

nevertheless G_n does not have any cycle isomorphic to C_n.

We deal now with the case $n = 6$. Let us consider a cycle graph C_4 and fix $[v, w] \in E(C_4)$ and $u_1, u_2, u_3 \notin C_4$. Let G_6 be the graph with

$$V(G_6) := V(C_4) \cup \{u_1, u_2, u_3\},$$

$$E(G_6) := E(C_4) \cup \{[v, u_1], [u_1, u_2], [u_2, u_3], [u_3, w], [v, u_3]\}.$$

Note that G_6 has cycles isomorphic to C_4, C_5 and C_7, but it does not have any cycle isomorphic to C_6.

Denote by g the unique cycle in G_6 isomorphic to C_7. Let x be the point in C_4 at distance 2 from the midpoint of the edge $[v, w]$. Consider the geodesic triangle $T = \{x, v, u_2\}$ contained in the cycle g. If p is the point in $[xu_2]$ with $d(p, x) = 2$, then $\delta(G_6) \geq \delta(T) \geq d(p, [xv] \cup [vu_2]) = d(p, u_2) = \frac{6}{4}$.

Since $\operatorname{diam}(G_6) = \frac{7}{2}$, we have $\delta(G_6) \leq \frac{7}{4}$ by Theorem 2.4.4. In order to prove $\delta(G_6) = \frac{6}{4}$, it suffices to check that $\delta(G_6) \neq \frac{7}{4}$, by Theorem 2.4.2. Seeking for a contradiction, assume that $\delta(G_6) = \frac{7}{4}$. Theorem 2.4.3 gives that there exist a geodesic triangle $T = \{x, y, z\} \in \mathbb{T}_1$ in G_6 and $p \in [xy]$ such that $d(p, [xz] \cup [zy]) = \delta(T) = \frac{7}{4}$ and $x, y, z \in J(G_6)$. Since $d(p, \{x, y\}) \geq d(p, [xz] \cup [zy]) = \frac{7}{4}$, $L([xy]) \geq \frac{7}{2}$; the equality $\operatorname{diam}(G_n) = \frac{7}{2}$ implies that

$L([xy]) = \frac{7}{2}$. Then $L(T) = L([xy]) + L([xz] \cup [zy]) \geq 2L([xy]) = 7$. Since $L(g) = 7$ and g is the largest cycle in G_6, T is contained in the cycle g and $L([xz] \cup [zy]) = \frac{7}{2} = L([xy])$. Hence, without loss of generality we can assume that T is the geodesic bigon $T = \{x, y\}$. Denote by γ_1 and γ_2 the two geodesics in T with $\gamma_1 \cup \gamma_2 = g$ and $\gamma_1 \cap \gamma_2 = \{x, y\}$, then p is the midpoint of either γ_1 or γ_2. Since it is impossible to have $x, y \in C_4$ or $x, y \notin C_4$, we can assume that $x \in C_4$ and $y \notin C_4$. Hence, without loss of generality we can assume that $v \in \gamma_1$ and $w \in \gamma_2$. One can check that there are just two possibilities for y: we have that y is either u_2 or the midpoint of the edge $[u_1, u_2]$. Note that $d(v, \gamma_2) = d(v, w) = d(v, u_3) = 1 < \frac{3}{2} \leq d(v, y)$ and $d(w, \gamma_1) = d(w, v) = 1 < 2 \leq d(w, y)$, and this implies that the distance from p to the other side of T is less than $d(p, y) = \frac{7}{4}$, which is a contradiction. Therefore, $\delta(G_n) = \frac{6}{4}$. □

Theorem 3.1.8 below characterizes in a simple way the graphs G with $\delta(G) = 1$. Recall that Bandelt and Chepoi characterize in [6] the 1-hyperbolic graphs with respect to the 4-point condition. The following examples show that there is no relation between the set of 1-hyperbolic graphs satisfying the 4-point condition and the set of 1-hyperbolic graphs with respect to the Rips condition. Denote by $\delta_{4p}(G)$ the 4-point condition hyperbolicity constant of the graph G. One can check that $\delta(C_5) = 5/4$ and, by [6], $\delta_{4p}(C_5) = 1$. We also have $\delta(C_4) = 1$ and, by [6], $\delta_{4p}(C_4) \neq 1$.

The following result characterizes in a simple way the graphs G with $\delta(G) = 1$.

Theorem 3.1.8. *Let G be a graph. Then $\delta(G) = 1$ if and only if the following conditions hold:*

(1) *There exists a cycle isomorphic to C_4.*

(2) *For every cycle γ such that $L(\gamma) \geq 5$ and for every vertex $w \in \gamma$, it is satisfied* $\deg_\gamma(w) \geq 3$.

Proof. On one hand, if $\delta(G) = 1 < \frac{5}{4}$, by Proposition 3.1.4 we obtain (1) and, by Theorem 3.1.2 we have (2). On the other hand, if (1) holds, by Theorems 2.4.2 and 3.1.1 we have $\delta(G) \geq 1$. If (2) also holds, by Theorem 3.1.2 we know that $\delta(G) < \frac{5}{4}$. Finally, since $\delta(G)$ is a multiple of $\frac{1}{4}$ by Theorem 2.4.2, we conclude $\delta(G) = 1$. □

Now we give a sufficient conditions in order to have $\delta(G) = \frac{5}{4}$.

Proposition 3.1.9. *Let G be a graph. Assume that the following conditions hold:*

(1) *There exist a cycle g in G such that $L(g) \geq 5$ and a vertex $w \in g$ satisfying* $\deg_g(w) = 2$.

(2) *For every cycle γ we have* $\operatorname{diam}(\gamma) \leq \frac{5}{2}$.

Then we have $\delta(G) = \frac{5}{4}$.

Proof. By Theorem 3.1.2, if (1) holds we have $\delta(G) \geq \frac{5}{4}$. Let us consider now any geodesic triangle $T = \{x, y, z\}$ that is a cycle; by hypothesis, $\operatorname{diam}(T) \leq \frac{5}{2}$. Hence, for any $p \in [xy]$ we have $d(p, [xz] \cup [zy]) \leq d(p, \{x, y\}) \leq \frac{1}{2}\, d(x, y) \leq \frac{5}{4}$. Therefore, $\delta(T) \leq \frac{5}{4}$ for any geodesic triangle T that is a cycle; in consequence, $\delta(G) \leq \frac{5}{4}$ by Theorem 2.4.3. □

Proposition 3.1.9 gives sufficient conditions in a graph G to have $\delta(G) = \frac{5}{4}$, but condition (2) is not a necessary condition. The graph G showed in Figure 3.3, since $d(x, y) = 3$ and the midpoints of every geodesic from x to y is a vertex of degree 7, satisfies $\delta(G) = \frac{5}{4}$ (we can take the geodesic triangle $T = \{x, y, z\}$ and $p \in [xy]$ such that $d(p, y) = \frac{5}{4}$ to get $d(p, [xz] \cup [zy]) = \frac{5}{4}$). Moreover, this graph contains a cycle γ such that $L(\gamma) \geq 6$ and $\operatorname{diam}(\gamma) = 3$.

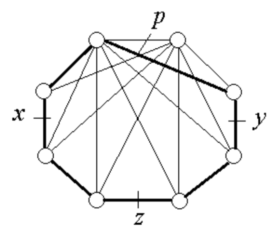

Figure 3.3: Graph G that satisfies $\delta(G) = \frac{5}{4}$.

Proposition 3.1.10. *Let G be a graph. If $\delta(G) \geq \frac{3}{2}$, then there exists a cycle g in G such that $L(g) \geq 6$ and $\operatorname{diam}(g) \geq 3$.*

Proof. By Theorem 2.4.3, if $\delta(G) \geq \frac{3}{2}$, then there exists a geodesic triangle $T = \{x, y, z\}$ that is a cycle, with $x, y, z \in J(G)$, and $p \in [xy]$ such that $\frac{1}{2} \leq d(p, [xz] \cup [zy]) \leq d(p, \{x, y\})$. Therefore, $d(x, y) \geq 3$ and, consequently, we can take $g = T$. □

3.2 Hyperbolicity constant and effective diameter

A graph with small hyperbolicity constant can have arbitrarily large diameter: the path graph with n vertices P_n verifies $\delta(P_n) = 0$ and $\operatorname{diam}(P_n) = \operatorname{diam} V(P_n) = n-1$ for every n. However, there is a concept related with the diameter, the *effective diameter*, which is small when the hyperbolicity constant is small, as we will prove in this chapter.

Definition 3.2.1. *We say that a vertex v of a graph G is a* cut-vertex *if $G \setminus \{v\}$ is not connected. A graph is* two-connected *if it does not contain cut-vertices. Given any edge in G, let us consider the maximal two-connected subgraph containing it. We call to the set of these maximal two-connected subgraphs $\{G_n\}_n$ the* canonical T-decomposition *of G.*

We will need the following result, which allows to obtain global information about the hyperbolicity constant of a graph from local information (see [14, Theorem 3]).

Lemma 3.2.2. *Let G be any graph with canonical T-decomposition $\{G_n\}_n$. Then*

$$\delta(G) = \sup_n \delta(G_n).$$

Definition 3.2.3. *Given a graph G and its canonical T-decomposition $\{G_n\}$, we define the* effective diameter *as*

$$\operatorname{diameff} V(G) := \sup_n \operatorname{diam} V(G_n), \qquad \operatorname{diameff}(G) := \sup_n \operatorname{diam}(G_n).$$

Lemma 3.2.2 and Theorem 2.4.4 have the following consequence.

Lemma 3.2.4. *Let G be any graph. Then*

$$\delta(G) \le \frac{1}{2} \operatorname{diameff}(G).$$

As a corollary of Theorems 2.4.2 and 3.1.1, every G_n in the canonical T-decomposition of a graph G with $\delta(G) < 1$ is isomorphic to either K_2 or K_3. Therefore, we have the following result.

Proposition 3.2.5. *Let G be any graph. If $\delta(G) < 1$, then* $\operatorname{diameff} V(G) = 1$ *and* $\operatorname{diameff}(G) \le \frac{3}{2}$. *Furthermore:*

(1) $\delta(G) = 0$ *if and only if* $\operatorname{diameff} V(G) = \operatorname{diameff}(G) = 1$.

(2) $\delta(G) = \frac{3}{4}$ *if and only if* $\operatorname{diameff} V(G) = 1$ *and* $\operatorname{diameff}(G) = \frac{3}{2}$.

We are going to obtain now sharp bounds for $\operatorname{diameff} V(G)$ and $\operatorname{diameff}(G)$ when $\delta(G) < \frac{3}{2}$. This is the only case where we can obtain them, since Remark 3.2.19 shows that it is not possible to obtain similar bounds if $\delta(G) \ge \frac{3}{2}$. We start with some lemmas which will be useful in order to simplify the proofs of the main results. Recall that $J(G)$ denotes the union of the set $V(G)$ and the midpoints of the edges of G.

Lemma 3.2.6. *Let G be any graph. If there exists a cycle C in G containing a geodesic $[vw]$ with $v, w \in V(G)$, $3 \le d(v, w) \le 5$ and $L(C) \le d(v, w) + 5$, then $\delta(G) \ge \frac{3}{2}$.*

Proof. Define
$$\mathcal{F} := \{\, \sigma \mid \sigma \text{ is a cycle in } G \text{ containing } [vw] \,\}.$$
We know that $\mathcal{F} \neq \emptyset$ by hypothesis. Let $C_0 \in \mathcal{F}$ such that $L(C_0) = \min_{\sigma\in\mathcal{F}} L(\sigma)$.

Let γ be the path joining v and w with $[vw] \cup \gamma = C_0$ and $[vw] \cap \gamma = \{v, w\}$. Therefore, $L(\gamma) \le 5$.

Let u be the midpoint of γ and γ', γ'' the two paths such that $v \in \gamma'$, $w \in \gamma''$, $\gamma' \cup \gamma'' = \gamma$ and $\gamma' \cap \gamma'' = \{u\}$. Thus $L(\gamma') = L(\gamma'') \le \frac{5}{2}$, and this implies that γ' is a geodesic, since otherwise there is an edge e joining v and a vertex of γ with e not contained in γ, and there exists a cycle C_1 containing $[vw]$ with $L(C_1) < L(C_0)$, which is a contradiction.

By symmetry, γ'' is a geodesic and $T = \{[vw], \gamma', \gamma''\}$ is a geodesic triangle in G that is a cycle. Let p be the point in $[vw]$ with $d(p, v) = \frac{3}{2}$. Then p is the midpoint of an edge and $\delta(G) \ge d(p, \gamma' \cup \gamma'') = d(p, v) = \frac{3}{2}$. □

Theorem 3.2.7. *Let G be any graph. If there exists a cycle in G containing a geodesic $[pq]$ with $p, q \in V(G)$ and $d(p, q) \ge 3$, then $\delta(G) \ge \frac{3}{2}$.*

Proof. Let $\mathcal{C}$ be the set of cycles in G containing a geodesic of length 3 joining two vertices. Since $\mathcal{C} \neq \emptyset$ by hypothesis, there exists $C_0 \in \mathcal{C}$ such that $L(C_0) \le L(C)$ for every $C \in \mathcal{C}$. Let $v, w \in V(G)$ such that $d(v, w) = 3$ and $[vw] \subset C_0$. Denote by $v_0, w_0, v_1, v_2, \dots, v_r$ the other vertices of C_0, with $\{[v, v_0], [v_0, w_0], [w_0, w], [v, v_1], [v_1, v_2], \dots, [v_{r-1}, v_r], [v_r, w]\} = E(G) \cap C_0$. The conditions on C_0 imply the following:

$$\text{if } g \text{ is a geodesic joining } x, y \in V(G) \cap (C_0 \setminus \{v_0, w_0\}) \quad (3.1)$$
$$\text{with } L(g) < d_{C_0}(x, y), \text{ then } g \cap \{v_0, w_0\} \neq \emptyset.$$

Since $d(v, w) = 3$, by Lemma 3.2.6, we can assume that $L(C_0) \ge 9$, since otherwise $\delta(G) \ge \frac{3}{2}$. We also know by triangle inequality that $2 \le d(v_1, w) \le 4$.

Seeking for a contradiction assume that $d(v_1, w) = 3$. Note that $d_{C_0}(v_1, w) = 4$ since $L(C_0) \ge 9$. Then $[v_1w] \cap (C_0 \setminus ([v_0, w_0] \cup [w_0, w])) = \{v_1\}$ by (3.1) and $d_{C_0}(v_1, w) = 4$. Therefore the cycle $C_0' := (C_0 \setminus ([v_1, v] \cup [v, v_0] \cup [v_0, w_0] \cup [w_0, w])) \cup [v_1w]$ belongs to $\mathcal{C}$ and $L(C_0') = L(C_0) - 1 < L(C_0)$, contradicting the minimality of C_0. Hence, $d(v_1, w) \neq 3$. The same argument proves that $d(v_r, v) \neq 3$. If $d(v_1, w) = 2$, then $[v_1, w_0] \in E(G)$ by (3.1) and $d_{C_0}(v_1, w) = 4$.

Seeking for a contradiction, assume that $d(v_1, w) = 2$ and $d(v_r, v) = 2$. The previous argument gives that $[v_1, w_0], [v_r, v_0] \in E(G)$. Consider the geodesics $\gamma_1 := [v, v_1] \cup [v_1, w_0] \cup [w_0, w]$ and $\gamma_2 := [v, v_0] \cup [v_0, v_r] \cup [v_r, w]$ joining v and w. Since $\gamma_1 \cup \gamma_2 \in \mathcal{C}$ and $L(\gamma_1 \cup \gamma_2) = 6 < 9 \le L(C_0)$, we obtain a contradiction.

Hence, we have $d(v_1, w) = 4$ or $d(v_r, v) = 4$. By symmetry we can assume that $d(v_1, w) = 4$. Then $d(v_1, w_0) = 3$ and $d(v_2, w) \geq d(w, v_1) - d(v_1, v_2) = 3$.

Recall that a geodesic η joining two vertices $a, b \in C_0$ is a shortcut if $d(a, b) = L(\eta) < d_{C_0}(a, b)$ and $\eta \cap C_0 = \{a, b\}$.

Since $d(v_1, w) = 4$ and $w_0 \in [v_1 w]$, there is no shortcut joining w_0 and v_1.

Seeking for a contradiction assume that there exists a shortcut η joining w_0 and v_j for some $2 \leq j \leq r$. Then $C_2 := [v_j, v_{j-1}] \cup \cdots \cup [v_2, v_1] \cup [v_1, v] \cup [v, v_0] \cup [v_0, w_0] \cup \eta$ is a cycle and $L(C_2) < L(C_0)$. Since $d(v_1, w_0) = 3$, we have $C_2 \in \mathcal{C}$, contradicting the minimality of C_0.

Hence, for each $1 \leq j \leq r$, there is no shortcut joining w_0 and v_j.

Since $d(v_1, w) = 4$, we have $d(v_1, v_0) = 2$ and triangle inequality gives $1 \leq d(v_2, v_0) \leq 3$. Seeking for a contradiction assume that $d(v_2, v_0) < 3$. Assume first that $d(v_2, v_0) = 1$. Then $[v_0, v_2] \in E(G)$, and $C_3 := (C_0 \setminus ([v_2, v_1] \cup [v_1, v] \cup [v, v_0])) \cup [v_0, v_2]$ is a cycle such that $L(C_3) < L(C_0)$. Since $d(v_2, w) \geq 3$, we have $C_3 \in \mathcal{C}$, contradicting the minimality of C_0. Assume now that $d(v_2, v_0) = 2$. Then there exists a vertex u with $[v_2, u], [u, v_0] \in E(G)$. Since $d(v_1, v_0) = 2$ and there is no shortcut joining w_0 and v_2, by (3.1) we have either $u = v_3$ or $u \notin C_0$. Since there is no shortcut joining w_0 and v_j ($1 \leq j \leq r$), we deduce from (3.1) that $d(v_2, w) = 4$ and, hence, $d(v_3, w) \geq 3$. Let us define $C_4 := (C_0 \setminus ([v_2, v_1] \cup [v_1, v] \cup [v, v_0])) \cup [v_0 v_2]$ if $u \notin C_0$, and $C_4 := (C_0 \setminus ([v_3, v_2] \cup [v_2, v_1] \cup [v_1, v] \cup [v, v_0])) \cup [v_0, v_3]$ if $u = v_3$. Then C_4 is a cycle and $L(C_4) < L(C_0)$. Since $d(v_3, w) \geq 3$, we have $C_4 \in \mathcal{C}$, contradicting the minimality of C_0.

Therefore, $d(v_2, v_0) = 3$ and there is no shortcut joining v_0 and v_2. Since $d(v_1, v_0) = 2$, there is no shortcut joining v_0 and v_1. Seeking for a contradiction assume that there exists a shortcut μ joining v_0 and v_j for some $3 \leq j \leq r$. Then $C_5 := [v_j, v_{j-1}] \cup \cdots \cup [v_2, v_1] \cup [v_1, v] \cup [v, v_0] \cup \mu$ is a cycle and $L(C_5) < L(C_0)$. Since $d(v_2, v_0) = 3$, we have $C_5 \in \mathcal{C}$, contradicting the minimality of C_0.

Hence, for each $1 \leq j \leq r$, there is no shortcut joining v_0 and v_j. Since we have proved the similar result for w_0, by (3.1), C_0 is isometric to a cycle graph with the same length than it and, consequently, C_0 is an isometric subgraph of G. Then Lemmas 2.4.5 and 2.4.6 give $\delta(G) \geq \delta(C_0) = \frac{1}{4} L(C_0) \geq \frac{9}{4} > \frac{3}{2}$.

□

Proposition 3.2.8. *Let G be any graph. If G does not have cut-vertices and $\delta(G) \leq \frac{5}{4}$, then* $\operatorname{diam} V(G) \leq 2$ *and* $\operatorname{diam}(G) \leq 3$.

Proof. Note that $\operatorname{diam}(G) \leq 3$ is a direct consequence of $\operatorname{diam} V(G) \leq 2$. Seeking for a contradiction, assume that $\operatorname{diam} V(G) \geq 3$. Let γ be a geodesic in G joining v_1 and v_4 whose vertices are $\{v_1, v_2, v_3, v_4\}$, in this order; then $L(\gamma) = 3$. Since v_2 is not a cut-vertex, there is a path γ_1 joining v_1 and v_3 with $v_2 \notin \gamma_1$. Without loss of generality we can assume that γ_1 is minimal, i.e., $L(\gamma_1) \leq L(g_1)$ for every path g_1 joining v_1 and v_3 with $v_2 \notin g_1$. Similarly, there is a minimal path γ_2 joining v_2 and v_4 with $v_3 \notin \gamma_2$.

Case 1. If $v_1 \in \gamma_2$ or $v_4 \in \gamma_1$, then by symmetry we can assume that $v_1 \in \gamma_2$. Hence,

$C = \gamma \cup \gamma_2$ is a cycle containing a geodesic with length 3 and Theorem 3.2.7 gives $\delta(G) \geq \frac{3}{2}$. This is a contradiction and then we can assume that $v_1 \notin \gamma_2$ and $v_4 \notin \gamma_1$.

Case 2. Assume that $\gamma_1 \cap \gamma_2 \neq \emptyset$. Then there exists $x \in \gamma_1 \cap \gamma_2 \cap V(G)$ such that if γ_1' is the subcurve of γ_1 joining v_1 and x, and γ_2' is the subcurve of γ_2 joining x and v_4, then $\gamma_1' \cup \gamma_2'$ joins v_1 and v_4, and $v_2, v_3 \notin \gamma_1' \cup \gamma_2'$. Then $C = \gamma_1' \cup \gamma_2' \cup \gamma$ is a cycle and Theorem 3.2.7 gives $\delta(G) \geq \frac{3}{2}$, which is a contradiction.

Case 3. Finally, assume that $\gamma_1 \cap \gamma_2 = \emptyset$, $v_1 \notin \gamma_2$ and $v_4 \notin \gamma_1$. If $L(\gamma_2) = 2$, then $[v_1, v_2] \cup \gamma_2$ is a geodesic with length 3, and therefore we can apply Theorem 3.2.7 taking the cycle $C = [v_1, v_2] \cup \gamma_2 \cup [v_3, v_4] \cup \gamma_1$ and the geodesic $[v_1, v_2] \cup \gamma_2$, obtaining a contradiction. If $L(\gamma_2) > 2$, then consider the path $\gamma_2' = \{v_1, v_2, w_1, w_2\}$ with $w_1, w_2 \in \gamma_2$, $d_{\gamma_2}(v_2, w_1) = 1$ and $d_{\gamma_2}(v_2, w_2) = 2$. If γ_2' is a geodesic, again applying Theorem 3.2.7 to the cycle $(\gamma \setminus [v_2, v_3]) \cup \gamma_1 \cup \gamma_2$ we have a contradiction. Otherwise, considerer a geodesic γ_2'' joining v_1 and w_2. The minimality of γ_2 gives that $[v_2, w_2] \notin V(G)$. Assume that $[v_1, w_1] \in E(G)$, then $\gamma \cup [v_1, w_1] \cup (\gamma_2 \setminus [v_2, w_1])$ is a cycle and Theorem 3.2.7 provides a contradiction. Hence, $[v_1, w_1] \notin E(G)$, $v_2, w_1 \notin \gamma_2''$ and $\gamma_2'' \cap \gamma = \{v_1\}$; then we have a cycle $\gamma \cup \gamma_2'' \cup (\gamma_2 \setminus \{[v_2, w_1] \cup [w_1, w_2]\})$ in G and Theorem 3.2.7 gives a contradiction.

Thus $\operatorname{diam} V(G) \leq 2$ and, consequently, $\operatorname{diam}(G) \leq 3$. □

Finally, we obtain an upper bound of $\operatorname{diameff} V(G)$ and $\operatorname{diameff}(G)$ for every graph G with $\delta(G) = \frac{5}{4}$.

Theorem 3.2.9. *Let G be any graph. If $\delta(G) = \frac{5}{4}$, then* $\operatorname{diameff} V(G) \leq 2$ *and* $\operatorname{diameff}(G) \leq 3$*, and the inequalities are sharp.*

Proof. Note that $\operatorname{diameff}(G) \leq 3$ is a direct consequence of $\operatorname{diameff} V(G) \leq 2$. Seeking for a contradiction, assume that $\operatorname{diameff} V(G) > 2$. If $\{G_n\}_n$ is the canonical T-decomposition of G, then there exists n_0 with $\operatorname{diam}(G_{n_0}) > 2$. Since G_{n_0} does not have cut-vertices, Proposition 3.2.8 gives $\delta(G_{n_0}) > \frac{5}{4}$ and Lemma 3.2.2 gives $\delta(G) > \frac{5}{4}$, which is a contradiction. Thus $\operatorname{diameff} V(G) \leq 2$ and, consequently, $\operatorname{diameff}(G) \leq 3$.

Note that $\operatorname{diameff}(G) \leq 3$ is a direct consequence of $\operatorname{diameff} V(G) \leq 2$. Seeking for a contradiction, assume that $\operatorname{diameff} V(G) > 2$. If $\{G_n\}_n$ is the canonical T-decomposition of G, then there exists n_0 with $\operatorname{diam}(G_{n_0}) > 2$. Since G_{n_0} does not have cut-vertices, Proposition 3.2.8 gives $\delta(G_{n_0}) > \frac{5}{4}$ and Lemma 3.2.2 gives $\delta(G) > \frac{5}{4}$, which is a contradiction. Thus $\operatorname{diameff} V(G) \leq 2$ and, consequently, $\operatorname{diameff}(G) \leq 3$.

In order to show that the inequalities are sharp, consider two graphs G_1 and G_2 isomorphic to the complete graph K_4. Fix $e_j \in E(G_j)$ ($j = 1, 2$) and consider the unique edge $e_j' \in E(G_j)$ with $e_j \cap e_j' = \emptyset$; let x_j be the midpoint of e_j'. Define G as the graph obtained from G_1 and G_2 by identifying e_1 and e_2. Denote by e the edge of G obtained by identifying e_1 and e_2. One can check that $\operatorname{diameff} V(G) = \operatorname{diam} V(G) = 2$, $\operatorname{diameff}(G) = \operatorname{diam}(G) = 3$ and $d_G(x, y) = 3$ if and only if $\{x, y\} = \{x_1, x_2\}$. Thus $\delta(G) \leq \frac{3}{2}$ by Theorem 2.4.4. Let γ_1 and γ_2 be two geodesics in G joining x_1 and x_2 with $\gamma_1 \cap \gamma_2 = \{x_1, x_2\}$, and B the geodesic bigon $B = \{\gamma_1, \gamma_2\}$. One can check that $\delta(G) \geq \delta(B) = \frac{5}{4}$. Hence, Theorem 2.4.2 implies $\delta(G) \in \{\frac{5}{4}, \frac{3}{2}\}$. Seeking for a contradiction assume that $\delta(G) = \frac{3}{2}$. By Theorem 2.4.3, there exist a geodesic triangle $T = \{x, y, z\}$ that is a cycle and $p \in [xy]$ such that $d_G(p, [xz] \cup [zy]) = \frac{3}{2}$. It is clear that $d_G(p, x), d_G(p, y) \geq \frac{3}{2}$; thus, $d_G(x, y) \geq 3$ and $L(T) \geq 6$. Since $\operatorname{diam}(G) = 3$, $d_G(x, y) = 3$ and we conclude $\{x, y\} = \{x_1, x_2\}$, $p \in V(G)$ and $p \in e$. Thus, $e = [p, q]$ for some vertex q. Since G has six vertices and T is a cycle, $L(T) = 6$ and T is a geodesic bigon $T = \{x_1, x_2\} = \{g_1, g_2\}$. Without loss of generality we can assume that $p \in g_1$, and then $q \in g_2$ and $d_G(p, g_2) = d_G(p, q) = 1$, which is a contradiction. Hence, $\delta(G) \neq \frac{3}{2}$ and we conclude $\delta(G) = \frac{5}{4}$. Since $\operatorname{diameff} V(G) = 2$ and $\operatorname{diameff}(G) = 3$, the inequalities are sharp.

□

We prove now a similar result to Theorem 3.2.9 for graphs with hyperbolicity constant 1. We need some previous results.

Proposition 3.2.10. *Let G be any graph. If there exists a cycle in G containing a geodesic $[vw]$ with $v, w \in J(G)$ and $d(v, w) \geq 3$, then $\delta(G) \geq \frac{5}{4}$.*

Proof. Without loss of generality we can assume that $d(v, w) = 3$, since otherwise we can take a subset of $[vw]$. Let C_0 be a cycle in G containing $[vw]$ and minimal, i.e., such that $L(C_0) \leq L(C)$ for every cycle C containing $[vw]$. Since $d(v, w) = 3$, $L(C_0) \geq 6$.

Note that we have either $v, w \in V(G)$ or $v, w \in J(G) \setminus V(G)$.
If $v, w \in V(G)$, then Theorem 3.2.7 gives $\delta(G) \geq \frac{3}{2} > \frac{5}{4}$.

Assume now $v, w \in J(G) \setminus V(G)$. Let v_1, v_2, v_3 be the vertices in $[vw]$ with $d(v, v_j) = (j - \frac{1}{2})$. Since $w \notin V(G)$ there exists $v_4 \in V(G)$ with $w \in [v_3, v_4]$. Define γ as $\gamma := (C_0 \setminus [v_1 w]) \cup \{v_1\}$. Let $v_1, w_1, \ldots, w_r$ be the vertices in γ with $d_\gamma(v_1, w_j) = j$ for $1 \leq j \leq r$ and $w_r = v_4$. Define $s := \max\{1 \leq j \leq r - 1 \,|\, [v_1, w_j] \in E(G)\}$ and C as the cycle $C := [v_1, v_2] \cup [v_2, v_3] \cup [v_3, w_r] \cup [w_r, w_{r-1}] \cup \ldots \cup [w_{s+1}, w_s] \cup [w_s, v_1]$. Note that $[v_1, v_3], [v_1, w_r] \notin E(G)$ since $d(v, w) = 3$, furthermore, $[v_1, w_j] \notin E(G)$ for $j > s$; hence, $\deg_C(v_1) = 2$ and $L(C) \geq 5$, since $d(v_1, w) = \frac{5}{2}$, and Theorem 3.1.2 gives $\delta(G) \geq \frac{5}{4}$. □

The argument in the proof of Proposition 3.2.10 also gives the following result.

Corollary 3.2.11. *Let G be any graph. If there exists a cycle in G containing a geodesic $[vw]$ with $v, w \in V(G)$, $d(v, w) = 2$ and $\delta(G) = 1$, then $[vw]$ is contained in a cycle with length 4.*

Proposition 3.2.12. *Let G be any graph. If G does not have cut-vertices and $\delta(G) \leq 1$, then* $\operatorname{diam} V(G) \leq 2$ *and* $\operatorname{diam}(G) \leq \frac{5}{2}$.

Proof. By Proposition 3.2.8 we know that $\operatorname{diam} V(G) \leq 2$ and $\operatorname{diam}(G) \leq 3$. Seeking for a contradiction, assume that $\operatorname{diam}(G) > \frac{5}{2}$, that is, $\operatorname{diam}(G) = 3$.

Then there are $v, w \in J(G) \setminus V(G)$ such that $d(v, w) = 3$. Let γ be a geodesic in G joining v and w. Assume that v and w are the midpoints of $[v_1, v_2]$ and $[w_1, w_2]$, respectively (with $v_1, w_1 \in \gamma$). Since $d(v_2, w_2) = 2$, there is a geodesic γ_1 joining v_2 and w_2 with $L(\gamma_1) = 2$.
Case 1. If $\gamma \cap \gamma_1 = \emptyset$, then we have a cycle $C := [v_1 w_1] \cup [w_1, w_2] \cup \gamma_1 \cup [v_2, v_1]$ with the geodesic $\gamma \subset C$ and Proposition 3.2.10 gives $\delta(G) \geq \frac{5}{4}$, which is a contradiction.
Case 2. Assume that $\gamma \cap \gamma_1 \neq \emptyset$. Let $x \in V(G)$ be the midpoint of γ. Since $d(v_i, w_j) = 2$ for $i, j \in \{1, 2\}$, $v_1, w_1 \notin \gamma_1$ and $\gamma_1 = [v_2, x] \cup [x, w_2]$. Since x is not a cut-vertex, there is a path γ_2 joining v_1 and w_1 with $x \notin \gamma_2$. Without loss of generality we can assume that γ_2 is a shortest path.
Case 2.1. If $v_2 \notin \gamma_2$ and $w_2 \notin \gamma_2$, then consider the cycle $C := [v_1, v_2] \cup \gamma_1 \cup [w_2, w_1] \cup \gamma_2$ with the geodesic $\gamma' := [v v_2] \cup \gamma_1 \cup [w_2 w] \subset C$. Proposition 3.2.10 gives $\delta(G) \geq \frac{5}{4}$, which is a contradiction.
Case 2.2. Assume that $v_2, w_2 \in \gamma_2$. In this case we have a cycle $C := \gamma \cup [w w_2] \cup (\gamma_2 \setminus \{[v_1, v_2], [w_1, w_2]\}) \cup [v_2 v]$ with $\gamma \subset C$. Proposition 3.2.10 gives $\delta(G) \geq \frac{5}{4}$, which is a contradiction.
Case 2.3. Finally, assume that either $v_2 \in \gamma_2$ and $w_2 \notin \gamma_2$, or $v_2 \notin \gamma_2$ and $w_2 \in \gamma_2$. By symmetry we can assume that $v_2 \in \gamma_2$ and $w_2 \notin \gamma_2$. Consider the vertex $u \in \gamma_2$ such that $u \neq v_1$ and $[u, v_2] \in E(G)$, and let v' be the midpoint of $[u, v_2]$.
Case 2.3.1. If $d(v', w) = 3$, then $\sigma := [v' v_2] \cup \gamma_1 \cup [w_2 w]$ is a geodesic joining v' and w, and therefore we can apply Proposition 3.2.10 taking the cycle $C := (\gamma_2 \setminus [v_1, v_2]) \cup [w_1, w_2] \cup \gamma_1$ and the geodesic $\sigma \subset C$, obtaining a contradiction.
Case 2.3.2. If $d(v', w) < 3$, then $d(v', w) = 2$. Since $d(v_2, w_j) = 2$ for $j \in \{1, 2\}$, u must be adjacent to w_1 or w_2.
If $d(u, w_2) = 1$, then $C := [u, w_2] \cup [w_2 w] \cup \gamma \cup [v v_2] \cup [v_2, u]$ is a cycle containing the geodesic γ. Therefore, we obtain a contradiction by applying Proposition 3.2.10.
If $d(u, w_2) > 1$, then $d(u, w_1) = 1$. Consider the cycle $C := [u, w_1] \cup [w_1, w_2] \cup [w_2, x] \cup [x, v_2] \cup [v_2, u]$. Since $d(u, w_2) > 1$ and $d(v_2, w_2) = 2$, $[u, w_2], [v_2, w_2] \notin E(G)$ and $\deg_C(w_2) = 2$. Therefore, $L(C) = 5$ and, by Theorem 3.1.2 we have $\delta(G) \geq \frac{5}{2}$, which is a contradiction. □

This result can be improved as follows.

Proposition 3.2.13. *Let G be any graph. If G does not have cut-vertices and $\delta(G) \leq 1$, then* $\operatorname{diam}(G) \leq 2$.

Proof. Seeking for a contradiction, assume that $\operatorname{diam}(G) > 2$, then $\operatorname{diam}(G) = \frac{5}{2}$ by Proposition 3.2.12. Therefore, $\operatorname{diam} V(G) = 2$.

Then there are $v \in V(G), w \in J(G) \setminus V(G)$ such that $d(v, w) = \frac{5}{2}$. Let γ be a geodesic in G joining v and w. Assume that w is the midpoint of $[w_1, w_2]$, with $w_1 \in \gamma$. Since $\operatorname{diam} V(G) = 2$ and $d(v, w) = \frac{5}{2}$, $d(v, w_2) = 2$ and there is a geodesic γ_1 joining v and w_2 with $L(\gamma_1) = 2$.

Case 1. If $\gamma \cap \gamma_1 = \{v\}$, then $\gamma \cup [ww_2] \cup \gamma_1$ is a cycle. Consider the geodesic bigon $B := \{\gamma, \gamma_1 \cup [w_2 w]\}$. If p is the midpoint of γ, then $\delta(G) \ge \delta(B) \ge d(p, \gamma_1 \cup [w_2 w]) = \frac{5}{4}$, which is a contradiction.
Case 2. Assume that $\gamma \cap \gamma_1 \neq \{v\}$. Let $x \in V(G)$ be the vertex of γ with $d(x, v) = 1$. Since $d(v, w_1) = d(v, w_2) = 2$, $\gamma_1 = [v, x] \cup [x, w_2]$ and $\gamma \cap \gamma_1 = [v, x]$. Since x is not a cut-vertex, there is a path γ_2 joining v and $\{w_1, w_2\}$ with $x \notin \gamma_2$. Without loss of generality we can assume that γ_2 has minimum length, i.e., $L(\gamma_2) \le L(g)$ for every path g joining v and the set $\{w_1, w_2\}$. By symmetry, we can assume that γ_2 joins v and w_1.
Case 2.1. If $L(\gamma_2) = 2$, then $\gamma_2 \cup [w_1, w_2] \cup \gamma_1$ is a cycle. Consider the geodesic bigon $B := \{\gamma_2 \cup [w_1 w], \gamma_1 \cup [w_2 w]\}$. If p is the midpoint of $\gamma_2 \cup [w_1 w]$, then $\delta(G) \ge \delta(B) \ge d(p, \gamma_1 \cup [w_2 w]) = \frac{5}{4}$, which is a contradiction.
Case 2.2. If $L(\gamma_2) \ge 3$, then consider the cycle $C := [w_1 v] \cup \gamma_2$ with $L(C) \ge 5$. The minimality of γ_2 gives $\deg_C(w_1) = 2$. Since $L(C) \ge 5$, $\delta(G) \ge \frac{5}{2}$ by Theorem 3.1.2, which is a contradiction.

Thus $\operatorname{diam}(G) \le 2$.

□

Finally, we obtain the precise value of diameff(G) for every graph G with $\delta(G) = 1$. Furthermore, the next result is another characterization of the graphs with $\delta(G) = 1$.

Theorem 3.2.14. *Let G be any graph. Then $\delta(G) = 1$ if and only if* diameff$(G) = 2$.

Proof. Assume that diameff$(G) = 2$. Lemma 3.2.4 gives $\delta(G) \le \frac{1}{2}$diameff$(G) = 1$. If $\delta(G) < 1$, then Proposition 3.2.5 gives diameff$(G) \le \frac{3}{2}$, which contradicts diameff$(G) = 2$. Hence, $\delta(G) = 1$.

Assume now $\delta(G) = 1$. Seeking for a contradiction, assume that diameff$(G) > 2$. If $\{G_n\}_n$ is the canonical T-decomposition of G, then there exists n_0 with $\operatorname{diam}(G_{n_0}) > 2$.

Since G_{n_0} does not have cut-vertices, Proposition 3.2.13 gives $\delta(G_{n_0}) > 1$ and Lemma 3.2.2 gives $\delta(G) > 1$, which is a contradiction. Thus diameff$(G) \le 2$. Furthermore, Lemma 3.2.4 gives $2 = 2\delta(G) \le$ diameff(G). Hence, diameff$(G) = 2$. □

Proposition 3.2.5 and Theorem 3.2.14 imply the following results.

Corollary 3.2.15. *Let G be any graph and $\frac{3}{2} \le$ diameff$(G) \le 2$. Then*

$$\delta(G) = \frac{1}{2} \operatorname{diameff}(G).$$

Corollary 3.2.16. *Let G be any graph. Then $\delta(G) \ge \frac{5}{4}$ if and only if* diameff$(G) \ge \frac{5}{2}$.

Corollary 3.2.17. *Let G be any graph and $\frac{3}{2} \le$ diameff$(G) \le \frac{5}{2}$. Then*

$$\delta(G) = \frac{1}{2} \operatorname{diameff}(G).$$

Proof. If $\frac{3}{2} \le \operatorname{diameff}(G) \le 2$, then Corollary 3.2.15 gives the equality. Assume now that $\operatorname{diameff}(G) = \frac{5}{2}$. We have $\delta(G) \ge \frac{5}{4}$ by Corollary 3.2.16. Finally, $\delta(G) \le \frac{5}{4}$ by Lemma 3.2.4. □

Lemma 3.2.4, Proposition 3.2.5 and Theorems 3.2.9 and 3.2.14 have the following consequence.

Corollary 3.2.18. *Let G be any graph. If $\delta(G) < \frac{3}{2}$, then*

$$\frac{5}{8}\big(\operatorname{diameff}(G) - 1\big) \le \delta(G) \le \frac{1}{2}\operatorname{diameff}(G).$$

Remark 3.2.19. *It is not possible to bound* $\operatorname{diameff} V(G)$ *or* $\operatorname{diameff}(G)$ *if* $\delta(G) \ge \frac{3}{2}$:

Let G be the Cayley graph of the group $\mathbb{Z} \times \mathbb{Z}_2$ (G has the shape of an infinite railway). We have $\delta(G) = \frac{3}{2}$ and the canonical T-decomposition of G has just a graph $G_1 = G$; hence, $\operatorname{diameff} V(G) = \operatorname{diam} V(G_1) = \infty$ *and* $\operatorname{diameff}(G) = \infty$.

For each $n > 6$ consider the cycle graph C_n, and fix vertices $v_1 \in V(G)$ and $v_2 \in V(C_n)$. The graph G_n obtained from G and C_n by identifying v_1 and v_2 has canonical T-decomposition $\{G, C_n\}$ and $\operatorname{diameff} V(G_n) = \operatorname{diameff} V(G) = \infty$ *and* $\operatorname{diameff}(G_n) = \infty$. *Furthermore, Lemmas 3.2.2 and 2.4.6 give*

$$\delta(G_n) = \max\big\{\delta(G),\, \delta(C_n)\big\} = \max\left\{\frac{3}{2}\,,\, \frac{n}{4}\right\} = \frac{n}{4}\,.$$

In order to characterize the graphs with hyperbolicity constant $\frac{5}{4}$ we define some families of graphs. Denote by C_n the cycle graph with $n \ge 3$ vertices and by $V(C_n) := \{v_1^{(n)}, \dots, v_n^{(n)}\}$ the set of their vertices such that $[v_n^{(n)}, v_1^{(n)}] \in E(C_n)$ and $[v_i^{(n)}, v_{i+1}^{(n)}] \in E(C_n)$ for $1 \le i \le n-1$.

Let $\mathcal{C}_6$ be the set of graphs obtained from C_6 by adding a (proper or not) subset of the set of edges $\{[v_2^{(6)}, v_6^{(6)}],\, [v_4^{(6)}, v_6^{(6)}]\}$. Let us define the set of graphs

$\mathcal{F}_6 :=$ {graphs containing, as induced subgraph, an isomorphic graph to some element of $\mathcal{C}_6$}.

Let $\mathcal{C}_7$ be the set of graphs obtained from C_7 by adding a (proper or not) subset of the set of edges $\{[v_2^{(7)}, v_6^{(7)}],\, [v_2^{(7)}, v_7^{(7)}],\, [v_4^{(7)}, v_6^{(7)}],\, [v_4^{(7)}, v_7^{(7)}]\}$.

Define

$\mathcal{F}_7 :=$ {graphs containing, as induced subgraph, an isomorphic graph to some element of $\mathcal{C}_7$}.

Let $\mathcal{C}_8$ be the set of graphs obtained from C_8 by adding a (proper or not) subset of the set $\{[v_2^{(8)}, v_6^{(8)}],\, [v_2^{(8)}, v_8^{(8)}],\, [v_4^{(8)}, v_6^{(8)}],\, [v_4^{(8)}, v_8^{(8)}]\}$. Also, let $\mathcal{C}_8'$ be the set of graphs obtained from C_8 by adding a (proper or not) subset of $\{[v_2^{(8)}, v_8^{(8)}],\, [v_4^{(8)}, v_6^{(8)}],\, [v_4^{(8)}, v_7^{(8)}],\, [v_4^{(8)}, v_8^{(8)}]\}$.

Define

$\mathcal{F}_8 :=$ {graphs containing, as induced subgraph, an isomorphic graph to some element of $\mathcal{C}_8 \cup \mathcal{C}'_8$}.

Let $\mathcal{C}_9$ be the set of graphs obtained from C_9 by adding a (proper or not) subset of the set of edges $\{[v_2^{(9)}, v_6^{(9)}], [v_2^{(9)}, v_9^{(9)}], [v_4^{(9)}, v_6^{(9)}], [v_4^{(9)}, v_9^{(9)}]\}$. Define

$\mathcal{F}_9 :=$ {graphs containing, as induced subgraph, an isomorphic graph to some element of $\mathcal{C}_9$}.

Finally, we define the set $\mathcal{F}$ by

$$\mathcal{F} := \mathcal{F}_6 \cup \mathcal{F}_7 \cup \mathcal{F}_8 \cup \mathcal{F}_9.$$

In [26, Lemma 3.21] appears the following result.

Lemma 3.2.20. *Let G be any graph. Then $G \in \mathcal{F}$ if and only if there is a geodesic triangle $T = \{x, y, z\}$ in G that is a cycle with $x, y, z \in J(G)$, $L([xy]), L([yz]), L([zx]) \leq 3$ and $\delta(T) = \frac{3}{2} = d(p, [yz] \cup [zx])$ for some $p \in [xy] \cap V(G)$.*

Finally, we obtain a simple characterization of the graphs G with $\delta(G) = \frac{5}{4}$.

Theorem 3.2.21. *Let G be any graph. Then $\delta(G) = \frac{5}{4}$ if and only if we have either* $\operatorname{diameff}(G) = \frac{5}{2}$ *or* $\operatorname{diameff} V(G) = 2$, $\operatorname{diameff}(G) = 3$ *and* $G \notin \mathcal{F}$.

Proof. Assume that $\delta(G) = \frac{5}{4}$. Theorem 3.2.9 and Lemma 3.2.20 give $\operatorname{diameff} V(G) \leq 2$, $\operatorname{diameff}(G) \leq 3$ and $G \notin \mathcal{F}$. Furthermore, $\operatorname{diameff}(G) \geq \frac{5}{2}$ by Corollary 3.2.16, and this implies $\operatorname{diameff} V(G) = 2$.

Assume now $\operatorname{diameff}(G) = \frac{5}{2}$. Corollary 3.2.17 gives $\delta(G) = \frac{5}{4}$.

Finally, assume that $\operatorname{diameff} V(G) = 2$, $\operatorname{diameff}(G) = 3$ and $G \notin \mathcal{F}$. Thus $\delta(G) \leq \frac{3}{2}$ by Theorem 2.4.4. Besides, $\delta(G) \geq \frac{5}{4}$ by Corollary 3.2.16. Hence, Theorem 2.4.2 implies $\delta(G) \in \left\{\frac{5}{4}, \frac{3}{2}\right\}$. Seeking for a contradiction assume that $\delta(G) = \frac{3}{2}$. By Theorem 2.4.3, there exists a geodesic triangle $T = \{x, y, z\}$ that is a cycle with $x, y, z \in J(G)$ and $\delta(T) = \frac{3}{2} = d(p, [yz] \cup [zx])$ for some $p \in [xy]$. Then $d(p, \{x, y\}) \geq d(p, [yz] \cup [zx]) = \frac{3}{2}$ and $d(x, y) \geq 3$. Since $\operatorname{diameff}(G) = 3$ and T is a cycle, we have $L([xy]) = 3$, $L([yz]), L([zx]) \leq 3$. Since $\operatorname{diameff} V(G) = 2$, $x, y \in J(G) \setminus V(G)$, p is the midpoint of $[xy]$ and it is a vertex of G. Thus Lemma 3.2.20 gives $G \in \mathcal{F}$, which is the contradiction we were looking for. Hence, $\delta(G) \neq \frac{3}{2}$ and we conclude $\delta(G) = \frac{5}{4}$. □

Chapter 4

Gromov hyperbolicity of minor graphs

4.1 Hyperbolicity and edge contraction on simple graphs

In this section we study the distortion of the hyperbolicity constant by contraction of one edge in any simple graph. If G is a graph and $e := [A, B] \in E(G)$, we denote by G/e the graph obtained by contracting the edge e from it (we remove e from G while simultaneously we merge A and B).

Definition 4.1.1. *Denote by V_e the vertex in G/e obtained by identifying A and B in G.*

Note that any vertex $v \in V(G) \setminus \{A, B\}$ can be seen as itself in $V(G/e)$. Also we can write any edge in $E(G/e)$ in terms of its endpoints, but we write V_e instead of A or B. If $[v, A]$ and $[v, B]$ are edges of G for some $v \in V(G)$, then we replace both edges by a single edge $[v, V_e] \in G/e$ (recall that we work with simple graphs), see Figure 4.1.

We define the map $h : G \to G/e$ in the following way: if x belongs to the edge e, then $h(x) := V_e$; if $x \in G$ does not belong to e, then $h(x)$ is the "natural inclusion map". Clearly h is onto, *i.e.*, $h(G) = G/e$. Besides, h is an injective map in the union of edges without endpoints in $\{A, B\}$.

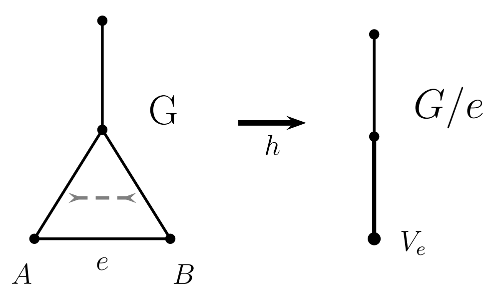

Figure 4.1: The map h.

Given $e \in E(G)$, denote by $\mathcal{C}(G,e)$ the set of cycles in G with length 3 containing e.

Proposition 4.1.2. *Let G be a graph and $e \in E(G)$. Then*

$$d_{G/e}\big(h(x),h(y)\big) \le d_G(x,y) \le d_{G/e}\big(h(x),h(y)\big) + \frac{3}{2}\,, \quad \forall x,y \in G. \tag{4.1}$$

Furthermore, if $y \in J(G)$ or x,y are not contained in the same cycle $C \in \mathcal{C}(G,e)$, then

$$d_{G/e}\big(h(x),h(y)\big) \le d_G(x,y) \le d_{G/e}\big(h(x),h(y)\big) + 1. \tag{4.2}$$

Proof. Without loss of generality we can assume that G is a connected graph, since otherwise we can consider each connected component.
Fix $x,y \in G$. Let $[xy]_G$ be a geodesic in G joining x and y. Clearly, $h([xy]_G)$ is a path joining $h(x)$ and $h(y)$ with length at most $L([xy]_G)$, thus, we obtain $d_{G/e}\big(h(x),h(y)\big) \le d_G(x,y)$. Hence, the first inequalities in (4.1) and (4.2) hold.

Let γ' be a geodesic in G/e joining $h(x)$ and $h(y)$. If x,y are not contained in the same cycle $C \in \mathcal{C}(G,e)$, then there is a path γ in G with $h(\gamma)=\gamma'$ and $L(\gamma) \le L(\gamma')+1$ since e (or a subset of e) can be included in γ. Therefore, $d_G(x,y) \le L(\gamma) \le L(\gamma')+1 = d_{G/e}\big(h(x),h(y)\big)+1$.

If $x,y \in C \in \mathcal{C}(G,e)$, then

$$d_G(x,y) \le d_C(x,y) \le \frac{1}{2}\,L(C) = \frac{3}{2} \le d_{G/e}\big(h(x),h(y)\big) + \frac{3}{2}\,.$$

Finally, consider $x,y \in C \in \mathcal{C}(G,e)$ with $y \in J(G)$.

We deal with several cases.

Case 1. If $d_{G/e}\big(h(x),h(y)\big) \ge 1/2$, then

$$d_G(x,y) \le \frac{1}{2}\,L(C) = \frac{1}{2} + 1 \le d_{G/e}\big(h(x),h(y)\big) + 1.$$

Case 2. Assume that $d_{G/e}\big(h(x),h(y)\big) < 1/2$.

Case 2.1. y is the midpoint of e. If $x \in e$, then $d_G(x,y) \le 1/2$. If $x \notin e$, then $d_G(x,y) = d_{G/e}\big(h(x),h(y)\big) + 1/2$.

Case 2.2. $y \in \{A,B\}$. If $x \in e$, then $d_G(x,y) \le 1$. If $x \notin e$, then we have either $d_G(x,y) = d_{G/e}\big(h(x),h(y)\big)$ or $d_G(x,y) = d_{G/e}\big(h(x),h(y)\big) + 1$.

Case 2.3. $y \in V(C) \setminus \{A,B\}$. Thus $d_G(x,y) = d_{G/e}\big(h(x),h(y)\big)$.

Case 2.4. $y \in J(G) \setminus \{V(C) \cup e\}$. We have either $d_G(x,y) = d_{G/e}\big(h(x),h(y)\big)$, $d_G(x,y) = d_{G/e}\big(h(x),h(y)\big) + 1$ or $d_G(x,y) = 1 - d_{G/e}\big(h(x),h(y)\big) \le 1$.

This finishes the proof. $\square$

Note that the inequalities in (4.1) and (4.2) are attained. If G is any graph, $[v,w] \in E(G)$ and $\{v,w\} \cap \{A,B\} = \emptyset$, then $d_{G/e}(h(v),h(w)) = 1 = d_G(v,w)$. Consider a cycle graph $G = C_3$ and $x,y \in C_3$ such that $x \neq y$ and there is $v \in V(C_3)$ with $d_{C_3}(x,v) = d_{C_3}(v,y) = 3/4$. Let e be the edge in C_3 with $x,y \notin e$. Hence, we have $d_{C_3}(x,y) = 3/2$ and $h(x) = h(y)$. Finally, consider a cycle graph $G = C_3$, x_0, y_0 two different midpoints of edges in C_3 and $e \in E(C_3)$ with $x_0, y_0 \notin e$. Thus, $d_{C_3}(x_0,y_0) = 1$ and $h(x_0) = h(y_0)$.

The previous lemma has the following consequence about the continuity of h.

Proposition 4.1.3. *The map h is an 1-Lipschitz continuous function.*

Let (X,d_X) and (Y,d_Y) be two metric spaces. A map $f : X \longrightarrow Y$ is said to be an (α,β)-*quasi-isometric embedding*, with constants $\alpha \geq 1,\ \beta \geq 0$ if for every $x,y \in X$:

$$\alpha^{-1} d_X(x,y) - \beta \leq d_Y(f(x),f(y)) \leq \alpha d_X(x,y) + \beta.$$

The function f is ε-*full* if for each $y \in Y$ there exists $x \in X$ with $d_Y(f(x),y) \leq \varepsilon$.

A map $f : X \longrightarrow Y$ is said to be a *quasi-isometry* if there exist constants $\alpha \geq 1,\ \beta,\varepsilon \geq 0$ such that f is an ε-full (α,β)-quasi-isometric embedding.

A fundamental property of hyperbolic spaces is the following:

Theorem 4.1.4 (Invariance of hyperbolicity)**.** *Let $f : X \longrightarrow Y$ be an (α,β)-quasi-isometric embedding between the geodesic metric spaces X and Y. If Y is hyperbolic, then X is hyperbolic. Furthermore, if Y is δ-hyperbolic, then X is δ'-hyperbolic, where δ' is a constant which just depends on α,β,δ.*

Besides, if f is ε-full for some $\varepsilon \geq 0$ (a quasi-isometry), then X is hyperbolic if and only if Y is hyperbolic. Furthermore, if X is δ-hyperbolic, then Y is δ'-hyperbolic, where δ' is a constant which just depends on $\alpha,\beta,\delta,\varepsilon$.

Remark 4.1.5. *The definition of $\delta(G)$ when G is a non-connected graph gives that 4.1.4 holds for non-connected graphs.*

Using the invariance of hyperbolicity (Theorem 4.1.4), we can obtain the main qualitative aim in this section.

Theorem 4.1.6. *Let G be any graph and $e \in E(G)$. Then G is hyperbolic if and only if G/e is hyperbolic. Furthermore, if G (respectively, G/e) is δ-hyperbolic, then G/e (respectively, G) is δ'-hyperbolic, where δ' is a constant which just depends on δ.*

Proof. Lemma 4.1.2 gives that h is a 0-full $(1,3/2)$-quasi-isometry from G onto G/e, and we obtain the result by Theorem 4.1.4. □

One can expect that the edge contraction is a monotone transformation for the hyperbolicity constant, *i.e.*, the hyperbolicity constant always decreases by edge contraction (for instance, if e is any edge of the cycle graph C_3, then C_3/e is the path graph P_2 and $\delta(P_2) = 0 < 3/4 = \delta(C_3)$; if e is any edge of the cycle graph C_n with $n \geq 4$, then C_n/e is the cycle graph C_{n-1} and $\delta(C_{n-1}) = (n-1)/4 < n/4 = \delta(C_n)$). However, the following example provides a family of graphs such that the hyperbolicity constant increases by contracting certain edge.

We need two definitions. Recall that the *girth* of a graph G is the minimum of the lengths of the cycles in G. Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two graphs with $V(G_1) \cap V(G_2) = \emptyset$. The *graph join* $G_1 \uplus G_2$ of G_1 and G_2 has $V(G_1 \uplus G_2) = V(G_1)V(G_2)$ and two different vertices u and v of $G_1 \uplus G2$ are adjacent (i.e., $[u, v]E(G_1 \uplus G2)$) if $u \in V(G_1)$ and $v \in V(G_2)$, or $[u, v] \in E(G_1)$ or $[u, v] \in E(G_2)$.

Example 4.1.7. *In [83, Theorem 11], the authors obtain the precise value of the hyperbolicity constant of the wheel graph with n vertices W_n: $\delta(W_4) = \delta(W_5) = 1$, $\delta(W_n) = 3/2$ for every $7 \leq n \leq 10$, and $\delta(W_n) = 5/4$ for $n = 6$ and for every $n \geq 11$. Note that we can obtain W_n from W_{n+1} by edge contraction, so, we have that $\delta(W_{11}) = 5/4$ and $\delta(W_{10}) = 3/2$. Furthermore, in [30] the authors obtain the value of the hyperbolicity constant of the graph join of E (the empty graph with just one vertex) and every graph. Thus, taking G as a graph join of E and any graph G^* with girth* 10, *then $\delta(G) = 5/4$, but contracting an edge e belonging to any cycle in G^* with length 10, G/e is the graph join of E and other graph with girth* 9, *so [30, Corollary 7] gives $\delta(G/e) = 3/2$.*

Other aim in this work is to obtain quantitative relations between $\delta(G/e)$ and $\delta(G)$. Since the proofs of these inequalities are long, in order to make the arguments more transparent, we collect some results in technical lemmas.

For any simple (non-selfintersecting) path γ' joining two different points in G/e which are not contained in an edge e_0 with $h^{-1}(e_0) \in \mathcal{C}(G, e)$, we define $\Gamma(\gamma')$ as the set of paths γ in G such that $h(\gamma) = \gamma'$ and

$$\Gamma_0(\gamma') = \big\{ g \in \Gamma(\gamma') \mid L(g) \leq L(\gamma) \quad \forall\, \gamma \in \Gamma(\gamma') \big\}.$$

Note that any curve in $\Gamma_0(\gamma')$ is a simple path. We denote by $h_0^{-1}(\gamma')$ any fixed choice of curve in $\Gamma_0(\gamma')$. If $t' \in \gamma' \setminus V_e$ we denote by $t = h_0^{-1}(t')$ the point in $h_0^{-1}(\gamma')$ such that $h(t) = t'$ (note that, since γ' is a simple path, any $t' \in \gamma' \setminus V_e$ defines an unique $t \in h_0^{-1}(\gamma')$). If $t' = V_e \in \gamma'$, then $h_0^{-1}(V_e) = h^{-1}(V_e) = e$. Hence, $h_0^{-1}(t') = h^{-1}(t') \cap h_0^{-1}(\gamma')$. Furthermore, for any geodesic γ' in G/e such that $V_e \notin \gamma'$ we have that

$$h|_{h^{-1}(\gamma')} \colon h^{-1}(\gamma') \longrightarrow \gamma'$$

is a bijective map and $\gamma = h^{-1}(\gamma') = h_0^{-1}(\gamma')$ is a geodesic in G with $L(\gamma) = L(\gamma')$.

Lemma 4.1.8. *Let G be a graph and $e \in E(G)$. Let $x, y \in G \setminus \{e\}$ such that there is no $C \in \mathcal{C}(G,e)$ with $x, y \in C$. Assume that there are two geodesics γ_G and $\gamma_{G/e}$ in G and G/e, respectively, joining x, y and $h(x), h(y)$, respectively, such that $L(\gamma_G) = L(\gamma_{G/e}) = L(h(\gamma_G))$ and $e \subset h_0^{-1}(\gamma_{G/e})$. Then we have*

$$d_{G/e}(h(\alpha), \gamma_{G/e}) \le \delta(G) \quad \forall\ \alpha \in \gamma_G \tag{4.3}$$

and

$$d_{G/e}\big(\alpha', h(\gamma_G)\big) \le 2\delta(G) \quad \forall\ \alpha' \in \gamma_{G/e}. \tag{4.4}$$

Remark 4.1.9. *Since there is no $C \in \mathcal{C}(G,e)$ with $x, y \in C$, we deduce that $h(x), h(y)$ are not contained in an edge e_0 with $h^{-1}(e_0) \in \mathcal{C}(G,e)$, and then $h_0^{-1}(\gamma_{G/e})$ is well defined.*

Proof. We can assume that G is connected. Without loss of generality we can assume that G is hyperbolic, since otherwise the inequalities trivially hold. Let z be the midpoint of $e = [A, B]$. By symmetry, we can assume that the closure of the connected components of $h_0^{-1}(\gamma_{G/e} \setminus \{V_e\})$ join x with A, and B with y. Clearly,

$$d_G(z, x) = d_G(z, A) + \tfrac{1}{2} = d_{G/e}(h(x), V_e) + \tfrac{1}{2},$$

$$d_G(z, y) = d_G(y, B) + \tfrac{1}{2} = d_{G/e}(h(y), V_e) + \tfrac{1}{2}.$$

So, there are geodesics $[xz]_G$ and $[zy]_G$ in G verifying the following: $[xz]_G$ contains the closure of the connected component of $h_0^{-1}(\gamma_{G/e} \setminus \{V_e\})$ joining x with A, and $[zy]_G$ contains the closure of the connected component of $h_0^{-1}(\gamma_{G/e} \setminus \{V_e\})$ joining B with y. Hence, $T := \{\gamma_G, [yz]_G, [zx]_G\}$ is a geodesic triangle in G and so,

$$d_G(\alpha, [yz]_G \cup [zx]_G) = d_G(\alpha, h_0^{-1}(\gamma_{G/e})) \le \delta(G)$$

for every $\alpha \in \gamma_G$, and (4.3) holds by Lemma 4.1.2.

In order to obtain (4.4), without loss of generality we can assume that $\alpha' \in h([yz]_G)$, since $\gamma_{G/e} = h([yz]_G) \cup h([zx]_G)$. Denote by α a point in $[yB]_G$ with $h(\alpha) = \alpha'$. If $d_G(y, e) = d_G(y, B) \le 2\delta(G)$, then we have

$$d_{G/e}(\alpha', h(\gamma_G)) \le d_G(\alpha, \gamma_G) \le d_G(\alpha, y) \le d_G(B, y) \le 2\delta(G).$$

Assume that $d_G(y, e) > 2\delta(G)$. Now we can take a point $w \in [yz]_G$ such that $d_G(w, e) = \delta(G)$. If $\alpha \in [wy] \setminus \{w\}$, then the hyperbolicity of G implies $d_G(\alpha, \gamma_G \cup [zx]_G) \le \delta(G)$; note that $d_G(\alpha, [zx]_G) > \delta(G)$ since $\gamma_{G/e}$ is a geodesic in G/e and $d_G(w, e) = \delta(G)$. Hence, $d_G(\alpha, \gamma_G) \le \delta(G)$ and thus Lemma 4.1.2 gives $d_{G/e}(\alpha', h(\gamma_G)) \le \delta(G)$. Assume now that $\alpha \in [wB] \setminus B$ (therefore, $\alpha' \ne V_e$). Thus, there exists $\alpha_1 \in [yz]_G$ such that $d_G(\alpha, \alpha_1) = \delta(G)$ and $d_G(\alpha_1, e) > \delta(G)$. Therefore, $\alpha_1 \in [wy] \setminus \{w\}$ and we have proved that $d_G(\alpha_1, \gamma_G) \le \delta(G)$. Hence, Lemma 4.1.2 gives

$$d_{G/e}(\alpha', h(\gamma_G)) \le d_G(\alpha, \gamma_G) \le d_G(\alpha, \alpha_1) + d_G(\alpha_1, \gamma_G) \le 2\delta(G).$$

If $\alpha = B$, then $\alpha' = V_e$ and the inequality for $\alpha' = V_e$ is obtained by continuity. □

In what follows, if x, y belong to some $C \in \mathcal{C}(G,e)$, then we denote by $h_0^{-1}([h(x)h(y)]_{G/e})$ any fixed choice of a geodesic in C with $[h(x)h(y)]_{G/e} \subseteq h(h_0^{-1}([h(x)h(y)]_{G/e}))$, and by $h_0^{-1}(\alpha')$ any point in $h^{-1}(\alpha') \cap h_0^{-1}([h(x)h(y)]_{G/e})$.

Lemma 4.1.10. *Let G be a graph and $e \in E(G)$ such that G/e is not a tree. Let $[xy]_G$ be a geodesic in G joining $x, y \in J(G)$. Assume that $h([xy]_G)$ is not a geodesic in G/e and let $[h(x)h(y)]_{G/e}$ be a geodesic in G/e joining $h(x)$ and $h(y)$. Then we have*

$$d_G(h_0^{-1}(\alpha'), [xy]_G) \le \delta(G/e) + 1 \le \frac{7}{3}\,\delta(G/e), \quad \forall\ \alpha' \in [h(x)h(y)]_{G/e} \tag{4.5}$$

and

$$d_G\big(\alpha, h_0^{-1}([h(x)h(y)]_{G/e})\big) \le 2\delta(G/e), \quad \forall\ \alpha \in [xy]_G. \tag{4.6}$$

Proof. Without loss of generality we can assume that G/e is hyperbolic, since otherwise the inequalities hold. Since G/e is not a tree, $\delta(G/e) \ge 3/4$ by [68, Theorem 11], and

$$\delta(G/e) + 1 \le \frac{7}{3}\,\delta(G/e).$$

We deal with several cases.

(a) There exists $C \in \mathcal{C}(G,e)$ with $x, y \in C$. For every $\alpha, \beta \in C$ we have

$$d_G(\alpha,\beta) \le \operatorname{diam} C = \frac{3}{2} = \min\Big\{\frac{3}{4} + 1, 2\frac{3}{4}\Big\} \le \min\big\{\delta(G/e)+1, 2\delta(G/e)\big\}.$$

Since $[xy]_G$ and $h_0^{-1}([h(x)h(y)]_{G/e}$ are contained in C, (4.5) and (4.6) hold.

(b) There is no $C \in \mathcal{C}(G,e)$ with $x, y \in C$. Thus $h(x), h(y)$ are not contained in an edge e_0 with $h^{-1}(e_0) \in \mathcal{C}(G,e)$, and then $h_0^{-1}([h(x)h(y)]_{G/e})$ is defined as before Lemma 4.1.8.

Let $[A,B] := e$. Since $h([xy]_G)$ is not a geodesic in G/e and $x, y \in J(G)$, we have $e \cap [xy]_G \subsetneq \{A,B\}$, $L([xy]_G) \ge 3/2$, $d_G(x,y) = d_{G/e}(h(x),h(y)) + 1$ and $V_e \in [h(x)h(y)]_{G/e}$.

$(b.1)$ Assume that there exists a cycle $C \in \mathcal{C}(G,e)$ with $L([xy]_G \cap C) > 1$. Since $[xy]_G$ is a geodesic in G, $h([xy]_G)$ is not a geodesic in G/e and $x, y \in J(G)$, we have $L([xy]_G \cap C) = 3/2$, $C \subset h_0^{-1}([h(x)h(y)]_{G/e}) \cup [xy]_G$, the closures of $h_0^{-1}([h(x)h(y)]_{G/e}) \setminus C$ and $[xy]_G \setminus C$ are two geodesics in G with the same endpoints, and the closures of $[h(x)h(y)]_{G/e} \setminus h(C)$ and $h([xy]_G) \setminus h(C)$ are two geodesics in G/e with the same endpoints. Since $L([xy]_G \cap C) = 3/2 = L(h_0^{-1}([h(x)h(y)]_{G/e}) \cap C)$ and $L(C) = 3$, we have

$$d_G(h_0^{-1}(\alpha'), [xy]_G) \le 3/4 \le \delta(G/e), \quad \forall\ \alpha' \in [h(x)h(y)]_{G/e} \cap h(C),$$
$$d_G\big(\alpha, h_0^{-1}([h(x)h(y)]_{G/e})\big) \le 3/4 \le \delta(G/e), \quad \forall\ \alpha \in [xy]_G \cap C.$$

Since the closures of $h_0^{-1}([h(x)h(y)]_{G/e}) \setminus C$ and $[xy]_G \setminus C$ are two geodesics in G with the same endpoints, and the closures of $[h(x)h(y)]_{G/e} \setminus h(C)$ and $h([xy]_G) \setminus h(C)$ are two geodesics in G/e with the same endpoints, we also have

$$d_G(h_0^{-1}(\alpha'), [xy]_G) \le d_G\big(h_0^{-1}(\alpha'), [xy]_G \setminus C\big) = d_{G/e}\big(\alpha', h([xy]_G) \setminus h(C)\big) \le \delta(G/e),$$

for every $\alpha' \in [h(x)h(y)]_{G/e} \setminus h(C)$, and

$$d_G\big(\alpha, h_0^{-1}([h(x)h(y)]_{G/e})\big) \le d_G\big(\alpha, h_0^{-1}([h(x)h(y)]_{G/e}) \setminus C\big) =$$

$$= d_{G/e}\big(h(\alpha), [h(x)h(y)]_{G/e} \setminus h(C)\big) \le \delta(G/e),$$

for every $\alpha \in [xy]_G \setminus C$.

$(b.2)$ Assume now that $L([xy]_G \cap C) \le 1$ for every $C \in \mathcal{C}(G,e)$.

Note that $L\big(h([xy]_G)\big) = L([xy]_G)$, since $e \cap [xy]_G \subset \{A, B\}$, so, for any $z \in [xy]_G$ we have $L\big(h([xz]_G)\big) = L([xz]_G)$ and $L\big(h([zy]_G)\big) = L([zy]_G)$. Consider the points $A', B' \in [xy]_G$ such that $d_G(x, A') = d_G(x, e)$ and $d_G(y, B') = d_G(y, e)$. Since $x, y \in J(G)$, we have $A', B' \in V(G)$. Since $L\big(h([xy]_G)\big) = L([xy]_G)$, $[A', B'] \in E(G)$ and $[A', B'] \subset [xy]_G$. Let z be the midpoint of $[A', B']$. Since $d(x, A') = d(x, e)$, $d_G(y, B') = d_G(y, e)$, $d_G(z, A') = d_G(z, B') = 1/2$ and $L([xy]_G \cap C) \le 1$ for every $C \in \mathcal{C}(G,e)$, we have that $h([xz]_G)$ and $h([zy]_G)$ are geodesics in G/e. Hence, $T = \{[h(x)h(y)]_{G/e}, h([yz]_G), h([zx]_G)\}$ is a geodesic triangle in G/e, and thus

$$d_G(h_0^{-1}(\alpha'), [xy]_G) \le d_{G/e}(\alpha', h([xy]_G)) + 1 = d_{G/e}(\alpha', h([xz]_G) \cup h([zy]_G)) + 1$$
$$\le \delta(T) + 1 \le \delta(G/e) + 1 \le \frac{7}{3}\,\delta(G/e),$$

for every $\alpha' \in [h(x)h(y)]_{G/e}$.

In order to obtain (4.6), without loss of generality we can assume that $\alpha \in [yz]_G$. If $L([yz]_G) \le 2\delta(G/e)$, then we have $d_G\big(\alpha, h_0^{-1}([h(x)h(y)]_{G/e})\big) \le d_G(\alpha, y) \le 2\delta(G/e)$. Assume that $L([yz]_G) > 2\delta(G/e)$. Let w be the point in $[yz]_G$ with $d_G(w, z) = \delta(G/e)$. If $\alpha \in [wy]_G \setminus \{w\}$, then the hyperbolicity of G/e implies $d_{G/e}\big(h(\alpha), [h(x)h(y)]_{G/e} \cup h([zx]_G)\big) \le \delta(G/e)$. Note that if $d_{G/e}\big(h(\alpha), h([zx]_G)\big) \le \delta(G/e)$, then a geodesic γ joining $h(\alpha)$ and $h([zx]_G)$ in G/e contains V_e and, since $V_e \in [h(x)h(y)]_{G/e}$, we obtain

$$d_{G/e}(h(\alpha), [h(x)h(y)]_{G/e}) \le d_{G/e}(h(\alpha), V_e) \le L(\gamma) \le \delta(G/e).$$

Thus, we have $d_{G/e}(h(\alpha), [h(x)h(y)]_{G/e}) \le \delta(G/e)$. Hence, we obtain

$$d_G\big(\alpha, h_0^{-1}([h(x)h(y)]_{G/e})\big) = d_{G/e}(h(\alpha), [h(x)h(y)]_{G/e}) \le \delta(G/e).$$

Assume now that $\alpha \in [zw]_G \setminus \{z\}$. Then, there exists $\alpha_1 \in [wy]_G \setminus \{w\}$ such that $d_G(\alpha, \alpha_1) = \delta(G/e)$, and we deduce

$$d_G\big(\alpha, h_0^{-1}([h(x)h(y)]_{G/e})\big) \le d_G(\alpha, \alpha_1) + d_G\big(\alpha_1, h_0^{-1}([h(x)h(y)]_{G/e})\big) \le 2\delta(G/e).$$

The inequality for $\alpha = z$ is obtained by continuity.

□

Remark 4.1.11. *Let G be any graph, $e \in E(G)$ and $T = \{\gamma_1, \gamma_2, \gamma_3\}$ a geodesic triangle in G. Then at least one of the curves $h(\gamma_1), h(\gamma_2), h(\gamma_3)$ is a geodesic in G/e, since otherwise there exists another geodesic triangle $T' = \{\gamma_1', \gamma_2', \gamma_3'\}$ with the same vertices that T and such that the edge e is contained in $\gamma_1' \cap \gamma_2' \cap \gamma_3'$.*

The following result will be useful.

Theorem 4.1.12. *[12, Theorem 2.7] For any hyperbolic graph G, there exists a geodesic triangle $T = \{x, y, z\}$ that is a cycle with $x, y, z \in J(G)$ and $\delta(T) = \delta(G)$.*

In order to prove Theorem 4.1.14 we will need the following technical result.

Lemma 4.1.13. *Let G be a graph and $e \in E(G)$. If G/e is a tree, then*

$$\delta(G) \le 1.$$

Proof. If G is a tree, then $\delta(G) = 0 \le 1$.

Assume now that G is not a tree. Since G/e is a tree, if an edge e_0 is contained in a cycle in G, then it is contained in some cycle $C_0 \in \mathcal{C}(G, e)$ and it contains A or B. Since any cycle in $\mathcal{C}(G, e)$ contains the edge e, if $[A, B] = e$, then every cycle in G contains the vertices A and B. Therefore, any cycle in G has length at most 4.

Theorem 4.1.6 gives that G is hyperbolic since $\delta(G/e) = 0$. Hence, by Theorem 4.1.12 there exist a geodesic triangle $T = \{x, y, z\}$ in G that is a cycle with $x, y, z \in J(G)$ and $p \in [xy]$ with $d_G(p, [yz] \cup [zx]) = \delta(T) = \delta(G)$. Since T is a cycle, we have seen that $L(T) \le 4$. Thus

$$\delta(G) = d_G(p, [yz] \cup [zx]) \le d_G(p, \{x, y\}) \le \frac{1}{2}\, d_G(x, y) \le \frac{1}{4}\, L(T) \le 1.$$

□

The previous results allow to obtain a quantitative version of Theorem 4.1.6.

Theorem 4.1.14. *Let G be a graph and $e \in E(G)$. Then*

$$\frac{1}{3}\, \delta(G/e) \le \delta(G) \le \frac{16}{3}\, \delta(G/e) + 1. \tag{4.7}$$

Proof. Without loss of generality we can assume that G is a connected graph, since otherwise we can consider each connected component. By Theorem 4.1.6 we have that G and G/e are hyperbolic or not simultaneously. If G and G/e are not hyperbolic, then $\delta(G) = \delta(G/e) = \infty$ and (4.7) holds. Assume now that both graphs are hyperbolic.

Let us prove the first inequality in (4.7). If G is a tree, then $\delta(G/e) = 0$ and the first inequality holds. Assume that G/e is not a tree, thus $\delta(G/e) > 0$. By Theorem

4.1.12 there exist a geodesic triangle $T' = \{[x'y'], [y'z'], [z'x']\}$ in G/e that is a cycle with $x', y', z' \in J(G/e)$ and $p' \in [x'y']$ with $d_{G/e}(p', [y'z'] \cup [z'x']) = \delta(T') = \delta(G/e)$.

Consider $T \subset h^{-1}([x'y'] \cup [y'z'] \cup [z'x'])$ such that T is a cycle with $h(T) = T'$. Define $x := h^{-1}(x') \cap T$, $y := h^{-1}(y') \cap T$ and $z := h^{-1}(z') \cap T$, if $V_e \notin \{x', y', z'\}$; otherwise, if $V_e = a'$ with $a' \in \{x', y', z'\}$, then we define a as the midpoint of e. Hence, we can define g_{ab} as the simple curve contained in T joining a and b, and such that $h(g_{ab}) = [a'b']$, for $a, b \in \{x, y, z\}$ (note that $g_{ab} = h_0^{-1}([a'b'])$ if $h_0^{-1}([a'b'])$ is defined as before Lemma 4.1.8, *i.e.*, if a', b' are not contained in an edge e_0 with $h_0^{-1}(e_0) \in \mathcal{C}(G, e)$). Then $x, y, z \in J(G)$ and T can be seen as the triangle $\{g_{xy}, g_{yz}, g_{zx}\}$. Note that if $V_e \in \{x', y', z'\}$, then T is a geodesic triangle in G.

We deal with several cases.

(a) If T is a geodesic triangle in G, then by Lemma 4.1.2 we have for any $p \in h^{-1}(p') \cap g_{xy}$

$$\delta(G/e) = \delta(T') = d_{G/e}(p', [y'z'] \cup [z'x']) \le d_G(p, g_{yz} \cup g_{zx}) \le \delta(T) \le \delta(G)$$

and so, the first inequality in (4.7) holds.

(b) Assume that T is not a geodesic triangle in G. Thus, we have $V_e \in T' \setminus \{x', y', z'\}$, $e \subset T$ and $L(T) = L(T') + 1$. Since $V_e \notin \{x', y', z'\}$, the edge e is contained in exactly one of g_{xy}, g_{yz}, g_{zx}, and the other two paths are geodesics in G by Proposition 4.1.2.

$(b.1)$ Assume that $e \subset g_{xy}$. Note that e is contained in the interior of g_{xy} (recall that $V_e \notin \{x', y', z'\}$); since $x, y \in J(G)$, we have $L(g_{xy}) \ge 2$. Therefore, $x, y \in G \setminus \{e\}$ and there is no $C \in \mathcal{C}(G, e)$ with $x, y \in C$. Hence, $h_0^{-1}([x'y'])$ is defined as before Lemma 4.1.8, $g_{xy} = h_0^{-1}([x'y'])$ and $e \subset h_0^{-1}([x'y'])$. Consider a geodesic $[xy]$ in G joining x and y. We have $L([xy]) = L([x'y']) = L(h([xy]))$. Note that g_{xy} is not a geodesic by hypothesis. By Lemma 4.1.8 there is $p \in [xy]$ such that $d_{G/e}(p', h(p)) \le 2\delta(G)$. Thus, since $\{[xy], g_{yz}, g_{zx}\}$ is a geodesic triangle in G, there is $p_1 \in g_{yz} \cup g_{zx}$ such that $d_G(p, p_1) \le \delta(G)$. Hence, Proposition 4.1.2 gives

$$\begin{aligned}\delta(G/e) &= d_{G/e}(p', [y'z'] \cup [z'x']) \le \delta_{G/e}(p', h(p_1)) \le d_{G/e}(p', h(p)) + d_{G/e}(h(p), h(p_1)) \\ &\le 2\delta(G) + d_G(p, p_1) \le 3\delta(G).\end{aligned}$$

$(b.2)$ Assume now that $e \subset g_{yz} \cup g_{zx}$. By symmetry, we can assume that $e \subset g_{yz}$. Note that g_{yz} is not a geodesic by hypothesis, and that g_{xy}, g_{zx} are geodesics. Consider a geodesic $[yz]$ in G joining y and z. Since $\{g_{xy}, [yz], g_{zx}\}$ is a geodesic triangle in G, there is $p \in [yz] \cup g_{zx}$ such that $d_G(h_0^{-1}(p'), p) \le \delta(G)$.

$(b.2.1)$ If $p \in g_{zx}$, then Proposition 4.1.2 gives

$$\delta(G/e) = d_{G/e}(p', [y'z'] \cup [z'x']) \le d_{G/e}(p', h(p)) \le d_G(h_0^{-1}(p'), p) \le \delta(G).$$

$(b.2.2)$ Assume that $p \in [yz]$. The argument in $(b.1)$ also gives that $y, z \in G \setminus \{e\}$, there is no $C \in \mathcal{C}(G, e)$ with $y, z \in C$, $e \subset g_{yz} = h_0^{-1}([y'z'])$, and $L([yz]) = L([y'z']) = L(h([yz]))$.

Therefore, by Lemma 4.1.8 there is $p_1' \in [y'z']$ such that $d_{G/e}(h(p), p_1') \le \delta(G)$. Thus, we have by Proposition 4.1.2

$$\begin{aligned}\delta(G/e) &= d_{G/e}(p', [y'z'] \cup [z'x']) \le d_{G/e}(p', p_1') \le d_{G/e}(p', h(p)) + d_{G/e}(h(p), p_1') \\ &\le d_G(h_0^{-1}(p'), p) + \delta(G) \le 2\delta(G).\end{aligned}$$

Hence, the first inequality in (4.7) holds.

Let us prove the second inequality in (4.7). By Theorem 4.1.12 there exist a geodesic triangle $T = \{x, y, z\}$ in G that is a cycle with $x, y, z \in J(G)$ and $p \in [xy]$ with $d_G(p, [yz] \cup [zx]) = \delta(T) = \delta(G)$. Since $x, y, z \in J(G)$ we have

$$d_G(p, [yz] \cup [zx]) = d_G(p, J(G) \cap ([yz] \cup [zx])),$$

and if $d_G(p, [yz] \cup [zx]) = d_G(p, q)$ with $q \in J(G) \cap ([yz] \cup [zx])$, then Lemma 4.1.2 gives

$$d_G(p, [yz] \cup [zx]) = d_G(p, q) \le d_{G/e}\big(h(p), h(q)\big) + 1.$$

If $\delta(G) \le 1$, then the second inequality in (4.7) holds. Hence, we can assume that $\delta(G) > 1$. Note that since $\delta(G) > 1$ we have that G/e is not a tree by Lemma 4.1.13.

Let n be the number of geodesics in G/e of the set $\big\{h([xy]), h([yz]), h([zx])\big\}$. By Remark 4.1.11 we have $n \in \{1, 2, 3\}$.

We consider several cases.

(A) Assume that $h([xy])$ is a geodesic in G/e.

$(A.1)$ If $n = 3$, then Lemma 4.1.2 gives

$$\delta(G) = d_G(p, [yz] \cup [zx]) \le d_{G/e}\big(h(p), h([yz]) \cup h([zx])\big) + 1 \le \delta(G/e) + 1.$$

$(A.2)$ Consider the case $n = 2$. By symmetry we can assume that $h([yz])$ is a geodesic in G/e and let $[h(z)h(x)]$ be a geodesic in G/e joining $h(z)$ and $h(x)$. Then there is $p' \in h([yz]) \cup [h(z)h(x)]$ with $d_{G/e}(h(p), p') \le \delta(G/e)$. By Lemma 4.1.2, we have $d_G(p, h_0^{-1}(p')) \le d_{G/e}\big(h(p), p'\big) + 1 \le \delta(G/e) + 1$. If $p' \in h([yz])$, then $h_0^{-1}(p') \in [yz]$ and

$$\delta(G) = d_G(p, [yz] \cup [zx]) \le d_G(p, h_0^{-1}(p')) \le \delta(G/e) + 1.$$

Assume that $p' \in [h(z)h(x)]$. By Lemma 4.1.10, there is $p_1 \in [zx]$ with $d_G(h_0^{-1}(p'), p_1) \le (7/3)\,\delta(G/e)$. Then we have

$$\delta(G) = d_G(p, [yz] \cup [zx]) \le d_G(p, p_1) \le d_G(p, h_0^{-1}(p')) + d_G(h_0^{-1}(p'), p_1) \le \frac{10}{3}\,\delta(G/e) + 1.$$

$(A.3)$ If $n = 1$, then let $[h(y)h(z)], [h(z)h(x)]$ be geodesics in G/e joining $h(y), h(z)$ and $h(z), h(x)$, respectively. Then there is $p' \in [h(y)h(z)] \cup [h(z)h(x)]$ with $d_{G/e}(h(p), p') \le \delta(G/e)$. By Lemma 4.1.2 we have $d_G(p, h_0^{-1}(p')) \le d_{G/e}\big(h(p), p'\big) + 1 \le \delta(G/e) + 1$. By symmetry we can assume that $p' \in [h(y)h(z)]$. By Lemma 4.1.10 there is $p_1 \in [yz]$ with $d_G(h_0^{-1}(p'), p_1) \le (7/3)\,\delta(G/e)$. Thus,

$$\delta(G) = d_G(p, [yz] \cup [zx]) \le d_G(p, p_1) \le d_G(p, h_0^{-1}(p')) + d_G(h_0^{-1}(p'), p_1) \le \frac{10}{3}\,\delta(G/e) + 1.$$

(B) Assume now that $h([xy])$ is not a geodesic in G/e. Let $[h(x)h(y)]$ be a geodesic in G/e joining $h(x)$ and $h(y)$. By Lemma 4.1.10 there is $p' \in [h(x)h(y)]$ with $d_G(p, h_0^{-1}(p')) \le 2\delta(G/e)$.

$(B.1)$ Consider the case $n = 2$, *i.e.*, $h([yz])$ and $h([zx])$ are geodesics in G/e. Then there is $p'' \in h([yz]) \cup h([zx])$ with $d_{G/e}(p', p'') \le \delta(G/e)$. By Lemma 4.1.2, we have

$$d_G\big(h_0^{-1}(p'), h_0^{-1}(p'')\big) \le d_{G/e}(p', p'') + 1 \le \delta(G/e) + 1.$$

Thus, $h_0^{-1}(p'') \in [yz] \cup [zx]$ and

$$\delta(G) = d_G(p, [yz] \cup [zx]) \le d_G(p, h_0^{-1}(p'')) \le d_G(p, h_0^{-1}(p')) + d_G\big(h_0^{-1}(p'), h_0^{-1}(p'')\big) \le 3\delta(G/e) + 1.$$

$(B.2)$ Consider the case $n = 1$. By symmetry we can assume that $h([yz])$ is a geodesic in G/e and let $[h(z)h(x)]$ be a geodesic in G/e joining $h(z)$ and $h(x)$. Then there is $p'' \in h([yz]) \cup [h(z)h(x)]$ with $d_{G/e}(p', p'') \le \delta(G/e)$. By Lemma 4.1.2, we have

$$d_G\big(h_0^{-1}(p'), h_0^{-1}(p'')\big) \le d_{G/e}(p', p'') + 1 \le \delta(G/e) + 1.$$

If $p'' \in h([yz])$, then we obtain

$$\delta(G) = d_G(p, [yz] \cup [zx]) \le d_G(p, h_0^{-1}(p'')) \le d_G(p, h_0^{-1}(p')) + d_G(h_0^{-1}(p'), h_0^{-1}(p'')) \le 3\delta(G/e) + 1.$$

If $p'' \in [h(z)h(x)]$, then by Lemma 4.1.10 there is $p_1 \in [zx]$ with $d_G(h_0^{-1}(p''), p_1) \le (7/3)\,\delta(G/e)$. Then we have

$$\begin{aligned}\delta(G) &= d_G\big(p, [yz] \cup [zx]\big) \le d_G(p, p_1) \\ &\le d_G\big(p, h_0^{-1}(p')\big) + d_G\big(h_0^{-1}(p'), h_0^{-1}(p'')\big) + d_G\big(h_0^{-1}(p''), p_1\big) \\ &\le \frac{16}{3}\,\delta(G/e) + 1.\end{aligned}$$

□

The bounds in Theorem 4.1.14 are sharp, as the following examples show.

Example 4.1.15. *Let G_0 be the diamond graph,* i.e.*, the complete graph with 4 vertices K_4 without one edge (see Figure 4.2). Let e be the edge joining the two vertices with degree 3 in G_0. Then G_0/e is isomorphic to the path graph with 3 vertices P_3. Clearly, we have $\delta(G_0) = 1$ and $\delta(G_0/e) = 0$. This fact allows to obtain many graphs G attaining the upper bound in Theorem 4.1.14: Consider any tree T and fix vertices $v \in V(T)$ and $u \in V(G_0 \setminus \{e\})$. Let G be the graph obtained from G_0 and T by identifying the vertices u and v. Then $\delta(G) = \delta(G_0) = 1$ and $\delta(G/e) = \delta(G_0/e) = 0$, since G/e and G_0/e are trees.*

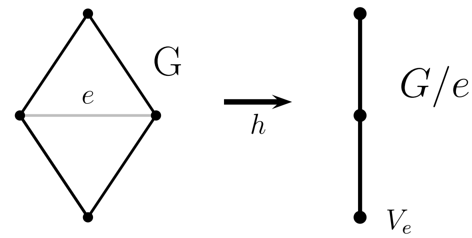

Figure 4.2: Example with upper bound in Theorem 4.1.14, *i.e.*, $\delta(G) = 1$ and $\delta(G/e) = 0$.

Example 4.1.16. *If T is any tree and e is any edge of T, then T/e is also a tree, $\delta(T) = \delta(T/e) = 0$, and so, the lower bound in Theorem 4.1.14 is attained.*

Theorem **??** below allows to improve Theorem 4.1.14 in some special cases. In order to do it we need some previous result.

We say that a vertex e in a graph G is a *cut-vertex* if $G \setminus v$ is not connected. We say that an edge $e \in E(G)$ is a *cut-edge* if $G \setminus e$ is not connected. A graph is *two-connected* if it is connected and it does not contain cut-vertices. Given a graph G, we say that a family of subgraphs $\{G_s\}_s$ of G is a *T-decomposition* of G if $\cup_s G_s = G$ and $G_s \cap G_r$ is either a *cut-vertex* or the empty set for each $s \neq r$. Every graph has a T-decomposition, as the following example shows. Given any edge in G, let us consider the maximal two-connected subgraph containing it. We call to the set of these maximal two-connected subgraphs $\{G_s\}_s$ the *canonical T-decomposition* of G.

In [14] the authors obtain the following result about T-decompositions.

Theorem 4.1.17. *[14, Theorem 3] Let G be a graph and $\{G_s\}_s$ be any T-decomposition of G, then*

$$\delta(G) = \sup_s \delta(G_s).$$

The following result improves Theorem 4.1.14 when e is a cut-edge.

Proposition 4.1.18. *Let G be a graph and e a cut-edge in G. Then*

$$\delta(G/e) = \delta(G) = \delta(G \setminus e).$$

Proof. Consider the T-decomposition $\{G_s\}_s$ of $G \setminus e$. Then $\{G_s\}_s \cup \{e\}$ is a T-decomposition of G and Proposition 4.1.17 gives

$$\delta(G) = \max\left\{ \sup_s \delta(G_s),\, \delta(e)\right\} = \max\left\{ \sup_s \delta(G_s),\, 0\right\} = \sup_s \delta(G_s) = \delta(G \setminus e).$$

For each s, let G'_s be the subgraph of G/e obtained from G_s by replacing the vertex in $\{a, b\}$ by v_e. Note that G'_s and G_s are isomorphic (and isometric) and, therefore, $\delta(G'_s) = \delta(G_s)$. Since $\{G'_s\}_s$ is a T-decomposition of G/e, Proposition 4.1.17 gives

$$\delta(G/e) = \sup_s \delta(G'_s) = \sup_s \delta(G_s) = \delta(G).$$

□

Recall that a *cactus* is a connected graph in which any two cycles have at most one vertex in common, i.e., every edge belongs to at most one cycle.
The *circumference* $c(G)$ of the graph G is the supremum of the lengths of cycles in G.

Proposition 4.1.19. *Let G be a cactus. Then*

$$\delta(G) = \frac{1}{4}c(G).$$

Proof. In order to bound $\delta(G)$, by Theorem 4.1.12 it suffices to consider geodesic triangles $T = \{x, y, z\}$ that are cycles. Hence, $L(T) \le c(G)$, $d_G(x, y) \le L(T)/2 \le c(G)/2$ and $d_G(p, [yz] \cup [zx]) \le d_G(p, \{x, y\}) \le d_G(x, y) \le c(G)/4$ for every $p \in [xy]$. Therefore, $\delta(G) \le c(G)/4$. Consider any fixed cycle C in G. Since G is a cactus, $d_C(x, y) = d_G(x, y)$ for every $x, y \in C$. Choose $x, y \in C$ with $d_C(x, y) = L(C)/2$ and denote by g_1, g_2 the geodesics in C joining x and y with $g_1 \cup g_2$ and $g_1 \cap g_2 = \{x, y\}$. Denote by B the geodesic bigon $B = \{g_1, g_2\}$. If p is the midpoint in g_1, then $\delta(B) \ge d_G(p, g_2) = d_G(p, \{x, y\}) = d_G(x, y)/2 = L(C)/4$. Thus $\delta(G) \ge c(G)/4$, and we conclude $\delta(G) = c(G)/4$ □

We denote by $\mathfrak{C}(G)$ the set of cycles C in G with $L(C) = c(G)$ if $c(G) < \infty$. The following result improves Theorem 4.1.14 for cacti.

Theorem 4.1.20. *Let G be a cactus.*

- *If either $c(G) = \infty$ or $\mathfrak{C}(G)$ contains at least two cycles, then $\delta(G/e) = \delta(G)$ for every $e \in E(G)$.*
- *If $\mathfrak{C}(G)$ contains just a single cycle C, then $\delta(G/e) = \delta(G)$ if and only if $e \notin E(G)$.*

Proof. Note that G/e is also a cactus for every $e \in E(G)$ and $c(G/e)$ is equal to either $c(G)$ or $c(G)1$. Assume first that either $c(G) = \infty$ o $\mathfrak{C}(G)$ contains at least two cycles, and fix any $e \in E(G)$. Thus, $c(G/e) = c(G)$ and Proposition **??** $\delta(G/e) = \delta(G)$. Assume now that $\mathfrak{C}(G)$ contains just a single cycle C. If $e \notin E(G)$, then $c(G/e) = c(G)$ and $\delta(G/e) = \delta(G)$. If $e \in E(G)$, then $c(G/e) = c(G)1$ and Proposition 4.1.19 gives $\delta(G/e) \neq \delta(G)$. □

4.2 Hyperbolicity of minor graphs

In order to obtain results on hyperbolicity of minor graphs we deal now with other transformation of graphs involved in the definition of minor: the deletion of edges. Let G be a graph and $\{e_j\}_{j\in J} \subseteq E(G)$. Recall that $G\setminus\{e_j\}_{j\in J}$ is the graph with $V(G\setminus\{e_j\}_{j\in J}) = V(G)$ and $E(G\setminus\{e_j\}_{j\in J}) = E(G)\setminus\{e_j\}_{j\in J}$. Theorem 4.2.3 below provides quantitative relations between $\delta(G\setminus e)$ and $\delta(G)$, where e is any edge of G.

One can expect that the edge deletion is a monotone transformation for the hyperbolicity constant. However, the following examples provide two families of graphs in which the hyperbolicity constant increases and decreases, respectively, by removing some edge.

Example 4.2.1. *Consider G_n as a cycle graph C_n with $n \geq 3$ vertices and fix $e_n \in E(G_n)$. Thus, $G_n \setminus e_n$ is isomorphic to a path graph P_n. Since $\delta(C_n) = n/4$ and $\delta(P_n) = 0$, we have $\delta(G_n) > \delta(G_n \setminus e_n)$ and*

$$\lim_{n\to\infty} \big(\delta(G_n) - \delta(G_n \setminus e_n)\big) = \infty.$$

Example 4.2.2. *Let $C_{a,b,c}$ be the graph with three disjoint paths joining two vertices with lengths $a \leq b \leq c$. Consider an edge e of $C_{a,b,c}$ contained in the path with length a. It is easy to check that $\delta(C_{a,b,c} \setminus e) = (c+b)/4$ and [68, Lemma 19] gives that $\delta(C_{a,b,c}) = (c + \min\{b, 3a\})/4$. If $3a < b$, then $\delta(C_{a,b,c}) < \delta(C_{a,b,c} \setminus e)$. In particular, consider $\Gamma_n = C_{n,4n,4n}$ and fix $e_n \in E(\Gamma_n)$ contained in the path with length n. Thus, $\delta(\Gamma_n) = 7n/4$ and $\delta(\Gamma_n \setminus e_n) = 2n$. So, $\delta(\Gamma_n) < \delta(\Gamma_n \setminus e_n)$ and*

$$\lim_{n\to\infty} \big(\delta(\Gamma_n \setminus e_n) - \delta(\Gamma_n)\big) = \infty.$$

In [28] the authors obtain quantitative information about the distortion of the hyperbolicity constant of the graph $G \setminus e$ obtained from the graph G by deleting an arbitrary edge e from it. The following theorem is a weak version of their main result.

Theorem 4.2.3. *[28, Theorem 3.15] Let G be a connected graph and $e = [a, b] \in E(G)$ with $G \setminus e$ connected. Then*

$$\max\left\{\frac{1}{5}\delta(G\setminus e), \frac{1}{4}\big(d_{G\setminus e}(a,b)+1\big)\right\} \leq \delta(G) \leq 6\delta(G\setminus e) + d_{G\setminus e}(a,b). \tag{4.8}$$

One can deduce from Proposition 4.1.18 and Theorem 4.2.3 that for any finite subset $\{e_1,\dots,e_k\} \subseteq E(G)$, we have that $G\setminus\{e_1,\dots,e_k\}$ is hyperbolic if and only if G is hyperbolic. This is not true for any infinite subset of edges (if G is any non-hyperbolic graph, then $G \setminus E(G)$ is hyperbolic). However, we can obtain a similar result for appropriate infinite subsets of edges.

Consider a subset $\{e_j\}_{j\in J} \subset E(G)$ with $e_j = [a_j, b_j]$ for any $j \in J$. We say that $\{e_j\}_{j\in J}$ is a *proper-removal subset* if $\mathcal{L}\big(G, \{e_j\}_{j\in J}\big) < \infty$, where

$$\mathcal{L}\big(G, \{e_j\}_{j\in J}\big) := \sup\Big\{d_{G\setminus\{e_j\}_{j\in J}}(a_k, b_k) \,\Big|\, k \in J \text{ with } a_k, b_k \text{ in the same connected component of } G\setminus\{e_j\}_{j\in J}\Big\}.$$

Proposition 4.2.4. *Let G be a graph and $\{e_j\}_{j\in J}$ a proper-removal subset of $E(G)$. Then $G\setminus\{e_j\}_{j\in J}$ is hyperbolic if and only if G is hyperbolic.*

Proof. Define $G' := G\setminus\{e_j\}_{j\in J}$ and consider any $M \ge \mathcal{L}\big(G,\{e_j\}_{j\in J}\big) < \infty$. Consider the canonical injection $i : G' \to G$.

Let us assume first that $G\setminus\{e_j\}_{j\in J}$ is connected. We are going to prove

$$\frac{1}{M}\, d_{G'}(x,y) - 1 \le d_G(i(x),i(y)) \le d_{G'}(x,y)\,, \qquad \text{for every } x,y\in G'\,. \tag{4.9}$$

Fix $x,y\in G'$ and let η be a geodesic in G' joining x and y. Since $i(\eta)$ is a path joining $i(x)$ and $i(y)$ with length $L(i(\eta)) = L(\eta)$, we have $d_{G'}(x,y) = L(\eta) = L(i(\eta)) \ge d_G\big(i(x),i(y)\big)$. Hence, the second inequality in (4.9) holds.

In order to prove the first inequality in (4.9), consider a geodesic γ in G from $i(x)$ to $i(y)$. If γ does not contain edges in $\{e_j\}_{j\in J}$, then $i^{-1}(\gamma)$ is also a geodesic in G' joining x with y and $d_{G'}(x,y) = L(i^{-1}(\gamma)) = L(\gamma) = d_G\big(i(x),i(y)\big)$. Assume now that γ contains some edge e_j with $j\in J$. Since γ is a compact set, it contains just a finite amount of edges in $\{e_j\}_{j\in J}$. Let $\{e_{j_0}, e_{j_1},\dots,e_{j_r}\}$ be these edges in γ, in this order. Thus, $i^{-1}(\gamma)$ is the union of $r+2$ geodesics in G'. Let $\{z_{k,1}, z_{k,2}\}$ be the endpoints of e_{j_k} for $0\le k\le r$ with

$$i^{-1}(\gamma) = [\,x\,z_{0,1}] \cup [z_{0,2}\,z_{1,1}] \cup [z_{1,2}\,z_{2,1}] \cup\cdots\cup [z_{r-1,2}\,z_{r,1}] \cup [z_{r,2}\,y\,].$$

Since $d_{G'}(z_{k,1},\, z_{k,2}) \le M$, we have

$$\begin{aligned}
d_{G'}(x,y) \le\ & d_{G'}(x,z_{0,1}) + d_{G'}(z_{0,1},z_{0,2}) + \sum_{k=1}^{r}\big(d_{G'}(z_{k-1,2},z_{k,1}) + d_{G'}(z_{k,1},z_{k,2})\big) + d_{G'}(z_{r,2},y)\\
\le\ & d_G(i(x),i(z_{0,1})) + M + \sum_{k=1}^{r}\big(d_G(i(z_{k-1,2}),i(z_{k,1})) + M d_G(i(z_{k,1}),i(z_{k,2}))\big) + d_G(i(z_{r,2}),i(y))\\
\le\ & M + M\Big(d_G(i(x),i(z_{0,1})) + \sum_{k=1}^{r}\big(d_G(i(z_{k-1,2}),i(z_{k,1})) + d_G(i(z_{k,1}),i(z_{k,2}))\big) + d_G(i(z_{r,2}),i(y))\Big)\\
=\ & M + M L(\gamma) = M + M d_G(i(x),i(y)),
\end{aligned}$$

and we conclude

$$\frac{1}{M}\, d_{G'}(x,y) - 1 \le d_G(i(x),i(y)).$$

Hence, i is a $(M,1)$-quasi-isometric embedding. Since G' is connected, i is $1/2$-full, i is a quasi-isometry and Theorem 4.1.4 gives that G' is hyperbolic if and only if G is hyperbolic. Furthermore, if G (respectively G') is δ-hyperbolic, then G' (respectively G') is δ'-hyperbolic, where δ' is a constant which just depends on δ and M.

Finally, assume that $G\setminus\{e_j\}_{j\in J}$ is not connected. Let J_0 the subset of J such that e_j is a cut-edge of G. Let $\{G^i\}_{i\in I}$ be the pairwise disjoint (connected) subgraphs of G such that $\big\{\{G^i\}_{i\in I}, \{e_j\}_{j\in J_0}\big\}$ is a T-decomposition of G. Define J^i as $J^i := \{j \in J \mid e_j \in E(G^i)\}$. Thus, $J = \big(\cup_{i\in I} J^i\big) \cup J_0$ and the sets in this union are pairwise disjoint. Also, $G' = \cup_{i\in I}\big(G^i \setminus \{e_j\}_{j\in J^i}\big)$. By Proposition 4.1.18, we have $\delta(G \setminus \{e_j\}_{j\in J_0}) = \delta(G) = \sup_{i\in I} \delta(G^i)$. Let us define $M := \mathcal{L}\big(G, \{e_j\}_{j\in J}\big)$.

Assume that G is hyperbolic. Thus, $\delta(G^i) \le \delta(G)$ for any $i \in I$, by Proposition 4.1.17. Since $G^i \setminus \{e_j\}_{j\in J^i}$ is connected and $\mathcal{L}\big(G^i, \{e_j\}_{j\in J^i}\big) \le M$ for each $i \in I$, we have proved that $\delta(G^i \setminus \{e_j\}_{j\in J^i}) \le \delta'$, where δ' is a constant which just depends on $\delta(G)$ and M. Since $\big\{G^i \setminus \{e_j\}_{j\in J^i}\big\}_{i\in I}$ are the connected components of G', we have $\delta(G') \le \delta'$.

A similar argument gives that if G' is hyperbolic, then $\delta(G) \le \delta'$, where δ' is a constant which just depends on $\delta(G')$ and M. □

By Theorem 4.1.14, given any finite subset $\{e_1, \dots, e_k\} \subseteq E(G)$, we have that $G/\{e_1, \dots, e_k\}$ is hyperbolic if and only if G is hyperbolic. This is not true for any infinite subset of edges (if G is any non-hyperbolic graph, then $G/E(G)$ is hyperbolic). However, we can obtain a similar result for appropriate infinite subsets of edges.

Consider a subset $\{e_j\}_{j\in J} \subset E(G)$ with connected components $\{K_i\}_{i\in I}$. We say that $\{e_j\}_{j\in J}$ is a *proper-contraction subset* if $\sup_{i\in I} \operatorname{diam}_G K_i < \infty$.

Proposition 4.2.5. *Let G be a graph and $\{e_j\}_{j\in J}$ a proper-contraction subset of $E(G)$. Then $G/\{e_j\}_{j\in J}$ is hyperbolic if and only if G is hyperbolic.*

Proof. Define $G' := G/\{e_j\}_{j\in J}$ and $M := \sup_{i\in I} \operatorname{diam}_G K_i < \infty$. For each $j \in J$, let v_{e_j} be the vertex in G' obtained by identifying the endpoints of e_j (note that $v_{e_{j_1}} = v_{e_{j_2}}$ if $e_{j_1}, e_{j_2} \in K_i$ for some $i \in I$). Consider the natural map $h : G \to G'$ with $h(x) := v_{e_j}$ for any $x \in e_j$ and $j \in J$. We are going to prove

$$\frac{1}{M+1}\, d_G(x,y) - 1 \le d_{G'}(h(x),h(y)) \le d_G(x,y)\,, \qquad \text{for every } x,y \in G\,. \tag{4.10}$$

Fix $x, y \in G$ and let η be a geodesic in G joining x and y. Since $h(\eta)$ is a path joining $h(x)$ and $h(y)$ with $L(h(\eta)) \le L(\eta)$, we have $d_{G'}\big(h(x),h(y)\big) \le L(h(\eta)) \le L(\eta) = d_G(x,y)$. Hence, the second inequality in (4.10) holds.

In order to prove the first inequality in (4.10), consider a geodesic γ in G' from $h(x)$ to $h(y)$. If γ does not contain vertices in $\{h(K_i)\}_{i\in I}$, then $h^{-1}(\gamma)$ is a geodesic in G joining x with y and $d_G(x,y) = L(h^{-1}(\gamma)) = L(\gamma) = d_{G'}\big(h(x),h(y)\big)$. Assume now that γ contains some vertex $h(K_i)$ with $i \in I$. Since γ is a compact set, it contains just a finite amount of vertices in $\{h(K_i)\}_{i\in I}$. Let $\{h(K_{i_0}), h(K_{i_1}), \dots, h(K_{i_r})\}$ be these vertices in γ, in this order. Thus, there is a union of $r+2$ geodesics g_0 in G such that $h(g_0) = \gamma$ and $L(g_0) = L(\gamma)$. Let $\{z_{k,1}, z_{k,2}\} = g_0 \cap K_{i_k}$ for $0 \le k \le r$ with

$$g_0 = [\,x\, z_{0,1}] \cup [z_{0,2}\, z_{1,1}] \cup [z_{1,2}\, z_{2,1}] \cup \cdots \cup [z_{r-1,2}\, z_{r,1}] \cup [z_{r,2}\, y\,]$$

(it is possible to have $z_{0,1} = z_{0,2}$ and/or $z_{r,1} = z_{r,2}$; if $z_{0,1} = z_{0,2}$ then $x \in K_{i_0}$, if $z_{r,1} = z_{r,2}$ then $y \in K_{i_r}$). Since $d_G(z_{k,1}, z_{k,2}) \le \operatorname{diam}_G K_{i_r} \le M$, we have

$$\begin{aligned}
d_G(x,y) \le\ & d_G(x,z_{0,1}) + d_G(z_{0,1},z_{0,2}) + \sum_{k=1}^{r} \big(d_G(z_{k-1,2},z_{k,1}) + d_G(z_{k,1},z_{k,2})\big) + d_G(z_{r,2},y) \\
\le\ & d_G(x,z_{0,1}) + M + \sum_{k=1}^{r} \big(d_G(z_{k-1,2},z_{k,1}) + M\, d_G(z_{k-1,2},z_{k,1})\big) + d_G(z_{r,2},y) \\
\le\ & M + (M+1)\Big(d_G(x,z_{0,1}) + \sum_{k=1}^{r} d_G(z_{k-1,2},z_{k,1}) + d_G(z_{r,2},y)\Big) \\
=\ & M + (M+1)\,L(g_0) = M + (M+1)\,L(\gamma) = M + (M+1)\,d_{G'}(h(x),h(y)),
\end{aligned}$$

and we conclude

$$\frac{1}{M+1}\, d_G(x,y) - 1 \le \frac{1}{M+1}\, d_G(x,y) - \frac{M}{M+1} \le d_{G'}(h(x),h(y)).$$

Hence, h is a $(M+1,1)$-quasi-isometric embedding. Since h is a surjective map (and then 0-full), h is a quasi-isometry and Theorem 4.1.4 gives that G' is hyperbolic if and only if G is hyperbolic. □

Finally, since the hyperbolicity constant of any isolated vertex is 0, Propositions 4.2.4 and 4.2.5 give the following qualitative result.

Theorem 4.2.6. *Let G be a graph, G_1 a minor graph of G obtained by contracting a proper-contraction subset of $E(G)$, G_2 a minor graph of G_1 obtained by deleting a proper-removal subset of $E(G_1)$, and G' a minor graph of G_2 (and of G) obtained by deleting any amount of isolated vertices. Then G is hyperbolic if and only if G' is hyperbolic.*

4.3 Hyperbolicity and minors of non-simple graphs

Simple graphs are the usual context in the study of hyperbolicity. However, the operation of contraction is naturally defined for non-simple graphs. For this reason, in this last section we study the distortion of the hyperbolicity constant by contraction of one edge in non-simple graphs.

Since we work with non-simple graphs, if there are $n_1 \geq 1$ edges in G joining v and A and $n_2 \geq 1$ edges joining v and B for some $v \in V(G)$, then we obtain $n_1 + n_2$ edges joining v and V_e in G/e, see Figure 4.3. Thus, in the context of non-simple graphs a cycle $C \in \mathcal{C}(G, e)$ is transformed in a double edge in G/e, as in Figure 4.3.

We define the map $H : G \to G/e$ in the following way: if x belongs to the edge e, then $H(x) := V_e$; if $x \in G$ does not belong to e, then $H(x)$ is the "natural inclusion map". Clearly H is onto, *i.e.*, $H(G) = G/e$. Besides, H is an injective map in $G \setminus \{e\}$.

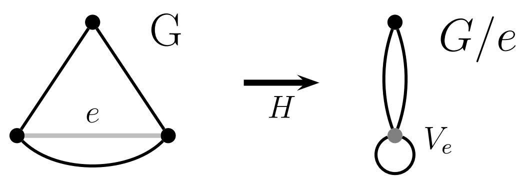

Figure 4.3: The map H.

We prove now a version of Proposition 4.1.2 for non-simple graphs.

Proposition 4.3.1. *Let G be a non-simple graph and $e \in E(G)$. Then*

$$d_{G/e}\big(H(x), H(y)\big) \leq d_G(x, y) \leq d_{G/e}\big(H(x), H(y)\big) + 1, \tag{4.11}$$

for every $x, y \in G$.

Proof. We can assume that G is connected. Fix $x, y \in G$ and let $[xy]_G$ be a geodesic in G joining x and y. Clearly, $H([xy]_G)$ is a path joining $H(x)$ and $H(y)$ with length at most $L([xy]_G)$. Hence, we obtain $d_{G/e}\big(H(x), H(y)\big) \leq d_G(x, y)$.

Let γ' be a geodesic in G/e joining $H(x)$ and $H(y)$. Then there is a path γ in G with $H(\gamma) = \gamma'$ and $L(\gamma) \leq L(\gamma') + 1$ since γ can contain e or a subset of e. Therefore, $d_G(x, y) \leq L(\gamma) \leq L(\gamma') + 1 = d_{G/e}\big(H(x), H(y)\big) + 1$. □

Note that the inequalities in (4.11) are attained. If G is any non-simple graph, $\{v, w\} \neq \{a, b\}$ and $[v, w] \in E(G)$, then $d_{G/e}(H(v), H(w)) = 1 = d_G(v, w)$. If G is any non-simple graph, then $d_G(a, b) = 1 = d_{G/e}(H(a), H(b)) + 1$.

For any simple path γ' joining two different points in G/e, $H^{-1}(\gamma')$ is either a simple path γ in G or the union of a simple path γ with e. In both cases, we define $H_0^{-1}(\gamma') := \gamma$. If $\alpha' \in \gamma'$, then we define $H_0^{-1}(\alpha') := H^{-1}(\alpha') \cap H_0^{-1}(\gamma')$ (note that $H_0^{-1}(\alpha') = H^{-1}(\alpha')$ for every $\alpha' \neq v_e$ and $H_0^{-1}(v_e)$ can be either a, b or e). Hence, if $v_e \notin \gamma'$, then $H_0^{-1}(\gamma') = H^{-1}(\gamma')$ and

$$H\,|_{H^{-1}(\gamma')} \colon H^{-1}(\gamma') \longrightarrow \gamma'$$

is a bijective map.

One can check that the following simpler versions for non-simple graphs of Lemmas 4.1.8, 4.1.10 and 4.1.13 hold.

Lemma 4.3.2. *Let G be a non-simple graph and $e \in E(G)$. Assume that for some $x, y \in G \setminus \{e\}$ there are two geodesics γ_G and $\gamma_{G/e}$ in G and G/e, respectively, joining x, y and $H(x), H(y)$, respectively, such that $L(\gamma_G) = L(\gamma_{G/e}) = L(H(\gamma_G))$ and $e \subset H_0^{-1}(\gamma_{G/e})$. Then we have*

$$d_{G/e}(H(\alpha), \gamma_{G/e}) \leq \delta(G) \quad \forall\ \alpha \in \gamma_G$$

and

$$d_{G/e}\big(\alpha', H(\gamma_G)\big) \leq 2\delta(G) \quad \forall\ \alpha' \in \gamma_{G/e}.$$

Lemma 4.3.3. *Let G be a non-simple graph and $e \in E(G)$ such that G/e is not a tree. Let $[xy]_G$ be a geodesic in G joining $x, y \in J(G)$. Assume that $H([xy]_G)$ is not a geodesic in G/e and let $[H(x)H(y)]_{G/e}$ be a geodesic in G/e joining $H(x)$ and $H(y)$. Then we have*

$$d_G(H_0^{-1}(\alpha'), [xy]_G) \leq \delta(G/e) + 1 \leq \frac{7}{3}\,\delta(G/e), \quad \forall\ \alpha' \in [H(x)H(y)]_{G/e}$$

and

$$d_G\big(\alpha, H_0^{-1}([H(x)H(y)]_{G/e})\big) \leq 2\delta(G/e), \quad \forall\ \alpha \in [xy]_G.$$

Hence, the main results also hold for non-simple graphs.

Theorem 4.3.4. *Let G be a non-simple graph and $e \in E(G)$. Then*

$$\frac{1}{3}\,\delta(G/e) \leq \delta(G) \leq \frac{16}{3}\,\delta(G/e) + 1.$$

Proposition 4.3.5. *Let G be a non-simple graph and e a cut-edge in G. Then*

$$\delta(G/e) = \delta(G) = \delta(G \setminus e).$$

Theorem 4.3.6. *Let G be a non-simple graph, G_1 a minor graph of G obtained by contracting a proper-contraction subset of $E(G)$, G_2 a minor graph of G_1 obtained by deleting a proper-removal subset of $E(G_1)$, and G' a minor graph of G_2 (and of G) obtained by deleting any amount of isolated vertices. Then G is hyperbolic if and only if G' is hyperbolic.*

Conclusions

The main aim of this work is to study the graphs with small hyperbolicity constants, i.e., the graphs which are like trees (in the Gromov sense). In Chapter 3 we give a partial answer to the question:

What is the structure of graphs with small hyperbolicity constant?
Two of our main results are Theorems 3.1.8 and 3.2.14, which characterize in two simple ways the graphs G with $\delta(G) = 1$ (the case $\delta(G) < 1$ is known, see Theorem 3.1.1).

We also characterize the graphs G with $\delta(G) = \frac{5}{4}$ in Theorem 3.2.21. Theorems 3.1.2 and 3.2.9, Corollary 3.2.16 and Proposition 3.1.9 give necessary conditions and a sufficient condition in order to have $\delta(G) = \frac{5}{4}$. Proposition 3.1.10 gives a necessary condition in order to have $\delta(G) = \frac{3}{2}$. (Recall that Theorem 2.4.2 shows that $\delta(G)$ is a multiple of $\frac{1}{4}$.)

Although it is not possible to obtain bounds for the diameter of graphs with small hyperbolicity constant, in Section 4 we obtain such bounds for the effective diameter if $\delta(G) < \frac{3}{2}$ (see Proposition 3.2.5 and Theorems 3.2.9 and 3.2.14).

This is the only case where we can obtain them, since Remark 3.2.19 shows that it is not possible to obtain similar bounds if $\delta(G) \geq \frac{3}{2}$.

Furthermore, Corollary 3.2.17 provides an explicit formula for the hyperbolicity constant of many graphs.

In Chapter 4 we obtain the invariance of the hyperbolicity under the contraction of a finite number of edges. Besides, we obtain quantitative information about the distortion of the hyperbolicity constant of the graph G/e obtained from the graph G by contracting an arbitrary edge e from it for simple and non-simple graphs, in Sections 4.1 and 4.3, respectively. In Sections 4.2 and 4.3 we obtain the invariance of the hyperbolicity on many minor graphs as a consequence of these results for simple and non-simple graphs, respectively.

Furthermore, this work provides information about the hyperbolicity constant of minor graphs.

Contributions of this work

The results in this work appear in [11, 10, 32]; these papers have been published or submitted to prestigious international mathematical journals. These results were presented in the following international and national conferences and Seminars:

- VIII Jornadas de Matemática Discreta y Algorítmica, July 2014, Tarragona, Spain.
- Seminar on Orthogonality, Approximation Theory and Applications. Group of Applied Mathematical Analysis (GAMA), May 2015, Universidad Carlos III de Madrid, Spain.
- III Congreso de Jóvenes Investigadores RSME, September 2015, Murcia, Spain.
- Seminario de Matemáticas, November 2015, Universidad Autónoma de Guerrero, México.
- X Workshop of Young Researchers in Mathematics, September 2016, Universidad Complutense de Madrid, Spain.

Open problems

In this work, we have studied the hyperbolicity of an important class of graphs: minor graphs.

We are interested in the study of Gromov hyperbolicity of other interesting class: Δ-regular graphs.

Hyperbolic cubic (3-regular) graphs are studied in some previous works, and we think that it is possible to extend many of these results to the context of Δ-regular graphs, for every $\Delta \geq 3$.

In particular, we want to characterize the Δ-regular graphs with small hyperbolicity constants, to use the T-decompositions in the study of Gromov hyperbolicity of Δ-regular graphs, and to find relationships between the hyperbolicity constant of a Δ-regular graph and its complement.

Another interesting problem is to obtain inequalities relating the hyperbolicity constant and other parameters used in Graph Theory. In particular, we want to relate the hyperbolicity constant and the differential of a graph.

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