Wave Equations in $\mathbb{R}^n$ and in Basic Control Theory

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Tiivistelmä — Referat — Abstract

Tutkielmassa esitellään aaltoyhtälöiden teoriaa käsitellen niitä sekä osittaisdifferentiaaliyhtälöiden alku- tai reuna-arvo-ongelmina että kontrolliteorian avulla. Aluksi käydään lävitse heikon derivaatan ja Sobolev-avaruuksien määritelmät, sekä tutustutaan lyhyesti keskeisiin tuloksiin Sobolev-funktioiden approksimoinnista, jatkamisesta ja upotuksista. Tutkielman keskeisin osio on omistettu aaltoyhtälön reuna- ja alkuarvo-ongelmien heikkojen ratkaisujen määrittelylle ja niiden olemassaolon sekä yksikäsitteisyyden todistamiselle yleisinä hyperbolisina osittaisdifferentiaaliyhtälöinä. Tutkielman viimeisessä osiossa määritellään joukko kontrolliteorian peruskäsitteitä ja erityisesti tarkastellaan lineaarisen systeemin reunakontrolliongelmaa tavoitteena todistaa, että tämä systeemi on kontrolloituva. Lopuksi tutkitaan, kuinka aaltoyhtälön reunakontrolliongelman voi muuntaa lineaariseksi systeemiksi, minkä seurauksena voidaan todeta myös yksinkertaistetun aaltoyhtälön olevan kontrolloituva.

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Abstract

The purpose of this thesis is to look from two different perspectives how wave equations can be solved. These are the forward problem of wave equations as partial differential equations of the initial or boundary-value type, and later in the framework of control theory. First, to provide a meaningful solution space, the basics of the theory of weak derivatives and Sobolev spaces are discussed along with some approximation, extension, and embedding theorems. Then, the initial or boundary value problem of the wave equation is defined, its weak solutions are constructed based on general hyperbolic partial differential equations, and the existence and uniqueness of said solutions is proved. The last part of this thesis concentrates on linear control theory: controllability of a linear system, and especially how it can be defined and proven form the first half of the last chapter. The other half is reserved for wave equations in control theory and why it is possible to reduce a wave control problem to solving the control problem of the aforementioned linear system.

Tiivistelmä

Tutkielmassa esitellään aaltoyhtälöiden teoriaa käsitellen niitä sekä osittaisdifferentiaaliyhtälöiden alkuarvo- tai reuna-arvo-ongelmina että kontrolliteorian avulla. Aluksi käydään lävitse heikon derivaatan ja Sobolev-avaruuksien määritelmät, sekä tutustutaan lyhyesti keskeisiin tuloksiin Sobolev-funktioiden approksimoinnista, jatkamisesta ja upotuksista. Tutkielman keskeisin osio on omistettu aaltoyhtälön reuna- ja alkuarvo-ongelmien heikkojen ratkaisujen määrittelylle ja niiden olemassaolon sekä yksikäsitteisyyden todistamiselle yleisinä hyperbolisina osittaisdifferentiaaliyhtälöinä. Tutkielman viimeisessä kappaleessa määritellään joukko kontrolliteorian peruskäsitteitä ja erityisesti tarkastellaan lineaarisen systeemin reunakontrolliongelmaa tavoitteena todistaa, että tämä systeemi on kontrolloituva. Lopuksi tutkitaan, kuinka aaltoyhtälön reunakontrolliongelman voi muuntaa lineaariseksi systeemiksi, minkä seurauksena voidaan todeta myös yksinkertaistetun aaltoyhtälön olevan kontrolloituva.

Contents

1 Introduction

From modelling a beam of sunlight to measuring tectonic movements, the natural world offers countless intriguing examples of phenomena which have inspired scientists to make amazing discoveries. Here, a specific subcategory of hyperbolic partial differential equations known as wave equations is taken under a microscope to understand its fundamental properties. First, we consider a wave equation as a forward problem: Given an initial value and boundary values, what kind of wave satisfies the equation? Is there a solution for this problem, and if the answer is yes, is it unique? The second question we try to find an answer for, is what happens when the initial state is known along with a desired outcome of the values the wave function produces at a given moment of time. Can fiddling with the input source produce this desired output?

This latter problem belongs to the field of control theory. The control theory itself was born from the mathematical tradition and technical advances of the industrialisation as a means of reasoning scientifically the process of independently limiting or directing the use of energy or resources within a system – often within a machine or a production process. This model of thinking is still relevant in comptemporary world where efficient automated processes are important in most engineering sciences.

As being said already, this thesis is an attempt to form a good basic understanding of what a wave equation is, and how it can be solved as a forward problem, or with an approach of control theory. For this purpose, the basics of Sobolev spaces and weak derivatives are also considered in order to work out solution spaces for hyperbolic equalities. The last chapters are reserved for control theory with the purpose to study how it applies to simple wave equations.

2 General Definitions, Results and Notations

This chapter contains a general list of notations and definitions the reader might find useful later on. If not otherwise stated the definitions and theorems are sourced from Evan's *"Partial Differential Equations"* [3].

Leigh defines *Cauchy sequences* in *"Functional Analysis and Linear Control Theory"* [10, p. 5] in the following way:

Definition 2.1. Let X be a vector space with a measure d. For a *Cauchy sequence* (x_n) the following holds true: for $\epsilon > 0$, there exists an integer N for which $d(x_n, x_m) < \epsilon$ when $m, n > N$.

Definition 2.2. Let X and Y be Banach spaces. A *linear operator* is a mapping $f : X \to Y$ with $f(ax + by) = af(x) + bf(y)$ when $x \in X$, $y \in Y$ and a, b are constants. When the space $Y = \mathbb{R}$ or $Y = \mathbb{C}$, the mapping is called a *linear functional*·

Definition 2.3. A *Banach space* is a *normed linear space* where all Cauchy sequences converge within the space itself. In other words, the space is a *complete normed linear space.*

Definition 2.4. A *Hilbert space* is a Banach space for which its norm is defined as

$$\|x\| := (x, x)^{1/2},$$

where $(,)$ marks its given inner product.

Definition 2.5. Let X and Y by two Banach spaces with $X \subset Y$ and $f \in X$. Now, if both conditions

(i) $\|f\|_Y \leq C\|f\|_X$, where C is a constant

(ii) each bounded sequence in X is precompact in Y, meaning that for each $r > 0$ there exists a finite cover which consists of sets of diameter smaller than r,

hold true, then X is *compactly embedded* in Y, $X \subset\subset Y$.

According to Adams [1, p. 3], *dual spaces* are defined followingly:

Definition 2.6. Let Y be a vector space. The *dual space of* Y is the set of all continuous linear functionals on Y. The dual space is denoted as Y^*.

Evans' book contains the following theorems and definitions [3]:

Theorem 2.7. *A space Y can be embedded to its double dual space Y^{**} by defining a set of mappings*

$$x \mapsto T_x,$$

where

$$T_x(\phi) = \phi(x).$$

Definition 2.8. The *weak**-*topology* is the weak topology of Y^* induced by the image $T(Y) \in Y^{**}$.

Definition 2.9. Let R be a commutative ring with unity and M, N and L R-modules. Then a *pairing* is an R-bilinear map $f : M \times N \to L$.

According to Logemann and Ryan [11, p. 286], differentiability in an open subset of $\mathbb{R}^n$ is defined as

Definition 2.10. Let U be an open subset of $\mathbb{R}^n$ with $U \neq \emptyset$. A function $f : U \to \mathbb{R}^m$ is *differentiable at* $x \in U$ if there exists a real $m \times n$ -matrix known as $(Df)(x)$ for which

$$\lim_{y \to 0} \frac{\|f(x+y) - f(x) - (Df(x))y\|}{\|y\|} = 0.$$

If a matrix $(Df)(x)$ exists for every $x \in U$, the function f is *differentiable*. In such case, $Df : U \to \mathbb{R}^{m \times n}$ with $x \mapsto (Df)(x)$ is called the *derivative* of the function f.

A more general definition for a derivative in normed spaces is defined in Gen Nakamura and Roland Potthast's *Inverse Modeling* [14, p. 2-50]. This *Fréchet differential* coincides with the classical derivative for functions $f : \mathbb{R}^n \to \mathbb{R}^n$ as explained by Nakamura and Potthast.

Definition 2.11. Let X and Y be normed spaces and U an open subset of X. A mapping $f_U \to Y$ is called *Fréchet-differentiable* at $u \in U$ if there exists an operator f' in the normed space of bounded linear operators from X to Y and a mapping $g : U \to Y$ such that

(i) $\frac{\|g(h)\|}{\|h\|} \to 0$ when $\|h\| \to 0$,

(ii) $f(u+h) = f(u) + f'h + g(h)$.

If f is Fréchnet-differentiable at every $u \in U$, then f is Fréchnet-differentiable in U.

Notice that a Fréchnet differential f' defined before is unique.

Definition 2.12. Let U be an open subset of $\mathbb{R}^n$. The space of continuous functions $f : U \to \mathbb{R}^n$ is notated by $C(U)$.

Likewise, the spaces of continuously differentiable functions are defined as

$$C^k = \{f : U \to \mathbb{R}^n : f \text{ is } k\text{-times continuously differentiable}\}$$

and

$$C^\infty = \{f : U \to \mathbb{R}^n : f \text{ is infinitely differentiable}\}.$$

The space of infinitely differentiable functions $\psi : U \to \mathbb{R}$ which all have compact support in U is notated by $C_c^\infty(U)$. Any function $\psi \in C_c^\infty(U)$ is generally called a *test function*.

As a special case, when $U = [a, b] \subset \mathbb{R}$, the space is notated by $C_c^\infty(a, b)$.

Definition 2.13. Let U be an open subset of $\mathbb{R}^n$. The *Hölder space* $C^{0,\gamma}(\overline{U})$ is defined as the space of all functions $f \in C(\overline{U})$ for which the norm

$$\|f\|_{C^{0,\gamma}(\overline{U})} := \sum_{|\alpha|\leq k} \|D^\alpha f\|_{C(\overline{U})} + \sum_{|\alpha=k|} \sup_{\substack{x,y\in U\\ x\neq y}} \left\{ \frac{|f(x) - f(y)|}{|x - y|^\gamma} \right\} < \infty.$$

A couple of notices concerning Hölder spaces: they satisfy the criteria of the definition 2.5 and thus are Banach spaces. Oftentimes the constant is set to be $0 < \gamma < 1$.

The basics of measure theory are presented after Evans' book [3, pp. 729-731, 733].

Definition 2.14. Let M be a collection of subsets of $\mathbb{R}^n$. If

(i) $\emptyset, \mathbb{R}^n \in M$,

(ii) if $A \in M$, then $\mathbb{R}^n - A \in M$,

(iii) if $\{A_n\}_{n=1}^\infty \subset M$, then both $\bigcup_{n=1}^\infty A_n$ and $\bigcap_{n=1}^\infty A_n$ are in M,

and M is called a *σ-algebra*.

Theorem 2.15. *There exists a σ-algebra M and a mapping $f : M \to [0, \infty]$ for which the following conditions apply:*

(i) Every open and every closed subset of $\mathbb{R}^n$ belongs to M.

(ii) For any ball $B \in \mathbb{R}^n$ the value of $f(B)$ is the n-dimensional volume of B.

(iii) If $\{A_n\}_{n=1}^{\infty} \subset M$ and the sets A_n are pairwise disjoint for every index $n = 1, 2, ...,$ then

$$f\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} f(A_n).$$

(iv) Let $B \in M$ and $f(B) = 0$. If $A \subseteq B$, then $A \in M$ and $f(A) = 0$.

The function f is most often notated by $|\cdot|$ and is called the n-dimensional *Lebesgue measure*. Similarly, the sets in M are *Lebesgue measurable sets.*

Definition 2.16. Let $f : \mathbb{R}^n \to R$. The function f is a *measurable* function if

$$f^{-1}(U) \in M$$

for each open subset $U \subset \mathbb{R}$.

Definition 2.17. Let X be a real Banach space. A function $f : [0, T] \to X$ is *strongly measurable* if there exist functions $s_n : [0, T] \to X$ such that

(i) every function s_n is *simple*, that is

$$s_n(t) = \sum_{i=1}^{k} \chi_{U_i}(t) x_i$$

for each $n = 1, 2, ...$ when $0 \leq t \leq T$, every U_i is a Lebesgue measurable subset of $[0, T]$ and $x_i \in X$ when $i = 1, ..., k$,

(ii) $s_n(t) \to f(t)$ for almost every $0 \leq t \leq T$.

Definition 2.18. For a function $s(t) = \sum_{i=1}^{k} \chi_{U_i}(t) x_i$ the integral

$$\int_0^T s(t)\, dt := \sum_{i=1}^{k} |U_i| x_i,$$

if s is simple.

A strongly measurable function $f : [0, T] \to X$ is *summable* if there exists a sequence of simple functions $\{s_n\}_{n=1}^{\infty}$ for which the integral

$$\int_0^T \|s_n(t) - f(t)\|\, dt \to 0 \text{ when } n \to \infty.$$

Furthermore, if a function $f : [0, T] \to X$ is summable, then its integral

$$\int_0^T f(t)\, dt = \lim_{n\to\infty} \int_0^T s_n(t)\, dt.$$

Robert Adams defines L^p-spaces in *"Sobolev Spaces"* [1, p. 22] in the following way:

Definition 2.19. Let $1 \leq p < \infty$. The space of measurable functions in a domain $U \subset \mathbb{R}^n$ for which

$$\int_U |f(x)|^p \, dx < \infty$$

is defined as $L^p(U)$. More specifically, the space consists of equivalence classes of measurable functions – a function f is likened to a representative of a class f_0 if it is measurable and its values differ from f_0 only on a set of a measure zero.

Similarly, according to Evans [3, p. 702]

Definition 2.20. Let U be an open subset of $\mathbb{R}^n$. Define $L^p_{loc}(U)$ as the space of measurable functions $f : U \to \mathbb{R}$ such that $f \in L^p(V)$ for all $V \subset\subset U$.

Definition 2.21. Let U be an open subset of $\mathbb{R}^n$. If a convergence happens locally in $L^p(U)$, that is

$$f_n \to f \text{ in } L^p(V),$$

for each $V \subset\subset U$, the notation used is

$$f_n \to f \text{ in } L^p_{loc}(U), \text{ as } n \to \infty.$$

Theorem 2.22. *Let U be an open subset of $\mathbb{R}^n$. The space $L^p(U)$ is complete if $1 \leq p \leq \infty$.*

The proof of the theorem 2.10 in Adam's book includes a proof for the completeness of $L^p(U)$ [1, pp. 26-27].

Definition 2.23. Let U be an open subset of $\mathbb{R}^n$. The space $L^\infty(U)$ is defined as

$$\{f : U \mapsto \mathbb{R}^n : \sup |f(x)| < \infty \text{ for all } x \in U\}.$$

The norm

$$\|f\|_{L^\infty(U)} = \|f\|_\infty := \sup_{x \in U} |f(x)|.$$

According to Evans [3, p. 731],

Definition 2.24. The *essential supremum* of a measurable function f over set U is defined by

$$\text{ess } \sup_U f = \inf\{y \in \mathbb{R} : \mu(\{x \in U : f(x) > y\}) = 0\}.$$

Definition 2.25. The *support* of a function f is the set where f is nonzero,

$$\operatorname{supp}(f) = \overline{\{x : f(x) \neq 0\}}.$$

Definition 2.26. The value of the *floor function* of f is notated by $\lfloor f(x) \rfloor$. The floor function returns the integer part of a value $f(x)$ for all x in the domain of f.

Definition 2.27. A vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, where for every index i it holds $a_i \in \mathbb{N}$, is a *multi-index* of *order*

$$|\alpha| = \alpha_1 + \ldots + \alpha_n.$$

The α-*derivative* of a function u is defined as

$$D^\alpha u(x) := \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} u.$$

As stated in Vladimir Maz'ya's *Sobolev Spaces with Applications to Elliptic Partial Differential Equations"* [13, pp. 3-4], multi-indeces have the following properties:

Definition 2.28. Let $n, m \in \mathbb{N}$ with $m \leq n$. For multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$,

$$\alpha! = \alpha_1!\, \alpha_2! \, \ldots \alpha_n!$$

and

$$\binom{\alpha}{\beta} = \sum_{\alpha \geq \beta} \frac{\alpha!}{\beta!(\alpha - \beta)!}.$$

The following theorem and proof are presented in Evans' [3, p. 706].

Theorem 2.29 (Cauchy's inequality)**.** *Let $a, b \in \mathbb{R}^n$. Then*

$$ab \leq \frac{1}{2}\left(a^2 + b^2\right).$$

Proof. Since $0 \leq (a-b)^2 = a^2 - 2ab + b^2$, the result is gained by adding $2ab$ and then dividing the inequality by 2. $\square$

The next two *Gronwall's inequalities* are formulated and proven according to Evans [3, pp. 708-709].

Proposition 2.30 (Gronwall's inequality)**.** *Let $\eta(\cdot)$ be a nonnegative, absolutely continuous function on $[0,T]$, which satisfies the differential inequality*

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t)$$

for almost every $t \in [0,T]$, when ϕ and ψ are also nonnegative, summable functions on $[0,T]$. Then

$$\eta(t) \leq e^{\int_0^t \phi(s)\,ds} \left[\eta(0) + \int_0^t \psi(s)\,ds\right] \tag{1}$$

for all $0 \leq t \leq T$.

Proof. Suppose that the function η is nonnegative and absolutely continuous on interval $[0,T]$. Also, suppose that functions ϕ and ψ are nonnegative and summable on $[0,T]$, and that for the derivative of η the following inequality holds

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t),$$

for almost every $t \in [0,T]$. Now for almost every $s \in [0,T]$

$$\frac{d}{ds}\left(\eta(s)e^{-\int_0^s \phi(r)\,dr}\right) = e^{-\int_0^s \phi(r)\,dr}\left(\eta'(s) - \phi(s)\eta(s)\right) \leq e^{-\int_0^s \phi(r)\,dr}\psi(s).$$

Next, consider the previous function before derivation:

$$\eta(t)e^{-\int_0^t \phi(r)\,dr} \leq \eta(0) + \int_0^t e^{-\int_0^s \phi(r)\,dr}\psi(s)\,ds \leq \eta(0) + \int_0^t \psi(s)\,ds.$$

The inequality (1) now follows since it was assumed that ϕ and ψ are nonnegative and summable on $[0,T]$. $\square$

Proposition 2.31 (Gronwall's inequality 2)**.** *Let ξ be a nonnegative, summable function on $[0,T]$ which for almost every t satisfies the integral inequality*

$$\xi(t) \leq C_1 \int_0^t \xi(s)\,ds + C_2$$

for some nonnegative constants C_1 and C_2. Then

$$\xi(t) \leq C_2(1 + C_1 t e^{C_1 t})$$

for almost every $t \in [0,T]$.

Proof. Suppose ξ is a nonnegative, summable function on $[0,T]$ which for almost every t satisfies the inequality

$$\xi(t) \leq C_1 \int_0^t \xi(s)\, ds + C_2. \tag{2}$$

Let function $\eta(t) := \int_0^t \xi(s)\, ds$. Then its derivative $\eta' \leq C_1\eta + C_2$ for almost every $t \in [0,T]$. Now, the Gronwall's inequality 2.30 implies

$$\begin{aligned}\eta(t) &\leq e^{\int_0^t C_1\, ds}\left[\eta(0) + \int_0^t C_2\, ds\right] \\ &= e^{C_1 t}(\eta(0) + C_2 t) \\ &= C_2 t e^{C_1 t},\end{aligned}$$

since η is defined by the definite integral of ξ. Next, continue approximating η from (2):

$$\xi(t) \leq C_1\eta(t) + C_2 \leq C_1 C_2 t e^{C_1 t} + C_2 \leq C_2(1 + C_1 t e^{C_1 t}). \qquad \square$$

According to Arfken and Weber's *"Mathematical Methods for Physicists"* [2, p. 590], the *Leibniz's integral rule* can be defined by the following theorem.

Theorem 2.32 (Leibniz' integral rule)**.** *Let U be an open subset of $\mathbb{R}^n$ with $g, h \in C^1(U)$. For a Lebesgue integral with a measurable integrand $f = f(x,\alpha) \in C^1(U)$ and limits g and h which both may depend on a variable α, the derivative of an integral*

$$\frac{d}{d\alpha}\int_{g(\alpha)}^{h(\alpha)} f(x,\alpha) dx = \int_{g(\alpha)}^{h(\alpha)} \frac{\partial f(x,\alpha)}{\partial \alpha} dx + f[h(\alpha),\alpha]\frac{dh(\alpha)}{d\alpha} - f[g(\alpha),\alpha]\frac{dg(\alpha)}{d\alpha}.$$

Theorem 2.33 (Minkovski's inequality)**.** *Let $1 \leq p \leq \infty$ and $f, g \in L^p(U)$, where U is an open subset of $\mathbb{R}^n$. Now*

$$\|f + g\|_{L^p(U)} \leq \|f\|_{L^p(U)} + \|g\|_{L^p(U)}.$$

The proof is presented in Evans' book [3, p. 707].

Definition 2.34. Let A be an $n \times n$-matrix.

(i) A is *symmetric* if $A_{ij} = A_{ji}$ for every index $i, j = 1, \ldots, n$.

(ii) A is *positively semi-definite* if all of its eigenvalues are nonnegative.

(iii) A is *invertible* if there exists an $n \times n$ matrix A' for which

$$AA' = I \qquad \text{and} \qquad A'A = I,$$

where I is the $n \times n$ identity matrix.

Definition 2.35. Let A be a $n \times m$-matrix. A is *right invertible* if $A'A = I$ holds true. Likewise, if $AA' = I$, then A is *left invertible.*

As a notice, for a square matrix either left or right invertibly implies full invertibly and vice versa.

For these and more results of matrix properties, I recommend David Poole's *"Linear Algebra: A Modern Introduction"* [15].

The next two theorems and their proofs are presented in Logemann and Ryan's *"Ordinary Differential Equations"* [11, pp. 262-266].

Theorem 2.36. *Let $M \in \mathbb{R}^{m \times n}$. Now the following hold true*

(i) $(\operatorname{im} M)^{\perp} = \ker M^* = \{x^* : x^* M = 0\}$.

(ii) The rank of M, or $\operatorname{rank} M$, is equal to its maximum number of linearly independent rows, or equivalently, colums.

(iii) If $m \leq n$, there exists $M^{\sharp} \in \mathbb{R}^{n \times m}$ for which $MM^{\sharp}z = z$ for all $z \in \operatorname{im} M$.

(iv) If $m > n$, there exists $M^{\sharp} \in \mathbb{R}^{n \times m}$ for which $M^{\sharp}Mz = z$ for all $z \in (\ker M)^{\perp}$.

Theorem 2.37 (The Cayley-Hamilton theorem)**.** *Let $M \in \mathbb{R}^{n \times n}$. Then it satisfies its characteristic equation*

$$M^n + \lambda_{n-1}M^{n-1} + \cdots + \lambda_1 M + \lambda_0 I = 0,$$

where λ_i are the eigenvalues of the matrix M.

3 Weak Derivatives and Sobolev Spaces

The motivation for *weak derivatives* and *Sobolev spaces* rises from the need of having function spaces with more lenient definitions for well-behaving in terms of derivatives. In the case of partial differential equations not all solutions are perfectly smooth – there can be troublesome behaviour appearing in the solutions of even the simplest physical problems. The weak derivative is especially designed to work with these types of solution functions, as compared to ordinary derivatives, the set of functions for which weak derivatives exist is larger. The Sobolev spaces are defined by weak derivatives just as the C^n*-spaces* are defined by ordinary or partial derivatives. In this paper Sobolev spaces are used as the main domains while proving existence and uniqueness of the solutions of wave equations.

This chapter follows the chapter "5. Sobolev Spaces" of Evans' book [3, pp. 253-309]. Any material from other sources is cited separately.

3.1 Weak derivatives

Definition 3.1. Let U be an open subset of $\mathbb{R}^n$. When $f, g \in L^1_{loc}(U)$ and α is a multi-index, the function g is defined as the α^{th}*-weak derivative of* f,

$$D^\alpha f = g,$$

if

$$\int_U f D^\alpha \psi \, dx = (-1)^{|\alpha|} \int_U g\psi \, dx$$

for every test function $\psi \in C_c^\infty(U)$.

Lemma 3.2 (Uniqueness of weak derivatives)**.** *If there exists a weak* α^{th}*-partial derivative of function* f*, then it is uniquely defined almost everywhere.*

Proof. Let $U \subset \mathbb{R}^n$ be open, α a multi-index and $f \in L^1_{loc}(U)$. Suppose that there are two different functions g and h which satisfy the criteria of the definition 3.1.

Now,

$$(-1)^{|\alpha|} \int_U g\psi \, dx = \int_U f D^\alpha \psi \, dx = (-1)^{|\alpha|} \int_U h\psi \, dx$$

for every test function $\psi \in C_c^\infty(U)$. Thus,

$$(-1)^{|\alpha|} \int_U (g - h)\psi \, dx = 0,$$

and consequently $g = h$ almost everywhere. □

3.2 Sobolev spaces

Here is a short collection of definitions concerning *Sobolev spaces*, their norms and the *dual space* of one of the most frequently used Sobolev space. The notation is done according to Evans' book [3], but other conventions also exist in literature. As an example, Ladyzhenskaya's *The Boundary Value Problems of Mathematical Physics* [7, pp. 19-24] offers an another, equivalent way of defining Sobolev spaces and consequently a different approach to proving some of the theorems presented in this chapter.

Definition 3.3. Let U be an open subset of $\mathbb{R}^n$. The *Sobolev space*

$$W^{k,p}(U)$$

consists of all locally summable or, in other words, locally integrable functions $f : U \to \mathbb{R}$ for which the following holds true: for each multi-index $\alpha \leq k$, where k is a nonnegative integer, $D^\alpha f$ exists in the sense of the definition 3.1 and belongs to $L^p(U)$, where $1 \leq p \leq \infty$.

The *closure* of $C_c^\infty(U)$ in $W^{k,p}(U)$ is notated by $W_0^{k,p}(U)$.

A function f is in $W_{loc}^{k,p}(U)$ if $f\chi \in W^{k,p}(V)$ for every $V \subset\subset U$ when χ is in $C_c^\infty(U)$.

The following technical theorem shows how weak derivatives function with the definition of Sobolev spaces.

Theorem 3.4. *Let U be an open subset of $\mathbb{R}^n$ and $f, g \in W^{k,p}(U)$. Assume α is a multi-index with $|\alpha| \leq k$. Then for each constant $\lambda, \mu \in \mathbb{R}$, the linear combination $\lambda f + \mu g \in W^{k,p}(U)$ and the weak derivative is also linear,*

$$D^\alpha(\lambda f + \mu g) = \lambda D^\alpha f + \mu D^\alpha g.$$

Proof. By the definition of the functions f and g, and the linearity of $L_{loc}^1(U)$, the linear combination $\lambda f + \mu g \in L_{loc}^1(U)$. Next, consider the definition of weak derivatives and let $\psi \in C_c^\infty(U)$. Now,

$$\int_U f D^\alpha \psi \, dx = (-1)^{|\alpha|} \int_U f_\alpha \psi \, dx$$

and

$$\int_U g D^\alpha \psi \, dx = (-1)^{|\alpha|} \int_U g_\alpha \psi \, dx.$$

Consequently,

$$\begin{aligned}
\int_U (\lambda f + \mu g) D^\alpha \psi \, dx &= \int_U \lambda f D^\alpha \psi \, dx + \int_U \mu g D^\alpha \psi \, dx \\
&= \lambda \int_U f D^\alpha \psi \, dx + \mu \int_U g D^\alpha \psi \, dx \\
&= \lambda (-1)^{|\alpha|} \int_U f_\alpha \psi \, dx + \mu (-1)^{|\alpha|} \int_U g_\alpha \psi \, dx \\
&= (-1)^{|\alpha|} \int_U \lambda f_\alpha \psi \, dx + (-1)^{|\alpha|} \int_U \mu g_\alpha \psi \, dx \\
&= (-1)^{|\alpha|} \int_U (\lambda f_\alpha + \mu g_\alpha) \psi \, dx,
\end{aligned}$$

which proves

$$D^\alpha (\lambda f + \mu g) = \lambda D^\alpha f + \mu D^\alpha g.$$

□

Definition 3.5. Let U be an open subset of $\mathbb{R}^n$. The *Sobolev norm of a function* $f \in W^{k,p}(U)$ is

$$\|f\|_{W^{k,p}(U)} := \begin{cases} \left(\sum_{|\alpha| \le k} \int_U |D^\alpha f|^p \, dx \right)^{1/p}, & \text{when } 1 \le p < \infty \\ \sum_{|\alpha| \le k} \text{ess } \sup_U |D^\alpha f|, & \text{when } p = \infty. \end{cases}$$

It is notable that the sums go through all possible multi-indices limited only by the index k while the rest of the norm is the relative to the l^p- or L^p-norm of the derivative $D^\alpha f$.

Theorem 3.6. *The formulae defined earlier impose norms in spaces* $W^{k,p}(U)$.

Proof. Let $U \subset \mathbb{R}^n$, functions $f, g \in W^{k,p}(U)$ and λ be a constant. The purpose of this proof is to verify that the formula $\| \cdot \|_{W^{k,p}(U)}$ satisfies the criteria set for a norm:

(i)
$$\begin{aligned}
\|\lambda f\|_{W^{k,p}(U)} &= \begin{cases} \left(\sum_{|\alpha| \le k} \int_U |D^\alpha (\lambda f)|^p \, dx \right)^{1/p}, & \text{when } 1 \le p < \infty \\ \sum_{|\alpha| \le k} \text{ess } \sup_U |D^\alpha (\lambda f)|, & \text{when } p = \infty \end{cases} \\
&= \begin{cases} \left(\sum_{|\alpha| \le k} \int_U |\lambda|^p |D^\alpha f|^p \, dx \right)^{1/p}, & \text{when } 1 \le p < \infty \\ \sum_{|\alpha| \le k} \text{ess } \sup_U |\lambda| \, |D^\alpha f|, & \text{when } p = \infty. \end{cases} \\
&= |\lambda| \, \|f\|_{W^{k,p}(U)}.
\end{aligned}$$

(ii) Suppose $\|f\|_{W^{k,p}(U)} = 0$. By the definition of the Sobolev norm, the absolute value of $D^\alpha f$ is now zero almost everywhere for all multi-indeces α with $|\alpha| \leq k$. This implies $f = 0$ almost everywhere. This chain of implications applies also in the other direction, hence

$$\|f\|_{W^{k,p}(U)} = 0 \text{ if and only if } f = 0 \text{ almost everywhere.}$$

(iii) Suppose $1 \leq p < \infty$. By Minkovski's inequality 2.33

$$\begin{aligned}
\|f+g\|_{W^{k,p}(U)} &= \left(\sum_{|\alpha|\leq k} \|D^\alpha f + D^\alpha g\|_{L^p(U)}^p \right)^{1/p} \\
&\leq \left(\sum_{|\alpha|\leq k} (\|D^\alpha f\|_{L^p(U)} + \|D^\alpha g\|_{L^p(U)})^p \right)^{1/p} \\
&\leq \left(\sum_{|\alpha|\leq k} \|D^\alpha f\|_{L^p(U)}^p \right)^{1/p} + \left(\sum_{|\alpha|\leq k} \|D^\alpha g\|_{L^p(U)}^p \right)^{1/p} \\
&= \|f\|_{W^{k,p}(U)} + \|g\|_{W^{k,p}(U)}.
\end{aligned}$$

If $p = \infty$, then define sets

$$\begin{aligned}
A &= \{x \in U : f(x) \leq \|f\|_{W^{k,\infty}(U)}\} \\
B &= \{x \in U : g(x) \leq \|g\|_{W^{k,\infty}(U)}\}.
\end{aligned}$$

Notice that the measure $\mu\{(A \cap B)^c\} = 0$. Thus, if $x \in A \cap B$, then

$$|(f+g)(x)| \leq \|f\|_{W^{k,\infty}(U)} + \|g\|_{W^{k,\infty}(U)},$$

which implies

$$\|f+g\|_{W^{k,\infty}(U)} \leq \|f\|_{W^{k,\infty}(U)} + \|g\|_{W^{k,\infty}(U)}. \qquad \square$$

Later the dual space of $H_0^1(U)$ is needed to prove the existence of hyperbolic equations' weak solutions. The next definition shows how this dual space is found explicitly according to Evans [3, p. 722].

Definition 3.7. Let U be an open subset of $\mathbb{R}^n$. The *dual space of* $H_0^1(U)$ is $H^{-1}(U)$. A function f belongs to $H^{-1}(U)$ if it is a bounded linear functional on $H_0^1(U)$, which means that it satisfies

(i) $f(ax+by) = af(x) + bf(y)$ for $a, b \in \mathbb{R}$ and $x, y \in U$,

(ii) $|f(x)| \leq c|f(x)|$ for all $x \in U$.

To use the dual space of $H_0^1(U)$ efficiently, define a mapping known as a *pairing* between the space and its dual. This is denoted by $\langle \ , \ \rangle$, meaning

$$f^*(f) = \langle f^*, f \rangle,$$

when $f \in H_0^1(U)$ and $f^* \in H^{-1}(U)$.

Definition 3.8. Let U be an open subset of $\mathbb{R}^n$. Assume that a sequence $\{f_n\}_{n=1}^{\infty}$ and a function f are in $W^{k,p}(U)$. The sequence f_n *converges* to f in $W^{k,p}(U)$,

$$f_n \to f \text{ in } W^{k,p}(U), \text{ as } n \to \infty,$$

if

$$\lim_{n\to\infty} \|f_n - f\|_{W^{k,p}(U)} = 0.$$

3.3 Approximations, extensions and inequalities

One of the key steps in transferring the knowledge gained from working with Sobolev functions to the case of norm inequalities is to approximate Sobolev functions with smooth functions, which are considerably more straightforward to handle. Here, the basis of *global approximation* is given as an outline to the theory – more can be read in Evans' book [3, pp. 264-268], where the complete line of theorems and proofs is presented in detail. Later on, different ways to handle Sobolev functions are presented shortly, mostly omitting their proofs for the sake of easy reading. These inequalities, extensions, and other technical ways to simplify working with Sobolev spaces can be found with more details in Evans' book [3, pp. 268-271, 275-289], and Adams' book [1, chapters III-VI].

But first, consider the property of Sobolev spaces that makes them so useful when approximating solution functions: The following proof is done based on Evans' book [3, pp. 262-263].

Theorem 3.9. *Let $U \subset \mathbb{R}^n$. The Sobolev space $W^{k,p}(U)$ is a Banach space for each $k = 1, 2, \ldots$ and $1 \leq p \leq \infty$.*

Proof. It was already proven in the theorem 3.6 that Sobolev norms are true norms. Thus, the only thing left here to prove is the convergence of Cauchy sequences in $W^{k,p}(U)$.

Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $W^{k,p}(U)$. Then the sequence $\{D^\alpha f_n\}_{n=1}^{\infty}$ is also a Cauchy sequence in $L^p(U)$ for each $|\alpha| \leq k$ by the definition of

Sobolev spaces. From the completeness of $L^p(U)$ it follows that there exist functions $f_\alpha \in L^p(U)$ for which

$$D^\alpha f_n \to f_\alpha \text{ in } L^p(U) \text{ as } n \to \infty$$

for every multi-index $|\alpha| \leq k$. When $\alpha = (0, \ldots, 0)$,

$$f_n \to f_{(0,\ldots,0)} \text{ in } L^p(U) \text{ as } n \to \infty.$$

Next, show that $f_{(0,\ldots,0)} \in W^{k,p}(U)$ and $D^\alpha f = f_\alpha$ for each $|\alpha| \leq k$. Let ϕ be an arbitrary test function in $C_c^\infty(U)$. Then consider the definition of a weak derivative to get

$$\begin{aligned}
\int_U f D^\alpha \phi \, dx &= \lim_{n\to\infty} \int_U f_n D^\alpha \phi \, dx \\
&= \lim_{n\to\infty} (-1)^{|\alpha|} \int_U D^\alpha f_n \phi \, dx \\
&= (-1)^{|\alpha|} \int_U f_\alpha \phi \, dx.
\end{aligned}$$

This implies

$$D^\alpha f_n \to D^\alpha f \text{ in } L^p(U) \text{ as } n \to \infty$$

for every multi-index $|\alpha| \leq k$. Hence,

$$f_n \to f_{(0,\ldots,0)} \text{ in } W^{k,p}(U) \text{ as } n \to \infty. \qquad \square$$

With an applicable definition of an inner product, some of the Sobolev spaces can be proved to satisfy the criteria set for a complete inner product space. This follows as a corollary to the previous theorem.

Corollary 3.9.1. *Let U be an open subset of $\mathbb{R}^n$ and $f, g \in W^{k,2}(U)$. A Sobolev space $W^{k,2}(U)$ is a Hilbert space with its inner product defined by*

$$(f, g)_{W^{k,2}(U)} := \sum_{|\alpha|\leq k} \int_U D^\alpha f D^\alpha g \, dx$$

Proof. Let $1 \leq k < \infty$. Assume $f, g, h \in W^{k,2}(U)$ and $\lambda, \mu \in \mathbb{R}^n$. Then

(i)
$$\begin{aligned}
(f, g)_{W^{k,2}(U)} &= \sum_{|\alpha|\leq k} \int_U D^\alpha f D^\alpha g \, dx \\
&= \sum_{|\alpha|\leq k} \int_U D^\alpha g D^\alpha f \, dx \\
&= (g, f)_{W^{k,2}(U)}
\end{aligned}$$

(ii)
$$
\begin{aligned}
(\lambda f + \mu g, h)_{W^{k,2}(U)} &= \sum_{|\alpha|\le k} \int_U D^\alpha(\lambda f + \mu g) D^\alpha h \, dx \\
&= \sum_{|\alpha|\le k} \int_U (\lambda D^\alpha f + \mu D^\alpha g) D^\alpha h \, dx \\
&= \sum_{|\alpha|\le k} \int_U \lambda D^\alpha f D^\alpha h + \mu D^\alpha g D^\alpha h \, dx \\
&= \sum_{|\alpha|\le k} \int_U \lambda D^\alpha f D^\alpha h \, dx + \sum_{|\alpha|\le k} \int_U \mu D^\alpha g D^\alpha h \, dx \\
&= \lambda (f,h)_{W^{k,2}(U)} + \mu (g,h)_{W^{k,2}(U)}
\end{aligned}
$$

(iii)
$$
\begin{aligned}
(f,f)_{W^{k,2}(U)} &= \sum_{|\alpha|\le k} \int_U D^\alpha f D^\alpha f \, dx \\
&= \sum_{|\alpha|\le k} \int_U (D^\alpha f)^2 \, dx \ge 0 \text{ for all } f \in W^{k,2}(U).
\end{aligned}
$$

(iv) Suppose $(f,f)_{W^{k,2}(U)} = 0$. Then for all $|\alpha| \le k$ the weak derivative $D^\alpha f = 0$ almost everywhere, which implies $f = 0$ almost everywhere. For the opposite direction: if the function f is zero almost everywhere, then $D^\alpha f = 0$ almost everywhere for all $|\alpha| \le k$, and thus $(f,f)_{W^{k,2}(U)} = 0$. □

Being Hilbert spaces, these Sobolev spaces deserve a special notion used from now on, namely
$$H^k(U) := W^{k,2}(U).$$
Also, the pairing between $H_0^1(U)$ and $H^{-1}(U)$ is notated by $\langle \ , \ \rangle$.

The theory of approximations is built in several steps generalising the previous results, starting from local approximation: when $U \subset \mathbb{R}^n$ is divided into subsets $U_\epsilon = \{x \in U : dist(x, \partial U) > \epsilon\}$ for $\epsilon > 0$, the following theorem applies as stated in Evans' book [3, pp. 264-265]:

Theorem 3.10 (Local approximation by smooth functions). *Let U be an open subset of $\mathbb{R}^n$. Assume that for some $1 \le p < \infty$ and $k \in \mathbb{N}$ a function $f \in W^{k,p}(U)$. Set*
$$f^\epsilon = \eta_\epsilon * f \ \text{ in } U_\epsilon.$$
Then $f^\epsilon \in C^\infty(U_\epsilon)$ for each $\epsilon > 0$, and $f^\epsilon \to u$ in $W^{k,p}_{loc}(U)$, as $\epsilon \to 0$.

This proof is written in detail in [3, pp. 264-265, 713].

The knowledge that all Cauchy sequences have a limit within the Sobolev space in question leads to it being possible to build approximating Cauchy sequences of smooth functions as $C^\infty(\overline{U}) \subset W^{k,p}(U)$ when the set U is 'nice' enough. The next theorem by Evans [3, p. 266] explains the specific criteria detailedly.

Theorem 3.11 (Global approximation up to the boundary)**.** *Let $U \subset \mathbb{R}^n$ be bounded with ∂U being C^1. Suppose that $f \in W^{k,p}(U)$ for some $1 \leq p < \infty$ and $k \in \mathbb{N}$. Then there exist functions $f_n \in C^\infty(\overline{U})$ for which*

$$f_n \to f \text{ in } W^{k,p}(U) \text{ as } n \to \infty.$$

Here, the proof is omitted; it can be found in Evans' book [3, pp. 266-268].

As the main topic of this paper is wave equations in $\mathbb{R}^n$, it is necessary to extend the previous approximation result to the whole $\mathbb{R}^n$. The following *extension theorem* by Evans [3, p. 268] is the key to this process.

Theorem 3.12 (Extension theorem)**.** *Let $1 \leq p \leq \infty$, and suppose that $U \subset \mathbb{R}^n$ is bounded with a C^1-smooth boundary. Select a bounded open set V for which $U \subset\subset V$. Then there exists a bounded linear operator*

$$E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$$

such that for each $f \in W^{1,p}(U)$ the following properties are satisfied

(i) $Ef = f$ *almost everywhere in U,*

(ii) Ef *has a compact support within V, and*

(iii)
$$\|Ef\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|f\|_{W^{1,p}(U)},$$
where the constant C depends only on p, U and V.

The proof can be found in [3, pp. 268-271].

Definition 3.13. The *extension* of function $f : U \mapsto \mathbb{R}^n$ to the whole $\mathbb{R}^n$ is defined as Ef, where the operator E is defined in the Extension theorem 3.12.

For more information on different Sobolev inequalities and the proof of the following, here only shortly presented theorems can be found in Evans' book [3, pp. 275-289].

Theorem 3.14 (Poincaré-Friedrichs inequality)**.** *Let U be a bounded, open subset of $\mathbb{R}^n$. Suppose $f \in W_0^{k,p}(U)$ with $1 \le p < n$. Then*

$$\|u\|_{L^q(U)} \le C\|Df\|_{L^p(U)}$$

for each

$$1 \le q \le p^* = \frac{np}{n-p}.$$

The constant C depends only on p, q, n and the set U.

The proof can be found in Evans' book [3, pp. 279-280].

Theorem 3.15 (Morrey's inequality)**.** *Let $n < p < \infty$. Then there exists a constant C, depending only on n and p, for which*

$$\|f\|_{C^{0,\gamma}(\mathbb{R}^n)} \le C\|f\|_{W^{1,p}(\mathbb{R}^n)}$$

for every $f \in C^1(\mathbb{R}^n)$, where

$$\gamma := 1 - n/p.$$

The proof can be read in Evans' book [3, pp. 280-283].

Aside from extending a function continuously to gain a new, essentially similar function to work on, it is also possible is to consider the *version* of a function.

Definition 3.16. A function f^* is a *version* of a function f if their values are the same everywhere except for a set of measure zero. Or in other words, $f^* = f$ *almost everywhere.*

The following theorem and its proof after Evans' [3, pp. 283-284] ties together several of the previously mentioned results.

Theorem 3.17. *Let U be a bounded, open subset of $\mathbb{R}^n$ with C^1-smooth boundary ∂U. Assume $n < p \le \infty$ and $f \in W^{1,p}(U)$. Then f has a version $f^* \in C^{0,\gamma}(\bar{U})$ where $\gamma = 1 - n/p$, with the estimate*

$$\|f^*\|_{C^{0,\gamma}(\bar{U})} \le C\|f\|_{W^{1,p}(U)},$$

where the constant C depends only on n, p and U.

Proof. Assume that ∂U is C^1-smooth and $f \in W^{1,p}(U)$. Then, by the Extension theorem 3.12, there exists an extension $Ef = g$ in $W^{1,p}(\mathbb{R}^n)$ which satisfies

$$\begin{cases} g = f \text{ almost everywhere in } U \\ g \text{ has a compact support} \\ \|g\|_{W^{1,p}(\mathbb{R}^n)} \le C\|f\|_{W^{1,p}(U)}. \end{cases}$$

First, prove the case $n < p < \infty$. Theorem 3.10 implies the existence of functions $g^\epsilon \in C_c^\infty(\mathbb{R}^n)$ for each $\epsilon > 0$. These functions converge

$$g^\epsilon \to g \text{ in } W^{1,p}(\mathbb{R}^n) \text{ when } \epsilon \to 0.$$

Form a sequence $\{g_m\}_{m=1}^\infty$ of these functions by choosing the index m to correspond to the values ϵ growing smaller.

Since the function g_m are in $C_c^\infty(\mathbb{R}^n)$, they satisfy the criteria for Morrey's inequality 3.15, and thus

$$\|g_m - g_k\|_{C^{0,1-n/p}(\mathbb{R}^n)} \le C\|g_m - g_k\|_{W^{1,p}(\mathbb{R}^n)}$$

for all indeces $k, m \ge 1$. This implies that there exists a function $g^* \in C^{0,1-n/p}(\mathbb{R}^n)$ for which

$$g_m \to g^* \text{ in } C^{0,1-n/p}(\mathbb{R}^n).$$

From the convergences in both $W^{1,p}(\mathbb{R}^n)$ and $C^{0,1-n/p}(\mathbb{R}^n)$ it follows that $g^* = f$ almost everywhere on the set U. In other words, g^* is a version of f.

Next prove the estimate: Morrey's inequality 3.15 implies

$$\|g_m\|_{C^{0,1-n/p}(\mathbb{R}^n)} \le C\|g_m\|_{W^{1,p}(\mathbb{R}^n)}.$$

The same convergence statements as before can now be used to take the limit within this inequality with the result of

$$\|g^*\|_{C^{0,1-n/p}(\mathbb{R}^n)} \le C\|g\|_{W^{1,p}(\mathbb{R}^n)}.$$

This can be further estimated by the estimate for the extension g to get

$$\|g^*\|_{C^{0,1-n/p}(\mathbb{R}^n)} \le C\|g\|_{W^{1,p}(\mathbb{R}^n)} \le C\|f\|_{W^{1,p}(U)},$$

which concludes the proof in the case $n < p < \infty$. □

The result also applies in the case $p = \infty$, a proof of which can be found in Kinnunen's lecture notes *"Sobolev spaces"* [6, pp. 59]. Here, it is omitted since the basics of the wave equations do not require it – there it is enough to consider only Sobolev spaces with a finite index p.

The following theorem makes it possible to consider a well-behaving version instead of a Sobolev function with potentially less continuity when proving general theorems. From here on, whenever a Sobolev function is presented, it is identified with its continuous version when $p > n$.

As the previous inequalities state, when conditions are right, a function in a Sobolev space may well belong to other function spaces also. To continue with this idea, according to Evans [3, p. 286], the following theorem explains closely the relationship between certain Sobolev and L^p-spaces. This compact embedding will later prove its worth in defining a basis that works for both spaces $H_0^1(U)$ and $L^2(U)$.

Theorem 3.18 (Rellich-Kondrachov Compactness Theorem)**.** *Let U be a bounded open subset of $\mathbb{R}^n$ and ∂U be C^1. Suppose that $1 \leq p < n$. Then*

$$W^{1,p}(U) \subset\subset L^q(U)$$

for each $1 \leq q < p^* = \frac{pn}{n-p}$.

Notice that the theorem applies also in the case of $p = \infty$, since $p^* > p$ always and $p^* \to \infty$ as $p \to n$.

The detailed proof can be found in Evans' book [3, pp. 286-289].

Ladyzhenskaya offers another formulation of this same theorem, where the space $W^{1,2}(U) = H^1(U)$ proves out to be *precompact* in $L^2(U)$ when U is a bounded domain [7, pp. 25-28].

3.4 Time-dependent Sobolev spaces

To prove the existence and uniqueness of weak solutions of wave equations in the next chapter it is necessary to add the time dimension to the definition of Sobolev spaces. Here this is done by defining the *time-dependent Sobolev spaces* by considering functions of a Sobolev space with a target Hilbert space at any point on time from 0 to T as described in Evans' book [3, pp. 301-302].

Definition 3.19. Let Y be a Hilbert space. The *time-dependant C^n-spaces* are defined as

$$\begin{aligned} C([0,T],Y) &:= \{f : [0,T] \to Y : f \text{ is continuous}\} \\ C^n([0,T],Y) &:= \{f : [0,T] \to Y : f \text{ is } n\text{-times continuously Fréchet differentiable}\}. \end{aligned}$$

From continuity and differentiability to measurable functions and weak differentiability; the next defintions follow the same pattern as $C^n(U)$-spaces.

Definition 3.20. Let Y be an Hilbert space. The *time-dependent L^p-space*

$$L^p([0,T],Y)$$

consists of all strongly measurable functions $f : [0,T] \to Y$ with

$$\|f\|_{L^p([0,T],Y)} := \begin{cases} \left(\int_0^T \|f(t,\cdot)\|_Y^p \, dt\right)^{1/p} < \infty & \text{for } 1 \leq p < \infty \\ \text{ess } \sup_{0 \leq t \leq T} \|f(t,\cdot)\|_Y < \infty & \text{for } p = \infty. \end{cases}$$

Definition 3.21. Let Y be a Hilbert space and $f \in L^1([0,T],Y)$. A function $g \in L^1([0,T],Y)$ is the *weak derivative* of f, if

$$\int_0^T \psi'(t)f(t)\,dt = -\int_0^T \psi(t)g(t)\,dt$$

for all scalar functions $\psi \in C_c^\infty(0,T)$. The derivative of f is notated by $f' = g$ or $f_t = g$.

Using the previous definitions it is again possible to extend the concept of ordinary derivatives to reach the actual time-dependent Sobolev spaces and their norms.

Definition 3.22. Let Y be a Hilbert space. The *time-dependent Sobolev space*

$$W^{1,p}([0,T],Y)$$

consists of all functions $f \in L^p([0,T],Y)$ for which there exists $f' \in L^p([0,T],Y)$ in the weak sense. The norm is defined by

$$\|f\|_{W^{1,p}([0,T],Y)} := \begin{cases} \left(\int_0^T \|f(t)\|_Y^p + \|f'(t)\|_Y^p\,dt\right)^{1/p} & \text{for } 1 \le p < \infty \\ \text{ess }\sup_{0\le t\le T}(\|f(t)\|_Y + \|f'(t)\|_Y) & \text{for } p = \infty. \end{cases}$$

For functions belonging to the space $W^{1,p}([0,T],Y)$ the connection with the space $L^p([0,T],Y)$ is the same as before with $W^{k,p}(U)$ and $L^p(U)$. Since now the index $k = 1$, the norms only include a function and its first derivative which makes it possible to write open the sum over multi-indices. The reason for this restriction in the case $k = 1$ is due to the fact that it is enough to consider only spaces of form $W^{1,p}(U)$ when searching for a solution for a wave equation as will be seen in the next chapter.

As for the earlier results concerning ordinary Sobolev spaces, the time-dependant Sobolev spaces are defined as function spaces. The space Y is an ordinary Sobolev space and hence, the inequalities and theorems apply for this space.

4 Wave Equations in $\mathbb{R}^n$

This chapter concentrates on shedding light on some of the basic properties of wave equations: it contains the proofs for the existence and uniqueness of solutions to these equations in the case where the coefficients determining the equation can depend both on time and space. The chapter mainly follows Evans' [3, pp. 398-417] and Ladyzhenskaya's [7, pp. 147-168] books as sources.

According to Evans [3, pp. 398-399],

Definition 4.1. The *wave operator* is defined as

$$\Box = \frac{\partial^2}{\partial t^2} + L,$$

where the operator L is a second-order partial differential operator for each time coordinate $t \geq 0$. The operator L is either of divergence form

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij}(x,t)u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x,t)u_{x_i} + c(x,t)u$$

or nondivergence form

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x,t)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x,t)u_{x_i} + c(x,t)u,$$

where the coefficients a^{ij}, b^i and c are given and the indeces $i, j = 1, \ldots, n$.

Actually, this is the general notation of second order hyperbolic partial differential operator. The classification of wave equations as a subcategory of hyperbolic equations is relatively difficult to do exactly, so this paper presents the proofs to general hyperbolic equations as they represent wave transmission in heterogenous and non-isotropic media such as sound waves in a room with furniture. A case like this leads to a general wave equation as solid, complex shaped objects affect the sound waves in a way that is hard to predict and intuitionally difficult to model.

The hyperbolic operator needs to satisfy the next characterising definition involving the coefficients of the operator L.

Definition 4.2. Partial differential operator $\frac{\partial^2}{\partial t^2} + L$ is *(uniformly) hyperbolic* if there exists a constant C such that

$$\sum_{i,j=1}^{n} a^{ij}(x,t)\xi_i\xi_j \geq C\,|\xi|^2$$

for all $(x,t) \in U_T = U\times]0,T]$ and $\xi \in \mathbb{R}^n$, where $U \subset \mathbb{R}^n$ is open and bounded, and $T > 0$.

Remark 1. The coefficients of the operator L are supposed to satisfy the following criteria

$$a^{ij}(x,t), b^i(x,t), c(x,t) \in C^1(\overline{U}_T)$$
$$\text{and } a^{ij} = a^{ji} \text{ for all } i,j = 1,\dots,n.$$

The actual equation studied from now on is the initial/boundary-value problem

$$\begin{aligned} u_{tt} + Lu &= f \text{ in } U_T \\ u &= 0 \text{ on } \partial U \times [0,T] \\ u = g, u_t &= h \text{ on } U \times \{t=0\}, \end{aligned} \tag{$\star$}$$

where functions $f : U_T \to \mathbb{R}$, $g,h : U \to \mathbb{R}$ are given and $u : \bar{U}_T \to \mathbb{R}$, $u = u(x,t)$ is the unknown function.

Remark 2. The function f, g and h satisfy the following criteria

$$f \in L^2(U_T),\ g \in H_0^1(U) \text{ and } h \in L^2(U).$$

As can be seen from the definition of L, this is a very general equation: it is inhomogenous and the coefficients depend both on time and n-dimensional space. In contrast to the basic homogenous wave equation in $\mathbb{R}$, which is relatively easy to solve with basic analysis, the general case leads to more complex structures. Hence, this chapter culminates in proofs of the existence and uniqueness of weak solutions which function as mappings from the given time slot to a specific Sobolev space.

4.1 On defining weak solutions

The next step for defining the initial/boundary value problem for wave equations is to solve such systems. Motivation for a *weak solution* stems from the need of finding an appropriate function of some known level of smoothness to solve a wave equation. It turns out that a time dependent L^2-space defined from an interval $[0,T]$ to Sobolev space $H_0^1(U)$ satisfy this criterion. This subchapter contains an outline of the thought process leading to this conclusion before the actual proofs following Evans' ideas [3, pp. 400-402].

Definition 4.3. Let U be an open subset of $\mathbb{R}^n$ and $x \in U$. The *associated mappings* of $(\star)$ are defined as

$$\mathbf{u} : [0,T] \to H_0^1(U),$$
$$[\mathbf{u}(t)]\,(x) := u(x,t)$$

and

$$\mathbf{f} : [0,T] \to L^2(U),$$
$$[\mathbf{f}(t)]\,(x) := f(x,t),$$

when $0 \le t \le T$.

What is the point of changing the original functions in the initial/boundary value problem $(\star)$ to the associated mappings? Shortly, this transforms the system to a form that makes the later proofs of uniqueness and existence easier to follow. The change is on the level of function spaces as this way the functions of interest are in $L^2([0,T], H_0^1(U))$ or $L^2([0,T], L^2(U))$.

The following steps show how this change is conducted.

First, fix any function $v \in H_0^1(U)$. Next, multiply the equation $u_{tt} + Lu = f$ by v to get

$$v(x)u_{tt}(x,t) + v(x)Lu(x,t) = v(x)f(x,t),$$

where $(x,t) \in U_T$. Then integrate the result by parts over U, which results in

$$\int_U \big(v(x)u_{tt}(x,t) + v(x)Lu(x,t)\big)\,dx = \int_U v(x)f(x,t)\,dx.$$

As for the term $Lu(x,t)$, it is possible to use the non-divergence form from the definition 4.1 to write

$$\begin{aligned}
&\int_U v(x)u_{tt}(x,t)\,dx \\
&+ \int_U \left(-\sum_{i,j=1}^{n} a^{ij}(x,t)u_{x_i}(x,t)v_{x_j}(x) + v(x)\left(\sum_{i=t}^{n} b^i(x,t)u_{x_i}(x,t) + c(x,t)u(x,t)\right)\right)\,dx \\
&= \int_U f(x,t)v(x)\,dx.
\end{aligned}$$

Here the function u and its derivatives can be interchanged with the aforementioned associate mappings,

$$\begin{aligned}
&\int_U \mathbf{u}_{tt}(t)v\,dx + \int_U \left(\sum_{i,j=1}^{n} a^{ij}(\cdot,t)\mathbf{u}_{x_i}(t)v_{x_j} + \sum_{i=1}^{n} b^i(\cdot,t)\mathbf{u}_{x_i}(t)v + c(\cdot,t)\mathbf{u}(t)v\right)dx \\
&= \int_U \mathbf{f}(t)v\,dx.
\end{aligned}$$

This equation includes two $L^2(U)$-norms and an integral that for now is shortened as B to get

$$(\mathbf{u}'', v)_{L^2(U)} + B\,[\mathbf{u}, v; t] = (\mathbf{f}, v)_{L^2(U)}.$$

On the other hand, the problem $(\star)$ implies also

$$u_{tt}(x,t) = g^0(x,t) + \sum_{j=1}^{n} g^j_{x_j}(x,t),$$

where the function g^0 is defined as

$$g^0(x,t) := f(x,t) - \sum_{i=1}^{n} b^i(x,t)u_{x_i}(x,t) - cu(x,t),$$

and for $j = 1, \dots, n$, a function g^j is based on the first term of the operator L and defined as

$$g^j(x,t) := \sum_{i=1}^{n} a^{ij}(x,t)u_{x_i}(x,t).$$

It seems now that it would be best to look for a weak solution $\mathbf{u}$ for which $\mathbf{u}'' \in H^{-1}(U)$ when $0 \le t \le T$. With this the term $(\mathbf{u}'', v)$ can be interpreted as $\langle \mathbf{u}'', v \rangle$, that is, as a *pairing* between $H_0^1(U)$ and $H^{-1}(U)$. This leads to the following definition by Evans [3, p. 400] that is the culmination of this process of determining how to actually define a weak solution for a wave equation in an accurate and useful way.

Definition 4.4. A function $\mathbf{u} \in L^2([0,T], H_0^1(U))$ for which $\mathbf{u}' \in L^2([0,T], L^2(U))$ and $\mathbf{u}'' \in L^2([0,T], H^{-1}(U))$ is a *weak solution* of the hyperbolic initial/boundary-value problem $(\star)$ if

$$\langle \mathbf{u}'', v \rangle + B\,[\mathbf{u}, v; t] = (\mathbf{f}, v)_{L^2(U)}$$

for each $v \in H_0^1(U)$ and almost every $0 \le t \le T$, and

$$\mathbf{u}(0) = g \text{ and } \mathbf{u}'(0) = h.$$

4.2 Existence of weak solutions

The earlier subsection shows the very idea behind solving a wave equation with the help of Sobolev spaces and L^2-norm. Still, this is not enough to actually prove the existence of solutions yet, but it is a foundation for the next results that will culminate in a proof of existence of weak solutions for equations of type ($\star$).

The approach here is based on *Galerkin's method* which is a way to construct a weak solution to the hyperbolic initial/boundary-value problem by first building a finite-dimensional approximation of the problem, then searching for its solution and lastly returning to solving the original case by taking limits. The approximations in this process are also known as the *Galerkin's approximations* according to Evans [3, p. 401].

As already said, proving the existence of weak solutions by Galerkin's method involves approximated solution functions and limits. First it is necessary to find an appropriate set of functions that simultaneously satisfy several criteria in two different spaces. Earlier it was shown that the Sobolev space $H_0^1(U)$ and $L^2(U)$-space are closely related and due to the *Rellich-Kondrachov Compactness Theorem* 3.18 it is possible to find a set of functions that works as a basis in both spaces.

How can the basis be of use in finding probable solutions for wave equations? Firstly, assume that functions w_k, where $k \in \mathbb{N}$, are smooth, and most importantly they satisfy these two criteria:

$$\{w_k\}_{k=1}^{\infty} \text{ is an orthogonal basis of } H_0^1(U)$$

and

$$\{w_k\}_{k=1}^{\infty} \text{ is an orthonormal basis of } L^2(U).$$

One example of such basis is an eigenbasis of a compact selfadjoint operator, also the familiar operator $L = -\Delta$ fits the criteria [3, p. 375].

Next, fix a positive integer m and consider a function $\mathbf{u}_m : [0,T] \to H_0^1(U)$ of a form

$$\mathbf{u}_m(t) := \sum_{k=1}^{m} d_m^k(t) w_k(x).$$

Select coefficients $d_m^k(t)$, when $0 \le t \le T$ and index $k = 1, \dots, m$, so that

$$\begin{aligned} d_m^k(0) &= (g, w_k)_{L^2(U)}, \text{ when } k = 1, \dots, m, \\ {d_m^k}'(0) &= (h, w_k)_{L^2(U)}, \text{ when } k = 1, \dots, m \end{aligned}$$

and

$$(\mathbf{u}_m'', w_k)_{L^2(U)} + B\left[\mathbf{u}_m, w_k; t\right] = (\mathbf{f}, w_k)_{L^2(U)} \qquad (\star\star)$$

with $0 \le t \le T$ and $k = 1, \dots, m$. In other words, for every m try to find a function $\mathbf{u}_m$, which is the projection of a function u onto the finite-dimensional subspace spanned by $\{w_k\}_{k=1}^{m}$, in order to satisfy the conditions set for the equation ($\star\star$).

This next theorem, and especially its proof, will show that this indeed is possible in practice as Evans shows in[3, pp. 401-402].

Theorem 4.5. *For each $m \in \mathbb{N}$ there exists a unique function $\mathbf{u}_m$ of the form*

$$\mathbf{u}_m(t) := \sum_{k=1}^{m} d_m^k(t) w_k(x)$$

satisfying both

$$\begin{aligned} d_m^k(0) &= (g, w_k)_{L^2(U)}, \quad \text{with } k = 1, \dots, m, \\ {d_m^k}'(0) &= (h, w_k)_{L^2(U)}, \quad \text{with } k = 1, \dots, m \end{aligned}$$

and

$$(\mathbf{u}_m'', w_k)_{L^2(U)} + B\left[\mathbf{u}_m, w_k; t\right] = (\mathbf{f}, w_k)_{L^2(U)},$$

where $0 \leq t \leq T$ and $k = 1, \dots, m$.

Proof. Assume that $\mathbf{u}_m$ is of the form $\sum_{k=1}^{m} d_k^m(t) w_k(x)$. Since $\{w_k\}_{k=1}^{\infty}$ is an orthogonal basis in $H_0^1(U)$, the product of $\mathbf{u}_m''(t)$ and w_k is

$$\begin{aligned} (\mathbf{u}_m''(t), w_k)_{L^2(U)} &= \left(\sum_{k=1}^{m} {d_m^k}''(t) w_k, w_k \right)_{L^2(U)} \\ &= {d_m^k}''(t). \end{aligned}$$

Next, simplify the bilinear form by defining coefficients $B\left[w_l, w_k; t\right]$ corresponding to specific coefficient functions defined earlier:

$$\begin{aligned} B\left[\mathbf{u}_m, w_k; t\right] &= \int_U \left(\sum_{i,j=1}^{n} a^{ij}(\cdot, t) \mathbf{u}_{x_i}(t) w_{k_{x_j}}(\cdot) + \sum_{i=1}^{n} b^i(\cdot, t) \mathbf{u}_{x_i}(t) w_k(\cdot) + c(\cdot, t) \mathbf{u}(t) w_k(\cdot) \right) dx \\ &:= \sum_{l=1}^{m} B\left[w_l, w_k; t\right] d_m^l(t) \end{aligned}$$

for all $k = 1, \dots, m$. Notice that $B\left[w_l, w_k; t\right]$ depends solely on t and the index n is the dimension of the subset $U \subset \mathbb{R}^n$.

To simplify the notations, define

$$f^k(t) := (\mathbf{f}(t), w_k)_{L^2(U)} \text{ with } k = 1, \dots, m.$$

Now the inner product equation $(\star\star)$ can be written as

$${d_m^k}''(t) + \sum_{l=1}^{m} B\left[w_l, w_k; t\right] d_m^l(t) = f^k(t), \text{ when } k = 1, \dots, m.$$

This is a linear system of ordinary differential equations. The method of turning it into a system of equations of first degree is explained in Hirsch and Smale's *"Differential equations, dynamical systems, and linear algebra"* [4, pp. 102-103]: First, define

$$\begin{cases} \mathbf{y}_1 := \mathbf{d} \\ \mathbf{y}_2 := \mathbf{y}_1' = \mathbf{d}', \end{cases}$$

where $\mathbf{d} = (d_m^1, d_m^2, \ldots, d_m^m)^T$. To simplify the notation, write

$$A := \begin{pmatrix} B\left[w_1, w_1; t\right] & B\left[w_1, w_2; t\right] & \ldots & B\left[w_1, w_m; t\right] \\ B\left[w_2, w_1; t\right] & B\left[w_2, w_2; t\right] & \ldots & B\left[w_2, w_m; t\right] \\ \vdots & \vdots & \ddots & \vdots \\ B\left[w_m, w_1; t\right] & B\left[w_m, w_2; t\right] & \ldots & B\left[w_m, w_m; t\right] \end{pmatrix}$$

and define $\mathbf{f} := (f^1(t), f^2(t), \ldots, f^m(t))^T$. Then, with the vector

$$\mathbf{y} = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix}$$

it is possible to rewrite the linear system as

$$\mathbf{y}' = \begin{pmatrix} \mathbf{y}_2 \\ \mathbf{f} - A\mathbf{y}_1 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{I} \\ -A & 0 \end{pmatrix} \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbf{f} \end{pmatrix}.$$

By taking these initial/boundary conditions into account, the theory of inhomogeneous ordinary differential equations in Logemann and Ryan's book [11, pp. 40-41] states that there exists a unique solution, namely a C^2-function $\mathbf{d}_m(t) = (d_m^1(t), ..., d_m^m(t))$ which satisfies the conditions

$$\begin{aligned} d_m^k(0) &= (g, w_k)_{L^2(U)}, \ \text{ when } k = 1, \ldots, m, \\ {d_m^k}'(0) &= (h, w_k)_{L^2(U)}, \ \text{ when } k = 1, \ldots, m \end{aligned}$$

and

$${d_m^k}''(t) + \sum_{l=1}^{m} B\left[w_l, w_k; t\right] d_m^l(t) = f^k(t), \ \text{ when } k = 1, \ldots, m,$$

for $0 \le t \le T$. Therefore the function $\mathbf{u}_m$ of the assumed form solves the equation $(\star\star)$ for $0 \le t \le T$. $\square$

The next phase is to show that a subsequence of the approximate solutions $\mathbf{u}_m$ converges to a weak solution of the hyperbolic initial/boundary-value problem. As usual, this convergence is shown by approximating the norms and finding an upper bound as shown in the following theory and its proof, which both are based on Evans' [3, pp. 402-404].

Theorem 4.6 (Energy estimates)**.** *There exists a constant C depending only on U, T and the coefficients of the operator L such that*

$$\max_{0\le t\le T}\left(\|\mathbf{u}_m(t)\|_{H_0^1(U)}+\|\mathbf{u}_m'(t)\|_{L^2(U)}\right)+\|\mathbf{u}_m''\|_{L^2([0,T],H^{-1}(U))}$$
$$\le\left(\|\mathbf{f}\|_{L^2([0,T],L^2(u))}+\|g\|_{H_0^1(U)}+\|h\|_{L^2(U)}\right)$$

for $m\in\mathbb{N}$.

Proof. This proof uses several already defined notations and presented equations from this chapter; the starting point is the inner product equation of the theorem 4.5.

Let $0\le t\le T$ and $k=1,\dots,m$. First, multiply the inner product equation

$$(\mathbf{u}_m'',w_k)_{L^2(U)}+B\left[\mathbf{u}_m,w_k;t\right]=(\mathbf{f},w_k)_{L^2(U)}$$

by ${d_m^k}'(t)$ and then sum over $k=1,\dots,m$. By theorem 4.5 the function $\mathbf{u}_m$ is of the form $\sum_{k=1}^m d_m^k(t)w_k(x)$, and hence the equation becomes

$$\left(\mathbf{u}_m'',\sum_{k=1}^m {d_m^k}'(t)w_k\right)_{L^2(U)}+B\left[\mathbf{u}_m,\sum_{k=1}^m {d_m^k}'(t)w_k;t\right]=\left(\mathbf{f},\sum_{k=1}^m {d_m^k}'(t)w_k\right)_{L^2(U)}.$$

This can be simplified by taking notice of the time derivatives of $\mathbf{u}_m$: by the definition of $\mathbf{u}_m$ the derivative

$$\mathbf{u}_m'(t)=\sum_{k=1}^m {d_m^k}'(t)w_k.$$

Combining the new formulas, the previous equation becomes

$$(\mathbf{u}_m'',\mathbf{u}_m')_{L^2(U)}+B\left[\mathbf{u}_m,\mathbf{u}_m';t\right]=(\mathbf{f},\mathbf{u}_m')_{L^2(U)}$$

for almost every $0\le t\le T$.

There are three terms to consider separately. From the left side of the equation take first the inner product

$$\begin{aligned}(\mathbf{u}_m'',\mathbf{u}_m')_{L^2(U)}&=\left(\frac{d}{dt}\mathbf{u}_m',\mathbf{u}_m'\right)_{L^2(U)}\\&=\int_U\left(\frac{d}{dt}\mathbf{u}_m'\right)\mathbf{u}_m'dx\\&=\frac{1}{2}\left(\frac{d}{dt}\int_U|\mathbf{u}_m'|^2dx\right)\\&=\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{u}_m'\|_{L^2(U)}^2\right),\end{aligned}$$

following by the bilinear form

$$\begin{aligned}
B\left[\mathbf{u}_m, \mathbf{u}_m'; t\right] &= \int_U \sum_{i,j=1}^{n} a^{ij} \mathbf{u}_{m,x_i} \mathbf{u}'_{m,x_j}\, dx \\
&+ \int_U \left(\sum_{i=1}^{n} b^i \mathbf{u}_{m,x_i} \mathbf{u}_m' + c \mathbf{u}_m \mathbf{u}_m' \right) dx \\
&=: B_1 + B_2.
\end{aligned}$$

The coefficients a^{ij} are symmetrical in the sense of translation of both i and j, in short $a^{ij} = a^{ji}$ when $i, j = 1, \ldots, n$. This fact can be used to define a symmetric bilinear form for $u, v \in H_0^1(U)$ as

$$A\left[u, v; t\right] := \int_U \sum_{i,j=1}^{n} a^{ij} u_{x_i} v_{x_j}\, dx.$$

Differentiate this to get a new representation for the term B_1:

$$\begin{aligned}
\frac{d}{dt} A\left[u, u; t\right] &= \frac{1}{2} \int_U \sum_{i,j=1}^{n} a_t^{ij} u_{x_i} u_{x_j}\, dx + \frac{1}{2} \int_U \sum_{i,j=1}^{n} a^{ij} \frac{\partial}{\partial t} (u_{x_i} u_{x_j})\, dx \\
&= \frac{1}{2} \int_U \sum_{i,j=1}^{n} a_t^{ij} u_{x_i} u_{x_j}\, dx + \frac{1}{2} \int_U \sum_{i,j=1}^{n} a^{ij} u'_{x_i} u_{x_j}\, dx + \frac{1}{2} \int_U \sum_{i,j=1}^{n} a^{ij} u_{x_i} u'_{x_j}\, dx \\
&= \frac{1}{2} \int_U \sum_{i,j=1}^{n} a_t^{ij} u_{x_i} u_{x_j}\, dx + \int_U \sum_{i,j=1}^{n} a^{ij} u'_{x_i} u_{x_j}\, dx
\end{aligned}$$

by the symmetry of the matrix A. Therefore,

$$B_1 = \frac{d}{dt} \left(\frac{1}{2} A\left[\mathbf{u}_m, \mathbf{u}_m; t\right] \right) - \frac{1}{2} \int_U \sum_{i,j=1}^{n} a_t^{ij} \mathbf{u}_{m,x_i} \mathbf{u}_{m,x_j}\, dx.$$

The absolute value of the latter term can be approximated with Cauchy's inequality 2.29

$$\begin{aligned}
\left| \int_U \sum_{i,j=1}^{n} a_t^{ij} \mathbf{u}_{m,x_i} \mathbf{u}_{m,x_j}\, dx \right| &\leq C \int_U \sum_{i,j=1}^{n} |\mathbf{u}_{m,x_i}| |\mathbf{u}_{m,x_j}|\, dx \\
&\leq C \sum_{i,j=1}^{n} \int_U |\mathbf{u}_{m,x_i}|^2\, dx \\
&\leq C \|\mathbf{u}_m\|^2_{H_0^1(U)}.
\end{aligned}$$

Combined with the previous equation for B_1 this implies

$$B_1 \geq \frac{d}{dt}\left(\frac{1}{2}A\left[\mathbf{u}_m, \mathbf{u}_m; t\right]\right) - C\|\mathbf{u}_m\|^2_{H^1_0(U)},$$

where C is a constant. As for the other part of the form $B\left[\mathbf{u}_m, \mathbf{u}'_m; t\right]$, it is possible to approximate it with Cauchy's inequality as before to get

$$\begin{aligned}
|B_2| &= \left|\int_U \left(\sum_{i=1}^n b^i \mathbf{u}_{m,x_i}\mathbf{u}'_m + c\mathbf{u}_m\mathbf{u}'_m\right) dx\right| \\
&\leq \int_U \sum_{i=1}^n \left|b^i \mathbf{u}_{m,x_i}\mathbf{u}'_m\right| \, dx + \int_U \sum_{i=1}^n \left|c\mathbf{u}_m\mathbf{u}'_m\right| \, dx \\
&\leq C\left(\|\mathbf{u}_m\|^2_{H^1_0(U)} + \|\mathbf{u}'_m\|^2_{L^2(U)}\right).
\end{aligned}$$

Now combine the results to discover a new inequality:

$$\begin{aligned}
&\frac{d}{dt}\left(\|\mathbf{u}'_m\|^2_{L^2(U)} + A\left[\mathbf{u}_m, \mathbf{u}_m; t\right]\right) \\
&\quad\leq C\left((\mathbf{u}''_m, \mathbf{u}'_m) + B_1 + C\|\mathbf{u}_m\|^2_{H^1_0(U)}\right) \\
&\quad\leq C\left((\mathbf{u}''_m, \mathbf{u}'_m) + B_1 + B_2 + C\|\mathbf{u}_m\|^2_{H^1_0(U)} - B_2\right) \\
&\quad\leq C\left((\mathbf{f}, \mathbf{u}'_m) + C'\|\mathbf{u}_m\|^2_{H^1_0(U)} + |B_2|\right) \\
&\quad\leq C\left(\|\mathbf{u}_m\|^2_{H^1_0(U)} + \|\mathbf{u}'_m\|^2_{L^2(U)} + \|\mathbf{f}\|_{L^2(U)}\|\mathbf{u}'_m\|_{L^2(U)}\right).
\end{aligned}$$

Continue with Cauchy's inequality 2.29 to find an upper bound for the product term

$$\begin{aligned}
&\leq C\left(\|\mathbf{u}_m\|^2_{H^1_0(U)} + \|\mathbf{u}'_m\|^2_{L^2(U)} + \frac{1}{2}\left[\|\mathbf{f}\|^2_{L^2(U)} + \|\mathbf{u}'_m\|^2_{L^2(U)}\right]\right) \\
&\leq C\left(\|\mathbf{u}'_m\|^2_{L^2(U)} + \|\mathbf{u}_m\|^2_{H^1_0(U)} + \|\mathbf{f}\|^2_{L^2(U)}\right).
\end{aligned}$$

Lastly, this can be approximated with

$$\leq C\left(\|\mathbf{u}'_m\|^2_{L^2(U)} + A\left[\mathbf{u}_m, \mathbf{u}_m; t\right] + \|\mathbf{f}\|^2_{L^2(U)}\right),$$

since by the Poincaré-Friedrichs inequality 3.14

$$\theta \int_U |Du|^2 \, dx \leq A\left[u, u; t\right]$$

when $u \in H_0^1(U)$ because the operator L is uniformly hyperbolic.

To make the rest of the proof more readable, define two new functions

$$\eta(t) := \|\mathbf{u}_m'(t)\|_{L^2(U)}^2 + A\left[\mathbf{u}_m(t), \mathbf{u}_m(t); t\right]$$

and

$$\xi(t) := \|\mathbf{f}(t)\|_{L^2(U)}^2.$$

With these the inequality involving the time derivative simplifies into

$$\eta'(t) \leq C_1 \eta(t) + C_2 \xi(t)$$

for $0 \leq t \leq T$ and some constants C_1, C_2. The functions η, η' are nonnegative because the term $A\left[\mathbf{u}_m(t), \mathbf{u}_m(t); t\right]$ is nonnegative due to the definition 4.2 of the hyperbolic operator. With ξ also being nonnegative, η absolutely continuous and η', ξ summable, the Gronwall's inequality 2.31 leads to an estimate

$$\eta(t) \leq e^{C_1 t}\left(\eta(0) + C_2 \int_0^t \xi(s) ds\right), \quad \text{when } 0 \leq t \leq T.$$

On the other hand,

$$\begin{aligned}
\eta(0) &= \|\mathbf{u}_m'(0)\|_{L^2(U)}^2 + A\left[\mathbf{u}_m(0), \mathbf{u}_m(0); 0\right] \\
&\leq C\left(\|g\|_{L^2(U)}^2 + \|h\|_{H_0^1}^2\right),
\end{aligned}$$

since due to the definition in theorem 4.5

$$\|\mathbf{u}_m(0)\|_{H_0^1(U)} \leq \|g\|_{H_0^1(U)}.$$

Returning to the original notation the combined estimate is

$$\begin{aligned}
&\|\mathbf{u}_m'(t)\|_{L^2(U)}^2 + A\left[\mathbf{u}_m(t), \mathbf{u}_m(t); t\right] \\
&\qquad \leq C\left(\|g\|_{H_0^1(U)}^2 + \|h\|_{L^2(U)} + \|\mathbf{f}\|_{L^2([0,T], L^2(U))}^2\right).
\end{aligned}$$

Because the time coordinate has been arbitrary during the proof up until now, the bound is in fact

$$\begin{aligned}
&\max_{0 \leq t \leq T}\left(\|\mathbf{u}_m'(t)\|_{L^2(U)}^2 + A\left[\mathbf{u}_m(t), \mathbf{u}_m(t); t\right]\right) \\
&\qquad \leq C\left(\|g\|_{H_0^1(U)}^2 + \|h\|_{L^2(U)} + \|\mathbf{f}\|_{L^2([0,T], L^2(U))}^2\right).
\end{aligned}$$

Then to the final stage in the proof: Fix any $v \in H_0^1(U)$ with $\|v\|_{H_0^1(U)} \leq 1$, and divide the function into two parts $v = v^1 + v^2$, where $v^1 \in \operatorname{span}\{w_k\}_{k=1}^m$

and $(v^2, w_k) = 0$, with $k = 1, \dots, m$. Clearly also $\|v^1\|_{H_0^1(U)} \leq 1$. Notice that using their definitions the pairing of $\mathbf{u}_m''$ and v equals to

$$\langle \mathbf{u}_m'', v \rangle = (\mathbf{u}_m'', v)_{L^2(U)} = (\mathbf{u}_m'', v^1)_{L^2(U)} = (\mathbf{f}, v^1)_{L^2(U)} - B\left[\mathbf{u}_m, v^1; t\right].$$

This means that in the sense of norms the upper bound is

$$|\langle \mathbf{u}_m'', v \rangle| \leq C \left(\|\mathbf{f}\|_{L^2(U)} + \|\mathbf{u}_m\|_{H_0^1(U)} \right).$$

Finally, the approximation, better known as the *energy estimate*, is

$$\begin{aligned} \int_0^T \|\mathbf{u}_m''\|_{H^{-1}(U)}^2 \, dt &\leq C \int_0^T \left(\|\mathbf{f}\|_{L^2(U)}^2 + \|\mathbf{u}_m\|_{H_0^1(U)}^2 \right) dt \\ &\leq C \left(\|g\|_{H_0^1(U)^2}^2 + \|h\|_{L^2(U)^2} + \|\mathbf{f}\|_{L^2([0,T], L^2(U))}^2 \right). \qquad \square \end{aligned}$$

The energy estimates are important because they allow taking the limits in the Galerkin approximations. Without these limits it would be impossible to speak of weak solutions in general cases; the main point is to try to find a sequence of functions with 'good behaviour' in $L^2(U)$ to lead to the final solution in the form of a limit function.

The next theorem is the first of the two main results for wave equations, since as partial differential equations the aim is to prove both the existence of solutions and later the uniqueness of them. Here, the proof follows Evans' text [3].

Theorem 4.7 (Existence of weak solutions)**.** *There exists a weak solution for the initial/boundary-value problem*

$$\begin{aligned} u_{tt} + Lu = f \;\; &\textit{in } U_T \\ u = 0 \;\; &\textit{on } \partial U \times [0, T] \\ u = g, u_t = h \;\; &\textit{on } U \times \{t = 0\}. \end{aligned}$$

Proof. This proof consists of three parts: The first uses the energy estimates to consider a sequence that forms a base for the rest of the proof. The second part is to check if its limit function satisfies the criteria of the inner product equation $(\star\star)$. The last work to do is to ensures that this function has the right initial conditions.

By the energy estimates theorem 4.6 the sequence $\{\mathbf{u}_m\}_{m=1}^\infty$ is bounded in $L^2([0,T], H_0^1(U))$, the sequence $\{\mathbf{u}_m'\}_{m=1}^\infty$ is bounded in $L^2([0,T], L^2(U))$ and $\{\mathbf{u}_m''\}_{m=1}^\infty$ is bounded in $L^2([0,T], H^{-1}(U))$. Thus there exists a subsequence $\{\mathbf{u}_{m_l}\}_{l=1}^\infty \subset \{\mathbf{u}_m\}_{m=1}^\infty$ and a function $\mathbf{u} \in L^2([0,T], H_0^1(U))$ with derivatives

$\mathbf{u}' \in L^2([0,T], L^2(U))$ and $\mathbf{u}'' \in L^2([0,T], H^{-1}(U))$. Notably these sequences have the following properties

$$\begin{cases} \mathbf{u}_{m_l} \rightharpoonup \mathbf{u} & \text{weakly in } L^2([0,T], H_0^1(U)) \\ \mathbf{u}'_{m_l} \rightharpoonup \mathbf{u}' & \text{weakly in } L^2([0,T], L^2(U)) \\ \mathbf{u}''_{m_l} \rightharpoonup \mathbf{u}'' & \text{weakly in } L^2([0,T], H^{-1}(U)). \end{cases}$$

Next consider the inner product equation $(\star\star)$. Fix an integer n and choose a function $\mathbf{v} \in C^1([0,T], H_0^1(U))$ that can be written in the form

$$\mathbf{v}(t) = \sum_{k=1}^{n} d^k(t) w_k(x)$$

where d^k are smooth functions for all $k = 1, \ldots, n$ and functions w_k form an orthonormal basis in $L^2(U)$ and an orthogonal basis in $H_0^1(U)$. Then select an integer $m \geq n$ and multiply the inner product equation $(\star\star)$ by $d^k(t)$ to get

$$\left(\mathbf{u}''_m, d^k(t) w_k\right)_{L^2(U)} + B\left[\mathbf{u}_m, d^k(t) w_k; t\right] = \left(\mathbf{f}, d^k(t) w_k\right)_{L^2(U)}$$

for $0 \leq t \leq T$ and $k = 1, \ldots, m$. Since the L^2-inner product and the pairing between $H_0^1(U)$ and $H^{-1}(U)$ are equivalent in this case, the equation can be written as

$$\left\langle \mathbf{u}''_m, \sum_{k=1}^{n} d^k(t) w_k \right\rangle + B\left[\mathbf{u}_m, \sum_{k=1}^{n} d^k(t) w_k; t\right] = \left(\mathbf{f}, \sum_{k=1}^{n} d^k(t) w_k\right)_{L^2(U)}.$$

Next, replace the sums with the function $\mathbf{v}$ and integrate with respect to t over the interval $(0,T)$ to get

$$\int_0^T \left(\langle \mathbf{u}''_m, \mathbf{v}\rangle + B\left[\mathbf{u}_m, \mathbf{v}; t\right]\right) dt = \int_0^T (\mathbf{f}, \mathbf{v})_{L^2(U)}\, dt.$$

Now set $m = m_l$ to pass to the limit $m \to \infty$ in the weak sense in the three different L^2 spaces mentioned earlier in the properties of the sequences $\{\mathbf{u}_{m_l}\}_{l=1}^{\infty}$ and $\{\mathbf{u}''_{m_l}\}_{l=1}^{\infty}$. As a consequence, the equation includes now only the limit function $\mathbf{u}$ as seen in the following line

$$\int_0^T \left(\langle \mathbf{u}'', \mathbf{v}\rangle + B\left[\mathbf{u}, \mathbf{v}; t\right]\right) dt = \int_0^T (\mathbf{f}, \mathbf{v})_{L^2(U)}\, dt.$$

Since functions of type $\mathbf{v}(t) = \sum_{k=1}^{n} d^k(t) w_k(x)$ are dense in $L^2([0,T], H_0^1(U))$ due to the definitions of $\{d^k\}_{k=1}^{\infty}$ and $\{w_k\}_{k=1}^{\infty}$, this equation holds for all such functions. This leads to the conclusion

$$\langle \mathbf{u}'', v\rangle + B\left[\mathbf{u}, v; t\right] = (\mathbf{f}, v)_{L^2(U)}$$

for all $v \in H_0^1(U)$ and almost every $0 \le t \le T$. From the weak convergence of the sequences it also follows that $\mathbf{u} \in C([0,T], L^2(U))$ and $\mathbf{u}' \in C([0,T], H^{-1}(U))$.

The only things still left to be verified are the initial values

$$\mathbf{u}(0) = g$$

and

$$\mathbf{u}'(0) = h.$$

This requires again strating from the integral form of the equation $(\star\star)$. Therefore, fix any function $\mathbf{v} \in C^2([0,T], H_0^1(U))$ with the initial values $\mathbf{v}(T) = \mathbf{v}'(T) = 0$. Next integrate the equation $(\star\star)$ from 0 to T with respect to t to get

$$\int_0^T \big(\langle \mathbf{u}'', \mathbf{v}\rangle + B\left[\mathbf{u}, \mathbf{v}; t\right]\big)\, dt = \int_0^T (\mathbf{f}, \mathbf{v})_{L^2(U)}\, dt.$$

Then integrate the first term of the integrand on the left by parts twice:

$$\int_0^T \langle \mathbf{u}'', \mathbf{v}\rangle\, dt + \int_0^T B\left[\mathbf{u}, \mathbf{v}; t\right]\, dt = \int_0^T (\mathbf{f}, \mathbf{v})_{L^2(U)}\, dt$$

The first integration results in

$$-\langle \mathbf{u}'(0), \mathbf{v}(0)\rangle + \langle \mathbf{u}'(T), \mathbf{v}(T)\rangle - \int_0^T \langle \mathbf{u}', \mathbf{v}\rangle\, dt + \int_0^T B\left[\mathbf{u}, \mathbf{v}; t\right]\, dt = \int_0^T (\mathbf{f}, \mathbf{v})_{L^2(U)}\, dt.$$

The second one

$$-\langle \mathbf{u}'(0), \mathbf{v}(0)\rangle + (\mathbf{u}(0), \mathbf{v}'(0))_{L^2(U)} - (\mathbf{u}(T), \mathbf{v}'(T))_{L^2(U)} + \int_0^T \big((\mathbf{v}'', \mathbf{u})_{L^2(U)} + B\left[\mathbf{u}, \mathbf{v}; t\right]\big)dt$$
$$= \int_0^T (\mathbf{f}, \mathbf{v})_{L^2(U)}\, dt.$$

Since $\mathbf{v}(T) = \mathbf{v}'(T) = 0$, some of the terms resulting from integration by parts equal zero. Moving the rest of the terms to the right side gives

$$\int_0^T \big((\mathbf{v}'', \mathbf{u})_{L^2(U)} + B\left[\mathbf{u}, \mathbf{v}; t\right]\big)dt = \int_0^T (\mathbf{f}, \mathbf{v})_{L^2(U)}\, dt$$
$$- (\mathbf{u}(0), \mathbf{v}'(0))_{L^2(U)} + \langle \mathbf{u}'(0), \mathbf{v}(0)\rangle.$$

As before, replace the function $\mathbf{u}$ with a function from the sequence $\mathbf{u}_m$ to prepare for going to the limit:

$$\int_0^T \left((\mathbf{v}'', \mathbf{u}_m)_{L^2(U)} + B\left[\mathbf{u}_m, \mathbf{v}; t\right]\right) dt = \int_0^T (\mathbf{f}, \mathbf{v})_{L^2(U)}\, dt \\ - \left(\mathbf{u}_m(0), \mathbf{v}'(0)\right)_{L^2(U)} + \left(\mathbf{u}_m'(0), \mathbf{v}(0)\right).$$

Next set $m = m_l$. Now, using the knowledge of the limits in terms of weak convergence, passing to the limit $l \to \infty$ the equation becomes

$$\int_0^T \left((\mathbf{v}'', \mathbf{u})_{L^2(U)} + B\left[\mathbf{u}, \mathbf{v}; t\right]\right) dt = \int_0^T (\mathbf{f}, \mathbf{v})_{L^2(U)}\, dt - \left(g, \mathbf{v}'(0)\right)_{L^2(U)} + \left(h, \mathbf{v}(0)\right)_{L^2(U)}$$

since $d_m^k(0) = (g, w_k)_{L^2(U)}$ and $d_m^k{}'(0) = (h, w_k)_{L^2(U)}$ when $1 \leq k \leq m$. Looking back to the previous two equations, the only differences between them are the terms from the partial integration on the right side. These have to be equivalent and thus lead to a new equation

$$- \left(\mathbf{u}(0), \mathbf{v}'(0)\right)_{L^2(U)} + \langle \mathbf{u}'(0), \mathbf{v}(0)\rangle = - \left(g, \mathbf{v}'(0)\right)_{L^2(U)} + \left(h, \mathbf{v}(0)\right)_{L^2(U)}.$$

Since the choosen $\mathbf{v}(0)$ and $\mathbf{v}'(0)$ are arbitrary, this concludes

$$\mathbf{u}(0) = g \text{ and } \mathbf{u}'(0) = h.$$

Thus $\mathbf{u}$ satisfies the criteria set for a weak solution of the hyperbolic initial/-boundary-value problem. □

4.3 Uniqueness of weak solutions

Since the existence of the weak solution for the hyperbolic initial/boundary value problem has now been proved, all that is left of the basic theory is the *uniqueness* of said solution. The following proof is not as straightforward as the proof of existence. The lack of knowledge of the properties of a derivative $\mathbf{u}'$ leads to certain complications that are solved with 'tricks'. The main frame of this proof is written according to Evans' book [3, pp. 406-408] and supporting information is gathered from Ladyzhenskaya's text [7, pp. 164-168].

Notice that due to a surplus of superscript and a need to differentiate between these and derivatives, time derivates are marked with subscripts in the following proof. As an example, $\mathbf{u}'$ is written as $\mathbf{u}_t$.

Theorem 4.8 (Uniqueness of weak solutions)**.** *A weak solution of the hyperbolic initial/boundary-value problem is unique.*

Proof. As often is the case in proving uniqueness, here also it is sufficient to show that the weak solution of the simplest version of the hyperbolic initial/boundary-value problem with an arbitrary operator L is

$$\mathbf{u} \equiv 0.$$

The equation is linear due to the linearity of the operator L. This means that if the function $\mathbf{u} \equiv 0$ is unique, then any other solution for a similar hyperbolic equation can be proved unique by considering a proof of contradiction: the difference of two simultaneous solutions defines another function which will also act as a solution for the original equation. But because of the linearity and properties of derivatives, this difference function will be zero almost everywhere, which implies that the assumption of having two different solutions produces a contradiction, and thus it is a false one.

With this deduction in mind, consider the wave problem ($\star$) with all the initial values and source functions $\mathbf{f} \equiv g \equiv h \equiv 0$. First, fix $0 \leq s \leq T$ and set

$$\mathbf{v}(t) := \begin{cases} \int_t^s \mathbf{u}(\tau)\, d\tau, & \text{if } 0 \leq t \leq s \\ \mathbf{0}, & \text{if } s \leq t \leq T. \end{cases}$$

Now $\mathbf{v}(t) \in H_0^1(U)$ for each $0 \leq t \leq T$, since $\mathbf{u}$ is in the closed space $H_0^1(U)$ for every τ in $(0, T)$, and hence

$$\int_0^s \big(\langle \mathbf{u}_{tt}, \mathbf{v}_t\rangle + B\left[\mathbf{u}, \mathbf{v}; t\right]\big)\, dt = 0.$$

Integrate the first term of the integrand by parts to get

$$\left(\mathbf{u}_t(s), \mathbf{v}(s)\right)_{L^2(U)} - \left(\mathbf{u}_t(0), \mathbf{v}(0)\right)_{L^2(U)} - \int_0^s \left(\mathbf{u}_t, \mathbf{v}_t\right)_{L^2(U)}\, dt + \int_0^s B\left[\mathbf{u}, \mathbf{v}; t\right]\, dt$$

where the two first terms on the left are zero because $\mathbf{u}_t(0) = \mathbf{v}(s) = 0$. What is left is

$$\int_0^s \big(-\left(\mathbf{u}_t, \mathbf{v}_t\right)_{L^2(U)} + B\left[\mathbf{u}, \mathbf{v}; t\right]\big)\, dt = 0,$$

since supposedly $\mathbf{f} \equiv 0$ in the equation ($\star\star$). Due to the definition of $\mathbf{v}$, the derivative $\mathbf{v}_t = -\mathbf{u}$ in the weak sense on the interval $0 \leq t \leq s$. Therefore, on this interval

$$\int_0^s \big(\langle \mathbf{u}_t, \mathbf{u}\rangle - B\left[\mathbf{v}_t, \mathbf{v}; t\right]\big)\, dt = 0. \qquad (\star\star\star)$$

To help with the somewhat complex notation, define two new functions, namely

$$C\left[u, v; t\right] := \int_U \left(\sum_{i=1}^n b^i v_{x_i} u + \frac{1}{2} b^i_{x_i} uv\right)\, dx$$

and

$$D[u, v; t] := \frac{1}{2} \int_U \left(\sum_{i,j=1}^{n} a_t^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^{n} b_t^i u_{x_i} v + c_t uv \right) dx$$

for $u, v \in H_0^1(U)$. The connection between these two and the bilinear form is based on the time derivative of $B[\mathbf{v}, \mathbf{v}; t]$

$$\frac{d}{dt}\frac{1}{2} B[\mathbf{v}, \mathbf{v}; t] = \frac{d}{dt}\frac{1}{2} \int_U \left(\sum_{i,j=1}^{n} a^{ij} \mathbf{v}_{x_i} \mathbf{v}_{x_j} + \sum_{i=1}^{n} b^i \mathbf{v}_{x_i} \mathbf{v} + c\mathbf{v}^2 \right) dx.$$

This can be differentiated term by term within the integral due to the properties of the functions a^{ij}, b^i, c and $\mathbf{v}$. Start with the first sum,

$$\begin{aligned}
\frac{d}{dt}\left(\sum_{i,j=1}^{n} a^{ij} \mathbf{v}_{x_i} \mathbf{v}_{x_j} \right) &= \sum_{i,j=1}^{n} a_t^{ij} \mathbf{v}_{x_i} \mathbf{v}_{x_j} + a^{ij} \frac{d}{dt}(\mathbf{v}_{x_i} \mathbf{v}_{x_j}) \\
&= \sum_{i,j=1}^{n} a_t^{ij} \mathbf{v}_{x_i} \mathbf{v}_{x_j} + a^{ij} (\mathbf{v}_{x_i t} \mathbf{v}_{x_j} + \mathbf{v}_{x_i} \mathbf{v}_{x_j t}) \\
&= \sum_{i,j=1}^{n} a_t^{ij} \mathbf{v}_{x_i} \mathbf{v}_{x_j} + 2a^{ij} \mathbf{v}_{x_i t} \mathbf{v}_{x_j}.
\end{aligned}$$

Likewise, differentiate the next sum

$$\begin{aligned}
\frac{d}{dt}\left(\sum_{i=1}^{n} b^i \mathbf{v}_{x_i} \mathbf{v} \right) &= \sum_{i=1}^{n} b_t^i \mathbf{v}_{x_i} \mathbf{v} + b^i \frac{d}{dt}(\mathbf{v}_{x_i} \mathbf{v}) \\
&= \sum_{i=1}^{n} b_t^i \mathbf{v}_{x_i} \mathbf{v} + b^i \mathbf{v}_{x_i t} \mathbf{v} + b^i \mathbf{v}_{x_i} \mathbf{v}_t.
\end{aligned}$$

And lastly,

$$\frac{d}{dt}\left(\sum_{i=1}^{n} c\mathbf{v}^2 \right) = \sum_{i=1}^{n} c_t \mathbf{v}^2 + 2c\mathbf{v}_t \mathbf{v}.$$

When they are integrated over U with multiplicator $1/2$ applied, the first terms of these derivatives can be put together to form

$$\frac{1}{2} \int_U \left(\sum_{i,j=1}^{n} a_t^{ij} \mathbf{v}_{x_i} \mathbf{v}_{x_j} + \sum_{i=1}^{n} b_t^i \mathbf{v}_{x_i} \mathbf{v} + c_t \mathbf{v}^2 \right) dx = D[v, v; t].$$

Subtracting $B[\mathbf{v}_t, \mathbf{v}; t]$ from the rest results in

$$\begin{aligned}
&\frac{1}{2}\int_U \left(\sum_{i,j=1}^{n} 2a^{ij}\mathbf{v}_{x_i t}\mathbf{v}_{x_j} + \sum_{i=1}^{n} b^i \mathbf{v}_{x_i t}\mathbf{v} + b^i \mathbf{v}_{x_i}\mathbf{v}_t + 2c\mathbf{v}_t\mathbf{v} \right) dx \\
&\qquad = B[\mathbf{v}_t, \mathbf{v}; t] + \frac{1}{2}\int_U \left(\sum_{i=1}^{n} -b^i \mathbf{v}_{x_i t}\mathbf{v} + b^i \mathbf{v}_{x_i}\mathbf{v}_t \right) dx.
\end{aligned}$$

Integrate this last integral by parts and remember that $\mathbf{v}_t = -\mathbf{u}$ to get $C[\mathbf{u}, \mathbf{v}; t]$.

Combine these results to get

$$\frac{d}{dt}\frac{1}{2}B\left[\mathbf{v}, \mathbf{v}; t\right] = D[v, v; t] + B[\mathbf{v}_t, \mathbf{v}; t] + C[\mathbf{u}, \mathbf{v}; t].$$

Since

$$\frac{d}{dt}\frac{1}{2}\|\mathbf{u}\|^2 = \langle \mathbf{u}_t, \mathbf{u} \rangle,$$

integrating this equation in terms of t leads to

$$\int_0^s \frac{d}{dt}\frac{1}{2}\|\mathbf{u}\|^2 \, dt = \int_0^s \langle \mathbf{u}_t, \mathbf{u} \rangle \, dt = \int_0^s B[\mathbf{v}_t, \mathbf{v}; t] \, dt.$$

Thus

$$\int_0^s \frac{d}{dt}\left(\frac{1}{2}\|\mathbf{u}\|^2_{L^2(U)} - \frac{1}{2}B\left[\mathbf{v}, \mathbf{v}; t\right] \right) dt = -\int_0^s \big(C\left[\mathbf{u}, \mathbf{v}; t\right] + D\left[\mathbf{v}, \mathbf{v}; t\right] \big) \, dt.$$

Due to the definitions of function $\mathbf{u}$ and $\mathbf{v}$, values $\mathbf{u}(0) = 0$ and $\mathbf{v}(s) = 0$. Hence integrating the left side the equation yields

$$\frac{1}{2}\|\mathbf{u}(s)\|^2_{L^2(U)} - \frac{1}{2}B\left[\mathbf{v}(0), \mathbf{v}(0); 0\right] = -\int_0^s \big(C\left[\mathbf{u}, \mathbf{v}; t\right] + D\left[\mathbf{v}, \mathbf{v}; t\right] \big) \, dt.$$

Next, approximate the newly defined forms C and D in terms of various norms of the funtions $\mathbf{u}$ and $\mathbf{v}$. Firstly,

$$\begin{aligned}
|C[\mathbf{u}, \mathbf{v}; t]| &\leq \int_U \left(\sum_{i=1}^{n} \left| b^i \mathbf{v}_{x_i}\mathbf{u} \right| + \frac{1}{2}\left| b^i_{x_i}\mathbf{u}\mathbf{v} \right| \right) dx \\
&\leq \int_U \left(\sum_{i=1}^{n} \|b^i\|_\infty \cdot \frac{1}{2}(\mathbf{v}^2_{x_i} + \mathbf{u}^2) + \frac{1}{2}\|b^i_{x_i}\|_\infty \cdot \frac{1}{2}(\mathbf{u}^2 + \mathbf{v}^2) \right) dx \\
&\leq C \int_U \left(|\nabla \mathbf{v}(t)|^2 + \mathbf{u}^2(t) + \mathbf{v}(t)^2 \right) dx \\
&= C \left(\|\mathbf{v}(t)\|^2_{H^1_0(U)} + \|\mathbf{u}(t)\|^2_{L^2(U)} \right),
\end{aligned}$$

since for all indeces i it was supposed that $b^i \in C^1(\overline{U}_T)$. Similarly, for the form

$$
\begin{aligned}
|D[\mathbf{v},\mathbf{v};t]| &\leq \frac{1}{2}\int_U \left(\sum_{i,j=1}^{n} \left|a_t^{ij}\mathbf{v}_{x_i}\mathbf{v}_{x_j}\right| + \sum_{i=1}^{n}\left|b_t^i\mathbf{v}_{x_i}\mathbf{v}\right| + \left|c_t\mathbf{v}^2\right| \right) dx \\
&\leq \frac{1}{2}\int_U \left(\sum_{i,j=1}^{n} \|a_t^{ij}\|_\infty \cdot \frac{1}{2}(\mathbf{v}_{x_i}^2 + \mathbf{v}_{x_j}^2) + \sum_{i=1}^{n} \|b_t^i\|_\infty \cdot \frac{1}{2}(\mathbf{v}_{x_i}^2 + \mathbf{v}^2) + \|c_t\|_\infty \cdot \mathbf{v}^2 \right) dx \\
&\leq C\int_U \left(|\nabla \mathbf{v}(t)|^2 + \mathbf{v}^2(t)\right) dx \\
&= C\|\mathbf{v}(t)\|^2_{H_0^1(U)}\,.
\end{aligned}
$$

These two imply

$$
\left| \int_0^s \left(C[\mathbf{u},\mathbf{v};t] + D[\mathbf{v},\mathbf{v};t]\right) dt \right| \leq C\int_0^s \left(\|\mathbf{v}\|^2_{H_0^1(U)} + \|\mathbf{u}\|^2_{L^2(U)} \right) dt\,.
$$

Futhermore, by hyperbolicity

$$
\begin{aligned}
\theta \int_U |\nabla \mathbf{v}(0)|^2 dx &\leq \int_U \sum_{i,j} a^{ij} v_{x_i}(0) v_{x_j}(0)\, dx \\
&= B[\mathbf{v},\mathbf{v};0] - \int_U \left(\sum_i b^i \mathbf{v}_{x_i}(0)\mathbf{v}(0) + c\mathbf{v}^2(0) \right) dx \\
&\leq B[\mathbf{v},\mathbf{v};0] - C\left(\int_U |\nabla \mathbf{v}(0)|\mathbf{v}(0)\, dx + \|\mathbf{v}(0)\|^2_{L^2(U)} \right)\,.
\end{aligned}
$$

Notice that for every $\epsilon > 0$ it holds

$$
|\nabla \mathbf{v}(0)|\, \mathbf{v}(0) \leq \frac{\epsilon}{2}\, |\nabla \mathbf{v}(0)\, |^2 + \frac{1}{2\epsilon}\mathbf{v}(0)^2\,.
$$

Choose an ϵ such that $C\epsilon = \theta$, where C is the constant from the previous inequality. Then

$$
C\int_U |\nabla \mathbf{v}(0)|\mathbf{v}(0)\, dx \leq \frac{\theta}{2}\|\nabla \mathbf{v}(0)\|^2_{L^2(U)} + C\|\mathbf{v}(0)\|^2_{L^2(U)}\,.
$$

Thus

$$
\frac{\theta}{2}\int_U |\nabla \mathbf{v}(0)|^2\, dx \leq B[\mathbf{v},\mathbf{v};t] + C\|\mathbf{v}(0)\|^2_{L^2(U)}
$$

and

$$
\begin{aligned}
\|\mathbf{u}(s)\|^2_{L^2(U)} + \|\mathbf{v}(0)\|^2_{H^1_0(U)} &\leq \|\mathbf{u}(s)\|^2_{L^2(U)} + \frac{2}{\theta}\left(B[\mathbf{v},\mathbf{v};t] + C\|\mathbf{v}(0)\|^2_{L^2(U)}\right) \\
&\leq C\left(\frac{1}{2}\|\mathbf{u}(s)\|^2_{L^2(U)} + \frac{1}{2}B[\mathbf{v},\mathbf{v};t] + \|\mathbf{v}(0)\|^2_{L^2(U)}\right) \\
&\leq C\left(\int_0^s \left(\|\mathbf{v}\|^2_{H^1_0(U)} + \|\mathbf{u}\|^2_{L^2(U)}\right) dt + \|\mathbf{v}(0)\|^2_{L^2(U)}\right).
\end{aligned}
$$

In the next phase of the proof it will be easier to use a new notation to simplify reading even further; the integral of the function $\mathbf{u}$ from zero to time t is

$$\mathbf{w}(t) := \int_0^t \mathbf{u}(\tau)\, dt,\ 0 \leq t \leq T.$$

With this the previous inequality becomes

$$
\begin{aligned}
&\|\mathbf{u}(s)\|^2_{L^2(U)} + \|\mathbf{w}(s)\|^2_{H^1_0(U)} \qquad (\star\star\star\star) \\
&\quad\leq C\left(\int_0^s \left(\|\mathbf{w}(t) - \mathbf{w}(s)\|^2_{H^1_0(U)} + \|\mathbf{u}\|^2_{L^2(U)} + \|\mathbf{w}(s)\|^2_{L^2(U)}\right) dt\right).
\end{aligned}
$$

By parallelogram the first term of the upper bound can be further estimated by

$$\|\mathbf{w}(t) - \mathbf{w}(s)\|^2_{H^1_0(U)} \leq \|\mathbf{w}(t)\|^2_{H^1_0(U)} + \|\mathbf{w}(s)\|^2_{H^1_0(U)}.$$

Also, from the properties of the function $\mathbf{u}$, L^2-norm, and Riemann integral in general it follows that

$$\|\mathbf{w}(s)\|_{L^2(U)} \leq \int_0^s \|\mathbf{u}(t)\|_{L^2(U)}\, dt.$$

Together with these notions the previous inequality $(\star\star\star\star)$ implies

$$\|\mathbf{u}(s)\|^2_{L^2(U)} + (1 - 2sC_1)\|\mathbf{w}(s)\|^2_{H^1_0(U)} \leq C_1 \int_0^s \left(\|\mathbf{w}\|^2_{H^1_0(U)} + \|\mathbf{u}\|^2_{L^2(U)}\right) dt.$$

Next choose a constant $T_1 \in [0, T]$ so that

$$1 - 2T_1C_1 \geq \frac{1}{2}.$$

In this case, if $0 \leq s \leq T_1$, then

$$\|\mathbf{u}(s)\|^2_{L^2(U)} + \|\mathbf{w}(s)\|^2_{H^1_0(U)} \leq C \int_0^s \left(\|\mathbf{u}\|^2_{L^2(U)} + \|\mathbf{w}\|^2_{H^1_0(U)}\right) dt.$$

Consider the left side of the inequality as a function of s. Because the left side forms a nonnegative, summable function on $[0, T_1]$ and the right side is of correct form, it is now possible to refer to Gronwall's inequality 2.31. In this case the constant $C_2 = 0$, so the left side of the inequality is zero. This implies that $\mathbf{u} \equiv \mathbf{0}$ on the interval $[0, T_1]$.

To finish the proof, apply the same argument to the other intervals $[(n-1)T_1, nT_1]$ with index $n \geq 2$ to cover the whole interval $[0, T]$ to deduce the wanted result. $\square$

5 Control Theory

This chapter introduces the basics of control theory. The questions that define this approach are "How to tell if a preferred state of a linear system can be reached or not?" and "How can a finite dimensional wave equation be understood in this framework?". The viewpoint is changed when trying to understand the inner workings of a model: In the previous chapter, the partial differential equation considered is a forward problem. This means that initial and boundary values are given with the information of the assumed dependencies between a function and its derivatives. In control theory, the *initial state of the system* and notably, the *observation* of the state derived from 'outside of the system itself' are presumed known. The object of interest is then the input source or *control* and whether or not it is possible to find one that produces the preferred end result.

The chapter is based on *"Inverse Boundary Spectral Problems"* by Katchalov, Kurylev and Lassas [5], the lecture notes of course *"Johdatus inversio-ogelmiin* by Lassas [8], and *"Ordinary Differential Equations"* by Logemann and Ryan [11].

5.1 Basics of control theory

According to Logemann and Ryan [11, p. 65], a linear system of an initial value problem is defined as

Definition 5.1. Let $I \subset \mathbb{R}$ be an interval containing zero and pair $(0, \psi) \in I \times \mathbb{R}^n$. A *linear system* can be written in the following form

$$\dot{x} = Ax + Bf,\ x(0) = \psi \in \mathbb{R}^n, \tag{$*$}$$
$$y = Cx, \tag{$**$}$$

where $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{p \times n}$.

The linear system consists of an *inhomogeneous initial value problem* on line $(*)$ and an *observation* data marked as the equation $(**)$. Since the focus of this chapter is on control theory, the *input* f of the system is called the *control*. Likewise, the *output* y is the *observation* which in the case of a real world application would be the measured data from the process. The function x describes the *state* of the system at any given time t.

The first step to solve this control problem is to notice that the equation $(*)$ can be solved by the formula known as *variation of parameters* as stated in Logemann and Ryan's book [11, p. 41-43]. But before continuing with this theorem, it is necessary to define the following function spaces: According to Logemann and Ryan [11, pp. 280-281],

Definition 5.2. The space of matrices $\mathbb{F}^{n\times m}$ is notated by $\mathbb{M}_{\mathbb{F}}$.

Definition 5.3. Let I be an interval in $\mathbb{R}$ and $Y \subset \mathbb{R}^n$. A function $f : I \to \mathbb{M}_{\mathbb{F}}$ is *piecewise continuous* if for every $a, b \in I$, where $a < b$, the interval $[a, b]$ has a finite partition $a < x_1 < x_2 < \cdots < x_k < b$ such that

(i) the function f is continuous on every interval (x_l, x_{l+1}) when $l = 1, 2, \ldots, k-1$,

(ii) the function f has a right limit at the point x_1,

(iii) the function f has a left limit at the point x_k,

(iv) the function f has a right and a left limit at the point x_i for every index $2, 3 \ldots, k-1$

The vector space of *piecewise continuous functions* from I to Y is notated by $PC(I, Y)$.

As a notice from Logemann and Ryan's book [11, p. 281]: a piecewise continuous function $f : I \to \mathbb{M}_{\mathbb{F}}$ is integrable on every interval $[a, b]$ when $a, b \in I$ and $a < b$.

Then to the actual theorem used to solve linear systems of initial value problems [11, pp. 41-43]:

Theorem 5.4 (Variation of parameters)**.** *For each input $f \in PC(I, \mathbb{R}^m)$ and initial value $\psi \in \mathbb{R}^n$ the unique solution of the problem $(*)$ on I is given by the formula of variation of parameters*

$$x(t; \psi, f) := e^{At}\psi + \int_0^t e^{A(t-s)} B f(s)\, ds$$

for all $t \in I$.

Proof. First, it is necessary to show that the formula actually produces a solution for the initial value problem. Start by derivating the formula in terms of the variable t: Suppose that the function f is at least piecewise continuous. Then

$$\begin{aligned}\dot{x}(t; \psi, f) =& \frac{d}{dt}\left(e^{At}\psi + \int_0^t e^{A(t-s)} B f(s)\, ds\right)\\ =& A e^{At}\psi + \frac{d}{dt}\int_0^t e^{A(t-s)} B f(s)\, ds,\end{aligned}$$

which by Leibniz's rule 2.32 can be written as

$$\begin{aligned}
\dot{x}(t;\psi,f) =& Ae^{At}\psi + e^{A(t-t)}Bf(t)\frac{d}{dt}t - e^{A(t-0)}Bf(0)\frac{d}{dt}\cdot 0 + \int_0^t \frac{\partial}{\partial t}e^{A(t-s)}Bf(s)\,ds \\
=& Ae^{At}\psi + Bf(t) + A\int_0^t e^{A(t-s)}Bf(s)\,ds \\
=& Ax(t;\psi,f) + Bf(t).
\end{aligned}$$

Then notice that the initial value

$$\begin{aligned}
x(0;\psi,f) =& e^{A\cdot 0}\psi + \int_0^0 e^{A(0-s)}Bf(s)\,ds \\
=& \psi.
\end{aligned}$$

This proves that the formula is a solution for $(*)$. Next, suppose that function y is also a solution for the linear system. Now,

$$\begin{aligned}
\frac{d(x-y)}{dt} &= \frac{dx}{dt} - \frac{dy}{dt} \\
&= Ax + Bf - (Ay + Bf) \\
&= A(x-y)
\end{aligned}$$

and this leads to an equation

$$\dot{x} - Ax = \dot{y} - Ay.$$

Then, a function $z = x - y$ should solve the corresponding homogeneous linear system

$$\dot{z} = Az, \qquad z(0) = 0$$

and thus, $x = y$. $\square$

In the previous chapter where the *forward problem* for wave equation was solved the input function was given. In comparison, this time the search is for an answer for what happens when the initial state is known and the input is something that can be chosen within some limits. Can a predetermined *target state* be reached from these settings? This problem type is known as a *controllability problem* [11, p. 67].

Definition 5.5. Let $\psi, \zeta \in \mathbb{R}^n$. If there exists a $T > 0$ and a control $f \in PC([0,T],\mathbb{R}^m)$ for which $x(T;\psi,f) = \zeta$, then ζ is *reachable* from the initial value ψ. Systems $(*)$ and $(*)$-$(**)$ are *controllable* if for all pairs $(\psi,\zeta) \in \mathbb{R}^{n\times n}$, ζ is reachable from ψ.

The main information of any linear system of type $(*)$ can be summarised as a pair of matrices $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$. As before, this pair is controllable if the actual system is controllable. Further on, the *controllability matrix* can be formulated as

$$\mathcal{C}(A, B) := (B, AB, A^2B, \dots, A^{n-1}B) \in \mathbb{R}^{n\times(mn)},$$

where $\mathcal{C}(A, B)$ defines a mapping from $\mathbb{R}^{mn}$ to $\mathbb{R}^n$ [11, p. 69]. The *controllability Gramian* of the same set of matrices (A, B) is

$$Q_T(A, B) := \int_0^T e^{At}BB^*e^{A^*t}\,dt \in \mathbb{R}^{n\times n},$$

where the interval of integration depends on parameter $T > 0$. The matrix Q_T is both symmetric and positive semi-definite [11, p. 69]. In the following theorems these two definitions prove to be extremely useful when it comes to understanding the set of reachable states and its connections with the pair (A, B).

Remark 3. The *image* of a matrix $M \in \mathbb{R}^{m\times n}$ is notated by $\operatorname{im} M$.

The next theorem by Logemann and Ryan [11, pp. 69-70] states the first common characteristics between the matrices, but before it is useful to consider the following lemma:

Lemma 5.6. *For a pair of matrices A, B derived from a linear system of type $(*)$ the following equivalence holds*

$$z \in (\operatorname{im}\mathcal{C}(A, B))^\perp \Leftrightarrow z^*e^{At}B = 0 \textit{ for all } t \in \mathbb{R}.$$

Proof. Suppose $z \in (\operatorname{im}\mathcal{C}(A, B))^\perp$. By theorem 2.36 this is equivalent with $z^*\mathcal{C}(A, B) = 0$. From the definition of $\mathcal{C}(A, B)$ it follows that $z^*A^kB = 0$ for all $k \in [0, n-1]$. This can be generalised by the Cayley-Hamilton theorem 2.37 to cover every $k \in \mathbb{N}\cup\{0\}$, since it states that the operator A^k is a linear combination of operators $I = A^0, A, A^2, \dots, A^{n-1}$ for every $k \in \mathbb{N} \cup \{0\}$. Hence continuing inductively from $k = n-1$ yields the result $z^*A^kB = 0$ for $k \in \mathbb{N} \cup \{0\}$. This implies

$$0 = \sum_{k=0}^{\infty} \frac{t^k}{k!} z^*A^kB = z^*e^{At}B \text{ for all } t \in \mathbb{R}.$$

For the other direction in the equivalence, suppose that $z^*e^{At}B = 0$ for all $t \in \mathbb{R}$. When $t = 0$ this becomes

$$z^*e^{A\cdot 0}B = z^*B = z^*A^0B = 0.$$

Next, proceed to differentiate the function $z^* e^{At} B$ in terms of t:

$$\frac{d}{dt} z^* e^{At} B = z^* A e^{At} B,$$

and evaluate this again at $t = 0$ to get

$$z^* A e^{A \cdot 0} B = z^* A^1 B = 0,$$

since $z^* e^{At} B$ is identically zero for all $t \in \mathbb{R}$. Continuing this process inductively shows that $z^* A^k B = 0$ for all $k = 0, \ldots, n-1$. Thus, $z^* \mathcal{C}(A, B) = 0$, and now theorem 2.36 implies $z \in (\operatorname{im} \mathcal{C}(A, B))^\perp$. □

Theorem 5.7. *For images of a pair of matrices A, B derived from a linear system of type $(*)$,*

$$\operatorname{im} \mathcal{C}(A, B) = \operatorname{im} Q_T(A, B) \textit{ for all } T > 0.$$

Proof. For all $T > 0$, the claim is equivalent with

$$(\operatorname{im} \mathcal{C}(A, B))^\perp = (\operatorname{im} Q_T(A, B))^\perp,$$

since for both $\mathbb{R}^{n \times (mn)}$ and $\mathbb{R}^{n \times n}$ the orthocomplements are unique. The proof is done in two parts proving inclusions to both directions separately.

Suppose $T > 0$ and $z \in (\operatorname{im} \mathcal{C}(A, B))^\perp$. By lemma 5.6, for all $t \in \mathbb{R}$ $z^* e^{At} B = 0$. This implies that

$$z^* Q_T = \int_0^T z^* e^{At} B B^* e^{A^* t} dt = 0 \text{ for all } t \in \mathbb{R},$$

which in turn implies according to theorem 2.36 that $(\operatorname{im} Q_T(A, B))^\perp$. This proves that

$$(\operatorname{im} \mathcal{C}(A, B))^\perp \subset (\operatorname{im} Q_T(A, B))^\perp.$$

Then suppose $z \in (\operatorname{im} Q_T(A, B))^\perp$. Theorem 2.36 implies now that $z^* Q_T = 0$, and since Q_T is symmetrical, also $Q_T z = 0$ follows from this. Thus $\langle z, Q_T z \rangle = 0$. Now,

$$0 = \int_0^T \langle z, e^{At} B B^* e^{A^* t} z \rangle dt = \int_0^T \| B^* e^{A^* t} z \|^2 dt.$$

The norm integrand is a continuous, non-negative function, which implies that $B^* e^{A^* t} z = 0$ for all $t \in [0, T]$. From this it follows that $z^* e^{At} B = 0$ for all $t \in \mathbb{R}$, and consequently by lemma 5.6 $z \in (\operatorname{im} \mathcal{C}(A, B))^\perp$. This means that

$$(\operatorname{im} Q_T(A, B))^\perp \subset (\operatorname{im} \mathcal{C}(A, B))^\perp.$$

□

One of the direct consequences of this theorem and theorem 2.36 is that for all $T > 0$ there exists a *left inverse* mapping $Q_T^\sharp \in \mathbb{R}^{n\times n}$ for which

$$Q_T Q_T^\sharp z = z \text{ for all } z \in \operatorname{im} \mathcal{C}(A, B).$$

To continue with the assisting definitions, the *input-to-state map*

$$\mathbf{C}_T : PC([0, T], \mathbb{R}^m) \to \mathbb{R}^n,\ u \mapsto \int_0^T e^{A(T-t)} Bf(t)\, dt$$

depends on parameter $T > 0$. Its image $\operatorname{im} \mathbf{C}_T(PC([0, T], \mathbb{R}^m))$ is the set of all reachable states from 0 in time T as its name suggests. With this linear map, the variation of parameter formula from theorem 5.4 simplifies to the form

$$x(T; \psi, f) := e^{AT}\psi + \mathbf{C}_T f.$$

Next consider the set of states of system $(*)$ which are reachable from 0. With maps $\mathbf{C}_T$ this set is

$$\begin{aligned} R :&= \bigcup_{T>0} \operatorname{im} \mathbf{C}_T(PC([0, T], \mathbb{R}^m)) \qquad (***) \\ &= \{x(T; 0, f) = \mathbf{C}_T f : T > 0, f \in PC([0, T], \mathbb{R}^m)\}. \end{aligned}$$

To better understand the set R it is purposeful to look closer into optional ways of defining it. In [11, pp. 69-71] this is done by considering images of other sets defined earlier in this chapter.

Theorem 5.8.

$$R = \operatorname{im} \mathcal{C}(A, B) = \operatorname{im} \mathbf{C}_T(PC([0, T], \mathbb{R}^m)) \textit{ for all } T > 0.$$

Proof. Suppose z is an arbitrary element in the subspace R. Then it is possible to write $z = \mathbf{C}_T f$ with some $T > 0$ and $f \in PC([0, T], \mathbb{R}^m)$. Next, consider R divided into two parts: namely $\operatorname{im} \mathcal{C}(A, B)$ and its ortho-complement $(\operatorname{im} \mathcal{C}(A, B))^\perp$. Now $z = z_1 + z_2$, where $z_1 \in \operatorname{im} \mathcal{C}(A, B)$ and $z_2 \in (\operatorname{im} \mathcal{C}(A, B))^\perp$. By lemma 5.6, for all $t \in \mathbb{R}$

$$z \in (\operatorname{im} \mathcal{C}(A, B))^\perp \Leftrightarrow z^* e^{At} B = 0.$$

By this $z_2^* e^{T-t} B = 0$ for all $t \in [0, T]$, which leads to an equation

$$0 = \int_0^T z_2^* e^{T-t} Bf(t)\, dt = \langle z_2, \mathbf{C}_T f\rangle = \langle z_2, z\rangle = \|z_2\|^2.$$

Here it follows that $z = z_1$ which was defined as an element of $\operatorname{im} \mathcal{C}(A, B)$. As z was arbitrary, $R \subset \operatorname{im} \mathcal{C}(A, B)$.

The next goal is to prove the other inclusion. Suppose $z \in \operatorname{im} \mathcal{C}(A, B)$, with $T > 0$ arbitrary and set $u = \mathbf{C}_T^\sharp z$, where

$$\left(\mathbf{C}_T^\sharp z\right) := B^* e^{A^*(T-t)} Q_T^\sharp z \text{ for all } t \in [0, T]$$

is a mapping from $\mathbb{R}^n$ into $PC([0,T], \mathbb{R}^m)$, and $Q_T^\sharp \in \mathbb{R}^{n \times n}$ satisfies

$$Q_T Q_T^\sharp z = z \text{ for all } z \in \operatorname{im} \mathcal{C}(A, B).$$

Note that by its definition $\mathbf{C}_T^\sharp$ is linear. Let $y \in \mathbb{R}^n$. Now

$$\mathbf{C}_T \mathbf{C}_T^\sharp y = \int_0^T e^{A(T-t)} B B^* e^{A^*(T-t)} Q_T^\sharp y \, dt = Q_T Q_T^\sharp y.$$

For $z \in \operatorname{im} \mathcal{C}(A, B)$ this is $\mathbf{C}_T \mathbf{C}_T^\sharp z = z$ since by the definition of $\mathbf{C}_T^\sharp$ the mapping $\mathbf{C}_T$ is right invertible. Continue the previous equation by applying the definitions of u and R $(***)$ to get

$$z = \mathbf{C}_T \mathbf{C}_T^\sharp z = \mathbf{C}_T f = x(T; 0, f),$$

which means that $z \in R$ and $z \in \operatorname{im} \mathbf{C}_T(PC([0,T], \mathbb{R}^m))$. Hence $R = \operatorname{im} \mathcal{C}(A, B)$ and $R \subset \operatorname{im} \mathbf{C}_T(PC([0,T], \mathbb{R}^m))$ for all $T > 0$. Since

$$R = \bigcup_{T>0} \operatorname{im} \mathbf{C}_T(PC([0,T], \mathbb{R}^m)) \text{ for all } T \leq 0,$$

it follows that

$$\operatorname{im} \mathbf{C}_T(PC([0,T], \mathbb{R}^m)) \subset R. \qquad \square$$

To tie up the previous results into a form that is useful when defining whether a system is controllable or not, this following theorem lists different equivalent ways of proving controllability [11, pp. 72-73].

Theorem 5.9. *The following are equivalent*

(i) The initial value problem $(*)$ is controllable.

(ii) $\operatorname{rank} \mathcal{C}(A, B) = n$.

(iii) Q_T is invertible for some $T > 0$.

(iv) Q_T is invertible for all $T > 0$.

(v) $\mathbf{C}_T$ *is surjective for some* $T > 0$.

(vi) $\mathbf{C}_T$ *is surjective for all* $T > 0$.

(vii) $\mathbf{C}_T$ *is right invertible for some* $T > 0$.

(viii) $\mathbf{C}_T$ *is right invertible for all* $T > 0$.

Proof. The proof follows the implication chart below.

$$
\begin{array}{ccccccc}
 & & (ii) & \Rightarrow & & (viii) \\
 & \Nearrow & & \Searrow & & \\
(i) & \Leftarrow & (iii) & \Leftarrow & (iv) & \Downarrow \\
 & & & \Nearrow & & \\
 & & & (iv) & & \\
 & & \Nwarrow & \Swarrow & & \\
 & & (v) & \Leftarrow & & (vii)
\end{array}
$$

$(iv) \Rightarrow (iii)$, $(vi) \Rightarrow (v)$, $(viii) \Rightarrow (vii)$

These implications follow immediately from their claims.

$(ii) \Leftrightarrow (iv) \Leftrightarrow (vi)$

By theorem 5.7 Q_T is invertible if and only if $\operatorname{rank} \mathcal{C}(A, B) = n$. Since theorem 5.8 can only hold true, when $n = \operatorname{rank}(\operatorname{im} \mathcal{C}(A, B)) = \operatorname{rank}(\operatorname{im} \mathbf{C}_T(PC([0, T], \mathbb{R}^m)))$ for all $T > 0$, it also implies that (iv) is equivalent with (vi).

$(vii) \Rightarrow (v)$

Suppose that (vii) holds. Then $\mathbf{C}_T \mathbf{C}_T^\sharp z = z$ for all $z \in \mathbb{R}^n$, which implies that $\operatorname{im} \mathbf{C}_T(PC([0, T])) = \mathbb{R}^n$, in other words $\mathbf{C}_T$ is surjective for some $T > 0$.

$(ii) \Rightarrow (viii)$

Suppose $\operatorname{rank} \mathcal{C}(A, B) = n$. Then $\operatorname{im} \mathcal{C}(A, B) = \mathbb{R}^n$ and by the definitions of $\mathbf{C}_T$ and $\mathbf{C}_T^\sharp$, it follows as in the proof of theorem 5.8 that $\mathbf{C}_T \mathbf{C}_T^\sharp z = z$ for all $z \in \mathbb{R}^n$. Thus $\mathbf{C}_T$ is right invertible for all $T > 0$.

$(v) \Rightarrow (i)$

Suppose that $\mathbf{C}_T$ is surjective for some $T > 0$. Choose state vectors $a, b \in \mathbb{R}^n$ to be arbitrary and then define a function

$$z := b - ae^{AT}.$$

Since $\mathbf{C}_T$ is surjective, there exists a function $f \in PC([0, T], \mathbb{R}^m)$ for which $\mathbf{C}_T f = z$. This can also be written as

$$
\begin{aligned}
x(T; a, f) &:= ae^{AT} + \int_0^T e^{A(T-t)} B f(t) dt \\
&= b - z + \mathbf{C}_T f = b.
\end{aligned}
$$

By definition 5.5, the state b is reachable form the state a for every pair $(a, b) \in \mathbb{R}^n$ and hence the initial value problem $(*)$ is controllable.

$(iii) \Rightarrow (i)$

Suppose that Q_T is invertible for some $T > 0$. As before, let $a, b \in \mathbb{R}^n$. Since Q_T is invertible, $\operatorname{im} Q_T(A, B) = \mathbb{R}^n$. By theorem 5.7 it also holds that $\operatorname{im} \mathcal{C}(A, B) = \mathbb{R}^n$ which leads to the deduction that $b - ae^{AT} \in \operatorname{im} \mathcal{C}(A, B)$. Next, define a function $u := \mathbf{C}_T^\sharp(b - ae^{AT})$. With this

$$\begin{aligned} x(T; a, f) &= ae^{AT} + \mathbf{C}_T f \\ &= ae^{AT} + \mathbf{C}_T \mathbf{C}_T^\sharp (b - ae^{AT}) = b, \end{aligned}$$

and as in before, the claim (i) holds.

$(i) \Rightarrow (ii)$

Suppose that the initial value problem $(*)$ is controllable. Then the set of reachable states of this system $R = \mathbb{R}^n$ by its definition $(***)$. By theorem 5.8, the rank of $\mathcal{C}(A, B)$ is equal with the dimension of $\operatorname{im} \mathcal{C}(A, B) = n$. Thus statement (ii) holds. □

5.2 Wave equation in control theory

And how does the control theory relate to wave equations? The actual formulation for a wave equation defined previously in $(\star)$ does not seem to satisfy the definitions for a linear system in this chapter. Hence, the next task is to show that a wave equation can be considered as such system and as a consequence, the earlier results of this chapter can be applied into controlling a wave system. The change to a linear system is done according to the examples given in the article *"Basic concepts of control theory"* by Markus [12] and the method provided by Hirsch [4, pp. 102-103]. Otherwise, the section follows the lecture material of the course "*Johdatus inversio-ongelmiin*" by Matti Lassas [8].

The following simplified wave operator will be considered in this section:

Definition 5.10. Let $l > 0$ and $q \in C^\infty([0, l])$ be a real valued function. The *boundary spectral data* of the wave operator $A : H_0^2(0, l) \to L^2(0, l)$, where

$$A = -D^2 + q(x) \tag{$****$}$$

is

$$((\lambda_i)_{i=1}^\infty, (\partial_x \phi_i(0))_{i=1}^\infty),$$

where λ_i are the eigenvalues of the operator A and ϕ_i are the corresponding eigenvectors for $i \in \mathbb{N}$.

Like earlier in this chapter, suppose that the boundary spectral data is known for the system (∗∗∗∗) and try to find appropriate *potential* function q. The following method used to reconstruct q is called the *boundary control method*. This requires certain definitions for function spaces used in following proofs.

Definition 5.11. Let

$$L^2(a,b) = \{f \in L^2(0,1) : f|_{(0,a)} = 0, f|_{(b,1)} = 0\},$$

where $0 \leq a < b \leq 1$.

Definition 5.12. An operator $S_T : C_c^\infty(\mathbb{R}_+) \mapsto L^2(0,1)$ is defined to

$$S_T f_0 := u^{f_0}(\cdot, T) \in L^2(0,1),$$

where the function $u^{f_0}(x,t)$ is the generalized solution for the wave equation

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + q(x)\right) u(x,t) &= 0, \ \text{ when } 0 \leq x \leq 1, t > 0 \\ u(0,t) &= f_0(t) \in C_c^\infty(\mathbb{R}_+) \\ u(1,t) &= 0 \\ u(x,0) &= 0, \ \frac{\partial u}{\partial t}(x,0) = 0, \end{aligned}$$

where the function $u \in C^2(\overline{\mathbb{R}}_+ \times [0,1])$. Here, the function f_0 is called the *control* of this equation.

This new operator is built from three blocks: the second partial time derivative, the second partial space coordinate derivative and a function q depending only on the variable x. Written with the notation from definition 4.1,

$$Lu = -1 \cdot u_{xx} + 0 \cdot u_x + q(x)u(x),$$

where the function $c(x,t) = q(x)$.

This simplified wave operator is a sub type of the general hyperbolic operator defined in the previous chapter. Consequently, the results derived earlier from it do still hold true.

The following lemma is crucial in showing that the wave equation defined earlier in this chapter is indeed controllable, or at least approximately controllable.

Lemma 5.13. *Let $0 < a \leq T < 1$ and $b = T - a \geq 0$. Now*

$$X = \{u^{f_0}(\cdot, T) : f_0 \in C_c^\infty(b, T)\} = S_T C_c^\infty(b, T)$$

is a dense subset in $L^2(0, a)$.

Proof. This proof consists of two parts. The main goal is to prove two separate inclusions: first $S_T f_0 \subset L^2(0, a)$ and later $C_c^\infty(0, a) \subset X \subset L^2(0, a)$.

Let $f_0 \in C_c^\infty(b, T)$. Suppose that the extension of u defined according to theorem 3.12 as $u^e = Eu$ is the classical solution for the equation

$$\begin{aligned}
\left(\partial_t^2 - \partial_x^2 + q(x)\right) u^e(x, t) = 0, \text{ when } 0 \leq t \leq T \text{ and } 0 \leq x \leq 2,\\
u^e|_{t=0} = 0,\ u_t^e|_{t=0} = 0,\\
u^e|_{x=0} = f_0,\ u^e|_{x=2} = 0,
\end{aligned}$$

where similarly to the extension u^e, the extension of function q is $q^e \in C^\infty([0, 2])$, and its values on the interval $[0, 1]$ are $q^e|_{[0,1]} = q$. It follows from the values on the boundaries and the definition of f_0 that u^e disappears on

$$S = \{(x, t) \in [0, 2] \times [0, T] : t - b < x \text{ and } t - b < 2 - x\}.$$

Next define the intersection of sets S and $[0, 1] \times [0, T]$, which written exactly is the set

$$U = \{(x, t) \in [0, 1] \times [0, T] : t - b < 2 - x\}.$$

Then, consider again another wave equation nearly identical with the previous one:

$$\begin{aligned}
\left(\partial_t^2 - \partial_x^2 + q(x)\right) u(x, t) = 0, \text{ when } 0 \leq t \leq T \text{ and } 0 \leq x \leq 1,\\
u|_{t=0} = 0,\ u_t|_{t=0} = 0,\\
u|_{x=0} = f_0,\ u|_{x=1} = 0.
\end{aligned}$$

Here, the function $u^e|_{(0,1)\times(0,T)}$ is the solution for this equation with $u \in C^2([0, 1] \times [0, T])$, and thus

$$u = u^{f_0} = S_t f_0, \text{ when } 0 \leq t \leq T \text{ and } u^{f_0}|_U \equiv 0.$$

This proves the first inclusion $S_T f_0 \subset L^2(0, a)$.

Now for the other inclusion. This is easiest to prove with a change of coordinates such that $y = T - t$ and $s = -x$ as demonstrated in the lecture notes [8], which essentially means that the wave is reflected back to its starting point.

Let $h \in C_c^\infty(0,a)$ and suppose that the function w is the solution for the equation

$$\begin{aligned}
\big(\partial_s^2 - \partial_y^2 + q(s)\big)\, w(s,y) &= 0, \text{ when } -1 < s < 0 \text{ and } 0 < y < T,\\
w|_{y=0} = h(-s),\ w|_{y=T} &= 0,\\
w|_{s=-1} = 0,\ w_s|_{s=-1} &= 0.
\end{aligned}$$

Since $h(-s) \in C_c^\infty(-a,0)$, it holds true that

$$\begin{aligned}
w &\in C^\infty([-1,0]\times[0,T]),\\
w(s,y) &= 0, \text{ when } s < -a+y.
\end{aligned}$$

Define the original functions u and f_0 as

$$\begin{aligned}
u(x,t) &= w(T-t,-x)\\
f_0(t) &= w(-t,0) \in C^\infty([0,T]).
\end{aligned}$$

This leads to the actual solution for the wave equation, for which

$$\begin{aligned}
u(x,T) &= w(0,-s)|_{s=-x} = h(x)\\
\operatorname{supp}(f_0) &\subset [b,T].
\end{aligned}$$

This implies

$$C_c^\infty(0,a) \subset X \subset L^2(0,a),$$

which proves that the set X is dense in $L^2(0,a)$. □

But what does this all mean in terms of control theory in practice? The final goal left is to connect the result of lemma 5.13 with the linear system defined in the beginning of this chapter. Start with a boundary control problem

$$\begin{aligned}
(\partial_t^2 - \Delta + q)u(x,t) &= 0\\
u|_{t=0} = 0, u_t|_{t=0} &= 0\\
u|_{\partial I\times\mathbb{R}_+} &= f(x,t), \quad f \in C_c^\infty(\partial I \times \mathbb{R}_+),
\end{aligned}$$

where the interval $I = [0,1]$. Next, consider u as a combination of two functions,

$$u(x,t) = v(x,t) + (Ef)(x,t),$$

where E is the extension operator from theorem 3.12 and v satisfies the following inhomogeneus wave equation

$$\begin{aligned}
(\partial_t^2 - \Delta + q)v(x,t) &= -(\partial_t^2 - \Delta + q)E(f)\\
v|_{t=0} = 0, v_t|_{t=0} &= 0\\
v|_{\partial I\times\mathbb{R}_+} &= 0.
\end{aligned}$$

What is notable in this equation is the fact that comparing to the previous wave equation for the function u, this one here has zero as the boundary value. This allows it to be written as a matrix formula according to the examples by Markus [12] and as was earlier done in the proof of theorem 4.5 [4, pp. 102-103]:

$$\partial_t \begin{pmatrix} v(t) \\ v_t(t) \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{I} \\ \Delta + q & 0 \end{pmatrix} \begin{pmatrix} v \\ v_t \end{pmatrix} + \begin{pmatrix} 0 \\ (-\partial_t^2 + \Delta - q)E \end{pmatrix} f.$$

Here, the vector

$$x(t) := (v(t), v_t(t))$$

is the *state of the system* at a moment $t \in [0, a]$. The state x belongs to the Sobolev space $H^1([0, a], L^2(0, 1))$. Now, this system is actually in the form defined as the linear system $(*)$ with

$$\begin{aligned} A &= \begin{pmatrix} 0 & \mathrm{I} \\ \Delta + q & 0 \end{pmatrix} \\ B &= \begin{pmatrix} 0 \\ (-\partial_t^2 + \Delta - q)E \end{pmatrix} \\ f &= f \\ x(0) &= \begin{pmatrix} v(0) \\ v_t(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Since this form exists, it is now possible to deduce that an initial/boundary value problem of a wave equation is controllable.

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