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Estimates and rigidity for stable solutions to some nonlinear elliptic problems

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Estimates and rigidity for stable solutions to some nonlinear elliptic problems

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Meglio piccamora non pinsaricci,
meglio starsene a taliare il mari
che, a Vigàta o a Boccadasse,
sempre mari è.

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Il campo del vasaio

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Contents

Summary

This thesis deals with the study of elliptic Partial Differential Equations. It is divided into two parts, the first one concerning a nonlinear elliptic equation involving the p-Laplacian, and the second one focused on a nonlocal problem, which can be formulated by means of a Dirichlet to Neumann operator related to the fractional Laplacian.

In the first part, we study the regularity of stable solutions to the nonlinear elliptic equation involving the p-Laplacian

$$-\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = f(u) \qquad \text{in } \Omega \subset \mathbb{R}^n, \tag{0.1}$$

where Ω is a bounded domain, $p \in (1, +\infty)$ and f is a C^1 nonlinearity. This equation is the nonlinear version of the widely studied semilinear elliptic equation $-\Delta u = f(u)$ in a bounded domain $\Omega \subset \mathbb{R}^n$. Stable solutions to semilinear equations have been very recently proved to be bounded, and therefore smooth, in dimension $n \leq 9$ by Cabré, Figalli, Ros-Oton, and Serra [38]. This result is optimal, since examples of unbounded stable solutions are well-known in dimension $n \geq 10$. Furthermore, the results in [38] give a complete answer to a long-standing open problem raised by Brezis and Vázquez [25] about the regularity of *extremal solutions* to $-\Delta u = \lambda f(u)$. These are nontrivial examples of stable solutions to semilinear equations, that can be bounded or unbounded in dependence of the dimension n, domain Ω, and nonlinearity f.

We investigate the boundedness of stable solutions to (0.1), which is conjectured to be true up to dimension $n < p + 4p/(p-1)$. Indeed, examples of unbounded stable solutions are known if $n \geq p + 4p/(p-1)$, even in the unit ball. Moreover, in the radial case or under strong assumptions on the nonlinearity, stable solutions to (0.1) are proved to be bounded in the optimal dimension range $n < p + 4p/(p-1)$.

We prove a new L^∞ a priori estimate for stable solutions to (0.1), under a new condition on n and p, which is optimal in the radial case, and more restrictive in the general one. However, it improves the known results in the field, and it is the first example of a technique providing both a result in the nonradial case and the optimal result in the radial case. To establish this result, we extend a technique developed by Cabré [30] for the classical case of the problem, with $p = 2$, to the framework of the p-Laplacian. The strategy is based on a Hardy inequality on the level sets of the solution and on a geometric inequality for stable solutions to (0.1).

In the first part of the thesis we also investigate Hardy-Sobolev inequalities on hypersurfaces of Euclidean space, all of them involving a mean curvature term and having universal constants independent of the hypersurface. Our motivation comes from several applications of these inequalities to the study of a priori estimates for stable solutions, both in the semilinear and in the nonlinear case.

First, we give a quick and easy to read proof of the celebrated Michael-Simon and

Allard inequality, which is a generalized Sobolev inequality on hypersurfaces of $\mathbb{R}^{n+1}$ involving a mean curvature term. Choosing a geometric point of view and focusing on the codimension one case, we follow mainly the ideas of Michael and Simon [110], but inserting a simplification given by Allard [10].

Concerning inequalities of the Hardy type, we prove two new forms of the Hardy inequality on hypersurfaces of $\mathbb{R}^{n+1}$, and an improved Hardy or Hardy-Poincaré inequality, all of them containing a mean curvature term. Our first Hardy inequality originates from an application to the regularity of stable solutions to semilinear elliptic equations. In the proof, we use the notion of tangential derivatives presented in Giusti's book [97], and an integration by parts formula for tangential derivatives. As a byproduct of this result, we prove the Hardy inequality on the level sets of a function that we use in the proof of our L^∞ a priori estimate for stable solutions to (0.1). Our second Hardy inequality is proved using the "ground state" substitution, a technique based on exploiting a specific positive solution of the Euler-Lagrange equation of the energy functional associated with the inequality. We then use a refinement of this procedure, combined with a Poincaré inequality with weights, to obtain a Hardy-Poincaré inequality on hypersurfaces. We point out that the ground state substitution has been widely applied to the study of Hardy-type inequalities in the Euclidean setting but, up to our knowledge, it has never been used before in the context of hypersurfaces of $\mathbb{R}^{n+1}$.

In the second part of this thesis, we deal with a Dirichlet to Neumann problem arising in a model for water waves. Considering the slab of fixed height $\mathbb{R}^n \times [0,1]$, a smooth bounded function u defined on $\mathbb{R}^n$, and a parameter $a \in (-1,1)$, we are interested in studying the following system of equations

$$\begin{cases} \operatorname{div}(y^a \nabla v) = 0 & \text{for} \quad x \in \mathbb{R}^n, y \in (0,1), \\ v_y(x,1) = 0 & \qquad\; x \in \mathbb{R}^n, y = 1, \\ v(x,0) = u(x) & \qquad\; x \in \mathbb{R}^n, y = 0, \\ -\lim\limits_{y \to 0} y^a v_y = f(v) & \qquad\; x \in \mathbb{R}^n, y = 0. \end{cases} \tag{0.2}$$

This system can also be reformulated as a nonlocal problem on the component endowed with the Dirichlet datum. Indeed, we can define the Dirichlet to Neumann operator $\mathcal{L}_a$ as

$$\mathcal{L}_a u(x) := -\lim_{y \to 0} y^a v_y(x,y), \tag{0.3}$$

and study the nonlocal equation $\mathcal{L}_a u = f(u)$ in $\mathbb{R}^n$. This problem is related to a water waves model and, in a suitable limit, it recovers a fractional Laplace operator.

Problem (0.2) was first studied in the case $a = 0$ by de la Llave and Valdinoci [70]. Their main result is a Liouville theorem, that establishes the one dimensional symmetry of monotone solutions, provided that a control on the growth of the energy associated to the problem is satisfied. As a byproduct of this result, the authors obtained the one dimensional symmetry of monotone solutions to problem (0.2) with $a = 0$ and $n = 2$.

The study of one dimensional symmetry of some classes of solutions to (0.2) is motivated by a long standing conjecture of De Giorgi for the monotone solutions of the classical Allen-Cahn equation. See the Introduction to Part II for the statement and the motivation of the conjecture, and also for the most important known results in the topic.

In the present work, we extend the Liouville theorem in [70] from $a = 0$ to all the fractional parameters $a \in (-1,1)$, also considering the wider class of stable solutions to (0.2) instead of monotone solutions. Indeed, we recall that it is a standard fact for this kind of problems that a solution u which is monotone in one direction is also stable, meaning that the second variation of the associated energy is nonnegative definite at u. As a consequence of our result, we obtain the one dimensional symmetry of stable solutions to (0.2) in dimension $n = 2$ for every parameter $a \in (-1, 1)$.

Moreover, we consider the three dimensional case of problem (0.2) with $a \in (-1, 1)$, establishing sharp energy estimates for both the energy minimizers and the monotone solutions to (0.2). These estimates lead to the one-dimensional symmetry of these classes of solutions when $n = 3$ for every $a \in (-1, 1)$, by an application of our Liouville theorem mentioned above. Indeed, both monotone and minimizing solutions are a subclass of stable solutions.

Concerning this nonlocal problem, we also investigate the nature of the operator $\mathcal{L}_a$ defined in (0.3). First, we deduce its expression as a Fourier operator for every $a \in (-1, 1)$, which was previously known only in the case $a = 0$. As a result, we obtain that the Fourier symbol of $\mathcal{L}_a$ is asymptotic to the one of the fractional Laplacian for high frequencies, but it is similar to the symbol of the classical Laplacian for small frequencies. This behavior highlights the mixed nature of the operator $\mathcal{L}_a$, which is nonlocal, but not of purely fractional type, and structurally depends on the fractional parameter a. We further investigate this aspect by studying the Γ-convergence of the energy functional associated to the equation $\mathcal{L}_a u = W'(u)$, where W is a double-well potential.

Specifically, we prove the Γ-convergence of our energy functional to a limit that corresponds to a mere interaction energy when $a \in (0, 1)$ and to the classical perimeter when $a \in (-1, 0]$. In terms of the corresponding fractional parameter $s = (1 - a)/2$, this dichotomy reflects a purely nonlocal behavior when $s \in (0, 1/2)$ and a purely classical asymptotics when $s \in [1/2, 1)$. We point out that the threshold $s = 1/2$ that we obtain here, as well as the Γ-limit behavior for the regime $s \in [1/2, 1)$, is common to other nonlocal problems treated in the literature, but the limit functional that we obtain in the strongly nonlocal regime $s \in (0, 1/2)$ appears to be new and structurally different from other nonlocal energy functionals that have been widely investigated.

The thesis is divided into two parts and each part is divided into two chapters. Each chapter corresponds to an article or a preprint, as follows.

Part I:

- [111] MIRAGLIO, P. Boundedness of stable solutions to nonlinear equations involving the p-Laplacian, preprint, arXiv:1907.13027 (2019);
- [39] CABRÉ, X.; MIRAGLIO, P. Universal Hardy-Sobolev inequalities on hypersurfaces of Euclidean space, *forthcoming*.

Part II:

- [57] CINTI, E.; MIRAGLIO, P.; VALDINOCI, E. One-dimensional symmetry for the solutions of a three-dimensional water wave problem, *J. Geom. Anal.* (2019), DOI 10.1007/s12220-019-00279-z.
- [112] MIRAGLIO, P.; VALDINOCI, E. Energy asymptotics of a Dirichlet to Neumann problem related to water waves, *preprint*, arXiv:1909.02429 (2019).

In addition, Appendix A and part of the Introduction to Part II are taken from the survey

- [75] DIPIERRO, S.; MIRAGLIO, P.; VALDINOCI, E. Symmetry results for the solutions of a partial differential equation arising in water waves, to appear in *2019 MATRIX Annals*, preprint: arXiv:1901.03581 (2019).

Riassunto

Questa tesi si occupa di equazioni differenziali alle derivate parziali di tipo ellittico. È divisa in due parti: la prima riguarda un'equazione nonlineare per il p-Laplaciano, mentre la seconda è incentrata su un problema nonlocale, che può essere formulato per mezzo di un operatore di Dirichlet-Neumann collegato con il Laplaciano frazionario.

Nella prima parte, studiamo la regolarità delle soluzioni stabili dell'equazione nonlineare per il p-Laplaciano

$$- \Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2} \nabla u \right) = f(u) \qquad \text{in } \Omega \subset \mathbb{R}^n, \tag{0.4}$$

dove Ω è un dominio limitato, $p \in (1, +\infty)$ e f è una nonlinearità C^1. Questa equazione è la versione nonlineare dell'equazione semilineare $-\Delta u = f(u)$ in un dominio limitato $\Omega \subset \mathbb{R}^n$, che è stata ampiamente studiata in letteratura. Molto recentemente, Cabré, Figalli, Ros-Oton, e Serra [38] hanno dimostrato che le soluzioni stabili delle equazioni semilineari sono limitate, e quindi regolari, in dimensione $n \leq 9$. Questo risultato è ottimale, dato che esempi di soluzioni illimitate e stabili sono noti in dimensione $n \geq 10$. Inoltre, i risultati in [38] forniscono una risposta completa ad un annoso problema aperto, proposto da Brezis e Vázquez [25], sulla regolarità delle *soluzioni estremali* dell'equazione $-\Delta u = \lambda f(u)$. Queste ultime sono infatti esempi non banali di soluzioni stabili di equazioni semilineari, che possono essere limitate o illimitate in dipendenza della dimensione n, del dominio Ω, e della nonlinearità f.

In questa tesi studiamo la limitatezza delle soluzioni stabili di (0.4), che si congettura essere vera fino alla dimensione $n < p + 4p/(p-1)$. Sono infatti noti esempi di soluzioni stabili e illimitate quando $n \geq p + 4p/(p-1)$, anche quando il dominio è la palla unitaria. Inoltre, nel caso radiale o assumendo ipotesi forti sulla nonlinearità, è stato dimostrato che le soluzioni stabili di (0.4) sono limitate quando $n < p + 4p/(p-1)$.

Nel Capitolo 1 della tesi dimostriamo una nuova stima L^∞ a priori per le soluzioni stabili di (0.4), assumendo una nuova condizione su n e p, che è ottimale nel caso radiale e più restrittiva nel caso generale. Il nostro risultato migliora ciò che è noto in letteratura e ed è il primo esempio di tecnica che produce sia un risultato nel caso non radiale sia il risultato ottimale nel caso radiale. Per ottenere questo risultato estendiamo al caso del p-Laplaciano una tecnica sviluppata da Cabré [30] per il caso classico del problema, con $p = 2$. La strategia si basa su una disuguaglianza di Hardy sugli insiemi di livello della soluzione, combinata con una disuguaglianza di tipo geometrico per le soluzioni stabili di (0.4).

Nella prima parte della tesi ci occupiamo anche di disuguaglianze funzionali di tipo Hardy e Sobolev, su ipersuperfici dello spazio euclideo. Nel fare ciò siamo motivati dalle varie applicazioni di questo tipo di risultati allo studio di stime a priori per le soluzioni stabili, sia nel caso semilineare che nel caso nonlineare.

Come prima cosa, forniamo una dimostrazione semplificata della diguaglianza di Michael-Simon e Allard, che è una disuguaglianza di Sobolev generalizzata su ipersuperfici di $\mathbb{R}^{n+1}$ con un termine di curvatura. Scegliendo un punto di vista geometrico, seguiamo principalmente la dimostrazione di Michael e Simon [110], inserendo una semplificazione presente nel lavoro di Allard [10].

Per quanto riguarda le disuguaglianze di tipo Hardy, in questo lavoro dimostriamo due forme diverse della disuguaglianza di Hardy su ipersuperfici di $\mathbb{R}^{n+1}$, oltre ad una disuguaglianza di Hardy-Poincaré, tutte contenenti un termine di curvatura media. Dimostriamo la nostra prima disuguaglianza di Hardy usando l'integrazione per parti e le derivate tangenti presentate nel libro di Giusti [97]. Inoltre, come conseguenza di questo risultato, otteniamo la disuguaglianza di Hardy sugli insiemi di livello di una funzione, che utilizziamo per dimostrare la limitatezza delle soluzioni stabili e di cui abbiamo parlato in precedenza. Per dedurre la nostra seconda disuguaglianza di Hardy usiamo invece una tecnica nota come sostituzione ground state, che è basata sull'utilizzo di una specifica soluzione positiva dell'equazione di Eulero-Lagrange del funzionale associato alla disuguaglianza. Usando un rifinimento di questa tecnica, combinato con una disuguaglianza di Poincaré con i pesi, otteniamo la nostra disuguaglianza di Hardy-Poincaré su ipersuperfici. È importante sottolineare che la sostituzione ground state è stata ampiamente applicata nell'ambiente di lavoro Euclideo ma, per quanto ci è noto, non è mai stata usata nel contesto delle ipersuperfici di $\mathbb{R}^{n+1}$.

Nella seconda parte di questa tesi ci occupiamo di un problema di Dirichlet - Neumann che emerge da un modello per le onde d'acqua. Considerando una striscia di altezza fissata $\mathbb{R}^n \times [0,1]$, una funzione regolare e limitata u definita su $\mathbb{R}^n$ e un parametro $a \in (-1,1)$, siamo interessati a studiare il sistema di equazioni

$$\begin{cases} \operatorname{div}(y^a \nabla v) = 0 & \text{for} \quad x \in \mathbb{R}^n, y \in (0,1), \\ v_y(x,1) = 0 & \qquad x \in \mathbb{R}^n, y = 1, \\ v(x,0) = u(x) & \qquad x \in \mathbb{R}^n, y = 0, \\ -\lim_{y \to 0} y^a v_y = f(v) & \qquad x \in \mathbb{R}^n, y = 0. \end{cases} \tag{0.5}$$

Questo sistema di equazioni può essere riformulato come un problema nonlocale sulla componente dotata del dato di Dirichlet. Possiamo infatti definire l'operatore di Dirichlet - Neumann $\mathcal{L}_a$ come

$$\mathcal{L}_a u(x) := -\lim_{y \to 0} y^a v_y(x,y), \tag{0.6}$$

e studiare l'equazione nonlocale $\mathcal{L}_a u = f(u)$ in $\mathbb{R}^n$. Questo problema è collegato ad un modello per le onde d'acqua e, ad un limite appropriato, restituisce un operatore di Laplace frazionario.

Il problema (0.5) è stato studiato per la prima volta nel caso $a = 0$ da de la Llave e Valdinoci [70]. Il loro risultato principale è un teorema di Liouville che garantisce la simmetria unidimensionale delle soluzioni monotone, sotto l'ipotesi di avere un controllo sulla crescita dell'energia associata al problema. Come conseguenza di questo risultato, gli autori ottengono la simmetria unidimensionale delle soluzioni monotone del problema (0.5) con $a = 0$ e $n = 2$.

Lo studio della simmetria unidimensionale di alcune classi di soluzioni di (0.5) è motivato da un'importante congettura di De Giorgi, formulata a proposito delle soluzioni monotone dell'equazione di Allen-Cahn — si veda l'Introduzione alla Parte II per la motivazione e l'enunciato della congettura, e anche per i risultati più importanti ottenuti nel campo.

In questo lavoro estendiamo il teorema di Liouville in [70] da $a = 0$ a tutti i parametri frazionari $a \in (-1, 1)$, considerando inoltre la più ampia classe delle soluzioni stabili di (0.5) al posto delle soluzioni monotone. È infatti noto in questo tipo di problemi che le soluzioni monotone siano anche stabili, ovvero che la seconda variazione dell'energia associata sia definita positiva. Come conseguenza di questo risultato, otteniamo la simmetria unidimensionale delle soluzioni stabili di (0.5) in dimensione $n = 2$ per ogni parametro $a \in (-1, 1)$.

Consideriamo inoltre il problema (0.5) nel caso tridimensionale e con $a \in (-1, 1)$, ottenendo stime di energia ottimali sia per le soluzioni che minimizzano l'energia sia per le soluzioni monotone di (0.5). Queste stime conducono alla simmetria unidimensionale di queste classi di soluzioni in dimensione $n = 3$, tramite un'applicazione del nostro teorema di Liouville menzionato in precedenza. Infatti, va ricordato che sia le soluzioni monotone che quelle minimizzanti sono in particolare stabili.

Per quanto riguarda questo problema nonlocale, siamo anche interessati a studiare la natura dell'operatore $\mathcal{L}_a$ definito in (0.6). Deduciamo innanzitutto la sua espressione come operatore di Fourier per ogni $a \in (-1, 1)$, che era nota precedentemente solo per $a = 0$. Da questo risultato otteniamo che il simbolo di Fourier di $\mathcal{L}_a$ è asintotico a quello del Laplaciano frazionario per frequenze alte, ma è simile al simbolo del Laplaciano classico per piccole frequenze. In questo modo si evidenzia la natura mista dell'operatore $\mathcal{L}_a$, che è nonlocale ma non puramente frazionaria. Approfondiamo ulteriormente la comprensione di questo fenomeno attraverso lo studio della Γ-convergenza del funzionale dell'energia associato all'equazione $\mathcal{L}_a u = W'(u)$, dove W è un potenziale a doppio pozzo.

In particolare, dimostriamo la Γ-convergenza del nostro funzionale energia ad un limite che corrisponde a un'energia di pura interazione quando $a \in (0, 1)$ e al perimetro classico quando $a \in (-1, 0]$. Rispetto al corrispondente parametro frazionario $s = (1 - a)/2$, questa dicotomia riflette un comportamento puramente nonlocale quando $s \in (0, 1/2)$ e un'asintotica puramente classica quando $s \in [1/2, 1)$. Evidenziamo che sia il limite $s = 1/2$ ottenuto qui sia il comportamento al Γ-limite nel regime $s \in [1/2, 1)$ sono comuni ad altri problemi nonlocali trattati in letteratura. Al contrario, il funzionale limite che otteniamo nel regime puramente nonlocale $s \in (0, 1/2)$ è nuovo e strutturalmente diverso da altri funzionali di energia nonlocali che sono stati ampiamente studiati.

Resumen

Mi tesis se encaja en el estudio de las Ecuaciones en Derivadas Parciales elípticas. Está dividida en dos partes: la primera trata una ecuación no-lineal con el p-Laplaciano, y la segunda está focalizada en un problema no-local, que puede formularse mediante un operador de Dirichlet-Neumann relacionado con el Laplaciano fraccionario.

En la primera parte de la tesis, estudiamos la regularidad de las soluciones estables de la ecuación no lineal con el p-Laplaciano

$$-\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = f(u) \qquad \text{en } \Omega \subset \mathbb{R}^n, \tag{0.7}$$

donde Ω es un dominio acotado, $p \in (1, +\infty)$ y f es una no-linealidad C^1. Esta ecuacion es la versión no-lineal de la ámpliamente estudiada ecuacion semi-lineal $-\Delta u = f(u)$ en un dominio acotado $\Omega \subset \mathbb{R}^n$. Cabré, Figalli, Ros-Oton, and Serra [38] han demostrado recientemente que las soluciones estables de las ecuaciones son acotadas, y por tanto regulares, en dimensión $n \leq 9$. Este resultado es optimal, dado que son conocidos ejemplos de soluciones no acotadas y estables en dimensión $n \geq 10$. Además, los resultados en [38] dan una respuesta completa a un problema abierto, presentado por Brezis and Vázquez [25] sobre la regularidad de las *soluciones extremales* de la ecuación $-\Delta u = \lambda f(u)$. Estas últimas son, de hecho, ejemplos no-triviales de soluciones estables de ecuaciones semilineales, que pueden ser o no ser acotadas dependiendo de la dimensión n, del dominio Ω, y de la no-linealidad f.

En esta tesis estudiamos la regularidad de las soluciones estables de (0.7), problema que se conjetura de ser cierta cuando $n < p + 4p/(p-1)$ y, de hecho, se conocen ejemplos de soluciones no regulares cuando $n \geq p + 4p/(p-1)$, incluso dentro de la bola unitaria. Además, se ha demostrado que, en el caso radial o assumiendo hipótesis fuertes sobre la no-linealidad, las soluciones estables de (0.7) son acotadas, y por tanto regulares, cuando se satisface que $n < p + 4p/(p-1)$.

En el primer capítulo, demostramos una nueva estimación L^∞ para las soluciones estables de (0.7), bajo una nueva condición en n y p, que es optimal en el caso radial, y más restrictiva en el caso general. Esta investigación mejora conocidos resultados del tema y es el primer ejemplo — contemplando el p-Laplacian — de un método que produce un resultado para el caso general y un resultado optimal en el caso radial. Para obtener este resultado, extendemos al caso del p-Laplaciano la técnica desarrollada por Cabré [30] para el caso clásico del problema, con $p = 2$. La estrategia se basa en una desigualdad de Hardy sobre los conjuntos de nivell de la solución, combinado con una desigualdad de tipo geométrico para las soluciones estables de (0.7).

En la primera parte de la tesis nos ocupamos también de las desigualdades funcionales del tipo Hardy y Sobolev, sobre hipersuperfícies del espacio Euclideo. Nuestra motivación proviene de varias apliaciones que tienen estas desigualdades en el estudio de estimaciones para las soluciones estables, tanto en el caso semilineal como en el no-lineal.

Primeramente, demostramos la conocida desigualdad de Michael-Simon y Allard, que es una desigualdad de Sobolev generalizada sobre la hipersuperfície de $\mathbb{R}^{n+1}$ con un término de curvatura media. Escogiendo un punto de vista geométrico, seguimos principalmente la demostración de Michael y Simon [110], añadiendo una simplifiación presentada en el trabajo de Allard [10].

Por lo que concierne a la desigualdad de tipo Hardy, en este trabajo demostramos dos formas diferentes de la desigualdad de Hardy sobre hipersuperfícies de $\mathbb{R}^{n+1}$, y otra desigualdad de Hardy-Poincaré, todas conteniendo un término de curvatura media. Demostramos nuestra primera desigualdad de tipo Hardy usando la integración por partes y las derivadas tangenciales presentadas en el libro de Giusti [97]. Además, como consecuencia de este resultado, obtenemos también una desigualdad de Hardy sobre los conjuntos de nivel de la función, que necesitamos para probar la acotación de las soluciones estables mencionadas previamente. Para demostrar la segunda desigualdad de Hardy usamos una técnica conocida como *sustitución ground state*, basada en el uso de una solución positiva específica de la ecuación de Euler-Lagrange del funcional asociado a la desigualdad. Sucesivamente, usando un refinamiento de esta técnica combinado con una desigualdad de Poincaré ponderada, obtenemos nuestra desigualdad de Hardy-Poincaré sobre hipersuperfícies. Es importante subrayar que la sustitución ground state ha sido ampliamente aplicada en el espacio euclideo, pero no se conoce en la literatura un utilizo de esta técnica en el contexto de las hipersuperficies $\mathbb{R}^{n+1}$ anterior a nuestro trabajo.

En la segunda parte de esta tesis, nos ocupamos de un problema de Dirichlet - Neumann que emerge de un modelo para las ondas en el agua. Considerando una tira de altura fija $\mathbb{R}^n \times [0,1]$, una función regular y acotada u definida en $\mathbb{R}^n$, y un parámetro $a \in (-1,1)$, nos interesamos en estudiar el siguiente sistema de ecuaciones

$$\begin{cases} \operatorname{div}(y^a \nabla v) = 0 & \text{para} \quad x \in \mathbb{R}^n, y \in (0,1), \\ v_y(x,1) = 0 & \qquad\quad\; x \in \mathbb{R}^n, y = 1, \\ v(x,0) = u(x) & \qquad\quad\; x \in \mathbb{R}^n, y = 0, \\ -\lim\limits_{y \to 0} y^a v_y = f(v) & \qquad\quad\; x \in \mathbb{R}^n, y = 0. \end{cases} \tag{0.8}$$

Este sistema de ecuaciones puede ser reformulado como un problema no-local sobre la componente dotada del dato de Dirichlet. De hecho, podemos definir el operador de Dirichlet-Neumann $\mathcal{L}_a$ como

$$\mathcal{L}_a u(x) := -\lim_{y \to 0} y^a v_y(x,y), \tag{0.9}$$

y estudiar la ecuación no-local $\mathcal{L}_a u = f(u)$ en $\mathbb{R}^n$. Este problema está relacionado con un modelo para las ondas en agua y, en un límite apropiado, recubre un operador de Laplace fraccionario.

El problema (0.8) fue estudiado por primera vez en el caso $a = 0$ por de la Llave y Valdinoci [70]. Su principal resultado es un teorema del tipo Liouville, que garantiza la simetría unidimensional de las soluciones monótonas, bajo la hipótesis de tener un control sobre el crecimiento de la energía asociada al problema. Como consecuencia de este resultado, los autores de [70] obtuvieron la simetría unidimensional de las soluciones monótonas del problema (0.8) con $a = 0$ y $n = 2$.

El estudio de la simetría unidimensional de algunas clases de soluciones de (0.8) está motivado por una importante conjetura de De Giorgi para las soluciones monótonas de la ecuación de Allen-Cahn — ver la Introducción a la Parte II para el enunciado y la motivación de la conjetura, y también para los resultados conocidos en este campo.

En la segunda parte de la tesis, extendemos el teorema de tipo Liouville de [70] desde $a = 0$ para todos los parámetros fraccionarios $a \in (-1, 1)$, considerando también la clase más amplia de soluciones estables de (0.8) en vez de las soluciones monótonas. De hecho, las soluciones monótonas de este tipo de problemas son también estables, es decir, la segunda variación de la energía asociada es definida positiva. Como consecuencia de este resultado, obtenemos la simetria unidimensional de las soluciones estables de (0.8) en dimensión $n = 2$ para cada parámetro $a \in (-1, 1)$.

Además, consideramos también el problema (0.8) en el caso tridimensional y con $a \in (-1, 1)$, obteniendo una estimación de la energía optimal tanto para las soluciones que minimizan la energía como para las soluciones monótonas de (0.8). Estas estimaciones nos conducen a la simetría unidimensional de estas clases de soluciones en dimensión $n = 3$, aplicando nuestro teorema del tipo Liouville mencionado anteriormente. De hecho, ambas las clases de soluciones, las monótonas y las minimales, son una subclase de las soluciones estables.

Relativo a este problema no-local, estamos también interesados en estudiar la naturaleza del operador $\mathcal{L}_a$ definidio en (0.9). Primero, deducimos su expresión como operador de Fourier para cada valor de $a \in (-1, 1)$, que anteriormente solo se conocía para $a = 0$. Este resultado evidencia la naturaleza del operador $\mathcal{L}_a$, que es es no-local pero no puramente fraccionario. De hecho, el símbolo de Fourier de $\mathcal{L}_a$ es asintótico al Laplaciano fraccionario para frecuencias altas y al Laplaciano clasico en el caso de frecuencias bajas. Este comportamiento mixto del operador $\mathcal{L}_a$ está estudiado en profundiad en la tesis a través del estudio de la Γ-convergencia del funcional energía asociado a la ecuación $\mathcal{L}_a u = W'(u)$, donde W es un potencial de tipo "double well".

En particular, demostramos la Γ-convergencia de nuestro funcional de energía a un límite que corresponde a una energía de interacción pura cuando $a \in (0, 1)$ y al perímetro clásico cuando $a \in (-1, 0]$. Respecto al parámetro fraccionario $s = (1 - a)/2$, esta dicotomía refleja un comportamiento puramente no-local cuando $s \in (0, 1/2)$ y una asintótica puramente clásica cuando $s \in [1/2, 1)$. Remarcamos también que el límite $s = 1/2$ obtenido, así como el comportamiento del Γ-límite para el régimen $s \in [1/2, 1)$, es común a otros problemas no-locales tratados en la literatura. Al contrario, el funcional límite que obtenemos en el régimen puramente no-local $s \in (0, 1/2)$ es nuevo y estructuralmente diferente a otros funcionales de energía no-locales que han sido investigados.

Part I

Stable solutions to nonlinear elliptic equations

Introduction to Part I

The first part of this thesis is devoted to the study of the regularity of stable solutions to the nonlinear elliptic equation involving the p-Laplacian

$$-\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = f(u) \qquad \text{in } \Omega \subset \mathbb{R}^n,$$

as well as to the associated Dirichlet problem, under the assumption that Ω is a smooth bounded domain of $\mathbb{R}^n$. Our main result about a priori estimates for stable solutions is based on the application of a Hardy inequality on hypersurfaces of the Euclidean space. Moreover, other recent results concerning the regularity of stable solutions to semilinear or nonlinear elliptic equations have been obtained exploiting similar techniques. Motivated by this, we study some Hardy-Sobolev inequalities on hypersurfaces of the Euclidean space, all of them involving a mean curvature term and having universal constants independent from the hypersurface.

Background and known results

The regularity of the minimizers to elliptic equations is a classical topic in the Calculus of Variations, and more generally in Partial Differential Equations. Considering a PDE problem and the associated energy functional $\mathcal{E}$, a state of the system with lowest energy solves an equality, $\mathcal{E}' = 0$, and an inequality, $\mathcal{E}'' \geq 0$. However, we can consider critical points of the energy which are not necessarily global minimizers, but that have nonnegative second variation of the energy. The solutions to the problem belonging to this class, which of course includes the minimizers of the system, are called *stable solutions*.

For a bounded set $\Omega \subset \mathbb{R}^n$ and a C^1 nonlinearity f, we consider a solution u to the semilinear equation

$$-\Delta u = f(u) \qquad \text{in } \Omega, \tag{I.1}$$

or to the associated Dirichlet problem. If we consider a potential F such that $F' = f$, then (I.1) is the Euler-Lagrange equation for the energy functional

$$\mathcal{E}(u) := \int_\Omega \left(\frac{|\nabla u|^2}{2} - F(u)\right) dx, \tag{I.2}$$

Considering the second variation of the energy functional $\mathcal{E}$, we say that a solution u to (I.1) is *stable* if

$$\int_\Omega |\nabla \xi|^2\, dx - \int_\Omega f'(u)\xi^2\, dx \geq 0 \qquad \text{for every } \xi \in C_c^\infty(\Omega). \tag{I.3}$$

Very recently, Cabré, Figalli, Ros-Oton, and Serra [38] established that stable solutions to the semilinear Dirichlet problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{I.4}$$

are bounded in dimension $n \leq 9$, under some reasonable hypotheses on f, that include $f \geq 0$. Moreover, the regularity of stable solutions can be deduce from their boundedness using the classical theory of elliptic PDEs, and this result is optimal, since it is well-known that

$$\overline{u}(x) = \log |x|^{-2} \text{ solves (I.4) with } f(u) = 2(n-2)e^u \text{ and } \Omega = B_1, \tag{I.5}$$

and $\overline{u}$ is stable if $n \geq 10$.

The result in [38] gives a complete answer to the long-standing open problem of the regularity of stable solutions to semilinear elliptic equations in "low" dimensions. This problem is the PDE counterpart of the regularity of stable minimal[1] hypersurfaces, which is also conjectured to hold in "low" dimensions. Indeed, minimizing minimal hypersurfaces are proved to be smooth up to dimension seven — see [20, 97] — while in $\mathbb{R}^8$ the Simons cone

$$\mathcal{C} := \{x \in \mathbb{R}^8 \text{ s.t. } x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2\}$$

is a minimizing minimal surface with a singularity at the origin — see [20]. Considering the wider class of stable minimal surfaces instead of the minimizers of the area functional, the regularity has been proved to hold only in dimension $n = 3$ by Fisher-Colbrie and Schoen [91], and by do Carmo and Peng [78]. The problem remains open in dimensions $4 \leq n \leq 7$.

Going back to the PDE problem, the study of the regularity of stable solutions to semilinear equations started in the seventies with the seminal paper [61] by Crandall and Rabinowitz. In this work, the authors considered problem (I.4) for some special nonlinearities, including the exponential and the power type ones. This paper gave rise to an extensive literature, part of which is outlined in subsection 1.1.1. We refer to the brilliant monograph [79] for a complete introduction to the problem.

The importance of proving L^∞ a priori estimates for stable solutions to (I.1) has been stressed since the mid-nineties by Brezis, motivated by the study of *extremal solutions* to the semilinear problem (I.4). Indeed, under some suitable hypotheses on f, one can define extremal solutions, which are nontrivial examples of stable solutions to (I.4). To introduce them, we consider a positive parameter λ, and the Dirichlet problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{I.6}$$

Assuming that the nonlinearity f is positive, nondecreasing, and superlinear at infinity, it is proved that there exists a parameter $\lambda^* \in (0, +\infty)$ such that if $\lambda \in (0, \lambda^*)$, then there exists a positive minimal solution u_λ of (I.6) — here *minimal* means the smallest possible solution — while if $\lambda > \lambda^*$ there exists no weak solution to (I.6). In addition, every u_λ is stable, the family $(u_\lambda)_\lambda$ is increasing in λ, and one can define the limit

$$u^* := \lim_{\lambda \to \lambda^*} u_\lambda.$$

[1]Here and throughout the thesis, *minimal* hypersurface refers to a hypersurface which is a critical point (not necessarily a minimizer) of the area functional.

We refer to [79] and the references therein for the proof of these classical results. The function u^* is called the *extremal solution* of problem (I.6) and it is a weak solution to (I.6) with $\lambda = \lambda^*$, in the sense introduced by Brezis et al.[24]. Indeed, we say that $u \in L^1(\Omega)$ is a weak solution of (I.6) if $f(u)\mathrm{dist}(x, \partial\Omega) \in L^1(\Omega)$ and

$$\int_\Omega (u\Delta\varphi + \lambda f(u)\varphi)\; dx = 0 \qquad \text{for every} \quad \varphi \in C_c^2(\overline{\Omega}).$$

Assuming in addition that f is convex, Martel [109] established that u^* is the unique weak solution to (I.6) for $\lambda = \lambda^*$. In the late nineties, Brezis [23], and Brezis and Vázquez [25] posed several open question about the extremal solution, especially concerning its regularity, which can be deduced from its boundedness using the classical theory of elliptic PDEs.

After more than twenty years, in the very recent paper [38] the authors provide a complete answer to two important open questions about extremal solutions. In particular, they establish that $u^* \in W_0^{1,2}(\Omega)$ is a distributional solution to (I.6) with $\lambda = \lambda^*$ in every dimension, and it is a classical solution if $n \leq 9$. This last result is optimal, since explicit examples of unbounded extremal solutions are known if $n \geq 10$.

Choosing a parameter $p \in (1, +\infty)$, instead of the energy functional defined in (I.2) we consider

$$\mathcal{E}(u) := \int_\Omega \left(\frac{|\nabla u|^p}{p} - F(u) \right) dx.$$

Then its Euler-Lagrange equation involves the p-Laplacian and reads

$$-\Delta_p u := \mathrm{div}\left(|\nabla u|^{p-2}\, \nabla u \right) = f(u) \qquad \text{in } \Omega, \tag{I.7}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain and $f = F'$ is a C^1 nonlinearity. Specifically, we can still ask if stable solutions to the nonlinear equation (I.7) or to the associated Dirichlet problem

$$\begin{cases} -\Delta_p u &= f(u) \qquad \text{in } \Omega \\ \quad\;\; u &= 0 \qquad\;\;\; \text{on } \partial\Omega, \end{cases} \tag{I.8}$$

are bounded in "low" dimensions. Clearly, the optimal threshold dimension for the boundedness may depend on p.

A solution u of (I.7) is said to be stable if the second variation of the associated energy functional at u is nonnegative definite, i.e.,

$$\int_{\Omega \cap \{|\nabla u| > 0\}} \left\{ |\nabla u|^{p-2}\, |\nabla \xi|^2 + (p-2)\, |\nabla u|^{p-4}\, (\nabla u \cdot \nabla \xi)^2 \right\} dx \\ - \int_\Omega f'(u)\xi^2\, dx \geq 0,$$

for every $\xi \in \mathcal{T}_u$, where $\mathcal{T}_u$ is a suitable class of test functions introduced in [51, 85] — we refer to the beginning of Chapter 1 for the definition of $\mathcal{T}_u$.

The study of the boundedness of stable solutions to (I.8) has been initiated by Garciá-Azorero, Peral, and Puel [93, 94] considering $f(u) = e^u$. They established that for this choice of the nonlinearity, stable solutions are bounded whenever

$$n < p + \frac{4p}{p-1}, \tag{I.9}$$

proving also that this condition is optimal. Indeed, they showed that whenever $n \geq p + 4p/(p-1)$, the function $u(x) = \log(|x|^{-p})$ is a $W^{1,p}(B_1)$ singular stable solution to (I.8), with $\Omega = B_1$ and $f(u) = p^{p-1}(n-p)e^u$.

Stable solutions to (I.8) are known to be bounded in the optimal dimension range (I.9) also in the case of power-like nonlinearities and arbitrary domains, thanks to the result of Cabré and Sanchón [41], and in the radial case for every locally Lipschitz nonlinearity, as proved by Cabré, Capella, and Sanchón [33].

Concerning general smooth nonlinearities f in the nonradial case, some partial results are available under the following assumptions on the nonlinearity

$$f(0) > 0, \qquad f' \geq 0, \qquad \lim_{t \to +\infty} \frac{f(t)}{t^{p-1}} = +\infty, \tag{I.10}$$

and also that there exists $T \geq 0$ such that

$$(f(t) - f(0))^{\frac{1}{p-1}} \text{ is convex for all } t \geq T. \tag{I.11}$$

Under these hypotheses on f, Sanchón [118, 119] exploited some ideas developed by Nedev [116] for the semilinear problem, and proved that stable solutions are bounded whenever

$$\begin{cases} n < p + \frac{p}{p-1} & \text{and} \qquad p \geq 2; \\ n \leq p + \frac{2p}{p-1}(1 + \sqrt{2-p}) & \text{and} \qquad p \in (1,2). \end{cases}$$

Some years later, Castorina and Sanchón [51] obtained the boundedness of stable solutions in dimension

$$n \leq p + 2 \tag{I.12}$$

for every $p > 1$, assuming (I.10) and (I.11) for some $T \geq 0$. This was done extending the approach of Cabré [28] for $p = 2$.

The study of stable solutions to semilinear and nonlinear elliptic equations is related to the functional inequalities of the Sobolev and Hardy type. To give an example of this connection, we show how the stability in dimension $n \geq 10$ of $\overline{u}(x) = \log |x|^{-2}$ defined in (I.5) can be easily checked using the Hardy inequality on $\mathbb{R}^n$. Indeed, recalling that we assume $f(u) = 2(n-2)e^u$ in (I.5), we have that $\overline{u}$ satisfies the stability inequality (I.3) if and only if

$$2(n-2) \int_\Omega \frac{\xi^2}{|x|^2}\, dx \leq \int_\Omega |\nabla \xi|^2 \, dx \qquad \text{for every } \xi \in C_c^\infty(\Omega).$$

On the other hand, the Hardy inequality for functions with compact support in $\Omega \subset \mathbb{R}^n$ states that for every $\xi \in C_c^\infty(\Omega)$

$$\frac{(n-2)^2}{4} \int_\Omega \frac{|\xi|^2}{|x|^2}\, dx \leq \int_\Omega |\nabla \xi|^2 \, dx,$$

where the constant $(n-2)^2/4$ is sharp and never attained by any function in $C_c^\infty(\Omega)$. As a consequence, $\overline{u}$ is stable if and only if $2(n-2) \leq (n-2)^2/4$, and this forces $n \geq 10$.

Besides this simple application of the Hardy inequality, some a priori estimates for stable solutions to (I.1) or (I.7) have been obtained exploiting Hardy and Sobolev inequalities on hypersurfaces of Euclidean space.

The first result of this kind in due to Cabré [28], and consists of an L^∞ a priori estimate for stable solutions to semilinear equations of the form (I.1) in dimension $n \leq 4$. The proof in [28] relies on the Michael-Simon and Allard inequality, which is a Sobolev inequality on submanifolds of $\mathbb{R}^n$, applied on the level sets of the stable solution.

Even if the regularity of stable solutions to (I.4) has been very recently proved to hold in the optimal dimension range $n \leq 9$ — see [38] — the result in [28] has been the best one available in the topic for nearly a decade. Moreover, in [28] the a priori bound for stable solutions up to dimension 4 is proved for every nonlinearity f smooth enough, and in particular it does not depend on the sign of f. On the contrary, in [38], the nonlinearity is assumed to be nonnegative and this is exploited several times to obtain the sharp result in dimension $n \leq 9$. It is indeed an interesting open problem to establish whether the boundedness of stable solutions to (I.4) in dimension $n \leq 9$ can be proved without the assumption $f \geq 0$. Observe that this holds true in the radial case — see [32].

Very recently, Cabré [30] provided a new proof of the boundedness of stable solutions to semilinear equations up to dimension $n = 4$, based on a new Hardy inequality on the level sets of u. In a unified way with the general case, it gives also an alternative proof of the sharp result for the radial case in dimension $n \leq 9$. As in [28], the only hypotheses in [30] about the nonlinearity concern its regularity, and in particular the nonnegativity of f is not assumed. In Chapter 1, we extend this technique to the context of the p-Laplacian.

Finally, in the nonradial case and for general nonlinearities, the optimal result for the p-Laplacian will be achieved in the forthcoming paper [40] by Cabré, Sanchón, and the author, assuming that $p > 2$ and the domain is strictly convex. Specifically, we will prove that stable solutions are bounded in the optimal dimension range $n < p + 4p/(p-1)$ whenever $p > 2$ and Ω is strictly convex. This will be done by extending to the p-Laplacian framework some of the techniques used in [38]. In Chapter 1, and generally in this thesis, we do not use any idea or method developed in [38].

Results of the thesis (Part I)

In Chapter 1, we consider stable solutions to the nonlinear equation (I.7) involving the p-Laplacian, in a smooth bounded domain $\Omega \subset \mathbb{R}^n$ and for a C^1 nonlinearity f. Our main result establishes an L^∞ a priori estimate for stable solutions which holds for every $f \in C^1$, under a new condition on n and p. It is stated as follows.

Theorem I.1 ([111]). *Let f be any C^1 nonlinearity, $\Omega \subset \mathbb{R}^n$ a smooth bounded domain, $p \in (1, +\infty)$, and u a regular stable solution to* (I.7)*. Assume that*

$$\begin{aligned} & n \geq 4 \quad \text{and} \quad n < \frac{1}{2}\left(\sqrt{(p-1)(p+7)} + p + 5\right); \\ \text{or} \quad & n = 3 \quad \text{and} \quad p < 3. \end{aligned} \tag{I.13}$$

(i) Then, for every $\delta > 0$, we have that

$$\|u\|_{L^\infty(K_\delta)} \leq C\left(\|u\|_{L^1(\Omega)} + \|\nabla u\|_{L^p(\Omega\setminus K_\delta)}\right), \tag{I.14}$$

where

$$K_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq \delta\},$$

and C is a constant depending only on Ω, δ, and p.

(ii) *If in addition Ω is strictly convex, u is a positive solution of the Dirichlet problem* (I.8), *and f is positive, then*

$$\|u\|_{L^\infty(\Omega)} \leq C, \tag{I.15}$$

where C is a constant depending only on Ω, p, f, and $\|u\|_{L^1(\Omega)}$.

(iii) *If Ω is a ball and f is strictly positive in $(0, +\infty)$, then both* (I.14) *and* (I.15) *hold if*

$$\begin{aligned} n \geq 3 \quad &\textit{and} \quad n < p + \frac{4p}{p-1}; \\ n = 2 \quad &\textit{and} \quad p \in (1, 3). \end{aligned}$$

We remark that our a priori estimates (I.14) and (I.15) hold under a condition on n and p, which is optimal only in the radial case for $n \geq 3$, whereas it is more restrictive in the nonradial case. However, our condition (I.13) improves (I.12) for $n \geq 4$, since $p + 2 < (\sqrt{(p-1)(p+7)} + p + 5)/2$.

In Theorem I.1 we prove an interior estimate (I.14) and a global bound in strictly convex domains (I.15). It is worth observing that the former requires no assumptions on the domain, except its regularity and boundedness, nor on the values of u at the boundary of Ω. On the other hand, we prove the global result (I.15) assuming the domain to be strictly convex and u to be a positive solution to the Dirichlet problem (I.8). The reason is that we deduce the global bound (I.15) from a combination of the interior inequality (I.14) with some boundary estimates, which are available only if the domain is strictly convex and the function is a positive solution of the Dirichlet problem (I.8).

Concerning the nonlinearity f, the only hypothesis that we assume to prove the interior bound (I.14) concerns its regularity. Then, we have to assume f to be positive in order to prove the global bound (I.15). In the forthcoming paper [40], we will assume some additional assumptions on f, including $f \geq 0$, to prove the boundedness of stable solutions in the optimal dimension range $n < p + 4p/(p-1)$ when $p > 2$.

Theorem I.1, which is the main result in Chapter 1, is obtained as a consequence of the following proposition. It gives a control over the weighted L^p norm of the gradient of u in Ω, in terms of the L^p norm of the gradient of u in a small neighborhood of the boundary of the domain.

Proposition I.2 ([111]). *Let f be any C^1 nonlinearity, $\Omega \subset \mathbb{R}^n$ a smooth bounded domain, $p \in (1, +\infty)$, and u a regular stable solution to* (I.7). *Let $\alpha \in [0, n-1)$ satisfy*

$$\begin{aligned} 4(n-1-\alpha)^2 &> (\alpha-2)^2(n-1)(p-1) \qquad &\textit{if } n > p; \\ 4(n-1-\alpha)^2 &> (\alpha-2)^2(p-1)^2 \qquad &\textit{if } n \leq p. \end{aligned} \tag{I.16}$$

Then, for all $\delta > 0$ and $y \in K_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq \delta\}$, it holds that

$$\int_\Omega |\nabla u(x)|^p \, |x-y|^{-\alpha} \, dx \leq C \|\nabla u\|^p_{L^p(\Omega \setminus K_\delta)}, \tag{I.17}$$

where C is a constant depending only on Ω, δ, p, and α.

If Ω is a ball and f is strictly positive in $(0, +\infty)$, then (I.17) *holds with $y = 0$ if, instead of* (I.16), *we assume that $\alpha \in [0, n-1)$ satisfies*

$$\begin{aligned} 4(n-1) &> (\alpha-2)^2(p-1) \qquad &\textit{if } n > p; \\ 4(n-1)^2 &> (\alpha-2)^2(p-1)^2 \qquad &\textit{if } n \leq p. \end{aligned}$$

The proof of Proposition I.2 is based on two ingredients: a geometric inequality for stable solutions to (I.7), and a geometric Hardy inequality on the level sets of the stable solution. The former ingredient is due to Farina, Sciunzi, and Valdinoci [84,85] and it is an extension to the p-Laplacian framework of the famous geometric inequality by Sternberg and Zumbrun [131,132] for stable solutions to semilinear equations. Assuming that $p \in (1, +\infty)$, Ω is a smooth bounded domain of $\mathbb{R}^n$, f a C^1 nonlinearity and u a regular stable solution to (I.7), the result in [84,85] states that for every $\eta \in C_c^1(\Omega)$

$$\begin{aligned}\int_{\Omega\cap\{|\nabla u|>0\}} \left((p-1)\,|\nabla u|^{p-2}\,|\nabla_T\,|\nabla u||^2 + |\nabla u|^p\,|A|^2\right)\eta^2\,dx \\ \leq (p-1)\int_\Omega |\nabla u|^p\,|\nabla \eta|^2\,dx,\end{aligned} \tag{I.18}$$

where $|A|^2$ is the square of the second fundamental form of the level sets of u and the tangential gradient ∇_T is referred to the level sets of u as well. These geometric objects are introduced in detail in Chapter 1.

The second ingredient in the proof of Proposition I.2 is a geometric Hardy inequality on the foliation of hypersurfaces given by the level sets of the stable solution u, originally proved by Cabré [30]. To state it, for every point $y \in \mathbb{R}^n$, we consider $r_y = |x-y|$ and the radial derivative $u_{r_y} = \nabla u \cdot (x-y)/r_y$. For every smooth function u, parameter $\alpha \in [0, n-1)$, and $\varphi \in C_c^1(\mathbb{R}^n)$, it states that

$$\begin{aligned}&(n-1-\alpha)\int_\Omega |\nabla u|\,\varphi^2 r_y^{-\alpha}\,dx + \alpha\int_\Omega \frac{u_{r_y}^2}{|\nabla u|}\varphi^2 r_y^{-\alpha}\,dx \\ &\leq \left(\int_\Omega |\nabla u|\,\varphi^2 r_y^{-\alpha}\,dx\right)^{\frac{1}{2}}\left(\int_{\Omega\cap\{|\nabla u|>0\}} |\nabla u|\left(4\,|\nabla_T\varphi|^2 + \varphi^2\,|H|^2\right) r_y^{-\alpha+2}\,dx\right)^{\frac{1}{2}},\end{aligned} \tag{I.19}$$

where the mean curvature H and the tangential gradient ∇_T are both referred to the level sets of u. Recall that by Sard's theorem we know that almost every level set of u is a smooth embedded hypersurface of $\mathbb{R}^n$.

The proof of Proposition I.2 relies on the combination of (I.18) and (I.19). Without going into details, the first key point is a suitable choice of the test function φ in (I.19), that is

$$\varphi = |\nabla u|^{\frac{p-1}{2}}\,\zeta,$$

where ζ is a positive smooth cut-off function. In this way, from (I.19) we obtain an inequality for the weighted L^p norm of ∇u, in which the mean curvature appears in the right-hand side. Then, observing that $H^2 \leq (n-1)\,|A|^2$, we are able to control the right-hand side of this inequality using (I.18), for an appropriate choice of η. Finally, one of the terms in the right-hand side of the resulting inequality can be reabsorbed in the left-hand side, provided that α satisfies (I.16). We refer to Chapter 1 for the details.

In Chapter 2 we investigate some Hardy-Sobolev inequalities on hypersurfaces of Euclidean space, all of them involving the mean curvature and having universal constants independent from the hypersurface. As discussed above, our motivation comes from the various applications of these results to the problem of regularity of stable solutions to semilinear and nonlinear equations. Besides the result in Chapter 1, we refer to the previously mentioned [28,30,51].

Throughout Chapter 2, we consider M to be an n-dimensional C^2 hypersurface of $\mathbb{R}^{n+1}$, with mean curvature H and normal vector ν_M. For every C^1 function φ defined on M we define the tangential gradient of ψ with respect to M as

$$\nabla_T \varphi := \nabla \varphi - (\nabla \varphi \cdot \nu_M)\, \nu_M.$$

The Sobolev inequality in the Euclidean form was proved to hold on minimal hypersurfaces of $\mathbb{R}^{n+1}$, with different constant, by Miranda [113] in 1967. Some years later, a more general Sobolev inequality involving a mean curvature term was proved by Michael and Simon [110], and Allard [10], for k-dimensional submanifolds of the Euclidean space, not necessarily minimal. In the case of hypersurfaces, i.e., submanifolds of codimension one, their result can be stated as follows. Here and throughout the thesis, we denote with $C_c^1(M)$ the space of C^1-functions with compact support on M. In case M is a compact hypersurface without boundary, then $C_c^1(M) = C^1(M)$.

Theorem I.3 (Michael-Simon [110], Allard [10])**.** *Let M be a smooth n-dimensional hypersurface of $\mathbb{R}^{n+1}$, and E be a smooth domain with compact closure in M. Then,*

$$|E|^{\frac{n-1}{n}} \leq C\Big(\mathrm{Per}(E) + \int_E |H|\, dV\Big), \tag{I.20}$$

where H is the mean curvature of M, $\mathrm{Per}(E)$ is the perimeter of E, and C is a constant depending only on the dimension n of M.

Now, let $p \in [1, n)$ and $\varphi \in C_c^1(M)$. Then, there exists a positive constant C depending only on n and p such that

$$\|\varphi\|^p_{L^{p^*}(M)} \leq C \int_M \big(|\nabla_T \varphi|^p + |H\varphi|^p \big)\, dV, \tag{I.21}$$

where $p^* = np/(n-p)$ is the Sobolev exponent and ∇_T denotes the tangential gradient to M.

It is important to stress that the constants appearing in the right-hand side of (I.20) and (I.21) only depend on n and p, and not on the hypersurface M. Indeed, the geometry of M is involved only in the mean curvature H. In particular, when M is minimal, such term vanishes and we recover the Sobolev inequality proved by Miranda [113].

In Chapter 2, we give a quick and easy to read proof of the isoperimetric inequality (I.20), in the context of hypersurfaces of $\mathbb{R}^{n+1}$. More precisely, our proof follows mainly the strategy of Michael and Simon [110], but with a simplification given by Allard [10]. Then, we deduce (I.21) from (I.20) for real valued functions defined on M using a standard technique, that we include for the sake of completeness.

In the same chapter, we also investigate Hardy inequalities on hypersurfaces of Euclidean space, obtaining two new forms of the inequality and an improved Hardy inequality in the sense of Poincaré.

Our first result in this field is strictly related to the foliated Hardy inequality (I.19) proved by Cabré [30], and that we exploit in the proof of Theorem I.1. Indeed, one can deduce[2] from (I.19) the following Hardy inequality for a smooth hypersurface M of $\mathbb{R}^{n+1}$ — see the Introduction to Chapter 2 for more details. Here and throughout the thesis, $C_c^1(M)$ denotes the space of the C^1 functions with compact support on M. In case M is compact without boundary, then $C_c^1(M) = C^1(M)$.

[2]Here we are considering n-dimensional hypersurfaces of $\mathbb{R}^{n+1}$, while in (I.19) the level sets of u are $(n-1)$-dimensional hypersurfaces of the ambient space $\mathbb{R}^n$. Thus, we apply here (I.19) with n instead of $n-1$.

Theorem I.4 ([39]). *Let M be a smooth n-dimensional hypersurface of $\mathbb{R}^{n+1}$, and $a \in [0, n)$. Then, for every $\varphi \in C^1_c(M)$*

$$\begin{aligned}(n-a)\int_M \frac{\varphi^2}{|x|^a}\,dV + a\int_M \left(\frac{x}{|x|}\cdot \nu_M\right)^2 \frac{\varphi^2}{|x|^a}dV \\ \leq \left(\int_M \frac{\varphi^2}{|x|^a}\,dV\right)^{\frac{1}{2}} \left(\int_M \left(4\,|\nabla_T \varphi|^2 + |H\varphi|^2\right)|x|^{2-a}\,dV\right)^{\frac{1}{2}},\end{aligned} \tag{I.22}$$

where ν_M is the unit normal normal to M in $\mathbb{R}^{n+1}$.

In Chapter 2, we present a direct proof of Theorem I.4, not relying on the proof of (I.19) in [30]. Our proof is based on integration by parts and the use of tangential derivatives as presented in Giusti's book [97]. Then, using the coarea formula, from (I.22) we deduce inequality (I.19) on the level sets of stable solutions. Moreover, we prove a version of (I.22), and thus of (I.19), for an exponent $p \geq 1$, and not only for $p = 2$. Our general result is stated in Theorem 2.3.1, and the version of (I.19) for an exponent $p \geq 1$ is deduced in Corollary 2.3.2.

We point out that, as in the case of the Michael-Simon and Allard inequality, all the constants appearing in (I.22) do not depend on M, but only on the dimension n and the parameter a. Moreover, if we consider M to be a minimal hypersurface, then $H = 0$ and we obtain the Hardy inequality with the Euclidean sharp constant. In the case of $M = \mathbb{R}^n$ for every $\varphi \in C^1_c(\mathbb{R}^n)$ the inequality reads

$$\frac{(n-2)^2}{4}\int_{\mathbb{R}^n} \frac{\varphi^2}{|x|^2}\,dx \leq \int_{\mathbb{R}^n} |\nabla \varphi|^2\,dx. \tag{I.23}$$

Indeed, we recall that $(n-2)^2/4$ is known to be the best constant in (I.23), and the equality is not attained by any function $\varphi \in C^1_c(\mathbb{R}^n)$.

Knowing the best constant in this kind of inequalities is important for the applications to PDEs, as for instance in Theorem I.1 or in [30]. In addition, if u in (I.19) is a radial function, then $u_r^2 = |\nabla u|^2$, and we can add the two terms in the left-hand side of (I.19). Thus, the constant in front of the first integral in the left-hand side of (I.19) becomes $n-1$ instead of $n-1-a$. This fact is important in the application to nonlinear elliptic equations and, in particular, it gives a better result when we consider radial solutions — see Theorem I.1 and [30].

A related but different Hardy inequality on manifolds was proved by Carron [49] in 1997. Considering a smooth hypersurface M of $\mathbb{R}^{n+1}$ with $n \geq 3$, for every $\varphi \in C^1_c(M)$ it states that

$$\frac{(n-2)^2}{4}\int_M \frac{\varphi^2}{|x|^2}\,dV \leq \int_M \left(|\nabla_T \varphi|^2 + \frac{n-2}{2}\frac{|H|\,\varphi^2}{|x|}\right)dV. \tag{I.24}$$

This paper by Carron [49] gave rise to several articles about Hardy inequalities on manifolds, some of which are commented on in Chapter 2. We improve (I.24) in the context of the hypersurfaces of $\mathbb{R}^{n+1}$, by adding an extra term in its left-hand side. Moreover, our proof is new and totally different from the one in [49]. Our result is stated as follows.

Theorem I.5 ([39]). *Let M be a smooth n-dimensional hypersurface of $\mathbb{R}^{n+1}$, with $n \geq 3$, and $\varphi \in C_c^1(M)$. Then,*

$$\begin{aligned}\frac{(n-2)^2}{4} \int_M \frac{\varphi^2}{|x|^2}\, dV + \frac{n^2-4}{4} \int_M \left(\frac{x}{|x|} \cdot \nu_M\right)^2 \frac{\varphi^2}{|x|^2}\, dV \\ \leq \int_M \left(|\nabla_T \varphi|^2 + \frac{n-2}{2} \frac{|H|\, \varphi^2}{|x|}\right) dV,\end{aligned} \tag{I.25}$$

where ν_M is the unit normal normal to M in $\mathbb{R}^{n+1}$.

We prove Theorem I.5 using the ground state substitution, a technique that is completely different from the one used by Carron in [49]. In short, the ground state substitution consists of writing the function φ as $\varphi = v\omega$, where ω is a positive solution of the Euler-Lagrange equation of the energy functional associated with the inequality. More precisely, we take $\omega(x) = |x|^{-(n-2)/2}$. In Euclidean space, this technique has been applied by Brezis and Vázquez [25] to obtain an improved Hardy inequality. More recently, Frank and Seiringer [92] used it to prove both classical and fractional Hardy inequalities on the Euclidean space. We could not find in the literature the use of this method in the context of hypersurfaces of $\mathbb{R}^{n+1}$.

Even if we take $a = 2$ in (I.22), the two Hardy inequalities on hypersurfaces stated in Theorems I.4 and I.5 are different in their formulation, and completely independent in their proof. Their statements differ mainly in the mean curvature terms, as we have $|H|^2$ in (I.22) and $|H| / |x|$ in (I.25), also with different constants in front. Finally, our proof of Theorem I.4 works for every exponent $p \geq 1$ — see Theorem 2.3.1 for the general statement — while our proof of Theorem I.5 gives a meaningful result only for $p = 2$. This aspect is further commented on in Chapter 2.

We observe that our Theorem I.5 improves Carron's inequality (I.24) in the setting of hypersurfaces of $\mathbb{R}^{n+1}$, adding an extra term in the left-hand side of the inequality, which is the same term appearing also in (I.22). However, when M is a minimal hypersurface — thus in particular for $M = \mathbb{R}^n$ — both Carron's inequality (I.25) and our Theorems I.4 and (I.5) give the Hardy inequality with sharp Euclidean constant.

In the Euclidean setting several improvements of the Hardy inequality are provided, most of them consisting of adding a positive term in the left-hand side of (I.23). Clearly, this additional term has to be of lower order than the Hardy integral, by the sharpness of the constant $(n-2)^2/4$. For instance, Brezis and Vázquez [25] improved (I.23) in the Poincaré sense, i.e., by adding the L^2-norm of the function φ in the left-hand side of the inequality. For any bounded domain $\Omega \subset \mathbb{R}^n$, any dimension $n \geq 2$ and for every function $\varphi \in H_0^1(\Omega)$, their result states that

$$\frac{(n-2)^2}{4} \int_\Omega \frac{\varphi^2}{|x|^2}\, dx + \lambda_2 \left(\frac{\omega_n}{|\Omega|}\right)^{\frac{2}{n}} \int_\Omega \varphi^2\, dx \leq \int_\Omega |\nabla \varphi|^2\, dx, \tag{I.26}$$

where λ_2 is the first eigenvalue of the Laplacian in the unit ball of $\mathbb{R}^2$, hence positive and independent of n. Our last result in this part of the thesis is an analogue of (I.26) on hypersurfaces of $\mathbb{R}^{n+1}$. It states as follows.

Theorem I.6 ([39]). *Let M be a smooth n-dimensional hypersurface of $\mathbb{R}^{n+1}$, with $n \geq 2$, and $B_r = B_r(0) \subset \mathbb{R}^{n+1}$ be the open ball of radius r centered at the origin. Then, for every*

$\varphi \in C_c^1(B_r \cap M)$ we have

$$\begin{aligned} \frac{(n-2)^2}{4}\int_M \frac{\varphi^2}{|x|^2}\,dV + \frac{n^2-4}{4}\int_M \left(\frac{x}{|x|}\cdot \nu_M\right)^2 \frac{\varphi^2}{|x|^2}\,dV + \frac{1}{2r^2}\int_M \varphi^2\,dV \\ \le \int_M \left(|\nabla_T \varphi|^2 + \frac{n-2}{2}\,\frac{|H|\,\varphi^2}{|x|} + \frac{1}{4}\,|H\varphi|^2\right)\,dV, \end{aligned} \tag{I.27}$$

where ν_M is the unit normal normal to M in $\mathbb{R}^{n+1}$.

We prove this result exploiting again the ground state substitution, combining the proof of Theorem I.5 with a Poincaré inequality in hypersurfaces of $\mathbb{R}^{n+1}$, that we state in Proposition 2.4.2. The former argument brings the first mean curvature term in (I.27), while the latter brings the second one. Observe that they are the same curvature terms that appear in (I.22) and (I.25).

Chapter 1

Boundedness of stable solutions to nonlinear equations

We consider stable solutions to the equation $-\Delta_p u = f(u)$ in a smooth bounded domain $\Omega \subset \mathbb{R}^n$ for a C^1 nonlinearity f. Either in the radial case, or for some model nonlinearities f in a general domain, stable solutions are known to be bounded in the optimal dimension range $n < p + 4p/(p-1)$. In this chapter, under a new condition on n and p, we establish an L^∞ a priori estimate for stable solutions which holds for every $f \in C^1$. Our condition is optimal in the radial case for $n \geq 3$, whereas it is more restrictive in the nonradial case. This work improves the known results in the topic and gives a unified proof for the radial and the nonradial cases.

The existence of an L^∞ bound for stable solutions holding for all C^1 nonlinearities when $n < p + 4p/(p-1)$ has been an open problem over the last twenty years. The forthcoming paper [40] by Cabré, Sanchón, and the author will solve it when $p > 2$.

1.1 Introduction

For a smooth bounded domain $\Omega \subset \mathbb{R}^n$, a C^1 nonlinearity f and for every $p \in (1, +\infty)$, we consider the elliptic equation involving the p-Laplacian

$$-\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f(u) \qquad \text{in } \Omega, \tag{1.1.1}$$

and the associated Dirichlet problem

$$\begin{cases} -\Delta_p u &= f(u) & \text{in } \Omega \\ u &> 0 & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1.2}$$

Solutions $u \in W^{1,p}(\Omega)$ to equation (1.1.1) correspond to critical points of the functional

$$\mathcal{E}(u) := \int_\Omega \left(\frac{1}{p} |\nabla u|^p - F(u) \right) dx,$$

where $F'(t) = f(t)$, and the boundary condition in (1.1.2) is intended in the weak sense as $u \in W_0^{1,p}(\Omega)$. Stable solutions to (1.1.1) are those for which the second variation of energy $\mathcal{E}$ is nonnegative. More precisely, a solution $u \in C^1(\overline{\Omega})$ to (1.1.1) is said to be stable if

$$\int_{\Omega\cap\{|\nabla u|>0\}} \left\{ |\nabla u|^{p-2} |\nabla \xi|^2 + (p-2) |\nabla u|^{p-4} (\nabla u \cdot \nabla \xi)^2 \right\} dx \\ - \int_\Omega f'(u)\xi^2 \, dx \geq 0$$

for every $\xi \in \mathcal{T}_u$, defined in [51,85] as

$$\mathcal{T}_u := \begin{cases} W^{1,2}_{\sigma,0}(\Omega) & \text{if } p \geq 2; \\ \{\xi \in W^{1,2}_0(\Omega) : \ \|\nabla \xi\|_{L^2_\sigma(\Omega)} < \infty\} & \text{if } p \in (1,2). \end{cases}$$

Here and throughout the chapter, $\|\cdot\|_{L^2_\sigma(\Omega)}$ is the weighted $L^2(\Omega)$ norm with weight $\sigma = |\nabla u|^{p-2}$, and $W^{1,2}_{\sigma,0}(\Omega)$ is defined as the completion of $C^1_c(\Omega)$ with respect to the norm

$$\begin{aligned} \|\xi\|_{W^{1,2}_\sigma(\Omega)} &:= \|\xi\|_{L^2(\Omega)} + \|\nabla \xi\|_{L^2_\sigma(\Omega)} \\ &= \left(\int_\Omega \xi^2 \, dx \right)^{\frac{1}{2}} + \left(\int_\Omega |\nabla u|^{p-2} |\nabla \xi|^2 \, dx \right)^{\frac{1}{2}}. \end{aligned}$$

See the beginning of section 4 in [51] for more details about the class $\mathcal{T}_u$ of test functions. In short, it is important to stress that, with this definition, $\mathcal{T}_u$ is a Hilbert space. The difference in defining the class is due to the fact that if $p \geq 2$ then $\sigma \in L^\infty(\Omega)$, while this is not true when $p \in (1,2)$.

We say that u is a *regular solution* to (1.1.2) if it solves the equation in the distributional sense and $f(u) \in L^\infty(\Omega)$. Every regular solution is proved to be $C^{1,\beta}(\overline{\Omega})$ — see [72,106, 134] — and this is the best regularity that one can expect for solutions to nonlinear equations involving the p-Laplacian.

In this chapter we focus on the boundedness of stable solutions to (1.1.1), or to the associated Dirichlet problem (1.1.2), for general nonlinearities $f \in C^1$. The importance of this problem for the classical Laplacian — when $p = 2$ — has been stressed by Haïm Brezis since the mid-nineties — see [23,25]. Very recently, it has been completely solved by Cabré, Figalli, Ros-Oton, and Serra [38], proving that stable solutions are bounded whenever $n \leq 9$. This result is indeed optimal, since explicit examples of unbounded stable solutions to (1.1.1) with $p = 2$ are well-known when $n \geq 10$.

The boundedness of stable solutions to (1.1.2) is conjectured to hold under the assumption $n < p + 4p/(p-1)$. In fact, when $n \geq p + 4p/(p-1)$, Ω is a ball and $f(u) = e^u$, García Azorero, Peral, and Puel showed in [94] the existence of an unbounded stable solution to (1.1.2). On the other hand, considering radial solutions to (1.1.2) in a ball, Cabré, Capella, and Sanchón proved in [33] the boundedness of stable solutions, provided that $n < p + 4p/(p-1)$. In the nonradial case and for general nonlinearities, the optimal result will be achieved in the forthcoming paper [40] by Cabré, Sanchón, and the author, assuming that $p > 2$ and the domain is strictly convex. This is done extending to the p-Laplacian framework some of the techniques used in [38]. In the present work we do not use any idea or method developed in [38]. The papers [33,40,94] are part of an extensive literature on the topic, which is outlined in subsection 1.1.1.

The aim of the present chapter is to provide L^∞ a priori bounds for stable solutions to (1.1.2) under a certain condition over n and p. In the nonradial case, our condition over n and p is not optimal, but it improves the known results in the field. In the radial case, our proof gives for $n \geq 3$ the optimal result in [33] in an unified way with the one in general domains. Furthermore, our technique is based on a geometric Hardy inequality on the level sets of the stable solution. This approach — that we explain below in detail — has been introduced by Cabré in [30] for the classical version $p = 2$ of the problem and it has never been used before in the context of the p-Laplacian.

1.1.1 Available results

Let us describe first the large literature for the classical case $p = 2$, and then list the most important results for problem (1.1.2) with $p \in (1, +\infty)$.

The first paper about this topic is by Crandall and Rabinowitz in 1975 [61], in which they study problem (1.1.2) with $p = 2$ for smooth nonlinearities f satisfying

$$f(0) > 0, \qquad f' \geq 0, \qquad \lim_{t \to +\infty} \frac{f(t)}{t} = +\infty. \tag{1.1.3}$$

These assumptions are verified for instance by exponential and power-type nonlinearities, as discussed in [61].

Assuming that f satisfies (1.1.3), we can introduce extremal solutions, which are nontrivial examples of stable solutions to (1.1.2), sometimes unbounded. In order to define them in the classical case, let us consider a positive parameter $\lambda > 0$ and the Dirichlet problem

$$\begin{cases} -\Delta u &= \lambda f(u) \qquad \text{in } \Omega \\ \quad u &= 0 \qquad\quad\;\; \text{on } \partial\Omega. \end{cases} \tag{1.1.4}$$

It is known the existence of an extremal parameter $\lambda^* \in (0, +\infty)$ such that if $\lambda \in (0, \lambda^*)$, then problem (1.1.4) admits a regular solution u_λ which is minimal, while if $\lambda > \lambda^*$ then it admits no regular solution. In addition, the family $\{u_\lambda\}$ is increasing in λ, every u_λ is stable, and one can define the limit

$$u^* := \lim_{\lambda \to \lambda^*} u_\lambda. \tag{1.1.5}$$

The function u^* is a weak[1] solution of (1.1.4) with $\lambda = \lambda^*$ and it is stable. Assuming also that f is convex, u^* is the unique weak solution to (1.1.4) for $\lambda = \lambda^*$. It is called the *extremal solution* of problem (1.1.4) and its boundedness depends on the dimension, the domain and the nonlinearity. In [25] the authors raised several open question about the extremal solution, especially about its regularity, which can be deduced from its boundedness using classical tools in the theory of elliptic PDEs — see also the open problems raised by Brezis in [23].

When $f(u) = e^u$, Crandall and Rabinowitz prove in [61] that $u^* \in L^\infty(\Omega)$ if $n \leq 9$, while $u^*(x) = \log |x|^{-2}$ when $\Omega = B_1$ and $n \geq 10$ — see [103]. Similar results hold for $f(u) = (1 + u)^m$, and also for functions f such that

$$\lim_{t \to +\infty} \frac{f(t) f''(t)}{f'(t)} \qquad \text{exists,} \tag{1.1.6}$$

[1] In the sense introduced by Brezis et al.[24]: $u \in L^1(\Omega)$ is a weak solution of (1.1.4) if $f(u)\mathrm{dist}(x, \partial\Omega) \in L^1(\Omega)$ and

$$\int_\Omega (u \Delta\varphi + \lambda f(u) \varphi)\, dx = 0$$

for every $\varphi \in C^2_0(\overline{\Omega})$.

as proved also in [61].

We will describe now some L^∞ a priori estimates which have been proved for the smooth stable solutions u_λ to (1.1.4) with $\lambda < \lambda^*$, under different assumptions on f. The estimates are uniform in λ and they led, by letting $\lambda \nearrow \lambda^*$, to the boundedness of the extremal solution. Since the proofs work for every smooth stable solution to (1.1.2) with $p = 2$ under the same assumptions on f, we describe here the results in the framework of stable solutions to (1.1.2) with $p = 2$.

Nedev obtained in [116] an L^∞ bound for stable solutions in dimensions $n = 2, 3$, under the hypothesis that f is convex and satisfies (1.1.3). Some years later, Cabré and Capella [32] solved the radial case for every Lipschitz nonlinearity, proving the boundedness of stable solutions when $\Omega = B_1$ and $n \leq 9$.

In 2010 Cabré [28] proved that in dimensions $n \leq 4$ stable solutions are bounded in every convex domain and for every C^1 nonlinearity. A few years later, Villegas [138] removed the convexity hypothesis about Ω when $n = 4$, by further assuming that f is convex. Its proof uses both the results in [28] and [116],

The proof in [28] is rather delicate and it is based on the Michael-Simon and Allard inequality on the level sets of a stable solution u. The same result has been proved very recently in [30] also by Cabré, using this time a Hardy inequality on the level sets of u. This new method is not only simpler, but it also gives a unified proof of the radial case — in the optimal dimension range $n \leq 9$ — and of the nonradial case if $n = 3, 4$, obtaining boundedness of stable solutions to (1.1.2) with $p = 2$ when Ω is convex.

In [28, 30], the L^∞ a priori bounds for stable solutions are obtained through an estimate in the interior of the domain combined later with some estimates near the boundary. The interior bounds hold for every regular domain Ω and do not depend on the values of the stable solutions at the boundary. On the contrary, in order to have boundary estimates, the author needs to consider stable solutions to the Dirichlet problem (1.1.2) with $p = 2$ and also to assume the convexity of Ω. In the present chapter, we follow the strategy of the second paper, [30], extending it to the case of the p-Laplacian.

As we mentioned above, very recently Cabré, Figalli, Ros-Oton, and Serra [38] settled the problem, proving that stable solutions are bounded in dimension $n \leq 9$. The interior regularity applies to every nonnegative f, while the global result requires f to be nondecreasing and convex. This was done by the authors using new and different ideas from the ones in [28, 30]. For more details about the classical problem for the Laplacian we refer to the recent survey [29] and to the book [79].

Before outlining in detail our results, we comment on what is known about the boundedness of stable solutions for the p-Laplacian. Let us start by describing the extremal solutions for the problem

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1.7}$$

where λ is a positive parameter and f a C^1 nonlinearity. Under the assumptions

$$f(0) > 0, \qquad f' \geq 0, \qquad \lim_{t \to +\infty} \frac{f(t)}{t^{p-1}} = +\infty, \tag{1.1.8}$$

there exists an extremal parameter $\lambda^* \in (0, +\infty)$ such that if $\lambda \in (0, \lambda^*)$, then problem (1.1.7) admits a minimal regular solution u_λ, while if $\lambda > \lambda^*$ then it admits no regular solution. Furthermore, the family $\{u_\lambda\}$ is increasing in λ, every u_λ is stable and we can define u^* as in (1.1.5) — see [41] for these results about the extremal problem for the p-Laplacian.

For $p \neq 2$ and $f \in C^1$ satisfying (1.1.8), it is not known in general whether u^* is a distributional solution of (1.1.7) with $\lambda = \lambda^*$. However, when f is the exponential nonlinearity or it satisfies some strong assumptions — see [41, 93, 94, 119] — it has been proved that u^* is a distributional solution to (1.1.7) with $\lambda = \lambda^*$. In this cases, it is called the *extremal solution* of problem (1.1.7).

As in the classical case, also for $p > 1$ the integrability and regularity properties of u^* are obtained as a consequence of uniform estimates for the stable branch $\{u_\lambda\}$.

García Azorero, Peral, and Puel treated the exponential nonlinearity $f(u) = e^u$ for $p > 1$ in [93, 94]. They established the boundedness of stable solutions when

$$n < p + \frac{4p}{p-1}, \tag{1.1.9}$$

and showed that this condition is optimal. Indeed, they provided an example of unbounded stable solution to (1.1.2) with $f(u) = e^u$, $\Omega = B_1$ and $n \geq p + 4p/(p-1)$.

Some years later, Sanchón proved in [118] that stable solutions are bounded in the optimal dimension range (1.1.9), under the hypothesis that $f \in C^2$ is an increasing function, it satisfies (1.1.8) and also the strong assumption (1.1.6) on the behavior of f at infinity. The same result is obtained by Cabré and Sanchón in [41], assuming that the nonlinearity satisfies (1.1.8) and the power growth hypothesis $f(t) \leq c(1+t)^m$, where m is smaller than a "Joseph-Lundgren type" exponent which is optimal for the regularity of stable solutions.

As we mentioned above, the radial case of problem (1.1.2) was settled by Cabré, Capella, and Sanchón in [33] for every locally Lipschitz nonlinearity. Indeed, under this assumption they proved that radial stable solutions are bounded in the optimal range $n < p + 4p/(p-1)$.

Back to the nonradial case, the following works deal with general nonlinearities satisfying essentially (1.1.8). They are also the most recent results in the topic.

Sanchón in [118, 119] considers nonlinearities f that satisfy (1.1.8) and

$$\text{there exists } T \geq 0 \text{ s.t. } (f(t) - f(0))^{\frac{1}{p-1}} \text{ is convex for all } t \geq T. \tag{1.1.10}$$

Observe that when $p = 2$ this last condition becomes the standard convexity assumption on f made in [25] and appearing also in the recent paper [38].

In [118, 119] it is proved the boundedness of stable solutions whenever

$$\begin{cases} n < p + \frac{p}{p-1} & \text{and} \quad p \geq 2; \\ n \leq p + \frac{2p}{p-1}(1 + \sqrt{2-p}) & \text{and} \quad p \in (1,2). \end{cases}$$

Both results are obtained following the approach of Nedev in [116] for $p = 2$. Later, Castorina and Sanchón [51] extended Cabré's method in [28] for $p = 2$ to the case of the p-Laplacian, proving that stable solutions are bounded in the range

$$n \leq p + 2 \tag{1.1.11}$$

under the assumption that f is C^1, and satisfies (1.1.8) and (1.1.10).

In the forthcoming paper [40] by Cabré, Sanchón, and the author, the interior results in [38] for $p = 2$ will be extended to the case of the p-Laplacian. In particular, we will prove that stable solutions are bounded in the optimal dimension range $n < p + 4p/(p-1)$ whenever $p > 2$ and Ω is strictly convex.

1.1.2 New results and strategy of the proof

Theorem 1.1.1 below is the main result of the present chapter. It establishes, under a new condition on n and p, an L^∞ a priori estimate for stable solutions for every C^1 nonlinearity. This condition improves the one in [51], (1.1.11), when $n \geq 4$ and $p > 2$, even though it is not optimal.

Our result consists of an interior estimate for stable solutions which does not depend on the boundary values of the function and holds for every C^1 nonlinearity and every bounded domain — see (1.1.13) below. Up to our knowledge, ours is the first result of this kind for stable solutions to (1.1.1) in the setting of the p-Laplacian.

This interior estimate leads to a global L^∞ estimate under the further assumption that the domain is strictly convex and that u is a stable solution of the Dirichlet problem (1.1.2), and not only of equation (1.1.1) — see (1.1.14) below.

Theorem 1.1.1. *Let f be any C^1 nonlinearity, $\Omega \subset \mathbb{R}^n$ a smooth bounded domain, $p \in (1, +\infty)$, and u a regular stable solution to* (1.1.1). *Assume that*

$$\begin{aligned} & n \geq 4 \quad \textit{and} \quad n < \frac{1}{2}\left(\sqrt{(p-1)(p+7)} + p + 5\right); \\ \textit{or} \quad & n = 3 \quad \textit{and} \quad p < 3. \end{aligned} \tag{1.1.12}$$

(i) Then, for every $\delta > 0$, we have that

$$\|u\|_{L^\infty(K_\delta)} \leq C\left(\|u\|_{L^1(\Omega)} + \|\nabla u\|_{L^p(\Omega \setminus K_\delta)}\right), \tag{1.1.13}$$

where

$$K_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq \delta\},$$

and C is a constant depending only on Ω, δ, and p.

(ii) If in addition Ω is strictly convex, u is a positive solution of the Dirichlet problem (1.1.2), *and f is positive, then*

$$\|u\|_{L^\infty(\Omega)} \leq C, \tag{1.1.14}$$

where C is a constant depending only on Ω, p, f, and $\|u\|_{L^1(\Omega)}$.

(iii) If Ω is a ball and f is strictly positive in $(0, +\infty)$, then both (1.1.13) *and* (1.1.14) *hold if*

$$\begin{aligned} & n \geq 3 \quad \textit{and} \quad n < p + \frac{4p}{p-1}; \\ & n = 2 \quad \textit{and} \quad p \in (1, 3). \end{aligned} \tag{1.1.15}$$

Remark 1.1.2. We point out that condition (1.1.12) forces $p > 2$ for $n \geq 5$, and $p > 4/3$ for $n = 4$. Furthermore, our condition (1.1.12) improves (1.1.11) for $n \geq 4$, since $p + 2 < (\sqrt{(p-1)(p+7)} + p + 5)/2$.

The interior estimate (1.1.13) does not require any assumption on the values of u at the boundary of Ω, nor the strict convexity of the domain. On the other hand, passing from (1.1.13) to the global bound (1.1.14) requires some boundary estimates, which are available if we assume that the domain is strictly convex, u is a positive stable solution to the Dirichlet problem (1.1.2), and f is positive. We will introduce the boundary estimates in Section 1.2, before the proof of Theorem 1.1.1.

Our main result Theorem 1.1.1 is obtained as a consequence of the following proposition. It is an estimate of the weighted L^p norm of the gradient of u in Ω, being controlled by the L^p norm of the gradient of u in a small neighborhood of the boundary of the domain.

Proposition 1.1.3. *Let f be any C^1 nonlinearity, $\Omega \subset \mathbb{R}^n$ a smooth bounded domain, $p \in (1, +\infty)$, and u a regular stable solution to* (1.1.1)*. Let $\alpha \in [0, n-1)$ satisfy*

$$\begin{aligned} &4(n-1-\alpha)^2 > (\alpha-2)^2(n-1)(p-1) && \text{if } n > p; \\ &4(n-1-\alpha)^2 > (\alpha-2)^2(p-1)^2 && \text{if } n \leq p. \end{aligned} \tag{1.1.16}$$

Then, for all $\delta > 0$ and $y \in K_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq \delta\}$, it holds that

$$\int_\Omega |\nabla u(x)|^p \, |x-y|^{-\alpha} \, dx \leq C \|\nabla u\|^p_{L^p(\Omega \setminus K_\delta)}, \tag{1.1.17}$$

where C is a constant depending only on Ω, δ, p, and α.

If Ω is a ball and f is strictly positive in $(0, +\infty)$, then (1.1.17) *holds with $y = 0$ if, instead of* (1.1.16)*, we assume that $\alpha \in [0, n-1)$ satisfies*

$$\begin{aligned} &4(n-1) > (\alpha-2)^2(p-1) && \text{if } n > p; \\ &4(n-1)^2 > (\alpha-2)^2(p-1)^2 && \text{if } n \leq p. \end{aligned} \tag{1.1.18}$$

As we mentioned above, in order to prove Proposition 1.1.3 we follow the strategy used in [30] for the problem with the Laplacian. The main ingredients are a geometric inequality for stable solutions to (1.1.1) and a Hardy inequality on the level sets of the function u. The first tool is originally due to Sternberg and Zumbrun [131, 132] for the case of the Laplacian. We will use a generalization of this inequality to the p-Laplacian case, due to Farina, Sciunzi, and Valdinoci [84, 85] and stated in Theorem 1.1.4 below.

The Hardy inequality that we use is originally due to Cabré [30], but it can also be deduced from more general Hardy inequalities studied in [39] by Cabré and the author. In order to state these two results, we need to introduce some notation.

If u is a C^1 solution to (1.1.1) and we consider the set of regular points of u, defined by $\{x \in \Omega : |\nabla u(x)| > 0\}$, then u is C^2 in this set — see Corollary 2.2 of [64] — since the equation is uniformly elliptic in a neighborhood of every regular point.

Therefore, for any $x \in \Omega \cap \{|\nabla u| > 0\}$ we can define the level set of u passing through x as

$$\mathcal{L}_{u,x} := \{y \in \Omega : u(y) = u(x)\},$$

which is a C^2 embedded hypersurface of $\mathbb{R}^n$. In $\{x \in \Omega : |\nabla u(x)| > 0\}$ we can define

$$\nu := \frac{\nabla u}{|\nabla u|},$$

which is the normal vector to the level sets of u. Now, we can also introduce the notion of tangential gradient along the level sets. We define it for every function $\varphi \in C^1(\Omega)$ as the projection of $\nabla \varphi$ on the tangent space to the level sets passing through x, i.e.

$$\nabla_T \varphi := \nabla \varphi - \langle \nabla \varphi, \nu \rangle \, \nu.$$

For any $x \in \Omega \cap \{|\nabla u| > 0\}$ we denote with κ_i the $n-1$ principal curvatures of $\mathcal{L}_{u,x}$ and we recall that the mean curvature of the level sets is defined as

$$H := \sum_{i=1}^{n-1} \kappa_i.$$

In the statement of the geometric property of stable solutions, the square of the second fundamental form of the level sets appears. It is defined as

$$|A|^2 := \sum_{i=1}^{n-1} \kappa_i^2.$$

Now, we can state the geometric inequality for stable solutions to $-\Delta_p u = f(u)$.

Theorem 1.1.4 (Farina, Sciunzi, Valdinoci [84, 85]). *Let $p \in (1, +\infty)$, Ω be a smooth bounded domain of $\mathbb{R}^n$, f a C^1 nonlinearity and u a regular stable solution to* (1.1.1). *Then, for every $\eta \in C_c^1(\Omega)$ it holds that*

$$\begin{aligned} \int_{\Omega \cap \{|\nabla u| > 0\}} \Big((p-1) \, |\nabla u|^{p-2} \, |\nabla_T \, |\nabla u||^2 + |\nabla u|^p \, |A|^2 \Big) \eta^2 \, dx & \\ \leq (p-1) \int_\Omega |\nabla u|^p \, |\nabla \eta|^2 \, dx. & \end{aligned} \tag{1.1.19}$$

As we mentioned above, this result is originally due to Sternberg and Zumbrun [131, 132] for stable solutions to $-\Delta u = f(u)$ in a smooth bounded domain Ω with $f \in C^1$. In this case, for every $\eta \in C_c^1(\Omega)$, the inequality reads

$$\int_{\Omega \cap \{|\nabla u| > 0\}} \Big(|\nabla_T \, |\nabla u||^2 + |\nabla u|^2 \, |A|^2 \Big) \eta^2 \, dx \leq \int_\Omega |\nabla u|^2 \, |\nabla \eta|^2 \, dx. \tag{1.1.20}$$

The idea of obtaining L^∞ bounds for stable solutions to $-\Delta u = f(u)$ using (1.1.20) was used for the first time in [28]. The key point in [28] is the combination of (1.1.20) with the Michael-Simon and Allard inequality, applied on every level set of u.

A similar but simpler strategy is used in [30], still for the classical problem with the Laplacian. It is based on a new geometric Hardy inequality on the foliation of hypersurfaces given by the level sets of u, a much simpler tool than the Michael-Simon and Allard inequality. In the present chapter we extend this idea to the case of the p-Laplacian. We need both Theorem 1.1.4 and the new Hardy inequality provided in [30] to prove Proposition 1.1.3, which is the key estimate to prove Theorem 1.1.1.

We need to introduce some further notation in order to state the Hardy inequality on hypersurfaces of $\mathbb{R}^n$. For every $y \in \mathbb{R}^n$, we define

$$r_y = r_y(x) = |x - y| \, ,$$

and for every function $\psi(x) \in C^1$ we write its radial derivative as

$$\psi_{r_y}(x) = \frac{x - y}{|x - y|} \cdot \nabla \psi(x).$$

The geometric Hardy inequality is stated in the following theorem. Recall that, in the statement, the mean curvature H and the tangential gradient ∇_T are referred to the level sets $\mathcal{L}_{u,x}$ of u, which are C^2 embedded hypersurfaces of $\mathbb{R}^n$ for every point $x \in \Omega \cap \{|\nabla u| > 0\}$.

Theorem 1.1.5 (Cabré [30]). *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, $u \in C^1_c(\mathbb{R}^n) \cap C^2_c(\mathbb{R}^n \cap \{|\nabla u| > 0\})$, $\alpha \in [0, n-1)$ and $y \in \mathbb{R}^n$. Then, for every $\varphi \in C^1_c(\mathbb{R}^n)$*

$$
\begin{aligned}
&(n-1-\alpha)\int_\Omega |\nabla u|\,\varphi^2 r_y^{-\alpha}\,dx + \alpha\int_\Omega |\nabla u|^{-1}\,u_{r_y}^2\varphi^2 r_y^{-\alpha}\,dx \\
&\quad \le \left(\int_\Omega |\nabla u|\,\varphi^2 r_y^{-\alpha}\,dx\right)^{\frac{1}{2}} \left(\int_{\Omega\cap\{|\nabla u|>0\}} |\nabla u| \left(4\,|\nabla_T\varphi|^2 + \varphi^2\,|H|^2\right) r_y^{-\alpha+2}\,dx\right)^{\frac{1}{2}}.
\end{aligned}
\tag{1.1.21}
$$

In particular, if u is radial, then

$$
(n-1)^2\int_\Omega \left|u_{r_y}\right| \varphi^2 r^{-\alpha}\,dx \le \int_{\Omega\cap\{|u_{r_y}|>0\}} \left|u_{r_y}\right| \left(4\,|\nabla_T\varphi|^2 + \varphi^2\,|H|^2\right) r^{-\alpha+2}\,dx.
$$

1.2 Proof of the L^∞ bounds

We prove in this section our main results, namely Proposition 1.1.3 and Theorem 1.1.1, using the geometric inequality for stable solutions and the Hardy inequality on level sets.

Proof of Proposition 1.1.3. We apply the geometric Hardy inequality of Theorem 1.1.5 to the function

$$
\varphi = |\nabla u|^{\frac{p-1}{2}}\,\zeta,
$$

where ζ is a positive smooth function that satisfies

$$
\zeta_{|\partial\Omega} = 0 \qquad \text{and} \qquad \zeta \equiv 1 \ \text{ in } \ K_{\delta/2}. \tag{1.2.1}
$$

To be completely rigorous, in the proof we should use

$$
\varphi_\varepsilon := \left(|\nabla u|^2 + \varepsilon^2\right)^{\frac{p-1}{4}}\zeta
$$

instead of φ, and then let $\varepsilon \to 0$. We omit the details of this simple argument.

To simplify notation, we define

$$
I := \int_\Omega |\nabla u|^p\, r_y^{-\alpha}\zeta^2\,dx;
$$

$$
I_r := \int_\Omega |\nabla u|^{p-2}\, u_{r_y}^2 r_y^{-\alpha}\zeta^2\,dx.
$$

Plugging φ into (1.1.21), we obtain

$$
\begin{aligned}
&((n-1-\alpha)I + \alpha I_r)^2 \\
&\qquad \le I\int_{\Omega\cap\{|\nabla u|>0\}} |\nabla u|\, r_y^{-\alpha+2}\left(4\,|\nabla_T\varphi|^2 + |\nabla u|^{p-1}\,\zeta^2\,|H|^2\right)dx,
\end{aligned}
\tag{1.2.2}
$$

with $\alpha \in [0, n-1)$ to be chosen. The tangential gradient of φ can be computed as

$$
\nabla_T\varphi = \frac{p-1}{2}\zeta\,|\nabla u|^{\frac{p-3}{2}}\,\nabla_T\,|\nabla u| + |\nabla u|^{\frac{p-1}{2}}\,\nabla_T\zeta,
$$

and the Cauchy-Schwarz inequality gives

$$4\,|\nabla_T\varphi|^2 \leq (1+\varepsilon)(p-1)^2\zeta^2\,|\nabla u|^{p-3}\,|\nabla_T\,|\nabla u||^2 + \frac{C}{\varepsilon}\,|\nabla u|^{p-1}\,|\nabla_T\zeta|^2\,,$$

where C is a positive universal constant, and $\varepsilon > 0$ will be chosen later. Therefore, we get

$$\begin{aligned}((n-1-\alpha)I + \alpha I_r)^2 \leq (1+\varepsilon)I \int_{\Omega\cap\{|\nabla u|>0\}} \Big((p-1)^2\,|\nabla u|^{p-2}\,|\nabla_T\,|\nabla u||^2 \\ + |\nabla u|^p\,H^2\Big) r_y^{-\alpha+2}\zeta^2\,dx + \frac{C}{\varepsilon}I \int_{\Omega\cap\{|\nabla u|>0\}} |\nabla u|^p\,r_y^{-\alpha+2}\,|\nabla_T\zeta|^2\,dx.\end{aligned} \tag{1.2.3}$$

Concerning the last integral in (1.2.3), we can control it in terms of the L^p-norm of the gradient of u in a neighborhood of the boundary of Ω, since $|\nabla_T\zeta|$ has support in $(\Omega \setminus K_{\delta/2}) \subset (\Omega \setminus K_\delta)$. We also use that, since $y \in K_\delta$, we have

$$\delta/2 < r_y(x) < \operatorname{diam}(\Omega) \qquad \text{for every } x \in \Omega \setminus K_{\delta/2}. \tag{1.2.4}$$

Therefore, we deduce the bound

$$\int_{\Omega\cap\{|\nabla u|>0\}} |\nabla u|^p\,r_y^{-\alpha+2}\,|\nabla_T\zeta|^2\,dx \leq C\int_{\Omega\setminus K_\delta} |\nabla u|^p\,dx, \tag{1.2.5}$$

for some positive constant C depending only on Ω, δ, and α. Observe that we need both the upper and the lower bound on $r_y(x)$ since a priori α in (1.2.5) can be greater or smaller than 2.

In the next step, we use that $H^2 \leq (n-1)\,|A|^2$ and apply the geometric stability inequality (1.1.19) in Theorem 1.1.4. Observe that, to apply it, we need to have $(p-1)$ instead of $(p-1)^2$ in the first term in the right-hand side of (1.2.3), and no constants in front of the term containing $|A|^2$. This will force us to make a bound which differs whether n is above or below p. For this reason, we distinguish the two cases.

When $n > p$, we have $p-1 < n-1$ and from (1.2.3) we deduce that

$$\begin{aligned}&((n-1-\alpha)I + \alpha I_r)^2 \\ &\leq (1+\varepsilon)(n-1)I \int_{\Omega\cap\{|\nabla u|>0\}} \Big((p-1)\,|\nabla u|^{p-2}\,|\nabla_T\,|\nabla u||^2 \\ &\qquad + |\nabla u|^p\,|A|^2\Big) r_y^{-\alpha+2}\zeta^2\,dx + \frac{C}{\varepsilon}I\int_{\Omega\setminus K_\delta} |\nabla u|^p\,dx.\end{aligned} \tag{1.2.6}$$

Now, we can control the right-hand side of (1.2.6) using the geometric stability inequality (1.1.19) with test function

$$\eta = r_y^{\frac{2-\alpha}{2}}\,\zeta. \tag{1.2.7}$$

The following computations must be done with a regularization of η in a small neighborhood of y, that we call η_ε. Since all terms in the rest of the proof are given by integrable functions, by dominated convergence we can let $\varepsilon \to 0$ in all the integrals. For this reason, we directly write the computations with η instead of η_ε.

Plugging η in (1.1.19) and combining it with (1.2.6), we obtain

$$
\begin{aligned}
&((n-1-\alpha)I+\alpha I_r)^2\\
&\quad\le (1+\varepsilon)(n-1)(p-1)I\int_\Omega |\nabla u|^p \left|\nabla\left(r_y^{\frac{2-\alpha}{2}}\zeta\right)\right|^2 dx+\frac{C}{\varepsilon}I\int_{\Omega\setminus K_\delta}|\nabla u|^p\,dx.
\end{aligned}
$$

Using again the Cauchy-Schwarz inequality, there exists a positive universal constant C such that

$$
\left|\nabla\left(r_y^{\frac{2-\alpha}{2}}\zeta\right)\right|^2\le(1+\varepsilon)\frac{(\alpha-2)^2}{4}r_y^{-\alpha}\zeta^2+\frac{C}{\varepsilon}r_y^{2-\alpha}|\nabla\zeta|^2,
$$

again for the same $\varepsilon>0$ that we will choose later. Since we have chosen ζ satisfying (1.2.1), if $n>p$ we get

$$
\begin{aligned}
(n-1-\alpha)^2 I^2 &\le ((n-1-\alpha)I+\alpha I_r)^2\\
&\le (1+\varepsilon)^2(n-1)(p-1)\frac{(\alpha-2)^2}{4}I^2+\frac{C}{\varepsilon}I\|\nabla u\|^p_{L^p(\Omega\setminus K_\delta)}.
\end{aligned}
\tag{1.2.8}
$$

If instead $n\le p$, the same procedure works — including the same choice of test function (1.2.7) in the stability inequality (1.1.19) — but we have a difference in the constants. Indeed, in (1.2.3) we use that $H^2\le(n-1)|A|^2\le(p-1)|A|^2$. In this way, we can take $(p-1)$ out of the integral and obtain the right constants to apply the geometric stability inequality (1.1.19). As a consequence, instead of (1.2.8) we get

$$
\begin{aligned}
(n-1-\alpha)^2 I^2 &\le ((n-1-\alpha)I+\alpha I_r)^2\\
&\le (1+\varepsilon)^2(p-1)^2\frac{(\alpha-2)^2}{4}I^2+\frac{C}{\varepsilon}I\|\nabla u\|^p_{L^p(\Omega\setminus K_\delta)}.
\end{aligned}
\tag{1.2.9}
$$

Summarizing, if $\alpha\in[0,n-1)$ satisfies condition (1.1.16), then we can choose $\varepsilon>0$ in (1.2.8) or (1.2.9) such that

$$
\int_{K_{\delta/2}}|\nabla u|^p r_y^{-\alpha}\,dx\le C\|\nabla u\|^p_{L^p(\Omega\setminus K_\delta)},
$$

for some positive constant C depending only on Ω, δ, p, and α. Finally, using (1.2.4) and that $K_\delta\subset K_{\delta/2}$ we can control the integral over $\Omega\setminus K_{\delta/2}$ with

$$
\int_{\Omega\setminus K_{\delta/2}}|\nabla u|^p r_y^{-\alpha}\,dx\le C\|\nabla u\|^p_{L^p(\Omega\setminus K_\delta)},
$$

proving (1.1.17).

Let us assume now that Ω is a ball and f is strictly positive in $(0,+\infty)$. Then Corollary 1.1 of [64] ensures that u is radially symmetric and decreasing in the radius r. Taking $y=0$, we have that $I=I_r$. Furthermore, $H^2=(n-1)|A|^2$ and $\nabla_T|\nabla u|=0$, since ∇u is orthogonal to the level sets. In this case, from (1.2.2) we deduce

$$
(n-1)I\le\int_{\Omega\cap\{|\nabla u|>0\}}|\nabla u|^p|A|^2 r_0^{-\alpha+2}\zeta^2\,dx,
$$

instead of (1.2.6). Therefore, under the less restrictive assumption (1.1.18), we obtain (1.1.17) with $y=0$ in the same way as in the nonradial case. □

Proposition 1.1.3 is the main tool in the proof of the interior estimate (1.1.13). In the following lemma, we introduce some boundary estimates that we will need to pass from (1.1.13) to the global bound (1.1.14) in strictly convex domains.

Proposition 1.2.1 (Castorina, Sanchón [51]). *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, f a positive C^1 nonlinearity, $p \in (1, +\infty)$ and u a positive regular solution to* (1.1.2).

If Ω is strictly convex, then there exist positive constants δ and γ depending only on the domain Ω, such that for every point x with $\operatorname{dist}(x, \partial\Omega) < \delta$*, there exists a set $I_x \subset \Omega$ of positive measure γ for which*

$$u(x) \le u(y) \qquad \text{for every } y \in I_x.$$

In particular,

$$\|u\|_{L^\infty(\Omega \setminus K_\delta)} \le \frac{1}{\gamma} \|u\|_{L^1(\Omega)},$$

where $K_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \ge \delta\}$.

The proof of this lemma — which can be found in [51] — is based on a moving planes procedure for the p-Laplacian developed in [64]. For this method to work, the strict convexity assumption about Ω is crucial.

We can now prove our main result.

Proof of Theorem 1.1.1. Let us assume that there exists a nonnegative exponent α satisfying (1.1.16) such that

$$n - p < \alpha < n - 1.$$

The existence of such an exponent α depends on the values of n and p and, in particular, it is ensured when we assume that n and p satisfy (1.1.12) — see Appendix 1.A.1 for all the details.

As a consequence of Proposition 1.1.3, for every $y \in K_\delta$ we obtain that

$$\int_\Omega |\nabla u(x)|^p \, r_y(x)^{-\alpha} \, dx \le C \|\nabla u\|^p_{L^p(\Omega \setminus K_\delta)}, \tag{1.2.10}$$

for some constant $C > 0$ depending only on Ω, δ, and p.

In the radial case, Proposition 1.1.3 gives (1.2.10) with $y = 0$ for some nonnegative exponent $\alpha \in (n - p, n - 1)$ satisfying the less restrictive condition (1.1.18). It can be checked that such an an exponent α exists whenever n and p satisfy (1.1.15) — see Remark 1.A.1.1 in the appendix to this chapter.

Summarizing, in both the radial and the nonradial case — under different assumptions on n and p — we have (1.2.10) for some nonnegative $\alpha > n - p$, and we want to deduce (1.1.13) and (1.1.14).

In order to prove the interior bound (1.1.13), for every point $y \in K_\delta$ we use [96, Lemma 7.16] for the set $B_y := B_{\delta/2}(y)$, obtaining

$$\left| u(y) - \overline{u}_{B_y} \right| \le C \int_\Omega |\nabla u| \, r_y^{1-n} \, dx.$$

Here C is a positive constant depending only on n and δ, and $\overline{u}_{B_y}$ is the mean of u over the set B_y, defined by

$$\overline{u}_{B_y} := \frac{1}{|B_y|} \int_{B_y} u \, dx.$$

Then, applying the Hölder inequality with exponents p and p' it follows that

$$\left|u(y)-\overline{u}_{B_y}\right| \leq C\left(\int_\Omega |\nabla u|^p\, r_y^{-\alpha}\, dx\right)^{\frac{1}{p}}\left(\int_\Omega r_y^{\frac{p-np+\alpha}{p-1}}\, dx\right)^{\frac{1}{p'}}. \tag{1.2.11}$$

The last integral is bounded, since $\alpha > n-p$ and

$$\int_\Omega r_y(x)^{\frac{p-np+\alpha}{p-1}}\, dx \leq \left|S^{n-1}\right| \frac{p-1}{\alpha-n+p}\operatorname{diam}(\Omega)^{\frac{\alpha-n+p}{p-1}}.$$

Now, using (1.2.10) and (1.2.11) we can conclude that

$$\|u\|_{L^\infty(K_\delta)} \leq C\left(\|u\|_{L^1(\Omega)} + \|\nabla u\|_{L^p(\Omega\setminus K_\delta)}\right), \tag{1.2.12}$$

which is (1.1.13), with C depending only on Ω, p, and δ.

Assume now that Ω is strictly convex, u is a positive solution of problem (1.1.2) and f is positive in $(0,+\infty)$. Then, Proposition 1.2.1 gives the boundary estimate

$$\|u\|_{L^\infty(\Omega\setminus K_{2\delta})} \leq \frac{1}{2\delta}\|u\|_{L^1(\Omega)},$$

where δ is a positive constant that depends only on Ω. We use this bound to control $f(u)$ in the set $\Omega \setminus K_{2\delta}$. By interior and boundary regularity[2] for problem (1.1.2), we deduce stronger estimates in the set $\Omega\setminus K_\delta$, which is contained in $\Omega\setminus K_{2\delta}$. In particular, we have $\|\nabla u\|_{L^\infty(\Omega\setminus K_\delta)} \leq C$, for some constant C which depends only on Ω, f, p and $\|u\|_{L^1(\Omega)}$. Combining this with (1.2.12) we obtain (1.1.14), since we also have that

$$\|u\|_{L^\infty(\Omega\setminus K_\delta)} \leq \|u\|_{L^\infty(\Omega\setminus K_{2\delta})} \leq C\|u\|_{L^1(\Omega)}.$$

If Ω is a ball and f is strictly positive in $(0,+\infty)$, then from Corollary 1.1 of [64] we know that u is radially symmetric and decreasing in the radius r. Therefore, it is sufficient to estimate $u(0)$. From (1.2.10) with $y=0$, we obtain

$$|u(0)| \leq C\left(\|u\|_{L^1(\Omega)} + \|\nabla u\|_{L^p(\Omega\setminus K_\delta)}\right).$$

Then, proceeding in the same way as in the nonradial case we deduce (1.1.14). $\square$

1.A.1 Appendix to Chapter 1

In this appendix we show the existence of a nonnegative $\alpha \in (n-p, n-1)$ satisfying (1.1.16) whenever n and p satisfy (1.1.12), completing in this way the argument of the proof of Theorem 1.1.1. We distinguish two cases.

Case 1, $n > p$. We take[3] $\alpha = n-p+\varepsilon$ for some $\varepsilon \in (0, p-1)$, and we plug it in (1.1.16), obtaining

$$4(p-1-\varepsilon)^2 > (p-n+2-\varepsilon)^2(n-1)(p-1).$$

[2] See Theorem 1 in [72] or Theorem 1 in [134] for interior $C^{1,\beta}$ regularity in the style of De Giorgi and Theorem 1 in [106] for boundary regularity. See also Appendix E in [117].

[3] The following argument gives that our condition (1.1.12) is the optimal one for the existence of some $\alpha \in (n-p, n-1)$ satisfying (1.1.16). To see this in the case $n \geq 4$ (otherwise it is simple), we may assume $n > p+2$, since (1.1.12) already includes $n \leq p+2$. But then, since $\alpha \in (n-p, n-1)$ we also have $\alpha \in (2, n-1)$ and therefore, if (1.1.16) is satisfied by some $\alpha \in (2, n-1)$, then it is also satisfied by any smaller α in this interval. Thus, in the argument, our choice $\alpha = n-p+\varepsilon$ for small $\varepsilon > 0$ imposes non restriction.

If the inequality holds with $\varepsilon = 0$, then it also holds for an arbitrary small $\varepsilon > 0$. In this way, we are reduced to check that whenever n and p satisfy (1.1.12), then

$$4(p-1) > (p-n+2)^2(n-1).$$

This inequality is cubic in n, but quadratic in p. Solving it with respect to p, and exploiting also a surprising cancellation in the discriminant, we obtain

$$\frac{n^2-5n+8}{n-1} < p < n.$$

Observe that this forces $n > 2$. For $n \geq 3$, we solve it with respect to n and find

$$\frac{1}{2}\left(p+5-\sqrt{(p-1)(p+7)}\right) < n < \frac{1}{2}\left(p+5+\sqrt{(p-1)(p+7)}\right). \tag{1.A.1.1}$$

If $n = 3$, both inequalities hold true for every $p < 3 = n$. If $n \geq 4$ instead, the lower bound in (1.A.1.1) is always verified, and the upper bound on n is the one appearing in (1.1.12).

Case 2, $n \leq p$. In this case, inequality (1.1.16) reads

$$4(n-1-\alpha)^2 > (\alpha-2)^2(p-1)^2. \tag{1.A.1.2}$$

For $n = 2, 3$ one can directly check that no nonnegative solutions $\alpha \in (n-p, n-1)$ exist. Indeed, when $n = 2$ we can take the square root of (1.A.1.2) and check that the solutions α are either strictly negative or greater than 1. When $n = 3$ instead, (1.A.1.2) contradicts the assumption $p \geq n = 3$.

For $n \geq 4$, we are going to see that, for every $p \geq n$, there exists a nonnegative $\alpha \in (n-p, n-1)$ satisfying (1.A.1.2). For this, it suffices to look for α belonging to $(2, n-1)$. We can now take the square root of (1.A.1.2) and solve the inequality with respect to α. In this way, we find

$$\alpha < \frac{2(n+p-2)}{p+1}$$

and one can easily check that $2(n+p-2)/(p+1) > 2$ for all $p > 1$, since we are assuming $n \geq 4$.

Remark 1.A.1.1. The same ideas — including the same choice of α when $n > p$ — can be used to check that in the radial case there exists a nonnegative $\alpha \in (n-p, n-1)$ satisfying (1.1.18) whenever n and p satisfy (1.1.15). The only difference is that in the case $n > p$ we get an inequality which is quadratic in n and cubic in p, and we can directly solve it with respect to n, finding $p < n < p + 4p/(p-1)$.

Chapter 2

Hardy-Sobolev inequalities on hypersurfaces of Euclidean space

In this chapter we study Hardy-Sobolev inequalities on hypersurfaces of $\mathbb{R}^{n+1}$, all of them involving a mean curvature term and having universal constants independent of the hypersurface. We first consider the celebrated Sobolev inequality of Michael-Simon and Allard, in our codimension one framework. Using their ideas, but simplifying their presentations, we give a quick and easy-to-read proof of the inequality. Next, we establish two new Hardy inequalities on hypersurfaces. One of them originates from an application to the regularity theory of stable solutions to semilinear elliptic equations. The other one, which we prove by exploiting a "ground state" substitution, improves the Hardy inequality of Carron. With this same method, we also obtain an improved Hardy or Hardy-Poincaré inequality.

2.1 Introduction

In this chapter we establish some new Hardy inequalities on hypersurfaces of Euclidean space. As the one of Carron [49] — for which we find an improved version — all of them involve a mean curvature term and have universal constants. Our inequalities have their origin in the recent work [30] by Cabré on the regularity theory of stable solutions to semilinear elliptic equations. The paper [30] established the regularity of such solutions up to dimension four, for all nonlinearities, by using a foliated version of one of our new Hardy inequalities — the one of Theorem 2.1.3 below. In this way, [30] succeeded to greatly simplify the 2010 proof of the same result found in [28] by Cabré[1]. In addition, [28] used the Michael-Simon and Allard Sobolev inequality, which is a more sophisticated tool than our Hardy inequality. In fact, one of the features of the current chapter is that proofs are rather elementary — even if they concern functions defined on hypersurfaces. In particular, in Section 2.2 we give a quick and easy-to-read proof of the Sobolev inequality of Michael-Simon and Allard, for completeness and since we believe it can be useful for potential readers.

Let us start presenting the inequality of Michael-Simon and Allard. In 1967, Miranda [113] established that the Sobolev inequality holds in its Euclidean form, but possibly with a different constant, on every minimal hypersurface of $\mathbb{R}^n$. Some years later, a

[1] In the case of nonnegative nonlinearities, regularity of stable solutions up to the optimal dimension nine has been recently obtained by Cabré, Figalli, Ros-Oton, and Serra [38].

more general Sobolev inequality for k-submanifolds of $\mathbb{R}^n$, not necessarily minimal, was proved independently by Michael and Simon [110] and by Allard [10]. This inequality was subsequently generalized by Hoffman and Spruck [101] to submanifolds of general Riemannian manifolds.

In the context of hypersurfaces of $\mathbb{R}^{n+1}$, i.e., submanifolds of the Euclidean space with codimension one, the Sobolev inequality reads as follows.

Theorem 2.1.1 (Allard [10], Michael-Simon [110]). *Let M be a smooth n-dimensional hypersurface of $\mathbb{R}^{n+1}$, $p \in [1, n)$, and $\varphi \in C^1(M)$ have compact support in M. If M is compact without boundary, any function $\varphi \in C^1(M)$ is allowed.*

Then, there exists a positive constant C, depending only on n and p, such that

$$\|\varphi\|^p_{L^{p^*}(M)} \leq C \int_M \left(|\nabla_T \varphi|^p + |H\varphi|^p \right) dV, \tag{2.1.1}$$

where $p^* = np/(n-p)$ is the Sobolev exponent, H is the mean curvature of M, and ∇_T denotes the tangential gradient to M.

The constant C in (2.1.1) is universal, in the sense that it depends only on the dimension n and on the exponent p, but not on M. Thus, the geometry of the hypersurface plays a role just through the term involving the mean curvature H appearing in the right-hand side of (2.1.1). In particular, when M is minimal[2], such term vanishes and we recover the Sobolev inequality proved earlier by Miranda [113].

The formulation of the Michael-Simon and Allard inequality stated in Theorem 2.1.1 can be easily deduced, using standard tools, from the following isoperimetric inequality.

Theorem 2.1.2 (Allard [10], Michael-Simon [110]). *Let M be a smooth n-dimensional hypersurface of $\mathbb{R}^{n+1}$ and $E \subset M$ a smooth domain with compact closure in M. Then*

$$|E|^{\frac{n-1}{n}} \leq C\Big(\mathrm{Per}(E) + \int_E |H|\, dV\Big), \tag{2.1.2}$$

where H is the mean curvature of M, $\mathrm{Per}(E)$ is the perimeter of E, and C is a constant depending only on the dimension n of M.

The inequalities presented in Theorems 2.1.1 and 2.1.2 were proven in the seventies in [10, 110], in independent works. In [10] the proof is based on establishing an isoperimetric inequality, like the one in Theorem 2.1.2, for k-dimensional varifolds of $\mathbb{R}^n$. From it, Theorem 2.1.1 can be easily deduced. Instead, in [110] the authors prove directly a Sobolev inequality for submanifolds of $\mathbb{R}^n$ of any codimension. A slight modification of the argument in [110], due to Leon Simon, is presented in the monograph [59, Theorem 3.11].

In the current chapter, where we focus on the case of hypersurfaces of $\mathbb{R}^{n+1}$, we first present a quick and easy-to-read proof of the Michael-Simon and Allard inequality. Our proof uses mainly the tools of Michael and Simon [110] but contains two simplifications: we target at the isoperimetric inequality (instead, [110] pursues the Sobolev inequality) and we use a quick Gronwall-type argument from Allard [10].

After [10, 110], alternative proofs of the Sobolev inequality have been found. In the case of two-dimensional minimal surfaces (with any codimension), Leon Simon gave a

[2]Here and throughout the thesis, *minimal* hypersurface refers to a hypersurface which is a critical point (not necessarily a minimizer) of the area functional, i.e., a hypersurface with zero mean curvature.

rather simple proof which, in addition, carries a constant optimal up to a factor of 2. This work remained unpublished, but is presented in [56, 135]. An improved version of it, which holds in any two-dimensional surface, not necessarily minimal, was found by Topping [135]. In the case of submanifolds of arbitrary dimension and codimension, Castillon [50] gave a new proof of the Michael-Simon and Allard Sobolev inequality by using optimal transport methods. Finally, an important result has been obtained very recently by Brendle [22], also in the case of arbitrary dimension and codimension. He finds a new proof of the Sobolev inequality that, in addition, carries the sharp constant in the case of minimal submanifolds of $\mathbb{R}^{n+1}$ of codimension at most two. This is the first time that the Michael-Simon and Allard inequality is proved in minimal submanifolds (or even minimal hypersurfaces) with the optimal Euclidean constant. Brendle's method is a clever extension of the proof of the sharp Euclidean isoperimetric inequality found by Cabré in [27]. In Appendix 2.A.2 we describe it in some more detail, together with other results about optimal constants in the Michael-Simon and Allard inequality — a topic that has been studied mainly in the case of submanifolds being either minimal or compact without boundary.

Our interest in the Michael-Simon and Allard inequality originates from an application of it to the regularity theory for semilinear elliptic equations. More precisely, in 2010 Cabré proved in [28] an a priori estimate for stable solutions to $-\Delta u = f(u)$ in bounded domains of $\mathbb{R}^{n+1}$, using as a key tool the Michael-Simon and Allard inequality (2.1.1) applied on every level set of u. The estimate in [28], whose proof was quite delicate, led to the regularity of stable solutions in dimensions $n+1 \leq 4$ for every smooth nonlinearity f.

An alternative and much simpler proof of this same result has been recently found by Cabré [30]. This new method does not use the Michael-Simon and Allard inequality, but it is based instead on a new Hardy inequality with sharp constant — also established in [30] — adapted to the level sets of a function u. In [30], this Hardy inequality is later used with u being a stable solution to $-\Delta u = f(u)$ in a bounded domain $\Omega \subset \mathbb{R}^{n+1}$. To describe the new inequality, for every smooth function u we consider its radial derivative $u_r = \nabla u \cdot x / |x|$. Then, for every $\varphi \in C^1_c(\Omega)$, with $\Omega \subset \mathbb{R}^{n+1}$ an open set, and every parameter $a \in [0, n)$, the Hardy inequality from [30] states that

$$\begin{aligned} (n-a) \int_\Omega |\nabla u| \frac{\varphi^2}{|x|^a}\, dx + a \int_\Omega \frac{u_r^2}{|\nabla u|} \frac{\varphi^2}{|x|^a}\, dx \\ \leq \left(\int_\Omega |\nabla u| \frac{\varphi^2}{|x|^a}\, dx \right)^{\frac{1}{2}} \left(\int_\Omega |\nabla u| \frac{4\, |\nabla_T \varphi|^2 + |H\varphi|^2}{|x|^{a-2}}\, dx \right)^{\frac{1}{2}}, \end{aligned} \tag{2.1.3}$$

where the tangential gradient ∇_T and the mean curvature H are referred to the level sets[3] of u.

Throughout the chapter, the mean curvature H is the sum, and not the arithmetic mean, of the principal curvatures. Therefore, when M is the n-dimensional unit sphere, we have $H = n$.

Using the coarea formula, from (2.1.3) one can deduce the following Hardy inequality on a single hypersurface[4] M. Here and throughout the chapter, $C^1_c(M)$ denotes the

[3] By Sard's theorem, if $u \in C^\infty$, almost every level set of u is a smooth embedded hypersurface of $\mathbb{R}^{n+1}$.

[4] For this, one applies (2.1.3) with $u(x) = \operatorname{dist}(x, M)$ in $\Omega_\varepsilon := \{0 < u < \varepsilon\} \cap B_R$ after extending $\varphi \in C^1_c(M \cap B_R)$ to be constant in the normal directions to M. Then one divides the inequality by ε and

space of C^1 functions with compact support on M. In case M is a compact hypersurface without boundary, then $C^1_c(M) = C^1(M)$.

Theorem 2.1.3. *Let M be a smooth hypersurface of $\mathbb{R}^{n+1}$ and $a \in [0, n)$. Then, for every $\varphi \in C^1_c(M)$ we have*

$$\begin{aligned}(n-a)\int_M \frac{\varphi^2}{|x|^a}\,dV + a\int_M \left(\frac{x}{|x|}\cdot\nu_M\right)^2 \frac{\varphi^2}{|x|^a}\,dV \\ \leq \left(\int_M \frac{\varphi^2}{|x|^a}\,dV\right)^{\frac{1}{2}} \left(\int_M \frac{4\,|\nabla_T\varphi|^2 + |H\varphi|^2}{|x|^{a-2}}\,dV\right)^{\frac{1}{2}},\end{aligned} \tag{2.1.4}$$

where ν_M is the unit normal to M in $\mathbb{R}^{n+1}$.

In this chapter we present a direct proof of Theorem 2.1.3 which does not rely on the more involved proof from [30] of its foliated version (2.1.3). Then, using the coarea formula, we deduce (2.1.3) from it — see Corollary 2.3.2 and its proof. Moreover, in Theorem 2.3.1 we give a version of (2.1.4), and thus of (2.1.3), for an arbitrary exponent $p \geq 1$ instead of $p = 2$. Our proof of Theorem 2.1.3 is elementary and based on the use of the tangential derivatives δ_i, which we recall in Appendix 2.A.1.

Note that when $M = \mathbb{R}^n$, $n \geq 3$, and $a = 2$, then (2.1.4) is the Euclidean Hardy inequality with best constant,

$$\frac{(n-2)^2}{4}\int_{\mathbb{R}^n} \frac{\varphi^2}{|x|^2}\,dx \leq \int_{\mathbb{R}^n} |\nabla\varphi|^2\,dx, \tag{2.1.5}$$

since the second term in the left-hand side of (2.1.4) vanishes. Instead, when M is close to a sphere in $\mathbb{R}^{n+1}$ centered at the origin, such term becomes important and could even make larger the constant $n-a$ in the first term in the left-hand side of (2.1.4). This is one of the interesting points of our result. Note, however, that (2.1.4) is trivial when $M = S^n$, since $H \equiv n$.

The foliated version (2.1.3) of our Hardy inequality was used in [30] to establish the boundedness of stable solutions to semilinear elliptic equations up to dimension $n+1 \leq 4$ for all nonlinearities. Thanks to our improved version, which includes the second term on its left-hand side, the same proof gave, in the radial case, regularity up to the optimal dimension $n+1 \leq 9$ — since one has $u_r^2 = |\nabla u|^2$ in its left hand side for radial solutions. In the nonradial case, the optimal result in dimension $n+1 \leq 9$ has been recently obtained, for nonnegative nonlinearities, by Cabré, Figalli, Ros-Oton, and Serra [38]. This result, whose proof does not rely on Hardy-Sobolev inequalities, gives a complete answer to a long standing open question posed by Brezis [23] and by Brezis and Vázquez [25].

The application of inequality (2.1.3) to the regularity theory of stable solutions has been extended by the author in [111] to nonlinear equations involving the p-Laplacian — see also Chapter 1. It is worth pointing out here that this is done using the quadratic version (2.1.3) of the Hardy inequality on the level sets, and not the one for a general exponent p stated in Corollary 2.3.2.

lets $\varepsilon \to 0$. This requires a more general version of (2.1.3) in which the part of $\partial\Omega = \partial\Omega_\varepsilon$ where $\varphi \neq 0$ is divided into two open subsets with u being constant on each of them (equals 0 and ε in our case). This version of (2.1.3) can be proved exactly as in [30], after checking that the foliated integration by parts formula of Lemma 2.1 in [30] also holds for these boundary conditions.

A related but different Hardy inequality on hypersurfaces of $\mathbb{R}^{n+1}$ was proved in 1997 by Carron [49]. It states that in every dimension $n \geq 3$ and for all $\varphi \in C_c^1(M)$ it holds that

$$\frac{(n-2)^2}{4} \int_M \frac{\varphi^2}{|x|^2} \, dV \leq \int_M \left(|\nabla_T \varphi|^2 + \frac{n-2}{2} \frac{|H| \, \varphi^2}{|x|} \right) dV. \tag{2.1.6}$$

In particular, this established that the Hardy inequality in its Euclidean form and with its best constant holds in every minimal hypersurface of $\mathbb{R}^{n+1}$. Observe that this also follows from our Theorem 2.1.3 by taking $a = 2$. Also in the context of minimal hypersurfaces, in Section 2.3 we will prove an analogue sharp Hardy inequality with exponent $p \neq 2$, namely, (2.3.2). Even if not explicitly mentioned in [104], inequality (2.3.2) also follows by the results of Kombe and Özaydin [104, Theorem 2.1].[5]

In [49] Carron proved also an intrinsic Hardy inequality on Cartan-Hadamard manifolds. His work gave rise to numerous papers in the topic of Hardy inequalities on manifolds, some of which are commented on next. Carron's work was extended to general Riemannian manifolds by Kombe and Özaydin [104, 105], who also included the case of a general exponent p instead of only $p = 2$. Some intrinsic Hardy inequalities with general weights, not necessarily of the power type, are studied by D'Ambrosio and Dipierro [62]. The case of the hyperbolic space $\mathbb{H}^n$ and related manifolds is treated by Berchio, Ganguly, Grillo, and Pinchover [17, 18], obtaining sharp constants and improved versions of the inequality. Finally, let us mention the recent work of Batista, Mirandola, and Vitório [16] improving Carron's inequality with power weights in the setting of manifolds isometrically immersed in Cartan-Hadamard manifolds.

In Theorem 2.1.4 below, we obtain an improved version of Carron's inequality (2.1.6) in the case of hypersurfaces of $\mathbb{R}^{n+1}$ by adding a nonnegative term on its left-hand side (the same term as in the inequality of Theorem 2.1.3 with $a = 2$). We could not find such additional term within the literature on Hardy's inequalities. In addition, our method of proof towards Hardy's inequalities is different from the ones in [16, 49], for instance.

Theorem 2.1.4. *Let M be a smooth hypersurface of $\mathbb{R}^{n+1}$ with $n \geq 3$. Then, for every $\varphi \in C_c^1(M)$ we have*

$$\begin{aligned} \frac{(n-2)^2}{4} \int_M \frac{\varphi^2}{|x|^2} \, dV + \frac{n^2-4}{4} \int_M \left(\frac{x}{|x|} \cdot \nu_M \right)^2 \frac{\varphi^2}{|x|^2} \, dV & \\ \leq \int_M \left(|\nabla_T \varphi|^2 + \frac{n-2}{2} \frac{|H| \, \varphi^2}{|x|} \right) dV&, \end{aligned} \tag{2.1.7}$$

where ν_M is the unit normal to M in $\mathbb{R}^{n+1}$.

As in Theorem 2.1.3, the second term in the left-hand side of (2.1.7) is of special interest when M is close to be a sphere of $\mathbb{R}^{n+1}$ centered at the origin.

We prove Theorem 2.1.4 using a technique which, in the case of the Euclidean space, is known as ground state substitution. It dates back at least to the time of Jacobi and it has been applied for instance in the spectral theory of Laplace and Schrödinger operators. It is based on writing the function φ as $\varphi = v\omega$, where typically ω is a positive solution of the Euler-Lagrange equation of the energy functional associated with the inequality. This method has been used in the Euclidean setting by Brezis and Vázquez [25]

[5] One uses [104, Theorem 2.1] with $\alpha = 0$ and $\rho = |x|$, together with the well-known inequality $\Delta\rho \geq (n-1)/\rho$ involving the Laplace-Beltrami operator, which holds if $H \equiv 0$ as we show in the beginning of subsection 3.2.

to obtain an improved Hardy inequality in $\mathbb{R}^n$, stated in (2.1.9) below. The ground state substitution is essentially equivalent to the use of a Picone identity, as done in Abdellaoui, Colorado, and Peral [2], where the authors also obtained some improved Hardy inequalities in domains of $\mathbb{R}^n$. More recently, Frank and Seiringer [92] used the ground state substitution to prove fractional Hardy inequalities in $\mathbb{R}^n$. We will use this method in the framework of functions defined on a hypersurface of the Euclidean space — something that we could not find in previous literature. In our proof we will take $\omega(x) = |x|^{-(n-2)/2}$.

The two inequalities of Hardy type in Theorems 2.1.3 and 2.1.4 are different in their formulations and independent in their proofs. Their statements differ mainly in the mean curvature term, containing H^2 versus $|H| / |x|$, respectively. At the same time, their proofs use distinct techniques. In addition, our proof of Theorem 2.1.3 works for an arbitrary exponent $p \geq 1$ — see Theorem 2.3.1 for the general statement — while the one of Theorem 2.1.4 gives a significant result only in the case $p = 2$. Indeed, with our technique one can prove a p-version of (2.1.7), but it is of less interest due to the presence of the second fundamental form in its right-hand side (instead of only the mean curvature). Moreover, its left-hand side contains some factors $(|x_T| / |x|)^{p-2}$, where x_T is the tangential part of the position vector x.

As a simple interpolation of the Michael-Simon and Allard inequality and of Theorem 2.1.3 with $a = 2$, we obtain the following Hardy-Sobolev inequality on hypersurfaces of $\mathbb{R}^{n+1}$.

Corollary 2.1.5. *Let M be a smooth hypersurface of $\mathbb{R}^{n+1}$ with $n \geq 3$, $b \in [0,1]$, and $\varphi \in C_c^1(M)$. Then, there exists a positive constant C depending only on the dimension n, such that*

$$\left(\int_M \frac{|\varphi|^{\frac{2(n-2b)}{n-2}}}{|x|^{2b}} \, dV \right)^{\frac{n-2}{n-2b}} \leq C \int_M \left(|\nabla_T \varphi|^2 + |H\varphi|^2 \right) dV. \tag{2.1.8}$$

Corollary 2.4.1, which is the general version of (2.1.8) with exponents $p \in [1, n)$, covers some possible choices of the parameters in Caffarelli-Kohn-Nirenberg type inequalities on hypersurfaces. Indeed, in [15], Batista, Mirandola, and Vitório prove a Caffarelli-Kohn-Nirenberg inequality for submanifolds of Riemannian manifolds, from which Corollary 2.4.1 can be deduced, perhaps with a different constant. However, the proof in [15] is delicate and relies on Riemannian geometry techniques, while we easily show Corollary 2.4.1 as an interpolation of our previous results in the setting of hypersurfaces of $\mathbb{R}^{n+1}$.

The classical Hardy's inequality has been improved in the Euclidean setting in many ways, see for instance [2,4,14,25,66,137]. Many of these improvements consist of adding a positive term on the left-hand side of the inequality. This additional term has to be of lower order than the Hardy integral, by the optimality of the constant $(n-2)^2/4$. This is done for example by Brezis and Vázquez in [25, Theorem 4.1], where they get an improvement in the Poincaré sense. Namely, they control both a Hardy-type integral and the L^2-norm of a function in terms of the L^2-norm of its gradient. For any bounded domain $\Omega \subset \mathbb{R}^n$, any dimension $n \geq 2$ and for every function $\varphi \in H_0^1(\Omega)$, their result states that

$$\frac{(n-2)^2}{4} \int_\Omega \frac{\varphi^2}{|x|^2} \, dx + H_2 \left(\frac{\omega_n}{|\Omega|} \right)^{\frac{2}{n}} \int_\Omega \varphi^2 \, dx \leq \int_\Omega |\nabla \varphi|^2 \, dx, \tag{2.1.9}$$

where H_2 is the first eigenvalue of the Laplacian in the unit ball of $\mathbb{R}^2$, hence positive and independent of n, and ω_n is the measure of the unit ball in $\mathbb{R}^n$.

Using the ground state substitution as in the proof of Theorem 2.1.4, we prove the following analogue of the improved Hardy inequality by Brezis and Vázquez, now on hypersurfaces of $\mathbb{R}^{n+1}$. We require functions to have compact support on the hypersurface M intersected with a ball of radius r in the ambient space.

Theorem 2.1.6. *Let M be a smooth hypersurface of $\mathbb{R}^{n+1}$ with $n \geq 2$, and $B_r = B_r(0) \subset \mathbb{R}^{n+1}$ be the $(n+1)$-dimensional open ball of radius r centered at the origin.*

Then, for every $\varphi \in C_c^1(B_r \cap M)$ we have

$$\begin{aligned}\frac{(n-2)^2}{4}\int_M \frac{\varphi^2}{|x|^2}\,dV + \frac{n^2-4}{4}\int_M \left(\frac{x}{|x|}\cdot \nu_M\right)^2 \frac{\varphi^2}{|x|^2}\,dV + \frac{1}{2r^2}\int_M \varphi^2\,dV \\ \leq \int_M \left(|\nabla_T \varphi|^2 + \frac{n-2}{2}\frac{|H|\,\varphi^2}{|x|} + \frac{1}{4}|H\varphi|^2\right)dV,\end{aligned} \tag{2.1.10}$$

where ν_M is the unit normal to M in $\mathbb{R}^{n+1}$.

The proof of this result combines the one of Theorem 2.1.4 (which uses the ground state substitution) with a Poincaré inequality in hypersurfaces of $\mathbb{R}^{n+1}$, stated in Proposition 2.4.2. The former argument brings the first mean curvature term in (2.1.10), while the latter brings the second one. Note that these are the same curvature terms that appear in (2.1.7) and (2.1.4).

2.1.1 Structure of the chapter

In Section 2.2 we give a quick and easy-to-read proof of the Michael-Simon and Allard inequality. In Section 2.3 we prove the Hardy inequalities stated in Theorems 2.1.3 and 2.1.4. Finally, Section 2.4 deals with the Hardy-Sobolev inequality of Corollary 2.1.5 and the improved Hardy-Poincaré inequality of Theorem 2.1.6. The appendices concern tangential derivatives and divergence theorems on hypersurfaces, as well as optimal constants in the Michael-Simon and Allard inequality.

2.2 The Michael-Simon and Allard inequality

In this section we present a proof of the Michael-Simon and Allard inequality on hypersurfaces of $\mathbb{R}^{n+1}$ stated in Theorem 2.1.2. This result is a generalization of the isoperimetric inequality on minimal surfaces of Miranda [113] and it is due to Michael and Simon [110] and independently to Allard [10]. Throughout the chapter, M is an n-dimensional smooth hypersurface of $\mathbb{R}^{n+1}$ with mean curvature H, while E is a bounded subset of M with n-dimensional Hausdorff measure $|E|$ and perimeter $\mathrm{Per}(E)$.

In the proof of Theorem 2.1.2, the notions of tangential derivatives and tangential divergence are crucial. We introduce them in Definition 2.A.1.1, following the book of Giusti [97]. We also use the following divergence formula on M — see (2.A.1.6) in Appendix 2.A.1 for details. If Z is a *tangent* vector field on M, Ω a smooth domain in M, $\mathrm{div}_T Z$ the tangential divergence with respect to the hypersurface M, and ν_Ω is the outer normal vector along $\partial\Omega$ to Ω, then

$$\int_\Omega \mathrm{div}_T Z\,dV = \int_{\partial\Omega} Z\cdot \nu_\Omega\,dA. \tag{2.2.1}$$

In the proof of Theorem 2.1.2, we apply (2.2.1) in the domain $E_\rho = E \cap B_\rho(y)$, where $B_\rho(y)$ is the ball of $\mathbb{R}^{n+1}$ with radius ρ and center $y \in E$. In general, the boundary of E_ρ is not smooth. However, applying Sard's theorem on ∂E to the function "distance to y" defined on ∂E, we deduce that almost all its values are regular on ∂E and, hence, that the boundary of E_ρ is Lipschitz for almost every $\rho > 0$. At the same time, it is possible to state (2.2.1) for a domain Ω with Lipschitz boundary, approximating it with a sequence of smooth sets.

By computing the tangential divergence of the position vector x, we can deduce an important equality which is the starting point of the proof of Theorem 2.1.2:

$$\mathrm{div}_T x = \sum_{i=1}^{n+1} \delta_i x^i = \sum_{i=1}^{n+1} \left(\partial_i x^i - \nu_M^i \sum_{j=1}^{n+1} (\partial_j x^i) \nu_M^j \right) = n + 1 - \sum_{i=1}^{n+1} (\nu_M^i)^2 = n, \tag{2.2.2}$$

where δ_i for $i = 1, \ldots, n+1$ denote the tangential derivatives defined in Appendix 2.A.1. Before starting the proof of Theorem 2.1.2, we also recall that

$$H = \mathrm{div}_T \nu_M,$$

where ν_M is the normal vector to M — not to be confused with ν_Ω in (2.2.1) —, and that the mean curvature vector is $\mathcal{H} = H\nu_M$.

Proof of Theorem 2.1.2. Let $y \in E$ and define $E_\rho := E \cap B_\rho(y)$, where $B_\rho(y)$ is the ball of $\mathbb{R}^{n+1}$ centered at y of radius $\rho > 0$. We start the proof by showing the validity for almost every $\rho > 0$ of the inequality

$$n|E_\rho| \le \rho \Big(\mathrm{Per}(E_\rho) + \int_{E_\rho} |H| \, dV \Big). \tag{2.2.3}$$

To prove it, for simplicity and without loss of generality, we may take $y = 0$. We denote by ν_{E_ρ} the outer normal vector along ∂E_ρ to E_ρ. We call x_T the tangential part of the position vector x with respect to the hypersurface M and thus, using (2.A.1.2), we have

$$\begin{aligned} \mathrm{div}_T x &= \mathrm{div}_T \big(x_T + (x \cdot \nu_M)\nu_M \big) \\ &= \mathrm{div}_T x_T + \nabla_T (x \cdot \nu_M) \cdot \nu_M + (x \cdot \nu_M) \mathrm{div}_T \nu_M \\ &= \mathrm{div}_T x_T + (x \cdot \nu_M) H. \end{aligned}$$

Integrating in E_ρ, and using (2.2.2) and (2.2.1), we deduce

$$\begin{aligned} n|E_\rho| &= \int_{E_\rho} \mathrm{div}_T x \, dV = \int_{\partial E_\rho} x_T \cdot \nu_{E_\rho} \, dA + \int_{E_\rho} (x \cdot \nu_M) H \, dV \\ &\le \rho \, \mathrm{Per}(E_\rho) + \rho \int_{E_\rho} |H| \, dV, \end{aligned}$$

proving (2.2.3).

Back to a general point $y \in E$, note that

$$\mathrm{Per}(E_\rho) = \mathrm{Per}(E, B_\rho(y)) + \mathrm{Per}(B_\rho(y), E),$$

where $\mathrm{Per}(E, B_\rho(y))$ is the relative perimeter of E inside of $B_\rho(y)$ and $\mathrm{Per}(B_\rho(y), E)$ is defined in an analogue way. Thus, we can rewrite (2.2.3) as

$$\frac{d}{d\rho} \Big(-\rho^{-n} |E_\rho| \Big) \le \rho^{-n} \Big(\mathrm{Per}(E, B_\rho(y)) + \int_{E_\rho} |H| \, dV \Big),$$

which is equivalent to

$$\frac{d}{d\rho}\left(\rho^{-n}|E_\rho| \exp \int_0^\rho \frac{\mathrm{Per}(E, B_\sigma(y)) + \int_{E_\sigma} |H|\, dV}{|E_\sigma|}\, d\sigma\right) \geq 0.$$

Thus, the function between parentheses is monotone nondecreasing in ρ, and hence

$$\rho^{-n}|E_\rho| \exp \int_0^\rho \frac{\mathrm{Per}(E, B_\sigma(y)) + \int_{E_\sigma} |H|\, dV}{|E_\sigma|}\, d\sigma \geq \lim_{\rho \to 0} \rho^{-n}|E_\rho| = \omega_n,$$

where ω_n is the volume of the unit ball of $\mathbb{R}^n$.

By choosing $\rho_0 := \left(2|E|\omega_n^{-1}\right)^{\frac{1}{n}}$, we deduce that

$$\exp \int_0^{\rho_0} \frac{\mathrm{Per}(E, B_\sigma(y)) + \int_{E_\sigma} |H|\, dV}{|E_\sigma|}\, d\sigma \geq \rho_0^n \omega_n |E_{\rho_0}|^{-1} \geq \rho_0^n \omega_n |E|^{-1} = 2.$$

Therefore, for every point $y \in E$, there exists a radius $r(y) \in (0, \rho_0)$ such that

$$\rho_0 \Big(\mathrm{Per}(E, B_{r(y)}(y)) + \int_{E_{r(y)}} |H|\, dV\Big) \geq |E_{r(y)}| \log 2.$$

If we substitute the chosen value for ρ_0, we find

$$|E_{r(y)}| \leq C|E|^{\frac{1}{n}} \left(\mathrm{Per}(E, B_{r(y)}(y)) + \int_{E_{r(y)}} |H|\, dV\right), \tag{2.2.4}$$

for some constant C depending only on the dimension n.

Now, since $y \in E$ is arbitrary, we have that every point in the set E is the center of a ball $B(y) = B_{r(y)}(y)$ for which (2.2.4) holds. Since the union of these balls covers E, the Besicovitch covering theorem gives the existence of a countable sub-collection of balls $\{B(y_i)\}_i$, with the same radii $r(y_i)$ as before, such that

$$E \subset \bigcup B(y_i)$$

and such that every point in E belongs at most to N_n of the balls $B(y_i)$, where N_n is a constant depending only on n. Combining this covering argument with (2.2.4), we conclude (2.1.2). □

Now, it is standard to deduce the Sobolev inequality of Theorem 2.1.1 from the isoperimetric inequality (2.1.2).

Proof of Theorem 2.1.1. Step 1. First, we prove that for every smooth φ it holds that

$$\left(\int_M |\varphi|^{\frac{n}{n-1}}\, dV\right)^{\frac{n-1}{n}} \leq C \int_M \big(|\nabla_T \varphi| + |H\varphi|\big)\, dV, \tag{2.2.5}$$

where C is a positive constant depending only on n.

Let μ be the measure on M defined by $d\mu = |\varphi|^{\frac{1}{n-1}}\, dV$. Then, by Cavalieri's principle it holds that

$$\begin{aligned}
\int_M |\varphi|^{\frac{n}{n-1}}\, dV &= \int_M |\varphi|\, d\mu = \int_0^{+\infty} \mu\left(\{|\varphi| > t\}\right) dt = \int_0^{+\infty} \int_{\{|\varphi|>t\}} |\varphi|^{\frac{1}{n-1}}\, dV\, dt \\
&\le \int_0^{+\infty} \left(\int_{\{|\varphi|>t\}} |\varphi|^{\frac{n}{n-1}}\, dV \right)^{\frac{1}{n}} |\{|\varphi| > t\}|^{\frac{n-1}{n}}\, dt \qquad (2.2.6) \\
&\le \left(\int_M |\varphi|^{\frac{n}{n-1}}\, dV \right)^{\frac{1}{n}} \int_0^{+\infty} |\{|\varphi| > t\}|^{\frac{n-1}{n}}\, dt,
\end{aligned}$$

where we used Hölder's inequality in the second line.

From the regularity of φ and Sard's theorem, we have that the set of singular values of φ has zero Lebesgue measure. Considering only regular values t in the last line of (2.2.6), we can apply Theorem 2.1.2 to the set $E = \{|\varphi| > t\}$. In this way, we obtain

$$\begin{aligned}
\left(\int_M |\varphi|^{\frac{n}{n-1}}\, dV \right)^{\frac{n-1}{n}} &\le \int_0^{+\infty} |\{|\varphi| > t\}|^{\frac{n-1}{n}}\, dt \\
&\le C \left(\int_0^{+\infty} |\{|\varphi| = t\}|\, dt + \int_0^{+\infty} \int_{\{|\varphi|>t\}} |H|\, dV\, dt \right).
\end{aligned} \qquad (2.2.7)$$

Now, in the first integral in the right-hand side of (2.2.7) we use the coarea formula on manifolds — see [54, Theorem VIII.3.3.] — to write

$$\int_0^{+\infty} |\{|\varphi| = t\}|\, dt = \int_M |\nabla_T \varphi|\, dV.$$

Finally, plugging this identity in (2.2.7) and applying Fubini's Theorem on the last integral in (2.2.7), we obtain (2.2.5).

Step 2. We can easily extend (2.2.5) to the case of an exponent $p \in [1, n)$, proving (2.1.1). In order to do this, we define $\psi = |\varphi|^{s-1} \varphi$, with $s = p^*/1^*$, and we apply (2.2.5) to ψ. We obtain

$$\left(\int_M |\varphi|^{\frac{ns}{n-1}}\, dV \right)^{\frac{n-1}{n}} \le C \int_M |\varphi|^{s-1} \left(s\, |\nabla_T \varphi| + |H\varphi| \right) dV.$$

Now, exploiting that $ns/(n-1) = 1^* s = p^*$, using a Hölder inequality in the right-hand side with exponents p and p', and taking into account that $(s-1)p' = p^*$, we get

$$\left(\int_M |\varphi|^{p^*}\, dV \right)^{\frac{n-1}{n}} \le C \left(\int_M |\varphi|^{p^*}\, dV \right)^{\frac{p-1}{p}} \left(\int_M \left(s\, |\nabla_T \varphi| + |H\varphi| \right)^p dV \right)^{\frac{1}{p}}.$$

This establishes Theorem 2.1.1. □

2.3 Hardy inequalities on hypersurfaces

In this section we establish the two Hardy inequalities on hypersurfaces of $\mathbb{R}^{n+1}$ stated in Theorems 2.1.3 and 2.1.4. For the first one, we also prove a general version with exponent $p \ge 1$, which is stated in Theorem 2.3.1 below.

2.3.1 Hardy inequality through integration by parts

In this subsection we prove the following Hardy inequality, which is the version of Theorem 2.1.3 for a general exponent $p \geq 1$.

Theorem 2.3.1. *Let M be a smooth hypersurface of $\mathbb{R}^{n+1}$, $p \geq 1$, and $a \in [0, n)$. Then, for every $\varphi \in C^1_c(M)$ we have*

$$\begin{aligned}(n-a)\int_M \frac{|\varphi|^p}{|x|^a}\,dV + a\int_M \left(\frac{x}{|x|}\cdot \nu_M\right)^2 \frac{|\varphi|^p}{|x|^a}\,dV \\ \leq \left(\int_M \frac{|\varphi|^p}{|x|^a}\,dV\right)^{\frac{p-1}{p}} \left(\int_M \frac{|p\nabla_T\varphi - \mathcal{H}\varphi|^p}{|x|^{a-p}}\,dV\right)^{\frac{1}{p}}.\end{aligned} \tag{2.3.1}$$

By throwing the second term in the left-hand side of (2.3.1) and taking $p = a < n$, we deduce that the Hardy inequality in its Euclidean form and with its best constant,

$$\frac{(n-p)^p}{p^p}\int_M \frac{|\varphi|^p}{|x|^p}\,dV \leq \int_M |\nabla\varphi|^p\,dV, \tag{2.3.2}$$

holds on every minimal hypersurface M for all $p \in [1, n)$. As mentioned in our comments following (2.1.6), this inequality also follows from a result in [104].

We recall that, when $M = \mathbb{R}^n$, for $1 < p < n$ the optimal constant in (2.3.2) is not achieved by any function in the homogeneous Sobolev space $\dot{W}^{1,p}(\mathbb{R}^n)$ — the completion of $C^1_c(\mathbb{R}^n)$ with respect to the right-hand side of (2.3.2); see [92]. On the contrary, if $p = 1$, every radially symmetric decreasing function realizes the equality in (2.3.2) — as it can be checked using the coarea formula, the layer cake representation for the function φ, and the fact that $\operatorname{div}(x/|x|) = (n-1)/|x|$.

Proof of Theorem 2.3.1. Using formula (2.2.2) for the tangential divergence of the position vector x, and then integrating by parts according to (2.A.1.5), we can write

$$\begin{aligned} n\int_M \frac{|\varphi|^p}{|x|^a}\,dV &= \int_M \frac{|\varphi|^p}{|x|^a}\operatorname{div}_T x\,dV \\ &= -\int_M \left(p\frac{|\varphi|^{p-2}\varphi}{|x|^a}\nabla_T\varphi\cdot x + |\varphi|^p\, x\cdot\nabla_T|x|^{-a} - \frac{|\varphi|^p}{|x|^a}\mathcal{H}\cdot x\right)dV.\end{aligned}$$

Now, recalling that the tangential part of the position vector x is $x_T = x - (x\cdot\nu_M)\nu_M$, we compute

$$x\cdot\nabla_T|x|^{-a} = -a|x|^{-a-2}\,x\cdot x_T = -a|x|^{-a} + a\left(\frac{x}{|x|}\cdot\nu_M\right)^2|x|^{-a}.$$

Hence, we have

$$\begin{aligned}(n-a)\int_M \frac{|\varphi|^p}{|x|^a}\,dV + a\int_M \left(\frac{x}{|x|}\cdot \nu_M\right)^2 \frac{|\varphi|^p}{|x|^a}\,dV \\ = -\int_M \frac{|\varphi|^{p-2}\varphi}{|x|^{a-1}}\left(p\nabla_T\varphi\cdot\frac{x}{|x|} - \varphi\mathcal{H}\cdot\frac{x}{|x|}\right)dV \\ \leq \int_M \frac{|\varphi|^{p-1}}{|x|^{a-1}}|p\nabla_T\varphi - \mathcal{H}\varphi|\,dV.\end{aligned} \tag{2.3.3}$$

Finally, we apply Hölder's inequality with exponents p and p' to the last integral in (2.3.3), obtaining

$$\begin{aligned}\int_M \frac{|\varphi|^{p-1}}{|x|^{a-1}} |p\nabla_T\varphi - \mathcal{H}\varphi| \, dV &= \int_M \frac{|\varphi|^{p-1}}{|x|^{a(p-1)/p}} \frac{|p\nabla_T\varphi - \mathcal{H}\varphi|}{|x|^{(a-p)/p}} \, dV \\ &\le \left(\int_M \frac{|\varphi|^p}{|x|^a} \, dV \right)^{\frac{p-1}{p}} \left(\int_M \frac{|p\nabla_T\varphi - \mathcal{H}\varphi|^p}{|x|^{a-p}} \, dV \right)^{\frac{1}{p}}.\end{aligned}$$

Plugging this bound in (2.3.3), we obtain (2.3.1) and finish the proof of Theorem 2.3.1. □

When $p = 2$ and $n \ge 3$, we exploit a nice simplification in (2.3.1) and prove Theorem 2.1.3.

Proof of Theorem 2.1.3. We use (2.3.1) with $p = 2$. Then, since the vectors $\nabla_T\varphi$ and $\mathcal{H}$ are orthogonal, we have

$$|2\nabla_T\varphi - \mathcal{H}\varphi|^2 = 4\,|\nabla_T\varphi|^2 + |\mathcal{H}\varphi|^2$$

and Theorem 2.1.3 follows directly from Theorem 2.3.1. □

From Theorem 2.3.1 we deduce a version with exponent p for the foliated Hardy inequality (2.1.3) that Cabré established for $p = 2$ in [30]. In the statement, we use the following notation for the radial derivative:

$$u_r = \nabla u \cdot \frac{x}{|x|}.$$

Recall that the mean curvature $\mathcal{H}$ and the tangential gradient ∇_T refer to the level sets of the function u. The result is the following.

Corollary 2.3.2. *Let Ω be a smooth bounded domain of $\mathbb{R}^{n+1}$, u a $C^\infty(\overline{\Omega})$ function, $p \ge 1$, and $a \in [0, n)$. Then, for every $\varphi \in C^1_c(\Omega)$ we have*

$$\begin{aligned}(n-a) \int_\Omega |\nabla u| \frac{|\varphi|^p}{|x|^a} \, dx &+ a \int_\Omega \frac{u_r^2}{|\nabla u|} \frac{|\varphi|^p}{|x|^a} \, dx \\ &\le \left(\int_\Omega |\nabla u| \frac{|\varphi|^p}{|x|^a} \, dx \right)^{\frac{p-1}{p}} \left(\int_\Omega |\nabla u| \frac{|p\nabla_T\varphi - \mathcal{H}\varphi|^p}{|x|^{a-p}} \, dx \right)^{\frac{1}{p}}.\end{aligned} \tag{2.3.4}$$

Proof. Using the coarea formula in Euclidean space for the two integrals in the left-hand side of (2.3.4), we see that

$$\begin{aligned}(n-a) \int_\Omega |\nabla u| \frac{|\varphi|^p}{|x|^a} \, dx &+ a \int_\Omega \frac{u_r^2}{|\nabla u|} \frac{|\varphi|^p}{|x|^a} \, dx \\ &= (n-a) \int_{\mathbb{R}} \int_{\{u=t\}} \frac{|\varphi|^p}{|x|^a} \, dV \, dt + a \int_{\mathbb{R}} \int_{\{u=t\}} \left(\frac{x}{|x|} \cdot \frac{\nabla u}{|\nabla u|} \right)^2 \frac{|\varphi|^p}{|x|^a} \, dV \, dt.\end{aligned} \tag{2.3.5}$$

Now, by Sard's theorem, $\{u = t\}$ is a smooth hypersurface of $\mathbb{R}^{n+1}$ for almost every $t \in \mathbb{R}$, and the normal vector ν_M of $M = \{u = t\}$ is

$$\nu_M = \frac{\nabla u}{|\nabla u|}.$$

Therefore, we can apply (2.3.1) to the function φ on each smooth hypersurface $M = \{u = t\}$ and then integrate in dt, obtaining

$$\begin{aligned}(n-a)\int_{\mathbb{R}}\int_{\{u=t\}} \frac{|\varphi|^p}{|x|^a}\,dV\,dt + a\int_{\mathbb{R}}\int_{\{u=t\}} \left(\frac{x}{|x|}\cdot\frac{\nabla u}{|\nabla u|}\right)^2 \frac{|\varphi|^p}{|x|^a}\,dV\,dt \\ \le \left(\int_{\mathbb{R}}\int_{\{u=t\}} \frac{|\varphi|^p}{|x|^a}\,dV\,dt\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}}\int_{\{u=t\}} \frac{|p\nabla_T\varphi - \mathcal{H}\varphi|^p}{|x|^{a-p}}\,dV\,dt\right)^{\frac{1}{p}},\end{aligned}$$

where we have used Hölder's inequality for an integral in dt. Finally, using again the coarea formula and combining this inequality with (2.3.5), we deduce (2.3.4). $\square$

2.3.2 Hardy inequality through a ground state substitution

In this subsection we prove Theorem 2.1.4 using a method known as the ground state substitution. Within the proof we will need that

$$\begin{aligned}\mathrm{div}_T x_T &= \mathrm{div}_T\big(x-(x\cdot\nu_M)\nu_M\big) = n-(x\cdot\nu_M)\mathrm{div}_T\nu_M - \big(\nabla_T(x\cdot\nu_M)\big)\cdot\nu_M \\ &= n-(x\cdot\nu_M)H,\end{aligned} \tag{2.3.6}$$

where we have used that $\mathrm{div}_T x = n$, by (2.2.2), and that $\mathrm{div}_T\nu_M = H$.

It is now easy to deduce the inequality

$$\Delta|x| \ge \frac{n-1}{|x|} - \left(\frac{x}{|x|}\cdot\nu_M\right)H$$

for the Laplace-Beltrami operator on M — a result mentioned in the Introduction within the context of minimal hypersurfaces. Indeed, we have

$$\begin{aligned}\Delta|x| &= \mathrm{div}_T\nabla_T|x| = \mathrm{div}_T(x_T/|x|) = (\mathrm{div}_T x_T)/|x| + x_T\cdot\nabla_T|x|^{-1} \\ &= \left(n-(x\cdot\nu_M)H\right)/|x| - |x|^{-3}|x_T|^2 \\ &\ge (n-1)/|x| - (x\cdot\nu_M)H/|x|,\end{aligned}$$

as claimed.

Proof of Theorem 2.1.4. We substitute $\varphi(x) = \omega(x)v(x)$, with $\omega(x) = |x|^{-\frac{n-2}{2}}$ and $v \in C^1_c(M)$, in the gradient term

$$\int_M |\nabla_T\varphi|^2\,dV = \int_M |v\nabla_T\omega + \omega\nabla_T v|^2\,dV. \tag{2.3.7}$$

Applying the convexity inequality $|a+b|^2 \ge |a|^2 + 2\,a\cdot b$, valid for all vectors $a, b \in \mathbb{R}^n$, we obtain

$$\int_M |\nabla_T\varphi|^2\,dV \ge \int_M v^2\,|\nabla_T\omega|^2\,dV + \int_M \omega\,\nabla_T\omega\cdot\nabla_T\left(v^2\right)\,dV.$$

Using the formula of integration by parts (2.A.1.5), we get

$$\begin{aligned}\int_M |\nabla_T\varphi|^2\,dV \ge \int_M v^2\,|\nabla_T\omega|^2\,dV - \int_M v^2\mathrm{div}_T\left(\omega\nabla_T\omega\right)\,dV \\ + \int_M \omega\,v^2\nabla_T\omega\cdot\mathcal{H}\,dV.\end{aligned} \tag{2.3.8}$$

Since $\nabla_T\omega$ is a tangent vector and the mean curvature vector $\mathcal{H}$ is normal to M, the last term in (2.3.8) vanishes. Exploiting an additional cancellation after developing the divergence in (2.3.8), we have

$$\int_M |\nabla_T\varphi|^2\,dV \geq -\int_M \omega\, v^2 \mathrm{div}_T\left(\nabla_T\omega\right)\,dV. \tag{2.3.9}$$

Next, we compute the tangential divergence of the vector field $\nabla_T\omega$, where $\omega(x) = |x|^\alpha$ with $\alpha = -(n-2)/2$. The tangential gradient of ω is

$$\nabla_T\omega = \alpha\,|x|^{\alpha-2}\,x_T = \alpha\,|x|^{\alpha-2}\left(x - (x\cdot\nu_M)\nu_M\right).$$

Hence, using (2.3.6), we have

$$\begin{aligned}-\mathrm{div}_T\left(\nabla_T\omega\right) &= -\alpha\,\mathrm{div}_T\left(|x|^{\alpha-2}\left(x-(x\cdot\nu_M)\nu_M\right)\right)\\ &= -\alpha\,|x|^{\alpha-2}\left(n - x\cdot\nu_M H\right) - \alpha(\alpha-2)\,|x_T|^2\,|x|^{\alpha-4}.\end{aligned}$$

We plug this into (2.3.9), recalling that $\omega(x) = |x|^\alpha$, and obtain

$$\begin{aligned}\int_M |\nabla_T\varphi|^2\,dV &\geq \alpha\int_M |x|^{2\alpha-2}\,v^2 x\cdot\mathcal{H}\,dV\\ &\quad - n\alpha\int_M |x|^{2\alpha-2}\,v^2\,dV - \alpha(\alpha-2)\int_M |x_T|^2\,|x|^{2\alpha-4}\,v^2\,dV.\end{aligned} \tag{2.3.10}$$

Now we move the first integral in the right-hand side of (2.3.10) to the left-hand side of the inequality, and observe that $|x|^{2\alpha-2}\,v^2 = \varphi^2/\,|x|^2$. Therefore, (2.3.10) reads

$$\begin{aligned}&\int_M\left(|\nabla_T\varphi|^2 + \frac{n-2}{2}\,\frac{\varphi^2}{|x|^2}\,x\cdot\mathcal{H}\right)\,dV\\ &\qquad\geq -n\alpha\int_M \frac{\varphi^2}{|x|^2}\,dV - \alpha(\alpha-2)\int_M \frac{|x_T|^2}{|x|^2}\,\frac{\varphi^2}{|x|^2}\,dV.\end{aligned}$$

In the last integral, we have $|x_T|^2 = |x|^2 - (x\cdot\nu_M)^2$ and thus the inequality becomes

$$\begin{aligned}&\int_M\left(|\nabla_T\varphi|^2 + \frac{n-2}{2}\,\frac{\varphi^2}{|x|^2}\,x\cdot\mathcal{H}\right)\,dV\\ &\qquad\geq -\alpha(n+\alpha-2)\int_M \frac{\varphi^2}{|x|^2}\,dV + \alpha(\alpha-2)\int_M\left(\frac{x}{|x|}\cdot\nu_M\right)^2\frac{\varphi^2}{|x|^2}\,dV.\end{aligned}$$

Finally, since $-\alpha\,(n+\alpha-2) = (n-2)^2/4$ and $\alpha(\alpha-2) = (n^2-4)/4$, we conclude (2.1.7). $\square$

2.4 Hardy-Sobolev and Hardy-Poincaré inequalities on hypersurfaces

In this section we prove the Hardy-Sobolev inequality stated in Corollary 2.1.5 and the Hardy-Poincaré inequality of Theorem 2.1.6.

We start from the Hardy-Sobolev inequality on hypersurfaces, that we obtain as an interpolation of the Michael-Simon and Allard inequality and the Hardy inequality of Theorem 2.3.1. We state and prove here our result for a general power $p \in [1, n)$.

Corollary 2.4.1. *Let M be a smooth hypersurface of $\mathbb{R}^{n+1}$, $p \in [1,n)$, and $b \in [0,1]$. Then, for every $\varphi \in C^1_c(M)$ we have*

$$\left(\int_M \frac{|\varphi|^{p\frac{n-bp}{n-p}}}{|x|^{bp}}\, dV\right)^{\frac{n-p}{n-bp}} \leq C \int_M \left(\, |\nabla_T \varphi|^p + |H\varphi|^p \,\right) dV, \tag{2.4.1}$$

for some positive constant C depending only on n and p.

Proof. First, from (2.3.1) with $a = p$ it follows that

$$\begin{aligned}(n-p)\int_M \frac{|\varphi|^p}{|x|^p}\, dV &\leq (n-p)\int_M \frac{|\varphi|^p}{|x|^p}\, dV + p\int_M \left(\frac{x}{|x|}\cdot \nu_M\right)^2 \frac{|\varphi|^p}{|x|^p}\, dV \\ &\leq \left(\int_M \frac{|\varphi|^p}{|x|^p}\, dV\right)^{\frac{p-1}{p}} \left(\int_M |p\nabla_T\varphi - \mathcal{H}\varphi|^p\, dV\right)^{\frac{1}{p}}.\end{aligned}$$

Raising the inequality to the power p and using the convexity inequality $|a+b|^p \leq 2^{p-1}\left(|a|^p + |b|^p\right)$, we obtain

$$\begin{aligned}(n-p)^p \int_M \frac{|\varphi|^p}{|x|^p}\, dV &\leq \int_M |p\nabla_T\varphi - \mathcal{H}\varphi|^p\, dV \\ &\leq 2^{p-1}\int_M \left(p^p\, |\nabla_T\varphi|^p + |H\varphi|^p\right)\, dV.\end{aligned} \tag{2.4.2}$$

Observe that, if $b = 0$ or $b = 1$, then (2.4.1) follows respectively from the Michael-Simon and Allard inequality (2.1.1) or from the Hardy inequality (2.4.2). Thus, we can assume $b \in (0,1)$ in the rest of the proof.

Now, we consider the integral in the left-hand side of (2.4.1). Using Hölder's inequality with exponents $1/b$ and $1/(1-b)$, the Hardy inequality (2.4.2), and Theorem 2.1.1, we get

$$\begin{aligned}\int_M \frac{|\varphi|^{p\frac{n-bp}{n-p}}}{|x|^{bp}}\, dV &= \int_M \left(\frac{|\varphi|}{|x|}\right)^{bp} |\varphi|^{(1-b)\frac{np}{n-p}}\, dV \\ &\leq \left(\int_M \frac{|\varphi|^p}{|x|^p}\, dV\right)^b \left(\int_M |\varphi|^{p^*}\, dV\right)^{1-b} \\ &\leq C\left(\int_M \left(\, |\nabla_T\varphi|^p + |H\varphi|^p \,\right) dV\right)^{\beta},\end{aligned}$$

where C is a positive constant depending only on n and p, while β is

$$\beta = b + \frac{(1-b)p^*}{p} = \frac{n-bp}{n-p}.$$

Finally, raising the inequality to the power $1/\beta$, (2.4.1) is established. Observe that, since $\beta > 1$, $C^{1/\beta} \leq C$ if we take $C \geq 1$. Hence, the final constant depends only on n and p. □

The remaining part of this section is devoted to the proof an improved Hardy inequality in the Poincaré sense, stated in Theorem 2.1.6. Its proof is based on a modification of the ground state substitution method, that we have used in Theorem 2.1.4, and on a Poincaré inequality with weights stated next.

The following is a Poincaré inequality with exponent $p \geq 1$ and a weight of the type $|x|^{-a}$, for functions with compact support on a hypersurface M (more precisely, with support in a ball of radius r).

Proposition 2.4.2. *Let M be a smooth hypersurface of $\mathbb{R}^{n+1}$, $B_r = B_r(0) \subset \mathbb{R}^{n+1}$ the open ball of radius r centered at the origin, $p \geq 1$, and $a \in [0, n)$. Then, for every $\varphi \in C^1_c(B_r \cap M)$ we have*

$$(n-a)^p \int_M \frac{|\varphi|^p}{|x|^a}\, dV \leq 2^{p-1} r^p \int_M \left(p^p \frac{|\nabla_T \varphi|^p}{|x|^a} + \frac{|H\varphi|^p}{|x|^a} \right) dV. \tag{2.4.3}$$

Proof. As in the proof of Corollary 2.4.1, but with $a \in [0, n)$ instead of $a = p$, from (2.3.1) we obtain

$$(n-a)^p \int_M \frac{|\varphi|^p}{|x|^a}\, dV \leq 2^{p-1} \int_M \frac{p^p\, |\nabla_T \varphi|^p + |H\varphi|^p}{|x|^{a-p}}\, dV.$$

Then, taking advantage of the fact that the support of φ is contained in $B_r(0)$, we can bound $|x|^p \leq r^p$ and obtain (2.4.3). □

Now, we can prove Theorem 2.1.6. Note that here we assume $p = 2$ and $n \geq 2$.

Proof of Theorem 2.1.6. As in the proof of Theorem 2.1.4, we use the ground state substitution $\varphi = v\omega$, where $\omega(x) = |x|^{-(n-2)/2}$. We proceed as in the proof of Theorem 2.1.4, but in the right-hand side of (2.3.7) we use the identity[6] $|a+b|^2 = |a|^2 + 2\, a \cdot b + |b|^2$ for vectors $a, b \in \mathbb{R}^n$. Therefore, we find

$$\begin{aligned} \int_M \left(|\nabla_T \varphi|^2 + \frac{n-2}{2} \frac{|H|\, \varphi^2}{|x|} \right) dV \geq{}& \frac{(n-2)^2}{4} \int_M \frac{\varphi^2}{|x|^2}\, dV \\ &+ \frac{n^2-4}{4} \int_M \left(\frac{x}{|x|} \cdot \nu_M \right)^2 \frac{\varphi^2}{|x|^2}\, dV + \int_M \frac{|\nabla_T v|^2}{|x|^{n-2}}\, dV. \end{aligned} \tag{2.4.4}$$

Next, to control the last integral in (2.4.4) from below, we use inequality (2.4.3) with $\varphi = v$, $p = 2$, and $a = n-2$. Observe that this forces $n \geq 2$. In this way, we have

$$\int_M \frac{|\nabla_T v|^2}{|x|^{n-2}}\, dV \geq \frac{1}{2r^2} \int_M \frac{v^2}{|x|^{n-2}}\, dV - \frac{1}{4} \int_M \frac{|Hv|^2}{|x|^{n-2}}\, dV. \tag{2.4.5}$$

Finally, combining (2.4.4) and (2.4.5), and using the fact that $v^2 / |x|^{n-2} = \varphi^2$, (2.1.10) is established. □

[6] For an exponent $p \neq 2$, here one would use a well-known convexity inequality instead of this identity (see Lemma 2.6 and Remark 2.7 in [92], or [107, Lemma 4.2]).

2.A.1 Notation for tangential derivatives

In the setting of hypersurfaces of Euclidean space, tangential derivatives can be defined in an elementary calculus way without using Riemannian geometry, for instance as presented in Giusti's book [97]. Throughout the chapter, we adopt this definition of tangential derivatives, that we recall next. From it, one can define the tangential divergence of a vector field. Alternatively, one can define the tangential divergence intrinsically using Riemannian geometry, as done for instance in [54]. In this appendix, and for completeness, we introduce and compare these two notions in the setting of hypersurfaces of $\mathbb{R}^{n+1}$. We start by giving the former definition, following [97].

Definition 2.A.1.1. Let M be a smooth hypersurface of $\mathbb{R}^{n+1}$ with normal vector ν_M.

(a) Let φ be a C^1 function defined on M. We define the $i-$th tangential derivative of φ, for $i = 1, \dots, n+1$, as

$$\delta_i \varphi := \partial_i \varphi - \nu_M^i \sum_{j=1}^{n+1} (\partial_j \varphi) \nu_M^j,$$

where ν_M^j is the $j-$th component of the normal vector ν_M to M and $\partial_j \varphi$ is the $j-$th partial derivative of φ, once the function φ has been extended to all of $\mathbb{R}^{n+1}$.

(b) With φ as in (a), we define the tangential gradient of φ as the vector

$$\nabla_T \varphi = \nabla \varphi - (\nabla \varphi \cdot \nu_M) \nu_M = (\delta_1 \varphi, \delta_2 \varphi, \dots, \delta_{n+1} \varphi).$$

Note that $\nabla_T \varphi \cdot \nu_M = 0$ for every C^1 function φ defined on M.

(c) Let Z be a C^1 vector field defined on M with values in $\mathbb{R}^{n+1}$, not necessarily tangent to M, and whose components are Z^i with $i = 1, \dots, n+1$. We define its tangential divergence as

$$\mathrm{div}_T Z = \sum_{i=1}^{n+1} \delta_i Z^i. \tag{2.A.1.1}$$

From the definitions, it easily follows that

$$\mathrm{div}_T(\varphi Z) = \nabla_T \varphi \cdot Z + \varphi \,\mathrm{div}_T Z. \tag{2.A.1.2}$$

Observe that this definition of tangential derivatives is extrinsic and it does not give a basis of the n-dimensional tangent space of M, as the tangential derivatives δ_i for $i = 1, \dots, n+1$ are linearly dependent. However, if one is familiar with Riemannian geometry, then it is possible to check that, in the case of hypersurfaces of $\mathbb{R}^{n+1}$, the intrinsic Riemannian notion of divergence coincides with div_T defined in (2.A.1.1). We recall that the divergence of a *tangent* vector field Y on a general Riemannian manifold (M, g) is defined in an intrinsic way as

$$\mathrm{div} Y = \mathrm{tr}\big(\xi \longmapsto \nabla_\xi Y\big), \tag{2.A.1.3}$$

where ∇ is the Levi-Civita connection of (M, g). Now, Proposition II.2.1 in [54] states that, given two Riemannian manifolds (M, g) and $(\overline{M}, \overline{g})$ with M isometrically embedded in $\overline{M}$ and whose Levi-Civita connections are ∇ and $\overline{\nabla}$, then for every $p \in M$, $\xi \in T_p M$, and vector field $Y \in TM$ on M, we have that

$$\nabla_\xi Y = (\overline{\nabla}_\xi Y)_T,$$

where $(\overline{\nabla}_\xi Y)_T$ denotes the tangential component of $\overline{\nabla}_\xi Y$ with respect to M. Therefore, if $\overline{M} = \mathbb{R}^{n+1}$, M is an isometrically embedded hypersurface of $\mathbb{R}^{n+1}$, and Y is a tangent vector field on M, then we have

$$\mathrm{div} Y = \mathrm{tr}\big(\xi \longmapsto \nabla_\xi Y\big) = \mathrm{tr}\left(\xi \longmapsto \left(\overline{\nabla}_\xi Y\right)_T\right) = \sum_{i=1}^{n+1} \delta_i Y^i = \mathrm{div}_T Y,$$

where div is defined in (2.A.1.3) and div_T in (2.A.1.1).

Next, adopting the notion of tangential derivatives from Definition 2.A.1.1, we report a formula of integration by parts proved in [97]. For all C^1 functions v and w such that at least one of them has compact support on M, we have that

$$\int_M (\delta_i v)\, w \, dV = -\int_M v\,(\delta_i w)\; dV + \int_M vwH\nu_M^i \, dV, \tag{2.A.1.4}$$

where $i \in \{1, \ldots, n+1\}$, ν_M is the normal vector to M, and H is the mean curvature of M. For the proof of (2.A.1.4) we refer to[7] [97, Lemma 10.8] or to [30, Lemma 2.1]. If instead we consider a C^1 function v and a C^1 vector field Z, such that at least one of them has compact support on M, then from (2.A.1.4) we easily deduce

$$\int_M v\, \mathrm{div}_T Z \, dV = -\int_M \nabla_T v \cdot Z \, dV + \int_M vZ \cdot \mathcal{H} \, dV, \tag{2.A.1.5}$$

where $\mathcal{H} = H\nu_M$ is the mean curvature vector of M. Indeed, to show (2.A.1.5) it is sufficient to write $\mathrm{div}_T Z = \sum_{i=1}^{n+1} \delta_i Z^i$ and apply (2.A.1.4) on every term of the sum.

Observe that, if Z is tangent then the mean curvature term in (2.A.1.5) vanishes — since $\mathcal{H}$ is normal to M.

The following divergence formula with a boundary term is the analogue result to (2.A.1.5) with $v \equiv 1$ when Z does not have compact support. Given a C^1 *tangent* vector field Z defined on M and a smooth domain $\Omega \subset M$, we have that

$$\int_\Omega \mathrm{div}_T Z \, dV = \int_{\partial\Omega} Z \cdot \nu_\Omega \, dA, \tag{2.A.1.6}$$

where $\nu_\Omega \in TM$ is the outward unit normal to Ω. This identity can be proved using a suitable modification of the argument in [97, Lemma 10.8]. One can also deduce (2.A.1.6) from [54, Theorem III.7.5], i.e., the divergence formula on Riemannian manifolds. To this end, one must recall that in [54] the tangential divergence is defined as in (2.A.1.3) and, in the setting of hypersurfaces of $\mathbb{R}^{n+1}$, definition (2.A.1.3) is equivalent to the one we gave in Definition 2.A.1.1.

2.A.2 Optimal constants in the Michael-Simon and Allard inequality

For an integer $k \in [2, n]$, a k-dimensional submanifold M of $\mathbb{R}^{n+1}$ with mean curvature H, and a smooth domain $E \subset M$ with compact closure in M, the Michael-Simon and Allard inequality states that

$$|E|^{\frac{k-1}{k}} \leq C_1 \mathrm{Per}(E) + C_2 \int_E |H| \, dV, \tag{2.A.2.1}$$

[7] We point out two typos in [97, Lemma 10.8]: first, the mean curvature H is missing in the statement, but not in the proof; second, there is a sign error in front of the integral in the right-hand side, both in the statement and in the proof. The correct statement is (2.A.1.4).

for some positive constants C_1 and C_2 depending only on k. Most of the literature on the topic of sharp constants for (2.A.2.1) is focused on one of two important particular cases: either when the submanifolds M are minimal or when they are compact without boundary and we take $E = M$. The proofs in [10, 110] do not give sharp constants in any of these two situations.

In the former case the mean curvature of M is identically zero, and the problem is finding the optimal constant C_1 in the isoperimetric inequality on minimal submanifolds of $\mathbb{R}^{n+1}$. Under the additional assumption that the submanifold is area minimizing, Almgren [11] proved that the isoperimetric inequality with the Euclidean constant holds, i.e., for every smooth domain $E \subset M$ with compact closure in M, one has

$$k\,\omega_k^{\frac{1}{k}}\,|E|^{\frac{k-1}{k}} \leq \operatorname{Per}(E), \tag{2.A.2.2}$$

where ω_k is the volume of the k-dimensional unit ball. Back to the general context of non minimizers, in the case of two-dimensional minimal surfaces of $\mathbb{R}^{n+1}$ (i.e., with $k = 2$) some partial results have been available for a good number of years. Leon Simon obtained the desired inequality with half of the expected constant

$$2\pi\,|E| \leq \operatorname{Per}(E)^2.$$

He never published the proof of this result, but it can be found in the papers [56, 135]. In [135], Topping improved it to give a simple proof of the Michael-Simon and Allard inequality for 2-dimensional submanifolds of $\mathbb{R}^{n+1}$, not necessarily minimal. The constant 2π in Simon's inequality on minimal surfaces was improved by Stone [133] (the same improvement is attributed in [56] also to A. Ros), but still without achieving the constant 4π conjectured in (2.A.2.2). See the survey [56] for a detailed exposition of the problem. Finally, the conjecture for arbitrary dimension k has been very recently proved by Brendle [22] in the case of codimension 1 and 2. His method uses a clever extension of the proof of the sharp Euclidean isoperimetric inequality found by Cabré in [27]. Thus, both proofs use the solution of a Neumann problem, together with the ABP method. In addition, Brendle's proof allows to characterize flat disks as the only cases in which equality is achieved.

The second particular case of (2.A.2.1) consists of M being a compact manifold without boundary and $E = M$. Then, inequality (2.A.2.1) reads

$$|M|^{\frac{k-1}{k}} \leq C_2 \int_M |H|\, dV \tag{2.A.2.3}$$

with $2 \leq k \leq n$, and the problem of finding the optimal constant C_2 is still open. If $M = \partial A$ and $A \subset \mathbb{R}^{n+1}$ is a smooth bounded domain which is also assumed to be convex, then (2.A.2.3) holds with $k = n$ and equality is only achieved when A is a ball, as a consequence of the classical Aleksandrov-Fenchel inequality [8, 9]. More recently, Guan and Li [99], and Huisken and Ilmanen [102], relaxed the convexity assumption with weaker hypothesis on A, obtaining the sharp result in their settings. For a survey on the subject, see [53].

Part II

A Dirichlet to Neumann problem arising in water waves

Introduction to Part II

The second part of this thesis is devoted to the study of a problem arising from the theory of irrotational and inviscid fluids. Sometimes, in the literature people refer to it as a water wave problem. From the mathematical point of view, it is an elliptic Partial Differential Equation which is prescribed on a domain whose boundary possesses two connected components, one endowed with a Dirichlet datum, and the other one with a Neumann datum. The problem can also be reformulated as a nonlocal equation on the component endowed with the Dirichlet datum, as we are going to see in the following.

Broadly speaking, a PDE is a mathematical equation relating the values of an unknown function and its derivatives of different orders. We can check whether a function satisfies the equation at a certain point, provided we know the values of the function in an arbitrarily small neighborhood. On the contrary, in order to check if a function verifies a *nonlocal equation* at a certain point, one needs to know the values of the function at every point of the domain.

However, some types of nonlocal equations are proved to be equivalent to a boundary reaction problem of local nature in one more dimension. For instance, given a function u defined on $\mathbb{R}^n$, one can compute its harmonic extension in $\mathbb{R}^n_+$, as the solution v of $\Delta v = 0$ in $\mathbb{R}^n_+$ which satisfies $v = u$ on $\partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n$. Then, one can easily check that the normal derivative $\partial_\nu v$ on $\mathbb{R}^n$ is $(-\Delta)^{1/2}u$, namely the half-Laplacian of the Dirichlet datum u.

The possibility of writing the fractional Laplacian $(-\Delta)^s$, which is the prototype of nonlocal operators, as a Dirichlet to Neumann operator in one more dimension was established by Caffarelli and Silvestre [46] for all the fractional powers $s \in (0,1)$. The opportunity of studying a fractional problem as a local PDE in one more dimension quickly became a standard tool in the study of nonlocal equations, making possible extensive developments in the field.

From the mathematical point of view, our problem can be stated in the following way. We consider the slab $\mathbb{R}^n \times [0,1]$, with coordinates $x \in \mathbb{R}^n$ and $y \in [0,1]$, a parameter $a \in (-1,1)$, a smooth bounded function u defined on $\mathbb{R}^n$, and a nonlinearity $f \in C^{1,\gamma}(\mathbb{R})$, for some $\gamma > 0$. Then, we are interested in the solutions to the following system of equations:

$$\begin{cases} \operatorname{div}(y^a \nabla v) = 0 & \text{in } \mathbb{R}^n \times (0,1), \\ v_y(x,1) = 0 & \text{on } \mathbb{R}^n \times \{y = 1\}, \\ v(x,0) = u(x) & \text{on } \mathbb{R}^n \times \{y = 0\}, \\ -\lim\limits_{y\to 0} y^a v_y = f(v) & \text{on } \mathbb{R}^n \times \{y = 0\}. \end{cases} \tag{II.1}$$

As mentioned above, this problem can be formulated on the trace as a nonlocal equation of the type

$$\mathcal{L}_a u = f(u) \qquad \text{in } \mathbb{R}^n,$$

for a suitable linear operator $\mathcal{L}_a$, that we will define in the following.

Concerning problem (II.1), we are interested in the study of symmetry properties for some classes of solutions. Specifically, we will study the three dimensional case of the problem, dealing with monotone and minimizing solutions to (II.1). Furthermore, we also focus on the Dirichlet to Neumann operator $\mathcal{L}_a$, clarifying its mixed nature (local and nonlocal). To this end, we deduce the expression of $\mathcal{L}_a$ in Fourier modes, highlighting different asymptotic behaviors for small and large frequencies, that make the problem particularly interesting. Last, we will study the Γ-convergence of an energy functional related to the equation $\mathcal{L}_a u = W'(u)$, where W is a double well potential.

Background and known results

Problem (II.1) is related to a model for water waves. In Appendix A, we recall some basic fluid dynamics motivations to give a description of problem (II.1) in terms of classical physics. In short, we look at $\mathbb{R}^2 \times (0, H)$ as "the sea", where $\{y = 0\}$ is the surface of the sea and $\{y = H\}$ is its bottom. In this setting, we use the irrotationality of the fluid to write the velocity V of the fluid as $V = \nabla v$, where v is a velocity potential in the whole slab $\mathbb{R}^2 \times [0, H]$. Given the density ρ of the fluid, and the values of v on $\{y = 0\}$ denoted by u, we consider the velocity potential v in the whole slab $\mathbb{R}^2 \times [0, H]$ that solves

$$\begin{cases} 0 = \operatorname{div}(\rho V) = \operatorname{div}(\rho \nabla v) & \text{in } \mathbb{R}^2 \times (0, H), \\ 0 = V_3\big|_{y=H} = v_y\big|_{y=H} & \text{on } \mathbb{R}^2 \times \{y = H\}, \\ v = u\big|_{y=0} & \text{on } \mathbb{R}^2 \times \{y = 0\}. \end{cases} \tag{II.2}$$

We point out that the first equation in (II.2) models the mass conservation and the irrotationality of the fluid, and the second one is a consequence of the impenetrability of the matter — we refer to Appendix A for a complete description of the model. Then, given the datum of the velocity potential v on the surface, we are interested in studying the weighted vertical velocity at $y = 0$, which is responsible for the formation of a wave starting from the rest position of a "flat sea". Thus, the operator that we want to study is

$$\mathcal{L}_a u(x) := -\lim_{y \to 0} \rho(y) v_y(x, y). \tag{II.3}$$

When $\rho := 1$ and $H \to +\infty$, which is the case of a fluid with constant density and an "infinitely deep sea", the operator $\mathcal{L}_a$ is the square root of the Laplacian, see e.g. [46]. For finite values of H the operator described in (II.3) is nonlocal, but also not of purely fractional type, as we are going to see.

In the following, we choose

$$\rho(y) := y^a \tag{II.4}$$

as a density, where $a \in (-1, 1)$. We notice that, in this case,

$$\text{the limit as } H \to +\infty \text{ corresponds to the } s\text{-th root of the Laplacian,} \tag{II.5}$$

with $s := (1 - a)/2$, but for a finite value of H the problem is not of purely fractional type. From now on, we normalize the domain by setting $H := 1$. From a physical point of view, the choice in (II.4) corresponds to the situation in which the density of the fluid at a point depends only on the depth, in a power-like fashion, and it is constant in the horizontal directions.

After generalizing the physical setting $\mathbb{R}^2 \times [0,1]$ to the mathematically interesting case $\mathbb{R}^n \times [0,1]$ — with coordinates $x \in \mathbb{R}^n$ and $y \in [0,1]$ — the extension problem in (II.2) reads

$$\begin{cases} \operatorname{div}(y^a \nabla v) = 0 & \text{in } \mathbb{R}^n \times (0,1), \\ v_y(x,1) = 0 & \text{on } \mathbb{R}^n \times \{y=1\}, \\ v(x,0) = u(x) & \text{on } \mathbb{R}^n \times \{y=0\}. \end{cases} \tag{II.6}$$

Therefore, in light of (II.4), the Dirichlet to Neumann operator $\mathcal{L}_a$ in (II.3) is given by

$$\mathcal{L}_a u(x) = -\lim_{y\to 0} y^a v_y(x,y), \tag{II.7}$$

and, for a given nonlinearity $f \in C^{1,\gamma}(\mathbb{R})$, we want to study the nonlocal equation

$$\mathcal{L}_a u(x) = f(u) \qquad \text{in } \mathbb{R}^n. \tag{II.8}$$

As a technical remark, we notice that, in order to have the operator $\mathcal{L}_a$ well defined for every smooth function u defined on $\mathbb{R}^n$, we need to choose the extension v in (II.6) in a unique way. Indeed, for example, if v is a solution of (II.6) with $a = 0$, then so is the function $v(x,y) + e^{\pi x/2} \sin(\pi y/2)$. To overcome this problem and uniquely determine v in (II.6), we choose among all the possible solutions of (II.6) the one which is a minimizer of the Dirichlet energy

$$\mathcal{E}_K(w) := \int_{\mathbb{R}^n \times (0,1)} y^a \, |\nabla w(x,y)|^2 \, dx\, dy, \tag{II.9}$$

in the class of all the functions $w \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n \times (0,1), y^a)$ such that $w(x,0) = u(x)$. Such a minimizer v exists, it is unique, due to the convexity of the energy functional in (II.9), and it solves the problem in (II.6) — see Lemma 4.2.1 for the detailed proof of this existence and uniqueness result.

With the setting in (II.6), problem (II.8) can be formulated in the following way:

$$\begin{cases} \operatorname{div}(y^a \nabla v) = 0 & \text{in } \mathbb{R}^n \times (0,1), \\ v_y(x,1) = 0 & \text{on } \mathbb{R}^n \times \{y=1\}, \\ -\lim\limits_{y\to 0} y^a v_y = f(v) & \text{on } \mathbb{R}^n \times \{y=0\}, \end{cases} \tag{II.10}$$

where $f \in C^{1,\gamma}(\mathbb{R})$ with $\gamma > 0$.

Problem (II.10) has a variational structure, since the solutions of (II.10) correspond to critical points of the energy functional

$$\mathcal{E}(v) := \frac{1}{2} \int_{\mathbb{R}^n \times (0,1)} y^a |\nabla v(x,y)|^2 \, dx\, dy + \int_{\mathbb{R}^n \times \{y=0\}} F(v(x,0))\, dx, \tag{II.11}$$

where the associated potential F is such that $F' = -f$. Since problem (II.10) is set in a slab of fixed height, it is technically convenient to localize the energy functional on cylinders. Namely, we define the cylinder

$$C_R := B_R \times (0,1), \tag{II.12}$$

where $B_R \subset \mathbb{R}^n$ denotes the ball of radius R centered at 0. Then, by (II.11), the localized energy functional associated to problem (II.10) reads

$$\mathcal{E}_R(v) := \frac{1}{2} \int_{C_R} y^a |\nabla v(x,y)|^2 \, dx \, dy + \int_{B_R \times \{y=0\}} F(v(x,0)) \, dx.$$

In particular, the potential F is naturally defined up to an additive constant, hence, focusing on bounded solutions, we can also suppose that $F \geq 0$. For this kind of problems, the model case is the nonlinearity $f(t) := t - t^3$, which arises in the study of phase transitions and it is the derivative of the double-well potential

$$F(t) = \frac{1}{4} \left(1 - t^2\right)^2.$$

The usual notions of minimizer of the energy and of stable solution to problem (II.10) can be defined in a standard way. We say that a bounded function $v \in C^1(\mathbb{R}^n \times (0,1))$ is a *minimizer* for (II.10) if

$$\mathcal{E}_R(v) \leq \mathcal{E}_R(w)$$

for every $R > 0$ and for every bounded competitor w such that $v \equiv w$ on $\partial B_R \times (0,1)$.

We say that a bounded solution v of (II.10) is *stable* if the second variation of the energy is non-negative, i.e.

$$\int_{\mathbb{R}^n \times [0,1]} y^a |\nabla \xi|^2 \, dx \, dy - \int_{\mathbb{R}^n \times \{y=0\}} f'(v) \xi^2 \, dx \geq 0$$

for every function $\xi \in W^{1,2}(\mathbb{R}^n \times [0,1], y^a)$.

Clearly, if v is a minimizer for (II.10), then in particular it is a stable solution. Another important subclass of stable solutions that we consider is the one of *monotone solutions* to (II.10). We say that a solution v of (II.10) is monotone if it is strictly monotone in one horizontal direction, say $\partial_{x_n} v > 0$. For this kind of problems, it is possible to prove that monotone solutions are stable using a non-variational characterization of stability — see Lemma 3.3.1 for all the details.

Problem (II.10) was initially studied by de la Llave and Valdinoci in [70] with constant density, i.e. with $a = 0$. The main result in [70] is a Liouville theorem that assures the one-dimensional symmetry of monotone solutions on the trace, provided that a suitable energy estimate for the functional associated to the problem holds true. Since this energy estimate in dimension $n = 2$ is a direct consequence of a classical gradient bound, they obtain that monotone solutions to (II.10) with $a = 0$ depend on only one horizontal variable if $n = 2$.

The main motivation to study symmetry and rigidity properties for solutions to (II.10) comes from a conjecture formulated by Ennio De Giorgi about the one-dimensional symmetry of monotone solutions to the classical Allen-Cahn equation. This celebrated conjecture opened indeed a long-lasting line of investigation, as it was extended also to the nonlocal version of the Allen-Cahn equation, where the diffusion is driven by the fractional Laplacian.

In the following, we introduce and motivate the De Giorgi conjecture, and we outline the most important known results in the topic, both in the classical and in the fractional case.

Symmetry properties for the Allen-Cahn equation

In 1979 De Giorgi posed the following question.

Conjecture II.1. *Let u be a bounded and smooth solution of the Allen-Cahn equation*

$$-\Delta u = u - u^3 \qquad \textit{in } \mathbb{R}^n,$$

such that $\partial_{x_n} u > 0$. Is it true that, if $n \leq 8$, then u is one-dimensional?

A heuristic motivation of the conjecture can be formulated in light of the work of Modica and Mortola [115]. Indeed, they proved that a proper rescaling of the energy functional associated to the Allen-Cahn equation Γ-converges to the perimeter functional, as the rescaling parameter goes to zero. In short, Γ-convergence is a variational notion of convergence for functionals, introduced in [67, 69], that is compatible with the minimizing properties of the energy — we refer to [21, 63] for a complete introduction to this topic.

Heuristically, the result of Modica and Mortola means that a proper rescaling of the minimizers of the Allen-Cahn equation converges to characteristic functions of sets of minimal perimeter. The threshold dimension $n = 8$ in Conjecture II.1 comes from the fact that super-level sets of monotone functions are *locally* epigraphs, and minimal graphs are flat if $n - 1 \leq 7$. Observe that this is a tricky point, since the monotonicity hypothesis in Conjecture II.1 does not ensure that the level sets of u are complete graphs — see for instance the counterexample provided by Farina and Valdinoci in [86, formula (5)]. For a complete discussion of minimal surfaces, see the illuminating monograph [97].

Summing up, the above heuristic argument would give that, at least in dimension $n \leq 8$, if we look at monotone solutions "from very far" (through a rescaling), their level sets are close to hyperplanes. The question in Conjecture II.1 asks if, for this to hold, the level sets of the function must be necessarily parallel hyperplanes.

The conjecture of De Giorgi remained unanswered in every dimension n for almost twenty years. In the late nineties it was proved to hold if $n = 2$ by Ghoussoub and Gui [95] and by Berestycki, Caffarelli and Nirenberg [19]. A few years later, Ambrosio and Cabré [12] gave a positive answer to Conjecture II.1 in dimension $n = 3$. Regarding dimensions $4 \leq n \leq 8$, Savin proved in [120] the conjecture by assuming the following additional hypothesis about the limits in the monotone direction

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1. \tag{II.13}$$

Condition (II.13) can be weakened by assuming two-dimensional symmetry of the profiles at infinity, see [87]. More precisely, a number of symmetry results hold true under appropriate assumptions of geometric type. Without claiming to be exhaustive, we mention, for example the following results from [87]:

Symmetry from the profiles. Let $-\Delta u = u - u^3$ in $\mathbb{R}^n$, with $\partial_{x_n} u > 0$. Let

$$\underline{u}(x') := \lim_{x_n \to -\infty} u(x', x_n) \qquad \text{and} \qquad \overline{u}(x') := \lim_{x_n \to +\infty} u(x', x_n). \tag{II.14}$$

Then:

- If both $\underline{u}$ and $\overline{u}$ depend on (at most) two Euclidean variables, then $\underline{u}$ is identically -1 and $\overline{u}$ is identically $+1$; if also $n \leq 8$, then u is one dimensional;

- If either $\underline{u}$ or $\overline{u}$ depends on (at most) two Euclidean variables and $n \leq 4$, then u is one dimensional.

Symmetry from level sets being graphs. Let $-\Delta u = u - u^3$ in $\mathbb{R}^n$, with $\partial_{x_n} u > 0$, and let the notation in (II.14) hold true. Assume that one level set of u is a graph in the nth Euclidean direction. Then $\underline{u}$ is identically -1 and $\overline{u}$ is identically $+1$; if also $n \leq 8$, then u is one-dimensional.

Symmetry for monotone minimizers. Let u be a local minimizer of the energy functional

$$\int \frac{1}{2}|\nabla u(x)|^2 + \frac{1}{4}(1 - u^2(x))^2 \, dx,$$

and assume that $\partial_{x_n} u > 0$. Suppose that $n \leq 8$. Then u is one-dimensional.

Symmetry for minimizers with uniform limits. Let u be a local minimizer of the energy functional

$$\int \frac{1}{2}|\nabla u(x)|^2 + \frac{1}{4}(1 - u^2(x))^2 \, dx,$$

and assume that either

$$\lim_{x_n \to -\infty} u(x', x_n) = -1 \qquad \text{or} \qquad \lim_{x_n \to +\infty} u(x', x_n) = +1,$$

uniformly for $x' \in \mathbb{R}^{n-1}$. Then u is one-dimensional.

As a counterpart of the results giving positive answers to Conjecture II.1 (possibly under additional assumptions), del Pino, Kowalczyk and Wei provided in [71] an example of a monotone solution to the Allen-Cahn equation in dimension $n = 9$ which is not one-dimensional. In this way, they proved that dimension $n = 8$ in Conjecture II.1 is the optimal one.

We refer the reader to [53, 86] for some detailed surveys on topics related to Conjecture II.1.

Symmetry properties for the fractional Allen-Cahn equation

The fractional analogue of Conjecture II.1 can be formulated as follows:

Conjecture II.2. *Let $s \in (0,1)$ and u be a bounded and smooth solution of the fractional Allen-Cahn equation*

$$(-\Delta)^s u = u - u^3 \qquad \text{in } \mathbb{R}^n, \tag{II.15}$$

such that $\partial_{x_n} u > 0$. Is it true that, if n is sufficiently small, then u is one-dimensional?

This question is also motivated by an analogue in the fractional setting of the Γ-convergence result by Modica and Mortola, provided by Savin and Valdinoci in [123]. More precisely, for a given bounded set $\Omega \subset \mathbb{R}^n$, they consider a proper rescaling of the energy associated to (II.15)

$$\mathcal{I}_\varepsilon(u, \Omega) := \varepsilon^{2s} \mathcal{K}(u, \Omega) + \int_\Omega V(u) \, dx, \tag{II.16}$$

where $V = \frac{1}{4}(1 - u^2)^2$ is a double well potential, and $\mathcal{K}(u, \Omega)$ is the Ω contribution in the H^s-seminorm, defined as

$$\mathcal{K}(u, \Omega) := \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy + 2 \iint_{\Omega \times \mathscr{C}\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy.$$

The main result in [123] establishes that a proper rescaling of the energy $\mathcal{I}_\varepsilon$ converges in the Γ-sense to the classical perimeter if $s \geq 1/2$ and to the nonlocal area functional if $s \in (0, 1/2)$.

The nonlocal area functional was introduced by Caffarelli, Roquejoffre and Savin in [45], and — without going into the details — can be thought as a nonlocal version of the classical perimeter, counting the interactions between points which lie in the two separated sides of the boundary of the set.

In light of the Γ-convergence result in [123] and in analogy with the classical case, one could relate, at least at a level of motivations, the validity of Conjecture II.2 to the regularity and rigidity properties of the minimizers of the limit energy functional. Specifically, the Γ-limit objects are the classical minimal surfaces when $s \in [1/2, 1)$, and the nonlocal minimal surfaces when $s \in (0, 1/2)$. With respect to this, we recall that nonlocal minimal surfaces are known to be smooth only in dimension 2 — see [124] — and up to dimension 7 provided that $s \in [1/2 - \epsilon_0, 1/2)$ and ϵ_0 is sufficiently small — see [48]. Nonlocal minimal surfaces that are entire graphs are known to be necessarily hyperplanes only in dimension 2 and 3, and up to dimension 8 provided that $s \in [1/2 - \epsilon_0, 1/2)$ and ϵ_0 is sufficiently small — see [90]. Till now, no singular minimal surface is known — see however [65] for the construction of a singular cone in dimension 7, which is a stable critical point of the fractional perimeter when s is sufficiently small.

Of course, this lack of knowledge for the nonlocal minimal surfaces — when compared to the classical minimal surfaces — provides a series of conceptual difficulties when dealing with Conjecture II.2, especially in the regime $s \in (0, 1/2)$.

The problem posed by Conjecture II.2 was solved in dimension $n = 2$ by Cabré and Solà-Morales in [44] for $s = 1/2$, and then by Cabré, Sire, and Valdinoci in [43, 130] for every $s \in (0, 1)$.

A positive answer in dimension $n = 3$ was given by Cabré and Cinti in [35] and [36] in the cases $s = 1/2$ and $s \in (1/2, 1)$, respectively. Regarding the strongly nonlocal regime, namely when $s \in (0, 1/2)$, recently the conjecture has been proved in dimension $n = 3$ by Dipierro, Farina, and Valdinoci in [74], using an improvement of flatness result by [76]. An alternative proof of the same result has been announced by Cabré, Cinti, and Serra [37], using a different approach which relies on some sharp energy estimates and a blow-down convergence result for stable solutions.

Very recently, Figalli and Serra proved in [92] Conjecture II.2 to be true for $s = 1/2$ and $n = 4$, also providing one-dimensional symmetry of stable solutions to (II.15) with $s = 1/2$ in dimension $n = 3$. The result in [92] is quite surprising, since Conjecture II.1 without the extra assumption (II.13) is still open for $n = 4$. To achieve the result, the authors exploit in a brilliant way some fractional techniques developed in [58] and [124]. This is indeed a nice example of how purely nonlocal methods can lead to new and important results in PDEs.

Concerning higher dimensions, Savin proved in [121, 122] the conjecture for $4 \leq n \leq 8$ and $s \in [1/2, 1)$ under the additional assumption (II.13). Moreover, in [76] it has been proved that Conjecture II.2 is true in dimensions $4 \leq n \leq 8$ if $s \in [1/2 - \epsilon_0, 1/2)$, for some ϵ_0 sufficiently small, under the additional assumption (II.13).

We also recall that, similarly to what happens in the classical case, it is possible to obtain one-dimensional symmetry from the geometry of the profiles of the monotone solutions, defined in (II.14). More precisely, it has been proved in [74] that monotone solutions with two-dimensional limit profiles are necessarily one-dimensional in dimension $n \leq 8$, as long as $s \in [1/2 - \epsilon_0, 1/2)$, for a sufficiently small $\epsilon_0 \in (0, 1/2)$.

Besides these results, Conjecture II.2 is also open in its generality, and the critical dimension might depend on the fractional parameter s.

Results of the thesis (Part II)

Since in our framework we are dealing with the nonlocal operator $\mathcal{L}_a$, which is related to the fractional Laplacian, a natural counterpart of Conjecture II.2 is the following one:

Conjecture II.3. *Let $a \in (-1, 1)$ and u be a bounded and smooth solution of the fractional Allen-Cahn equation*

$$\mathcal{L}_a u = u - u^3 \qquad \text{in } \mathbb{R}^n,$$

such that $\partial_{x_n} u > 0$. Is it true that, if n is sufficiently small, then u is one-dimensional?

Conjecture II.3 is related to, but structurally different from, Conjecture II.2. This is due to the fact that the operator $\mathcal{L}_a$ defined in (II.7) is not purely nonlocal, as it attains a fractional Laplace operator only in the limit, according to (II.5). We will deepen the nature of the operator $\mathcal{L}_a$ in Chapter 4.

In this setting, Conjecture II.3 was first addressed by de la Llave and Valdinoci in [70], for the special case $n = 2$ and $a = 0$. As mentioned above, the main result in [70] is a Liouville theorem, that gives one-dimensional symmetry of monotone solutions under an assumption about the growth of the Dirichlet energy of the solution. In this way, they established Conjecture II.3 for $n = 2$ and $a = 0$ — see in particular Theorem 1 in [70].

In Chapter 3, we study Conjecture II.3 in the cases $n = 2$ and $n = 3$. Our first result extends the Liouville theorem in [70] from $a = 0$ to all the fractional parameters $a \in (-1, 1)$, also considering the broader class of stable solutions instead of monotone solutions. We state it as follows.

Theorem II.4 ([57]). *Let $f \in C^{1,\gamma}(\mathbb{R})$, with $\gamma > \max\{0, -a\}$, and let v be a bounded and stable solution of* (II.10).

Suppose that there exists $C > 0$ such that

$$\int_{C_R} y^a |\nabla_x v(x,y)|^2 \, dx \, dy \leq CR^2 \tag{II.17}$$

for any $R \geq 2$, where the notation in (II.12) *has been used for C_R.*

Then, there exist $v_0 : \mathbb{R} \times (0, 1) \to \mathbb{R}$ and $\omega \in S^{n-1}$ such that

$$v(x, y) = v_0(\omega \cdot x, y) \qquad \text{for any } (x, y) \in \mathbb{R}^n \times (0, 1).$$

In particular, the trace u of v on $\{y = 0\}$ can be written as $u(x) = u_0(\omega \cdot x)$.

Finally, either $u_0' > 0$ or $u_0' \equiv 0$.

For this kind of elliptic problems, it is a standard fact that bounded solutions have bounded gradients, see for example [96]. For this reason, if we assume $n = 2$, then hypothesis (II.17) is trivially verified by any bounded stable solution. Therefore, we deduce that bounded stable solutions to (II.10) — and not only monotone solutions — are one-dimensional on the trace if $n = 2$. In particular, this implies the validity of Conjecture II.3 in dimension $n = 2$ for all $a \in (-1, 1)$, as a corollary of Theorem II.4.

Concerning the case $n = 3$, in order to deduce Conjecture II.3 from Theorem II.4, we need to have suitable energy estimates which accomplish hypothesis (II.17). To this aim, in Chapter 3 we prove a control on the growth of the energy for some subclasses of stable solutions. These results, combined with Theorem II.4, give a positive answer to Conjecture II.3 when $n = 3$.

First, we consider solutions to (II.10) which are minimizers of the associated energy, and we obtain energy estimates. Our result states as follows.

Theorem II.5 ([57]). *Let $f \in C^{1,\gamma}(\mathbb{R})$, with $\gamma > \max\{0, -a\}$, and let v be a bounded minimizer for problem* (II.10).

Then, we have

$$\mathcal{E}_R(v) = \frac{1}{2}\int_{C_R} y^a |\nabla v|^2 \,dx\,dy + \int_{B_R\times\{y=0\}} F(v)\,dx \leq CR^{n-1}, \qquad \text{(II.18)}$$

for any $R \geq 2$, where the notation in (II.12) *has been used for C_R.*

We point out that (II.18) holds in general for minimizers of the energy associated to problem (II.10) in every dimension n, but only when $n \leq 3$ we can combine it with Theorem II.4 to deduce the one dimensional symmetry of minimizers.

Let us mention that the proof of Theorem II.5 is based on a direct comparison argument, a classical strategy in the Calculus of Variations. The idea is to build a suitable admissible competitor, for which we can explicitly compute or estimate the energy. Then, bound (II.18) follows directly from the fact that we are considering minimizers of the energy. We refer to Section 3.4 for the detailed proof of Theorem II.5.

After dealing with minimizers, we consider bounded solutions whose traces on $\{y = 0\}$ are monotone in some direction. Restricting to the case $n = 3$, we prove the following result, which establishes an energy estimate for monotone solutions to (II.10).

Theorem II.6 ([57]). *Let $f \in C^{1,\gamma}(\mathbb{R})$, with $\gamma > \max\{0, -a\}$, and let v be a bounded solution of* (II.10) *with $n = 3$ such that its trace $u(x) = v(x, 0)$ is monotone in some direction.*

Then, we have

$$\mathcal{E}_R(v) = \frac{1}{2}\int_{C_R} y^a |\nabla v|^2 \,dx\,dy + \int_{B_R\times\{y=0\}} F(v)\,dx \leq CR^2,$$

for any $R \geq 2$, where the notation in (II.12) *has been used for C_R.*

We stress that — on the contrary of (II.18), that holds in every dimension n — we prove this last energy estimate for monotone solutions to (II.10) only in the case $n = 3$. This restriction is due to the strategy used in the proof of Theorem II.6. Indeed, we use as a key tool that stable solutions enjoy rigidity properties in one dimension less, i.e. when $n = 2$, and this forces us to assume $n = 3$. We refer to Section 3.5 for a complete proof of Theorem II.6.

As a corollary of the energy estimates in Theorems II.5 and II.6, we obtain the following rigidity result for minimizers and monotone solutions in dimension $n = 3$. Indeed, for these classes of solutions, hypothesis (II.17) of Theorem II.4 is fulfilled, and the application is straightforward. We state the result in the following corollary.

Corollary 2.2.1 ([57]). *Let $f \in C^{1,\gamma}(\mathbb{R})$, with $\gamma > \max\{0, -a\}$ and let $n = 3$. Assume that one of the two following condition is satisfied:*

- *v is a bounded minimizer for problem* (II.10);
- *v is a bounded solution of* (II.10) *such that its trace $u(x) = v(x,0)$ is monotone in some direction.*

Then, there exist $v_0 : \mathbb{R} \times (0,1) \to \mathbb{R}$ and $\omega \in S^2$ such that:

$$v(x,y) = v_0(\omega \cdot x, y) \qquad \text{for all } (x,y) \in \mathbb{R}^3 \times (0,1).$$

In particular, the trace u of v on $\{y = 0\}$ can be written as $u(x) = u_0(\omega \cdot x)$.

In particular, Corollary 2.2.1 establishes the validity of Conjecture II.3 when $n = 3$. The case of dimension $n \geq 4$ remains open.

In Chapter 4 we further investigate the differences between problem (II.8)-(II.10) treated in this part of the thesis and its analogue for the fractional Laplacian. Our first goal is obtaining the expression of the Dirichlet to Neumann operator $\mathcal{L}_a$ in Fourier modes. This representation was already known for the special case $a = 0$ in [25,70], as

$$\widehat{\mathcal{L}_0 u}(\xi) = \frac{e^{|\xi|} - e^{-|\xi|}}{e^{|\xi|} + e^{-|\xi|}} \, |\xi| \, \widehat{u}(\xi), \tag{II.19}$$

where the symbol $\widehat{u}$ denotes the Fourier transform of u, as customary.

In our first result in Chapter 4, we extend expression (II.19) from $a = 0$ to every $a \in (-1,1)$, and we highlight interesting asymptotic properties of the Fourier symbol of $\widehat{\mathcal{L}_a}$, that hold for every value of the parameter a. We state the result as follows.

Theorem II.7 ([112]). *For every smooth bounded function u defined on $\mathbb{R}^n$ which is integrable, we can write the operator $\mathcal{L}_a$ defined in* (II.7) *via Fourier transform, as*

$$\widehat{\mathcal{L}_a u}(\xi) = c_1(s) \frac{J_{1-s}(-i\,|\xi|)}{J_{s-1}(-i\,|\xi|)} \, |\xi|^{2s} \, \widehat{u}(\xi),$$

where $1 - a = 2s$, J_k is the Bessel function of the first kind of order k, and

$$c_1(s) := i \left(\frac{1-i}{2} \right)^{4s-2} \frac{\Gamma(1-s)}{\Gamma(s)}.$$

Moreover, the symbol

$$S_s(\xi) := c_1(s) \frac{J_{1-s}(-i\,|\xi|)}{J_{s-1}(-i\,|\xi|)} \, |\xi|^{2s} \tag{II.20}$$

is a positive and increasing function of $|\xi|$, and enjoys the following asymptotic properties. There exist two positive constants C_1 and C_2 depending only on s such that

$$\begin{aligned} \lim_{|\xi| \to 0} \frac{S_s(\xi)}{|\xi|^2} &= C_1; \\ \lim_{|\xi| \to +\infty} \frac{S_s(\xi)}{|\xi|^{2s}} &= C_2. \end{aligned} \tag{II.21}$$

To better understand the implications of the behavior outlined in (II.21), we should remind that $|\xi|^2$ is the symbol of the classical Laplacian, and that the fractional Laplacian can be also written in the Fourier setting as

$$\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s}\,\widehat{u}(\xi),$$

see for example [73, Proposition 3.3].

Observe that the asymptotic behavior highlighted in (II.21) was already evident in the simpler case $s = 1/2$, from expression (II.19). Our result extends it to every fractional power $s \in (0,1)$, clarifying also the connection of $\mathcal{L}_a$ with the fractional Laplacian. Indeed, looking at the asymptotics (II.21), it becomes evident that the operator $\mathcal{L}_a$ is not of purely fractional type, as it shows a nonlocal behavior for high frequencies, but it becomes similar to the Laplacian for small frequencies.

To further investigate the mixed nature (local and nonlocal) of the operator $\mathcal{L}_a$, we study the Γ-convergence of a proper rescaling of the functional associated to the equation $\mathcal{L}_a(u) = W'(u)$, where W is a double-well potential.

From Theorem II.7 we deduce that we can write the Dirichlet energy associated to the operator $\mathcal{L}_a$ as

$$\mathcal{E}_K(v) = \frac{1}{2}\int_{\mathbb{R}^n\times(0,1)} y^a\,|\nabla v|^2\,dx\,dy = \frac{1}{2(2\pi)^n}\int_{\mathbb{R}^n} S_s(\xi)\,|\widehat{u}(\xi)|^2\,d\xi.$$

Therefore, we can associate to the equation $\mathcal{L}_a(u) = W'(u)$ the functional

$$\mathcal{J}(u) := \int_{\mathbb{R}^n} S_s(\xi)\,|\widehat{u}(\xi)|^2\,d\xi + \int_{\mathbb{R}^n} W(u)\,dx.$$

In line with the Γ-convergence results for the Allen-Cahn equation [115, 123] — both classical and fractional — we assume that W is a double-well potential, namely that it satisfies

$$\begin{gathered} W \in C^{2,\gamma}([0,1]), \qquad W(0) = W(1) = 0, \qquad W > 0 \text{ in } (0,1), \\ W'(0) = W'(1) = 0, \qquad \text{and} \qquad W''(0) = W''(1) \geq 0. \end{gathered}$$

We consider the following partial rescaling of the functional $\mathcal{J}$:

$$\mathcal{J}_\varepsilon(u) := \varepsilon^{2s}\int_{\mathbb{R}^n} S_s(\xi)\,|\widehat{u}(\xi)|^2\,d\xi + \int_{\mathbb{R}^n} W(u)\,dx,$$

and we point out that we work in the function space X defined as

$$X := \{u \in L^\infty(\mathbb{R}^n) \text{ s.t. } u \text{ has compact support and } 0 \leq u \leq 1\}.$$

In addition, we say that a sequence $u_j \in X$ converges to u in X if $u_j \to u$ in $L^1(\mathbb{R}^n)$ — note that, according to the definition, $X \subset L^1(\mathbb{R}^n)$.

Observe that the energy $\mathcal{J}_\varepsilon$ differs in the Dirichlet part from $\mathcal{I}_\varepsilon$ defined in (II.16), which is the energy considered by Savin and Valdinoci in [123]. However, the two Dirichlet energies are partially related, due to the asymptotic behavior of the symbol $S_s(\xi)$. This fact will indeed play an important role in our Γ-convergence result, as explained in more detail in Chapter 4.

In order to obtain an interesting result in terms of Γ-convergence, we need to rescale $\mathcal{J}_\varepsilon$ and consider $\mathcal{F}_\varepsilon : X \to \mathbb{R} \cup \{+\infty\}$ defined as

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \varepsilon^{-2s}\mathcal{J}_\varepsilon(u) & \text{if } s \in (0,1/2); \\ |\varepsilon \log \varepsilon|^{-1}\, \mathcal{J}_\varepsilon(u) & \text{if } s = 1/2; \\ \varepsilon^{-1}\mathcal{J}_\varepsilon(u) & \text{if } s \in (1/2,1). \end{cases} \tag{II.22}$$

When $s \in [\frac{1}{2}, 1)$, we define the limit functional $\mathcal{F} : X \to \mathbb{R} \cup \{+\infty\}$ as

$$\mathcal{F}(u) := \begin{cases} c_\# \mathrm{Per}(E) & \text{if } u = \chi_E \text{ for some bounded set } E \subset \mathbb{R}^n; \\ +\infty & \text{otherwise,} \end{cases} \tag{II.23}$$

where $c_\#$ is a positive constant depending only on n and s.

In the case $s \in (0,1/2)$, $\mathcal{F} : X \to \mathbb{R} \cup \{+\infty\}$ is defined as

$$\mathcal{F}(u) := \begin{cases} \displaystyle\int_{\mathbb{R}^n} S_s(\xi)\, |\widehat{u}(\xi)|^2\, d\xi & \text{if } u = \chi_E, \text{ for some set } E \subset \mathbb{R}^n; \\ +\infty & \text{otherwise.} \end{cases} \tag{II.24}$$

We remark that the limit functional $\mathcal{F}$ for $s \in (0,1/2)$ is well defined when $u = \chi_E$. Indeed, this is a consequence of the fact that the difference between $\mathcal{F}$ and the H^s seminorm of $u = \chi_E$ is finite — see Lemma 4.4.1 — and that the nonlocal area functional of a bounded set is always well defined for $s \in (0,1/2)$.

We stress that the limit functional $\mathcal{F}$ that we obtain in the strongly nonlocal regime $s \in (0,1/2)$ appears to be new in the literature, and really different from other energy functionals of nonlocal type that have been investigated before our work. We will further comment on it in Proposition 2.2.2 below.

Now, we can state the second main result in Chapter 4 in the following theorem. It establishes the Γ-convergence of the rescaled functional (II.22) to $\mathcal{F}$ defined in (II.24)-(II.23).

Theorem II.8 ([112]). *Let $s \in (0,1)$. Then the functional $\mathcal{F}_\varepsilon$ defined in* (II.22) *Γ-converges to the functional $\mathcal{F}$ defined in* (II.24)-(II.23), *i.e. for any u in X*

(i) for any u_ε converging to u in X

$$\liminf_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon) \geq \mathcal{F}(u);$$

(ii) there exists a sequence $(u_\varepsilon)_\varepsilon$ converging to u in X such that

$$\limsup_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon) \leq \mathcal{F}(u).$$

We stress that the Γ-limit functional $\mathcal{F}$ is defined in two different ways depending on whether s is above or below $1/2$. When $s \in [1/2, 1)$, the Γ-limit is the classical perimeter, as in the case of the energy associated to the fractional Laplacian, treated in [123]. Thus, similarly to the fractional Laplacian for $s \in [1/2, 1)$, the nonlocal operator $\mathcal{L}_a$ loses its nonlocal nature in the Γ-limit and recovers the classical perimeter functional.

On the other hand, when $s \in (0,1/2)$ the Γ-limit is a nonlocal functional, which is different from the fractional perimeter, and we are interested in having more information about $\mathcal{F}$ defined in (II.24).

We consider the special case $n = 1$, in order to be able to make explicit computations with the Fourier transform, and we study the Γ-limit functional $\mathcal{F}$ defined in (II.24). Since it is (possibly) finite only when $u = \chi_E$ for some set $E \subset \mathbb{R}$, we consider a connected interval $I_r \subset \mathbb{R}$ of length r and the characteristic function χ_{I_r}. Then, the square modulus of the Fourier transform of χ_{I_r} is

$$|\widehat{\chi_{I_r}}(\xi)|^2 = \frac{4\sin^2(r\xi)}{\xi^2},$$

which depends only on the length of the interval. Thus, we can define a function $\mathcal{T}_s(r) : [0,+\infty) \longrightarrow [0,+\infty)$ as

$$\mathcal{T}_s(r) := \mathcal{F}(\chi_{I_r}) = \int_{\mathbb{R}} S_s(\xi)\, |\widehat{\chi_{I_r}}(\xi)|^2 \, d\xi, \tag{II.25}$$

where $I_r \subset \mathbb{R}$ is a connected interval of length r. Observe that $\mathcal{T}_s$ depends on $s \in (0,1/2)$, as the symbol $S_s(\xi)$ defined in (II.20) depends on s. We collect in the following proposition some properties of the function $\mathcal{T}_s$, that allow us to relate it to the common notions of classical and fractional perimeter in one dimension.

Proposition 2.2.2 ([112])**.** *Let $s \in (0,1/2)$ and $n = 1$. The function $\mathcal{T}_s(r)$ defined in* (II.25) *is positive and enjoys the following asymptotic properties. There exist two positive constants C_1 and C_2 depending only on s such that*

$$\lim_{r\to 0^+} \frac{\mathcal{T}_s(r)}{r^{1-2s}} = C_1;$$
$$\lim_{r\to +\infty} \mathcal{T}_s(r) = C_2.$$

We recall that from the definition of nonlocal perimeter it follows that an interval of length r has fractional perimeter of order r^{1-2s}. Therefore, Proposition 2.2.2 makes clear that $\mathcal{F}$ behaves like the fractional perimeter for intervals of small length, while for big values of r it goes to a constant, counting the finite number of discontinuities of χ_{I_r}. In this sense, the limit functional defined in (II.24) interpolates the classical and the fractional perimeter, at least in dimension one.

Finally, we remark that the restriction $n = 1$ in Proposition 2.2.2 is due to the possibility of making explicit calculations with the Fourier transform, and not to some intrinsic feature of the functional $\mathcal{F}$ defined in (II.24). For this reason, a generalization of the results in Proposition 2.2.2 could hold in higher dimension. We believe that this is an interesting open question to be studied in future works.

Chapter 3

One dimensional symmetry for some classes of solutions

We prove a one-dimensional symmetry result for a weighted Dirichlet to Neumann problem arising in a model for water waves in dimensions 2 and 3. More precisely, we prove that stable solutions in dimension 2 and minimizers and monotone solutions in dimension 3 depend on only one Euclidean variable. Monotone solutions in the 2-dimensional case without weights were studied in [70]. In this chapter, a crucial ingredient in the proof is given by an energy estimate for minimizers obtained via a comparison argument.

3.1 Introduction

3.1.1 A water wave model

In this chapter, we establish one-dimensional symmetry results for solutions of a Dirichlet to Neumann problem which arises in a model for water waves.

A classical water wave model is that of considering an ideal fluid with density ϱ and velocity V in the spatial region $\mathbb{R}^2 \times [0, H]$ — that is the "sea", which is assumed to be of depth $H > 0$. For convenience, one can endow $\mathbb{R}^2 \times [0, H]$ with coordinates $x \in \mathbb{R}^2$ and $y \in [0, H]$. We consider the level $\{y = H\}$ as the "bottom of the sea" and the level $\{y = 0\}$ as the "surface of the sea"; in this notation, y represents the "depth of the sea". Here, we briefly introduce the framework in the stationary case. For a complete discussion of the model in the general case, see [75].

The mass conservation law gives that $\operatorname{div}(\varrho V) = 0$ and the irrotationality condition that $V = \nabla v$ in $\mathbb{R}^2 \times (0, H)$. Assuming that the bottom of the sea is made of solid material, the impenetrability of the matter gives that the vertical velocity vanishes along $\{y = H\}$. Then, given the values of v along the surface of the sea (and denoting such datum by u), one is interested in finding the vertical velocity on the surface, possibly weighted by the density of the fluid. This vertical velocity is, roughly speaking, responsible for the formation of a wave starting from the rest position of a "flat sea".

The problem turns out to be linear with respect to the derivatives of the datum u and semilinear in virtue of the nonlinearity $f(u)$, so it is convenient to denote the vertical velocity on the surface by $\mathcal{L}u$ (with a minus sign that we introduce for later convenience). In this setting, writing $V := (V_1, V_2, V_3) \in \mathbb{R}^3$, and denoting by v_y the derivative of v

with respect to the vertical variable y, the problem can be formulated as

$$\begin{cases} 0 = \operatorname{div}(\varrho V) = \operatorname{div}(\varrho \nabla v) & \text{in } \mathbb{R}^2 \times (0, H) \\ 0 = V_3\big|_{y=H} = v_y\big|_{y=H} & \text{on } \mathbb{R}^2 \times \{y = H\} \\ u = v\big|_{y=0} & \text{on } \mathbb{R}^2 \times \{y = 0\} \\ \mathcal{L}u = f(v) & \text{on } \mathbb{R}^2 \times \{y = 0\}, \end{cases} \tag{3.1.1}$$

where $\mathcal{L}u = -\varrho v_y\big|_{y=0}$ and $f : \mathbb{R} \to \mathbb{R}$ is a given, smooth function. When $\varrho := 1$ and $H \to +\infty$ (which is the case of a fluid with constant density and an "infinitely deep sea"), the problem in (3.1.1) is related to the square root of the Laplacian, see e.g. [46]. For finite values of H the operator described in (3.1.1) is nonlocal, but also not of purely fractional type. Hence, in the sequel, we normalize the domain by setting $H := 1$.

3.1.2 Dirichlet to Neumann operators

More specifically, in this chapter, motivated by (3.1.1), we consider the slab $\mathbb{R}^n \times [0, 1]$ with coordinates $x \in \mathbb{R}^n, y \in [0, 1]$ and a function $u : \mathbb{R}^n \to \mathbb{R}$. We then consider $v = v(x, y)$ as the bounded extension of $u : \mathbb{R}^n \to \mathbb{R}$ in the slab $\mathbb{R}^n \times (0, 1)$ satisfying the following problem with density $\varrho(y) = y^a$, where $a \in (-1, 1)$:

$$\begin{cases} \operatorname{div}(y^a \nabla v) = 0 & \text{in } \mathbb{R}^n \times (0, 1) \\ v(x, 0) = u(x) & \text{on } \mathbb{R}^n \times \{y = 0\} \\ v_y(x, 1) = 0 & \text{on } \mathbb{R}^n \times \{y = 1\}. \end{cases} \tag{3.1.2}$$

Then, in view of the physical description in (3.1.1), the problem in (3.1.2) naturally leads to the study of the Dirichlet to Neumann operator $\mathcal{L}_a$ given by

$$\mathcal{L}_a u(x) = -\lim_{y \to 0} y^a v_y(x, y)$$

Notice that[1] the operator $\mathcal{L}_a$ is given by the operator $\mathcal{L}$ appearing in (3.1.1) for the choice $\varrho(y) = y^a$, which is the weight that we consider throughout the chapter. We also observe that the operator $\mathcal{L}_a$ is closely related to the fractional Laplacian $(-\Delta)^s$ with $s = (1 - a)/2$, see e.g. [46], though it is not equal to any purely fractional operator. The case $a = 0$, which corresponds to v being the harmonic extension of u in $\mathbb{R}^n \times (0, 1)$, was considered in [70], where the authors write explicitly the Fourier symbol of the operator $\mathcal{L}_0$ in this specific case and show that, for large frequencies, the Fourier symbol of $\mathcal{L}_0$ is asymptotic to the Fourier symbol of the half-Laplacian (observe that for $a = 0$ we have $s = 1/2$).

The issue of understanding the local or non-local nature of the operator $\mathcal{L}_a$ depending on the parameter $a \in (-1, 1)$ is the main purpose of the forthcoming work [112]. To this end, a Fourier representation of $\mathcal{L}_a$ will be provided in [112] for every $a \in (-1, 1)$.

[1] In this chapter we will always work with the extended problem satisfied by v (see problem (3.1.4) below), hence we do not actually need to define the operator $\mathcal{L}_a$ nor to discuss under which conditions we have uniqueness for solutions to (3.1.2). We have chosen to introduce the Dirichlet-to-Neumann operator for the sake of completeness and to make a comparison with some well known related results for nonlocal equations involving the fractional Laplacian.

We study here the one-dimensional symmetry of certain bounded solutions to the problem

$$\mathcal{L}_a u = f(u) \qquad \text{in } \mathbb{R}^n, \tag{3.1.3}$$

where $f \in C^{1,\gamma}(\mathbb{R})$, with $\gamma > \max\{0, -a\}$, $n = 3$ and $a \in (-1, 1)$.

The results obtained in this chapter extend a known result by de la Llave and Valdinoci in [70]. In that paper, one-dimensional symmetry of monotone solutions of (3.1.3) is established when $n = 2$ and $a = 0$. With the extension (3.1.2), our problem can be formulated in the following local way, that we are going to consider throughout the chapter:

$$\begin{cases} \operatorname{div}(y^a \nabla v) = 0 & \text{in } \mathbb{R}^n \times (0,1) \\ -\lim\limits_{y \to 0} y^a v_y = f(v) & \text{on } \mathbb{R}^n \times \{y = 0\} \\ v_y(x, 1) = 0 & \text{on } \mathbb{R}^n \times \{y = 1\}. \end{cases} \tag{3.1.4}$$

We observe that this formulation of our problem is exactly (3.1.1) for the general n-dimensional case, with $H = 1$ and the weight $\varrho(y) = y^a$. We will show that, if v is a minimizer (in the sense of Definition 3.1.1 below) or a bounded monotone solution for problem (3.1.4) with $n = 3$, then there exist a function $v_0 : \mathbb{R} \times (0,1) \to \mathbb{R}$ and a vector $\omega \in S^2$ such that

$$v(x, y) = v_0(\omega \cdot x, y) \qquad \text{for every } x \in \mathbb{R}^3,\ y \in (0,1).$$

In particular, the trace u of v on $\{y = 0\}$ exhibits one-dimensional symmetry, i.e. it is a function of only one Euclidean variable. See also Chapter 3 of [26] for additional discussions.

It is interesting to point out that the results that we give here are new even in the case $a := 0$, corresponding to uniform density of the fluid. Nevertheless, we provided a general setting for the problem in (3.1.4) and we believe that such generality is worthwhile for a series of reasons:

- From a pure mathematical perspective, weights of the type y^a belong to the Muckenhoupt[2] class A_2, see [82,83], which plays a special interest in the analysis of partial differential equations with weights, since, in a sense, these weights constitute the fundamental example of nontrivial, possibly singular or degenerate, weights, for which a "good elliptic theory" is still possible;

- With respect to fractional operators, it is important to study different values of a, corresponding to different values of the fractional parameter (and, in this case, the value $a := 0$ is often a fundamental threshold dividing "local" and "nonlocal" behaviors at large scales, see e.g. Theorem 1.5 in [123]);

[2] As customary, one says that a weight w belongs to the Muckenhoupt class A_2 if there exists $C > 0$ such that, for all balls B, it holds that

$$\fint_B w(x)\, dx \fint_B \frac{1}{w(x)}\, dx \le C,$$

with $\fint$ denoting average. Roughly speaking, Muckenhoupt weights may be singular or degenerate, but they cannot be "too singular or too degenerate", in an integral sense. Also, w belongs to A_2 if and only if so does $1/w$.

- In applications, weights of the type y^a can model laminated materials (see e.g. [55]) and, in the context of fluids, describe situations in which the density of the fluid only depends on the depth;

- In other situations, equations as in (3.1.4) can be related to models in biological mathematics, in which v represents for instance the density of a given population: in this setting, many real-world experiments have confirmed that different populations exhibit anomalous diffusion, and that the diffusion parameters vary from one species to another (see e.g. [80] and the references therein), therefore it is relevant for concrete models to study nonlocal equations for all the parameter values;

- Most importantly, in our perspective, different values of a allow us more easily to (at least formally) interpolate between classical partial differential equations (in a sense, corresponding to the case $a \to 1$) and strongly nonlocal equations (corresponding to the case $a \to -1$). Hence, since the case $a := 0$ is of course extremely important to address, it is also crucial to comprise in the analysis all the values $a \in (-1, 1)$, so to develop a much better intuition of the problem and to permit the use of continuity and bifurcation methods. In this way, the investigation of nonlocal problems produces results for classical questions which would have not been available with other techniques. For instance, a very neat example in which fractional methods lead to new and important results in classical cases is embodied by the recent work [92], in which the authors brilliantly exploit nonlocal tools developed e.g. in [125] and [58] to obtain symmetry result in a Peierls-Nabarro model;

- The investigation of fractional problems in the full range of the fractional exponent cases $s \in \left(\frac{1}{2}, 1\right)$, $s = \frac{1}{2}$ and $s \in \left(0, \frac{1}{2}\right)$ is also important to understand the different behaviors of the energy contributions (see e.g. [123]). As a matter of fact, typically, when $s \in \left(\frac{1}{2}, 1\right)$, in spite of the nonlocal character of the problem, the major contribution is "of local type", in the sense that it comes from a very well delimited region of the space in which "all the action takes place". Conversely, when $s \in \left(0, \frac{1}{2}\right)$ the major contribution comes "from infinity" and long-range interactions become predominant. In this spirit, the case $s = \frac{1}{2}$ keeps a balance between these two energy tendencies and, in fact, when $s = \frac{1}{2}$ the energy contributions typically "repeat themselves at each dyadic scale", and, in practice, this special additional invariance often produces logarithmic energy terms that are characteristic for the case $s = \frac{1}{2}$.

3.1.3 Connection with the Allen-Cahn equation and a conjecture by De Giorgi

As we mentioned above, our operator $\mathcal{L}_a$ is related to the s-Laplacian $(-\Delta)^s$, for $s := (1-a)/2$, and the fractional Laplacian can be seen as a Dirichlet-to-Neumann operator for a local problem in the halfspace $\mathbb{R}^{n+1}_+$. More precisely, Caffarelli and Silvestre in [46] proved that one can study a semilinear nonlocal problem of the form

$$(-\Delta)^s u = f(u) \qquad \text{in } \mathbb{R}^n, \tag{3.1.5}$$

by studying the associated local problem

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla v) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ -\lim\limits_{y\to 0} y^a v_y = f(u) & \text{on } \mathbb{R}^n \times \{y=0\}. \end{cases}$$

The problem of finding one-dimensional symmetry results for monotone solutions to (3.1.5) is the counterpart, in the fractional setting, of a celebrated conjecture stated in 1978 by E. De Giorgi about bounded and monotone solutions of the classical Allen-Cahn equation $-\Delta u = u - u^3$, see [68]. The nonlinearity $u - u^3$, whose primitive (up to a sign) has a double well potential structure, arises in the study of phase transitions problem.

In dimension $n = 2$ the fractional De Giorgi conjecture has been proved in [44] for $s = \frac{1}{2}$ and in [43, 123, 130] for every $s \in (0,1)$. The same result in dimension $n = 3$ has been established by Cabré and Cinti in [35] and [36] with respectively $s = \frac{1}{2}$ and $s \in (\frac{1}{2}, 1)$. Then, Savin proved in [121] the conjecture in dimensions $4 \leq n \leq 8$ for $s \in (\frac{1}{2}, 1)$ and with the additional assumption

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1. \tag{3.1.6}$$

Very recently, the conjecture has been proved in dimension $n = 3$ and with $s \in (0, \frac{1}{2})$ independently and with different methods by Dipierro, Farina and Valdinoci in [74] (using an improvement of flatness result by [76]) and by Cabré, Cinti, and Serra in [37] (by a different approach which relies on some sharp energy estimates and a blow-down convergence result for stable solutions).

In another very recent result, Figalli and Serra proved in [92] the conjecture for monotone solutions of the half-Laplacian in dimension four without assumption (3.1.6) and we plan to further investigate this new method in the setting of water waves.

Focusing on the dimension that we take into account in this chapter, i.e. $n = 3$, we want to stress an important difference between the water wave problem and the fractional De Giorgi conjecture. As mentioned above, a different approach is needed to prove the one-dimensional symmetry of solutions to (3.1.5) when the parameter s crosses the value $\frac{1}{2}$. This is due to fact that the optimal energy estimates for solutions of (3.1.5) change depending whether s is above or below $1/2$, as shown in [36]. In particular, only when $s \in [1/2, 1)$ these energy estimates are enough to apply a Liouville type result and hence to obtain one-dimensional symmetry. As we are going to see, this does not happen in our case, since the framework is $\mathbb{R}^n \times (0,1)$ and the weight y^a is integrable between 0 and 1. This fact gives us some energy estimates that do not depend on a and allows us to prove one-dimensional symmetry of certain solutions to (3.1.3) with the same method for all the powers $a \in (-1, 1)$, namely $s \in (0,1)$.

For similar results in further dimensions, both in the classical and in the nonlocal case, see also [5, 12, 36, 43, 44, 76, 120, 123, 130].

We stress that these types of nonlocal or fractional problems usually present several sources of additional difficulties with respect to the classical cases, such as:

- Lack of explicit barriers and impossibility of performing straightforward calculations;
- Slow decay of the solutions at infinity;

- Long range interactions and contributions coming from infinity;
- Infinite energy amounts;
- Formation of new types of interfaces (such as "nonlocal minimal surfaces").

In general, we also stress that nonlocal operators may present important differences with respect to the classical ones, also at a very basic level (see e.g. the introductory discussion in Section 2.1 of [1]).

3.1.4 Variational formulation

As one can easily observe, problem (3.1.4) has a variational structure. Let $B_R \subset \mathbb{R}^n$ denote the ball of radius R centered at 0, and C_R the cylinder

$$C_R := B_R \times (0,1). \tag{3.1.7}$$

The (localized) energy functional associated to problem (3.1.4) is given by

$$\mathcal{E}_R(v) = \frac{1}{2}\int_{C_R} y^a |\nabla v|^2\, dx\, dy + \int_{B_R\times\{y=0\}} G(v)\, dx,$$

where the associated potential G is such that $G' = -f$. In particular, the potential G is naturally defined up to an additive constant. To appropriately gauge such constant, we set

$$c_u := \min\{G(s) \text{ s.t. } s \in [\inf u, \sup u]\} \tag{3.1.8}$$

in order to replace $G(u)$ with $G(u) - c_u$ and work with a positive potential.

We can now give the definitions of *minimizer* and of *stable solution* for problem (3.1.4) (problem (3.1.3) respectively) in a standard way.

Definition 3.1.1. We say that a bounded $C^1(\mathbb{R}^n \times (0,1))$ function v is a minimizer for (3.1.4) if

$$\mathcal{E}_R(v) \le \mathcal{E}_R(w)$$

for every $R > 0$ and for every bounded competitor w such that $v \equiv w$ on $\partial B_R \times (0,1)$. We say that a bounded $C^1(\mathbb{R}^n)$ function u is a minimizer for (3.1.3) if its extension v satisfying (3.1.2) is a minimizer for (3.1.4).

Definition 3.1.2. We say that a bounded solution v of (3.1.4) is *stable* if

$$\int_{\mathbb{R}^n\times[0,1]} y^a |\nabla \xi|^2\, dx\, dy - \int_{\mathbb{R}^n\times\{y=0\}} f'(u)\xi^2\, dx \ge 0$$

for every function $\xi \in C^1_0(\mathbb{R}^n \times [0,1])$.

We say that a bounded function u is a *stable* solution for (3.1.3) if its extension v satisfying (3.1.2) is a stable solution for (3.1.4).

Clearly, if v is a minimizer for (3.1.4) then, in particular, it is a stable solution. As we will observe later on in Section 3.3 (see Remark 3.3.2), also a monotone solution is stable, hence stability is a weaker notion of both minimality and monotonicity.

The one-dimensional symmetry result in two dimensions for the particular case $a = 0$ obtained in [70] follows as a corollary of a more general result (see Theorem 1 in [70]), which states that a bounded monotone solution satisfying a certain energy estimate is necessarily one-dimensional.

More precisely, Theorem 1 in [70] requires the existence of a positive constant C such that:

$$\int_{C_R} |\nabla_x v(x,y)|^2 \, dx \, dy \leq CR^2,$$

where ∇_x denotes the gradient in the x-variables.

This is trivially true in the case $n = 2$, thanks to the fact that the gradient of v is bounded, by standard elliptic estimates (see [96]).

3.1.5 Main results

The first result of this chapter generalizes Theorem 1 of [70] to the class of stable solutions. As it will become clear from the proof, this generalization in itself is not too difficult but it will be technically crucial for the purpose of this chapter, and in particular to prove the one-dimensional symmetry of monotone solutions in $\mathbb{R}^3$. Therefore, we state explicitly this result as follows:

Theorem 3.1.3. *Let $f \in C^{1,\gamma}(\mathbb{R})$, with $\gamma > \max\{0, -a\}$, and let v be a bounded and stable solution of (3.1.4).*

Suppose that there exists $C > 0$ such that

$$\int_{C_R} y^a |\nabla_x v(x,y)|^2 \, dx \, dy \leq CR^2 \tag{3.1.9}$$

for any $R \geq 2$.

Then, there exist $v_0 : \mathbb{R} \times (0,1) \to \mathbb{R}$ and $\omega \in \mathrm{S}^{n-1}$ such that

$$v(x,y) = v_0(\omega \cdot x, y) \qquad \textit{for any } (x,y) \in \mathbb{R}^{n+1}_+.$$

In particular, the trace u of v on $\{y = 0\}$ can be written as $u(x) = u_0(\omega \cdot x)$.

Moreover, $u_0' > 0$ or $u_0' \equiv 0$.

From this Theorem, we can directly obtain as a Corollary the one-dimensional symmetry of stable solutions of (3.1.4), when $n = 2$ and for every $a \in (-1, 1)$. This extends the result of de la Llave and Valdinoci in [70], in which they consider $n = 2$, $a = 0$ and v as a monotone solution of (3.1.4).

Corollary 3.1.4. *Let $f \in C^{1,\gamma}(\mathbb{R})$, with $\gamma > \max\{0, -a\}$ and let $n = 2$. Assume that v is a bounded stable solution for problem (3.1.4). Then, there exist $v_0 : \mathbb{R} \times (0,1) \to \mathbb{R}$ and $\omega \in S^2$ such that:*

$$v(x,y) = v_0(\omega \cdot x, y) \qquad \textit{for all } (x,y) \in \mathbb{R}^3 \times (0,1).$$

In particular, the trace u of v on $\{y = 0\}$ can be written as $u(x) = u_0(\omega \cdot x)$.

It is an open problem whether the energy estimate (3.1.9) holds for stable solutions when $n = 3$. In the following two results, we establish it for minimizers and for monotone solutions that, as observed before, are in particular stable solutions.

Next result is an energy estimate for minimizers in any dimension n. This type of results are essential in order to check energy conditions as in (3.1.9) and so apply Theorem 3.1.3.

Theorem 3.1.5 (Energy estimate for minimizers). *Let $f \in C^{1,\gamma}(\mathbb{R})$, with $\gamma > \max\{0, -a\}$, and let v be a bounded minimizer for problem* (3.1.4).

Then, we have

$$\frac{1}{2}\int_{C_R} y^a |\nabla v|^2 \, dx \, dy + \int_{B_R \times \{y=0\}} (G(v) - c_u) \, dx \le CR^{n-1}, \tag{3.1.10}$$

for any $R \ge 2$, where c_u is the constant introduced in (3.1.8).

When $n = 3$ we can prove the same estimate for bounded solutions whose traces on $\{y = 0\}$ are monotone in some direction.

Theorem 3.1.6 (Energy estimate for monotone solutions for $n = 3$). *Let $f \in C^{1,\gamma}(\mathbb{R})$, with $\gamma > \max\{0, -a\}$, and let v be a bounded solution of (3.1.4) with $n = 3$ such that its trace $u(x) = v(x, 0)$ is monotone in some direction.*

Then, we have

$$\frac{1}{2}\int_{C_R} y^a |\nabla v|^2 \, dx \, dy + \int_{B_R \times \{y=0\}} (G(v) - c_u) \, dx \le CR^2, \tag{3.1.11}$$

for any $R \ge 2$, where c_u is the constant introduced in (3.1.8).

As mentioned in Subsection 3.1.3, it is worth to stress a crucial difference between these energy estimates and the ones for the fractional Laplacian obtained in [35, 36]. While in our case we can control the energy with a term that does not depend on the exponent a of the weight, for the fractional Laplace problem this is not true when s is small and belongs to the strongly nonlocal range of exponents: in particular, the sharp energy estimates proved in [36] for $s < \frac{1}{2}$ are not enough to obtain one-dimensional symmetry of special solutions via a Liouville type argument.

As a consequence of Theorems 3.1.3, 3.1.5, and 3.1.6 we deduce the following result, which can be seen as the main result of this chapter and provides the one-dimensional symmetry for minimizers and monotone solutions of a three-dimensional water wave problem.

Theorem 3.1.7. *Let $f \in C^{1,\gamma}(\mathbb{R})$, with $\gamma > \max\{0, -a\}$ and let $n = 3$. Assume that one of the two following condition is satisfied:*

- *v is a bounded minimizer for problem* (3.1.4)*;*
- *v is a bounded solution of (3.1.4) such that its trace $u(x) = v(x, 0)$ is monotone in some direction.*

Then, there exist $v_0 : \mathbb{R} \times (0, 1) \to \mathbb{R}$ and $\omega \in S^2$ such that:

$$v(x, y) = v_0(\omega \cdot x, y) \qquad \text{for all } (x, y) \in \mathbb{R}^3 \times (0, 1).$$

In particular, the trace u of v on $\{y = 0\}$ can be written as $u(x) = u_0(\omega \cdot x)$.

When $n = 2$ and $a = 0$, the analogue of Theorem 3.1.7 was established in [70]: the improvement in our case comes from the enhanced energy estimates in Theorem 3.1.6. We stress once again that the result in Theorem 3.1.7 holds true for all $a \in (-1, 1)$ and, as we are going to see, we can perform a unified proof for all $a \in (-1, 1)$, without having to distinguish different regimes. The fact that the results and the methods are common for all $a \in (-1, 1)$ is indeed a special feature for our problem, and it is related to the fact that the equation in (3.1.4) is set in a slab (differently, for instance, from the cases in [35] and [36], in which the energy behavior of minimal solutions is completely different in dependence of a).

3.1.6 Technical comments and strategy of the proofs

It is interesting to point out that the results of this chapter are new not only in the three-dimensional case, but also in the two-dimensional case when $a \neq 0$. As mentioned above, in the two-dimensional case studied in [70] the energy estimate (3.1.9) follows easily by standard elliptic estimates which ensure that the gradient of any bounded solution to (3.1.4) is bounded. Of course, just using an L^∞ bound on the gradient of the solution, would imply that the energy in cylinders C_R grows like R^n, which, for $n = 3$ would not be enough to apply Theorem 3.1.3.

Our first energy estimate for minimizers (Theorem 3.1.5) is obtained via a comparison argument, similar to the one used in [5], based on the construction of a competitor which is constant in the smaller cylinder C_{R-1}.

The proof of the energy estimate for monotone solutions is, instead, more involved and it follows the strategy of [35,36], in which a similar estimate is proved for the fractional Laplacian.

Observe that in Theorem 3.1.6 we restrict the statement to the case $n = 3$. This is due to the fact that, after taking the limit at $\pm\infty$ in the direction of monotonicity of the solution, we reduce our problem to the classification of stable solutions in one dimension less. Such a classification (more precisely the one-dimensional symmetry and the monotonicity of the limit functions) is known only in dimension 2.

We point out that several important differences arise comparing the settings in [70] and in [35,36] with the one considered in this chapter. In particular:

- In [70], only the two-dimensional case is taken into account (and only the non-singular and nondegenerate case $a := 0$). The lower dimensionality assumption is important in [70] since it gives for free the appropriated bounds on the energy growth;

- In [35,36], the case of purely fractional operators are taken into account, while the operators treated here are nonlocal, but nonfractional as well, and these special features require here, among the other technical bounds, new energy estimates and a new set of regularity results, that are tailored for the case under consideration. On the other hand, as a byproduct of the sharp energy estimates that we find, we are able to obtain symmetry results for all values of $a \in (-1, 1)$ (while the energy estimates in [35,36] cannot be applied beyond the range $(-1, 0]$, thus reflecting the important difference between the water wave problem studied here and the fractional Laplace problem in [35,36]).

3.1.7 Organization of the chapter

The chapter is organized as follows:

- In Section 2 we collect some preliminary results on regularity and gradient estimates for solutions to (3.1.4);

- In Section 3 we give the proof of Theorem 3.1.3, which is based on two preliminary results: a characterization of stability (Lemma 3.3.1) and a Liouville type theorem (Lemma 3.3.3). We also deduce directly Corollary 3.1.4;

- In Section 4 we prove the energy estimate for minimizers (Theorem 3.1.5);

- In Section 5 we prove the energy estimate for monotone solutions (Theorem 3.1.6) which needs several ingredients (mainly Lemma 3.5.2 and Lemma 3.5.6).

3.2 Regularity results and gradient bounds

In this section we collect some regularity results and gradient estimates for solutions to problem (3.1.4).

We start by observing that the weight y^a, with $a \in (-1, 1)$ belongs to the so-called Muckenhoupt class A_2 and hence the theory developed by Fabes, Jerison, Kenig, and Serapioni [82,83] applies to the operator $\operatorname{div}(y^a\nabla)$.

More precisely in [82,83] a Poincaré inequality, a Harnack inequality, and the Hölder regularity for weak solutions of $\operatorname{div}(y^a\nabla) = 0$ are established . This theory gives interior regularity for solutions of our problem (3.1.4). In the sequel we will need regularity up to the boundary $\{y = 0\} \cup \{y = 1\}$ and some global L^∞ estimates for the derivatives of solutions to (3.1.4). For these results some care is needed, due to the presence of the weight y^a.

We define the weighted Sobolev spaces (recall (3.1.7))

$$L^2(C_R, y^a) := \{v : C_R \to \mathbb{R} \,|\, y^a v^2 \in L^1(C_R)\}.$$

$$H^1(C_R, y^a) := \{v : C_R \to \mathbb{R} \,|\, y^a(v^2 + |\nabla v|^2) \in L^1(C_R)\}.$$

In the sequel we will consider the following localized (in the x-variable) linear problem

$$\begin{cases} \operatorname{div}(y^a\nabla v) = 0 & \text{in } C_R \\ \partial_y v = 0 & \text{on } B_R \times \{y = 1\} \\ -y^a\partial_y v = g & \text{on } B_R \times \{y = 0\}. \end{cases} \tag{3.2.1}$$

We start by giving the definition of weak solution for (3.2.1).

Definition 3.2.1. Let $R > 0$, and let $g \in L^1(B_R)$. We say that a function $v \in H^1(C_R, y^a)$ is a *weak solution* of problem (3.2.1) if

$$\int_{C_R} y^a\nabla v \cdot \nabla\xi \, dx\, dy - \int_{B_R\times\{y=0\}} g\,\xi\, dx = 0,$$

for every $\xi \in C_0^\infty(B_R \times [0, 1])$.

Later on, we will need the following duality principle which is the analogue, for our problem, of Proposition 3.6 in [42] (see also [46]).

Lemma 3.2.2. *Let $g \in C(\mathbb{R}^n)$, $v \in C^2(\mathbb{R}^n \times (0, 1))$, and $y^a\partial_y v \in C(\mathbb{R}^n \times [0, 1])$. If v is a classical solution of*

$$\begin{cases} \operatorname{div}(y^a\nabla v) = 0 & \text{in } \mathbb{R}^n \times (0, 1) \\ \partial_y v = 0 & \text{on } \mathbb{R}^n \times \{y = 1\} \\ -y^a\partial_y v = g & \text{on } \mathbb{R}^n \times \{y = 0\}, \end{cases}$$

then the function $w = -y^a\partial_y v$ is a classical solution of the Dirichlet problem

$$\begin{cases} \operatorname{div}(y^{-a}\nabla w) = 0 & \text{in } \mathbb{R}^n \times (0, 1) \\ w = 0 & \text{on } \mathbb{R}^n \times \{y = 1\} \\ w = g & \text{on } \mathbb{R}^n \times \{y = 0\}. \end{cases}$$

The result in Lemma 3.2.2 follows by a simple computation and we refer to [46] for its proof.

We can now give a regularity result for the localized linear problem (3.2.1).

Proposition 3.2.3. *Let $g \in L^\infty(B_R)$ and let v be a bounded weak solution of* (3.2.1).
Then, there exists $\beta \in (0,1)$ (depending only on n and a) such that $v \in C^\beta(C_{R/2})$ with the following estimate

$$\|v\|_{C^\beta(\overline{C_{R/2}})} \le c_R^1,$$

for some c_R^1 depending on n, a, R, $\|g\|_{L^\infty(B_R)}$, $\|v\|_{L^\infty(C_R)}$.
Moreover, we have

$$\|y^a \partial_y v\|_{L^\infty(\overline{C_{R/2}})} \le c_R^2, \tag{3.2.2}$$

for some c_R^2 depending on the same quantities as above.

Proof. To prove the C^β regularity of the solution v in $B_{R/2} \times [0,1)$, we follow the argument used by Cabré and Sire to prove Lemma 4.5 in [42]. We need to modify such argument since, in our case, the solution of (3.2.1) is not directly related to the fractional Laplacian and a localization method needs to be exploited. First, we set $\bar{g} = g\,\eta$ where $\eta \in C_0^\infty(\mathbb{R}^n)$ is a cut-off function which is identically 1 in $B_{\frac{3}{4}R}$, so that $\bar{g}$ is now defined on the whole $\mathbb{R}^n$ and agrees with g in $B_{\frac{3}{4}R}$. Let now $\bar{v}$ be the bounded solution of

$$\begin{cases} \mathrm{div}(y^a \nabla \bar{v}) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ -y^a \partial_y \bar{v} = \bar{g} & \text{on } \mathbb{R}^n \times \{y=0\}, \end{cases}$$

which is precisely the local problem in the halfspace $\mathbb{R}^{n+1}_+$ associated to the nonlocal equation $(-\Delta)^{\frac{1-a}{2}}\bar{u} = \bar{g}$, where $\bar{u} = \bar{v}(x,0)$ ($\bar{v}$ is the so-called Caffarelli-Silvestre extension of $\bar{u}$, see [46]). By Remark 3.10 in [42], we have that $\bar{v}$ is continuous and bounded in $\overline{\mathbb{R}^{n+1}_+}$. Hence, by Proposition 2.9 in [127], we have that $\bar{u} \in C^\beta(\mathbb{R}^n)$ for some $\beta \in (0,1)$ depending only on n and a.

Let now $\widetilde{v} := v - \bar{v}$. Then, in $C_{\frac{3}{4}R} \subset \mathbb{R}^{n+1}_+$, the function $\widetilde{v}$ solves

$$\begin{cases} \mathrm{div}(y^a \nabla \widetilde{v}) = 0 & \text{in } C_{\frac{3}{4}R} \\ -y^a \partial_y \widetilde{v} = 0 & \text{on } B_{\frac{3}{4}R} \times \{y=0\}. \end{cases}$$

Since now we have reduced our problem to a problem with zero Neumann condition on $\{y=0\}$ we can do an even reflection of the solution $\widetilde{v}$ with respect to $\{y=0\}$ in order to get a bounded weak solution of

$$\mathrm{div}(|y|^a \nabla \widetilde{v}) = 0 \quad \text{in } B_{\frac{3}{4}R} \times (-1,1).$$

Now, we can apply the regularity theory in [83] (we recall that the weight $|y|^a$ belongs to the Muckenhoupt class A_2) to get that $\widetilde{v}$, and thus v, is $C^\beta(\overline{B_R} \times [0,1))$ for some $\beta \in (0,1)$ depending only on n and a.

The C^β regularity for v up to the top boundary $\{y=1\}$ follows in a standard way, again by even reflection with respect to $\{y=1\}$, observing that the weight y^a is non degenerate for $y=1$ and we have zero Neumann condition on this part of the boundary. This conclude the proof of the first part of the statement.

We now prove (3.2.2). By Lemma 3.2.2, the function $w := -y^a \partial_y v$ solves

$$\begin{cases} \mathrm{div}(y^{-a}\nabla w) = 0 & \text{in } C_R \\ w = 0 & \text{on } B_R \times \{y = 1\} \\ w = g & \text{on } B_R \times \{y = 0\}. \end{cases}$$

We introduce the function

$$\overline{w} := P_{\bar{s}}(\cdot, y) * \bar{g}$$

where $\bar{g} = g\eta$ is defined in the first part of the proof, $\bar{s}$ is such that $1 - 2\bar{s} = -a$ and $P_{\bar{s}}$ is the Poisson kernel for the fractional Laplacian (see Proposition 3.7 and Remark 3.8 in [42]). We have that $\overline{w} \in L^\infty(\overline{\mathbb{R}^{n+1}_+})$ and satisfies

$$\begin{cases} \mathrm{div}(y^{-a}\nabla \overline{w}) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ w = \bar{g} & \text{on } \mathbb{R}^n \times \{y = 0\}. \end{cases}$$

Now, we can define $\widetilde{w} := w - \overline{w}$. Arguing as in the first part of the proof, we have that $\widetilde{w}$ has zero (weighted) Neumann condition on $\{y = 0\}$ and hence its odd reflection across $\{y = 0\}$ satisfies

$$\mathrm{div}(|y|^{-a}\nabla \widetilde{w}) = 0 \quad \text{in } B_{\frac{3}{4}R} \times (-1, 1).$$

Using again the regularity theory in [83] (we recall that the weight $|y|^{-a}$ belongs to the Muckenhoupt class A_2) we get that $\widetilde{w}$ is $C^\beta(\overline{B_{R/2}} \times [0, 1))$ with $\beta \in (0, 1)$ depending only on n and a. Hence the function $w = \widetilde{w} + \overline{w}$ is bounded in $\overline{B_{R/2}} \times [0, 1]$ with a bound that only depends on the quantities specified in the statement of the proposition. This concludes the proof.

□

As a consequence of Proposition 3.2.3, we get the following estimate for solutions to the semilinear (localized) problem.

Corollary 3.2.4. *Let f be a function in $C^{1,\gamma}(\mathbb{R})$, with $\gamma > \max\{0, -a\}$ and let v be a bounded solution of*

$$\begin{cases} \mathrm{div}(y^a\nabla v) = 0 & \text{in } C_R \\ \partial_y v = 0 & \text{on } B_R \times \{y = 1\} \\ -y^a\partial_y v = f(v) & \text{on } B_R \times \{y = 0\}. \end{cases}$$

Then, there exists $\beta \in (0, 1)$ (depending only on n and a) such that $v \in C^\beta(C_{R/2})$ with the following estimates

$$\|v\|_{C^\beta(\overline{C_{R/2}})} \le c^1_R,$$

for some c depending on n, a, R, $\|f\|_{C^{1,\gamma}}$, $\|v\|_{L^\infty(C_R)}$.
Moreover, we have

$$\|y^a\partial_y v\|_{L^\infty(\overline{C_{R/2}})} \le c^2_R,$$

for some c depending on n, a, R, $\|f\|_{C^{1,\gamma}}$, $\|v\|_{L^\infty(C_R)}$.

Proof. It is enough to observe that, since $f \in C^{1,\gamma}$ and v is bounded then $f(v)$ is bounded and hence Proposition 3.2.3 applies to v. □

In the following proposition, we establish global gradient estimates for solutions to (3.1.4) (the semilinear problem in the infinite slab), which will be crucial to establish our main result.

Proposition 3.2.5. *Let f be a function in $C^{1,\gamma}(\mathbb{R})$, with $\gamma > \max\{0, -a\}$ and let v be a bounded solution of* (3.1.4).

Then,

$$\|\nabla_x v\|_{L^\infty(\mathbb{R}^n\times[0,1])} + \|y^a\partial_y v\|_{L^\infty(\mathbb{R}^n\times[0,1])} \leq C_1, \tag{3.2.3}$$

for some C_1 depending only on n, a, $\|f\|_{C^{1,\gamma}}$, $\|v\|_{L^\infty}$.

Proof. We start with the estimate for $|\nabla_x v|$. Let us define the function

$$v_1(x,y) := \frac{v(x+he,y) - v(x,y)}{|h|^\beta}$$

where $e \in S^{n-1}$, $h \in \mathbb{R}$ and β given by Proposition 3.2.3 (and, without loss of generality, possibly reducing β, we can assume that β is of the form $1/k$ for some integer k). By Corollary 3.2.4, we have that $v \in C^\beta(\overline{C_{R/2}})$ and hence that v_1 is bounded in $C_{R/4}$. In addition, v_1 solves

$$\begin{cases} \operatorname{div}(y^a\nabla v_1) = 0 & \text{in } C_{R/4} \\ \partial_y v_1 = 0 & \text{on } B_{R/4}\times\{y=1\} \\ -y^a\partial_y v_1 = \frac{f(v((x+he,0)) - f(v(x,0))}{|h|^\beta} & \text{on } B_{R/4}\times\{y=0\}. \end{cases} \tag{3.2.4}$$

Since $f \in C^{1,\gamma}(\mathbb{R})$ and $v \in C^\beta(\overline{C_{R/2}})$, the right-hand side in the third equation of (3.2.4) is bounded and we can apply Proposition 3.2.3 to v_1 in the cylinder $C_{R/4}$. Hence we obtain that v_1 is $C^\beta(\overline{C_{R/8}})$ and, using Lemma 5.6 in [31] and the fact that the direction e is arbitrary, that v is $C^{2\beta}(\overline{C_{R/8}})$. We have that

$$\|v\|_{C^{2\beta}(\overline{C_{R/8}})} \leq c_R$$

with c_R depending on n, a, R, $\|f\|_{C^{1,\gamma}(\mathbb{R})}$, $\|v\|_{L^\infty(C_R)}$.

Now, we can iterate this procedure for a finite number (namely, $k-1$) of times such that $k\beta \geq 1$ (this is possible since β is a fix strictly positive number depending only on the quantities specified in Proposition 3.2.3). In this way, we deduce that $\|\nabla_x v\|_{L^\infty(\overline{C_{R/8^k}})}$ is bounded. Moreover, since problem (3.1.4) is invariant under translations in the x-direction, we can obtain uniform estimates for $\|\nabla_x v\|_{L^\infty}$ in any (closed) cylinder $\overline{C_{R/8^k}(z,0)} = \overline{B_{R/8^k}(z)} \times [0,1]$ with $z \in \mathbb{R}^n$. Observe that the bound c_R^1 in Proposition 3.2.3 depends on the radius but not on the center of the balls B_R. Hence, by a covering argument we obtain the global bound (3.2.3).

To prove the second part of the statement, we use the bound (3.2.2) of Corollary 3.2.4. Again, after fixing the radius $R = 1$ and using a covering argument as before, we deduce that $\|y^a\partial_y v\|_{L^\infty(\mathbb{R}^n\times[0,1])} \leq C_1$, with C_1 depending only on n, a, $\|f\|_{C^{1,\gamma}}$, $\|v\|_{L^\infty}$, which concludes the proof. □

3.3 Proof of Theorem 3.1.3

In this section we establish Theorem 3.1.3 with a proof based on two main ingredients. The first one is the following characterization of stability, which is the analogue for our problem of Lemma 6.1 in [43].

Lemma 3.3.1. *Let d be a bounded, Hölder continuous function on $\mathbb{R}^n$. Then the inequality*

$$\int_{\mathbb{R}^n\times(0,1)} y^a|\nabla\eta|^2\,dx\,dy + \int_{\mathbb{R}^n\times\{y=0\}} d(x)\eta^2\,dx \geq 0 \tag{3.3.1}$$

holds true for any $\eta \in C_0^1(\mathbb{R}^n\times[0,1])$ if and only if there exists a Hölder continuous function $\varphi \in H^1_{loc}(\mathbb{R}^n\times[0,1], y^a)$, such that

$$\begin{cases} \operatorname{div}(y^a\nabla\varphi) = 0 & \text{in } \mathbb{R}^n\times(0,1) \\ -y^a\partial_y\varphi + d(x)\varphi = 0 & \text{on } \mathbb{R}^n\times\{y=0\} \\ \partial_y\varphi = 0 & \text{on } \mathbb{R}^n\times\{y=1\} \end{cases} \tag{3.3.2}$$

with

$$\varphi > 0 \text{ in } \mathbb{R}^n\times[0,1].$$

Proof. We first assume the existence of φ and we prove (3.3.1). Taken a test function η as in the statement of Lemma 3.3.1, we can multiply (3.3.2) by $\frac{\eta^2}{\varphi}$ and then integrate over $\mathbb{R}^n\times(0,1)$. We obtain:

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n\times(0,1)} \operatorname{div}(y^a\nabla\varphi)\frac{\eta^2}{\varphi} \\ &= -\int_{\mathbb{R}^n\times\{y=0\}} y^a\partial_y\varphi\frac{\eta^2}{\varphi} - 2\int_{\mathbb{R}^n\times(0,1)} y^a\frac{\eta}{\varphi}\nabla\eta\cdot\nabla\varphi + \int_{\mathbb{R}^n\times(0,1)} y^a\frac{|\nabla\varphi|^2\eta^2}{\varphi^2} \\ &\geq -\int_{\mathbb{R}^n\times\{y=0\}} d(x)\eta^2 - \int_{\mathbb{R}^n\times(0,1)} y^a|\nabla\eta|^2, \end{aligned}$$

where in the last estimate, we have used the boundary data of (3.3.2) and Cauchy-Schwarz inequality. This establishes (3.3.1).
The other implication is more delicate to prove. We first define

$$Q_R(\xi) := \int_{C_R} y^a|\nabla\xi|^2\,dx\,dy + \int_{B_R\times\{y=0\}} d(x)\xi^2\,dx$$

and we take λ_R as the infimum of $Q_R(\xi)$ in the set

$$\begin{aligned} S_R := &\left\{\xi \in H^1(C_R, y^a) : \xi \equiv 0 \text{ on } \partial B_R\times(0,1), \int_{B_R}\xi^2 = 1\right\} \\ &\subset H_0(C_R, y^a) = \{\xi \in H^1(C_R, y^a) : \xi \equiv 0 \text{ on } \partial B_R\times(0,1)\}. \end{aligned}$$

From the stability assumption and Definition 3.1.2, we know that $\lambda_R \geq 0$. We want to prove that λ_R is strictly decreasing in R, in order to deduce that

$$\lambda_R > 0. \tag{3.3.3}$$

To show that λ_R is decreasing in R, we observe that from the hypothesis λ_R is nonincreasing and Q_R is bounded below in S_R, since d is a bounded function. Now, if we take a minimizing sequence $(\xi_k)_k \subset S_R$, we have that $(\nabla\xi_k)$ is uniformly bounded in $L^2(C_R, y^a)$. Using also the compactness of the inclusion $H_0(C_R, y^a) \subset L^2(B_R)$ (see the proof of Lemma 4.1 in [43]), we can state that the infimum of Q_R in S_R is achieved by a function $\varphi_R \in S_R$. We observe also that, up to take $|\varphi_R|$ instead of φ_R, we can choose $\varphi_R \geq 0$. We remark that the function φ_R solves

$$\begin{cases} \operatorname{div}(y^a \nabla \varphi_R) = 0 & \text{in } C_R \\ -y^a \partial_y \varphi_R + d(x)\varphi_R = \lambda_R \varphi_R & \text{on } B_R \times \{y = 0\} \\ \partial_y \varphi_R = 0 & \text{on } B_R \times \{y = 1\} \\ \varphi_R = 0 & \text{on } \partial B_R \times (0,1). \end{cases} \tag{3.3.4}$$

Hence, from the strong maximum principle, we have that $\varphi_R > 0$ in C_R.
Now, we take $R_1 < R_2$ and our goal is to show that $\lambda_{R_1} > \lambda_{R_2}$. Since $\lambda_{R_1} \geq \lambda_{R_2}$ due to the inclusion of the domains, we argue by contradiction and suppose that $\lambda_{R_1} = \lambda_{R_2}$. The strategy is then to integrate by parts the quantity

$$\int_{C_{R_1}} \varphi_{R_2} \operatorname{div}(y^a \nabla \varphi_{R_1})$$

in order to obtain a contradiction. Indeed, by using (3.3.4) and the fact that $\lambda_{R_1} = \lambda_{R_2}$, we find that

$$\int_{\partial B_{R_1} \times (0,1)} y^a \varphi_{R_2} \frac{\partial \varphi_{R_1}}{\partial \nu} = 0. \tag{3.3.5}$$

Since $\varphi_{R_2} > 0$ and $\frac{\partial \varphi_{R_1}}{\partial \nu} < 0$ on $\partial B_R \times (0,1)$, the identity in (3.3.5) cannot hold true, thus we have reached the desired contradiction. Hence λ_R is strictly decreasing in R and the proof of (3.3.3) is complete.

Using the definition of λ_R and the fact that λ_R is strictly positive, we obtain that

$$Q_R(\xi) \geq \lambda_R \int_{B_R} \xi^2 \geq -\delta_R \int_{B_R} d(x)\xi^2 \qquad \text{for all } \xi \in S_R$$

with $0 < \delta_R := \frac{\lambda_R}{\|d\|_\infty}$, and therefore

$$Q_R(\xi) \geq \varepsilon_R \int_{C_R} y^a |\nabla \xi|^2, \tag{3.3.6}$$

with $\varepsilon_R := 1 - \frac{1}{1+\delta_R} > 0$. Now we are able to prove that, fixed $c_R > 0$, there exists a solution φ_R to the problem

$$\begin{cases} \operatorname{div}(y^a \nabla \varphi_R) = 0 & \text{in } C_R \\ -y^a \partial_y \varphi_R + d(x)\varphi_R = 0 & \text{on } B_R \times \{y = 0\} \\ \partial_y \varphi_R = 0 & \text{on } B_R \times \{y = 1\} \\ \varphi_R = c_R & \text{on } \partial B_R \times (0,1). \end{cases} \tag{3.3.7}$$

Setting $\varphi_R := \psi_R + c_R$, this problem is equivalent to the following one

$$\begin{cases} \operatorname{div}(y^a \nabla \psi_R) = 0 & \text{in } C_R \\ -y^a \partial_y \psi_R + d(x)\psi_R + c_R d(x) = 0 & \text{on } B_R \times \{y = 0\} \\ \partial_y \psi_R = 0 & \text{on } B_R \times \{y = 1\} \\ \psi_R = 0 & \text{on } \partial B_R \times (0,1). \end{cases}$$

We notice that we can solve the latter system by minimizing in the space $H_0(C_R, y^a)$ the functional

$$\begin{aligned} D(\xi) &= \frac{1}{2}\int_{C_R} y^a|\nabla\xi|^2 + \int_{B_R\times\{y=0\}}\left[\frac{1}{2}d(x)\xi^2 + c_R d(x)\xi\right] \\ &= \frac{1}{2}Q_R(\xi) + c_R\int_{B_R\times\{y=0\}} d(x)\xi. \end{aligned}$$

Since this functional is bounded from below and coercive in $H_0(C_R, y^a)$, thanks to (3.3.6), and since the inclusion $H_0(C_R, y^a) \subset L^2(B_R)$ is compact, there exists a minimizer of D in $H_0(C_R, y^a)$.

We want now to show that φ_R is strictly positive. To do this, we consider its negative part φ_R^-. By definition, it vanishes on $\partial B_R \times (0,1)$, and we can compute that $Q_R(\varphi_R^-)$=0.

Since the first eigenvalue λ_R of Q_R is positive, we have that $\varphi_R^- \equiv 0$ and so $\varphi_R \geq 0$. Hence, using the Hopf Lemma (see Lemma 4.11 in [42]), we deduce that $\varphi_R > 0$ in C_R.

Now that we have found a positive solution of (3.3.7), next step is proving that for a fixed $\delta > 0$

$$\sup_{C_R} \varphi_S \leq \widetilde{c}_R \qquad \text{for all } S > R+\delta, \tag{3.3.8}$$

for some $\widetilde{c}_R > 0$ (we stress that $\widetilde{c}_R$ depends on R but not on S). To do that, we choose c_R in (3.3.7) such that $\varphi_R(0) = 1$. [3] Let now φ_S be a solution of (3.3.7) in C_S, with $S > R+\delta$. We take a family of half balls $\{\mathcal{B}^+_{r,i}\}_i \subset \mathbb{R}^n \times [0,1]$, centered in $(x,0)$ with $x \in \overline{B}_R$, of radius $r \in (0, \frac{\delta}{4})$, in such a way that they cover $\overline{B}_R \times \{y=0\}$ and they have finite mutual intersection. They are in finite number and we call this number k (such number depends on R, but not on S). Since these balls cover $\overline{B}_R \times \{y=0\}$, there exists $j \in \{1,\dots,k\}$ such that $0 \in \mathcal{B}^+_{r,j}$. Since $\mathcal{B}^+_{4r,j} \subset C_S$, we can use the Harnack inequality of Lemma 4.9 in [42] and obtain:

$$\sup_{\mathcal{B}^+_{r,j}} \varphi_S \leq K_R \inf_{\mathcal{B}^+_{r,j}} \varphi_S \leq K_R \qquad \text{for every } S > R+\delta,$$

where K_R is a constant depending only on R. Now, using again the Harnack inequality in every ball $\mathcal{B}^+_{r,i}$ of the covering, and using the fact that the balls intersect two-by-two, we obtain the boundedness of φ_S over $B_R \times \{y=0\}$. Thanks to the Neumann condition on the top of the slab, we can extend this bound, using the maximum principle, to the whole cylinder of radius R, obtaining (3.3.8).

Using now the regularity result for the linear problem established in Proposition 3.2.3, we have a uniform bound on $\|\varphi_S\|_{C^\beta(B_{R/2}\times[0,1])}$ for every $S > R+\delta$, therefore we can find a subsequence of (φ_S) that converges locally to a function $\varphi \in C^\beta_{\mathrm{loc}}(\mathbb{R}^n \times [0,1])$ that is positive and solves (3.3.2).

□

Remark 3.3.2. Let v be a solution of (3.1.4) such that $\partial_{x_n} v(x,y) > 0$ for any $(x,y) \in \mathbb{R}^n \times [0,1)$. Then, we can apply Lemma 3.3.1 with the choice $d := -f'(u)$ and $\varphi := \partial_{x_n} v$, to deduce that v is stable. The stability of monotone solutions for this kind of problems

[3] To see that this is possible, consider φ^1_R to be the solution of (3.3.7) with $c_R = 1$. Hence, by the Hopf Lemma, $\varphi^1_R(0) \neq 0$. It is then enough to divide φ^1_R by the value $\varphi^1_R(0)$ to get a solution of (3.3.7) (corresponding to $c_R = (\varphi^1_R(0))^{-1}$) which takes value 1 at 0.

has already been observed in Lemma 7 in [70] for the case $a = 0$. We stress that in this chapter we also need the existence of a positive solution to the linearized problem as a necessary (and not only sufficient) condition for stability.

The second ingredient in the proof of Theorem 3.1.3 is the following Liouville-type result, which is the analogue of Theorem 4.10 in [42]. For its proof, we refer to Section 4.4 of [42], where a similar result is proven for some semilinear equations in the half-space. In this case, the adaptation to our framework is straightforward.

Lemma 3.3.3. *Let φ be a positive function in $L^\infty_{loc}(\mathbb{R}^n \times [0,1])$, $\sigma \in H^1_{loc}(\mathbb{R}^n \times [0,1], y^a)$ such that:*

$$\begin{cases} -\sigma \mathrm{div}(y^a \varphi^2 \nabla \sigma) \leq 0 & \text{in } \mathbb{R}^n \times (0,1) \\ y^a \sigma \frac{\partial \sigma}{\partial \nu} \leq 0 & \text{on } \mathbb{R}^n \times (\{y=0\} \cup \{y=1\}) \end{cases}$$

in the weak sense. If in addition:

$$\int_{B_R \times (0,1)} y^a \left(\varphi\sigma\right)^2 \leq CR^2$$

holds for every $R > 1$, then σ is constant.

We can now give the

Proof of Theorem 3.1.3. Let v be a stable solution of (3.1.4). Lemma 3.3.1 implies that there exists a positive function φ that solves (3.3.2) with $d(x) = -f'(u(x))$. We can define the functions

$$\sigma_i = \frac{\partial_{x_i} v}{\varphi} \qquad \text{for } i = 1, \ldots, n.$$

Our goal is to prove that they are constant. For every fixed i, since $\varphi^2 \nabla \sigma_i = \varphi \nabla v_{x_i} - v_{x_i} \nabla \varphi$ and using that both v_{x_i} and φ satisfy the linearized problem (3.3.2) with $d(x) = -f'(u(x))$, we deduce

$$\mathrm{div}(y^a \varphi^2 \nabla \sigma_i) = 0.$$

Moreover, using again that v_{x_i} and φ satisfy the same linearized problem (in particular they have the same Neumann condition on $\{y = 0\}$), we have

$$y^a \sigma_i \partial_y \sigma_i = y^a \frac{v_{x_i}}{\varphi^2} v_{x_i y} - y^a \frac{v_{x_i}^2}{\varphi^2} \frac{\varphi_y}{\varphi} = 0 \qquad \text{on } (\{y=0\} \cup \{y=1\}) \times \mathbb{R}^n.$$

Finally, assumption (3.1.9) gives

$$\int_{C_R} y^a (\varphi \sigma_i)^2 = \int_{C_R} y^a |\partial_{x_i} v|^2 \leq CR^2,$$

and hence we can apply Lemma 3.3.3 to deduce that σ_i is constant for every $i \in \{1, \ldots, n\}$ and we call these constants c_i. If $c_i = 0$ for every $i = 1, \ldots n$ then v only depends on y (it is constant in the x-variables). Otherwise, the solution v only depends on the variable y and on the one parallel to the vector $(c_1, \ldots, c_n, 0)$. We call this horizontal variable $\tilde{x}$:

$$\widetilde{x} = \frac{\sum_{i=1}^n c_i x_i}{(\sum_{i=1}^n c_i^2)^{\frac{1}{2}}}.$$

We have thus proven that the trace u of v on $\{y = 0\}$ is a function of only one Euclidean variable and hence can be written in the form

$$u(x_1, \dots, x_n) = u_0(\widetilde{x}),$$

where u_0 is a function defined on $\mathbb{R}$. We can compute the derivative of u_0 to get

$$u_0' = \left(\sum_{i=1}^{n} c_i^2 \right)^{\frac{1}{2}} \varphi.$$

If $c_i = 0$ for every $i \in \{1, \dots, n\}$ then $u_0' \equiv 0$, otherwise $u_0' > 0$. This concludes the proof of Theorem 3.1.3.

□

From Theorem 3.1.3 we can directly deduce the

Proof of Corollary 3.1.4. Since we are considering the case $n = 2$, we have from the gradient estimate in Proposition 3.2.5 that

$$\mathcal{E}_R(v) \le CR^2.$$

This energy estimate allows us to apply Theorem 3.1.3 and to obtain that there exist $v_0 : \mathbb{R} \times (0, 1) \to \mathbb{R}$ and $\omega \in S^1$ such that

$$v(x, y) = v_0(\omega \cdot x, y) \qquad \text{for any } (x, y) \in \mathbb{R}^2 \times (0, 1).$$ □

3.4 Energy estimate for minimizers

This section is devoted to the proof of Theorem 3.1.5. Here we prove the energy estimate (3.1.10) for solutions of (3.1.3) which minimize the associated energy, and we argue in an arbitrary dimension n (instead of taking $n = 3$ as we are going to do in the next section). It is worth noting that even if the estimate has no dimensional constraint in the case of minimal solutions, this will not give one-dimensional symmetry of minimizers in further dimensions by applying our method, unless one is willing to take additional assumption on the energy growth of the solutions. Indeed, in order to prove Theorem 3.1.7 we will use Theorem 3.1.3, which requires the energy in C_R to grow like R^2.
We consider v as a bounded minimizer of the functional:

$$\mathcal{E}_R(v) = \frac{1}{2} \int_{C_R} y^a |\nabla v|^2 \, dx \, dy + \int_{B_R \times \{y=0\}} \{G(u) - c_u\} \, dx,$$

such that $v(x, 0) = u(x)$. The function v solves (3.1.4), and the potential G is such that $G'(u) = -f(u)$. Recall that the potential G is naturally defined up to an additive constant and that we have set c_u to be the minimum of G in the range of u (see (3.1.8)), so that $G(u) - c_u \ge 0$. Moreover, we call τ the minimum point of G: in this way, we have that $G(\tau) - c_u = 0$.

As we are going to see, we can directly prove Theorem 3.1.5 using a comparison argument with a suitable choice of the competitor.

Proof of Theorem 3.1.5. Since v is a minimizer of $\mathcal{E}$, for every admissible competitor w (i.e. $w = v$ on $\partial B_R \times (0,1)$) we have

$$\mathcal{E}_R(v) \le \mathcal{E}_R(w).$$

We define

$$w(x,y) := \eta_R(x)\tau + (1 - \eta_R(x))v(x,y), \tag{3.4.1}$$

where $\eta_R : \mathbb{R}^n \to [0,1]$ is a smooth function that is equal to 1 inside B_{R-1} and that vanishes outside B_R. In this way, w is constantly equal to τ in the cylinder $B_{R-1} \times [0,1]$ and it is equal to $v(x,y)$ on the lateral boundary $\partial B_R \times [0,1]$, so w is an admissible competitor.

Now we recall the fact that if u is a bounded solution of (3.1.3), then from Proposition 3.2.5

$$\|\nabla_x v\|_{L^\infty(\mathbb{R}^n \times [0,1])} \le C \qquad \text{and} \qquad \|y^a \partial_y v\|_{L^\infty(\mathbb{R}^n \times [0,1])} \le C. \tag{3.4.2}$$

Using (3.4.2) and the definition (3.4.1) of w, we can control the energy of w (up to subtracting the constant c_u in the potential term) as

$$\begin{aligned} \frac{1}{2}\int_{C_R} y^a |\nabla v|^2 \, dx\, dy &+ \int_{B_R} \{G(w) - c_u\}\, dx \\ &\le C \int_{C_R \setminus C_{R-1}} y^a |\nabla v|^2 \, dx\, dy + C \int_{C_R \setminus C_{R-1}} y^a \left\{ |v|^2 + \tau^2 \right\} dy \\ &\quad + \int_{(B_R \setminus B_{R-1}) \times \{y=0\}} \{G(w) - c_u\}\, dx \\ &\le CR^{n-1} \left(\int_0^1 y^a\, dy + \int_0^1 y^{-a}\, dy \right) + CR^{n-1} \le CR^{n-1}. \end{aligned}$$

This concludes the proof of Theorem 3.1.5. □

3.5 Energy estimates for monotone solutions

The main goal of this section is to prove the energy estimate of Theorem 3.1.6 in dimension three for monotone solutions of (3.1.3). We first give the following result, which is the counterpart of Corollary 6 in [70] in presence of a weight.

Lemma 3.5.1. *Let v be a solution of* (3.1.4) *such that $\partial_{x_n} v(x,0) > 0$ for any $x \in \mathbb{R}^n$.*
Then, $\partial_{x_n} v(x,y) > 0$ for any $(x,y) \in \mathbb{R}^n \times [0,1)$.

Proof. We start by observing that the weak and strong maximum principle hold for weak solutions of problem (3.1.4). This follows exactly as in Remark 4.2 in [42], with the only difference that now we have a Neumann condition on the bottom boundary $\{y = 1\}$. In this part of the boundary it is then enough to apply Hopf's Lemma to a possible minimum of the solution v to get the result. With maximum principles at hand, the proof of the desired result follows exactly the proof of Lemma 5 in [70]. □

Let now $n = 3$ and let v be a solution of (3.1.4) whose trace u on $\{y = 0\}$ is monotone in the last direction x_3. By Lemma 3.5.1, v is monotone in x_3 in the whole slab $\mathbb{R}^3 \times [0,1]$, hence we can define two limit profiles of v as

$$\begin{aligned} \overline{v}(x',y) &:= \lim_{x_3 \to +\infty} v(x,y), \\ \underline{v}(x',y) &:= \lim_{x_3 \to -\infty} v(x,y), \end{aligned}$$

where $x' = (x_1, x_2)$. Notice that $\overline{v}$ and $\underline{v}$ are defined in $\mathbb{R}^2 \times [0,1]$, namely we reduced the problem by one dimension by taking the limit in x_3. This fact allows us to deduce good energy estimates for both $\overline{v}$ and $\underline{v}$ which, in turn, implies the one-dimensional symmetry and the monotonicity of $\overline{u}$ and $\underline{u}$ on $\{y = 0\}$. With these properties for $\underline{v}$ and $\overline{v}$, we are able to characterize the potential G associated to equation (3.1.3) (see Lemma 3.5.5 below).

Then, the proof of the energy estimates for monotone solutions follows by these two steps:

- if v is a bounded monotone solution to (3.1.4), then it is in particular a minimizer for the associated energy in a restricted class of functions S_R (basically functions $\widetilde{w}$ such that $\underline{v} \le \widetilde{w} \le \overline{v}$);
- the characterization of G implies that the competitor w constructed in the previous section belongs to the class S_R.

Some of these results are well known in the classical case or for the fractional Laplacian. Here, we need to prove them for our water waves problem, which offers a series of specific complications also due to the fact that the Poisson kernel is not explicit. For the sake of completeness, we are going to explain all the details in this section.

Using Theorem 3.1.3, we are able to prove some important properties of the two limit profiles. This also gives a characterization of the potential G as a corollary.

Lemma 3.5.2. *Let $f \in C^{1,\gamma}(\mathbb{R})$ with $\gamma > \max\{0, -a\}$ and v a bounded solution of (3.1.4) whose trace u on $\{y = 0\}$ is monotone in x_3.*

Then $\underline{v}$ and $\overline{v}$ are bounded and stable solutions of (3.1.4) with $n = 2$, and each of them is either constant or one-dimensional and monotone in the (x_1, x_2)-plane.

From Lemma 3.5.2, one also obtains:

Corollary 3.5.3. *Set $m = \inf \underline{u} \le \widetilde{m} = \sup \underline{u}$ and $\widetilde{M} = \inf \overline{u} \le M = \sup \overline{u}$. Then $G > G(m) = G(\widetilde{m})$ in $(m, \widetilde{m})$, $G'(m) = G'(\widetilde{m}) = 0$ and $G > G(M) = G(\widetilde{M})$ in $(M, \widetilde{M})$, $G'(M) = G'(\widetilde{M}) = 0$.*

Proof of Lemma 3.5.2. We prove the desired result for $\overline{v}$, clearly the same proof can be replied for $\underline{v}$.

The fact that $\overline{v}$ is a solution follows from seeing it as the limit of a sequence of functions in four variables, that is

$$\overline{v}(x', y) = \lim_{t \to \infty} v^t(x', x_3, y),$$

where $v^t(x', x_3, y) = v(x', x_3 + t, y)$. By Corollary 3.2.4, we have that v^t uniformly converges up to subsequences to $\overline{v}$ in the C^β sense on compact sets.
Now we want to prove that

$$\overline{v} \quad \text{is stable} \tag{3.5.1}$$

and then apply Theorem 3.1.3. By Remark 3.3.2, we have that if v is a monotone solution of (3.1.4) in dimension $n = 3$, then v is stable in $\mathbb{R}^3 \times (0,1)$, hence

$$\int_{\mathbb{R}^3 \times (0,1)} y^a |\nabla \xi|^2 + \int_{\mathbb{R}^3 \times \{y=0\}} f'(u) \xi^2 \ge 0, \tag{3.5.2}$$

for all $\xi \in C_0^\infty(\mathbb{R}^3 \times (0,1))$. Following an idea in [35], we define a special test function ξ in order to get the stability inequality for $\overline{v}$. We take $\rho > 0$ and a function $\phi_\rho \in C_0^\infty(\mathbb{R}, [0,1])$ such that $\phi_\rho = 0$ in $(-\infty, \rho) \cup (2\rho + 2, +\infty)$ and $\phi_\rho = 1$ in $(\rho + 1, 2\rho + 1)$. For every $\eta \in C_0^\infty(\mathbb{R}^2 \times (0,1))$ we define $\xi(x,y) = \eta(x', y)\phi_\rho(x_3)$. So (3.5.2) becomes, after dividing it by $\alpha_\rho = \int_{\mathbb{R}} \phi_\rho^2$:

$$\int_{\mathbb{R}^2 \times (0,1)} y^a |\nabla \eta|^2 \, dx' \, dy + \int_{\mathbb{R}^2 \times (0,1)} y^a \eta^2 \, dx' \, dy \int_{\mathbb{R}} \frac{(\phi_\rho')^2}{\alpha_\rho} \, dx_3 +$$
$$- \int_{\mathbb{R}^2 \times \{y=0\}} \eta^2 \, dx' \, dy \int_{\mathbb{R}} f'(v) \frac{\phi_\rho^2}{\alpha_\rho} \, dx_3 \geq 0.$$

When $\rho \to +\infty$ the second term vanishes, because of the definition of ϕ_ρ. In the third term, thanks to the fact that $f \in C^1(\mathbb{R})$, we have that $f'(v) \to f'(\overline{v})$, hence

$$\int_{\mathbb{R}^2 \times (0,1)} y^a |\nabla \eta|^2 - \int_{\mathbb{R}^2 \times \{y=0\}} f'(\overline{v}) \eta^2 \geq 0.$$

So we proved (3.5.1). In order to conclude that $\overline{v}$ is one-dimensional and monotone in the (x_1, x_2)-plane it is enough to apply Theorem 3.1.3, after observing that, from Proposition 3.2.5, $|\nabla_x \overline{v}| \in L^\infty(\mathbb{R}^2 \times [0,1])$ and hence assumption (3.1.9) is satisfied. □

Before proving Corollary 3.5.3, we define the notion of *layer* solution of (3.1.3) and we give a sufficient condition for the potential G to have a double-well structure.

Definition 3.5.4. We say that v is a layer solution for (3.1.4) if it satisfies (3.1.4),

$$v_{x_n}(x, 0) > 0 \qquad \text{for all } x \in \mathbb{R}^n$$

and

$$\lim_{x_n \to \pm\infty} v(x, 0) = \pm 1 \qquad \text{for every } x' \in \mathbb{R}^{n-1},$$

where $x = (x', x_n) \in \mathbb{R}^n$.

The following result gives a necessary condition to the existence of layer solutions of (3.1.4) with $n = 1$.

Lemma 3.5.5. *Let $f \in C^{1,\gamma}(\mathbb{R})$, with $\gamma > \max\{0, -a\}$ and $G' = f$. Let v be a bounded layer solution of*

$$\begin{cases} \operatorname{div}(y^a \nabla v) = 0 & \text{in } \mathbb{R}^n \times (0,1) \\ -y^a v_y(x, 0) = f(v) & \text{on } \mathbb{R}^n \times \{y = 0\} \\ v_y(x, 1) = 0 & \text{on } \mathbb{R}^n \times \{y = 1\}. \end{cases} \tag{3.5.3}$$

Then

$$G'(1) = G'(-1) = 0. \tag{3.5.4}$$

Moreover, if $n = 1$, we also have

$$G > G(1) = G(-1) \quad \text{in } (-1, 1). \tag{3.5.5}$$

Potentials satisfying (3.5.4), (3.5.5) are called "double-well potentials".

Proof of Lemma 3.5.5. The proof combines some ideas contained in the proofs of Lemmas 4.8 and 5.3 in [42].

First we prove that $G'(1) = G'(-1) = 0$, which holds in any dimension n. We take η smooth and nonnegative with compact support in $B_1 \times [0,1)$ and with strictly positive integral over B_1. For $R > 0$ we define $\eta_R := \eta(\frac{x}{R}, y)$. We slide the function v in the x_n direction by considering

$$v^t(x,y) = v(x', x_n + t, y),$$

which is also a solution of (3.5.3). So we have:

$$\begin{aligned} 0 &= \int_{C_R} \operatorname{div}(y^a \nabla v^t)\eta_R = \int_{B_R \times \{y=0\}} f(u^t)\eta_R - \int_{C_R} y^a \nabla v^t \cdot \nabla \eta_R \\ &= \int_{B_R \times \{y=0\}} f(u^t)\eta_R - \int_{B_R \times \{y=0\}} y^a u^t \partial_y \eta_R + \int_{C_R} v^t \operatorname{div}(y^a \nabla \eta_R). \end{aligned}$$

We have that the first integral converges as $t \to \infty$ to $f(1)R^n \int_{B_1} \eta$ and the other two integrals are bounded by CR^{n-1}. Hence $|f(1)| \leq \frac{C}{R}$ for every R and we get $f(1) = 0$ by letting $R \to \infty$. In the same way we can prove that $G'(-1) = 0$.

We prove now the second part of the statement. Let $n = 1$; we claim that

$$\int_0^1 \frac{t^a}{2}\left(v_x^2(x,t) - v_y^2(x,t)\right) dt = G(u(x,0)) - G(1).$$

First, we define the function w as:

$$w(x) = \int_0^1 \frac{t^a}{2}\left(v_x^2 - v_y^2\right)(x,t)\, dt.$$

We remark that w is well defined and bounded thanks to (3.2.3). In addition, Proposition 3.2.5 allows us to derive under the integral sign in the definition of $w(x)$, so we can compute the derivative of w as

$$\begin{aligned} \partial_x w(x) &= \int_0^1 t^a (v_x v_{xx} - v_y v_{xy})(x,t)\, dt \\ &= \lim_{y \to 0} y^a v_y(x,y) v_x(x,y) = \frac{d}{dx} G(u(x,0)), \end{aligned} \tag{3.5.6}$$

where we have used an integration by parts and the fact that v is a solution of (3.5.3). Using (3.5.6), we obtain that

$$w(x) - [G(v(x,0)) - G(1)] = C$$

for some constant C. Our next goal is proving that $C = 0$.

To this end, first we point out the estimate

$$|w(x)| \leq C \int_0^1 t^a |\nabla v(x,t)|^2\, dt.$$

We prove now that for every fixed $R > 0$

$$\|\nabla_x v\|_{L^\infty(C_R(x,0))} + \|y^a \partial_y v\|_{L^\infty(C_R(x,0))} \longrightarrow 0 \qquad \text{as } x \to +\infty, \tag{3.5.7}$$

where $C_R(x,0) = B_R(x,0) \times (0,1) \subset \mathbb{R} \times (0,1)$.

Suppose by contradiction that there exist $R > 0$, $y \in \mathbb{R}$, $\varepsilon > 0$ and a sequence $t_m \to +\infty$ such that:

$$\|\nabla_x v\|_{L^\infty(C_R(x,0))} + \|y^a \partial_y v\|_{L^\infty(C_R(x,0))} \geq \varepsilon \qquad \text{for every } m. \tag{3.5.8}$$

Notice that v^{t_m} is a solution of (3.5.3) for every m. Also, the sequence v^{t_m} is uniformly bounded. Consequently, we obtain $C^\beta(\overline{C_S})$ estimates for v^{t_m} from Proposition 3.2.3, and we stress that these estimates are uniform in m for every $S > 0$. From this fact we have that, up to subsequences, v^{t_m} converges to a bounded function $v \in C^\beta_{\mathrm{loc}}(\mathbb{R} \times [0,1])$ such that

$$\operatorname{div}(y^a \nabla v) = 0.$$

Since $v \equiv 1$, we get a contradiction with (3.5.8) and we obtain (3.5.7).
Letting now $x \to +\infty$ and using (3.5.7), we deduce that $C = 0$. Moreover taking the limit for $x \to -\infty$, we also have that

$$G(1) = G(-1).$$

Now, we are left with proving that

$$G > G(1) \qquad \text{in } (-1,1).$$

In order to do that, we want to prove that for every $x \in [0,1)$

$$\int_0^x \frac{t^a}{2}\big(v_y^2(t,y) - v_x^2(t,y)\big)\, dt < G(u(0,y)) - G(1). \tag{3.5.9}$$

We define for every $y \in \mathbb{R}$ and $x \in [0,1]$

$$\eta(x,y) := \int_0^x \frac{t^a}{2}\big(v_y^2(t,y) - v_x^2(t,y)\big)\, dt$$

and

$$\varphi(x,y) := G(u(x,0)) - G(1) - \eta(x,y).$$

First, we observe that φ can not be constant. Indeed, since $\eta(x,0) = 0$, the fact that φ is constant would imply that G is also constant. This would give $f \equiv 0$ and so u would be constant, in contradiction with the monotonicity property $u_x > 0$.
Using the fact that $\operatorname{div}(y^a \nabla v) = 0$, we can compute the derivatives of φ as

$$\begin{aligned} &\partial_y \varphi(x,y) = -\frac{y^a}{2}(v_x^2 - v_y^2) \\ \text{and} \qquad &\partial_x \varphi(x,y) = y^a v_x(x,y) v_y(x,y). \end{aligned}$$

Hence, after some computations (see also the proof of Lemma 5.3 in [42]) we see that $\varphi(x,y)$ is bounded and satisfies

$$\operatorname{div}(y^a \nabla \varphi(x,y)) = -a y^{2a-1} v_x^2(x,y) \tag{3.5.10}$$

in $\mathbb{R} \times (0,1)$. Our last claim is that

$$\varphi \text{ is strictly positive on } \mathbb{R} \times [0,1). \tag{3.5.11}$$

Notice that this claim implies (3.5.9). In order to prove (3.5.11), we assume by contradiction that there exists (x_0, y_0) in $\mathbb{R} \times [0, 1)$ such that $\varphi(x_0, y_0) \leq 0$.

Let us divide the proof in two cases, considering at a first attempt $a \geq 0$. From (3.5.10) it follows that

$$\operatorname{div}(y^a \nabla \varphi(x, y)) \leq 0 \tag{3.5.12}$$

and, using also the Hopf Lemma 4.11 in [42], we can say that $y_0 = 0$. Hence there exists $x_0 \in \mathbb{R}$ such that

$$G(u(x_0, 0)) - G(1) \leq 0.$$

But $\psi(x) = G(u(x, 0)) - G(1)$ goes to zero when $x \to \pm\infty$, so we can take x_0 as a global minimum for ψ. It follows that

$$0 = \frac{d}{dx} G(u(x_0, 0)) = \lim_{y \to 0} y^a v_y(x_0, y) v_x(x_0, y)$$

and from the monotonicity property of v, see Remark 3.3.2, we have

$$0 = -\lim_{y \to 0} y^a v_y(x_0, y).$$

By the maximum principle, the point $(x_0, 0)$ is also the minimum of $\varphi(x, y) = G(u(x, 0)) - G(1) - \eta(x, y)$. Since φ is the extension of ψ satisfying (3.5.12), we have that $(x_0, 0)$ is a strict minimum for φ and we get a contradiction by considering

$$\begin{aligned} 0 &> -y^a \partial_y \varphi(x_0, y)_{|_{\{y=0\}}} = y^a \partial_y \eta(x, y)_{|_{\{y=0\}}} \\ &= \lim_{y \to 0} \frac{y^{2a}}{2} \big(v_y^2(x_0, y) - v_x^2(x_0, y) \big) = \lim_{y \to 0} \frac{y^{2a}}{2} v_x^2(x_0, y) \geq 0. \end{aligned}$$

This contradiction proves (3.5.11) in the case $a \geq 0$. Now we deal with the case $a < 0$. First, we compute

$$\operatorname{div}(y^{-a} \nabla \varphi(x, y)) = -a y^{-1} v_y^2(x, y). \tag{3.5.13}$$

Recalling that we are supposing that a negative minimum of φ is achieved at $(x_0, y_0) \in \mathbb{R}^n \times [0, 1)$, we want to show that $y_0 = 0$. Since now a is negative, we have to add an extra term from (3.5.13) and consider

$$0 = \operatorname{div}(y^{-a} \nabla \varphi) + a y^{-1} v_y^2 = \operatorname{div}(y^{-a} \nabla \varphi) + \big(a y^{-a-1} \frac{v_y}{v_x} \big) \varphi_x.$$

From the fact that this last operator is uniformly elliptic with continuous coefficients in compact sets of $\mathbb{R}^n \times (0, 1)$, it follows that $y_0 = 0$. Now, we can obtain a contradiction by considering

$$\begin{aligned} 0 &\geq -\liminf_{y \to 0^+} y^{-a} \partial_y \varphi(x_0, y) = y^a \partial_y \eta(x, y)_{|_{\{y=0\}}} \\ &= \liminf_{y \to 0} \frac{1}{2} \big(v_x^2(x_0, y) - v_y^2(x_0, y) \big) = \frac{1}{2} v_x^2(x_0, 0) > 0. \end{aligned}$$

Notice that we have used also the fact that, from (3.2.3), $|v_y(x_0, y)| \leq C y^{-a} \to 0$ as $y \to 0^+$. This proves (3.5.11) also when a is negative and finishes the proof. $\square$

Now that we have characterized the potential G in presence of a layer solution, we are able to deduce Corollary 3.5.3 from Lemma 3.5.2.

Proof of Corollary 3.5.3. We want to find a layer solution of (3.1.4) in order to use the characterization given by Lemma 3.5.5. Our candidate is the function $\overline{w}$ defined as

$$\overline{w} := 2\left(\frac{\overline{v} - \widetilde{M}}{M - \widetilde{M}}\right) - 1.$$

We take the function $h(\overline{w}) := \frac{2f(\overline{v})}{M-\widetilde{M}}$, and we call H the potential associated to h, so $H' = -h$. Then, $\overline{w}$ is a solution of problem (3.1.4) with $n = 2$ with f replaced by the new nonlinearity $h(\overline{w})$. By Lemma 3.5.2 $\overline{w}$ is either constant or one-dimensional in the $\{x_1, x_2\}$-plane, and it is monotone if it is not constant. According to Definition 3.5.4, we have that $\overline{w}$ is a layer solution of problem (3.1.4) (with the new nonlinearity h). Now we can apply Lemma 3.5.5, and obtain that H is a double-well potential. Restating this result for G, we have that G is forced to satisfy $G'(\widetilde{M}) = G'(M) = 0$ and $G > G(\widetilde{M}) = G(M)$ in $(\widetilde{M}, M)$.
Using $\underline{v}$ instead of $\overline{v}$, we can prove with the same argument that $G'(\widetilde{m}) = G'(m) = 0$ and $G > G(\widetilde{m}) = G(m)$ in $(m, \widetilde{m})$. $\square$

As a final step before giving the proof of Theorem 3.1.6, we need to prove the following result, which ensures that if v is a bounded monotone solution for problem (3.1.4), then it is a minimizer in a particular class of functions. This result can be seen as the counterpart of Proposition 6.2 in [36] for the case into consideration here, in which we have to take into account the singularity and degeneracy of the weights, the different domain of the equation and the different boundary conditions.

Lemma 3.5.6. *Let $f \in C^{1,\gamma}(\mathbb{R})$, with $\gamma > \max\{0, -a\}$, v a bounded solution of (3.1.4) with $n = 3$, such that its trace $u(x) = v(x, 0)$ is monotone in its third variable.*
Then
$$\mathcal{E}_R(v) \leq \mathcal{E}_R(w)$$
for every $w \in H^1(C_R, y^a)$ such that $w = v$ on $\partial B_R \times (0, 1)$ and $\underline{v} \leq w \leq \overline{v}$ in C_R.

Proof. The proof of this property follows the proof of Proposition 6.2 in [36] and is based on two results:

(i) Uniqueness of solutions to the problem

$$\begin{cases} \operatorname{div}(y^a \nabla w) = 0 & \text{in } C_R \\ w = v & \text{on } \partial B_R \times (0,1) \\ -y^a \partial_y w = f(w) & \text{on } B_R \times \{y = 0\} \\ \partial_y w = 0 & \text{on } B_R \times \{y = 1\} \\ \underline{v} \leq w \leq \overline{v} & \text{in } B_R \times (0,1) \end{cases} \tag{3.5.14}$$

We give here a proof of this result that uses the idea of sliding the function v in the x_n-direction. Keeping in mind that $\underline{u}$ and $\overline{u}$ are respectively the trace of $\underline{v}$ and $\overline{v}$ on $\{y = 0\}$, let w be a solution of (3.5.14).

By Hopf Lemma (see Lemma 4.11 in [42]) and the maximum principle, we have that

$$\underline{v} < w < \overline{v} \qquad \text{in } \overline{C_R}. \tag{3.5.15}$$

Now we slide the function v in the direction of monotonicity x_n. We take

$$v^t(x,y) := v(x_1,\ldots,x_{n-1},x_n+t,y).$$

Since $v_t \to \overline{v}$ uniformly in $\overline{C}_R$ and by (3.5.15), we have that $w < v^t$ in $\overline{C}_R$ for t large enough. We take s as

$$s := \inf\left\{t > 0 \quad \text{s.t.} \quad w < v^t \quad \text{in } \overline{C}_R\right\}.$$

We need to prove that

$$s = 0. \tag{3.5.16}$$

Suppose by contradiction that $s > 0$. Then we would have $w \le v^s$ in all $\overline{B_R} \times [0,1]$ and there must be a point $(\bar{x},\bar{y})$ in which the two functions coincide. But $(\bar{x},\bar{y}) \notin \partial B_R \times (0,1)$ because along $\partial B_R \times (0,1)$ it holds that $w = v$, and we have the monotonicity hypothesis on v. So $(\bar{x},\bar{y})$ must be either in C_R or in $B_R \times \{y = 0\} \cup B_R \times \{y = 1\}$, but we get a contradiction either with the maximum principle or with the Hopf Lemma applied to the positive function $v^s - w$. Hence we proved (3.5.16).

(ii) Existence of a minimizer for $\mathcal{E}_R$ in the set:

$$S_R = \{w \in H^1(C_R, y^a) : w \equiv v \text{ on } \partial B_R \times (0,1),\ \underline{v} \le w \le \overline{v} \text{ in } C_R\}.$$

This result is the analogue of the one obtained in Lemma 4.1 of [43] for layer solutions of the fractional Laplacian. The proof can be adapted by substituting -1 and $+1$, which are the limits of the layer solution in [43], with $\underline{v}$ and $\overline{v}$, which are respectively a subsolution and a supersolution for problem (3.1.4).

We already know that v is a solution to problem (3.5.14) and, in view of point (i), we have uniqueness of this solution. So this solution must coincide with the minimizer for $\mathcal{E}_R$ in S_R. □

Now we are able to prove the energy estimate of Theorem 3.1.6 and to deduce the one-dimensional symmetry of monotone solutions from it.

Proof of Theorem 3.1.6. We follow the idea in the proof of Theorem 5.2 of [5] and Theorem 1.3 of [35], that is we show that the comparison function $\overline{w}$ defined in the previous section satisfies

$$\underline{v} \le \overline{w} \le \overline{v}. \tag{3.5.17}$$

In this way, we have that $\overline{w}$ belongs to S_R, which is the class of functions where v minimizes the energy. We recall that $\overline{w}$ is defined as:

$$\overline{w}(x,y) = \tau\eta_R(x) + \big(1 - \eta_R(x)\big)v(x,y).$$

If we prove that $\tau \in [\sup \underline{u}, \inf \overline{u}]$, we also have (3.5.17) from the maximum principle.

In order to prove this, we use Corollary 3.5.3. We set $m = \inf \underline{u}$, $\widetilde{m} = \sup \underline{u}$ and $\widetilde{M} = \inf \overline{u}$, $M = \sup \overline{u}$. We have

$$G > G(m) = G(\widetilde{m}) \quad \text{in } (m, \widetilde{m}) \quad \text{if } \underline{u} \text{ is not constant;}$$

$$G > G(M) = G(\widetilde{M}) \quad \text{in } (\widetilde{M}, M) \quad \text{if } \overline{u} \text{ is not constant.}$$

Suppose that $m \neq \widetilde{M}$: In all possible cases we have $\widetilde{m} \leq \widetilde{M}$, so there exists τ in $[\widetilde{m}, \widetilde{M}]$ such that $G(\tau) = c_u$, where c_u is the infimum of G in the range of u. Hence

$$\sup \underline{v} = \sup \underline{u} = \widetilde{m} \leq \tau \leq \widetilde{M} = \inf \overline{u} = \inf \overline{v}$$

and (3.5.17) is proved. Hence, $\overline{w} \in S_R$ and from Lemma 3.5.6 we can conclude that (3.1.11) holds true.

We have still to consider the special case in which $m = \widetilde{M}$ and $M = \widetilde{m}$. From Corollary 3.5.3, it follows that $G \geq G(m) = G(M)$ in (m, M) and all the solutions v^t are obtained by translation of v. Hence, they produce a foliation and we can use this fact together with the strong maximum principle to prove that v is a minimizer of the energy. In this way, we conclude that (3.1.11) follows by Theorem 3.1.5 in this special case. □

From the energy estimates in (3.1.10) and (3.1.11) we obtain the one-dimensional symmetry of both minimizers and monotone solutions by a direct application of Theorem 3.1.3.

Proof of Theorem 3.1.7. Either if v is a bounded minimizer or is a bounded solution whose trace on $\{y = 0\}$ is monotone, we have from Theorem 3.1.6 and Theorem 3.1.5 that

$$\frac{1}{2} \int_{C_R} y^a |\nabla v|^2 \, dx \, dy + \int_{B_R \times \{y=0\}} (G(v) - c_u) \, dx \leq CR^2.$$

This energy estimate is enough to apply Theorem 3.1.3 and to obtain that there exist $v_0 : \mathbb{R} \times (0,1) \to \mathbb{R}$ and $\omega \in S^2$ such that

$$v(x, y) = v_0(\omega \cdot x, y) \qquad \text{for any } (x, y) \in \mathbb{R}^3 \times (0,1). \qquad \square$$

Chapter 4

Asymptotics of the energy

We consider a Dirichlet to Neumann operator $\mathcal{L}_a$ arising in a model for water waves, with a nonlocal parameter $a \in (-1, 1)$. We deduce the expression of the operator in terms of the Fourier transform, highlighting a local behavior for small frequencies and a nonlocal behavior for large frequencies.

We further investigate the Γ-convergence of the energy associated to the equation $\mathcal{L}_a(u) = W'(u)$, where W is a double-well potential. When $a \in (-1, 0]$ the energy Γ-converges to the classical perimeter, while for $a \in (0, 1)$ the Γ-limit is a new nonlocal operator, that in dimension $n = 1$ interpolates the classical and the nonlocal perimeter.

4.1 Introduction

In this chapter, we consider a possibly singular or degenerate elliptic problem with weights, which is set on the infinite domain $\mathbb{R}^n \times (0, 1)$, endowed with mixed boundary conditions. When $n = 2$, such a problem is related to the formation of water waves from a steady ocean, the case of homogeneous density of the fluid corresponding to a Laplace equation in $\mathbb{R}^2 \times (0, 1)$ with mixed boundary conditions, and the weighted equation arising from power-like fluid densities.

We provide here two types of results. The first set of results focuses on the operator acting on $\mathbb{R}^n \times \{0\}$ produced by the associated Dirichlet to Neumann problem. That is, we consider the weighted Neumann derivative of the solution along the portion of the boundary that is endowed with a Dirichlet datum, which corresponds, in the homogeneous fluid case, to the determination of the vertical velocity field on the surface of the ocean. In this setting, we provide an explicit expression of this Dirichlet to Neumann operator in terms of the Fourier representation, and we describe the asymptotics of the corresponding Fourier symbols.

The second set of results deals with the energy functional associated to the Dirichlet to Neumann operator. Namely, we consider an energy built by the combination of a suitably weighted interaction functional of Dirichlet to Neumann type in the Fourier space with a double-well potential. In this setting, choosing the parameters in order to produce significant asymptotic structures, we describe the Γ-limit configuration.

The results obtained are new even in the case $n = 2$ and even for the Laplace equation. Interestingly, however, the fluid density plays a decisive role as a bifurcation parameter, and the case of uniform density is exactly the threshold separating two structurally different behaviors. Therefore, understanding the "more general" case of variable densities also provides structural information on the homogeneous setting. Specifically, we prove convergence of the energy functional to a Γ-limit corresponding to a mere interaction energy when $a \in (0,1)$ and to the classical perimeter when $a \in (-1,0]$. In terms of the corresponding fractional parameter $s = \frac{1-a}{2}$, this dichotomy reflects a purely nonlocal behavior when $s \in (0,1/2)$ and a purely classical asymptotics when $s \in [1/2,1)$. Interestingly, the threshold $s = 1/2$ corresponds here to the homogeneous density case, the strongly nonlocal regime corresponds to degenerate densities y^a with $a > 0$, and the weakly nonlocal regime to singular densities y^a with $a < 0$.

We also point out that the threshold $s = 1/2$ that we obtain here, as well as the limit behavior for the regime $s \in [1/2,1)$, is common to other nonlocal problems, such as the ones in [47, 123, 126]. On the other hand, the limit functional that we obtain in the strongly nonlocal regime $s \in (0,1/2)$ appears to be new in the literature, and structurally different from other energy functionals of nonlocal type that have been widely investigated.

The precise mathematical formulation of the problem under consideration is the following. We consider the slab $\mathbb{R}^n \times [0,1]$ with coordinates $x \in \mathbb{R}^n$ and $y \in [0,1]$, a smooth bounded function $u : \mathbb{R}^n \to \mathbb{R}$, and its bounded extension v in the slab $\mathbb{R}^n \times [0,1]$, which is the bounded function satisfying the mixed boundary value problem

$$\begin{cases} \operatorname{div}(y^a \nabla v) = 0 & \text{in } \mathbb{R}^n \times (0,1) \\ v_y(x,1) = 0 & \text{on } \mathbb{R}^n \times \{y=1\} \\ v(x,0) = u(x) & \text{on } \mathbb{R}^n \times \{y=0\}, \end{cases} \tag{4.1.1}$$

where $a \in (-1,1)$. Problem (4.1.1) naturally leads to the study of the Dirichlet to Neumann operator $\mathcal{L}_a$ defined as

$$\mathcal{L}_a u(x) = -\lim_{y \to 0} y^a v_y(x,y). \tag{4.1.2}$$

The operator $\mathcal{L}_a$, which is the main object of the present work, arises in the study of a water wave model. With respect to the physical motivation, one can consider $\mathbb{R}^n \times (0,1)$ as “the sea”, where $\{y = 0\}$ corresponds the surface of the sea (assumed to be at rest) and $\{y = 1\}$ is its bottom (assumed to be made of concrete and impenetrable material). More specifically, the first equation in (4.1.1) models the mass conservation and the irrotationality of the fluid, and the second one is a consequence of the impenetrability of the matter. The scalar function v plays the role of a velocity potential, that is the gradient of v corresponds to the velocity of the fluid particles. Given the datum of the velocity potential v on the surface — i.e. the Dirichlet condition on $\{y = 0\}$ in (4.1.1) — we are interested in studying the weighted vertical velocity on the surface, which is responsible for the formation of a wave emanating from the rest position of a “flat sea". The operator $\mathcal{L}_a$ defined in (4.1.2) models indeed this vertical velocity. We refer to [75] for a complete description of this model and for detailed physical motivations.

We observe that the energy functional associated to (4.1.1) can be written as

$$\mathcal{E}_K(v) := \frac{1}{2} \int_{\mathbb{R}^n \times (0,1)} y^a \, |\nabla v|^2 \, dx \, dy.$$

In what follows, we will consider the energy minimization in the class of functions

$$\mathcal{H}_u := \{ w \in H^1_{\text{loc}}(\mathbb{R}^n \times (0,1), y^a) \text{ s.t. } w(x,0) = u(x) \text{ for a.e. } x \in \mathbb{R}^n \}. \tag{4.1.3}$$

Such a minimizer exists and it is unique — see Lemma 4.2.1 below for a detailed proof — and we can define the interaction energy associated to u as the interaction energy of its minimal extension v. Namely, with a slight abuse of notation, we write

$$\mathcal{E}_K(u) := \inf_{v \in \mathcal{H}_u} \mathcal{E}_K(v).$$

Notice that the minimizer $v \in \mathcal{H}_u$ of the energy $\mathcal{E}_K$ solves the mixed boundary problem (4.1.1) in the weak sense, i.e.

$$\int_{\mathbb{R}^n \times (0,1)} y^a \nabla v \cdot \nabla \varphi = 0 \tag{4.1.4}$$

for every $\varphi \in C^\infty(\mathbb{R}^n \times [0,1])$ with compact support contained in $\mathbb{R}^n \times (0,1]$.

We observe that, thanks to the existence of a unique minimizer of the energy $\mathcal{E}_K$ in the class $\mathcal{H}_u$, the operator $\mathcal{L}_a$ is actually well-defined. Indeed, among all the (possibly many) solutions to (4.1.1), we can uniquely choose the one which minimizes $\mathcal{E}_K$ in $\mathcal{H}_u$, and define $\mathcal{L}_a u$ as its weighted vertical derivative evaluated at $y = 0$, according to (4.1.2).

In the case $a = 0$, which corresponds to v being the harmonic extension of u in $\mathbb{R}^n \times (0,1)$, the operator $\mathcal{L}_a$ defined in (4.1.2) was considered by de la Llave and Valdinoci in [70]. In particular, they studied the equation

$$\mathcal{L}_0(u) = f(u) \qquad \text{in } \mathbb{R}^n, \tag{4.1.5}$$

where $f \in C^{1,\beta}(\mathbb{R})$, and $\mathcal{L}_0$ is the operator defined in (4.1.2) with $a = 0$. The main result in [70] is a Liouville theorem for monotone solutions to (4.1.5), which leads in dimension $n = 2$ to the one-dimensional symmetry of monotone solutions.

Some years later, Cinti, Valdinoci, and the author [57] generalized this Liouville theorem to stable[1] solutions to

$$\mathcal{L}_a(u) = f(u) \qquad \text{in } \mathbb{R}^n, \tag{4.1.6}$$

where $f \in C^{1,\beta}(\mathbb{R})$ and $a \in (-1,1)$. More precisely, in [57] the rigidity of monotone and minimizing solutions to (4.1.6) is obtained in the case $n = 3$ for every $a \in (-1,1)$. This is done by combining the Liouville theorem for stable solutions with some new energy estimates for monotone and minimizing solutions to (4.1.6).

The problem of proving one-dimensional symmetry of some special classes of solutions to (4.1.6) is strictly related to a conjecture of De Giorgi for the classical Allen-Cahn equation, and also to an analogue conjecture for the fractional Laplacian. These conjectures are also related to a classical question posed by Gary W. Gibbons which originated from cosmological problems. We refer to the recent survey [75] for more details about these connections and for an outline of the most important recent results in these fields.

In [70] the operator $\mathcal{L}_0$ is written via Fourier transform as

$$\mathcal{L}_0 u = \mathcal{F}^{-1}\left(\frac{e^{|\xi|} - e^{-|\xi|}}{e^{|\xi|} + e^{-|\xi|}} \, |\xi| \, \widehat{u}(\xi)\right), \tag{4.1.7}$$

[1] We say that a solution u to (4.1.6) is stable if the second variation of the associated energy is nonnegative definite at u. We also remind that, for this kind of problems, monotone solutions are stable — see [57]. Clearly, minimizing solutions to (4.1.6) are also stable.

where $\widehat{u}$ denotes the Fourier transform of u and $\mathcal{F}^{-1}$ the inverse Fourier transform.

From expression (4.1.7), one can easily observe that for large frequencies the Fourier symbol of $\mathcal{L}_0$ is asymptotic to $|\xi|$, which is the Fourier symbol of the half-Laplacian (hence, the high-frequency wave formation is related, at least asymptotically, to the operator $\sqrt{-\Delta}$).

The first main result of the present chapter extends (4.1.7) to every $a \in (-1, 1)$, providing the Fourier representation of the operator $\mathcal{L}_a$ for every value of the parameter a in terms of special functions of Bessel type.

Theorem 4.1.1. *For every smooth bounded function u defined on $\mathbb{R}^n$ which is integrable, we can write the operator $\mathcal{L}_a$ defined in* (4.1.2) *via Fourier transform, as*

$$\widehat{\mathcal{L}_a u}(\xi) = c_1(s) \frac{J_{1-s}(-i\,|\xi|)}{J_{s-1}(-i\,|\xi|)}\, |\xi|^{2s}\, \widehat{u}(\xi), \tag{4.1.8}$$

where $1 - a = 2s$, J_k is the Bessel function of the first kind of order k, and

$$c_1(s) := i \left(\frac{1-i}{2}\right)^{4s-2} \frac{\Gamma(1-s)}{\Gamma(s)}. \tag{4.1.9}$$

Moreover, the symbol

$$S_s(\xi) := c_1(s) \frac{J_{1-s}(-i\,|\xi|)}{J_{s-1}(-i\,|\xi|)}\, |\xi|^{2s} \tag{4.1.10}$$

is a positive and increasing function of $|\xi|$, and enjoys the following asymptotic properties. There exist two positive constants C_1 and C_2 depending only on s such that

$$\begin{aligned} \lim_{|\xi|\to 0} \frac{S_s(\xi)}{|\xi|^2} &= C_1; \\ \lim_{|\xi|\to +\infty} \frac{S_s(\xi)}{|\xi|^{2s}} &= C_2. \end{aligned} \tag{4.1.11}$$

We remind that $|\xi|^2$ is the Fourier symbol of the classical Laplacian and that the fractional Laplacian can be expressed for a smooth function u defined in $\mathbb{R}^n$ as

$$(-\Delta)^s\, u(x) = \mathcal{F}^{-1}\left(|\xi|^{2s}\, \widehat{u}(\xi)\right).$$

As a consequence, from Theorem 4.1.1 we have that the operator $\mathcal{L}_a$ defined in (4.1.2) is somewhat asymptotically related to the fractional Laplacian, but it is not equal to any purely fractional operator. In this spirit, the asymptotic behaviors in (4.1.11) reveal an important difference between the problem considered here and several other fractional problems widely investigated in the literature. Namely, in light of (4.1.11), we have that for large frequencies the Fourier symbol of the operator $\mathcal{L}_a$ is asymptotic to the Fourier symbol of the fractional Laplacian $(-\Delta)^s$ with $s = \frac{1-a}{2}$, but for small frequencies it is always asymptotic to the Fourier symbol of the classical Laplacian, and this lack of homogeneity, combined with a significant structural difference "between zero and infinity", suggests a new and interesting interplay between local and nonlocal phenomena at different scales.

From (4.1.8) we also deduce an alternative formulation of the Dirichlet energy $\mathcal{E}_K$, that we state in the following result.

Corollary 4.1.2. *Let u be a smooth bounded function defined on $\mathbb{R}^n$ which is integrable, and v the solution of* (4.1.1) *obtained as the unique minimizer of $\mathcal{E}_K$ in the class $\mathcal{H}_u$. Then,*

$$\mathcal{E}_K(v) = \frac{1}{2}\int_{\mathbb{R}^n\times(0,1)} y^a\,|\nabla v|^2\,dx\,dy = \frac{1}{2(2\pi)^n}\int_{\mathbb{R}^n} S_s(\xi)\,|\widehat{u}(\xi)|^2\,d\xi, \tag{4.1.12}$$

where $a = 1-2s$ and $S_s(\xi)$ is defined in (4.1.10).

For later convenience, we introduce the notation

$$\begin{aligned} & S_s(\xi) = |\xi|^{2s}\,\widetilde{S}_s(\xi), \\ \text{where}\qquad & \widetilde{S}_s(\xi) := c_1(s)\frac{J_{1-s}(-i\,|\xi|)}{J_{s-1}(-i\,|\xi|)}, \end{aligned} \tag{4.1.13}$$

and $c_1(s)$ is defined in (4.1.9). When $s = 1/2$, from (4.1.7) we know that $\widetilde{S}_{1/2}$ is the hyperbolic tangent of $|\xi|$. In general, $\widetilde{S}_s$ is expressed in terms of Bessel functions of the first kind, and its behavior at zero and at infinity can be easily deduced by (4.1.11). Indeed, $\widetilde{S}_s$ converges to a finite constant at infinity, while it behaves like $|\xi|^{2-2s}$ near zero. This can be seen also in Figure 4.1, where the plots of $\widetilde{S}_s$ are displayed for some values of $s \in (0,1)$.

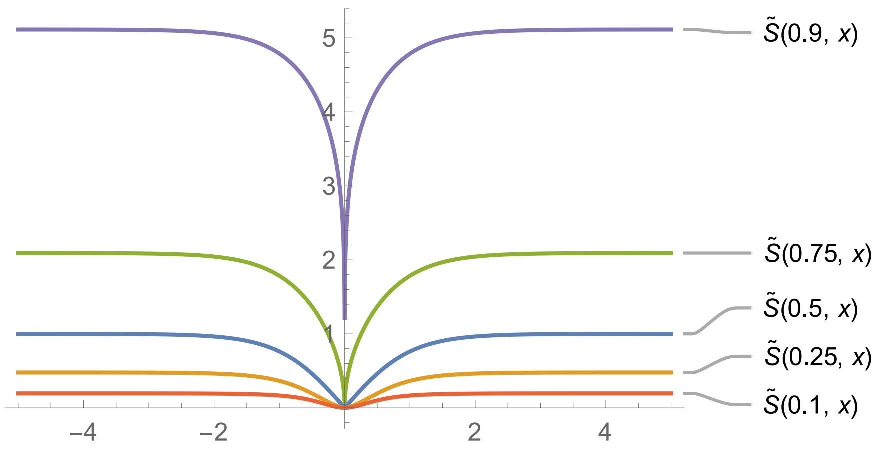

Figure 4.1: *The symbols $\widetilde{S}_s$ for different values of $s \in (0,1)$.*

Heuristically, on the one hand, the connection of $\mathcal{L}_a$ with the fractional Laplacian was already evident from the formulation (4.1.1)-(4.1.2) of the operator, using the extension problem. Indeed, if we consider a solution v of (4.1.1) in the whole half-space and not only in a strip of fixed height, then the associated Dirichlet to Neumann operator is the fractional Laplacian $(-\Delta)^s$ with $s = (1-a)/2$ — see [46].

On the other hand, the asymptotic properties outlined in (4.1.11) make more clear the different nature of $\mathcal{L}_a$ in dependence of the parameter a, which is a very specific feature of this operator. In order to further investigate this twofold behavior, we study the Γ-convergence of the energy associated to the equation $\mathcal{L}_a u = W'(u)$, where W is a double-well potential.

As well-known, the Γ-convergence is a variational notion of convergence for functionals, which was introduced in [67, 69] and that captures the minimizing features of the energy — see also [114] for a classical example of Γ-convergence in the context of phase transitions. In the recent years, there have been an increasing interest towards

Γ-convergence results for nonlocal functionals, and some important results in this topic have been obtained, see for instance [6,7,13,47,98,123,126]. For a complete introduction to topic of Γ-convergence, we refer the reader to [23,63].

Since the operator $\mathcal{L}_a$ is strictly related to the fractional Laplacian, we are particularly interested in the paper [123] by Savin and Valdinoci, in which they consider a proper rescaling of the energy

$$\mathcal{I}_\varepsilon(u,\Omega) := \varepsilon^{2s}\mathcal{K}(u,\Omega) + \int_\Omega V(u)\,dx, \tag{4.1.14}$$

where V is a double-well potential, Ω a bounded set, and $\mathcal{K}(u,\Omega)$ is defined as

$$\mathcal{K}(u,\Omega) := \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + 2\iint_{\Omega\times\mathscr{C}\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy.$$

Observe that $\mathcal{K}(u,\Omega)$ is the "Ω-contribution" of the H^s seminorm of u, where

$$[u]^2_{H^s(\mathbb{R}^n)} := \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy.$$

The main result in [123] — that we describe in more detail in Section 4.4 before the proof of Theorem 4.1.3 — establishes that a proper rescaling of $\mathcal{I}_\varepsilon$ converges in the Γ-sense to the classical perimeter when $s \geq 1/2$ and to the nonlocal area functional for $s \in (0,1/2)$.

For some set $E \subset \mathbb{R}^n$, the nonlocal area functional of ∂E in Ω is defined as $\mathcal{K}(u,\Omega)$ for $u = \chi_E - \chi_{\mathscr{C}E}$. This notion was introduced by Caffarelli, Roquejoffre and Savin in [45], and takes into account the interactions between points which lie in the set E and points which lie in its complement, thus producing a functional which can be thought as a nonlocal version of the classical perimeter. For an introduction to this topic, we refer to [25, Chapter 5], [60], [77], and [108].

We also recall[2] that the H^s seminorm can be written via Fourier transform as

$$[u]^2_{H^s(\mathbb{R}^n)} = \frac{2C(n,s)^{-1}}{(2\pi)^n}\int_{\mathbb{R}^n} |\xi|^{2s}\,|\widehat{u}(\xi)|^2\,d\xi, \tag{4.1.15}$$

where

$$C(n,s) := \left(\int_{\mathbb{R}^n} \frac{1-\cos(\zeta_1)}{|\zeta|^{n+2s}}\,d\zeta\right)^{-1}. \tag{4.1.16}$$

The alternative form (4.1.15) of the H^s seminorm highlights the similarity between $\mathcal{E}_K$ and the Dirichlet energy $\mathcal{K}(u,\Omega)$ in (4.1.14). This is evident after comparing (4.1.15) with expression (4.1.12) for $\mathcal{E}_K$, taking also into account that the symbol $S_s(\xi)$ behaves like $|\xi|^{2s}$ for high frequencies — see (4.1.11).

This fact, together with the results in [123], leads to the natural question of studying the Γ-convergence of a proper rescaling of

$$\mathcal{J}(u) := \int_{\mathbb{R}^n} S_s(\xi)\,|\widehat{u}(\xi)|^2\,d\xi + \int_{\mathbb{R}^n} W(u)\,dx,$$

[2]See [73, Proposition 3.4] for the proof of (4.1.15), and observe that $(2\pi)^{-n}$ is missing in the proof when they apply the Plancherel theorem.

where W is a double-well potential $W(t)$. In particular, throughout the chapter we assume that $W(t)$ satisfies

$$\begin{aligned} &W \in C^{2,\gamma}([0,1]), \qquad W(0) = W(1) = 0, \qquad W > 0 \text{ in } (0,1), \\ &W'(0) = W'(1) = 0, \qquad \text{and} \qquad W''(0) = W''(1) > 0. \end{aligned} \tag{4.1.17}$$

Observe also that the fact of being a double-well potential is invariant under a multiplicative constant.

The energy functional $\mathcal{J}$ is similar to $\mathcal{I}_\varepsilon$ considered in [123], with the important structural difference of replacing $\mathcal{K}(u,\Omega)$ with the Dirichlet energy associated to the operator $\mathcal{L}_a$, expressed with the Fourier transform.

For every $s \in (0,1)$, we consider the partial rescaling of $\mathcal{J}$ given by

$$\mathcal{J}_\varepsilon(u) := \varepsilon^{2s} \int_{\mathbb{R}^n} S_s(\xi)\, |\widehat{u}(\xi)|^2\, d\xi + \int_{\mathbb{R}^n} W(u)\, dx. \tag{4.1.18}$$

We also define the function space in which we work as

$$X := \{u \in L^\infty(\mathbb{R}^n) \text{ s.t. } u \text{ has compact support and } 0 \le u \le 1\}, \tag{4.1.19}$$

and we say that a sequence $u_j \in X$ converges to u in X if $u_j \to u$ in $L^1(\mathbb{R}^n)$. Observe indeed that, according to the definition, $X \subset L^1(\mathbb{R}^n)$.

In order to obtain an interesting result in terms of Γ-convergence, we take the rescaling of (4.1.18) given by $\mathcal{F}_\varepsilon : X \to \mathbb{R} \cup \{+\infty\}$, where

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \varepsilon^{-2s} \mathcal{J}_\varepsilon(u) & \text{if } s \in (0,1/2); \\ |\varepsilon \log \varepsilon|^{-1}\, \mathcal{J}_\varepsilon(u) & \text{if } s = 1/2; \\ \varepsilon^{-1} \mathcal{J}_\varepsilon(u) & \text{if } s \in (1/2,1). \end{cases} \tag{4.1.20}$$

It is important to point out that the rescaling of $\mathcal{J}_\varepsilon$ that we consider here is the same as the one used for the functional $\mathcal{I}_\varepsilon$ in [123], and it is chosen to produce a significant Γ-limit from the interplay of interaction and potential energies.

When $s \in (0,1/2)$, the limit functional $\mathcal{F} : X \to \mathbb{R} \cup \{+\infty\}$ is defined as

$$\mathcal{F}(u) := \begin{cases} \displaystyle\int_{\mathbb{R}^n} S_s(\xi)\, |\widehat{u}(\xi)|^2\, d\xi & \text{if } u = \chi_E, \text{ for some set } E \subset \mathbb{R}^n; \\ +\infty & \text{otherwise.} \end{cases} \tag{4.1.21}$$

We point out that the limit functional $\mathcal{F}$ for $s \in (0,1/2)$ is well-defined when $u = \chi_E$. This is a consequence of the fact that its difference with the H^s seminorm of $u = \chi_E$ is finite — see the forthcoming Lemma 4.4.1 — and that the nonlocal area functional of a bounded set is always well-defined for $s \in (0,1/2)$. Moreover, as stated explicitly in Lemma 4.4.1 means that we can see $\mathcal{F}$ as a perturbation of the nonlocal area functional. We will further comment on the functional $\mathcal{F}$ for $s \in (0,1/2)$ in Proposition 4.1.4 below.

In the case $s \in [1/2,1)$, we define $\mathcal{F} : X \to \mathbb{R} \cup \{+\infty\}$ as

$$\mathcal{F}(u) := \begin{cases} c_\# \mathrm{Per}(E) & \text{if } u = \chi_E \text{ for some set } E \subset \mathbb{R}^n; \\ +\infty & \text{otherwise,} \end{cases} \tag{4.1.22}$$

where $c_\#$ is a positive constant depending only on n and s, and $\mathrm{Per}(E)$ denotes the classical perimeter of the set E, in the sense described e.g. in [97].

The following is the second main result of the present chapter. It establishes the Γ-convergence of the rescaled functional (4.1.20) to $\mathcal{F}$ defined in (4.1.21)-(4.1.22).

Theorem 4.1.3. *Let $s \in (0,1)$. Then the functional $\mathcal{F}_\varepsilon$ defined in* (4.1.20) *Γ-converges to the functional $\mathcal{F}$ defined in* (4.1.21)-(4.1.22)*, i.e. for any u in X*

(i) for any u_ε converging to u in X

$$\liminf_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon) \geq \mathcal{F}(u); \tag{4.1.23}$$

(ii) there exists a sequence $(u_\varepsilon)_\varepsilon$ converging to u in X such that

$$\limsup_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon) \leq \mathcal{F}(u). \tag{4.1.24}$$

We stress that the Γ-limit functional $\mathcal{F}$ is defined in two different ways depending on whether s is above or below $1/2$, showing a purely local behavior when $s \in [1/2,1)$ and a purely nonlocal behavior when $s \in (0,1/2)$. In view of the different structure of the problem in terms of the nonlocal parameter s, we prove Theorem 4.1.3 in two different ways depending on the parameter range. For $s \in [1/2,1)$ the proof is presented in Section 4.4, while for $s \in (0,1/2)$ we include it in Section 4.5.

When $s \in [1/2,1)$, we recover the classical perimeter in the Γ-limit, as in the case of the energy associated to the fractional Laplacian treated in [123]. Moreover, the result in [123] plays a key role in our proof of Theorem 4.1.3 for $s \geq 1/2$. Indeed, in this case we "add and subtract" the square of the H^s-seminorm — properly rescaled — to the functional $\mathcal{F}_\varepsilon$. In this way, we write $\mathcal{F}_\varepsilon$ as the nonlocal area functional plus a remainder term. We then show that the remainder term goes to zero in the limit, and deduce the proof of Theorem 4.1.3 for $s \in [1/2,1)$ from a proper application of [123, Theorem 1.4].

On the other hand, when $s \in (0,1/2)$, the Γ-limit is the functional $\mathcal{F}$ defined in (4.1.21), that has a nonlocal feature. As a technical remark, we also point out that, in our framework, the case $s \in [1/2,1)$ is conceptually harder to address than the case $s \in (0,1/2)$, and the computational complications arising when $s \in [1/2,1)$ are often motivated by the fact that one has to relate a nonlocal behavior at a given configuration with a local asymptotic pattern.

When $n = 1$, we are able to make explicit computations with the Fourier transform, and obtain additional information on the Γ-limit functional $\mathcal{F}$ defined in (4.1.21).

To this end, since the limit functional $\mathcal{F}$ is (possibly) finite only when $u = \chi_E$ for some set $E \subset \mathbb{R}$, we consider a connected interval $I_r \subset \mathbb{R}$ of length r and the characteristic function χ_{I_r}. Then, the squared modulus of the Fourier transform of χ_{I_r} is

$$|\widehat{\chi_{I_r}}(\xi)|^2 = \frac{4\sin^2(r\xi)}{\xi^2}.$$

For the sake of completeness we included this computation in Appendix 4.A.1 — see Lemma 4.A.1.1. We also remark that the squared modulus of the Fourier transform of χ_{I_r} depends only on the length of the interval, thus $\mathcal{F}(\chi_{I_r})$ only depends on r.

Therefore, we can define a function $\mathcal{T}_s(r) : [0,+\infty) \longrightarrow [0,+\infty)$ as

$$\mathcal{T}_s(r) := \mathcal{F}(\chi_{I_r}) = \int_{\mathbb{R}} S_s(\xi)\, |\widehat{\chi_{I_r}}(\xi)|^2 \, d\xi, \tag{4.1.25}$$

where $I_r \subset \mathbb{R}$ is a connected interval of length r. Observe that $\mathcal{T}_s$ depends on $s \in (0,1/2)$, as the symbol $S_s(\xi)$ defined in (4.1.10) depends on s. The following result contains some properties of the function $\mathcal{T}_s$ that allow us to relate it to the common notions of classical and fractional perimeter in one dimension.

Proposition 4.1.4. *Let $s \in (0, 1/2)$ and $n = 1$. The function $\mathcal{T}_s(r)$ defined in* (4.1.25) *is positive and enjoys the following asymptotic properties. There exist two positive constants C_1 and C_2 depending only on s such that*

$$\lim_{r \to 0^+} \frac{\mathcal{T}_s(r)}{r^{1-2s}} = C_1; \tag{4.1.26}$$

$$\lim_{r \to +\infty} \mathcal{T}_s(r) = C_2. \tag{4.1.27}$$

We recall that from the definition of nonlocal perimeter it follows that an interval of length r has fractional perimeter of order r^{1-2s}. In this sense, Proposition 4.1.4 tells us that the limit functional defined in (4.1.21) interpolates the classical and the fractional perimeter, at least in dimension one. Indeed, for intervals of small length $\mathcal{T}_s(r)$ behaves like the fractional perimeter, while for large values of r it converges to a constant, counting the finite number of discontinuities of χ_{I_r}.

We remark that the restriction $n = 1$ in Proposition 4.1.4 is only due to the possibility of making explicit calculations with the Fourier transform. For this reason, we think that it is an interesting question to understand how the functional $\mathcal{F}$ defined in (4.1.21) for $s \in (0, 1/2)$ interpolates classical and nonlocal objects in any dimension.

Structure of the chapter

In Section 4.2 we prove that there exists a unique minimizer of the energy $\mathcal{E}_K$ in the class $\mathcal{H}_u$. In Section 4.3 we prove Theorem 4.1.1 about the Fourier representation of the operator $\mathcal{L}_a$. In Section 4.4 we prove the Γ-convergence result of Theorem 4.1.3 when $s \geq 1/2$. In Section 4.5 we assume $s \in (0, 1/2)$ and we prove both Theorem 4.1.3 and Proposition 4.1.4 about the limit functional. Finally, we collect in Appendix 4.A.1 some ancillary computations and technical results.

4.2 Existence and uniqueness of the minimizer for the Dirichlet energy

This section concerns the existence and the uniqueness of the minimizer of the energy $\mathcal{E}_K$ in the class of functions $\mathcal{H}_u$ defined in (4.1.3) for a given a smooth function u. We state the existence and uniqueness result as follows.

Lemma 4.2.1. *If u is a bounded smooth function defined in $\mathbb{R}^n$, then there exists a unique minimizer of the functional $\mathcal{E}_K$ in the class $\mathcal{H}_u$.*

Proof. **Step 1.** First, using a classical convexity argument, we prove that if such a minimizer exists, then it is *unique*. If we assume that v and w are two minimizers of $\mathcal{E}_K$ in $\mathcal{H}_u$, then considering the energy of their arithmetic mean we find that

$$\begin{aligned} \mathcal{E}_K\left(\frac{v+w}{2}\right) &= \frac{1}{2}\int_{\mathbb{R}^n \times (0,1)} y^a \frac{|\nabla v|^2 + |\nabla w|^2 + 2\nabla v \cdot \nabla w}{4}\, dx\, dy \\ &\leq \frac{1}{2}\mathcal{E}_K(v) + \frac{1}{2}\mathcal{E}_K(w) = \mathcal{E}_K(v). \end{aligned} \tag{4.2.1}$$

Since v and w are minimizers for $\mathcal{E}_k$, the Cauchy-Schwarz inequality in (4.2.1) is an equality, hence

$$\nabla v = \lambda \nabla w.$$

Now, since $\mathcal{E}_K(v) = \mathcal{E}_K(w)$, then $\lambda = \pm 1$. If $\lambda = +1$, then v and w are equal up to an additive constant, but this constant must be zero since both functions are equal to $u(x)$ when $y = 0$. If instead $\lambda = -1$, then from (4.2.1) we deduce that $\mathcal{E}_K(v) = \mathcal{E}_K(w) = 0$, therefore v and w are constant, and these constants must coincide since they agree when $y = 0$.

Step 2. Let us now prove *existence*. First, we observe that this is equivalent to proving that there exists a minimizer of the energy

$$\mathcal{E}_{K,2}(v) := \frac{1}{2} \int_{\mathbb{R}^n \times (0,2)} y^a |\nabla v|^2 \, dx \, dy,$$

in the class of functions

$$\mathcal{H}_{u,2} := \{ w \in H^1_{\mathrm{loc}}(\mathbb{R}^n \times (0,2), y^a) \text{ s.t. } w(x,0) = w(x,2) = u(x) \text{ for a.e. } x \in \mathbb{R}^n \}.$$

Indeed, let us suppose for the moment that such a minimizer exists and let us denote it with $\overline{v}$. Then, we can deduce that it is unique, using the same argument as in Step 1.

Furthermore, since $\overline{v}$ is a minimizer, then it is symmetric with respect to $\{y = 1\}$. To see this, let us consider the competitor

$$\widetilde{v}(x,y) := \begin{cases} \overline{v}(x,y) & \text{if } 0 < y < 1 \\ \overline{v}(x, 2-y) & \text{if } 1 < y < 2, \end{cases}$$

for which we have $\mathcal{E}_{K,2}(\widetilde{v}) = \mathcal{E}_{K,2}(\overline{v})$ and $\widetilde{v} \in \mathcal{H}_{u,2}$. By the uniqueness of the minimizer of $\mathcal{E}_{K,2}$ in $\mathcal{H}_{u,2}$, we deduce that $\widetilde{v} \equiv \overline{v}$, and therefore that $\overline{v}$ is symmetric with respect to $\{y = 1\}$. Now, if we consider the restriction $\overline{v}_{|\mathbb{R}^n \times (0,1)}$, then it belongs to $\mathcal{H}_u$. In addition, using the minimality and symmetry properties of $\overline{v}$, we deduce by a reflection argument that $\overline{v}_{|\mathbb{R}^n \times (0,1)}$ minimizes $\mathcal{E}_K$ in $\mathcal{H}_u$.

Summarizing, to prove Lemma 4.2.1 we are reduced to show that

$$\text{there exists a minimizer of the energy } \mathcal{E}_{K,2} \text{ in the class } \mathcal{H}_{u,2}. \tag{4.2.2}$$

In order to prove (4.2.2), we minimize the localized functional $\mathcal{E}_{K,2}$ on $B_R \times (0,2)$ and then take the limit as $R \to +\infty$. More precisely, we want to prove that there exists a minimizer of

$$\mathcal{E}^R_{K,2}(v) := \frac{1}{2} \int_{B_R \times (0,2)} y^a |\nabla v|^2 \, dx \, dy,$$

in the space

$$\mathcal{H}^R_{u,2} := \{ w \in H^1(B_R \times (0,2), y^a) \text{ s.t. } w(x,0) = w(x,2) = u(x) \text{ for a.e. } x \in B_R \},$$

and then take the limit as $R \to \infty$.

The existence of local minimizers for this problem follows from classical tools in the calculus of variations. Indeed, the lower boundedness of $\mathcal{E}^R_{K,2}$ and the convexity with respect to the gradient give the weak lower semi-continuity of the functional — see

[81, Theorem 1, p.446]. In addition, $\mathcal{E}^R_{K,2}$ is coercive[3] in the $H^1(B_R \times (0,2), y^a)$-norm and this, together with weak lower semicontinuity, is enough to conclude the existence of a minimizer of $\mathcal{E}^R_{K,2}$ in the class $\mathcal{H}^R_{u,2}$.

Furthermore, the local minimizer is unique for every $R > 0$, again by the standard convexity argument of Step 1. Therefore, for every $R > 0$ we know that there exists a unique minimizer v_R of $\mathcal{E}^R_{K,2}$ in $\mathcal{H}_{u,2}$ and we want to deduce (4.2.2), passing to the limit as $R \to \infty$.

To this end, we first observe that v_S solves $\operatorname{div}(y^a \nabla v_S) = 0$ in the weak sense in C_R, whenever $S \geq R$. We choose $\varphi = v_S \eta^2$ in the weak formulation (4.1.4) of the equation, where $\eta \in C^\infty_c(C_R, [0,1])$ and $\eta \equiv 1$ in $C_{R/2}$. Using also a Cauchy-Schwarz inequality, we obtain the Caccioppoli bound

$$\int_{B_{R/2}\times(0,2)} y^a \left|\nabla v_S\right|^2 \leq C \int_{B_{R/2}\times(0,2)} y^a \left|v_S\right|^2, \tag{4.2.3}$$

for a constant C depending only on R.

We then observe that, thanks to the maximum principle, every minimizer v_S of the energy functional attains its maximum at a boundary point. This maximum has to be less or equal than $\|u\|_{L^\infty(\mathbb{R}^n)}$, where u is the Dirichlet datum on the top and the bottom of the cylinder. Indeed, if this is not the case, then we can build a competitor with lower energy than v_S by simply truncating v_S when its absolute value exceeds $\|u\|_{L^\infty(\mathbb{R}^n)}$.

Therefore, we can bound the right-hand side of (4.2.3) with a constant depending only on n, R and $\|u\|_{L^\infty(\mathbb{R}^n)}$. This gives a uniform bound on the $H^1(B_{R/2} \times (0,2), y^a)$-norm of v_S for every $S > R$. Hence, we can find a subsequence of (v_S) that converges locally to a function $\overline{v} \in \mathcal{H}_{u,2}$. Finally, $\overline{v}$ minimizes $\mathcal{E}_{K,2}$ in $\mathcal{H}_{u,2}$ since v_S are local minimizers, and this proves (4.2.2). This concludes the proof of Lemma 4.2.1. □

4.3 The energy via Fourier transform

In this section we want to prove the representation via Fourier transform of the operator $\mathcal{L}_a$, outlined in Theorem 4.1.1. We start by considering the simplest case $a = 0$. To this end, we observe that problem (4.1.1) with $a = 0$ reads

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^n \times (0,1) \\ \partial_y v = 0 & \text{on } \mathbb{R}^n \times \{y=1\} \\ v(x,y) = u(x) & \text{on } \mathbb{R}^n \times \{y=0\}, \end{cases} \tag{4.3.1}$$

and the Dirichlet to Neumann operator is

$$\mathcal{L}_0 u = -\partial_y v(x,y)_{|_{\{y=0\}}}. \tag{4.3.2}$$

In this case, the representation via Fourier transform already appears in [70] by de la Llave and Valdinoci. We state here explicitly this result and give a simple proof of it. We will then use the same strategy, combined with a suitable special functions analysis, to prove Theorem 4.1.1 in the general case $a \in (-1,1)$.

[3] As a technical observation, we point out that the coercivity in this setting follows from the Poincaré inequality with Muckenhoupt weights — see [100, Chapter 15]. We also observe that, for this inequality to hold, it is enough to assume the Dirichlet datum on a portion of the boundary with nonnegative Hausdorff measure.

Proposition 4.3.1 (de la Llave, Valdinoci [70]). *For every smooth bounded function u defined on $\mathbb{R}^n$ which is integrable, we can write the operator $\mathcal{L}_0$ defined in* (4.3.2) *via Fourier transform as*

$$\widehat{\mathcal{L}_0 u} = S_{1/2}(\xi)\widehat{u}(\xi) = \frac{e^{|\xi|} - e^{-|\xi|}}{e^{|\xi|} + e^{-|\xi|}}\,|\xi|\,\widehat{u}(\xi). \tag{4.3.3}$$

Proof. Taking the Fourier transform of the first equation in (4.3.1), we find an ODE in the variable y, that is

$$-|\xi|^2\widehat{v} + \widehat{v}_{yy} = 0.$$

This equation is solved by

$$\widehat{v}(\xi, y) = \alpha(\xi)e^{|\xi|y} + \beta(\xi)e^{-|\xi|y},$$

where α and β are functions depending only on ξ. In order to determine α and β, we consider the Fourier transform of the second and third equations in (4.3.1). The Dirichlet condition on $\{y = 0\}$ gives

$$\alpha(\xi) + \beta(\xi) = \widehat{u}(\xi),$$

while the Neumann condition on $\{y = 1\}$ gives

$$\alpha(\xi)|\xi|e^{|\xi|} - \beta(\xi)|\xi|e^{-|\xi|} = 0.$$

Therefore, we find

$$\alpha(\xi) = \frac{e^{-2|\xi|}}{1 + e^{-2|\xi|}}\widehat{u}(\xi) \qquad \text{and} \qquad \beta(\xi) = \frac{1}{1 + e^{-2|\xi|}}\widehat{u}(\xi).$$

Finally, computing the Fourier transform of $\mathcal{L}_0 u$, we find

$$\widehat{\mathcal{L}_0 u}(\xi) = -\partial_y\widehat{v}(\xi, y)_{|_{\{y=0\}}} = (\beta(\xi) - \alpha(\xi))\,|\xi| = \frac{e^{|\xi|} - e^{-|\xi|}}{e^{|\xi|} + e^{-|\xi|}}\,|\xi|\,\widehat{u}(\xi),$$

and this proves (4.3.3). □

Now, we consider problem (4.1.1) for a general parameter $a \in (-1, 1)$ and we complete the proof of Theorem 4.1.1. For this, we use the same strategy as in the proof of Proposition 4.3.1, but extra computations are required, together with a set of useful identities involving special functions.

Proof of Theorem 4.1.1. As we did in the case $a = 0$, we start by considering the Fourier transform of the first equation in (4.1.1), that is

$$-|\xi|^2 y^a\widehat{v} + ay^{a-1}\widehat{v}_y + y^a\widehat{v}_{yy} = 0.$$

This is an ODE with respect to the variable y and it is solved by

$$\widehat{v}(\xi, y) = \alpha(\xi)y^{\frac{1-a}{2}}J_{\frac{a-1}{2}}(-i|\xi|y) + \beta(\xi)y^{\frac{1-a}{2}}Y_{\frac{a-1}{2}}(-i|\xi|y),$$

where J_m and Y_m are Bessel functions of order m of the first and second kind respectively, while α and β are functions depending only on ξ.

In order to determine $\alpha(\xi)$ and $\beta(\xi)$, we consider the Fourier transform of the second and third equations in (4.1.1). The equation on $\{y = 0\}$ gives

$$\widehat{u}(\xi) = \alpha(\xi) \lim_{y\to 0} y^{\frac{1-a}{2}} J_{\frac{a-1}{2}}(-i|\xi|y) + \beta(\xi) \lim_{y\to 0} y^{\frac{1-a}{2}} Y_{\frac{a-1}{2}}(-i|\xi|y). \tag{4.3.4}$$

We recall the two following properties of Bessel functions

$$\lim_{x\to 0} \frac{J_m(-ix)}{x^m} = \frac{2^{-2m}(1-i)^{2m}}{\Gamma(m+1)}; \tag{4.3.5}$$

$$\text{for non integer } m, \qquad Y_m(x) = \frac{J_m(x)\cos(m\pi) - J_{-m}(x)}{\sin(m\pi)}. \tag{4.3.6}$$

Now, using (4.3.5) and (4.3.6), we can write (4.3.4) as

$$\widehat{u}(\xi) = \alpha(\xi)(1-i)^{a-1} \frac{2^{1-a}}{\Gamma(\frac{a+1}{2})} |\xi|^{\frac{a-1}{2}} + \beta(\xi)(1-i)^{a-1} \frac{2^{1-a}}{\Gamma(\frac{a+1}{2})} \frac{\cos\left(\frac{a-1}{2}\pi\right)}{\sin\left(\frac{a-1}{2}\pi\right)} |\xi|^{\frac{a-1}{2}}.$$

Using the relation $1-a=2s$, the equation on $\{y=0\}$ can be finally written as

$$|\xi|^s\, \widehat{u}(\xi) = \frac{(1-i)^{-2s}2^{2s}}{\Gamma(1-s)} \left\{ \alpha(\xi) - \frac{\cos(s\pi)}{\sin(s\pi)} \beta(\xi) \right\}. \tag{4.3.7}$$

Now, we want to use the equation on $\{y=1\}$. First, we compute the derivative of $\widehat{v}(\xi, y)$ with respect to y

$$\begin{aligned} \partial_y \widehat{v}(\xi, y) = {} & \alpha(\xi) \frac{1-a}{2} y^{\frac{-1-a}{2}} J_{\frac{a-1}{2}}(-i|\xi|y) - \alpha(\xi) y^{\frac{1-a}{2}} i|\xi| J'_{\frac{a-1}{2}}(-i|\xi|y) \\ & + \beta(\xi) \frac{1-a}{2} y^{\frac{-1-a}{2}} Y_{\frac{a-1}{2}}(-i|\xi|y) - \beta(\xi) y^{\frac{1-a}{2}} i|\xi| Y'_{\frac{a-1}{2}}(-i|\xi|y). \end{aligned}$$

We can simplify this expression using the following formulas for the derivatives of Bessel functions

$$\begin{aligned} J'_{\frac{a-1}{2}}(x) &= \frac{a-1}{2x} J_{\frac{a-1}{2}}(x) - J_{\frac{a+1}{2}}(x), \\ Y'_{\frac{a-1}{2}}(x) &= \frac{a-1}{2x} Y_{\frac{a-1}{2}}(x) - Y_{\frac{a+1}{2}}(x). \end{aligned}$$

This gives

$$\partial_y \widehat{v}(\xi, y) = \alpha(\xi) i|\xi| y^{\frac{1-a}{2}} J_{\frac{a+1}{2}}(-i|\xi|y) + \beta(\xi) i|\xi| y^{\frac{1-a}{2}} Y_{\frac{a+1}{2}}(-i|\xi|y). \tag{4.3.8}$$

Using again the relation $1-a=2s$, we write the Neumann condition over $\{y=1\}$ as

$$0 = J_{1-s}(-i|\xi|)\alpha(\xi) + Y_{1-s}(-i|\xi|)\beta(\xi). \tag{4.3.9}$$

To determine α and β, we put together the information given by (4.3.7) and (4.3.9) — which are deduced from the second and third equation in (4.1.1). In this way, we obtain the system

$$\begin{cases} J_{1-s}(-i|\xi|)\alpha(\xi) + Y_{1-s}(-i|\xi|)\beta(\xi) = 0 \\ \alpha(\xi) - \frac{\cos(s\pi)}{\sin(s\pi)}\beta(\xi) = \left(\frac{1-i}{2}\right)^{2s} \Gamma(1-s)\, |\xi|^s\, \widehat{u}(\xi). \end{cases} \tag{4.3.10}$$

Solving (4.3.10), we find

$$\begin{aligned}\alpha(\xi) &= -\widetilde{c}(s)\frac{Y_{1-s}(-i|\xi|)}{\cos(s\pi)\,J_{1-s}(-i|\xi|)+\sin(s\pi)\,Y_{1-s}(-i|\xi|)}\,|\xi|^s\,\widehat{u}(\xi),\\ \beta(\xi) &= \widetilde{c}(s)\frac{J_{1-s}(-i|\xi|)}{\cos(s\pi)\,J_{1-s}(-i|\xi|)+\sin(s\pi)\,Y_{1-s}(-i|\xi|)}\,|\xi|^s\,\widehat{u}(\xi)\end{aligned} \tag{4.3.11}$$

where

$$\widetilde{c}(s) := -\left(\frac{1-i}{2}\right)^{2s}\sin(s\pi)\Gamma(1-s).$$

Using formula (4.3.8) for the y-derivative of $\widehat{v}$, we can compute the Fourier transform of $\mathcal{L}_a u$ and find

$$\begin{aligned}\widehat{\mathcal{L}_a u}(\xi) &= -y^a\partial_y\widehat{v}(\xi,y)_{|_{\{y=0\}}}\\ &= -i|\xi|\left[\alpha(\xi)\lim_{y\to 0}y^{1-s}J_{1-s}(-i|\xi|y)+\beta(\xi)\lim_{y\to 0}y^{1-s}Y_{1-s}(-i|\xi|y)\right].\end{aligned} \tag{4.3.12}$$

Using the properties in (4.3.5)-(4.3.6) of Bessel functions, we see that the first limit in (4.3.12) is zero, and the second one gives a nontrivial contribution. More specifically, we have that

$$\widehat{\mathcal{L}_a u}(\xi) = \frac{i}{\sin(s\pi)\Gamma(s)}\left(\frac{1-i}{2}\right)^{2s-2}|\xi|^s\,\beta(\xi).$$

We can simplify this expression, also using (4.3.6) in (4.3.11), and write it as

$$\widehat{\mathcal{L}_a u}(\xi) = c_1(s)\frac{J_{1-s}(-i\,|\xi|)}{J_{s-1}(-i\,|\xi|)}\,|\xi|^{2s}\,\widehat{u}(\xi),$$

where

$$c_1(s) = i\left(\frac{1-i}{2}\right)^{4s-2}\frac{\Gamma(1-s)}{\Gamma(s)}.$$

This proves (4.1.8), and we are left with showing the asymptotic properties (4.1.11) of the symbol $S_s(\xi)$ defined in (4.1.10).

First, we recall the notation in (4.1.13). From the Taylor expansion near 0 of the Bessel functions of the first kind expressed in (4.3.5), we easily deduce that $\widetilde{S}_s(0)=0$ and

$$\lim_{|\xi|\to 0}\frac{S_s(\xi)}{|\xi|^2} = C_1,$$

where C_1 is a positive constant depending only on s. Moreover, $\widetilde{S}_s(\xi)$ is radially monotone increasing, since

$$\widetilde{S}_s'(\xi) = c_2(s)\frac{\xi}{|\xi|^2}\frac{1}{J_{s-1}^2(-i\,|\xi|)}, \tag{4.3.13}$$

where $c_2(s) = 2c_1(s)\sin(s\pi)/\pi$, and this also proves that $S_s(\xi)$ is radially monotone increasing in ξ.

Finally, from the properties of the Bessel function, we also know that $\widetilde{S}_s(\xi)$ is bounded, and we easily deduce that

$$\lim_{|\xi|\to+\infty}\frac{S_s(\xi)}{|\xi|^{2s}} = C_2,$$

where C_2 is a positive constant depending only on s. This proves (4.1.11) and finishes the proof of Theorem 4.1.1. □

We observe that if we take $a = 0$ in (4.1.8), then $c_1(1/2) = i$, and

$$J_{-1/2}(-i\,|\xi|) = \frac{1+i}{\sqrt{\pi\,|\xi|}} \cosh(|\xi|) \qquad J_{1/2}(-i\,|\xi|) = \frac{1-i}{\sqrt{\pi\,|\xi|}} \sinh(|\xi|).$$

Therefore

$$\widehat{\mathcal{L}_0 u}(\xi) = \frac{e^{|\xi|} - e^{-|\xi|}}{e^{|\xi|} + e^{-|\xi|}}\,|\xi|\,\widehat{u}(\xi),$$

and we recover the special case (4.3.3).

To conclude this section, we deduce Corollary 4.1.2 from Theorem 4.1.3, providing an alternative form of the Dirichlet energy $\mathcal{E}_K$ associated to $\mathcal{L}_a$.

Proof of Corollary 4.1.2. Using the integration by parts formula and the fact that v is a weak solution of (4.1.1), we have

$$\mathcal{E}_K(v) = \frac{1}{2}\int_{\mathbb{R}^n\times(0,1)} y^a \nabla v\cdot\nabla v\,dx\,dy = \frac{1}{2}\int_{\mathbb{R}^n\times\{y=0\}} u\,\mathcal{L}_a(u)\,dx.$$

Applying Plancherel theorem and formula (4.1.8) for the Fourier transform of $\mathcal{L}_a(u)$, we conclude that

$$\mathcal{E}_K(v) = \frac{1}{2(2\pi)^n}\int_{\mathbb{R}^n} \widehat{\mathcal{L}_a u}(\xi)\overline{\widehat{u}(\xi)}\,d\xi = \frac{1}{2(2\pi)^n}\int_{\mathbb{R}^n} S_s(\xi)\,|\widehat{u}(\xi)|^2\,d\xi,$$

that concludes the proof of Corollary 4.1.2. $\square$

4.4 Γ-convergence for $s \in [1/2, 1)$

This section is mainly devoted to the proof of Theorem 4.1.3 in the case $s \geq 1/2$, that concerns the Γ-convergence of the functional $\mathcal{F}_\varepsilon$ defined in (4.1.20).

In the proof of the Γ-convergence result for $s \geq 1/2$ we use the following Lemma 4.4.1 which establishes that the difference between the rescaled H^s seminorm and the Dirichlet energy functional associated to the operator $\mathcal{L}_a$ is finite for every $u \in L^1(\mathbb{R}^n)$. This result is valid for all $s \in (0,1)$ and it will turn out to be useful not only when $s \in [1/2,1)$ to prove Theorem 4.1.3, but also when $s \in (0,1/2)$ to ensure that $\mathcal{F}$ is well-defined by (4.1.21).

Lemma 4.4.1. *For every $s \in (0,1)$ and $u \in L^1(\mathbb{R}^n)$, there exists a positive constant C depending only on n and s such that*

$$\int_{\mathbb{R}^n} \left(\widetilde{S}_s(\xi) - \mathcal{C}_s\right)|\xi|^{2s}\,|\widehat{u}(\xi)|^2\,d\xi \leq C\|u\|^2_{L^1(\mathbb{R}^n)},$$

where

$$\mathcal{C}_s := \lim_{\xi\to+\infty} \widetilde{S}_s(\xi) = 2^{1-2s}\frac{\Gamma(1-s)}{\Gamma(s)}. \tag{4.4.1}$$

Proof. First, we observe that $\|\widehat{u}\|_{L^\infty(\mathbb{R}^n)} \leq C\|u\|_{L^1(\mathbb{R}^n)}$, for some positive constant C depending only on n. Therefore, we have that

$$\int_{\mathbb{R}^n} \left(\widetilde{S}_s(\xi) - \mathcal{C}_s\right)|\xi|^{2s}\,|\widehat{u}(\xi)|^2\,d\xi \leq C\|u\|^2_{L^1(\mathbb{R}^n)}\int_{\mathbb{R}^n}\left(\widetilde{S}_s(\xi) - \mathcal{C}_s\right)|\xi|^{2s}\,d\xi,$$

and we want to show that the integral in the right-hand side is finite.

If $s = 1/2$, the expression of $\widetilde{S}_{1/2}(\xi)$ is simpler, and one can directly check that $\mathcal{C}_{1/2} = 1$ and

$$\int_{\mathbb{R}^n} \left(\frac{e^{|\xi|} - e^{-|\xi|}}{e^{|\xi|} + e^{-|\xi|}} - 1 \right) |\xi| \, d\xi = C \in (0, +\infty),$$

where the constant C depends only on n.

For the general case of any $s \in (0,1)$, we want to show that there exists a positive constant C depending only on n and s such that

$$\int_{\mathbb{R}^n} \left(\widetilde{S}_s(\xi) - \mathcal{C}_s \right) |\xi|^{2s} \, d\xi = C \in (0, +\infty). \tag{4.4.2}$$

To this end, we can use polar coordinates and write the integral as

$$\begin{aligned}
\int_{\mathbb{R}^n} \left(\widetilde{S}_s(\xi) - \mathcal{C}_s \right) |\xi|^{2s} \, d\xi &= \omega_{n-1} \int_0^{+\infty} \left(\widetilde{S}_s(r) - \widetilde{S}_s(+\infty) \right) r^{n-1+2s} \, dr \\
&\le \omega_{n-1} \int_0^{+\infty} r^{n-1+2s} \, dr \int_r^{+\infty} \left| \widetilde{S}'_s(t) \right| \, dt \\
&= \omega_{n-1} \int_0^{+\infty} \left| \widetilde{S}'_s(t) \right| \, dt \int_0^t r^{n-1+2s} \, dr = \frac{\omega_{n-1}}{n+2s} \int_0^{+\infty} t^{n+2s} \left| \widetilde{S}'_s(t) \right| \, dt.
\end{aligned} \tag{4.4.3}$$

Using (4.3.13) to compute $\widetilde{S}'_s(t)$, from (4.4.3) we deduce

$$\int_{\mathbb{R}^n} \left(\widetilde{S}_s(\xi) - \mathcal{C}_s \right) |\xi|^{2s} \, d\xi \le C \, c_2(s) \int_0^{+\infty} \frac{t^{n-1+2s}}{J^2_{s-1}(-it)} \, dt,$$

where C is a positive constant depending only on n and s. Finally, the last integral is finite, since the integrand is bounded, and goes to zero at infinity faster than every power. This shows (4.4.2) and concludes the proof of Lemma 4.4.1. □

Before proving Theorem 4.1.3 for $s \ge 1/2$, we recall the setting in [123] used by Savin and Valdinoci to state their Γ-convergence result. Indeed, we prove Theorem 4.1.3 for $s \ge 1/2$ by showing that the difference between the rescaled Dirichlet energies goes to zero at the limit, and then applying [123, Theorem 1.4].

We recall that the energy functional considered in [123] is $\mathcal{I}_\varepsilon$ defined in (4.1.14). After a rescaling, we can assume that the double-well potential V in (4.1.14) satisfies (4.1.17), and that the function space in [123] is defined as $Y := \{u \in L^\infty(\mathbb{R}^n) : 0 \le u \le 1\}$. Following [123], we say that u_ε converges to u in Y if $u_\varepsilon \to u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Observe that our function space X is contained in Y and X is equipped with the convergence in $L^1(\mathbb{R}^n)$. Thus, every time we consider a function u in X and a sequence u_ε converging to u in X, we are also in the setting considered in [123], and thus we are able to exploit useful results from the existing literature.

In respect to this matter, we recall that in [123] the functional $\mathcal{I}_\varepsilon$ in (4.1.14) is rescaled as

$$\mathcal{G}_\varepsilon(u, \Omega) := \begin{cases} \varepsilon^{-2s} \mathcal{I}_\varepsilon(u, \Omega) & \text{if } s \in (0, 1/2); \\ |\varepsilon \log \varepsilon|^{-1} \, \mathcal{I}_\varepsilon(u, \Omega) & \text{if } s = 1/2; \\ \varepsilon^{-1} \mathcal{I}_\varepsilon(u, \Omega) & \text{if } s \in (1/2, 1). \end{cases} \tag{4.4.4}$$

Theorem 1.4 in [123] establishes that $\mathcal{G}_\varepsilon$ converges in the Γ-sense to the classical perimeter if $s \in [1/2, 1)$ and to the nonlocal area functional if $s \in (0, 1/2)$. More precisely, the Γ-limit functional in [123] is defined for $s \in (0, 1/2)$ as

$$\mathcal{G}(u, \Omega) := \begin{cases} \mathcal{K}(u, \Omega) & \text{if } u_{|\Omega} = \chi_E \text{ for some set } E \subset \Omega; \\ +\infty & \text{otherwise,} \end{cases}$$

and for $s \in [1/2, 1)$ as

$$\mathcal{G}(u, \Omega) := \begin{cases} c_* \mathrm{Per}(E, \Omega) & \text{if } u_{|\Omega} = \chi_E \text{ for some set } E \subset \Omega; \\ +\infty & \text{otherwise,} \end{cases} \tag{4.4.5}$$

where c_* is a constant depending only on n, s and the double-well potential V — see [123] for more details.

We are now able to prove Theorem 4.1.3 for $s \geq 1/2$.

Proof of Theorem 4.1.3 for $s \in [1/2, 1)$. First, considering the functional $\mathcal{F}_\varepsilon$ defined in (4.1.20), we introduce the following notation for the ε-weights

$$\lambda(\varepsilon) := \begin{cases} |\log \varepsilon|^{-1} & \text{if } s = 1/2; \\ \varepsilon^{2s-1} & \text{if } s \in (1/2, 1); \end{cases}$$

and

$$\kappa(\varepsilon) := \begin{cases} |\varepsilon \log \varepsilon|^{-1} & \text{if } s = 1/2; \\ \varepsilon^{-1} & \text{if } s \in (1/2, 1). \end{cases}$$

Observe that the same ε-weights appear in the functional $\mathcal{G}_\varepsilon$ defined in (4.1.14), which is treated in [123]. In this proof, we will exploit several times the fact that $\lambda(\varepsilon) \to 0$ as $\varepsilon \to 0^+$.

We recall that the square of the H^s-seminorm can be written as

$$[u]^2_{H^s(\mathbb{R}^n)} = \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy = \frac{2C(n,s)^{-1}}{(2\pi)^n} \int_{\mathbb{R}^n} |\xi|^{2s} \, |\widehat{u}(\xi)|^2 \, d\xi,$$

where $C(n, s)$ is defined in (4.1.16).

We consider $\mathcal{F}_\varepsilon(u_\varepsilon)$ and we use the notation in (4.1.13). The limit at infinity of $\widetilde{S}_s$ is denoted with $\mathcal{C}_s$ — see (4.4.1) and also Figure 4.1 — and it is finite and positive for every $s \in (0, 1)$, then in particular in our case. We define

$$\overline{C}_s := 2^{n-1} \pi^n C(n, s) \, \mathcal{C}_s, \tag{4.4.6}$$

and we add and subtract $\lambda(\varepsilon) \overline{C}_s \, [u]^2_{H^s(\mathbb{R}^n)}$ to $\mathcal{F}_\varepsilon(u_\varepsilon)$. In this way we obtain

$$\begin{aligned} \mathcal{F}_\varepsilon(u_\varepsilon) = {} & \lambda(\varepsilon) \int_{\mathbb{R}^n} |\xi|^{2s} \left(\widetilde{S}_s(\xi) - \mathcal{C}_s \right) |\widehat{u}_\varepsilon(\xi)|^2 \, d\xi \\ & + \lambda(\varepsilon) \overline{C}_s \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x - y|^{n+2s}} \, dx \, dy + \kappa(\varepsilon) \int_{\mathbb{R}^n} W(u_\varepsilon) \, dx. \end{aligned}$$

Using Lemma 4.4.1 and the fact that $\lambda(\varepsilon) \to 0$ as $\varepsilon \to 0^+$, we deduce that for every $u \in X$ and for every sequence $(u_\varepsilon)_\varepsilon$ converging to u in $L^1(\mathbb{R}^n)$, it holds that

$$\lim_{\varepsilon\to 0^+} \lambda(\varepsilon) \int_{\mathbb{R}^n} |\xi|^{2s} \left(\widetilde{S}_s(\xi) - \mathcal{C}_s\right) |\widehat{u}_\varepsilon(\xi)|^2 \, d\xi = 0,$$

Therefore, for every $u \in X$, if $u_\varepsilon \to u$ in $L^1(\mathbb{R}^n)$, we have that

$$\lim_{\varepsilon\to 0^+} \left(\mathcal{F}_\uparrow(u_\varepsilon) - \widetilde{\mathcal{F}}_\varepsilon(u_\varepsilon)\right) = 0, \tag{4.4.7}$$

where

$$\begin{aligned} \widetilde{\mathcal{F}}_\varepsilon(w) &:= \lambda(\varepsilon)\, \mathcal{C}_s \int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{w}(\xi)|^2 \, d\xi + \kappa(\varepsilon) \int_{\mathbb{R}^n} W(w)\, dx \\ &= \lambda(\varepsilon)\, \overline{C}_s \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|w(x)-w(y)|^2}{|x-y|^{n+2s}}\, dx\, dy + \kappa(\varepsilon) \int_{\mathbb{R}^n} W(w)\, dx, \end{aligned} \tag{4.4.8}$$

and $\overline{C}_s$ is defined in (4.4.6).

Now, we use (4.4.7) and the Γ-convergence result in [123] to deduce the claims in (i) and (ii) of Theorem 4.1.3. To this end, we start from the liminf inequality in (i).

For every function $u \in X$ we can choose a radius $R > 0$ such that the ball $B_R \subset \mathbb{R}^n$ contains the support of u. Moreover, for any sequence $(u_\varepsilon)_\varepsilon$ that converges to u in $L^1(\mathbb{R}^n)$, from Theorem 1.4 in [123] we know that

$$\liminf_{\varepsilon\to 0^+} \mathcal{G}_\varepsilon(u_\varepsilon, B_R) \geq \mathcal{G}(u, B_R), \tag{4.4.9}$$

where $\mathcal{G}(u,\Omega)$ and $\mathcal{G}_\varepsilon(u,\Omega)$ are defined respectively in (4.4.5) and (4.4.4). In addition, by the definition of $\widetilde{\mathcal{F}}_\varepsilon$ in (4.4.8), for every $R > 0$, we have that

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \geq \mathcal{K}(u, B_R),$$

where $\mathcal{K}(u, B_R)$ appears in the definition of $\mathcal{G}_\varepsilon(u, B_R)$ given in (4.4.4). In particular, it follows that

$$\widetilde{\mathcal{F}}_\varepsilon(u_\varepsilon) \geq \overline{C}_s\, \mathcal{G}_\varepsilon(u_\varepsilon, B_R), \tag{4.4.10}$$

where $B_R \subset \mathbb{R}^n$ is the ball of radius R containing the support of u. We observe that both $\widetilde{\mathcal{F}}_\varepsilon$ and $\mathcal{G}_\varepsilon$ contain a double-well potential, and without loss of generality we can assume that

$$W = \overline{C}_s V, \tag{4.4.11}$$

where V is the potential function in the definition of $\mathcal{G}_\varepsilon$ (recall (4.1.14) and (4.4.4)).

Then, using (4.4.7), (4.4.9), and (4.4.10), it follows that

$$\begin{aligned} \liminf_{\varepsilon\to 0^+} \mathcal{F}_\uparrow(u_\varepsilon) &= \liminf_{\varepsilon\to 0^+} \widetilde{\mathcal{F}}_\varepsilon(u_\varepsilon) \\ &\geq \overline{C}_s \liminf_{\varepsilon\to 0^+} \mathcal{G}_\varepsilon(u_\varepsilon, B_R) \geq \overline{C}_s\, \mathcal{G}(u, B_R) = \mathcal{F}(u), \end{aligned}$$

which is the liminf inequality (4.1.23) for a sequence $(u_\varepsilon)_\varepsilon$ converging to u in X.

Now, we prove the limsup inequality in claim (ii) of Theorem 4.1.3. For this, we can assume that

$$u = \chi_E \text{ for some set } E \subset \mathbb{R}^n, \text{ and } \mathcal{F}(u) < +\infty, \tag{4.4.12}$$

otherwise the claim in (ii) is automatically satisfied.

In light of the definition of X given in (4.1.19), since u has compact support in $\mathbb{R}^n$, we can choose $R > 2$ large enough such that

$$\text{the support of } u \text{ is compactly contained in } B_{R/2}. \tag{4.4.13}$$

Moreover, from Theorem 1.4 in [123] we know the existence of a sequence u_ε that converges to u in B_R such that

$$\limsup_{\varepsilon\to 0^+} \overline{C}_s\, \mathcal{G}_\varepsilon(u_\varepsilon, B_R) \le \overline{C}_s\, \mathcal{G}(u, B_R) = \mathcal{F}(u), \tag{4.4.14}$$

where the last equality follows from the definitions of $\mathcal{G}$ and $\mathcal{F}$, the fact that $u = \chi_E$, and that the support of u is contained in B_R.

Besides, since u_ε converges to u in $L^1(B_R)$, for every k there exists $\varepsilon_k \in (0, 1/k)$ such that

$$\int_{B_R} |u - u_{\varepsilon_k}|\, dx \le \frac{1}{k}. \tag{4.4.15}$$

In view of (4.4.14) we can also suppose that

$$\overline{C}_s\, \mathcal{G}_{\varepsilon_k}(u_{\varepsilon_k}, B_R) \le \mathcal{F}(u) + \frac{1}{k}. \tag{4.4.16}$$

Now, for every $k \in \mathbb{N} \setminus \{0\}$, we define

$$\rho_k := \frac{1}{kR^{n-1}} \tag{4.4.17}$$

and

$$u_k^* := u_{\varepsilon_k} \psi_k,$$

where ψ_k is a smooth function defined on $\mathbb{R}^n$ with values in $[0, 1]$, such that

$$\psi_k \equiv 1 \text{ in } B_{R-\rho_k}, \qquad \psi_k \equiv 0 \text{ outside } B_R, \qquad \text{and} \qquad |\nabla \psi_k| \le \frac{C}{\rho_k}. \tag{4.4.18}$$

Then, $u_k^* \in X$, and we claim that

$$u_k^* \text{ converges to } u \text{ in } L^1(\mathbb{R}^n). \tag{4.4.19}$$

Indeed, using (4.4.15) and that the support of u is contained in B_R, we know that

$$\begin{aligned}
\int_{\mathbb{R}^n} |u_k^* - u|\, dx &= \int_{B_{R-\rho_k}} |u_{\varepsilon_k} - u|\, dx + \int_{B_R \setminus B_{R-\rho_k}} |u_{\varepsilon_k}\psi_k - u|\, dx \\
&\le \frac{1}{k} + 2\big|B_R \setminus B_{R-\rho_k}\big| \\
&\le \frac{1}{k} + C\big(R^n - (R-\rho_k)^n\big),
\end{aligned}$$

for some $C > 0$ depending only on n.

This and (4.4.17) yield that

$$\int_{\mathbb{R}^n} |u_k^* - u| \, dx \le \frac{1}{k} + CR^n \left(1 - \left(1 - \frac{\rho_k}{R}\right)^n\right) \le \frac{1}{k} + CR^{n-1}\rho_k = \frac{C}{k}.$$

From this, we plainly obtain (4.4.19), as desired.

Now, we recall that

$$\limsup_{k\to+\infty} \mathcal{G}_{\varepsilon_k}(u_{\varepsilon_k}, B_R) < +\infty,$$

thanks to (4.4.14) and the assumption in (4.4.12). We claim that

$$\limsup_{k\to\infty} \overline{C}_s \, \mathcal{G}_{\varepsilon_k}(u_{\varepsilon_k}, B_R) \ge \limsup_{k\to\infty} \widetilde{\mathcal{F}}_{\varepsilon_k}(u_k^*). \tag{4.4.20}$$

To this end, recalling also (4.4.11), we observe that

$$\begin{aligned} &\overline{C}_s \, \mathcal{G}_{\varepsilon_k}(u_{\varepsilon_k}, B_R) - \widetilde{\mathcal{F}}_{\varepsilon_k}(u_k^*) \\ &\qquad = I_k + II_k + III_k + IV_k + \kappa(\varepsilon_k) \int_{B_R\setminus B_{R-\rho_k}} \left(W(u_{\varepsilon_k}) - W(u_{\varepsilon_k}\psi_k)\right) dx, \end{aligned} \tag{4.4.21}$$

where I_k, II_k, III_k, and IV_k are defined as

$$\begin{aligned} I_k &:= 2\overline{C}_s\lambda(\varepsilon_k) \iint_{B_{R-\rho_k}\times(B_R\setminus B_{R-\rho_k})} \frac{(u_{\varepsilon_k}(x) - u_{\varepsilon_k}(y))^2}{|x-y|^{n+2s}} \\ &\qquad\qquad\qquad\qquad - \frac{(u_{\varepsilon_k}(x) - \psi_k(y)u_{\varepsilon_k}(y))^2}{|x-y|^{n+2s}} \, dx\,dy; \\ II_k &:= 2\overline{C}_s\lambda(\varepsilon_k) \iint_{B_{R-\rho_k}\times\mathscr{C}B_R} \frac{u_{\varepsilon_k}(y)\,(u_{\varepsilon_k}(y) - 2u_{\varepsilon_k}(x))}{|x-y|^{n+2s}} \, dx\,dy; \\ III_k &:= 2\overline{C}_s\lambda(\varepsilon_k) \iint_{(B_R\setminus B_{R-\rho_k})\times\mathscr{C}B_R} \frac{(u_{\varepsilon_k}(x) - u_{\varepsilon_k}(y))^2 - u_{\varepsilon_k}^2(x)\psi_k^2(x)}{|x-y|^{n+2s}} \, dx\,dy; \\ IV_k &:= \overline{C}_s\lambda(\varepsilon_k) \iint_{(B_R\setminus B_{R-\rho_k})^2} \frac{(u_{\varepsilon_k}(x) - u_{\varepsilon_k}(y))^2}{|x-y|^{n+2s}} \\ &\qquad\qquad\qquad\qquad - \frac{(u_{\varepsilon_k}(x)\psi_k(x) - u_{\varepsilon_k}(y)\psi_k(y))^2}{|x-y|^{n+2s}} \, dx\,dy. \end{aligned}$$

First, we consider the difference of the potential energies in (4.4.21). We claim that

$$\limsup_{k\to\infty} \kappa(\varepsilon_k) \int_{B_R\setminus B_{R-\rho_k}} \left(W(u_{\varepsilon_k}) - W(u_{\varepsilon_k}\psi_k)\right) dx \ge 0. \tag{4.4.22}$$

To show this, first we recall that we are assuming that $\mathcal{F}(u)$ is finite, therefore $u = \chi_E$ for some set $E \subset \mathbb{R}^n$. We also remind that the recovery sequence $(u_\varepsilon)_\varepsilon$ is defined in [123] as

$$u_\varepsilon := u_0\left(\frac{\mathrm{dist}(x)}{\varepsilon}\right),$$

where u_0 is the heteroclinic connecting the zeros of the potential W, i.e. 0 and 1, and $\mathrm{dist}(x)$ is the signed distance of x to ∂E, with the convention that $\mathrm{dist}(x) \ge 0$ inside E and $\mathrm{dist}(x) \le 0$ outside E (see in particular [123, page 497]).

We remark that, in view of (4.4.12) and (4.4.13), we have that $\chi_E = u = 0$ outside $B_{R/2}$, hence $E \subseteq B_{R/2}$. In particular, if x lies outside $B_{3R/4}$, we have that $\mathrm{dist}(x) \leq -R/4$. Hence, for k big enough, we can assume that u_{ε_k} is arbitrarily close to zero in $B_R \setminus B_{R-\rho_k}$. On the other hand, since W is a double-well potential — see (4.1.17) — it follows that $W'(t) \geq 0$ for t near zero. Therefore, since $u_{\varepsilon_k}\psi_k \leq u_{\varepsilon_k}$, for k big enough we have that

$$W(u_{\varepsilon_k}) - W(u_{\varepsilon_k}\psi_k) \geq 0 \qquad \text{in} \quad B_R \setminus B_{R-\rho_k},$$

and this shows (4.4.22).

Considering now the integral in I_k in (4.4.21), we observe that

$$\begin{aligned}(u_{\varepsilon_k}(x) - u_{\varepsilon_k}(y))^2 &- (u_{\varepsilon_k}(x) - \psi_k(y)u_{\varepsilon_k}(y))^2 \\ &= \left(1-\psi_k^2(y)\right) u_{\varepsilon_k}^2(y) - 2u_{\varepsilon_k}(x)u_{\varepsilon_k}(y)\,(1-\psi_k(y)) \\ &= u_{\varepsilon_k}(y)(1-\psi_k(y))\big((1+\psi_k(y))u_{\varepsilon_k}(y) - 2u_{\varepsilon_k}(x)\big)\end{aligned}$$

Since in this case we are integrating x over $B_{R-\rho_k}$, we have that $\psi_k(x) = 1$. Hence, we can write that

$$\begin{aligned}(1-\psi_k(y))&\big((1+\psi_k(y))u_{\varepsilon_k}(y) - 2u_{\varepsilon_k}(x)\big) \\ &= (\psi_k(x)-\psi_k(y))\big((2-\psi_k(x)+\psi_k(y))u_{\varepsilon_k}(y) - 2u_{\varepsilon_k}(x)\big) \\ &= (\psi_k(x)-\psi_k(y))\big(2(u_{\varepsilon_k}(y)-u_{\varepsilon_k}(x)) - (\psi_k(x)-\psi_k(y))u_{\varepsilon_k}(y)\big) \\ &= 2(\psi_k(x)-\psi_k(y))(u_{\varepsilon_k}(y)-u_{\varepsilon_k}(x)) - (\psi_k(x)-\psi_k(y))^2 u_{\varepsilon_k}(y).\end{aligned}$$

Consequently, using also that u_{ε_k} is uniformly bounded, we see that

$$\begin{aligned}I_k \leq C\lambda(\varepsilon_k) \iint_{B_{R-\rho_k}\times(B_R\setminus B_{R-\rho_k})} &\frac{(\psi_k(x)-\psi_k(y))\,(u_{\varepsilon_k}(x)-u_{\varepsilon_k}(y))}{|x-y|^{n+2s}}\,dx\,dy \\ +C\lambda(\varepsilon_k) \iint_{B_{R-\rho_k}\times(B_R\setminus B_{R-\rho_k})} &\frac{(\psi_k(x)-\psi_k(y))^2}{|x-y|^{n+2s}}\,dx\,dy.\end{aligned} \tag{4.4.23}$$

To estimate the second integral in (4.4.23), we use (4.4.18) to deduce that

$$(\psi_k(x)-\psi_k(y))^2 \leq \frac{C}{\rho_k^2}\,|x-y|^2,$$

and we obtain that

$$\begin{aligned}\iint_{B_{R-\rho_k}\times(B_R\setminus B_{R-\rho_k})} &\frac{(\psi_k(x)-\psi_k(y))^2}{|x-y|^{n+2s}}\,dx\,dy \\ &\leq \frac{C}{\rho_k^2}\iint_{B_{R-\rho_k}\times(B_R\setminus B_{R-\rho_k})} \frac{1}{|x-y|^{n+2s-2}}\,dx\,dy =: \mu_k \in (0,+\infty).\end{aligned} \tag{4.4.24}$$

Observe that μ_k is finite since $s \in [1/2, 1)$ and $|x-y|^{-n-2s+2}$ is integrable. Accordingly, we can choose ε_k so small that $\lambda(\varepsilon_k) \leq (k\mu_k)^{-1}$ and we conclude that

$$\begin{aligned}\lim_{k\to+\infty} C\lambda(\varepsilon_k) &\iint_{B_{R-\rho_k}\times(B_R\setminus B_{R-\rho_k})} \frac{(\psi_k(x)-\psi_k(y))^2}{|x-y|^{n+2s}}\,dx\,dy \\ &\leq \lim_{k\to+\infty} C\lambda(\varepsilon_k)\mu_k \leq \lim_{k\to+\infty}\frac{C}{k} = 0.\end{aligned} \tag{4.4.25}$$

This controls the second integral in (4.4.23). Instead, for the first integral in (4.4.23), we can use the Cauchy-Schwarz inequality and (4.4.14), to write that

$$\begin{aligned}
&\lambda(\varepsilon_k) \iint_{B_{R-\rho_k}\times(B_R\setminus B_{R-\rho_k})} \frac{(\psi_k(x)-\psi_k(y))\,(u_{\varepsilon_k}(x)-u_{\varepsilon_k}(y))}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad \le \left(\lambda(\varepsilon_k) \iint_{B_{R-\rho_k}\times(B_R\setminus B_{R-\rho_k})} \frac{(\psi_k(x)-\psi_k(y))^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{\frac12} \\
&\qquad\quad \times \left(\lambda(\varepsilon_k) \iint_{B_{R-\rho_k}\times(B_R\setminus B_{R-\rho_k})} \frac{(u_{\varepsilon_k}(x)-u_{\varepsilon_k}(y))^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{\frac12} \\
&\quad \le \left(\lambda(\varepsilon_k) \iint_{B_{R-\rho_k}\times(B_R\setminus B_{R-\rho_k})} \frac{(\psi_k(x)-\psi_k(y))^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{\frac12} \left(\mathcal{F}(u)+\frac{1}{k}\right)^{\frac12}.
\end{aligned}$$

Hence, in view of (4.4.12) and (4.4.24)-(4.4.25), we write that

$$\lim_{k\to+\infty} \lambda(\varepsilon_k) \iint_{B_{R-\rho_k}\times(B_R\setminus B_{R-\rho_k})} \frac{(\psi_k(x)-\psi_k(y))\,(u_{\varepsilon_k}(x)-u_{\varepsilon_k}(y))}{|x-y|^{n+2s}}\,dx\,dy = 0.$$

From this and (4.4.25), we conclude that

$$\lim_{k\to+\infty} I_k = 0. \tag{4.4.26}$$

Considering now the integral in II_k, we exploit that u_{ε_k} is uniformly bounded and that

$$\iint_{B_{R-\rho_k}\times\mathscr{C}B_R} \frac{1}{|x-y|^{n+2s}}\,dx\,dy =: \widetilde{\mu}_k \in (0,+\infty).$$

In this way, we conclude that

$$II_k \le C\,\widetilde{\mu}_k\lambda(\varepsilon_k).$$

Consequently, choosing ε_k so small that $\lambda(\varepsilon_k) \le (k\widetilde{\mu}_k)^{-1}$, we conclude that

$$\lim_{k\to+\infty} II_k \le \lim_{k\to+\infty} \frac{C}{k} = 0. \tag{4.4.27}$$

Now, we consider the integral in III_k and we claim that

$$\limsup_{k\to\infty} III_k \ge 0. \tag{4.4.28}$$

To this end, it is sufficient to show that

$$\lim_{k\to+\infty} \lambda(\varepsilon_k) \iint_{(B_R\setminus B_{R-\rho_k})\times\mathscr{C}B_R} \frac{u_{\varepsilon_k}^2(x)\psi_k^2(x)}{|x-y|^{n+2s}}\,dx\,dy = 0, \tag{4.4.29}$$

since the other part of the integral in III_k is positive.

We use that u_{ε_k} is uniformly bounded and that $\psi_k(y) = 0$ since we are integrating y over $\mathscr{C}B_R$, to write that

$$\begin{aligned} \lambda(\varepsilon_k) \iint_{(B_R\setminus B_{R-\rho_k})\times \mathscr{C}B_R} & \frac{u^2_{\varepsilon_k}(x)\psi_k^2(x)}{|x-y|^{n+2s}}\,dx\,dy \\ & \le C\lambda(\varepsilon_k) \iint_{(B_R\setminus B_{R-\rho_k})\times (B_{R+1}\setminus B_R)} \frac{(\psi_k(x)-\psi_k(y))^2}{|x-y|^{n+2s}}\,dx\,dy \\ & \quad + C\lambda(\varepsilon_k) \iint_{(B_R\setminus B_{R-\rho_k})\times \mathscr{C}B_{R+1}} \frac{1}{|x-y|^{n+2s}}\,dx\,dy. \end{aligned} \tag{4.4.30}$$

To control the first integral in the right-hand side of (4.4.30), we use that $(\psi_k(x) - \psi_k(y))^2 \le C\,|x-y|^2/\rho_k^2$, obtaining

$$\begin{aligned} \iint_{(B_R\setminus B_{R-\rho_k})\times (B_{R+1}\setminus B_R)} & \frac{(\psi_k(x)-\psi_k(y))^2}{|x-y|^{n+2s}}\,dx\,dy \\ & \le \frac{C}{\rho_k^2} \iint_{(B_R\setminus B_{R-\rho_k})\times (B_{R+1}\setminus B_R)} \frac{1}{|x-y|^{n+2s-2}}\,dx\,dy =: \nu_k \in (0,+\infty). \end{aligned}$$

Therefore, we can choose ε_k so small that $\lambda(\varepsilon_k) \le (k\nu_k)^{-1}$ and we deduce that

$$\begin{aligned} \lim_{k\to+\infty} C\lambda(\varepsilon_k) \iint_{(B_R\setminus B_{R-\rho_k})\times (B_{R+1}\setminus B_R)} & \frac{(\psi_k(x)-\psi_k(y))^2}{|x-y|^{n+2s}}\,dx\,dy \\ & \le \lim_{k\to+\infty} C\nu_k\lambda(\varepsilon_k) \le \lim_{k\to+\infty} \frac{C}{k} = 0. \end{aligned}$$

Concerning the last integral in (4.4.30), we integrate first y over $\mathscr{C}B_{R+1}$, and then x over $B_R \setminus B_{R-\rho_k}$, to obtain

$$\begin{aligned} \lim_{k\to+\infty} C\lambda(\varepsilon_k) \iint_{(B_R\setminus B_{R-\rho_k})\times \mathscr{C}B_{R+1}} & \frac{1}{|x-y|^{n+2s}}\,dx\,dy \\ & \le \lim_{k\to+\infty} C\lambda(\varepsilon_k)\left|B_R\setminus B_{R-\rho_k}\right| \le \lim_{k\to+\infty} \frac{C}{k}\lambda(\varepsilon_k) = 0. \end{aligned}$$

This shows the validity of (4.4.29), and concludes the proof of (4.4.28) about III_k.

Now, we consider the integral in IV_k. Using the expression

$$u_{\varepsilon_k}(x)\psi_k(x) - u_{\varepsilon_k}(y)\psi_k(y) = u_{\varepsilon_k}(x)\left(\psi_k(x)-\psi_k(y)\right) + \psi_k(y)\left(u_{\varepsilon_k}(x)-u_{\varepsilon_k}(y)\right),$$

we write IV_k as

$$\begin{aligned} IV_k = {} & \lambda(\varepsilon_k) \iint_{(B_R\setminus B_{R-\rho_k})^2} \frac{\left(1-\psi_k^2(y)\right)\left(u_{\varepsilon_k}(x)-u_{\varepsilon_k}(y)\right)^2}{|x-y|^{n+2s}}\,dx\,dy \\ & - \lambda(\varepsilon_k) \iint_{(B_R\setminus B_{R-\rho_k})^2} \frac{u^2_{\varepsilon_k}(x)\left(\psi_k(x)-\psi_k(y)\right)^2}{|x-y|^{n+2s}}\,dx\,dy \\ & - \lambda(\varepsilon_k) \iint_{(B_R\setminus B_{R-\rho_k})^2} \frac{2u_{\varepsilon_k}(x)\psi_k(y)\left(u_{\varepsilon_k}(x)-u_{\varepsilon_k}(y)\right)\left(\psi_k(x)-\psi_k(y)\right)}{|x-y|^{n+2s}}\,dx\,dy \end{aligned} \tag{4.4.31}$$

Since $|\psi_k(y)| \le 1$, the first integral in the right-hand side of (4.4.31) is nonnegative. To control the second term, we use that $(\psi_k(x) - \psi_k(y))^2 \le C\,|x-y|^2/\rho_k^2$, and write

$$\begin{aligned}\iint_{(B_R\setminus B_{R-\rho_k})^2} &\frac{(\psi_k(x)-\psi_k(y))^2}{|x-y|^{n+2s}}\,dx\,dy\\ &\le C\rho_k^{-2}\iint_{(B_R\setminus B_{R-\rho_k})^2}\frac{1}{|x-y|^{n+2s-2}}\,dx\,dy =: \widetilde{\nu}_k \in (0,+\infty).\end{aligned}$$

Hence, choosing ε_k so small that $\lambda(\varepsilon_k) \le (k\widetilde{\nu}_k)^{-1}$, it follows that

$$\lim_{k\to+\infty}\lambda(\varepsilon_k)\iint_{(B_R\setminus B_{R-\rho_k})^2}\frac{(\psi_k(x)-\psi_k(y))^2}{|x-y|^{n+2s}}\,dx\,dy \le \lim_{k\to+\infty}\frac{C}{k} = 0. \tag{4.4.32}$$

In the last integral in (4.4.31), we exploit that $u_{\varepsilon_k}(x)\psi_k(y)$ is uniformly bounded and we use the Cauchy-Schwarz inequality to write

$$\begin{aligned}\lambda(\varepsilon_k)&\iint_{(B_R\setminus B_{R-\rho_k})^2}\frac{(u_{\varepsilon_k}(x)-u_{\varepsilon_k}(y))\,(\psi_k(x)-\psi_k(y))}{|x-y|^{n+2s}}\,dx\,dy\\ &\le\left(\lambda(\varepsilon_k)\iint_{(B_R\setminus B_{R-\rho_k})^2}\frac{(u_{\varepsilon_k}(x)-u_{\varepsilon_k}(y))^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{\frac12}\\ &\qquad\times\left(\lambda(\varepsilon_k)\iint_{(B_R\setminus B_{R-\rho_k})^2}\frac{(\psi_k(x)-\psi_k(y))^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{\frac12}\\ &\le\left(\lambda(\varepsilon_k)\iint_{(B_R\setminus B_{R-\rho_k})^2}\frac{(\psi_k(x)-\psi_k(y))^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{\frac12}\left(\mathcal{F}(u)+\frac{1}{k}\right)^{\frac12}.\end{aligned}$$

Recalling (4.4.12) and (4.4.32), we thereby see that

$$\lim_{k\to+\infty}\lambda(\varepsilon_k)\iint_{(B_R\setminus B_{R-\rho_k})^2}\frac{(u_{\varepsilon_k}(x)-u_{\varepsilon_k}(y))\,(\psi_k(x)-\psi_k(y))}{|x-y|^{n+2s}}\,dx\,dy = 0. \tag{4.4.33}$$

Now, putting together (4.4.32), (4.4.33), and the fact that the first integral in the right hand side of (4.4.31) is positive, we deduce that

$$\limsup_{k\to\infty} IV_k \ge 0. \tag{4.4.34}$$

Finally, from (4.4.26), (4.4.27), (4.4.28) and (4.4.34) we deduce the desired claim in (4.4.20). Now, in light of (4.4.7), (4.4.14), (4.4.16), and (4.4.20), we have that

$$\limsup_{k\to\infty}\mathcal{F}_k(u_k^*) = \limsup_{k\to\infty}\widetilde{\mathcal{F}}_{\varepsilon_k}(u_k^*) \le \limsup_{k\to\infty}\overline{C}_s\,\mathcal{G}_{\varepsilon_k}(u_{\varepsilon_k},B_R) \le \mathcal{F}(u),$$

that is the claim in (ii) of Theorem 4.1.3.

This completes the proof of Theorem 4.1.3 for $s \in [1/2, 1)$. For completeness, we observe that the constant $c_\#$ appearing in the Γ-limit (4.1.22) can be written as

$$c_\# = \overline{C}_s\,c_*,$$

where $\overline{C}_s$ is defined in (4.4.6) and c_* is the constant appearing in (4.4.5), which in turn is related to the Γ-limit functional in [123] for $s \ge 1/2$. □

4.5 Γ-convergence for $s \in (0, 1/2)$

This section is focused on the Γ-convergence for the case $s \in (0, 1/2)$. First, we prove Theorem 4.1.3 in this case, and then we prove Proposition 4.1.4.

Proof of Theorem 4.1.3 for $s \in (0, 1/2)$. We consider any $u \in X$ and we start by proving the claim in (i), which is the liminf inequality for every sequence u_ε converging to u in X. Let u_ε be a sequence of functions in X that converges to u in $L^1(\mathbb{R}^n)$. If

$$\liminf_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(u) = +\infty,$$

then (4.1.23) is obvious. Hence, we assume that

$$\liminf_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(u) = l < \infty.$$

We take $(u_{\varepsilon_k})_k$ as a subsequence of (u_ε) that attains the limit l, and $(u_{\varepsilon_{k_j}})_j$ as a subsequence that converges to u almost everywhere. Then,

$$l = \liminf_{k \to \infty} \mathcal{F}_{\varepsilon_k}(u_{\varepsilon_k}) = \lim_{j \to \infty} \mathcal{F}_{\varepsilon_{k_j}}(u_{\varepsilon_{k_j}}) \geq \lim_{j \to \infty} \frac{1}{\varepsilon_{k_j}^{2s}} \int_{\mathbb{R}^n} W(u_{\varepsilon_{k_j}})\, dx.$$

Therefore,

$$\int_{\mathbb{R}^n} W(u)\, dx = \lim_{j \to +\infty} \int_{\mathbb{R}^n} W(u_{\varepsilon_{k_j}})\, dx = 0,$$

and $u(x) \in \{0; 1\}$ almost everywhere. Thus, we deduce that $u = \chi_E$ for some set $E \subset \mathbb{R}$. Using Fatou's lemma and the definition of $\mathcal{F}(u)$ in (4.1.21), we can conclude that

$$\liminf_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon) \geq \liminf_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} S_s(\xi)\, |\widehat{u}_\varepsilon(\xi)|^2\, d\xi \geq \int_{\mathbb{R}^n} S_s(\xi)\, |\widehat{u}(\xi)|^2\, d\xi = \mathcal{F}(u).$$

This completes the proof of the claim in (i).

Now, we prove the claim in (ii). We assume that $u = \chi_E$ for some set $E \subset \mathbb{R}^n$ — otherwise (4.1.24) is obvious — and we define the constant sequence $u_\varepsilon := u$.

Since $\mathcal{F}_\varepsilon(u)$ is defined for $u = \chi_E$ as

$$\mathcal{F}_\varepsilon(u) = \mathcal{F}(u) = \int_{\mathbb{R}^n} S_s(\xi)\, |\widehat{u}(\xi)|^2\, d\xi,$$

then we trivially have (4.1.24) for the constant sequence $(u_\varepsilon)_\varepsilon$. □

Now, we prove Proposition 4.1.4. This result gives important information about the limit functional $\mathcal{F}$ defined in (4.1.21) for $s \in (0, 1/2)$ in the case $n = 1$, showing that it interpolates the classical and the nonlocal perimeter.

Proof of Proposition 4.1.4. We recall that the function $\mathcal{T}_s(r) : [0, +\infty) \longrightarrow [0, +\infty)$ is defined as

$$\mathcal{T}_s(r) := \mathcal{F}(\chi_{I_r}) = \int_{\mathbb{R}} S_s(\xi)\, |\widehat{\chi_{I_r}}(\xi)|^2\, d\xi, \tag{4.5.1}$$

and the squared modulus of the Fourier transform of χ_{I_r} is

$$|\widehat{\chi_{I_r}}(\xi)|^2 = \frac{4 \sin^2(r\xi)}{\xi^2}.$$

This last computation is done in detail in Lemma 4.A.1.1. Since the squared modulus of $\widehat{\chi_{I_r}}$ depends only on the length of the interval, then $\mathcal{F}(\chi_{I_r})$ only depends on r and $\mathcal{T}_s$ is a well-defined function of $r \in [0, +\infty)$.

Plugging the expression of $|\widehat{\chi_{I_r}}(\xi)|^2$ in (4.5.1), we have

$$\mathcal{T}_s(r) = 4\int_{\mathbb{R}} \widetilde{S}_s(\xi)\frac{\sin^2(r\xi)}{|\xi|^{2-2s}}\,d\xi, \tag{4.5.2}$$

where $\widetilde{S}_s(\xi)$ is defined in (4.1.13).

We want to show the asymptotic behavior of $\mathcal{T}_s$ at zero, as stated in (4.1.26). To this end, we change variable $r\xi = \eta$ in (4.5.2) and we get the following expression for $\mathcal{T}_s(r)$

$$\mathcal{T}_s(r) = 4r^{1-2s}\int_{\mathbb{R}} \widetilde{S}_s\left(\frac{\eta}{r}\right)\frac{\sin^2(\eta)}{|\eta|^{2-2s}}\,d\eta. \tag{4.5.3}$$

From the dominated convergence theorem and the fact that $\sin^2(\eta)/\,|\eta|^{2-2s}$ is integrable in $\mathbb{R}$ when $s \in (0, 1/2)$, we deduce that

$$\lim_{r\to 0}\int_{\mathbb{R}} \widetilde{S}_s\left(\frac{\eta}{r}\right)\frac{\sin^2(\eta)}{|\eta|^{2-2s}}\,d\eta = C_1,$$

where C_1 is a positive constant depending only on s. Thus, from this bound and (4.5.3) we obtain (4.1.26), as desired.

Now, we want to prove (4.1.27), which describes the asymptotic behavior of $\mathcal{T}_s$ at infinity. We use the expression in (4.5.2) for $\mathcal{T}_s(r)$ and Lemma 4.A.1.2 to write

$$\lim_{r\to\infty}\mathcal{T}_s(r) = \lim_{r\to\infty} 4\int_{\mathbb{R}} \widetilde{S}_s(\xi)\frac{\sin^2(r\xi)}{|\xi|^{2-2s}}\,d\xi = 2\int_{\mathbb{R}} \frac{\widetilde{S}_s(\xi)}{|\xi|^{2-2s}}\,d\xi. \tag{4.5.4}$$

The function in the last integral is controlled by a constant near the origin — see (4.1.13) and (4.3.5) — and by $C/\,|\xi|^{2-2s}$ far from the origin, which is an integrable function at infinity, since $s \in (0, 1/2)$.

Therefore, the last integral in (4.5.4) is finite and this proves the desired claim in (4.1.27). The proof of Proposition 4.1.4 is thereby complete. $\square$

4.A.1 Appendix to Chapter 4

For the sake of completeness, we collect here two simple technical lemmata that we use in Chapter 4. Let us start with a very standard computation, that is the Fourier transform of the characteristic function of one interval.

Lemma 4.A.1.1. *Let $u(x) : \mathbb{R} \to [0, 1]$ be defined as $u(x) = \chi_I$, where I is a finite interval of $\mathbb{R}$, i.e. $I = (a_1, a_2) \subset \mathbb{R}$. Then,*

$$|\widehat{u}(\xi)|^2 = 4\frac{\sin^2(r\xi)}{\xi^2},$$

where $r = \frac{a_2-a_1}{2}$ is the width of the intervals I.

Proof. First, we compute the Fourier transform of the function u.

$$\begin{aligned}\widehat{u}(\xi) = \int_{a_1}^{a_2} e^{-ix\xi}\, dx = \frac{i}{\xi}\left(e^{-ia_2\xi} - e^{-ia_1\xi}\right) \\ = \frac{1}{\xi}\left\{\sin(a_2\xi) - \sin(a_1\xi) + i\left(\cos(a_2\xi) - \cos(a_1\xi)\right)\right\}.\end{aligned}$$

Then, we compute its square modulus.

$$\begin{aligned}|\widehat{u}(\xi)|^2 = \frac{1}{\xi^2}\left\{2 - 2\left(\sin(a_2\xi)\sin(a_1\xi) + \cos(a_2\xi)\cos(a_1\xi)\right)\right\} \\ = \frac{1}{\xi^2}\left\{2 - 2\cos\left((a_2 - a_1)\xi\right)\right\} = \frac{4}{\xi^2}\sin^2\left(\frac{a_2 - a_1}{2}\xi\right),\end{aligned}$$

and this concludes the proof of Lemma 4.A.1.1. □

We prove now a convergence result that we use in Section 4.5.

Lemma 4.A.1.2. *If $f \in L^1(\mathbb{R})$, then*

$$\lim_{\omega\to+\infty}\int_{\mathbb{R}} f(\eta)\sin^2(\omega\eta)\, d\eta = \frac{1}{2}\int_{\mathbb{R}} f(\eta)\, d\eta.$$

Proof. Let us assume first that $f \in C_c^1(\mathbb{R})$. We start from the identity

$$\int_{\mathbb{R}} f(\eta)\, d\eta = \int_{\mathbb{R}} f(\eta)\sin^2(\omega\eta)\ d\eta + \int_{\mathbb{R}} f(\eta)\cos^2(\omega\eta)\ d\eta,$$

and we want to show that

$$\lim_{\omega\to+\infty}\int_{\mathbb{R}} f(\eta)\cos^2(\omega\eta)\ d\eta = \lim_{\omega\to+\infty}\int_{\mathbb{R}} f(\eta)\sin^2(\omega\eta)\ d\eta. \tag{4.A.1.1}$$

We remark indeed that the claim in Lemma 4.A.1.2 follows once we establish (4.A.1.1). In order to prove (4.A.1.1), we change variables $\omega\eta = \omega\theta - \pi/2$ and we obtain

$$\begin{aligned}&\int_{\mathbb{R}} f(\eta)\cos^2(\omega\eta)\ d\eta = \\ &\quad\int_{\mathbb{R}} f(\theta)\sin^2(\omega\theta)\ d\theta + \int_{\mathbb{R}}\left\{f\left(\theta - \frac{\pi}{2\omega}\right) - f(\theta)\right\}\sin^2(\omega\theta)\ d\theta.\end{aligned} \tag{4.A.1.2}$$

Taking the limits as $\omega \to +\infty$ in (4.A.1.2), the last term goes to zero thanks to the Vitali convergence theorem and we obtain (4.A.1.1) if $f \in C_c^1(\mathbb{R})$. In general, when $f \in L^1(\mathbb{R})$, the result follows from the density of $C_c^1(\mathbb{R})$ in $L^1(\mathbb{R})$. □

Appendix A

Physical considerations

In this appendix, we give a detailed description of the physical considerations that are leading to the study of the partial differential equation

$$\begin{cases} \operatorname{div}(y^a \nabla v) = 0 & \text{for} \quad x \in \mathbb{R}^n, y \in (0,1), \\ v_y(x,1) = 0 & \qquad\ \ x \in \mathbb{R}^n, y = 1, \\ v(x,0) = u(x) & \qquad\ \ x \in \mathbb{R}^n, y = 0, \\ -\lim\limits_{y\to 0} y^a v_y = f(v) & \qquad\ \ x \in \mathbb{R}^n, y = 0. \end{cases} \tag{A.0.1}$$

To this end, we consider a possible physical description of an irrotational and inviscid fluid (the "ocean") in $\mathbb{R}^{n+1}$, though we commonly take $n = 2$ in the "real world". The position of a fluid particle at time t will be denoted by $X(t) = (x(t), y(t)) \in \mathbb{R}^n \times \mathbb{R}$. We suppose that, at time t, the region occupied by the ocean lies above the graph of a function $b(\cdot, t)$ (the "bottom of the ocean") and below the graph of a function $h(\cdot, t)$ (the "surface of the ocean"). Therefore, in this model, the ocean can be described by the time-dependent domain

$$\Omega(t) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \text{ s.t. } b(x, t) \leq y \leq h(x, t)\}, \tag{A.0.2}$$

see Figure A.1.

Given a point $X \in \Omega(t)$, we denote by $v(X, t)$ the velocity of the fluid particle at X at time t. We denote by $\Phi^t(X)$ the evolution produced by the vector field v at time t starting at the point X at time zero, that is the solution of the initial value problem

$$\begin{cases} \dfrac{d}{dt}\Phi^t(X) = v(\Phi^t(X), t) \qquad \text{for (small) } t > 0, \\ \qquad\qquad\qquad \Phi^0(X) = X. \end{cases} \tag{A.0.3}$$

We suppose that the density of the water is described by a positive function $\rho = \rho(X, t)$. Then, the mass of the fluid lying in a region $\widetilde{\Omega} \subset \mathbb{R}^{n+1}$ at time t is described by the quantity

$$\int_{\widetilde{\Omega}} \rho(X, t)\, dX. \tag{A.0.4}$$

The rate at which a fluid mass enters in $\widetilde{\Omega}$ through an infinitesimal portion of $\partial\widetilde{\Omega}$ in the vicinity of a point $X \in \partial\widetilde{\Omega}$ is given by the density times the velocity at X in the direction

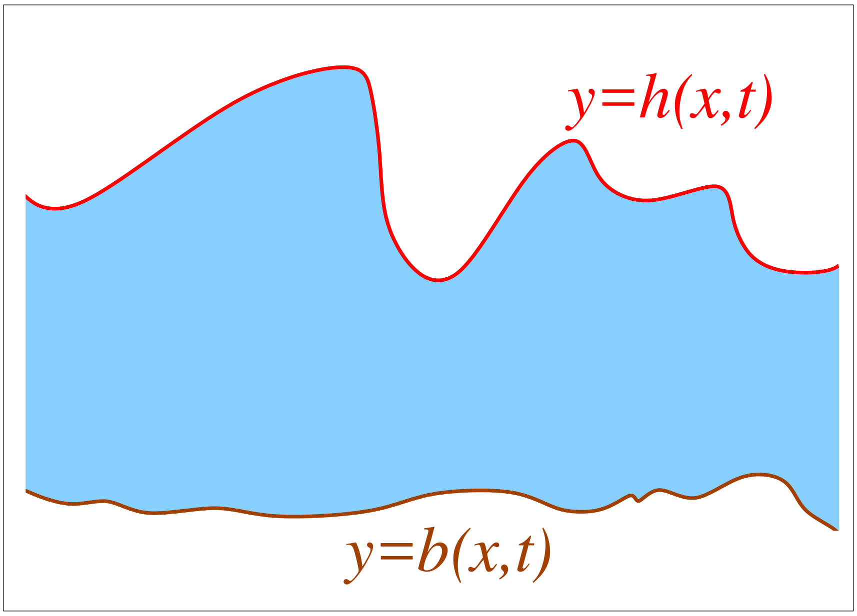

Figure A.1: *The domain* $\Omega(t)$ *in* (A.0.2).

of the inner normal of $\partial\widetilde{\Omega}$ at X. That is, if $\nu(X)$ denotes the exterior normal of $\partial\widetilde{\Omega}$ at X, we find that the rate at which a fluid mass enters in $\widetilde{\Omega}$ is given by

$$-\int_{\partial\widetilde{\Omega}} \rho(X,t)\, v(X,t)\cdot\nu(X)\, d\mathcal{H}^n(X).$$

Comparing with (A.0.4), and using the Divergence Theorem, this leads to

$$\begin{aligned}\int_{\widetilde{\Omega}} \partial_t\rho(X,t)\, dX &= \frac{d}{dt}\int_{\widetilde{\Omega}} \rho(X,t)\, dX \\ &= -\int_{\partial\widetilde{\Omega}} \rho(X,t)\, v(X,t)\cdot\nu(X)\, d\mathcal{H}^n(X) \\ &= -\int_{\widetilde{\Omega}} \operatorname{div}_X\big(\rho(X,t)\, v(X,t)\big)\, dX.\end{aligned}$$

From this, since the volume region $\widetilde{\Omega}$ is arbitrary, we obtain the "mass conservation law" (also known as "continuity equation") given by

$$\partial_t\rho(X,t) + \operatorname{div}_X\big(\rho(X,t)\, v(X,t)\big) = 0 \qquad \text{in } \Omega(t). \tag{A.0.5}$$

Let us now analyze the conditions occurring at the bottom and at the surface of the fluid. At the bottom, we assume that the fluid cannot penetrate inside the ground, hence its velocity is tangent to the seabed. Recalling the notation in (A.0.2), we have that v needs to be orthogonal to the normal direction of the graph of b, and thus, using the notation $X = (x,y) \in \mathbb{R}^n \times \mathbb{R}$,

$$v(X,t)\cdot\big(\nabla_x b(x,t), -1\big) = 0 \qquad \text{if } y = b(x,t). \tag{A.0.6}$$

We can therefore collect the results in (A.0.5) and (A.0.6) by writing

$$\begin{cases} \partial_t\rho(X,t) + \operatorname{div}_X\big(\rho(X,t)\, v(X,t)\big) = 0 & \text{in } \Omega(t), \\ v(X,t)\cdot\big(\nabla_x b(x,t), -1\big) = 0 & \text{on } \{y = b(x,t)\}. \end{cases} \tag{A.0.7}$$

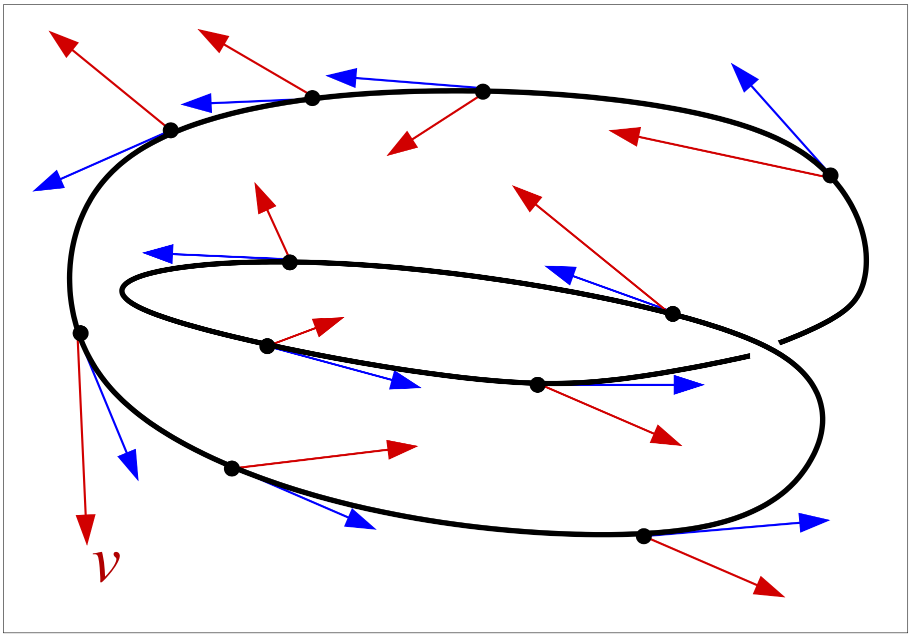

Figure A.2: *The velocity field v has always a positive component along the tangential direction of the closed curve, hence it is not irrotational.*

From (A.0.7) one sees that the vector field ρv has perhaps more physical meaning than v alone, since it represents the density speed of the flow, and it is somehow more meaningful to prescribe a bound on ρv rather than on v itself. For instance, the situation in which v becomes unbounded becomes physically realistic if ρv remains bounded, since, in this case, roughly speaking, only a very negligible amount of fluid would travel at exceptionally high speed. Therefore, though the equations are perfectly equivalent in case of "nice" vector fields v and densities ρ, we prefer to write (A.0.7) in a form which makes appear directly the quantity ρv rather than v alone. This is done by multiplying the identity on the bottom of the ocean by the density, to find

$$\begin{cases} \partial_t \rho(X,t) + \mathrm{div}_X\big(\rho(X,t)\, v(X,t)\big) = 0 & \text{in } \Omega(t), \\ \rho(X,t)\, v(X,t) \cdot \big(\nabla_x b(x,t), -1\big) = 0 & \text{on } \{y = b(x,t)\}. \end{cases} \tag{A.0.8}$$

We also assume that the fluid particles do not "circulate in a cyclone way", namely that the fluid is irrotational, see Figure A.2. To formalize this notion in an arbitrarily large number of dimensions in an elementary geometric way (without using the notion of higher dimensional curls), we assume that, for every fixed time, the integral of the velocity vector field along any closed one-dimensional curve in $\mathbb{R}^n$ vanishes. As a matter of fact, it would be enough to require such a condition along polygonal lines, and in fact it would be sufficient to require it along triangular connections.

This irrotationality condition implies (and, in fact, it is equivalent to) that the velocity field admits a potential, namely that there exists a scalar function $u = u(X,t)$ such that

$$v(X,t) = \nabla_X u(X,t). \tag{A.0.9}$$

We stress that (A.0.9) is a rather striking formula, since it reduces the knowledge of a vector valued function (namely, v) to the knowledge of (the derivatives of) a single scalar function. The construction of the potential u is standard, and can be performed along the following argument: we let Γ_X be the oriented segment starting at the origin and arriving at X, and we set

$$u(X,t) := \int_{\Gamma_X} v := \int_0^1 v(\vartheta X, t) \cdot X \, d\vartheta.$$

To prove (A.0.9), let $j \in \{1, \ldots, n\}$ and $\delta \neq 0$, to be taken arbitrarily small in what follows. We also denote by $\Gamma_{X,\delta,j}$ the oriented segment from X to $X + \delta e_j$. Also, given two adjacent segments Γ_1 and Γ_2, we denote by $\Gamma_1 \cup \Gamma_2$ the broken line joining the initial point of Γ_1 to the end point of Γ_1 (which coincides with the initial point of Γ_2) and that to the end point of Γ_2. Furthermore, we denote by $-\Gamma_1$ the segment Γ_1 run in the opposite direction. With this notation, we have that $\Gamma_{X+\delta e_j} \cup (-\Gamma_{X,\delta,j}) \cup (-\Gamma_X)$ forms a close triangle and accordingly, by the irrotationality condition,

$$\begin{aligned} 0 &= \int_{\Gamma_{X+\delta e_j} \cup (-\Gamma_{X,\delta,j}) \cup (-\Gamma_X)} v = \int_{\Gamma_{X+\delta e_j}} v - \int_{\Gamma_{X,\delta,j}} v - \int_{\Gamma_X} v \\ &= u(X + \delta e_j, t) - \delta \int_0^1 v(X + \vartheta \delta e_j, t) \cdot e_j \, d\vartheta - u(X,t). \end{aligned}$$

Dividing by δ and sending $\delta \to 0$, we obtain (A.0.9), as desired.

Then, inserting (A.0.9) into (A.0.8), we conclude that

$$\begin{cases} \partial_t \rho(X,t) + \operatorname{div}_X\big(\rho(X,t) \, \nabla_X u(X,t)\big) = 0 & \text{in } \Omega(t), \\ \rho(X,t) \, \nabla_x u(X,t) \cdot \nabla_x b(x,t) - \rho(X,t) \, \partial_y u(X,t) = 0 & \text{on } \{y = b(x,t)\}. \end{cases} \tag{A.0.10}$$

We observe that the setting in (A.0.1) is a particular case of that in (A.0.10), in which one considers the steady case of stationary solutions (i.e. ρ does not depend on time), with $X = (x,y) \in \Omega = \mathbb{R}^n \times (0,1)$, and $\rho(X) = y^a$, with $a \in (-1,1)$.

Remark A.0.1. Concerning the setting in (A.0.3), we recall that in the literature one also considers the "streamlines" of the fluid, described by a parameter $\tau \in \mathbb{R}$, which are (local) solutions of the differential equation (for fixed time t)

$$\frac{d}{d\tau} X(\tau, t) = v(X(\tau,t), t).$$

Notice that, if the velocity field v is independent of time, we can actually identify the curve parameter τ with the usual time t and then the streamlines describe the physical trajectories of the fluid particle. But in general, for velocity fields which depend on time, streamlines do not represent the physical trajectories. Nevertheless, streamlines are always instantaneously tangent to the velocity field of the flow and therefore they indicate the direction in which the fluid particle at a given point travels in time. We maintain the distinction between streamlines and physical trajectories of the flow, and in this note only the latter objects will be taken into account for the main computations.

Remark A.0.2. We point out that in the literature one often assumes that the fluid is "incompressible", that is, fixed any reference domain $\widetilde{\Omega}$,

$$\frac{d}{dt} \int_{\widetilde{\Omega}} \rho(\Phi^t(X), t) \, dX = 0.$$

This condition together with (A.0.3) leads to

$$\partial_t \rho(\Phi^t(X), t) + \nabla_X \rho(\Phi^t(X), t) \cdot v(\Phi^t(X), t) = 0, \tag{A.0.11}$$

or, equivalently, changing the name of the space variable

$$\partial_t \rho(X, t) + \nabla_X \rho(X, t) \cdot v(X, t) = 0. \tag{A.0.12}$$

The incompressibility condition (A.0.12) may be also understood from a "discrete analogue" by thinking that the density $\rho(X, t)$ of a gas formed by indistinguishable molecules at a point X at time t is measured by "counting the number of molecules" in the vicinity of X at time t. That is, fixing $r > 0$, the gas density could be defined as the number of molecules lying in $B_r(X)$ at time t. If the gas is incompressible, we expect that the number of molecules around the evolution $\Phi^t(X)$ of X remains the same. This gives that

$$\rho(\Phi^t(X), t) = \rho(X, 0),$$

which leads to (A.0.11) and (A.0.12).

To appreciate the structural difference between the mass conservation law in (A.0.5) and the incompressibility condition in (A.0.12), let us consider two examples. In the first example, let

$$v(X, t) := -X \qquad \text{and} \qquad \rho(X, t) := e^{nt},$$

with $n > 0$. In this case, the velocity field pushes all the fluid towards the origin, preserving the mass according to (A.0.5): as a consequence, the particles of the fluid get "packed" and their density increases, and the incompressibility condition (A.0.12) is indeed violated.

As a second example, let us consider the case in which

$$v(X, t) := -X \qquad \text{and} \qquad \rho(X, t) := 1.$$

In this case, the fluid elements are still pushed towards the origin, but the density remains constant. This means that there must be a leak somewhere, from which the fluid escapes. In this situation, the incompressibility condition in (A.0.12) is satisfied, but the mass is lost and accordingly (A.0.5) does not hold.

We also point out that if the the mass conservation law in (A.0.5) and the incompressibility condition in (A.0.12) are both satisfied, then

$$\begin{aligned} 0 &= \partial_t \rho(X, t) + \operatorname{div}_X \big(\rho(X, t)\, v(X, t)\big) \\ &= \partial_t \rho(X, t) + \nabla_X \rho(X, t) \cdot v(X, t) + \rho(X, t) \operatorname{div}_X v(X, t) \\ &= \rho(X, t) \operatorname{div}_X v(X, t), \end{aligned}$$

and, as a consequence,

$$\operatorname{div}_X v(X, t) = 0 \qquad \text{in } \Omega(t).$$

In this note, we will not explicitly take into account incompressibility assumptions, but merely the conservation of mass in (A.0.5).

Remark A.0.3. Concerning the top surface of the fluid, in the literature it is often assumed that fluid particles on this surface remain there forever (i.e., there is no "mixing effect" between the top surface of the sea and the rest of the water mass). This condition, in the notation of (A.0.2) and (A.0.3), would translate into

$$\Phi_2^t(X) = h(\Phi_1^t(X), t),$$

as long as $X = (x, y)$ and $y = h(x, 0)$, where $\Phi^t(X) = \big(\Phi_1^t(X), \Phi_2^t(X)\big) \in \mathbb{R}^n \times \mathbb{R}$. Hence, in view of (A.0.3),

$$0 = \frac{d}{dt}\Big(h(\Phi_1^t(X), t) - \Phi_2^t(X)\Big) = v(\Phi^t(X), t) \cdot \big(\nabla_x h(\Phi_1^t(X), t), \, -1\big) + \partial_t h(\Phi_1^t(X), t).$$

In this model, we do not need to assume this additional no mixing condition.

Remark A.0.4. Part II of this thesis is devoted to an elliptic problem related to the stationary case of the model that we just outlined. Besides assuming no dependence on time t, we also consider the simplification of a "flat ocean", by taking $b(x) = H > 0$ and $h(x) = 0$ — recall the notation in (A.0.2). This choice implies that we consider the sea as

$$\Omega = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \text{ s.t. } 0 \le y \le H\},$$

and that we are "reversing the vertical direction", in order to have the ocean surface on $\{y = 0\}$. This last simplification is done for pure mathematical convenience and does not affect the model.

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