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Research Article

On the Existence of a Cyclic Near-Resolvable $(6n+4)$-Cycle System of $2K_{12n+9}$

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In this article, we prove the existence of a simple cyclic near-resolvable $((v-1)/2)$-cycle system of $2K_v$ for $v \equiv 9 \pmod{12}$ by the method of constructing its starter. Then, some new properties and results related to this construction are formulated.

1. Introduction

Throughout this paper, all graphs are considered undirected with vertices in $\mathbb{Z}_v$ where v is odd. As usual, K_v will denote the complete graph of order v, and λK_v will denote the complete multigraph of order v and multiplicity λ in which every two vertices are joined by λ parallel edges.

A k-cycle system of a graph $G = (V, E)$ is a multiset $\mathscr{C}$ of k-cycles of G whose edge sets partition E. $\mathscr{C}$ is said to be cyclic if $V = \mathbb{Z}_v$ and for each k-cycle $C = (c_1, c_2, \ldots, c_k)$ in $\mathscr{C}$ we have that $C + 1 = (c_1 + 1, c_2 + 1, \ldots, c_k + 1) \pmod{v}$ is also in $\mathscr{C}$, and it is said to be simple if all its cycles are distinct. A starter of cyclic k-cycle system of G is a multiset $\mathscr{S}$ of k-cycles that generates the multiset $\mathscr{C}$ by repeated addition of 1 modulo v. A near-k-factor of G is a spanning k-regular subgraph of $G-a$ for some vertex a in G.

A k-cycle system $\mathscr{C}$ of $2K_v$ is said to be near-resolvable if its cycles can be partitioned into near-2-factors $\mathscr{N}_0, \mathscr{N}_1, \ldots, \mathscr{N}_{v-1}$ and $\mathscr{C}$ is denoted by $(v, k, 2)$-*NRCS*. In general, it has been shown that there exists a near-resolvable k-cycle system of λK_v if and only if λ is even and $v \equiv 1 \pmod{k}$ [1]. Such a near-resolvable k-cycle system is cyclic if it is possible to label the vertices of $2K_v$ with the elements of the cyclic group $\mathbb{Z}_v$ in such a way that $\mathscr{N}_i = \mathscr{N}_0 + i$ for $0 \le i \le v-1$, where $\mathscr{N}_0 + i$ denotes the near-2-factor of $2K_v$ obtained from $\mathscr{N}_0$ by adding i modulo v to all its vertices. The near-2-factor $\mathscr{N}_0$ is called a starter of cyclic near-resolvable k-cycle system of $2K_v$.

The existence problem of k-cycle systems of the complete multigraph λK_v has received much attention in recent years; this existence problem has been completely solved by Alspach and Gavlas [2] and by Šajna [3] for the important case when $\lambda = 1$, and by Alspach et al. [4] for the case $\lambda = 2$. An easier proof of the existence of odd cycle systems of K_v using the difference method has been reproved by Buratti [5]. Then, Wu and Buratti [6] provided an algorithm to construct an explicit odd k-cycle system of K_v whenever it exists. In particular, the existence of cyclic k-cycle systems of K_v has been solved when $v \equiv 1$ or $k \pmod{2k}$ [7, 8], $k = v$ [9], k is even with $v > 2k$ [10], k is a prime with the exception of $(v, k) = (9, 3)$ [7], $k \le 32$ or k is twice a prime power [11], and k is thrice a prime [12]. Further results on cycle systems are in the surveys [13–15].

The necessary and sufficient conditions for the existence of cyclic v-cycle system of λK_v and for the existence of simple cyclic p-cycle system of λK_p, where p is a prime, have been proved by Buratti et al. [16]. For $v \ge 3$ odd, the necessary and sufficient conditions for decomposing λK_v into λ-cycles, and into cycles of prime length have been established by Smith in [17]. Shortly later, Bryant et al. [18] proved that the necessary and sufficient conditions for the existence of a k-cycle system of λK_v for all $3 \le k \le v$ are that $\lambda(v-1)$ is even and k divides the number of edges in λK_v. More general results such as the existence problem for decomposing λK_v into cycles of varying lengths have been presented in [19, 20].

The problem of constructing near-resolvable k-cycle system of K_v has been contributed by many authors. A near-resolvable k-cycle system of K_v has been constructed for $k = 4$ with $v \equiv 1 \pmod{8}$ except possibly values $v = 33, 41, 57$ and except $v = 9$ (for which such a system does not exist) [21], $k = 10$ with $v \equiv 5 \pmod{20}$ or $v = 41$ [22], $k \geq 11$ with $v = 4k + 1$ [23]. Recently, the existence of a near-resolvable k-cycle system of K_{2km+1} for all $m \geq 1$ and $k \equiv 2 \pmod{4}$ except possibly for $m = 2$ and $k \geq 14$ has been proved by Wang and Cao [24]. Previously, it has been proved that there exists a $(rm+1, r, 2)$-*NRCS* for all odd $r \geq 3$ and all $m \geq 1$ [25]. In 2018, Matsubara and Kageyama [26] proved that a cyclic $(v, 4, 2)$-*NRCS* exists if and only if $v \equiv 1 \pmod{4}$.

In Section 2, we review some well-known definitions and preliminary results. Some introductory results are formulated in Section 3. Then, in Section 4, we explicitly construct a simple cyclic $(v, (v-1)/2, 2)$-*NRCS* for the case $v \equiv 9 \pmod{12}$ using a difference method. Moreover, we formulate some properties which are related to this construction. Finally, Section 5 discusses the conclusions and future work.

2. Preliminaries

In this section, we recall briefly some definitions and preliminary results that we used in the sequel. We start with the following definitions.

Definition 1 (see [27]). A path cover of a graph G is a collection of vertex-disjoint paths of G that covers the vertex set of G.

Definition 2 (see [5]). Let G be a graph and xy be an edge in G. The difference of an edge xy is defined as $d(x, y) = \pm|y-x|$.

Definition 3 (see [5, 28]). Let $G = (V(G), E(G))$ be a graph. The multiset

$$\Delta G = \{\pm |y - x| \mid x, y \in V(G), xy \in E(G)\} \tag{1}$$

is called the list of differences from G. More generally, for a set $\mathscr{G} = \{G_1, G_2, \ldots, G_n\}$ of graphs, the list of differences from $\mathscr{G}$ is the multiset $\Delta\mathscr{G} = \Delta G_1 \uplus \Delta G_2 \uplus \ldots \uplus \Delta G_n$ which is obtained by linking together the (ΔG_i)'s.

Definition 4 (see [6]). Let C be a k-cycle in λK_v. A cycle orbit of C, denoted as $Orb(C)$, is a set of distinct k-cycles in $\{C + i \mid i \in \mathbb{Z}_v\}$. A cycle orbit of C is called full if its cardinality is v; otherwise, the cycle orbit of C is short.

For convenience, we say C is a full (short) cycle.

Definition 5 (see [5, 6]). Let C be a k-cycle in λK_v. The type of C is the cardinality of the set $\{z \in \mathbb{Z}_v \mid C + z = C\}$.

From the above definition, it is obvious that if a cycle C is of type 1 $(d > 1)$, then C is a full (short) cycle.

Lemma 6 (see [5]). *If C is a k-cycle in λK_v, then the type of C is a common divisor of k and v.*

The following lemma is a consequence of the theory developed in [16]. It will be crucial for proving our main results.

Lemma 7. *Let $\mathscr{S}$ be a multiset of k-cycles of λK_v. Then, $\mathscr{S}$ is a starter of cyclic k-cycle system of λK_v if and only if $\Delta\mathscr{S}$ covers $\mathbb{Z}_v^* = \mathbb{Z}_v - \{0\}$ exactly λ times.*

3. Introductory Results

In this section, we introduce some definitions, notations, and introductory results required to establish our main results in the next section. We begin with defining relative path, relative cycle, and alternating arithmetic path that will be the basis for constructing the starter of simple cyclic near-resolvable $(6n + 4)$-cycle system of $2K_{12n+9}$.

Definition 8. Let G be a graph of order v, $P = [x_1, x_2, \ldots, x_k]$ be a k-path of G, and $C = (x_1, x_2, \ldots, x_k)$ be a k-cycle of G.

(1) The k-path $\overline{P} = [v - x_1, v - x_2, \ldots, v-x_k]$ is called the relative path of P.

(2) The k-cycle $\overline{C} = (v - x_1, v - x_2, \ldots, v - x_k)$ is called the relative cycle of C.

Lemma 9. *Let G be a graph of order v.*

(1) If P is a k-path of G and $\overline{P}$ is the relative path of P, then $\Delta P = \Delta\overline{P}$.

(2) If C is a k-cycle of G and $\overline{C}$ is the relative cycle of C, then $\Delta C = \Delta\overline{C}$.

Proof.

(1) Suppose $P = [x_1, x_2, \ldots, x_k]$ and $\overline{P} = [y_1, y_2, \ldots, y_k]$ are k-path of G and its relative path, respectively. The list of differences from P and $\overline{P}$ can be defined as

$$\Delta P = \{\pm |x_i - x_{i-1}| \mid i = 2, 3, \ldots, k\}, \tag{2}$$

$$\Delta\overline{P} = \{\pm |y_i - y_{i-1}| \mid i = 2, 3, \ldots, k\}. \tag{3}$$

Since $\overline{P}$ is the relative path of P, then $y_i = v - x_i$, for all $i = 1, 2, \ldots, k$. Hence, substituting $y_i = v - x_i$ into (3), we obtain

$$\begin{aligned}\Delta\overline{P} &= \{\pm |(v - x_i) - (v - x_{i-1})| \mid i = 2, 3, \ldots, k\} \\ &= \{\pm |x_i - x_{i-1}| \mid i = 2, 3, \ldots, k\} = \Delta P.\end{aligned} \tag{4}$$

(2) The proof is similar to part (1).

□

Lemma 10. *Let G be a graph of order v. If C_1 is a k-cycle of G and C_2 is the relative cycle of C_1; then, $orb(C_1) \neq orb(C_2)$.*

Proof. Let $C_1 = (c_{1,1}, c_{1,2}, \ldots, c_{1,k})$ be a k-cycle of G and let $C_2 = (c_{2,1}, c_{2,2}, \ldots, c_{2,k})$ be the relative cycle of C_1. Assume on the contrary that $orb(C_1) = orb(C_2)$; then, there exists an integer $i \in \mathbb{Z}_v$ such that $C_2 = i + C_1$. This implies that

$$c_{2,j} = i + c_{1,j} \quad \text{for all } j = 1, 2, \ldots, k. \tag{5}$$

Since C_2 is the relative cycle of C_1, then

$$c_{2,j} = v - c_{1,j} \quad \text{for all } j = 1, 2, \ldots, k. \tag{6}$$

Solving (5) and (6) for $c_{1,j}$ and $c_{2,j}$ yields

$$c_{1,j} = \frac{v-i}{2}$$
$$\text{and } c_{2,j} = \frac{v+i}{2} \tag{7}$$
$$\text{for all } j = 1, 2, \ldots, k.$$

This contradicts the fact that C_1 and C_2 are actually k-cycles. Thus, C_1 and C_2 must have different orbits, so $orb(C_1) \neq orb(C_2)$. □

An alternating arithmetic path is a path with two sets of vertices satisfying certain conditions, as defined below.

Definition 11. Let m and n be positive integers with $n \leq m \leq n+1$. An $(m+n)$-alternating arithmetic path, denoted by $AAP(m+n)$, is a path of length $m+n$ with vertex set $V = \{x_1, x_2, \ldots, x_m\} \cup \{y_1, y_2, \ldots, y_n\}$ and edge set $E = \{\{x_i, y_i\} \mid i = 1, 2, \ldots, n\} \cup \{\{y_i, x_{i+1}\} \mid i = 1, 2, \ldots, m-1\}$, such that the following properties are satisfied:

(1) $x_i - x_{i-1}$ is constant, for all $2 \leq i \leq m$.
(2) $y_i - y_{i-1}$ is constant, for all $2 \leq i \leq n$.

Definition 12. Let $AAP(m+n)$ be an $(m+n)$-alternating arithmetic path. The list of differences from $AAP(m+n)$ is the multiset

$$\Delta(AAP(m+n)) = \{\pm |y_i - x_i| \mid 1 \leq i \leq n\} \underline{\cup} \{\pm |x_{i+1} - y_i| \mid 1 \leq i \leq m-1\}. \tag{8}$$

According to Definition 11, the $(m+n)$-alternating arithmetic path either has odd order $(2n+1)$ when $m = n+1$ or has even order $(2n)$ when $m = n$. Throughout, we use the following notations for $(m+n)$-alternating arithmetic path of odd order and even order, respectively:

$$\begin{aligned} AAP(2n+1) &= [x_1, y_1, x_2, y_2, \ldots, x_n, y_n, x_{n+1}] \\ &= [x_i, y_i]_{2n+1}, \\ AAP(2n) &= [x_1, y_1, x_2, y_2, \ldots, x_n, y_n] \\ &= [x_i, y_i]_{2n}. \end{aligned} \tag{9}$$

In the following, we define a modulo scalar multiplication on paths and cycles in a finite graph of order v, and then we prove some lemmas that will be used later in order to investigate some properties related to our construction.

Definition 13. Let v, k, and h be positive integers with $1 \leq h < v$ and $gcd(h, v) = 1$. Let G be a graph of order v, $P = [x_1, x_2, \ldots, x_k]$ be a k-path of G, and $C = (x_1, x_2, \ldots, x_k)$ be a k-cycle of G.

(1) The modulo v multiplication of P and h is the k-path $h \cdot P = [hx_1, hx_2, \ldots, hx_k] \pmod{v}$.
(2) The modulo v multiplication of C and h is the k-cycle $h \cdot C = (hx_1, hx_2, \ldots, hx_k) \pmod{v}$.

Lemma 14. *Let G be a graph of order v and C be a k-cycle of G. If h is any integer such that $1 \leq h < v$ and $gcd(h, v) = 1$, then*

(1) $h \cdot C = (v-h) \cdot \overline{C}$.
(2) $(v-h) \cdot C = h \cdot \overline{C}$.

Proof.

(1) Suppose that $C = (x_1, x_2, \ldots, x_k)$ is a k-cycle of G. Then,

$$h \cdot C = (hx_1, hx_2, \ldots, hx_k) \pmod{v}. \tag{10}$$

Since $v^2 - (x_i + h)v$ is divisible by v, then $v^2 - (x_i + h)v \equiv 0 \pmod{v}$. Hence,

$$\begin{aligned} h \cdot C &= (v^2 - (x_1 + h)v + hx_1, v^2 - (x_2 + h)v \\ &\quad + hx_2, \ldots, v^2 - (x_k + h)v + hx_k) \pmod{v} \\ &= ((v - x_1)(v - h), (v - x_2)(v - h), \ldots, (v - x_k) \\ &\quad \cdot (v - h)) \pmod{v} \\ &= (v - h) \cdot ((v - x_1), (v - x_2), \ldots, (v - x_k)) \\ &\qquad \pmod{v} \\ &= (v - h) \cdot \overline{C}. \end{aligned} \tag{11}$$

(2) From the definition of modulo v multiplication of C and $(v-h)$, we obtain

$$\begin{aligned} (v - h) \cdot C &= ((v - h)x_1, (v - h)x_2, \ldots, (v - h)x_k) \\ &\qquad \pmod{v} \\ &= (vx_1 - hx_1, vx_2 - hx_2, \ldots, vx_k - hx_k) \\ &\qquad \pmod{v}. \end{aligned} \tag{12}$$

But vx_i is divisible by v, and this implies that $vx_i \equiv 0 \pmod{v}$. Hence,

$$\begin{aligned} (v - h) \cdot C &= (-hx_1, -hx_2, \ldots, -hx_k) \pmod{v} \\ &= h \cdot (-x_1, -x_2, \ldots, -x_k) \pmod{v} \\ &= h \cdot (v - x_1, v - x_2, \ldots, v - x_k) \\ &\qquad \pmod{v} \\ &= h \cdot \overline{C}. \end{aligned} \tag{13}$$

□

Lemma 15. *Let v and h be integers with $1 \leq h < v$ and $gcd(v,h) = 1$. Then, $h \cdot \mathbb{Z}_v^* = \{hi \ (mod\ v) \mid i \in \mathbb{Z}_v^*\}$ covers $\mathbb{Z}_v^*$.*

Proof. Let $x, y \in \mathbb{Z}_v^*$ with $x \neq y$. Assume on the contrary that $hx \pmod{v} = hy \pmod{v} = z$. Then, we get $hx = z + m_1 v$ and $hy = z + m_2 v$ for some integers m_1 and m_2.

Subtracting the above equations, we obtain $hx - hy = m_1 v - m_2 v$. This implies

$$x - y = \frac{v(m_1 - m_2)}{h}. \tag{14}$$

Since $, y \in \mathbb{Z}_v^*$, then $x - y < v$ and then from (14) we get $v(m_1 - m_2)/h < v$. This implies that $(m_1 - m_2)/h < 1$ and therefore $(m_1 - m_2) < h$.

On the other hand, since $\gcd(v,h) = 1$ and $(m_1 - m_2) < h$, then from (14) it follows that $(x - y)$ is a noninteger rational number. This contradicts the fact that $(x - y)$ is an integer. Thus, there are no $x, y \in \mathbb{Z}_v^*$ such that $hx \pmod{v} = hy \pmod{v}$, so $h \cdot \mathbb{Z}_v^*$ covers $\mathbb{Z}_v^*$. □

Lemma 16. *Let $n \geq 0$ be an integer; then, $12n + 9$ and $6n + 4$ are relatively prime.*

Proof. Let c be an integer such that c divides both $12n+9$ and $6n + 4$. Then, there exists $x, y \in \mathbb{Z}$ such that

$$cx = 12n + 9 \tag{15}$$

$$cy = 6n + 4. \tag{16}$$

From the equations above, we obtain $cx = 12n + 9 = 2(6n + 4) + 1 = 2cy + 1$. This implies that $cx - 2cy = 1$, and then

$$x - 2y = \frac{1}{c}. \tag{17}$$

Since $x - 2y \in \mathbb{Z}$, then either $c = 1$ or $c = -1$. Therefore, 1 is the only positive integer which divides both $12n + 9$ and $6n + 4$. □

Now, we define a way of writing the cycle as linked vertex-disjoint paths. This way will be used mainly to prove the existence results in the following section.

Definition 17. Let C be a k-cycle, $r \geq 2$ be a positive integer, and $\mathscr{P} = \{P_1, P_2, \ldots, P_r\}$ be a path cover of C. The set of r edges in C that links the end of P_i with the start of P_{i+1}, for all $i = 1, 2, \ldots, r$ where $P_{r+1} = P_1$, is called the link set of $\mathscr{P}$.

Remark 18. Let C be a k-cycle, $\mathscr{P} = \{P_1, P_2, \ldots, P_r\}$ be a path cover of C, and $E' = \{e_1, e_2, \ldots, e_r\}$ be a link set of $\mathscr{P}$. The cycle C can be expressed as linked vertex-disjoint paths as follows:

$$C = (P_1, P_2, \ldots, P_r). \tag{18}$$

Lemma 19. *Let C be a k-cycle, $\mathscr{P} = \{P_1, P_2, \ldots, P_r\}$ be a path cover of C, and $E' = \{e_1, e_2, \ldots, e_r\}$ be a link set of $\mathscr{P}$. Then, we have $\Delta C = \Delta\mathscr{P} \uplus \Delta E'$.*

Proof. Let $V(\mathscr{P}) = \bigcup_{i=1}^{r} V(P_i)$ be the set of vertices of $\mathscr{P}$ and $E(\mathscr{P}) = \bigcup_{i=1}^{r} E(P_i)$ the set of edges of $\mathscr{P}$. Based on Definition 3, the list of differences from C is defined as a multiset consisting of the difference for each edge in C as follows:

$$\Delta C = \{d(a,b) \mid a, b \in V(C), ab \in E(C)\}. \tag{19}$$

Since $\mathscr{P}$ is a path cover of C, then

$$V(C) = V(\mathscr{P}). \tag{20}$$

Also, from the definition of link set of $\mathscr{P}$, we obtain

$$E(C) = E(\mathscr{P}) \cup E'. \tag{21}$$

Substituting (20) and (21) into (19) yields

$$\begin{aligned} \Delta C &= \left\{d(a,b) \mid a, b \in V(\mathscr{P}), ab \in E(\mathscr{P}) \cup E'\right\} \\ &= \{d(a,b) \mid a, b \in V(\mathscr{P}), ab \in E(\mathscr{P})\} \\ &\quad \uplus \left\{d(e_i) \mid e_i \in E'\right\} = \Delta\mathscr{P} \uplus \Delta E'. \end{aligned} \tag{22}$$

□

To close this section, we provide an example below to demonstrate the concepts discussed in this section.

Example 20. Let $G = 2K_{21}$, $C = (4, 1, 3, 2, 6, 11, 14, 12, 13, 16)$ be a 10-cycle of G. Then, the cycle C can be written as linked vertex-disjoint paths as follows:

$$C = (AAP_1(4), Q_1, AAP_2(4), Q_2), \tag{23}$$

where $AAP_1(4) = [4, 1, 3, 2] = [5 - i, i]_4$ and $AAP_2(4) = [11, 14, 12, 13] = [10 + i, 15 - i]_4$ are 4-alternating arithmetic paths and $Q_1 = [6]$ and $Q_2 = [16]$ are trivial paths. In addition, the set of four edges $E' = \{\{2, 6\}, \{6, 11\}, \{13, 16\}, \{16, 4\}\}$ that links the paths $AAP_1(4), Q_1, AAP_2(4)$ and Q_2, respectively, along the cycle C is considered the link set for the path cover $\mathscr{P} = \{AAP_1(4), Q_1, AAP_2(4), Q_2\}$.

Based on Definition 8, the relative cycle of C is $\overline{C} = (17, 20, 18, 19, 15, 10, 7, 9, 8, 5)$. It is easy to see that the sum of each pair of corresponding vertices of C and its relative cycle is equal to 21 (the order of G).

Since $gcd(20, 21) = 1$, then

$$\begin{aligned} 20 \cdot C &= (80, 20, 60, 40, 120, 220, 280, 240, 260, 320) \pmod{21} \\ &= (17, 20, 18, 19, 15, 10, 7, 9, 8, 5). \end{aligned} \tag{24}$$

In other words, $\overline{C} = 20 \cdot C$ as shown in part (2) of Lemma 14.

4. Simple Cyclic Near-Resolvable $(6n+4)$-Cycle System of $2K_{12n+9}$

In this section, we prove, explicitly and directly, the existence of a simple cyclic near-resolvable $(6n + 4)$-cycle system of $2K_{12n+9}$ by constructing its starter.

Table 1: The list of differences from $\mathcal{S} = \{C_1, C_2\}$.

4-cycles	The list of differences
$C_1 = (1, 6, 5, 7)$	$\{5, 4, 1, 8, 2, 7, 6, 3\}$
$C_2 = (8, 3, 4, 2)$	$\{5, 4, 1, 8, 2, 7, 6, 3\}$

Table 2: A simple cyclic near-resolvable 4-cycle system of $2K_9$.

Focus	$Orb(C_1)$	$Orb(C_2)$
$i = 0$	$(1, 6, 5, 7)$	$(8, 3, 4, 2)$
$i = 1$	$(2, 7, 6, 8)$	$(0, 4, 5, 3)$
$i = 2$	$(3, 8, 7, 0)$	$(1, 5, 6, 4)$
$i = 3$	$(4, 0, 8, 1)$	$(2, 6, 7, 5)$
$i = 4$	$(5, 1, 0, 2)$	$(3, 7, 8, 6)$
$i = 5$	$(6, 2, 1, 3)$	$(4, 8, 0, 7)$
$i = 6$	$(7, 3, 2, 4)$	$(5, 0, 1, 8)$
$i = 7$	$(8, 4, 3, 5)$	$(6, 1, 2, 0)$
$i = 8$	$(0, 5, 4, 6)$	$(7, 2, 3, 1)$

To construct a simple cyclic near-resolvable $(6n + 4)$-cycle system of $2K_{12n+9}$, it is enough to exhibit a starter of cyclic k-cycle system of $2K_v$ which satisfies a near-2-factor and contains no two cycles in the same orbit. Let us provide an example to illustrate the above definition.

Example 21. Let $G = 2K_9$ and $\mathcal{S} = \{C_1, C_2\}$ be a set of 4-cycles of G such that $C_1 = (1, 6, 5, 7)$ and $C_2 = (8, 3, 4, 2)$.

Easily, it can be observed that the 4-cycles of $\mathcal{S}$ are vertex-disjoint and cover each nonzero element of $\mathbb{Z}_9$ exactly once. Hence, we can say that $\mathcal{S}$ is a 2-regular graph satisfying the near-2-factor with focus zero.

In order to show that $\mathcal{S} = \{C_1, C_2\}$ is a starter of cyclic 4-cycle system of $2K_9$, we need to calculate the list of differences from $\mathcal{S}$ as illustrated in Table 1.

As listed in Table 1, each nonzero element of $\mathbb{Z}_9$ occurs twice in $\Delta\mathcal{S} = \Delta C_1 \uplus \Delta C_2$. Then, by Lemma 7, $\mathcal{S} = \{C_1, C_2\}$ is a starter set of cyclic 4-cycle system of $2K_9$.

Since the sum of each pair of corresponding vertices of C_1 and C_2 is equal to 9 the order of G, then C_2 is the relative cycle of C_1, and so, by Lemma 10, $orb(C_1) \neq orb(C_2)$. From Definition 4, we conclude that all the generated cycles by repeated addition of 1 modulo 9 to $\mathcal{S} = \{C_1, C_2\}$ contain no repetitions.

Now, $\mathcal{S}$ satisfies all the conditions to be a starter of simple cyclic near-resolvable 4-cycle system of $2K_9$. Once the starter set has been provided, all cycles of simple cyclic $(9, 4, 2)$-*NRCS* can be generated by repeated addition of 1 modulo 9 as shown in Table 2.

In the following, we construct a simple cyclic near-resolvable $(6n + 4)$-cycle system of $2K_{12n+9}$. Since the construction is different depending on whether n is odd or even, we classify the construction into two cases: when n is odd and when n is even.

Lemma 22. *For any positive odd integer n, there exists a simple cyclic near-resolvable $(6n + 4)$-cycle system of $2K_{12n+9}$.*

Proof. Let $v = 12n+9$, where n is a positive odd integer. Let C_1 and C_2 be the $(6n+4)$-cycles of $2K_v$ defined as linked vertex-disjoint paths as follows:

$$\begin{aligned} C_1 &= \left(AAP_1(3n+3), AAP_2(3n+1)\right) \\ C_2 &= \left(\overline{AAP_1}(3n+3), \overline{AAP_2}(3n+1)\right), \end{aligned} \tag{25}$$

where

$$\begin{aligned} AAP_1(3n+3) &= [4i-3, 12n-4i+10]_{3n+3}, \\ AAP_2(3n+1) &= [6n-4i+6, 6n+4i+5]_{3n+1}, \\ \overline{AAP_1}(3n+3) & \\ &= [v-(4i-3), v-(12n-4i+10)]_{3n+3} = [12n \\ &\quad -4i+12, 4i-1]_{3n+3}, \\ \overline{AAP_2}(3n+1) & \\ &= [v-(6n-4i+6), v-(6n+4i+5)]_{3n+1} = [6n \\ &\quad +4i+3, 6n-4i+4]_{3n+1}. \end{aligned} \tag{26}$$

Since n is a positive odd integer, then any $(3n+3)$-alternating arithmetic path and $(3n+1)$-alternating arithmetic path have even order. As illustrated in Figure 1, the construction of C_1 and C_2 can be described in terms of their vertices as $C_i = (c_{i,1}, c_{i,2}, \ldots, c_{i,6n+4})$ for $i = 1, 2$.

In this way, we note that in the cycle C_1 the $c_{1,i}$'s with i odd and the $c_{1,i}$'s with i even form the following increasing sequences, respectively:

$c_{1,1} < c_{1,6n+3} < c_{1,3} < c_{1,6n+1} < \cdots < c_{1,3n+4} < c_{1,3n+2}$ in the interval $I = [1, 6n+3]$ and

$c_{1,3n+3} < c_{1,3n+1} < c_{1,3n+5} < c_{1,3n-1} < \cdots < c_{1,2} < c_{1,6n+4}$ in the complement of I in $\mathbb{Z}_v$.

In contrast, in C_2 the $c_{2,i}$'s with i odd and the $c_{2,i}$'s with i even form the following decreasing sequences, respectively:

$c_{2,1} > c_{2,6n+3} > c_{2,3} > c_{2,6n+1} > \cdots > c_{2,3n+4} > c_{2,3n+2}$ in $J = [6n+6, 12n+8]$ and

$c_{2,3n+3} > c_{2,3n+1} > c_{2,3n+5} > c_{2,3n-1} > \cdots > c_{2,2} > c_{2,6n+4}$ in the complement of J in $\mathbb{Z}_v$.

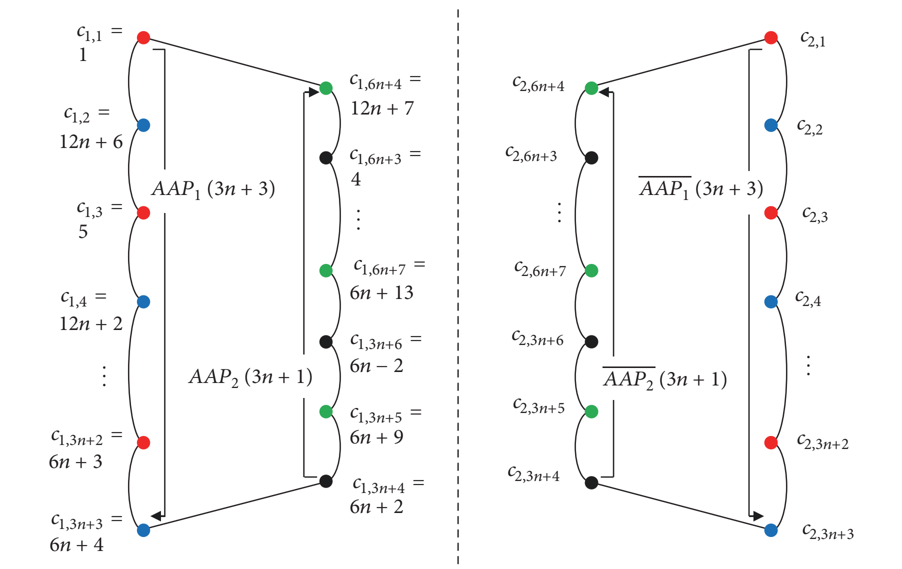
Thus, for $i = 1, 2$, the vertices in C_i are pairwise distinct and hence C_i is actually $(6n+4)$-cycle.

In the rest of this proof, three parts are considered to prove that the set of cycles $\mathcal{S} = \{C_1, C_2\}$ satisfies the conditions to be a starter of simple cyclic near-resolvable $(6n+4)$-cycle system of $2K_v$.

Part 1. In this part, we prove that $\mathcal{S} = \{C_1, C_2\}$ satisfies the near-two-factor condition. This will be verified by proving that the union of vertex sets of C_1 and C_2 covers each element of $\mathbb{Z}_v^*$ exactly once. The vertex sets of C_1 and C_2 can be enumerated by the union of vertex sets of all linked paths in both C_1 and C_2, respectively.

$$V(C_1) = V\left(AAP_1(3n+3)\right) \cup V\left(AAP_2(3n+1)\right), \tag{27}$$

$$V(C_2) = V\left(\overline{AAP_1}(3n+3)\right) \cup V\left(\overline{AAP_2}(3n+1)\right), \tag{28}$$



$AAP_1\,(3n+3) = [4i-3, 12n-4i+10]_{3n+3}$.

$AAP_2\,(3n+1) = [6n-4i+6, 6n+4i+5]_{3n+1}$.

$c_{2,i} = v - c_{1,i}$ for $i = 1, 2, \ldots, 6n+4$.

Figure 1: The construction of C_1 and C_2 in $2K_v$. n is a positive odd integer.

where

$$\begin{aligned}
V\left(AAP_1\,(3n+3)\right) &= \bigcup_{i=1}^{(3n+3)/2} \{4i-3\} \bigcup \bigcup_{i=1}^{(3n+3)/2} \{12n-4i+10\} \\
&= \{1, 5, \ldots, 6n+3\} \\
&\quad \cup \{12n+6, 12n+2, \ldots, 6n+4\}, \\
V\left(AAP_2\,(3n+1)\right) &= \bigcup_{i=1}^{(3n+1)/2} \{6n-4i+6\} \bigcup \bigcup_{i=1}^{(3n+1)/2} \{6n+4i+5\} \\
&= \{6n+2, 6n-2, \ldots, 4\} \\
&\quad \cup \{6n+9, 6n+13, \ldots, 12n+7\}, \\
V\left(\overline{AAP_1}\,(3n+3)\right) &= \{v-i \mid i \in V\left(AAP_1\,(3n+3)\right)\} \\
&= \{12n+8, 12n+4, \ldots, 6n+6\} \\
&\quad \cup \{3, 7, \ldots, 6n+5\}, \\
V\left(\overline{AAP_2}\,(3n+1)\right) &= \{v-i \mid i \in V\left(AAP_2\,(3n+1)\right)\} \\
&= \{6n+7, 6n+11, \ldots, 12n+5\} \\
&\quad \cup \{6n, 6n-4, \ldots, 2\}.
\end{aligned} \tag{29}$$

According to the above vertex sets, it can be easily noted that each nonzero element of $\mathbb{Z}_v$ occurs exactly once in $V(C_1) \uplus V(C_2)$. Since any cycle is a 2-regular graph and $V(C_1) \uplus V(C_2) = \mathbb{Z}_v^*$, then the set of cycles $\mathcal{S} = \{C_1, C_2\}$ forms near-two-factor with focus zero.

Part 2. This part shows that the set of cycles $\mathcal{S} = \{C_1, C_2\}$ is a starter of cyclic $(6n+4)$-cycle system of $2K_v$. For this part, it is sufficient to prove that the list of differences from $\mathcal{S}$ covers $\mathbb{Z}_v^*$ exactly twice.

Based on Definition 3, the list of differences from $\mathcal{S}$ is defined as $\Delta\mathcal{S} = \Delta(C_1) \uplus \Delta(C_1)$. Then, from Lemma 19 and Definition 12, the list of differences from C_1 is

$$\begin{aligned}
\Delta\left(C_1\right) = {}& \Delta\left(AAP_1\,(3n+3)\right) \uplus \{d\,(6n+4, 6n+2)\} \\
& \uplus \Delta\left(AAP_2\,(3n+1)\right) \uplus \{d\,(12n+7, 1)\},
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
\Delta\left(AAP_1\,(3n+3)\right) &= \left\{\pm\left|y_i - x_i\right| \mid 1 \le i \le \frac{3n+3}{2}\right\} \\
&\quad \uplus \left\{\pm\left|x_{i+1} - y_i\right| \mid 1 \le i \le \frac{3n+1}{2}\right\} \\
&= \left\{\pm\left|12n-8i+13\right| \mid 1 \le i \le \frac{3n+3}{2}\right\} \\
&\quad \uplus \left\{\pm\left|12n-8i+9\right| \mid 1 \le i \le \frac{3n+1}{2}\right\} \\
&= \{12n+5, 12n-3, \ldots, 1\} \uplus \{4, 12, \ldots, 12n+8\} \\
&\quad \uplus \{12n+1, 12n-7, \ldots, 5\} \\
&\quad \uplus \{8, 16, \ldots, 12n+4\},
\end{aligned}$$

$$\begin{aligned}
&\Delta\left(AAP_2\left(3n+1\right)\right)\\
&\quad=\left\{\pm\left|y_i-x_i\right| \mid 1\le i\le \frac{3n+1}{2}\right\}\\
&\quad\quad\underline{\cup}\left\{\pm\left|x_{i+1}-y_i\right| \mid 1\le i\le \frac{3n-1}{2}\right\}\\
&\quad=\left\{\pm\left|8i-1\right| \mid 1\le i\le \frac{3n+1}{2}\right\}\\
&\quad\quad\underline{\cup}\left\{\pm\left|8i+3\right| \mid 1\le i\le \frac{3n-1}{2}\right\}\\
&\quad=\{7,15,\ldots,12n+3\}\,\underline{\cup}\,\{12n+2,12n-6,\ldots,6\}\\
&\quad\quad\underline{\cup}\{11,19,\ldots,12n-1\}\\
&\quad\quad\underline{\cup}\{12n-2,12n-10,\ldots,10\},\\
&\{d\left(6n+4,6n+2\right)\}=\{2,12n+7\},\\
&\{d\left(12n+7,1\right)\}=\{12n+6,3\}.
\end{aligned}\tag{31}$$

As shown above, each nonzero element of $\mathbb{Z}_v$ appears exactly once in $\Delta(C_1)$.

From (25), we can deduce that C_2 is the relative cycle of C_1. Hence, by part (2) of Lemma 9, we obtain $\Delta(C_1)=\Delta(C_2)$. Now, we conclude that each nonzero element of $\mathbb{Z}_v$ appears exactly twice in $\Delta\mathcal{S}$. Based on Lemma 7, the set of cycles $\mathcal{S}=\{C_1,C_2\}$ is a starter of cyclic $(6n+4)$-cycle system of $2K_v$ for all odd positive integer n.

Part 3. We check that all the generated cycles from the starter $\mathcal{S}=\{C_1,C_2\}$ contain no repetitions by showing that all the cycles of $\mathcal{S}$ have different orbit.

Since C_2 is the relative cycle of C_1, then by Lemma 10, $orb(C_1)\neq orb(C_2)$. Thus, all the generated cycles by repeated addition of 1 modulo v to $\mathcal{S}=\{C_1,C_2\}$ contain no repetitions.

By summing up the former three parts, we have proved that, for any positive odd integer n, the set of cycles $\mathcal{S}=\{C_1,C_2\}$ is a starter of simple cyclic near-resolvable $(6n+4)$-cycle system of $2K_v$. □

Lemma 23. *For any nonnegative even integer n, there exists a simple cyclic near-resolvable* $(6n+4)$*-cycle system of* $2K_{12n+9}$.

Proof. Let $v=12n+9$, where n is a nonnegative even integer. Let C_1 and C_2 be the $(6n+4)$-cycles of $2K_v$ defined as linked vertex-disjoint paths as follows:

$$\begin{aligned}
C_1&=\left(AAP_1\left(3n+3\right),AAP_2\left(3n+1\right)\right)\\
C_2&=\left(\overline{AAP_1}\left(3n+3\right),\overline{AAP_2}\left(3n+1\right)\right),
\end{aligned}\tag{32}$$

where

$$\begin{aligned}
AAP_1\left(3n+3\right)&=\left[4i-3,12n-4i+10\right]_{3n+3},\\
AAP_2\left(3n+1\right)&=\left[6n+4i+3,6n-4i+4\right]_{3n+1},\\
\overline{AAP_1}\left(3n+3\right)&\\
&=\left[v-\left(4i-3\right),v-\left(12n-4i+10\right)\right]_{3n+3}\\
&=\left[12n-4i+12,4i-1\right]_{3n+3},\\
\overline{AAP_2}\left(3n+1\right)&\\
&=\left[v-\left(6n-4i+6\right),v-\left(6n+4i+5\right)\right]_{3n+1}\\
&=\left[6n-4i+6,6n+4i+5\right]_{3n+1}.
\end{aligned}\tag{33}$$

Since n is a nonnegative even integer, then any $(3n+3)$-alternating arithmetic path and $(3n+1)$-alternating arithmetic path have odd order. As shown in Figure 2, the construction of C_1 and C_2 can be described in terms of their vertices as $C_i=(c_{i,1},c_{i,2},\ldots,c_{i,6n+4})$ for $i=1,2$.

The rest of this proof is similar to the proof of Lemma 22, hence omitted. □

Theorem 24. *For each positive* $v\equiv 9\ (mod\ 12)$*, there exists a simple cyclic near-resolvable* $((v-1)/2)$*-cycle system of* $2K_v$.

Proof. The proof is immediate from Lemmas 22 and 23. □

By reviewing the construction of a starter of simple cyclic near-resolvable $(6n+4)$-cycle system of $2K_{12n+9}$, as shown in Figures 1 and 2, the construction has a butterfly shape in which each cycle represents a side of symmetrical butterfly wings. If given one cycle C of the starter set, the other is the relative cycle of C.

Next, some related properties of the starter of simple cyclic near-resolvable $(6n+4)$-cycle system of $2K_{12n+9}$ will be formulated.

Lemma 25. *Let n, h, and* $v=12n+9$ *be integers such that* $1\le h<v$ *and* $gcd(h,v)=1$*. If* $\mathcal{S}=\{C_1,C_2\}$ *is a starter of simple cyclic near-resolvable* $(6n+4)$*-cycle system of* $2K_v$*, then* $h\cdot V(\mathcal{S})=\{hi\ (mod\ v)\mid i\in(V(C_1)\cup V(C_2))\}$ *covers* $\mathbb{Z}_v^*$.

Proof. Since $\mathcal{S}=\{C_1,C_2\}$ is a starter of simple cyclic near-resolvable $(6n+4)$-cycle system of $2K_v$, then $\mathcal{S}=\{C_1,C_2\}$ satisfies the near-two-factor with focus zero. This implies that $V(C_1)\cup V(C_2)=\mathbb{Z}_v^*$. Then, by Lemma 15, we obtain the notion that $h\cdot V(\mathcal{S})$ covers $\mathbb{Z}_v^*$. □

Theorem 26. *Let n, h, and* $v=12n+9$ *be integers with* $1\le h<v$ *and* $gcd(h,v)=1$*. If* $\mathcal{S}=\{C_1,C_2\}$ *is a starter of simple cyclic near-resolvable* $(6n+4)$*-cycle system of* $2K_v$ *that satisfies (25) or (32), then* $\{h\cdot C_1,h\cdot C_2\}$ *and* $\{(v-h)\cdot C_1,(v-h)\cdot C_2\}$ *are the same starter of simple cyclic near-resolvable* $(6n+4)$*-cycle system of* $2K_v$.

Proof. Suppose that $\mathcal{S}=\{C_i=(c_{i,1},c_{i,2},\ldots,c_{i,6n+4})\mid i=1,2\}$ is a starter of simple cyclic near-resolvable $(6n+4)$-cycle system of $2K_v$ that satisfies (25) or (32). Then, C_2 is the relative cycle of C_1 and hence, by part (1) and (2) of Lemma 14, we have

$$\begin{gathered}
h\cdot C_1=(v-h)\cdot C_2\\
\text{and } h\cdot C_2=(v-h)\cdot C_1.
\end{gathered}\tag{34}$$

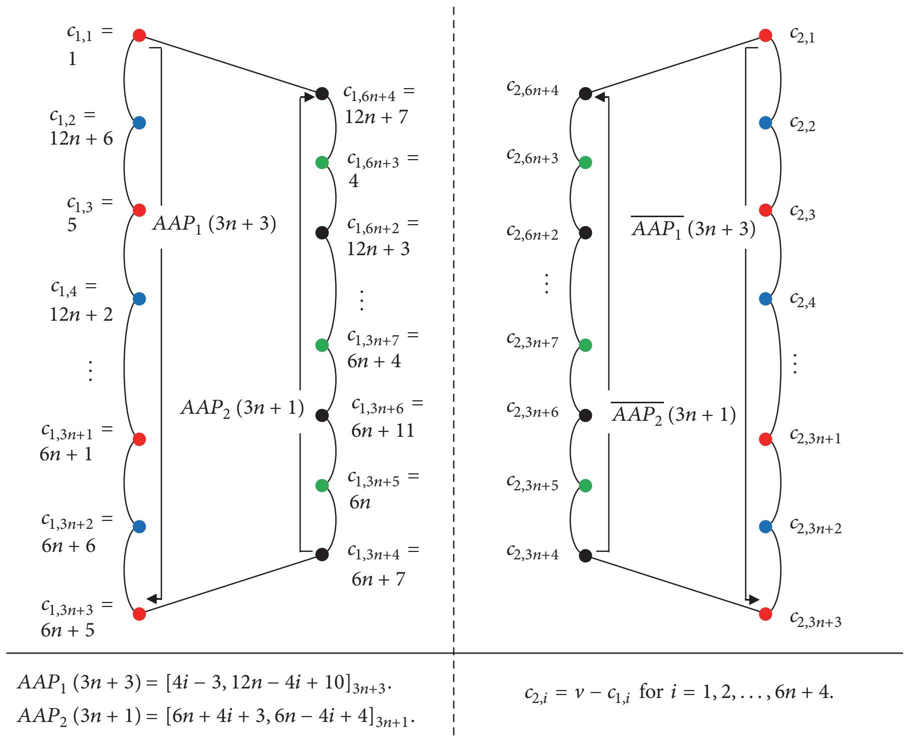


FIGURE 2: The construction of C_1 and C_2 in $2K_v$, n is a nonnegative even integer.

To prove that $\{h \cdot C_1, h \cdot C_2\}$ is a starter of simple cyclic near-resolvable $(6n+4)$-cycle system of $2K_v$, we need to prove the following:

(1) $h \cdot V(\mathcal{S})$ covers $\mathbb{Z}_v^*$.

(2) $\Delta(h \cdot C_1) \uplus \Delta(h \cdot C_2)$ covers $\mathbb{Z}_v^*$ exactly twice.

The first condition is satisfied from Lemma 25. Based on the definition of $h \cdot C_1$, for each edge $\{c_{1,i}, c_{1,i+1}\}$ in C_1 we have that $\{hc_{1,i}, hc_{1,i+1}\} \pmod{v} = \{x_{1,i}, x_{1,i+1}\}$ is an edge in $h \cdot C_1$.

Suppose $d_i = \pm|c_{1,i}, c_{1,i+1}|$ is the difference of the edge $\{c_{1,i}, c_{1,i+1}\}$ for $i = 1, 2, \ldots, 6n+4$, where $c_{1,6n+5} = c_{1,1}$. Then, the difference of the edge $\{x_{1,i}, x_{1,i+1}\}$ is $\pm|x_{1,i} - x_{1,i+1}| = hd_i \pmod{v}$, where $i = 1, 2, \ldots, 6n+4$ and $x_{1,6n+5} = x_{1,1}$.

Since $\mathcal{S} = \{C_1, C_2\}$ is a starter of simple cyclic near-resolvable $(6n+4)$-cycle system of $2K_v$, then $\Delta C_1 \uplus \Delta C_2$ covers $\mathbb{Z}_v^*$ exactly twice. But C_2 is the relative cycle of C_1; this implies that $\Delta C_1 = \Delta C_2 = \{d_i \mid i = 1, 2, \ldots, 6n+4\}$ covers $\mathbb{Z}_v^*$ exactly once. Therefore, $\Delta(h \cdot C_1) = \Delta(h \cdot C_2) = \{hd_i \pmod{v} \mid i = 1, 2, \ldots, 6n+4\}$ also covers $\mathbb{Z}_v^*$ exactly once (from Lemma 15); thus, $\Delta(h \cdot C_1) \uplus \Delta(h \cdot C_2)$ covers $\mathbb{Z}_v^*$ exactly twice. □

In the following example, we construct some of distinct starters of simple cyclic $(9,4,2)$-*NRCS* in accordance with the theorem above.

Example 27. Let h be an integer with $1 \le h < 9$ and $gcd(h,9) = 1$, $G = 2K_9$ with vertices in $\mathbb{Z}_9$ and $\mathcal{S} = \{C_1, C_2\}$ be a set of 4-cycles of G where $C_1 = (1,6,5,7)$ and $C_2 = (8,3,4,2)$.

In Example 21, it is proved that $\mathcal{S} = \{C_1, C_2\}$ is a starter of simple cyclic $(9,4,2)$-*NRCS*. The next table shows the possible values of h that make the set of cycles $\{h \cdot C_1, h \cdot C_2\}$ be a starter of simple cyclic $(9,4,2)$-*NRCS*.

In Table 3, it can be remarked that each set of cycles $\{h \cdot C_1, h \cdot C_2\}$ covers $\mathbb{Z}_9^*$ exactly once and the list of differences from $\{h \cdot C_1, h \cdot C_2\}$ covers $\mathbb{Z}_9^*$ exactly twice. Therefore, we conclude that for any $1 \le h < 9$ such that $\gcd(h,9) = 1$ both sets $\{h \cdot C_1, h \cdot C_2\}$ and $\{(9-h) \cdot C_1, (9-h) \cdot C_2\}$ are the same starter of simple cyclic $(9,4,2)$-*NRCS*.

The simple cyclic near-resolvable $(6n+4)$-cycle system of $2K_{12n+9}$ is a $v \times 2$ array if its starter is a set of full cycles. By the following theorem, we can answer the question whether the starter of simple cyclic near-resolvable $(6n+4)$-cycle system of $2K_{12n+9}$ is set of full cycles or not.

Theorem 28. *If n is a nonnegative integer, then any starter of simple cyclic near-resolvable $(6n+4)$-cycle system of $2K_{12n+9}$ is a set of full cycles.*

Proof. Let C_1 and C_2 be $(6n+4)$-cycles in $2K_{12n+9}$. Suppose that $\mathcal{S} = \{C_1, C_2\}$ is a starter of simple cyclic near-resolvable $(6n+4)$-cycle system of $2K_{12n+9}$.

From Lemma 16, we have that $6n+4$ and $12n+9$ are relatively prime, which implies that the only positive common divisor of $6n+4$ and $12n+9$ is 1. By Lemma 6, since the type of any $(6n+4)$-cycle in $2K_{12n+9}$ is a common divisor of $6n+4$ and $12n+9$, it follows that any cycle in $\mathcal{S}$ is of type 1. Consequently, any starter of simple cyclic near-resolvable $(6n+4)$-cycle system of $2K_{12n+9}$ is a set of full cycles. □

Table 3: A collection of distinct starters of simple cyclic $(9,4,2)$-*NRCS*.

h	$\{h\cdot C_1, h\cdot C_2\}$	$\Delta(h\cdot C_1)\uplus\Delta(h\cdot C_2)$
1 or 8	$\{(1,6,5,7),(8,3,4,2)\}$	$\{\pm5,\pm1,\pm2,\pm6\}\uplus\{\pm5,\pm1,\pm2,\pm6\}$
2 or 7	$\{(2,3,1,5),(7,6,8,4)\}$	$\{\pm1,\pm2,\pm4,\pm3\}\uplus\{\pm1,\pm2,\pm4,\pm3\}$
4 or 5	$\{(4,6,2,1),(5,3,7,8)\}$	$\{\pm2,\pm4,\pm1,\pm3\}\uplus\{\pm2,\pm4,\pm1,\pm3\}$

5. Conclusions

This article has proposed near-resolvable k-cycle system of $2K_v$ as an edge-decomposition of the complete multigraph $2K_v$ into v classes of k-cycles such that each class satisfies the near-2-factor. In particular, the difference method has been exploited to construct a simple cyclic near-resolvable $((v-1)/2)$-cycle system of $2K_v$ for the odd case $v \equiv 9 \pmod{12}$, and this construction has been exemplified for the case $v = 9$. Finally, we have formulated some properties of this construction. We expect that this study can be developed and extended to construct a simple cyclic near-resolvable k-cycle system of $2K_v$ for the case v odd.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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