Width-type graph parameters

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Szeged
2000.

Acknowledgements

I would like to express my gratitude to everybody who helped me to elaborate this thesis. Many thanks to my parents who encouraged me as a child to become a mathematician. Without them this work would never began.

Most of the credits go to my supervisor Péter Hajnal. His idea was to study the subject — Graph Minor Theory and path-width — which I really love now. I learnt a lot from him under the many pleasurable discussions we had. His positive and soft critics gave me some self-confidence. He helped me a lot with his 'natural' questions, and clarified my ideas. Many thanks for his scientific and human guidance.

Also here I would like to thank Robin Thomas for helpful correspondence and discussions. I used his class notes as a very good introduction and reference on the subject.

Finally, I am grateful to my wife Hajni for her love and support while this work was being done.

Financial support from OTKA grant No. F.026049 and F.030737 are grately acknowledged.

Short summary

The title of the thesis is *Width-type graph parameters*. The paper deals with the cornerstone of a modern subject, Graph Minor Theorem (GMT), the tree- and path-width, and the possible generalizations of them. These graph parameters are related to some cops-and-robber games on graphs.

The historical overview of the subject can be found in Chapter 1. We mention that any minor-closed class of graphs can be characterized by a finite list of so-called excluded minors. This fact is a consequence of the GMT (previously called Wagner's conjecture) proved by Robertson and Seymour. The complete proof of the theorem itself is very long and difficult. A key concept of the proof is a minor-monotone graph parameter. This measures the tree-likeness of the graph in some sense, thus it is called tree-width.

Kruskal's theorem says that the trees are well-quasi-ordered by the topological minor relation, hence by the minor relation too. Natural idea is to trace Wagner's conjecture back to Kruskal's theorem. Every graph can be considered as a tree-like structure. Tree-width measures the 'naturalness of this approach'. Trees itself have tree-width one. The main idea of one half in the more than 500 pages proof is to use induction on tree-width from here.

However, tree-width and its variations are important today irrespectively of the above, and began a life of their own.

In Chapter 2 we collected some necessary definitions, which will be used throughout the paper. Beside on that, it is also a 'warm-up' for the reader. We present some easy facts with easy proofs, which can help the reader to get aquinted to the concepts.

The main chapters are arranged as follows:

Chapter 3: Cops-and-robber games
3.1 Cops-and-robber games on graphs
3.2 Cops-and-robber games on directed graphs
3.3 Monotonicity for directed graphs
3.4 Directed path-width
3.5 Blockages for directed graphs

In Chapter 3, we present first the known results and methods related to cops-and-robber games on graphs. The importance of these games is that the arising minor-monotone graph parameters are equivalent to tree- and path-width. Moreover some proofs previously requiring many technical ideas became essentially simpler. In this chapter, we succeeded to generalize the known definitions and results on graphs to directed graphs. Among others we simplified the search of the cops in a game to monotone search. Also the equivalence of the arising cop-number and directed path-width is proven. Finally we could prove the non-existence of a blockage of order k if the directed path-width was less than k. In the undirected case the two claims are equivalent. We conjecture that the reverse implication does probably not hold for directed graphs.

In the chapter we consider two classes of digraphs, which seems to be interesting regarding cops-and-robber games. We give explicitly the exact values of their parameters. Moreover we state a conjecture saying that one of the classes is extremal in some sense.

Chapter 4 uses an equivalent definition of path-width. There are intervals of the real line assigned to the vertices of a graph. If two vertices are adjacent, then the corresponding intervals must intersect. (Not necessarily vice versa.) The width of a point is the number of intervals containing it. The width of such an interval-representation is the maximum width of the points. (Hence the maximum number of pairwise intersecting intervals.) The width of a graph is the minimum width of its interval-representations. This parameter is minor-monotone, and essentially equivalent to path-width. But this language allows us to prove some of the results in this chapter.

The excluded minor characterization of the graphs with path-width two was one of the goals of this thesis. This was considered as a difficult question, and only computer-aided proof existed before. The paper achieved this result by other methods. Introducing certain operations, it considers the minimal graphs respect to an ordering finer than the minor relation. We had to prove that the introduced operations preserve path-width. Instead of the 110 excluded minors, we describe the same class with 10 excluded non-reducible graphs. Also a linear-time recognition algorithm arose.

In Chapter 5 another graph-representation turns up. Here the vertices correspond to the arcs of a base circle. The width of a point on the base circle is the number of arcs containing it. The width of an arc-representation is the maximum of the width of the points. The arc-width of a graph is then the minimum width of such an arc-representation. This parameter is minor-monotone as well. Hence it is possible to formulate some excluded minor theorems.

Interesting and important is that in magnitude the arc-width of a given graph is between the path-width and its half. Equality holds e.g. for trees. The complete graphs realize the other end.

One of the curiosities of the chapter is the exact determination of the arc-width of the complete bipartite graph. Also here it is necessary to give — in some sense — good constructions.

Finally we present some of the possible excluded minor theorems. Important is that also non-connected graphs can be excluded minors. Among the obstructions for a certain class, unexpectedly the Kuratowski graphs turn up.

The results of Chapter 4 and 5 are partially joint with the supervisor.

In the last part of the thesis those open problems are collected, which the author met in his research. A lot of our own questions can be found among these.

Rövid összefoglaló

A doktori disszertáció címe *Szélesség típusú gráfparaméterek*. A dolgozat egy modern téma, a Gráf Minor Tétel (GMT) sarokkövének számító fogalommal a fa- és út-szélességgel, valamint annak lehetséges általánosításaival foglalkozik. Ezen gráfparaméterek kapcsolatban állnak gráfokon értelmezett rabló-pandúr játékokkal.

Az 1. fejezetben a téma történeti áttekintése található. Megemlítjük, hogy gráfok bármely minor-zárt osztálya karakterizálható úgynevezett kizárt minorok egy véges listájával. Ez a tény a Robertson és Seymour által bizonyított GMT (korábban Wagner-sejtés) következménye. Maga a tétel teljes bizonyítása nagyon hosszú és nehéz. A bizonyítás egyik kulcsfogalma egy minor-monoton gráfparaméter. Ez bizonyos értelemben a gráf faszerűségét méri, ezért is hívják fa-szélességnek.

Kruskal tétele azt mondja ki, hogy a fák jól-kvázi-rendezettek a topologikus részgráf relációra nézve, így a minor relációra nézve is. Természetes ötlet az, hogy a Wagner sejtést vezessük vissza Kruskal tételére. Minden gráf felfogható egy faszerű struktúrának. A fa-szélesség azt méri, hogy ez a 'felfogás mennyire természetes'. A fák fa-szélessége 1. A több mint 500 oldalas bizonyítás egyik felének alapötlete, hogy innen indulva teljes indukciót alkalmazzunk a fa-szélesség szerint.

A fa-szélesség és különböző változatai ma már ettől függetlenül is fontosak, és önálló életet élnek.

A 2. fejezetben a szükséges definíciókat gyűjtöttük össze, amelyeket a dolgozatban használunk. Ezen kívül ez egy 'bemelegítés' is az olvasónak. Néhány könnyű állítást mondunk itt ki könnyű bizonyításokkal, ami segíthet megszokni a fogalmakat.

A fő fejezetek a következő képpen tagolódnak:

3. Fejezet: Rabló-pandúr játékok

3.1 Rabló-pandúr játékok gráfokon

3.2 Rabló-pandúr játékok irányított gráfokon

3.3 Monotonitás irányított gráfokra

3.4 Irányított útszélesség

3.5 Blokádok irányított gráfokban

4. Fejezet: Kettő út-szélességű gráfok karakterizációja

4.1 Alapok

A 3. fejezetben először a gráfokra ismert rabló-pandúr játékokkal kapcsolatos eredményeket és módszereket ismertetjük. Ezen játékok fontosságát az adja, hogy a származtatott minor-monoton gráfparaméterek ekvivalensek a fa- ill. út-szélességgel. Ezen felül több, korábban sok technikai ötletet igénylő bizonyítást lényegesen leegyszerűsített. A fejezetben a gráfokra ismert definíciók és eredmények általánosítása sikerült irányított gráfokra. Többek között egyfajta játékban a pandúrok keresésének egyszerűsítése monoton kereséssé. Valamint ezen játékból származó pandúrszám ekvivalenciája az irányított út-szélességgel is bizonyításra kerül. Végül sikerült bizonyítani, hogy legfeljebb k irányított út-szélességű gráfban nem lehet k-nál nagyobb rendű blokád. Irányítatlan esetben a két állítás ekvivalens. Sejtésünk azonban az, hogy irányított gráfokra a fordított irány talán nem is igaz.

A fejezetben két olyan irányított gráfosztályt is vizsgálunk, ami érdekes lehet a rabló-pandúr játékok szempontjából. Explixcit megadjuk a megfelelő paraméterek értékeit. Továbbá megfogalmazunk egy sejtést, ami azt állítja hogy az egyik osztály extremális bizonyos szempontból.

A 4. fejezet az út-szélesség egy ekvivalens definícióját használja. Egy gráf csúcsainak a számegyenes bizonyos intervallumai felelnek meg. Ha két csúcs között van él, akkor a nekik megfelelő intervallumok metszik egymást. (Fordítva nem feltétlenül igaz.) Egy pont szélessége az őt tartalmazó intervallumok száma. Egy ilyen intervallum-reprezentáció szélessége a pontok szélességének maximuma. (Azaz az egymást páronként metsző intervallumok maximális száma.) Egy gráf szélessége pedig a minimális szélességű reprezentáció szélessége. Ezen paraméter minor-monoton, és lényegében ekvivalens az út-szélességgel. Ez a nyelvezet teszi azonban lehetővé a fejezet bizonyos eredményeinek bizonyítását.

A kettő út-szélességű gráfok kizárt minorokkal való karakterizációja volt a dolgozat egyik célja. Ez nehéz kérdésnek számított, és csak számítógéppel támogatott bizonyítása volt korábban. A dolgozat más módszerrel érte el az eredményt. Bizonyos operációk bevezetésével a minor relációnál finomabb rendezésben nézi a minimális gráfokat. A bevezetett operációkról bizonyítani kellett, hogy megőrzik az út-szélességet. Így 110 kizárt minor helyett 10 kizárt nem-redukálható gráffal írjuk le ugyanazt az osztályt. Egy lineáris idejű felismerési algoritmus is adódott.

Az 5. fejezetben a gráfok egy másik reprezentációja kerül elő. Itt a csúcsoknak egy alapkör ívei felelnek meg. Az alapkör egy pontjának szélessége az őt tartalmazó ívek száma. Egy ív-reprezentáció szélessége a pontok szélességének maximuma. Egy gráf ív-szélessége pedig a minimális szélességű ív-reprezentáció szélessége. Ezen paraméter is minor-monoton. Így lehetőség nyílik pl. kizárt minoros tételek megfogalmazására.

Érdekes és fontos, hogy nagyságrendileg egy adott gráf ív-szélessége az út-szélessége és az út-szélesség fele közé esik. Fákra pl. egyenlőség áll fenn. A teljes gráfok pedig a másik végletet realizálják.

A fejezet egyik különlegessége a teljes páros gráf ív-szélességének pontos meghatározása. Itt is szükség van bizonyos szempontból jó konstrukciók megadására is.

Végül a lehetséges kizárt minoros tételek közül mutatunk be néhányat. Fontos hogy nem-összefüggő gráfok is szerepelnek itt kizárt minorként. Az egyik osztály akadályai között váratlanul felbukkanak a Kuratowski gráfok is.

A 4-5. fejezetek eredményei részben a témavezetővel közösek.

A disszertáció utolsó részében azon megoldatlan problémák kerülnek felsorolásra, melyekkel a szerző kutatásai során találkozott. Ezek között nagy számban találhatók saját kérdések is.

Contents

Chapter 1

History and introduction

Kuratowski's theorem says, that a graph is planar iff it has no minor isomorphic to K_5 or $K_{3,3}$. This theorem gives a good characterization of planarity. There is a large collection of similar results in Graph Theory, called excluded minor theorems. Actually any minor-closed class of graphs can be characterized by a finite list of so called excluded minors. This fact is a consequence of the celebrated Graph Minor Theorem(GMT) of Robertson and Seymour [27]. The results of the Graph Minor project became known in the beginning of the 80's. The goal was to prove a theorem, often quoted as 'Wagner's Conjecture'. This states that a given infinite list of finite graphs always contains two graphs s.t. one is minor of the other. With other words, the finite graphs are well-quasi-ordered by the minor relation.

However, the proof of this single theorem is very involved. The project resulted in more than 20 deep and long papers. The above stated result seems to be completely proven in part XX. The sum of the length of the papers is certainly over 500 pages. But there are other results proven on the way to the main theorem. Until today 17 of the at least 20 papers have been published. This allows us to try to follow their ideas, and believe that the main result is true. A lot of special cases are already proven in the early papers (part I,III-V). For example if one assumes that at least one graph is planar, then they are done after the fifth paper. One key-ingredient of the proof is a minor-monotone graph parameter. In some sense, it measures the tree-likeness of the graph, and it is called tree-width. Kruskal's theorem says that the trees are well-quasi-ordered by the topological minor relation. Hence the concept of tree-width is very natural. The idea is to trace Wagner's conjecture back to Kruskal's theorem through tree-width. However, tree-width and its variations — we call them width-type parameters— began a life of their own too.

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We tried to write the paper in the following style. After the introduction of a new concept we always give some easy examples to make the understanding easier for the reader. We tried to give the proofs in more detail than in a journal paper. We attached lot of drawings, but even more can be necessary for a conscientious reader. We always tried to emphasize the clear heuristic of the proofs beside on the formalisms.

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In Chapter 2 we collected some necessary definitions, which will be used throughout the paper. Beside on that, Chapter 2 is also a 'warm-up' for the reader. We present some easy facts with easy proofs, which can help the reader to get aquinted to the concepts.

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Path-width is a variation of tree-width. This parameter is the most discussed one in our paper. The path-width is equivalent to a number arising from a kind of 'cops-and-robber' game. This approach is very nice. It can make the explicit determination of the path-width of graphs much easier. This aspect is discussed in Section 3.1. One important feature of such a game is monotonicity.

Recently many attempts were made on generalizations of the above concepts to directed graphs. The paper [2] is one of them, focusing on the cops-and-robber game point of view. In Section 3.2 we introduce several possible games on directed graphs. We show some of their properties, and the relations between them. The motivation is always the analogy to the undirected case. In one case we can prove a monotonicity result too. This result is described in Section 3.3.

We also define directed path-width (dpw), first suggested by Reed, Seymour and Thomas. We can associate a game to this parameter too. This approach is discussed in Section 3.4.

The last attempt is the generalization of the results of [7]. Here we show that if dpw of D is at most $k-1$, then there can not be an obstruction structure called blockage in D of order k. This is Section 3.5.

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Minor-closed families of graphs naturally arise in graph theory. There are natural classes like outerplanar, series-parallel, planar, linklessly embeddable graphs. Any minor-monotone graph parameter (e.g. the genus, the tree-width, the path-width, or the Colin de Verdière number of the graphs) defines a sequence of minor closed families by limiting from above the value of the parameter by any natural number t.

As we increase the value of t, the class grows and usually the number of excluded minors increases rapidly with t. So the determination of the excluded minors becomes harder and harder. The graphs with tree-width at most three requires four excluded minors [1]. D.P. Sanders in his Ph.D. thesis determined more than 75 minimal forbidden minors for tree-width at most four, but the list was still incomplete. Planar graphs require two excluded minors, nobody undertook the task of explicitly list the excluded minors for toroidal graphs. The class of graphs with path-width at most one require two excluded minors, N.G. Kinnersley in her Ph.D. thesis determined the 110 minimal forbidden minors for path-width at most two. It is known [30], that the number of excluded tree minors for the class of graphs with path-width at most t grows superexponentially.

It looks like that a very small limit on natural minor-monotone graph parameters creates classes with extreme complexity. This happens although the considered classes are important, since many NP-hard optimization problems can be solved for their members efficiently. Their recognition and representation problems are important.

One of our goals is to exhibit that characterizations by excluded minors are very often not appropriate for certain classes. This task will be considered mainly in Chapter 4. We introduce new operations for the class of graphs with path-width at most two. (Most of the operations work in general.) Based on these operations, we can significantly reduce the number of excluded structures.

The novelty of our approach is, that it gives a much better insight into the considered class, than an overwhelming list of more than 100 graphs. There is another point. In [15], Kinnersley and Langston wrote the following about finding the excluded minors for path-width two: "To assist in this heroic undertaking, massive computational power was used to verify that each obstruction represents a circuit that has no three-track layout, and to check that each proper minor of each obstruction represents a circuit that does have a three-track layout." Checking by computers proves the correctness of the result, but it does not give a good understanding of the considered theorem. Unlike this, we do not need computers to handle huge lists. Our operations do the job for us. On the list of Kinnersley and Langston many graphs look alike, the computer handles each of them independently. Our operations exhibit them as simple variations of some base graphs.

Another reason for publishing a new proof for a forbidden minor characterization is that our proof is more systematic, and hence it leads to a linear algorithm for testing the property 'having path-width at most two'.

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In Chapter 4 we represent the vertices of graphs with intervals on the real line. This lead us to path-width. In Chapter 5 we represent the vertices of a graph with arcs of a base circle. This is a very natural modification of path-width, called arc-width. We describe some of its basic properties. These two measures are similar in some sense, but different in other cases. Interesting and important is that in magnitude the arc-width of a given graph is between the path-width and its half. Equality holds e.g. for trees. On the other hand the arc-width of K_n is roughly $\frac{n}{2}$ and its path-width is $n-1$. We also present a proof of the most attractive result in this field so far. Namely we determine the arc-width of the complete bipartite graph.

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Throughout the paper we state some conjectures. But at the end we collect them together, and also add some open questions of other researchers.

Chapter 2

Notations, definitions, and basic facts

We only consider finite graphs. In most cases when we say graph, we mean a simple, connected, undirected graph. In our case it is very natural to do so. The vertex set of G is referred to as $V(G)$, its edge set as $E(G)$. $|V(G)| = n$ denotes the number of vertices. $uv \in G$ is an edge of G with endvertices u and v.

Let G be a graph and X is a set of vertices or edges. $G \setminus U$ is the graph that we obtain by deleting X. If U is a vertex set, $G|_U$ is the *induced subgraph* of G by U. $E(U)$ is the set of edges incident to any element of U. Let F be a set of edges. $V(F)$ is the set of vertices incident to at least one member of F. $G|_F$ is the graph induced by the edge set F, i.e. its vertex set is $V(F)$ and its edges set is F.

A graph H is a *minor* of a graph G, $G \succeq H$ in notation, if H can be obtained from a subgraph of G by contracting edges. While contracting an edge multiple edges can arise. To keep the graph simple, we only keep one edge in those cases. A class of graphs is called *minor-closed* if for every graph G in the class, every minor of G is also a member of the class. A *graph parameter* is a graph property which is expressed by natural numbers. A graph parameter π is called *minor-monotone* if $G \succeq H$ implies $\pi(G) \geq \pi(H)$.

A *path-decomposition* of a graph G is a pair (P, W), where P is a path and $W = (W_p : p \in V(P))$ is a family of subsets of $V(G)$, satisfying

(1) $\bigcup_{p \in V(P)} W_p = V(G)$, and every edge of G has both ends in some W_p, and

(2) if $p, p', p'' \in V(P)$ and p' lies on the path from p to p'', then $W_p \cap W_{p''} \subseteq W_{p'}$.

(the sets W_i are usually called bags)

The *width* of a path-decomposition is $\max(|W_p| - 1 : p \in V(P))$, and the *path-width* of G ($pw(G)$ in notation) is the minimum width over all path-decompositions of G. The -1 in the definition of width is only by tradition. (In this way any path has path-width one.) Hence we use the notation $pw^*(G)$ for $\max(|W_p| : p \in V(P))$, as well.

There are several alternative definitions for path-width (see [20], [22], [32]). For us one is especially important. An *interval representation of a graph* G (or simply representation of G) is a function ϱ, that assigns closed intervals of the real line l to the vertices of G, such that adjacent vertices correspond to intersecting intervals. We say that a point P of l is covered m times in a representation iff P is an element of exactly m intervals assigned to vertices. The *width* of the representation is the maximal m such that there exists a point which is covered m times. The path-width of the graph G is one less than the width of its minimal width representation. That is why we sometimes refer to $pw(G) + 1$ as $pw^*(G)$. A minimal width representation is also called *optimal*.

We will use the above description as the definition of path-width. Let us show its equivalence to the original definition by Robertson and Seymour.

Lemma 2.1 *The following are equivalent:*

(i) $pw(G) \leq k$

(ii) G is a subgraph of an interval graph H, which has maximum clique-size at most $k + 1$

(iii) G has an interval representation with width at most $k + 1$.

The equivalence of (ii) and (iii) is obvious. To prove (i) $\Leftrightarrow$ (iii), first we need a classical lemma:

Lemma 2.2 *(Gilmore and Hoffman 1964) [11] A graph G is an interval graph iff the maximal cliques of G can be linearly ordered $C_1, \ldots, C_t$ s.t. if $v \in C_i \cap C_{i''}$ and $1 \leq i \leq i' \leq i'' \leq t$, then $v \in C_{i'}$.*

Using this result, the equivalence in Lemma 2.1 is fairly easy to see.

Lemma 2.3 $pw(G) \leq k$ *iff G is a subgraph of an interval graph H that has maximum clique-size $k + 1$.*

Proof: If H is given, the maximal cliques $C_1, \ldots, C_t$ of H can play the role of the bags in the definition of a path-decomposition. Then by Lemma 2.2 the crucial (2) of the definition is automatically satisfied. The maximum size of a bag will be $k + 1$, hence $pw(G) \leq pw(H) \leq k$.

If $pw(G) \leq k$, then every vertex v of G appears in consecutive bags, and the maximum size of a bag is $k+1$. Let us define an interval graph as follows. If v appears in bags $W_i, \ldots, W_j$, then let the interval $[i, j]$ correspond to v. In this way we get an interval graph H with maximum clique-size $k+1$. If $uv \in E(G)$, then u and v are in a common bag, W_j say. Hence j is in both intervals corresponding to u and v. Hence G is a subgraph of H. □

A *tree-decomposition* of a graph G is a pair (T, W), where T is a tree and $W = (W_t : t \in V(T))$ is a family of subsets of $V(G)$, satisfying

(1) $\bigcup_{t \in V(T)} W_t = V(G)$, and every edge of G has both ends in some W_t, and

(2) if $t, t', t'' \in V(T)$ and t' lies on the path from t to t'', then $W_t \cap W_{t''} \subseteq W_{t'}$.

The *width* of a tree-decomposition is $\max(|W_t| - 1 : t \in V(T))$, and *tree-width* of G is the minimum width over all tree-decompositions of G. Trees themselves have tree-width one. This is the base of the traditional -1 in the definition of width.

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A *separation* of a graph G is a triple $(A, B, (v_1, v_2, \ldots, v_k))$ where A, B are subgraphs of G with $A \cup B = G$, $E(A) \cap E(B) = \emptyset$, and $V(A) \cap V(B) = \{v_1, v_2, \ldots, v_k\}$. We should think of a separation as representing G as a result of a gluing: we can obtain G from A by gluing to it a copy of B by identifying the corresponding vertices.

Let a *structure* S be a pair $(G(S), (u_1, u_2, \ldots, u_j))$, where $G(S)$ is a graph, and $u_1, u_2, \ldots, u_j$ are distinct vertices of $G(S)$, called the *vertices of attachment* of S. A graph G has a structure S, if G has a separation $(A, B, (v_1, v_2, \ldots, v_k))$ where $(B, (v_1, v_2, \ldots, v_k))$ is isomorphic to S.

A *reduction* R is a pair of structures, S_R and T_R, with the same sequence of vertices of attachment and $|V(S_R)| > |V(T_R)|$. Let G be a graph which has the structure S_R. Then we say that the reduction R can be performed on G. The result of the reduction is a graph H that can be obtained from a proper separation of G by replacing the side isomorphic to S_R by T_R.

We need some relaxations of the formal definition above. A *variation* R is a pair of structures, S_R and T_R, with the same sequence of vertices of attachment and $|V(S_R)| = |V(T_R)|$. Sometimes we allow the performance of a reduction or a variation only in case the corresponding separation satisfies some additional requirements. (For example some degree conditions on the vertices of attachment, or containment of a cycle on the side of the separation that is not changed during the reduction or variation). We can call them *conditional* reduction, conditional variation. From now on we use the notion of *operation* instead of the words reduction, variation, conditional reduction, conditional variations.

For an operation R, define the following partial order: $H \leq_R G$ if there is a sequence of graphs $H = G_1, G_2, \ldots, G_k = G$ such that for every $i < k$, G_i is obtained from G_{i+1} by performing R. Naturally we can define $\leq_{\mathcal{R}}$ if we have a set $\mathcal{R}$ of operations by $H \leq_{\mathcal{R}} G$ if $G_i \leq_R G_{i+1}$ for some $R \in \mathcal{R}$ for each i. If A is a class of graphs, an operation R is *A-monotone*, if for all graphs G, H satisfying $H \leq_R G$, if $G \in A$ then $H \in A$. An operation R is *A-safe* if for all graphs G, H satisfying $H \leq_R G$, $G \in A$ if and only if $H \in A$.

By saying a graph *has* or *contains* an operation R, we mean that the operation can be performed on it. A graph G is reducible if G has a reduction. $R(G)$ abbreviates the result of a performance of R on G.

$\star$

It is well-known, that "being identical or being on the same cycle" is an equivalence relation on the edges of a graph. Its equivalence classes span the so called *blocks* of the graph. The one element classes are the loops (what we do not have since we consider simple graphs) and the cut edges.

$\star$

A *rooted graph* (G, r) is a graph G with a specific node $r \in V(G)$, that is called the root of G.

We denote a directed edge by (u, v). This abbreviates that u is the tail and v is the head of the directed edge. If we have both (u, v) and (v, u) in a digraph G, then to simplify notation, we substitute them by an undirected edge uv. In this sense an undirected graph G can correspond to a directed graph, where every edge of G is replaced by two oppositely directed edges.

Chapter 3

Cops-and-robber games

3.1 Cops-and-robber games on graphs

In this section we consider different type of cops-and-robber games. We describe their relation to the different width-type parameters. In some cases the very strong monotonicity condition hold. This makes life much easier. Especially if the graph has some symmetry.

There are a number of variations of the game. The graph itself can be directed or undirected, the robber can be visible or invisible. (In this paper we restrict ourselves to finite graphs. The infinite case was also investigated in a number of papers.)

Definition 3.1 *Let G be a graph. There is a robber standing on a vertex of G. There are k cops willing to capture the robber. The robber can run at any time to another vertex along edges with great speed. The movement of the cops is only possible by helicopter, but they can fly to an arbitrary vertex. Let the robber be invisible for the cops. (We can think that the surface is covered by forests.) So the cops cannot see the robber from the helicopter. The meaning of the great speed is as follows. When the robber see a helicopter approaching a vertex, he can still decide to run somewhere. However the robber cannot run through a vertex which is occupied by a cop. The cops can only capture the robber if they occupy all neighbors of the vertex where the robber is standing, and then with one extra cop they capture the robber.*

If there is a winning strategy for k cops, we say that 'there is a capture with k cops', or 'k cops can search the graph'.

The goal is to decide how many cops are necessary to capture the robber. This minimum is denoted by $\overline{cn}(G)$. (cn stands for cop number, overline for the invisible case.)

The above statements can be formalized as follows.

A *position* is a pair (X, R), where X is the set of vertices occupied by cops, and R is the set of vertices, where the robber could be. A game (capture) is a sequence of positions. We set (X_0, R_0) to an initial position. In a normal game $X_0 = \emptyset$, and R_0 is the hole vertex set of G. In general at the start of step i, we have a position (X_{i-1}, R_{i-1}). The cop player chooses a new set X_i s.t. either $X_i \subseteq X_{i-1}$ or $X_{i-1} \subseteq X_i$. Then the robber's place satisfy $R_{i-1} \subseteq R_i$ or $R_i \subseteq R_{i-1}$ respectively. More precisely when $X_{i-1} \subseteq X_i$, then $R_i = R_{i-1} \setminus (X_i \setminus X_{i-1})$. When $X_i \subseteq X_{i-1}$, then $R_i = \{v :$ there is a path in $V(G) \setminus V(X_i)$ from v to a vertex a, where $a \in R_{i-1}\}$. If any time $R_i = \emptyset$, then the cop player won, moreover if $|X_i| \leq k$ for every i, then k cops are enough to capture the invisible robber. The minimal such k is called $\overline{cn}(G)$.

Lemma 3.2 *$\overline{cn}(G)$ is a minor-monotone graph parameter.*

Proof: Let G' be a graph arising from G by an edge-deletion or edge-contraction. If $\overline{cn}(G) \leq k$, then we know that there is a search with at most k cops. Consider the same search for the vertices of G'. Formally in the case of a contraction, some vertices $u, v \in G$ become a new vertex $n \in G'$. So whenever $u \in X_i$ or/and $v \in X_i$ for some i, then put $n \in X_i$ instead. This way we get a capture for G' with at most k cops. □

Definition 3.3 *We say that a capture (game/search) is monotone, if the cops have not visited any vertex more than once. Formally if $\emptyset = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n = V(G)$.*

Remark 3.4 *A graph G is called a caterpillar iff it is a path with pendant edges attached to some of its vertices.*

Lemma 3.5 *Two cops can capture an invisible robber in a connected graph G iff G is a caterpillar. Equivalently iff $G \not\succeq K_3, Y_1$, where Y_1 is the graph on Figure 3.1.*

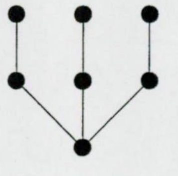


Figure 3.1:

Proof: **Proof:** Assume $P = v_1, \ldots, v_n$ is a longest path in G, and let $N_1(v_i)$ denote the 1-valent neighbors of v_i. In step 0. put the first cop onto v_1, the second cop is now called free. In the beginning of step i, put the free cop to v_{i+1}, where v_i is the last occupied vertex of P; this cop is not free now, and call the other cop free instead. In step i., the free cop subsequently occupies all the vertices of $N_1(v_{i+1})$. After doing this, step i. finishes. By this process the robber will be captured after step $n-1$.

The opposite direction is trivial, because K_3 and Y_1 require at least three cops each. □

Lemma 3.6 *If G has path-width at most $k-1$, then k cops can capture an invisible robber in G. In notation: $pw(G) \leq k-1 (\Leftrightarrow pw^*(G) \leq k) \Rightarrow \overline{cn}(G) \leq k$*

Proof: Mimic the proof of the case, when G is a path and there are two cops. $pw(G) \leq k-1$ means that every bag has size at most k ($|W_i| \leq k$), and we have k cops. Occupy the first bag with the cops. From the definition of path-decomposition it follows, that if we leave the cops in $W_1 \cap W_2$, and move the cops from $W_1 \setminus W_2$ to $W_2 \setminus W_1$, then the robber cannot move back to $W_1 \setminus W_2$. If we iterate this process, finally the robber will be captured. □

The opposite direction is proved by the following equivalence theorem. In this form it was published and proved by Bienstock et al. in [7]. We recall this without proof. Hence $pw^*(G) = \overline{cn}(G)$. At the same time, the crucial implication $(i) \Rightarrow (iii)$ was also proved by Bienstock and Seymour in [6]. Their proof idea will be heavily used and followed in Section 3.3.

Theorem 3.7 *For a graph G and a positive integer k, the following are equivalent:*

(i) k cops are enough to capture the invisible robber,

(ii) G has path-width at most $k-1$,

(iii) there is a monotone capture with at most k cops.

Monotonicity is very useful for us. Let us color by red the vertices which are reachable for the robber. Let the other vertices have blue color. In any step we can only move such a cop, whose vertex has only blue neighbors. Call such a cop, or a cop which is temporarily not in the game, free. If we want to decide whether k cops are enough, we have to do the following. In the beginning we have to occupy a vertex and all its neighbors. Then in any step we can move with the free cops. If we are stack (i.e. there is no free cop available), and there are still red vertices, then k cops were not enough with this starting vertex. When the graph has not too many vertices, or it has a kind of symmetry, then we can check all starting points easily. By this checking we determined the path-width of the graph, which is usually a difficult problem. In Chapter 4 we consider the case with three cops. Already that one is a hard nut.

⋆

Modify the previous game — defined by Definition 3.1 — only at one condition. This small difference drastically changes the situation.

Definition 3.8 *Copy Definition 3.1 in mind, but assume the cops can see the robber all the time. The number of necessary cops to capture the robber is denoted by $cn(G)$.*

Remark 3.9 *Minor-monotonicity of $cn(G)$ can be proved similarly as it was done for $\overline{cn}(G)$.*

Lemma 3.10 *Two cops can capture a visible robber in a connected graph G iff G is a tree. Equivalently iff G has no K_3-minor.*

Proof: If there is a cycle C in G, then K_3 is a minor of G. Trivially in K_3 the robber can only be captured with three cops. Hence two cops cannot capture the robber in G, by the minor-monotonicity.

If G is a tree, then we can define a sequence of cop-moves after which the robber is captured. Put one cop on a vertex v. See which component of $G \setminus v$ contains the robber. Transport the second cop to the neighbor of v in that component. Iterate this process with exchanging the role of the cops. In a moment the robber will be forced to be on a leaf, and be captured in the next move. □

The next statement formalizes the nice connection between tree-width and the cop-parameter *cn*.

Lemma 3.11 *If G has tree-width at most $k-1$, then k cops can capture a visible robber. Hence $tw(G) \leq k-1$ implies $cn(G) \leq k$.*

Proof: Mimic the proof for trees. □

The opposite direction can again be proved via an equivalence theorem. This is a result by Seymour and Thomas [29]. We recall it without proof.

Theorem 3.12 *[29] Let G be a graph, and k be a positive integer. Then the following are equivalent:*

(i) k cops are enough to capture a visible robber

(ii) k cops can monotonely capture a visible robber

(iii) G has tree-width at most $k-1$

Recently some of the above (well-studied) concepts were generalized to directed graphs. We try now to analyze this effort in the next section from the cops-and-robber game point of view.

3.2 Cops-and-robber games on directed graphs

There are several possible ways to define cops-and-robber games on directed graphs. For us the most natural one is the following:

Definition 3.13 *Let a directed graph D be given. The robber can run along the directed edges in the indicated direction. The cops move by helicopters. Assume in this version, that the robber is invisible. The goal is to decide how many cops are necessary to capture the robber. Denote this minimum by $\overline{cn^*}(D)$. (cn stands for cop number, overline for the invisible case, * indicates that the robber's move is not restricted like in Definition 3.19.)*

First we mention some basic properties of this new parameter.

Lemma 3.14 *One cop is enough to capture the invisible robber in D, iff D has no directed circuit as a subgraph, hence D is acyclic.*

Proof: Trivially the robber escapes from one cop on a directed circuit. Hence the condition is necessary.

No directed circuit means that we can number the vertices in such a way, that any directed edge goes from a bigger number to a smaller one. Hence one cop travelling in decreasing order on the vertices captures the robber. □

Lemma 3.15 *(a) If $A \subset V(D)$, $|A| = k$ is a set of vertices s.t. $V \setminus A$ has no directed circuit, then $\overline{cn^*}(D) \leq k+1$*

(b) Let D be a directed circuit with at least two vertices. Then $\overline{cn^*}(D) = 2$

Proof: (a) Occupy the set A with k cops. $V \setminus A$ is now a graph which can be searched with one cop by the previous lemma. Hence $k+1$ cops are enough to capture the invisible robber in D.

(b) One cop is not enough by the previous lemma. Two cops are enough by part (a). □

To be able to consider minor-monotonicity, one wishes to extend the minor operation to digraphs. This can be done in the natural way. Any edge can be deleted, any edge can be contracted. After the contraction of an edge, we delete the multiple edges.

Definition 3.16 *A directed graph D is a minor of a directed graph F, if D can be obtained from F by using edge-deletions and/or edge-contractions.*

We can now ask ourselves whether $\overline{cn^*}$ is minor-monotone? This is clearly not. Take a directed circuit. Revert one edge. The graph obtained in this way requires only one cop. But if we contract the reverted edge, we got a directed circuit which needs two cops. Of course there are more involved examples. In this situation there are two ways to choose from. Either we introduce a different minor operation for which the cop-number is monotone. Or we keep the minor operation as it is natural, and consider this cop-number parameter however. We concentrate on the second variation.

The notion of directed path-width came up in a joint work of Bruce Reed, Paul Seymour and Robin Thomas:

Definition 3.17 *One can define a directed path-decomposition (dpd) as a sequence W_1, $W_2, \ldots, W_k$ such that*

(i) the union of W_i is $V(D)$, and

(ii) if $i < j < k$, then $W_i \cap W_k$ is a subset of W_j, and

(iii) an edge either has both endpoints in the same W_i or has its head in W_i and tail in W_j, where $i \le j$.

The width of a dpd is the maximum size of a W_i minus one. (The W_i's are called bags again.) The directed path-width (dpw) of a digraph D is the minimum width over all possible dpd's.

$$dpw = \min_{W_i \text{ is a } dpd} \left(\max_{1 \le i \le k} (|W_i| - 1) \right)$$

We can call dpw a generalization of path-width to directed graphs, more precisely the following is true:

Lemma 3.18 *Let G be a graph, and let D be the graph obtained from G by replacing every edge by two directed edges in opposite directions. Then the path-width of G is equal to the directed path-width of D.*

Proof: Assume first that a path-decomposition of G is given. Every edge of G is in some W_i by definition. Hence if we make the replacing to get D keeping the W_i's unchanged, we get a directed path-decomposition as well with the same width.

Assume now that a dpd of D is given. Suppose there is a directed edge (u, v), $u \in W_i$ and $v \in W_j$ s.t. $i < j$. But also (v, u) is a directed edge by assumption, contradicting (iii) of Definition 3.17. Hence every directed edge is in some W_i, so the W_i's give a path-decomposition of G with the same width. □

Analog to the undirected case, if $dpw(D) \leq k-1$, then k cops are enough to capture an invisible robber in D. Hence $dpw(D) \leq k-1$ implies $\overline{cn}^*(D) \leq k$. The opposite implication will be discussed in Section 3.3.

The visible robber version can also be easily defined. The minimum number of necessary cops in that case is denoted by $cn^*(D)$.

Let us consider another version of cops-and-robber games appearing in [14].

Definition 3.19 *Let a directed graph D be given. The cops are either standing on a vertex or in a helicopter (temporarily removed from the game). The robber stands on a vertex of D, and can at any time with great speed run to another vertex in the same strong component of $D \setminus Z$, where Z is the set of vertices occupied by the cops. In other words, the robber can only move from a to b, if there is also a cop-free directed path from b to a. The goal is to decide how many cops are necessary to capture the robber. Denote this minimum by $cn(D)$ if the robber is visible, and $\overline{cn}(D)$ if the robber is invisible.*

In the aforementioned [14] only the visible case was considered. There are some trivial connections between the so far defined parameters, which we indicate on the next figure.

Remark 3.20

$$\begin{array}{ccc} cn^*(D) & \leq & \overline{cn}^*(D) \\ \vee\!\mid & & \vee\!\mid \\ cn(D) & \leq & \overline{cn}(D) \end{array}$$

These facts are easy to see, and these inequalities will be used henceforth without any further references.

In the study of cops-and-robber games on directed graphs, some special class of graphs turned out to be interesting. Let us discuss these examples:

Definition 3.21 *[14] For $k = 1, 2, \ldots$ let J_k be the union of k directed circuits $C_1, C_2, \ldots, C_k$ of length $2k$, and $2k$ directed paths $P_1, P_2, \ldots, P_k$ of length k resp. $Q_1, Q_2, \ldots, Q_k$ of length k. Here for $i = 1, 2, \ldots k$ C_i has vertex set $\{u_{i,1}, u_{i,2}, \ldots, u_{i,k}, v_{i,1}, v_{i,2}, \ldots, v_{i,k}\}$ (in order), P_i has vertex set $\{u_{i,1}, u_{i,2}, \ldots, u_{i,k}\}$ (in order), and Q_i has vertex set $\{v_{i,1}, v_{i,2}, \ldots, v_{i,k}\}$ (in order). Thus J_k has a planar drawing, where the circuits are concentric, the P's are disjoint paths linking C_1 to C_k, and the Q's are disjoint paths linking C_k to C_1. (See Figure 3.2.)*

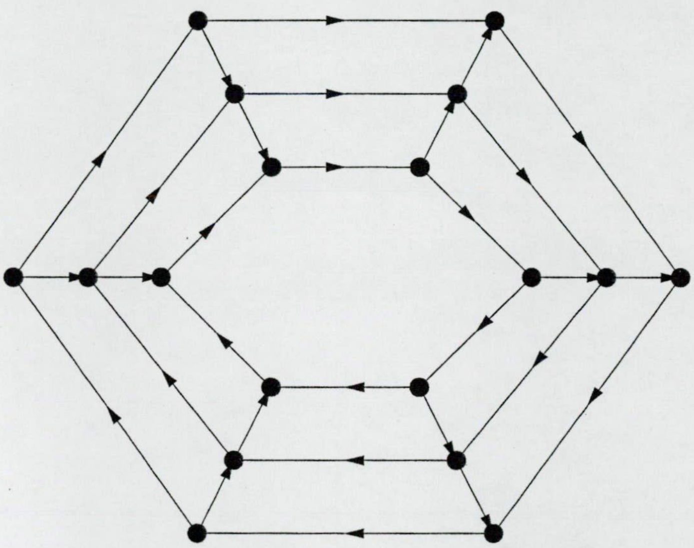


Figure 3.2: The 'candidate' directed grid; case $k = 3$

Lemma 3.22 $cn(J_k) = \overline{cn}(J_k) = k$

Proof: It is enough to prove that $k \leq cn(J_k)$ and $\overline{cn}(J_k) \leq k$.

If $X \subseteq V(J_k)$ and $|X| < k$, then there exist indices i, p, q such that $C_i \cup P_p \cup Q_q$ is disjoint from X. Let $\beta(X)$ be the strong component of $J_k \setminus X$ that includes C_i. It follows that β is well-defined. The winning tactic for the robber is to go the $\beta(X)$, when the cops are landing to X. Hence $k \leq cn(J_k)$.

Let us show now explicitly how k cops can capture the invisible robber (i.e. $\overline{cn}(J_k) \leq k$).

Land $k-1$ cops onto $v_{k,1}, \dots, v_{2,1}$. Now the robber cannot move if he stayed in $C_2 \cup \ldots \cup C_k$. So we can search this part completely with the remaining one cop; simply by landing to each point, one after the other. After this process land the last cop to $u_{1,1}$. Now the robber must be somewhere in $C_1 \setminus u_{1,1}$. We can thereby lift the cop from $v_{k,1}$. There is no strongly connected piece including $C_1 \setminus u_{1,1}$ for the moment. So the robber cannot move. So the rest of the search is again landing the free cop onto each vertex of $C_1 \setminus u_{1,1}$. □

Lemma 3.23 $cn^*(J_k) = \overline{cn^*}(J_k) = k+1$

Proof: It is enough to prove that $\overline{cn^*}(J_k) \leq k+1$, and $cn^*(J_k) \geq k+1$. First we prove that $k+1$ cops are enough to capture the invisible robber. Put k cops on Q_1. Then the rest of the graph has no directed circuit. Hence one can search through $J_k \setminus Q_1$ with one cop.

Secondly we prove $cn^*(J_k) \geq k+1$. Assume to the contrary that we have only k cops (the robber is visible). Then whenever there is a cop in the air (helicopter), there will be a cop-free directed circuit. That gives a winning strategy for the robber: "stay on the cop-free directed circuit". □

Remark 3.24 *(By an undirected edge of a directed graph, we mean two edges between the same two points, one edge in each direction.) If we take an undirected graph G, then $cn(G) = cn^*(G)$ and $\overline{cn}(G) = \overline{cn^*}(G)$.*

Definition 3.25 *For $k = 1, 2, \ldots$ let I_k^s be the union of k directed circuits $C_1, C_2, \ldots, C_k$ of length s, and s copies of the complete undirected graph on k vertices, $K_k^1, K_k^2, \ldots, K_k^s$, where C_i has vertex set $\{u_{i,1}, u_{i,2}, \ldots, u_{i,s}\}$, and K_k^j has vertex set $\{u_{1,j}, u_{2,j}, \ldots, u_{k,j}\}$.*

Lemma 3.26 *Assume that $s \gg k$ ($s > 2k$ say).*

> *(i)* $cn^*(I_k^s) = \overline{cn^*}(I_k^s) = 2k$
>
> *(ii)* $cn(I_k^s) = \overline{cn}(I_k^s) = k+1$

Proof: First we show explicitly that $\overline{cn}(I_k^s) \leq k+1$. Put k cops onto the vertices of K_k^1. Now the strongly connected components will be disjoint K_k's. So the robber is in one of them. Our goal is to move the cops in such a way, that the robber cannot escape from that special component. Put the $(k+1)$st cop onto $u_{1,2}$. Lift now the cop from $u_{1,1}$. Observe that the block structure did not change with these two moves. Hence the robber remained in the

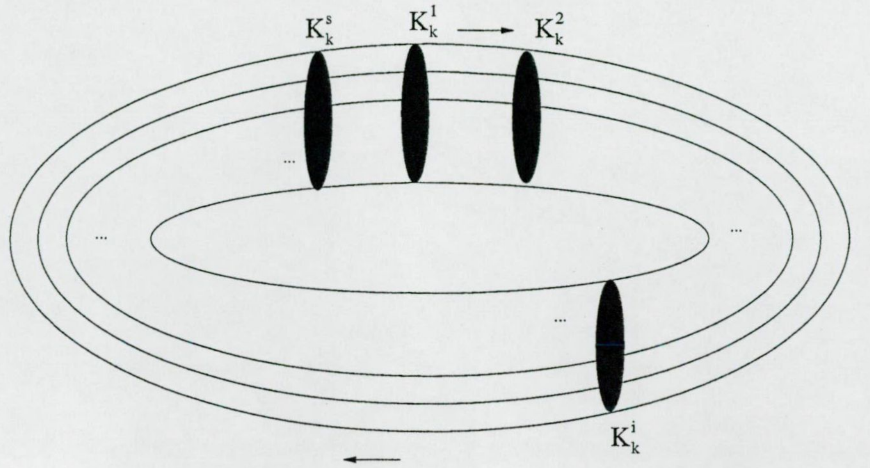


Figure 3.3:

same component. Put the free cop onto $u_{2,2}$, then lift the cop from $u_{2,1}$. We are basicly 'sliding' a cop along directed edges of the circuits. Continuing in this way we can search through the entire graph. So we capture the visible robber, hence also if he is invisible.

There is a K_k in our graph, so clearly $cn(I_k^s) \geq k$. But even k cops do not suffice. Because the robber can stay in a K_k as long as we do not put one cop to each of its vertices. When we do so, the robber can move in the last moment to another – now cop-free – K_k. In this way, the robber can always escape. Hence $\overline{cn}(I_k^s) \geq cn(I_k^s) \geq k+1$.

To see that $\overline{cn^*}(I_k^s) \leq 2k$, one has to mimic the game on a directed circuit with 2 cops.

Assume now that $cn^*(I_k^s) \leq 2k-1$, and we have only at most $2k-1$ cops. Then however we place them on I_k^s, there will be a directed circuit, C_j say, which contains only one cop. (Call such a circuit good.) If this cop stands on $u_{j,i}$, then the robber runs to $u_{j,m}$, s.t. $m-i$ is minimal modulo s. The robber remains in his place until he is 'attacked'. Which means that a new (second) cop approaching the directed circuit C_j. In that moment the robber runs forward to a K_k, where he can move to the good circuit. In this way the robber escapes. (Here we used the fact $s \gg k$.) □

The above Lemmas prove that the different cop-parameters behave different. However we conjecture, that Lemma 3.26 is best possible in some sense.

Conjecture 3.27 *$cn^*(D) \leq 2(cn(D)-1)$ and $\overline{cn^*}(D) \leq 2(\overline{cn}(D)-1)$*

3.3 Monotonicity for directed graphs

In this section we prove a monotone graph searching result for directed graphs. The main line of the proof is adopting a proof of Bienstock and Seymour, [6]. The difficulty here was to find the appropriate definitions for directed graphs. After that one has to check that the proofs go over nicely.

Consider the game described in Definition 3.13. We would like to prove that if k cops can capture the invisible robber, then they can do it in a nice monotone way too.

First we describe a slightly more general game. The robber is invisible in that case too. We will deduce the required result from the monotonicity of that game. In this part of the paper the word 'capture' will be only used for the cops' winning tactic in the sense of Definition 3.13. While the word 'search' is restricted to the game described next. Hence a 'search' will clear all the edges, while a 'capture' clears all the vertices of a digraph. The connection between the two concepts is indicated in Remark 3.29.

Definition 3.28 *A mixed-search in a directed graph D is a sequence of pairs*

$$(A_0, Z_0), \ldots, (A_n, Z_n)$$

(intuitively Z_i is the set of vertices occupied by the cops immediately before the (i+1)st step, and A_i is the set of clear edges) such that

(I) $0 \leq i \leq n$, $A_i \subseteq E(D)$, $Z_i \subseteq V(D)$,

(II) $0 \leq i \leq n$, any vertex which is a head of an edge in $E(D) \setminus A_i$ and tail of an edge in A_i is in Z_i,

(III) $A_0 = \emptyset$, $A_n = E(D)$,

(IV) (List of possible moves) for $1 \leq i \leq n$, either

(a) (placing new cops) $Z_i \supseteq Z_{i-1}$, and $A_i = A_{i-1}$, or

(b) (removing cops) $Z_i \subseteq Z_{i-1}$, and A_i is the set of edges e, s.t. every directed path containing an edge of $E(D) \setminus A_{i-1}$ before e in order, has an internal vertex in Z_i, and $A_i \subseteq A_{i-1}$, or

(c) (node searching e) $Z_i = Z_{i-1}$ and $A_i \subseteq A_{i-1} \cup \{e\}$ for some edge $e \in E(D) \setminus A_{i-1}$ with both ends in Z_{i-1}, or

(d) (sliding) $Z_i = (Z_{i-1} \setminus \{u\}) \cup \{v\}$ for some $u \in Z_{i-1}$ and $v \in V(D) \setminus Z_{i-1}$ and $e = (v, u) \in E(D)$, s.t. every other in-edge to u belongs to A_i, and $A_i = A_{i-1} \cup \{e\}$, or

(e) (clearing an edge with one cop) $Z_i = Z_{i-1}$, and $A_i = A_{i-1} \cup \{e\}$ for some edge $e = (u,v) \in E(D) \setminus A_{i-1}$ with head v in Z_{i-1} and every (possibly 0) edge with head u in A_{i-1}.

If $|Z_i| \leq k$ for $1 \leq i \leq n$, then $\overline{cn}_m(D) \leq k$ in notation.

Remark 3.29 *The list of possible moves is a very detailed description of what is going on. A move of a cop in the sense of Section 3.1 is now 'atomized' into three moves. First (b), then (a), then (c).*

The game described in Definition 3.13 is a version of mixed-search, where (IV.d) is not allowed. Because after every landing of a cop, we can declare those edges clear, which has now both endpoints occupied (i.e. (IV.c)), or whose tail is a clear vertex and its head is occupied by a cop (i.e. (IV.e)). Hence if $\overline{cn}^*(D) \leq k$, then $\overline{cn}_m(D) \leq k$.

The sets A_i satisfy $|A_i \setminus A_{i-1}| \leq 1$.

Definition 3.30 *A mixed-search of D is called monotone, if every edge of D is cleared exactly once. This is the same as saying that the cleared edges form a monotone increasing set.*

Our goal is to show that the existence of a mixed-search of D with k cops implies a monotone mixed-search of D with the same number of cops.

Definition 3.31 *If $X \subseteq E(D)$, let $\delta(X)$ be the set of those vertices which are the tail of an edge in X and also the head of an edge in $E(D) \setminus X$. Call these points dangerous.*

Lemma 3.32 *$|\delta|$ satisfies the submodular inequality, i.e.*

$$|\delta(X \cap Y)| + |\delta(X \cup Y)| \leq |\delta(X)| + |\delta(Y)|$$

for any vertex sets X and Y.

Proof: We have to prove that every dangerous vertex counted in the left-hand side (LHS) with certain multiplicity is also counted at least as many times in the right-hand side (RHS).

If $v \in \delta(X \cap Y)$, then by definition there is an edge e with head v and not in $X \cap Y$, and also an edge f with tail v and in $X \cap Y$. Hence $f \in X$, $f \in Y$, but $e \notin X$ and/or $e \notin Y$.

If $v \in \delta(X \cup Y)$, then with similar notation $e \notin X$, $e \notin Y$, and $f \in X$ and/or $f \in Y$.

Hence if a vertex v is counted on the LHS, then it is counted on the RHS too.

Moreover if $v \in \delta(X \cap Y)$ and $v \in \delta(X \cup Y)$, then $v \in \delta(X)$ and $v \in \delta(Y)$ too.

So if v is counted twice on the LHS, then it is counted twice on the RHS too. □

Definition 3.33 *A raid in D is a sequence $(X_0, X_1, \ldots, X_n)$ of subsets of $E(D)$, s.t. $X_0 = \emptyset$, $X_n = E(D)$, and $|X_i \setminus X_{i-1}| \leq 1$, for $1 \leq i \leq n$ (i.e. at most one new clear edge).*

The raid uses at most k cops if $|\delta(X_i)| \leq k$ for $0 \leq i \leq n$.

Lemma 3.34 *If $\overline{cn}_m(D) \leq k$, then there is a raid in D using at most k cops.*

Proof: Let $(A_0, Z_0), \ldots, (A_n, Z_n)$ be a mixed-search in D with each $|Z_i| \leq k$. Then each $\delta(A_i) \subseteq Z_i$, hence each $|\delta(A_i)| \leq k$, and also $|A_i \setminus A_{i-1}| \leq 1$ by definition, so $(A_0, \ldots, A_n)$ is a raid using at most k cops. □

Definition 3.35 *A raid is progressive if $X_0 \subseteq \ldots \subseteq X_n$, and $|X_i \setminus X_{i-1}| = 1$ (always a new clear edge).*

Lemma 3.36 *Suppose there is a raid in D using at most k cops. Then there is a progressive raid in D using at most k cops.*

Proof: Choose a raid $X_0 \subseteq \ldots \subseteq X_n$ with at most k cops s.t.

(1) $\sum_{i=0}^{n} |\delta(X_i)|$ is minimum,

and subject to (1),

(2) $\sum_{i=0}^{n} |X_i|$ is minimum.

We are going to show that $X_0 \subseteq \ldots \subseteq X_n$ is progressive.

(3) $|X_j \setminus X_{j-1}| = 1$, for $1 \leq j \leq n$.

For $|X_j \setminus X_{j-1}| \leq 1$, and if $|X_j \setminus X_{j-1}| = 0$, then $X_j \subseteq X_{j-1}$, and $(X_0, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n)$ is a raid with at most k cops, contradicting (1)-(2).

(4) $|\delta(X_{j-1} \cup X_j)| \geq |\delta(X_j)|$.

For otherwise $|\delta(X_{j-1} \cup X_j)| < k$, hence $(X_0, \ldots, X_{j-1}, X_{j-1} \cup X_j, X_{j+1}, \ldots, X_n)$ is a raid with at most k cops, contradicting (1).

(5) $X_{j-1} \subseteq X_j$.

From the submodularity

$$|\delta(X_{j-1} \cap X_j)| + |\delta(X_{j-1} \cup X_j)| \leq |\delta(X_{j-1})| + |\delta(X_j)|$$

From (4) it follows that $|\delta(X_{j-1} \cap X_j)| \leq |\delta(X_{j-1})|$. Hence $(X_0, \ldots, X_{j-2}, X_{j-1} \cap X_j, X_j, \ldots, X_n)$ is a raid with at most k cops. From (2) $|X_{j-1} \cap X_j| \geq |X_{j-1}| \Rightarrow X_{j-1} \subseteq X_j$. □

Lemma 3.37 *Let $(X_0, X_1, \ldots, X_n)$ be a progressive raid with at most k cops, and for $1 \leq j \leq n$ let $X_j \setminus X_{j-1} = \{e_j\}$. Then there is a monotone mixed-search of D using at most k cops, s.t. the edges of D are cleared in the order $e_1, \ldots, e_n$.*

Proof: We construct the monotone mixed-search inductively. Suppose that $1 \leq j \leq n$, and we have cleared the edges $e_1, \ldots, e_{j-1}$ in order, in such a way that no other edges have been cleared yet. Let A be the set of all vertices $v \in V(D)$ s.t. every edge having v as its head is in X_{j-1} (these are the non-dangerous vertices). Certainly each vertex in $\delta(X_{j-1})$ is currently occupied by a cop. Remove all other cops. Since $e_j \notin X_{j-1}$, its head is not in A. Let $N = \{u, v\}$ be the set of ends of e_j.

If $|N \cup \delta(X_{j-1})| \leq k$, we may place new cop(s) on the ends of e_j, and declare it cleared by (IV.c).

So assume $|N \cup \delta(X_{j-1})| > k$. W.m.a. $v \in N \setminus \delta(X_{j-1})$.

If $(u, v) = e_{j-1}$, then the tail u must be in A, and there is one free cop by the previous assumption. Hence e_{j-1} can be declared clear by (IV.e) putting a cop on v.

If $(v, u) = e_{j-1}$, then e_{j-1} is the only edge of $E(D) \setminus A_{j-1}$ having u as the head. Hence e_{j-1} can be declared clear by (IV.d). □

Summarizing the previous lemmas, we get the monotonicity result.

Lemma 3.38 *If there is a mixed-search of D with at most k cops, then there is a monotone mixed-search of D with at most k cops.*

Proof: By Lemma 3.34 there is a raid in D with at most k cops. Then by Lemma 3.36 there is a progressive raid in D with at most k cops. Hence by Lemma 3.37 there is a monotone mixed-search in D with at most k cops as required. □

We have shown the equivalence below:

Theorem 3.39 *For $k \geq 1$, the following are equivalent:*

(i) there is a mixed-search in D with at most k cops,

(ii) there is a raid in D with at most k cops,

(iii) there is a progressive raid in D with at most k cops,

(iv) there is a monotone mixed-search in D with at most k cops.

We would like to deduce another equivalence theorem from the above one, which includes directed path-width. First we need a Lemma which translates mixed-search's edge-monotonicity to 'vertex-monotonicity'.

Lemma 3.40 *If there exists a monotone (i.e. no edge is cleared twice) mixed-search of D without (IV.d) using at most k cops, then there also exists a monotone (i.e. where no vertex is revisited by the cops) capture in D with at most k cops.*

Proof: Consider all of these special monotone mixed-searches existing by the assumption. The cop-moves of these searches can be interpreted as a capture by Remark 3.29. From now on, consider only these captures.

Assume to the contrary that in every such capture, there is a vertex v which is revisited. This is only satisfied if the following two conditions hold:

(i) There was a cop on v, and he left v in step i.

(ii) A cop returned to v in step j, where $j > i$.

The first condition implies that after step i, there is some clear edge with head v, and the vertex v is not dangerous any more (by the monotonicity of the mixed-search).

The second condition implies that after step i there is still some non-cleared edges with head v. Knowing this, (i) implies that there is no clear edge with tail v, and hence there must be some clear edges $e_1, \dots, e_t$ with head v. Any e_l ($l = 1, \dots, t$) was either cleared by (IV.c) or (IV.e).

Assume the clearing of $e_i = (u, v)$ was done according to (IV.c). By (i) the cop on v left in step i. Hence u is not dangerous after step $i-1$. But then the clearing of (u, v) can be done after step $i-1$ by (IV.e). So the original clearing can be omitted, and replaced by the mentioned clearing after step $i-1$.

Hence it is enough to consider the case (IV.e). The first condition implies that u is not dangerous, so the clearing of (u, v) can be postponed until step j. In this way, we proved, that the cop's return to v can be ignored by rearranging the moves. Iterating this process a monotone capture arises. □

3.4 Directed path-width

After all of these preparations we can prove what we wanted. An equivalence showing that *dpw* corresponds to $\overline{cn}^*$, so Definition 3.13 and 3.17 lead to the same thing.

Theorem 3.41 *For $k \geq 1$ the following are equivalent:*

(v) $dpw(D) \leq k-1$,

(vi) $\overline{cn}^*(D) \leq k$,

(vii) there is a monotone capture of an invisible robber in D with at most k cops (as defined in Lemma 3.40)

We already mentioned in Remark 3.29 that $\overline{cn}^*(D) \leq k$ is equivalent to a mixed-search with at most k cops without using (IV.d). Hence we keep the mixed-search language and always show that (IV.d) was not used.

Proof: (v)$\Rightarrow$(vi). By assumption there exists a dpd where the bags have size at most k. First the at most k cops occupy W_1. Hence all the edges induced by W_1 can be cleared by (IV.c). Then the cops on $W_1 \setminus W_2$ take off and fly to $W_2 \setminus W_1$. Now the edges induced by W_2 can be cleared by (IV.c). Continuing in this way, the only non-cleared edges will be of form (u, v), where $u \in W_i \setminus W_j$ and $v \in W_j \setminus W_i$ and $i < j$. These edges can be cleared either by (IV.e) or with two cops and (IV.c). We did not use (IV.d), hence this is a capture with at most k cops.

(vi)$\Rightarrow$(vii). If D is a directed graph, let D^d denote the directed graph, where every edge of D is duplicated.

We claim that $\overline{cn}^*(D) = \overline{cn}_m(D^d)$. First of all $\overline{cn}^*(D) \geq \overline{cn}_m(D^d)$ is trivial. Also $\overline{cn}^*(D) \leq \overline{cn}_m(D^d)$ is true since in any mixed-search of D^d the duplicated edges cannot be cleared by (IV.d). So every edge of D^d must be cleared by (IV.c) or (IV.e).

By Theorem 3.39, we know that $\overline{cn}_m(D^d) \leq k$ implies the existence of a monotone mixed-search M of D^d too. Hence by Lemma 3.40, M is actually a monotone capture in D^d too. It is easy to see that M also yields a monotone capture in D itself.

(vii)$\Rightarrow$(v). Assume there is a monotone capture with at most k cops in D. If we simulate the moving of the cops, every vertex is occupied precisely once. The cop-moves can be arranged in such a way, that in every move one cop takes off and he lands immediately. Let the set of vertices occupied by the cops after the ith move be called W_i.

$\bigcup_i W_i$ is a dpd with width at most k.

To see this, we have to show the required (i)-(iii) properties of the definition of a dpd. This is easy, so we can omit it. □

3.5 Blockages for directed graphs

The notion of a blockage was introduced by Bienstock et al. in [7] as obstructions for having small pw. In this section we try to generalize the concepts of [7] i.e. blockages. After the results of the previous section one can additionally ask whether $dpw(D) \leq k-1$ implies the non-existence of a blockage of order k. As in the previous section the main task here was to define the concepts in the appropriate way for directed graphs. Then the proofs go the same way as in [7]. However one has to check the details behind the formalisms. In this section we use V as a shortening of $V(D)$.

First we define the attachment of a vertex set $X \subseteq V$. (The attachment includes the points occupied by the cops protecting the area X against the robber in $V \setminus X$.)

Definition 3.42 *Let $X \subseteq V$.*
$att(X) = \{x \in X : \exists y \in V \setminus X \text{ s.t. } (y, x) \in E(D)\}$.
$\alpha(X) = |att(X)|$.

Another crucial notion is the complement.

Definition 3.43 $X^c = (V \setminus X) \cup att(X)$.
Y is a complement of X if $X^c \subseteq Y$.
X and Y are complementary if $X^c \subseteq Y$ or/and $Y^c \subseteq X$ (at least one of them holds).

Remark 3.44 *$(X^c)^c \subseteq X$ is not always true.*
If X and Y are complementary and $|X \cap Y| \leq k$, then $\alpha(X) \leq k$ or/and $\alpha(Y) \leq k$.

Definition 3.45 *Let $k \geq 0$ be an integer. A blockage (in D, of order k) is a set $\mathcal{B}$ s.t.*

> *(i) each $X \in \mathcal{B}$ is a subset of V with $\alpha(X) \leq k$,*
>
> *(ii) if $X \in \mathcal{B}$ and $Y \subseteq X$ and $\alpha(Y) \leq k$, then $Y \in \mathcal{B}$,*
>
> *(iii) if X_1 and X_2 are complementary and $|X_1 \cap X_2| \leq k$, then $\mathcal{B}$ contains exactly one of X_1, X_2.*

We call these the blockage axioms.

Remark 3.46 *Considering axiom (iii), it can happen that $\alpha(X_1) \leq k$ but $\alpha(X_2) > k$ or vice versa. In such a case axiom (i) determines which one of X_1, X_2 is the set in $\mathcal{B}$.*

Lemma 3.47 *Let $\mathcal{B}$ be a blockage of order k in D, let $X \in \mathcal{B}$, and let $Y \subseteq V$ with $\alpha(Y) \leq k$ and $|(Y \setminus X) \cup att(X)| \leq k$. Then $Y \in \mathcal{B}$.*

Proof: Since X, X^c are complementary, $|X \cap X^c| = \alpha(X) \leq k$, and $X \in \mathcal{B}$, axiom (iii) implies that $X^c \notin \mathcal{B}$. $att(X \cup Y) \subseteq att(X) \cup (V \setminus X) = X^c$, hence $X \cup Y$ and X^c are complementary. $|(X \cup Y) \cap X^c| = |(Y \setminus X) \cup att(X)| \leq k$, hence axiom (iii) implies that $X \cup Y \in \mathcal{B}$. Now $\alpha(Y) \leq k$, hence $Y \in \mathcal{B}$ by axiom (ii). $\square$

We can now prove one implication regarding blockages and *dpw*.

Lemma 3.48 (v) *implies* (viii).

(v) $dpw(D) \leq k - 1$,

(viii) *there is no blockage of order k in D.*

Proof: Assume to the contrary that there is a blockage $\mathcal{B}$ of order k in D. By assumption $dpw(D) < k$. Let $(W_1, \ldots, W_m)$ be a dpd, where each $|W_i| \leq k$. Since $\emptyset$ and V are complementary and $\emptyset \subset V$, it follows from axioms (ii) and (iii) that $\emptyset \in \mathcal{B}$. From Lemma 3.47 $W_1 \in \mathcal{B}$ too. For $1 \leq i \leq k$, let $X_i := W_1 \cup \ldots \cup W_i$, and choose i maximum with $X_i \in \mathcal{B}$. Now $i \neq m$, because $V \notin \mathcal{B}$. Moreover, $att(X_i) \subseteq W_{i+1}$ by the definition of dpd. So $|(X_{i+1} \setminus X_i) \cup att(X_i)| \leq |W_{i+1}| \leq k$. By Lemma 3.47 $X_{i+1} \in \mathcal{B}$, contrary to the maximality of i. $\square$

The converse implication seems to be false for the author, but we could not find a counterexample. If this is the case, then there is no 'nice' way to describe why the cops can not succeed, as it was possible in the undirected case.

Chapter 4

Characterization of graphs with path-width two

In this Chapter we present a proof of the next
Theorem: The following statements are equivalent:

(i) G is a partial track;

(ii) G has path-width at most two;

(iii) G has no minor listed in the Appendix.

This is basicly a consequence of the reductions described in Section 4.2, and the fact that the ten non-reducible graphs have path-width four, see Section 4.12.

4.1 Basics

Let us first repeat the definitions:

Definition 4.1 *A path-decomposition of a graph G is a pair (P, W), where P is a path and $W = (W_p : p \in V(P))$ is a family of subsets of $V(G)$, satisfying*

(1) $\bigcup_{p \in V(P)} W_p = V(G)$, and every edge of G has both ends in some W_p, and

(2) if $p, p', p'' \in V(P)$ and p' lies on the path from p to p'', then $W_p \cap W_{p''} \subseteq W_{p'}$.

(Let us call the W_i's simply bags.)
The width of a path-decomposition is $\max(|W_p| - 1 : p \in V(P))$, *and the path-width of G ($pw(G)$ in notation) is the minimum width of all path-decompositions of G.*

Lemma 4.2 *If H is a minor of G, then the path-width of H is at most the path-width of G.*

Proof: All we have to show is that the decomposition can be modified without increasing the width, when we delete or contract an edge. For any edge e, the decomposition of G is proper for $G \setminus e$ too. Assume we contract an edge $e = xy$, and call the new vertex u. If we had a decomposition of G, then instead of x and y, put u into every bag, where x and/or y were present. This is clearly a proper decomposition of G/e □

Remark 4.3 $pw(K_n) = n - 1$.

Trees can have arbitrarily large path-width. To see this, let T_k denote the symmetric, ternary tree of height k. More exactly T_k has one specified vertex r of degree 3, all other vertices (except the leaves) have degree 4, and all leaves have distance k from r.

Lemma 4.4 $pw(T_k) \geq k$.

Proof: Observe first that from part (2) of the definition the next claims follow:
Every vertex $v \in G$ appears in consecutive bags. Moreover the similar statement holds for a connected subgraph: The vertices of a connected subgraph appear in consecutive bags. Built upon these previous remarks, we can prove the following:

If a graph G has a vertex v s.t. $G \setminus v$ has at least three connected components of path-width k or more, then $pw(G) \geq k + 1$.

Let H_1, H_2, H_3 be the above mentioned three connected components. W.m.a. that $pw(H_i) = k$, $(i = 1, 2, 3)$ otherwise we are done. Let $v_i \in H_i$ be the respective neighbors of v in G. Suppose to the contrary that $pw(G) \leq k$, and G has a path-decomposition (P, W) of width $\leq k$. $pw(H_i) = k$, so there must be a bag W_{j_i}, which only contains vertices from H_i, $(i = 1, 2, 3)$. W.m.a. $j_1 < j_2 < j_3$. $G \setminus H_2 = H_2'$ is a connected subgraph of G. So by the previous remark the vertices of H_2' should appear in consecutive bags. But this is false by $W_{j_1} \cap V(H_2') \neq \emptyset$, $W_{j_2} \cap V(H_2') = \emptyset$, $W_{j_3} \cap V(H_2') \neq \emptyset$. Thus $pw(G) \geq k + 1$.
Now the statement of the Lemma is a trivial application. □

There is another thing which can make path-width big. Namely a big grid-minor. This fact somehow means that the graph is 'highly' connected.

Definition 4.5 *Consider the graph on* $\{1, \ldots, n\}^2$ *with the edge set*

$$\{(i,j)(i',j') : |i - i'| + |j - j'| = 1\}.$$

This graph is called the $n \times n$ grid and denoted J_n. (Clearly the adjacency graph of an $n \times n$ chess-board.)

Lemma 4.6 *The* $n \times n$ *grid has path-width* n.

Proof: First we prove that $pw(J_n) \leq n$. Take the sets $(1, \ldots, n+1)$; $(2, \ldots, n+2)$ etc. $(n^2 - n, \ldots, n^2)$. It is easy to see that this is a path-decomposition with width n.

Now we prove that $tw(J_n) \geq n$, hence also $pw(J_n) \geq n$. More precisely, using Theorem 3.12, it is enough to prove that n cops are not enough to capture the visible robber in J_n. To see this, we describe a tactic (algorithm) for the robber. Using that he will never be captured. Before doing that, we make some comments. W.m.a. that in a general stage of the game either one cop takes off, or one cop lands. So we have to give a good tactic for the robber, which reacts on these movements. As we already remarked in the Definition 3.1, the cops can only capture the robber if they occupy all neighbors of the vertex where the robber is standing, and then with one extra cop they capture the robber. We will show that the robber can always stand on a vertex, which has at least one cop-free neighbor. Hence the robber can not be captured.

(1) If there are less than n cops on J_n, then there is a cop-free row, and a cop-free column. The intersection of these two objects is called a castle. There can be several castles at the same time of course, and there can exist a castle even if there are n cops on J_n. The robber's main tactic is: 'Go to a castle if there is any.'

(2) If there is a cop (not the nth one) approaching to the row (column) where the robber is, the robber can run away in the last moment to another castle.

(3) Assume the nth cop lands, and after his landing there would be no castle. If the cop attacks the robber's vertex, then the robber should move to a neighboring vertex. If the cop lands somewhere else, then the robber can stay where he is. In any case it is guaranteed that the robber's vertex has at least one cop-free neighbor.

By the previous remarks, with this tactic the robber escapes. □

The excluded minor characterization of graphs with small path-width is easy only for the value one. The value two is handled in the rest of this Chapter.

Lemma 4.7 *$pw(G) \leq 1$ iff G has neither K_3 nor Y_1 as a minor. This is equivalent to being a caterpillar graph.*

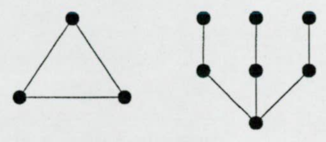


Figure 4.1: The excluded minors for path-width one

4.2 $PW2$-safe operations

Let $PW2$ denote the set of all simple graphs with path-width at most two. We will discuss $PW2$-monotone and $PW2$-safe operations. Some of them are well-known (for example taking minors), some of them are implicit in other works (compare Lemma 4.9, Lemma 4.10 and Lemma 4.11 with [4]), some of them are new. For sake of completeness we list all of them with proof. Our proofs are different from the previous results (and perhaps simpler), and provide a unified approach to these operations.

First we start with some simple observations on representations.

Let $\mathcal{I}$ denote the set of all closed intervals of the real line. Let $\phi : V(G) \to \mathcal{I}$ be a representation of G. Let $\phi(v) = [l(v), r(v)]$, i.e. $l(v), r(v)$ simply denotes the left, resp. right endpoint of the closed interval $\phi(v)$ representing the vertex v. If I is a closed interval, then $l(I)$ and $r(I)$ denotes the left, resp. right endpoint of I. The real line introduce a "new language". We can refer to the left side of a point or interval on our line. We can write $P < Q$ or $P < I$ iff P is on the left-hand side of point Q or interval I (i.e. $P < l(I)$). Similarly for two intervals, $I_1 < I_2$ means that $r(I_1) < l(I_2)$. $P + \varepsilon$ is the point on the right of P, distance ε from it. Q is between P and R iff $P < Q < R$ or $P > Q > R$. $[P]$ is the interval that contains only one point, P.

Observation 4.8 *(i) K_n has essentially one representation: the representing intervals are n intervals with non-empty intersection. Hence the minimal width of the representations is n.*

(ii) If G is connected, then for any representation, the union of the representing intervals is an interval (i.e. connected).
(iii) Assume there is a representation of a graph G. If there is a point p on the real line, s.t. $r(x) = p$ (resp. $l(x) = p$) for some vertex x, and the width at p is k, then for appropriately chosen small positive ε, the width at any point of the open interval $(p, p+\varepsilon)$ (resp. $(p-\varepsilon, p)$), is at most $k-1$.

The importance of (iii) is that it gives us some 'space' to deform a representation without increasing the width.

In a series of Lemmas we prove the main features of the operations. Unfortunately the formal proofs are hard to read. We advise the reader to have paper and pencil in hand, so every detail can be drawn up and be visualized.
Operation 1: O_1 = Deletion of an edge or isolated vertex. It preserves path-width at most two, so it is a $PW2$-monotone reduction.
Operation 2: O_2 = Contraction of an edge. It preserves path-width at most two, so it is a $PW2$-monotone reduction.
Operation 3: Let O_3 be the variation defined as follows. Let $S_3 = x, a, b, y$ be a path of length three. The attachments are x and y (the two endvertices of the path). Let T_3 be a star with three branches (a claw) with vertices x, a', b', y, where $deg(a') = deg(x) = deg(y) = 1$ and $deg(b') = 3$. The attachments are x and y (two leaves of the star). Then $O_3(G) = (G \setminus S_3) \cup T_3$. (See Figure 4.2, where the attachments are drawn as full circles.)

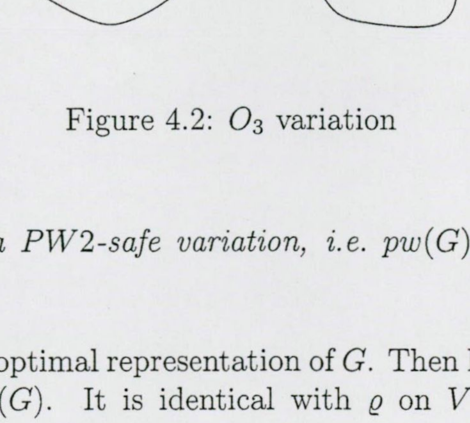


Figure 4.2: O_3 variation

Lemma 4.9 *O_3 is a $PW2$-safe variation, i.e. $pw(G) \leq 2$ if and only if $pw(O_3(G)) \leq 2$*

Proof: Let ϱ be an optimal representation of G. Then let ϱ' be the following representation of $O_3(G)$. It is identical with ϱ on $V(O_3(G)) \setminus \{a', b'\} =$

$V(G) \setminus \{a, b\}$, $\varrho'(b') = \varrho(a) \cup \varrho(b)$ and $\varrho'(a') = \varrho(a) \cap \varrho(b)$. The width of ϱ' is at most as much as the width of ϱ.

Let now ψ be an optimal representation of $O_3(G)$. Since the degree of a' is one, we can assume that $\psi(a')$ is an interval of one point: $l(a') = r(a') = P$. (This assumption is not vital for us, but it makes the picture behind the proof simpler.)

The natural way to modify ψ (without increasing the width) obtaining a representation ψ' of G is the following. Define $\psi'(a) = [l(b'), P]$, $\psi'(b) = [P, r(b')]$ and leave the intervals assigned by ψ in other cases (or do the same except exchanging the image of a and b). If none of these modifications works, then $\psi(x)$ and $\psi(y)$ are on the same (open) side of P. We can assume that $P < \psi(x), \psi(y)$, and $l(x) \leq l(y)$. $\psi(a') \cap \psi(b') \neq \emptyset$ and $\psi(y) \cap \psi(b') \neq \emptyset$, hence $P, l(y) \in \psi(b')$, which implies $[P, l(y)] \subset \psi(b')$, and $l(y)$ has width at least three. By Lemma 4.8(iii) ψ remains optimal by setting $\psi(a') = [l(y) - \varepsilon]$. This variation of ψ brings us to the case when the first (naive) approach works. This proves the claim. □

Operation 4: Instead of a formal description, let Figure 4.3 define operation O_4. (Do not forget that the full circles denote the attachments.)

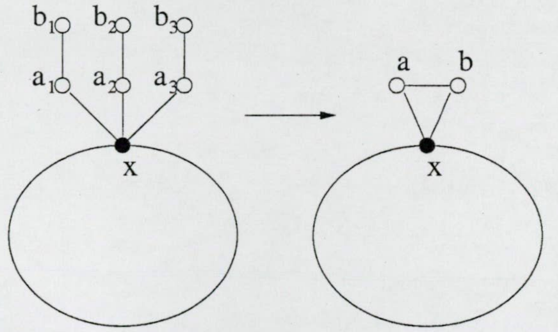


Figure 4.3: O_4 reduction

Lemma 4.10 *O_4 is a PW2-safe reduction, i.e. $pw(G) \leq 2$ if and only if $pw(O_4(G)) \leq 2$.*

Proof: Again name the vertices participating in the reduction as shown in Figure 4.3. ($V(O_4(G)) = V(G) \setminus \{a_1, a_2, a_3, b_1, b_2, b_3\} \cup \{a, b\}$.)

First let ψ be an optimal representation of $O_4(G)$. We can assume that $l(a) = r(a) = l(b) = r(b) = P \in \psi(x)$, i.e. $\psi(a) = \psi(b)$, a single point P. Then a small neighborhood of P is such, that only $\psi(x)$, $\psi(a)$ and $\psi(b)$ covers it. We can through away $\psi(a)$ and $\psi(b)$ and define $\psi'(a_1) = \psi'(b_1) = P - \varepsilon$,

$\psi'(a_2) = \psi'(b_2) = P$, $\psi'(a_3) = \psi'(b_3) = P + \varepsilon$. For any other vertex of G, its representing interval will be the same as in ψ. This way we obtained a representation ψ' of G with width at most three.

Now let ϱ be an optimal representation of G. If one of $\varrho(a_1) \cap \varrho(b_1)$, $\varrho(a_2) \cap \varrho(b_2)$ and $\varrho(a_3) \cap \varrho(b_3)$ intersects $\varrho(x)$, then there will be a point P of $\varrho(x)$ that is covered three times by the intervals playing roles in the reduction. This point can represent a and b. Leaving the representing intervals of other vertices of $O_4(G)$ (i.e. elements of $V(O_4(G)) \setminus \{a, b, c\} = V(G) \setminus \{a_1, a_2, a_3, b_1, b_2, b_3, x\}$) as in ϱ, we obtain a desirable representation.

If none of $\varrho(a_1) \cap \varrho(b_1)$, $\varrho(a_2) \cap \varrho(b_2)$ and $\varrho(a_3) \cap \varrho(b_3)$ intersects $\varrho(x)$, then at least two of the intersections (w.m.a. that $\varrho(a_1) \cap \varrho(b_1)$ and $\varrho(a_2) \cap \varrho(b_2)$) will be located on the same side (w.m.a. that on the left-hand side) of $\varrho(x)$. That implies that $l(x)$ will be covered by $\varrho(x)$, $\varrho(a_1)$ and $\varrho(a_2)$. That means that $l(x)$ can play the role of P, and the previous argument works. This proves the claim. □

Operation 5: Instead of a formal description, let Figure 4.4 define operation O_5. We stress that there is an important condition: we can perform O_5 only when the degree of x is at least three.

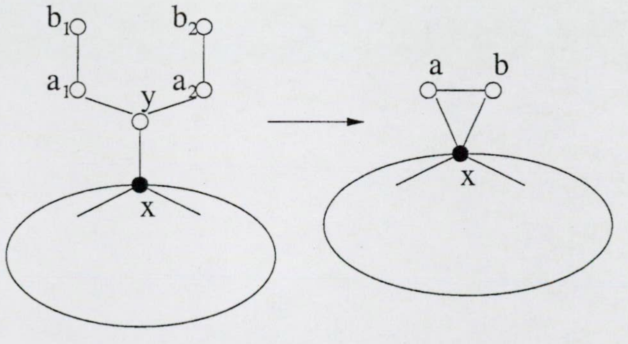


Figure 4.4: O_5 reduction

Lemma 4.11 *O_5 is a PW2-safe reduction, i.e. $pw(G) \leq 2$ if and only if $pw(O_5(G)) \leq 2$.*

Proof: Again name the vertices participating in the reduction as shown in Figure 4.4. ($V(O_5(G)) = V(G) \setminus \{a_1, a_2, b_1, b_2, y\} \cup \{a, b\}$.)

An optimal representation of $O_5(G)$ can be transformed to a width three representation of G as in Lemma 4.10.

Now we take an optimal representation ϱ of G. As in the previous proof, the only problem happens if neither $I_1 = \varrho(a_1) \cap \varrho(b_1)$ nor $I_2 = \varrho(a_2) \cap \varrho(b_2)$

intersects $\varrho(x)$. Hence I_1, I_2 and $\varrho(x)$ are pairwise disjoint. We can also assume that $\widehat{I_1} = \varrho(a_1) \cup \varrho(b_1)$ and $\widehat{I_2} = \varrho(a_2) \cup \varrho(b_2)$ does not intersect. (Otherwise ϱ would also represent $G + e$ — where e is an edge between the sets $\{a_1, b_1\}$ and $\{a_2, b_2\}$ — which clearly contains $O_5(G)$ as a minor.) We consider two cases.

1st case: I_1 and I_2 are on the same side of $\varrho(x)$, say $I_1 < I_2 < \varrho(x)$. Then I_2 separates $\widehat{I_1}$ from $\varrho(x)$, hence $\varrho(y)$ must contain I_2. Let P be any point of I_2. We can define ϱ', a representation of $O_5(G)$ (without increasing the width of ϱ) as $\varrho'(a) = \varrho'(b) = [P]$, $\varrho'(x) = \varrho(x) \cup \varrho(y)$ and all the other vertices has its representing intervals as in ϱ.

2nd case: $\varrho(x)$ separates I_1 and I_2, say $I_1 < \varrho(x) < I_2$. We are easily done if $\varrho(a_1)$ and $\varrho(a_2)$ intersects $\varrho(y)$ over $\varrho(x)$. Hence w.m.a. that $\varrho(x) \subset \varrho(y)$. There are at least two neighbors of x different from y, n and N say. Hence $\varrho(n) \not\supset \varrho(x)$. So one endpoint of $\varrho(n)$, $l(n) \in \varrho(x)$ say). Then we can take $\varrho'(x) = \varrho(x) \cup \varrho(y)$ and $\varrho'(a) = \varrho'(b) = [l(n) - \varepsilon]$ and leave all the other representing intervals as in ϱ. The constructed ϱ' is a representation of $O_5(G)$ having width at most three, if the width of ϱ did not exceed three. $\square$

For a moment we stop the flow of Lemmas for exhibiting the strength of our operations. In [30], the minimal acyclic forbidden minors (i.e. excluded trees) for path-width at most two were determined. The ten trees "look very similar". The last two operations can be used to reduce the excluded trees to one fundamental graph, D_3. Actually more can be said. Ten other graphs (altogether 20 out of the 110 excluded minors) are reducible to D_3. Figure 4.5 exhibits an example.

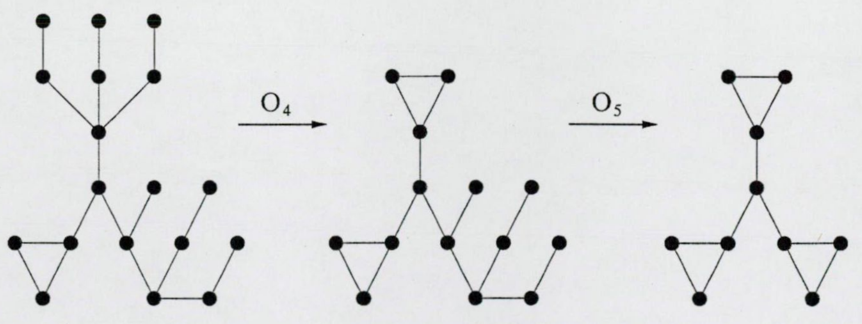


Figure 4.5: 20 excluded minors are reducible to D_3

Operation 6: Instead of a formal description, let Figure 4.6 define operation O_6. We point out that this is a conditional operation, i.e. we assume that the exchange of two structures can be done under certain assumption on the unchanged part: x must be contained in a cycle with a chord vw.

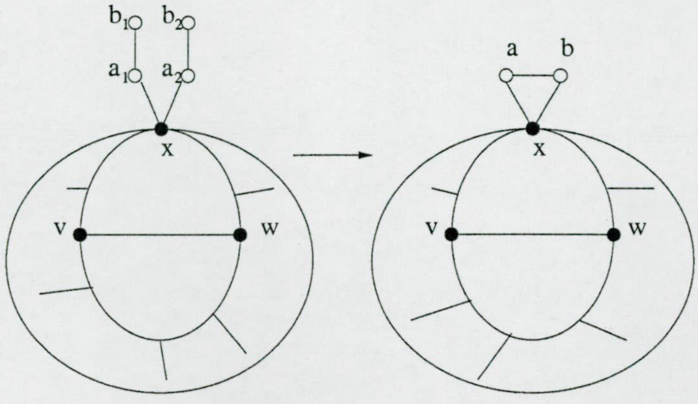


Figure 4.6: O_6 reduction

Lemma 4.12 *O_6 is a PW2-safe reduction, i.e. $pw(G) \leq 2$ if and only if $pw(O_6(G)) \leq 2$.*

Proof: Again name the vertices participating in the reduction as shown in Figure 4.6. ($V(O_6(G)) = V(G) \setminus \{a_1, a_2, b_1, b_2, \} \cup \{a, b\}$.)

An optimal representation of $O_6(G)$ can be transformed to a width three representation of G as in Lemma 4.10.

Now we take an optimal representation ϱ of G. As in the proof of Lemma 4.11 the only problem happens if neither $\varrho(a_1) \cap \varrho(b_1)$ nor $\varrho(a_2) \cap \varrho(b_2)$ intersects $\varrho(x)$, and $\varrho(x)$ separates $\varrho(a_1) \cap \varrho(b_1)$ and $\varrho(a_2) \cap \varrho(b_2)$, say $\varrho(a_1) \cap \varrho(b_1) < \varrho(x) < \varrho(a_2) \cap \varrho(b_2)$. We are also done if $\varrho(a_1) \cap \varrho(a_2) \cap \varrho(x) \neq \emptyset$. (both $\varrho(a_1)$ and $\varrho(a_2)$ must intersect $\varrho(x)$). Hence we can assume that $l(x) \leq r(a_1) < l(a_2) \leq r(x)$. Specially $l(x)$ and $r(x)$ are covered twice by the intervals corresponding to the vertices of S_{O_6}, the structure exchanged during the application of O_6.

Now we consider the remaining part of G. It contains a cycle C going through x, and having a vw chord. We partition C into four arcs (one of them consists of only one vertex, x), according to the Figure 4.7. For each arc we take the union of the intervals assigned to its vertices. Let $\varrho(x)$, I_1, I_2 and I_3 be the corresponding intervals ($I_i = \cup_{v \in A_i} \varrho(v)$). These four intervals cannot be pairwise intersecting (our representation has width at most three). The only possibility is that $\varrho(x)$ and I_3 are disjoint. Since the symmetry of our argument so far, we can assume that $I_3 < \varrho(x)$. I_1 and I_2 intersect both I_3 and $\varrho(x)$, hence they cover $l(x)$. This contradicts that $l(x)$ is covered by at most three intervals. This contradiction proves the claim. □

Operation 7: Instead of a formal description, let Figure 4.8 define operation

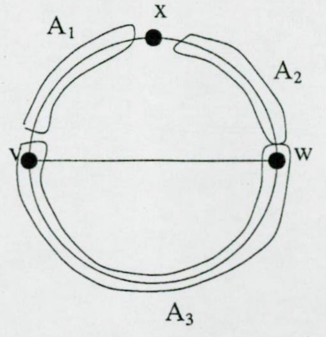


Figure 4.7: Four-arc partition

O_7. There is an important condition: the operation can be performed only when the degrees of x and y are at least three.

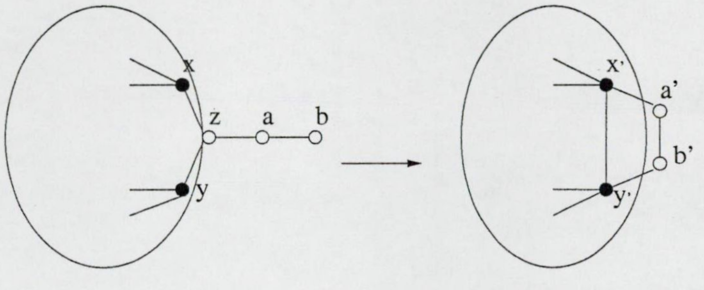


Figure 4.8: O_7 reduction

Lemma 4.13 *O_7 is a PW2-safe reduction, i.e. $pw(G) \leq 2$ if and only if $pw(O_7(G)) \leq 2$.*

Proof: Again name the vertices participating in the reduction as shown in Figure 4.8. ($V(O_6(G)) = V(G) \setminus \{z, a, b\} \cup \{a', b'\}$.)

An optimal representation ψ of $O_6(G)$ can be transformed to a width three representation of G as follows. $\psi(x) \cap \psi(b') \neq \emptyset$ and $\psi(y) \cap \psi(a') \neq \emptyset$ would mean that $\psi(x), \psi(y), \psi(a')$ and $\psi(b')$ are pairwise intersecting, that contradicts our assumption on the width of ψ. So w.m.a. that $\psi(x) \cap \psi(b') = \emptyset$, moreover $\psi(x) < \psi(b')$. Hence $\psi(a')$ and $\psi(y)$ both contains $r(x)$ and $l(b')$. W.m.a. that $l(b') = r(b') = P$ is covered by $\psi(a')$, $\psi(b')$ and $\psi(y)$. So by throwing away $\psi(a')$ and $\psi(b')$ we have "room" around P to define a representation of G: $\psi'(z) = [l(a'), P]$, $\psi'(a) = [P - \varepsilon, P + \varepsilon]$, and $\psi'(b) = [P + \varepsilon]$.

Now we take an optimal representation ϱ of G. If $\varrho(x)$ and $\varrho(y)$ are intersecting, then ϱ is a representation of $(G \setminus b) + xy$. O_3 can be applied on this graph to obtain $O_7(G)$ and we are done.

Henceforth we assume that $\varrho(x)$ and $\varrho(y)$ are disjoint, and $\varrho(x) < \varrho(y)$. If $\varrho(b)$ intersects $\varrho(x)$ or $\varrho(y)$ (let us assume that $\varrho(b) \cap \varrho(x) \neq \emptyset$), then we can easily obtain a representation of $O_7(G)$ by $\varrho'(x) = \varrho(x)$, $\varrho'(a) = \varrho(b)$, $\varrho'(b) = \varrho(a)$, $\varrho'(y) = \varrho(y) \cup \varrho(z)$, and of course taking all other representing intervals from ϱ. So from now on $\varrho(b)$, $\varrho(x)$ and $\varrho(y)$ are pairwise intersecting. We consider two cases.
1st case: $\varrho(b)$ is between $\varrho(x)$ and $\varrho(y)$. $\varrho(z)$ intersects both $\varrho(x)$ and $\varrho(y)$, hence $\varrho(b) \subset [r(x), l(y)] \subset \varrho(z)$. $\varrho(a)$ intersects $\varrho(b)$, hence there is a point P that is covered by $\varrho(z)$, $\varrho(a)$ and $\varrho(b)$. Let $\varrho'(x) = [l(x), P]$, $\varrho'(y) = [P, r(x)]$, $\varrho'(a') = [P - \varepsilon, P + \varepsilon]$, $\varrho'(b') = [P + \varepsilon]$, and all other representing intervals are from ϱ. This is a desired representation of $O_7(G)$.
2nd case: $\varrho(b)$ is an outside (we can assume that outside right) among $\varrho(x)$, $\varrho(y)$ and $\varrho(b)$. As before, we have $[r(x), l(y)] \subset \varrho(z)$. Now we use that y has two neighbors different from z. Since $l(y)$ is already covered by $\varrho(z)$ and $\varrho(y)$, at least one of the neighbors of y (n say) is such that $l(n) \in \varrho(y) \setminus \varrho(z)$. For suitable small ε the interval $[l(n) - \varepsilon, l(n))$ is covered only by $\varrho(x) \cup \varrho(a) \cup \varrho(b)$ and $\varrho(y)$. This allows us to define ϱ' similarly as in the first case. □

Operation 8: Instead of a formal description, let Figure 4.9 define operation O_8.

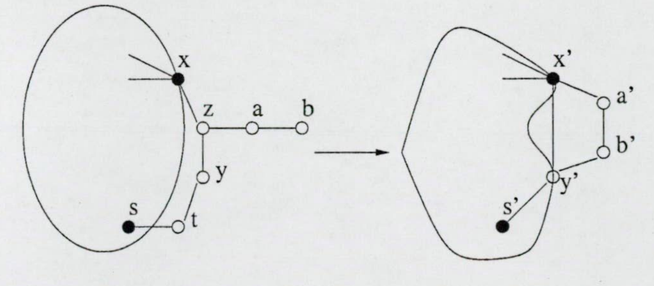


Figure 4.9: O_8 reduction

Lemma 4.14 *O_8 is a PW2-safe reduction, i.e. $pw(G) \leq 2$ if and only if $pw(O_8(G)) \leq 2$. Again there is a condition: we can perform O_8 only when the degree of x is at least three.*

Proof: O_8 can be simulated by an application of O_3 followed by an application of O_7 and an edge deletion. So one direction of our claim follows from Lemma 4.9 and Lemma 4.13.

The other direction is also easy. If $O_8(G)$ is given by an optimal representation (for notations see Figure 4.9), we can add a single neighbor (with degree one) to y' and still have a graph with a representation of width two. Indeed one of the three neighbors of y' has a representing interval with an extreme point inside the representing interval of y'. Just "next to" this extreme point we have enough room to represent the new neighbor. After this again we need to refer the reader to Lemma 4.13 and Lemma 4.9 to complete the proof. □

4.3 Non-reducible graphs characterization theorem

Some excluded minor theorems have the following structure: 'A graph G has property P iff G has no H-minor, where H is from the list L.' This type of characterization for the $PW2$-property was given in [15] (for an alternative solution see [3]). There are 110 excluded minors. We substitute this list with 10 minimal excluded graphs respect to $\leq_{\mathcal{R}}$, where $\mathcal{R} = \{O_1, \ldots, O_8\}$. One major point is that the appropriate theorem for path-width two is not the excluded minor theorem, but the one we present.

Theorem 4.15 *The following statements are equivalent*

(i) G has path width at most two,

(ii) G is not reducible to any of the graphs listed on Figure 4.10.

The (ii)⇒(i) can be done based on [15]. If one checks — and the authors did — that all the 110 excluded minors can be reduced to one of the graphs on our list, then it will be proven that any graph satisfying (ii) can not have any minor from the 110 list of Kinnersley and Langston. So their theorem implies (i). The formal description of this argument would be too long. But we ask the reader to take a look at [15]'s list. In a moment one can easily see several reductions. In most cases it is straightforward what one can do, what should be done. We give an example (See Figure 4.11) exhibiting a reduction of one of the "complicated" excluded minors to one of our graphs.

The (i)⇒(ii) implication easily follows if one sees that all graphs on our list has path-width more than two. This can be shown by brute force computation (as it was done in [15] in the case of 110 graphs) or one can refer to [15] again (one should believe their computation). In the next section we prove it by hand.

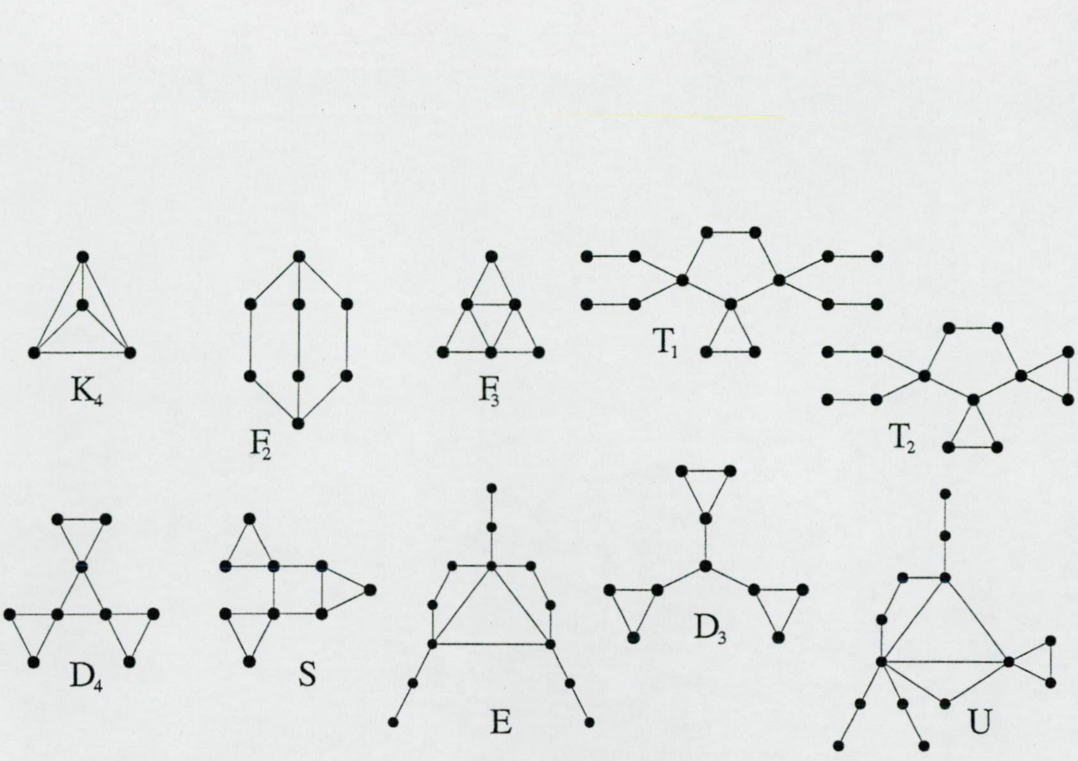


Figure 4.10: Non-reducible graphs

4.4 Path-width of the non-reducible graphs

The 110 excluded minors can be reduced to the ten fundamental graphs on Figure 4.10. These graphs are non-reducible in the sense, that they do not contain any reduction. However some of the ten graphs has the variation O_3. In such case, here we just picked out one of the variated graphs.

We prove that the ten non-reducible graphs have path-width at least three by proving that any representation of them has width at least four. As we mentioned earlier, this gives that all the 110 excluded minors of [15] have path-width at least three.

Lemma 4.16 *Any representation of K_4 has width at least four.*

Proof: Observation 4.8.

Lemma 4.17 *Any representation of F_2 has width at least four.*

Proof: Let ϱ be any representation of F_2. Let A_1, A_2 and A_3 be the three two-element sets as in Figure 4.12. Let $I_i = \bigcap_{v \in A_i} \varrho(v)$. W.m.a. that I_i's ($i = 1, 2, 3$) are pairwise disjoint (otherwise the width of ϱ is at least four). By symmetry assume that $I_1 < I_2 < I_3$. Then $\varrho(a_1) \cup \varrho(a) \cup \varrho(a_3)$ and $\varrho(b_1) \cup \varrho(b) \cup \varrho(b_3)$ covers I_2 (that is covered by $\varrho(a_2)$ and $\varrho(b_2)$ too). Hence the width of the representation is at least four in this case too. □

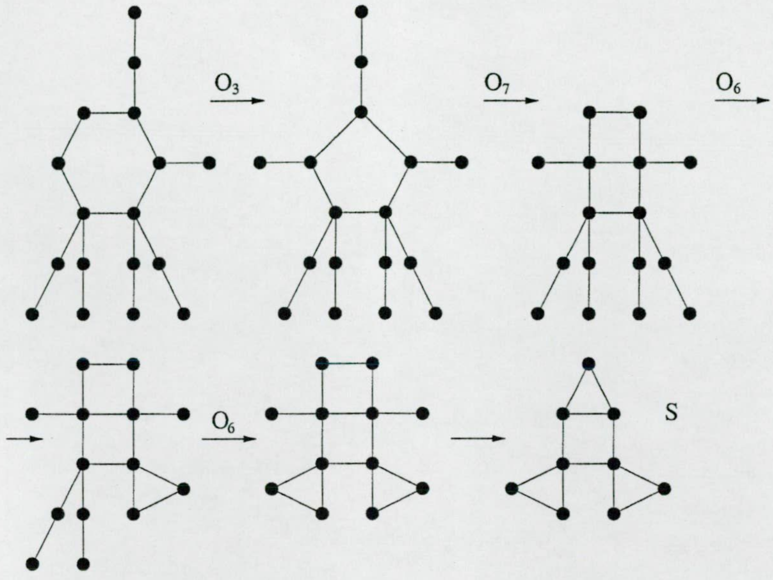


Figure 4.11: Reduction of an excluded minor

Lemma 4.18 *Let $A_i \subset V(G)$, $i = 1, 2, 3$ such that*
(i) $|A_i \cap A_j| \le 2$; $i \neq j \in \{1, 2, 3\}$
(ii) any representation of $G|_{A_i}$ has width at least three
(iii) $G \setminus A_i$ $(i = 1, 2, 3)$ is connected.
Then any representation of G has width at least four.

Proof: Let ϱ be any representation of G. This contains representations of the graphs spanned by A_1 A_2 and A_3. By our assumption, there are intervals I_1, I_2 and I_3 such that I_i $(i = 1, 2, 3)$ is covered by three intervals representing three vertices from A_i. Since two A_i's have at most two common elements, a non-empty intersection among the I_i's would mean width at least four. Hence we can assume that the I_i's are pairwise disjoint intervals, moreover that $I_1 < I_2 < I_3$. Let the union of the intervals assigned to the vertices of the connected $G \setminus A_2$ be I. $I \cap I_1 \neq \emptyset$ and $I \cap I_3 \neq \emptyset$. Hence I covers I_2 proving that the width is at least four. □

Lemma 4.19 *Any representation of the graphs F_3, T_1, T_2, D_4, S, D_3 has width at least four.*

Proof: Figure 4.12 marks the sets A_1, A_2, A_3 showing that the previous Lemma is applicable. □

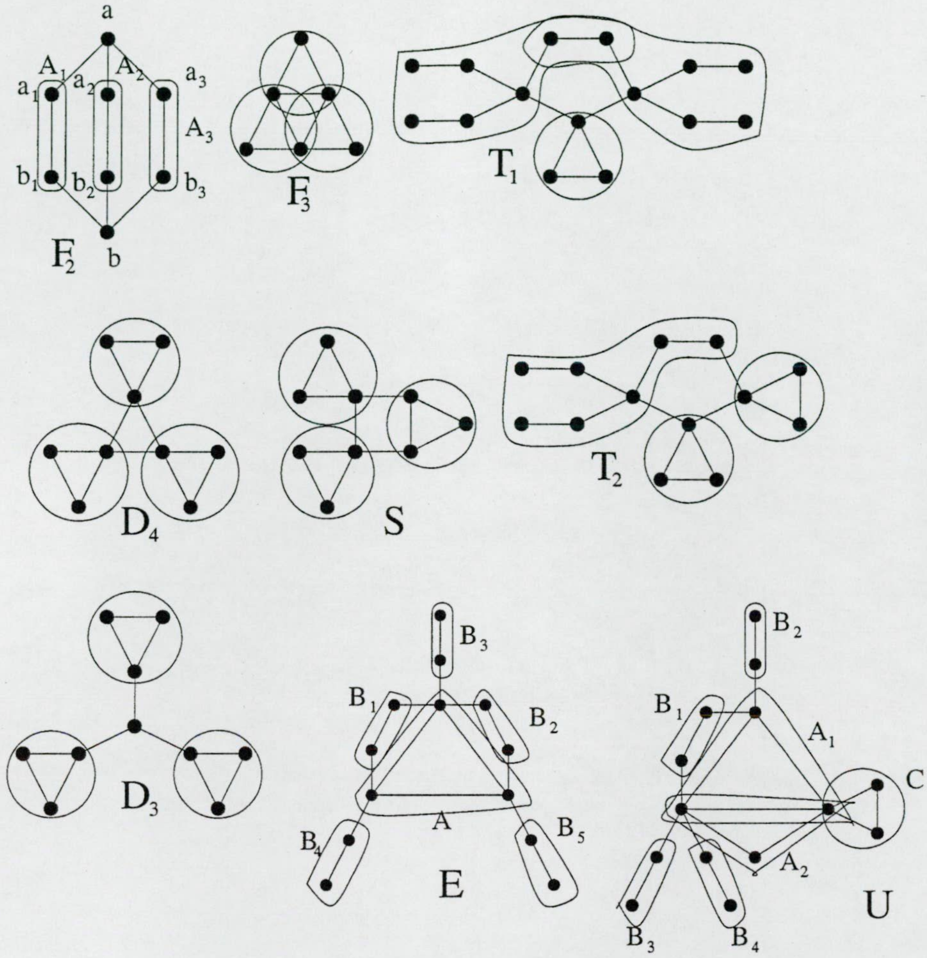


Figure 4.12: Non-reducible graphs

Lemma 4.20 *Any representation of E has width at least four.*

Proof: Let ϱ be a representation of E. A and the B_i's ($i = 1, 2, 3, 4, 5$) are cliques in E (see Figure 4.12, hence the intervals assigned to the vertices in each set are pairwise intersecting. Let $I = \bigcap_{v \in A} \varrho(v)$ and $I_i = \bigcap_{v \in B_i} \varrho(v)$ ($i = 1, 2, 3, 4, 5$). These intervals must be pairwise disjoint, otherwise the width is at least four.

Assume that $I_1 < I$. The two independent edges between A and B_1 implies that each point of $[r(I_1), l(I)]$ is covered by at least two intervals assigned to vertices of $A \cup B_1$. Hence if the width is fewer than four, there is no I_i between I and I_1. Similar argument is true for I_2. Hence $I_1 < I < I_2$. Furthermore there are no other I_i in the interval $[l(I_1), r(I_2)]$, i.e. I_3, I_4 and

I_5 falls into two categories: left from I_1 and right from I_2. One of these categories, say the former, contains two of these intervals. We can assume that two intervals from I_3, I_4 and I_5 are on the left side of I_1. Let $\widehat{B_3}$ be the vertex set we obtain by adding the only neighbor of B_3 to it. Let $J_3 = \bigcup_{v\in\widehat{A_3}} \varrho(v)$. J_3 covers I_3 and I. The same can be said for indices 4 and 5. This proves that I_1 will be covered four times. □

Lemma 4.21 *Any representation of U has width at least four.*

Proof: Let ϱ be a representation of U. A_1, A_2, the B_i's ($i = 1, 2, 3, 4$) and C are cliques in U, hence the intervals assigned to the vertices of each set are pairwise intersecting. Let $I_i = \bigcap_{v\in A_i} \varrho(v)$ ($i = 1, 2$) and $J_i = \bigcap_{v\in B_i} \varrho(v)$ ($i = 1, 2, 3, 4$) and $I = \bigcap_{v\in C} \varrho(v)$. We can assume that these intervals are pairwise disjoint.

First we consider the order of J_1, I_1 and I_2. As in the previous proof, we can assume that $J_1 < I_1 < I_2$ and there are no other J_i's in the interval $[l(J_1), r(I_2)]$. As before, one can show, that $J_2 \not> I_2$ (hence $I_2 < J_1$). Furthermore J_3 and J_4 cannot be on the left side of J_1. So $J_2 < J_1 < I_1 < I_2 < (J_3$ and $J_4)$ in some order (*).

Now we take a look at I. $U \backslash C$ spans a connected subgraph of U, hence all the I_i's and J_i's are on the same side of I. this leaves two possible positions for I in (*): first or last. Both possibilities implies width four, proving our claim. □

4.5 Partial tracks

First we describe a wide class of graphs with path-width at most two.

Definition 4.22 *A graph G is called track graph (or shortly a track) iff it can be represented in the following way. Let P and Q be two vertex disjoint paths. Their vertex sets are $V(P) = \{ p_1, p_2, \ldots, p_l \}$ and $V(Q) = \{ q_1, q_2, \ldots, q_k \}$ (the indices reflect the order of vertices along the path). The graph G contains a disjoint copy of P and Q, and some connections between them. We allow two types of connections. First we can have edges connecting a vertex of P to a vertex of Q. The p_iq_j edge is called ij-chord. Second we allow paths of length two connecting P and Q. We call these paths long chords. A long chord has three nodes, a p_i, a middle node m and a q_j. In this case we say that our long chord has type ij, it is a long ij-chord. We assume that for different long chords the middle nodes are different. We assume that if ij and $i'j'$ occur as types of chords or long chords, then $(i - i')(j - j') \geq 0$,* *i.e. the chords and long chords are not crossing. So G is a track graph if its vertex set is the disjoint union of $V(P)$, $V(Q)$ and M, and its edge set is the disjoint union of $E(P)$, $E(Q)$ and the edges of non-crossing chords, long chords (the last two types are called middle edges).*

Remark 4.23 *A track graph (and hence any subgraph of a track graph) has path-width at most two.*

Proof: Since in a track the chords are non-crossing, we can plan a simultaneous, synchronized discrete walk along the two sides such that

- in each move, on one of the sides, the walk advances to the next vertex on the corresponding path,
- for each ij, that is a type of a chord or a long chord, at some point of the walk, p_i and q_j are simultaneously visited.

After having this walk, one can easily define a sequence of pairs and triplets, proving the upper bound on the path-width: We code the walk by writing down (in order) the pairs of nodes describing the configurations of the walk (the two nodes visited at the same time), and the triplets describing the moves (the node that is not advancing, the old position of the advancing node and its new position). If a configuration reached the two endvertices of a long chord, then we add the triplets of the vertices on the long chord to our list. (One could also formulate the above walk in the language of a cops-and-robber game described in Section 3.1. Then the advance on a side would mean that a cop is standing on a vertex x_i, and a new cop is landing on x_{i+1}. Now the cop on x_i is free, and he can fly away.) □

Definition 4.24 *G is called a partial track graph iff it is a subgraph of a track graph.*

Theorem 4.25 *The following claims are equivalent:*

(i) G is a partial track;

(ii) G has path-width at most two;

(iii) G has no minor listed in the Appendix.

We have seen the implication $(i) \Rightarrow (ii)$. $(ii) \Rightarrow (iii)$ easily follows from the fact that each graph among our excluded minors has path-width at least three. This tedious work was done in Section 4.2 and Section 4.4, by introducing a few operations preserving the property 'having path-width more than two'. The most complicated part of the proof is $(iii) \Rightarrow (i)$. This will be presented in the next section.

4.6 The structure of graphs with path-width two

Let G be a connected graph, and C a cycle of it. The *bridges* of C (in G) are $G|_{E(N)}$, where N runs through the vertex sets of the components of $G-V(C)$, and the edges connecting two nodes of C (the *chords* of C). The *legs of a bridge* B are the common vertices of B and C. The set of legs are denoted by $L(B)$. Let $\{ a_1, a_2 \}$ and $\{ b_1, b_2 \}$ be two-element subsets of $V(C)$. The two pairs are *crossing* iff they are disjoint, and along the cycle, the "a vertices" alternate with the "b vertices". Let U and V be subsets of $V(C)$. We say that U and V are *crossing*, if there are two nodes of U and two nodes of V such that the two pairs are crossing. Two bridges are crossing iff their set of legs are crossing. A bridge is *simple* iff it is a path (and hence its legs are the endvertices of the path).

First we describe the forbidden minors for 2-connected partial tracks. The list of the forbidden graphs is on Figure 4.13. $F_1 \simeq K_4$

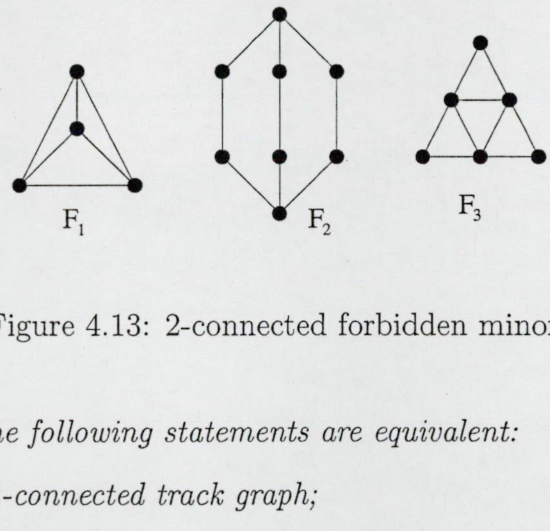


Figure 4.13: 2-connected forbidden minors

Lemma 4.26 *The following statements are equivalent:*

(a) G is a 2-connected track graph;

(b) G is a 2-connected partial track graph;

(c) G is 2-connected and has no F_1, F_2 or F_3 as a minor.

Proof: (a)$\Rightarrow$(b) is obvious.

(b)$\Rightarrow$(c) is easy.

To see (c)$\Rightarrow$(a), let us assume that G is a 2-connected graph without the excluded minors. Let C be the longest cycle of G. W.m.a. that G does not have two crossing bridges, otherwise it would contain a K_4 minor. If a bridge has two non-leg vertices, then there is path of length more than three connecting two vertices of C. Since C is the maximal length cycle, our graph G must contain F_2 as a minor. So we can conclude that all the bridges are chords, or they have one inner node (and since there is no K_4 minor, that inner node is adjacent to two vertices of C). Specially all the bridges are simple.

Each chord (long chord) determines two arcs of C. Let H_1 and H_2 be two chords (or long chords) of C determining different arc-pairs of C. Since we have no K_4 minor one is able to choose A_1 (one of the arcs belonging to H_1) and A_2 (one of the arcs belonging to H_2) such that they have no common inner nodes, and this choice is unique.

If C has no two chords with different arc-pairs then the claim is obvious. Otherwise we call an H chord (or long chord) *side chord* iff for every other chord H', the arcs assigned to H by the previous paragraph are the same. We call this arc the *side arc* of H. Otherwise we call H *a middle chord.* If we have three different side arcs, then we have F_3 as a minor.

Assume there are at most two (in this case there must be exactly two) side arcs. Then their complement on C consists of two paths, and G is a track graph based on these two paths. □

⋆

From now on, we will assume that our graph G does not have any minor from our excluded list (see Appendix). Very often we do not need all excluded minors, sometimes we emphasize this by pointing out which excluded minors we need for a specific claim.

Remark 4.27 *The graphs with path-width one are the caterpillars, see Lemma 4.7. There are two excluded minors for this class of graphs,* K_3 *and* T_1*. If we take three disjoint copies of any of these two, and join them to a new vertex, then we get an excluded minor for path-width two. (This operation was called Y-decomposition in [30].) There are* 20 *non-isomorphic graphs of this type. Let us call them* $Y_1 - Y_{20}$*.*

If we have a block of a graph G, then it is just a cut-edge, a cycle without chord, or a cycle (we call it *base cycle*) with two disjoint arcs (we call them *sides*), and non-crossing chords and/or long chords connecting the two sides. To make the two sides unique, we assume that they are minimal, i.e. their endpoints have an incident chord or long chord. One or both of the arcs can be *extreme* i.e. it is just one node. The inner nodes of the long chords are called *middle nodes.*

It would be good to call the two arcs of C connecting the two sides as *end-arcs.* Unfortunately sometimes this notion is ambiguous (as the notion of base cycle too). If the end-arc has length one or two, and its endpoints are connected by a chord or long chord, then we can modify the base cycle of B by exchanging this chord with the corresponding end-arc. So at the new look of B the old chord is the new end-arc, and the old end-arc now is "just" a chord or long chord. To overcome this problem, we consider the two pairs of end-vertices of sides (they might coincide, this is the case when both sides are extreme), and for each pair, consider the chords, long chords and (current) end-arc belonging to it. All of these items are paths. (See Figure 4.14.)

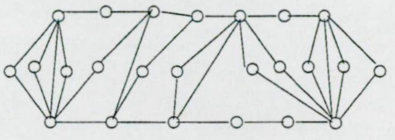


Figure 4.14: A block with no well defined end-arcs

First assume that the two pairs are different (i.e. at least one of the sides are not extreme). Consider the chords, long chords, the arc of the chosen base cycle connecting them. If one of them has length more than two, then it is a well-defined end-arc belonging to the pair. If none of them has length more than two, then all of these paths are considered as potential end-arcs. If the two pairs are not different, and there are two paths among the collected ones with length more than two, then they are the well-defined end-arcs. If just one path has length more than two, then it is a well-defined end-arc, and all the other paths are potential end-arcs. If there is no path with length more than two, then all the paths are potential end-arcs.

$\star$

Let B be a block of G. The *attachments of* B are the subgraphs $G|_{E(N)}$, where N runs through the vertex sets of the components of $G \setminus V(B)$. Since B is a block, each attachment has exactly one node from B. We call this node *the root of the attachment.* The union of the attachments having a common root is called a *bucket.* The *root of a bucket* is the common root of its attachments.

We will classify the attachments and buckets in the following way. An attachment is *complex* iff it has A_1, A_2, A_3 or A_4 as a rooted minor (minor with a root inherited from the initial rooted graph). See Figure 4.15.

A bucket is *wild* iff it contains A_2, A_5 or A_6 as a rooted minor. See Figure 4.16. (Hence any bucket with a complex attachment is wild.)

A bucket is *hard,* iff it has A_2 or A_7 as a rooted minor. See Figure 4.17. (Hence each wild bucket is hard at the same time.)

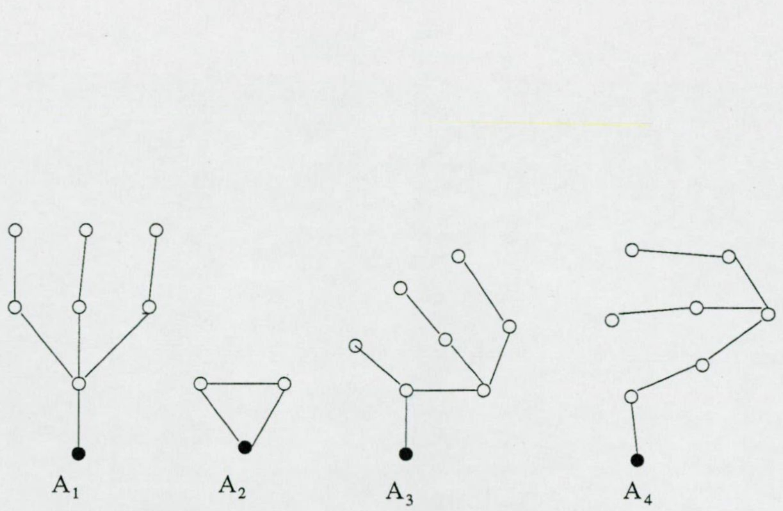


Figure 4.15: Complex attachments

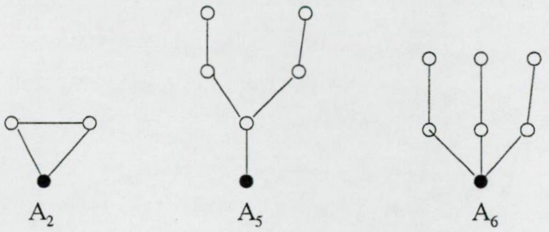


Figure 4.16: Wild buckets

A bucket is *foldable* iff it is hard but not wild. An attachment is important iff it has A_8 as a minor, i.e. it has at least two non-root vertices. An attachment is *tame*, if it is not important, i.e. it has a non-root node connecting by one edge to the root.

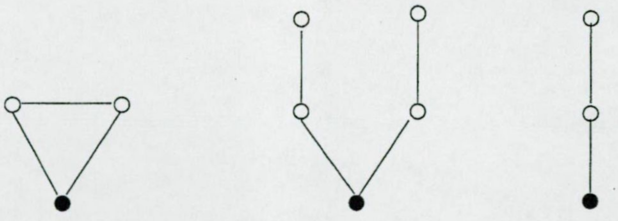
This classification can be extended to a classification of the vertices of the base cycle C of B.

A vertex on C is *empty*, if it is not a root of any attachment. A vertex v is important, if there is an important attachment with root v. I.e. a vertex v is *not important*, if all the attachments with root v are not important. A vertex is hard, (resp. wild, complex) if it has a hard, (resp. wild, complex) bucket rooted at it.

Now in a sequence of claims we establish the major properties of the attachments and buckets of a cycle in a subgraph of a track.

Question 4.28 *Let us assume that G does not have F_1, F_2, F_3, G_1, G_2 and G_3 as a minor. Let B be one of the blocks of G which is not an edge or a chordless cycle.*

(i) If none of the end-arcs of B is not well-defined, and we have two sets of paths as potential end-arcs (i.e. at least one of the sides is not extremal), then both sets have at most one path in it with nonempty inner node.



A$_2$ A$_7$ A$_8$

Figure 4.17: Hard buckets (A_2 and A_7) and an important attachment (A_8)

(ii) If none of the end-arcs is not well-defined, and we have one (two coinciding) set of paths as potential end-arcs, then the set has at most two paths in it with nonempty inner node.

(iii) If one of the end-arcs is not well-defined (we have a set of paths as potential end-arcs), then the set have at most one path in it with nonempty inner node.

Proof: The falsity of the claim immediately implies a minor from $\{G_1, G_2, G_3\}$. □

After this claim we can redefine the notion of end-arc. If among the possible end-arcs there are any with nonempty inner node, then we call them the end-arc. If some of the end-arcs is still undetermined (there is a set of possible end-arcs, all of them have empty inner nodes), then we can choose an arbitrary one and call it the end-arc. Of course this has an effect on the notion of middle nodes and base cycle. The inner nodes of the so far possible end-arcs, which were not chosen to be an end-arc, are called middle nodes from now on.

Question 4.29 *Let us assume that G does not have $F_1 - F_3$, $G_1 - G_3$ and H as a minor. Let B be one of the blocks of G, which is not an edge or a chordless cycle.*

(i) The middle nodes are empty.

(ii) The inner nodes of the sides are simple.

Proof: (i) If a middle node comes from a potential end-arc, then this middle node is definitely empty (otherwise it would have been declared to be an end-arc, and then not considered as a middle node). If it does not come from a potential end-arc, and it has nonempty middle node, then it would have G_1, G_2 or G_3 as a minor.

(ii) Let v be an inner node of one of the sides. Let us assume that v is not simple. Then G has F_7 as a minor. □

So we know, that all the important nodes of a block are on the end-arcs.

Our first goals are to show that after excluding the necessary minors, the following hold:

- $(G1)_0$ In a cycle block B, the sides, end-arcs and their buckets and attachments can be defined such a way, that the important nodes are among the endvertices of the end-arcs, and there is at most one hard bucket on each end-arc. (In order to have a unified notation (see the claim (G1)), we call the sides *extended sides*, and the end-arcs *reduced end-arcs*)

- (G1) In a non-edge block B, the sides can be extended (this way we obtain the notions *extended sides*, left and right reduced end-arcs), and their buckets, attachments can be defined such a way, that the important nodes are among the end-vertices of the reduced end-arcs, and there is at most one hard bucket on each reduced end-arc.

- (G2) An arc is called *short* iff it has length one, or length two with an empty node in the middle. Otherwise an arc will be called *long*. The reduced end-arcs are short.

- (G3) Each reduced end-arc contains at most one complex attachment.

There is one important point what we would like to stress: Even the reduced end-arcs can share an endpoint (i.e. even an extended side can be extreme). If the shared endpoint has attachment(s), then we have the freedom to classify these attachments as left or right. This freedom complicates the matter quite considerably.

Question 4.30 *Let us assume that G does not have F_8 or any of the graphs $I_1 - I_{14}$ as a minor. (See Figure 4.18, where W (resp. H) means any wild (resp. hard) attachment.) Let B be one of the blocks of G, which is not an edge, hence it has a base cycle C.*

(i) There are at most four important nodes in B.

(ii) There are at most two wild nodes on C.

(iii) If there are four important nodes, then there are at most two hard nodes.

Proof: (i) Five or more important nodes on the base cycle would result an I_1 minor.

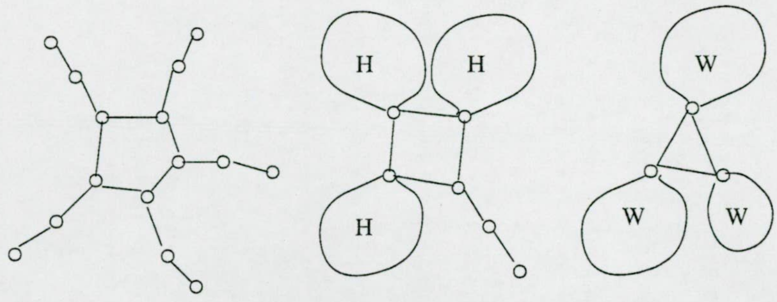


Figure 4.18: New excluded minors

(ii) At least three wild nodes in B would result a minor from $I_5 - I_{14}$.

(iii) At least four important nodes, and three or more wild nodes in B, would result a minor from $I_2 - I_4$. □

To establish our goals, first we handle the case when the block in question is just a cycle C.

First we prove $(\mathrm{G1})_0$. Let us assume that C has four important nodes, and two of them are hard. Then there are two possibilities to match the four important nodes by possible sides. One of them must work. If we have fewer than two hard nodes, then the definition of extended sides is easier: both matchings work. Let us assume that we have three important nodes. In the case when there are at most two hard nodes, we can do the same as before. If there are three hard nodes, then at least one of them is not wild, hence it is foldable. Then make it to be an extremal side, and partition its bucket, s.t. both the left and right end-arc would get a non-hard bucket. This can be easily done.

(G1) does not apply for the case we consider.

To see (G2), we consider the important nodes on our cycle. The excluded I_1 guarantees that there are at most four important nodes. First assume that there are three important nodes. They determine three arcs on our cycle.

- The excluded $L_1 - L_{10}$ guarantee, that there is no important node s.t. the two arcs meeting there are long, and the other two important nodes are hard.

- The excluded $L_{11} - L_{18}$ guarantee, that there is no foldable node s.t. one of the arcs incident to it is long, and the other two important nodes are wild.

- The excluded $L_{17} - L_{22}$ guarantee, that there are no two foldable nodes s.t. the arc determined by them is long, and the third important node is wild.

The three properties above guarantee (G2). If we have fewer than three important nodes, then (G2) is obvious. If we have exactly four of them, then the analysis is easier than the above one, without needing any new excluded minor.

The excluded minors $Y_1 - Y_{20}$ ensures us, that we do not have two complex attachments in a reduced end-arc.

The case when the block is a cycle with a chord (or long chord) is very similar. We just list the situations that must be excluded (and the reason for the exclusion), in order to be able to obtain (G1), (G2) and (G3).

The $M_1 - M_3$ excluded minors guarantee, that among the inner nodes of an end-arc, there are no two hard nodes. If among the inner nodes of an end-arc, there are two important ones, then the endvertices of the chord can not be both complex. This is guaranteed by N_1, N_2 and O_6. If there is a hard inner node in one of the end-arcs, then both end-vertices of the chord can not be wild. This is guaranteed by $O_1 - O_6$, and $I_1 - I_{14}$. If both end-arcs has a hard inner node, then none of the end-vertices of the chord can be wild. This is guaranteed by $P_1 - P_3$ and $I_1 - I_{14}$. This is enough to see that (G1) is true. (G2) is guaranteed by the excluded minors R_1, R_2, $S_1 - S_4$, $T_1 - T_4$, U_1, U_2, and $V_1 - V_5$. The checking of (G3) is not much different from the case of no chord.

The case when there are two chords with disjoint pair of end-vertices is even easier. In some sense, then all vertices of the end-arcs can be considered as inner nodes, and the excluded minors for the case of one chord is more than enough to guarantee (G1), (G2) and (G3).

In the case when there are at least two chords, no two of them having disjoint pair of end-vertices, require additional case analysis. We let the reader go through $W_1 - W_3$, $X_1 - X_9$ to check that these forbidden minors imply the truth of (G1), (G2) and (G3).

⋆

If v is a vertex such that each edge incident to v is a cut-edge, then we think of G as $(G_1, r_1), (G_2, r_2), \ldots, (G_k, r_k)$ rooted graphs are glued together by connecting $r_1, r_2, \ldots, r_k$ to a common neighbor v. We think about the rooted graphs (G_i, r_i) as buckets of v. The $Y_1 - Y_{20}$ excluded minors guarantee that at most two buckets are complex.

⋆

Now we are ready to complete the implication $(iii) \Rightarrow (i)$. Take any block with a cycle, and start a walk on the blocks (the block of the next step is always a block which shares a cut-point with the previous block) in the following manner. If there is a complex attachment, then follow that "direction". This walk connects some blocks in a path-like manner. The walk covers all the blocks with a cycle. The uncovered parts are very much controlled by the fact, that these attachments cannot be complex, and (G2) strictly describe their position. We have a global picture of G, and this picture shows that G is a partial track.

4.7 Recognition of graphs with path-width at most two

Now we sketch a linear time algorithm for deciding whether a given graph has path-width at most two. If the input graph has path-width greater than two, then the algorithm outputs a minor from our excluded list. The algorithm follows our proof of the excluded minor characterization.

First we determine the blocks of the graph. For a linear time algorithm doing so, see [31]. For each block, we check whether it has path-width at most two. This is durable by executing the following steps. First we detect all maximal paths such that each inner node of them has degree two. We group these paths according to their endvertices. For each pair of vertices we substitute the corresponding paths by a single edge connecting the two points. After this, the resulting graph must be a cycle (or in an extreme case a singleton of an edge) with a triangulation by diagonals such that no three triangles having two edges from the base cycle. If this is the case, then we are able to discover the sides and end-arcs, and check the desired properties. A linear time implementation of these steps is straightforward.

Now we start to investigate how these blocks are connected to each other. We build an auxiliary bipartite graph. Its nodes are the blocks, and the cut-vertices (those vertices which belong to more than one blocks) of G. A cut-vertex is connected to a block iff it belongs to it. It is well-known that the obtained graph has no cycle. Since we can assume that our input graph is connected, our auxiliary graph is a tree. For each block we decide whether it is an edge or it contains a cycle. This part of the algorithm can be easily implemented in linear time (again see [31]).

For each block with a cycle and a chord (or long chord), we check that all the middle nodes are empty, and the side nodes are simple. Any deviation gives us a minor from the forbidden list. Now we investigate the important nodes. We classify them as complex, wild, hard, foldable. If there is a cycle in the corresponding bucket, then the node is complex. If not, then the corresponding bucket is involved only at one block, and even a naive implementation gives a linear algorithm.

To check $(G1)_0$, $(G1)$, $(G2)$ and $(G3)$ is straightforward. Any deviation gives us a minor from the excluded list.

For each vertex with only cut-edges incident to it, one can easily characterize the corresponding buckets, as we did it for blocks with cycles. If there are at least three complex buckets, then we have a minor from $Y_1 - Y_{20}$.

Otherwise we are able to execute the walk at the end of the proof, and we obtain that the input graph is a partial track, its path-width is at most two.

We obtained the following theorem.

Theorem 4.31 *There is a linear time algorithm for recognizing graphs with path-width at most two.*

Chapter 5

New minor-monotone graph parameters

5.1 Arc-width of graphs

As we mentioned in the Introduction, any minor-closed class of graphs can be characterized by excluded minors. The theorems in Chapter 4 used the language of interval representations. That is why the following modification arises so naturally. we will indeed define a minor-monotone graph parameter. So if we bound this from above by one or two, we can get similar theorems as for path-width in Chapter 4.

Definition 5.1 *We might assign an arc of a base circle to each vertex of a graph such that adjacent vertices correspond to intersecting arcs. This is called an arc-representation of the graph. The width (in a representation) of a point p of the base circle is the number of arcs containing p. The width of an arc-representation is the maximum width of the points of the base circle. (This is not the maximum number of pairwise intersecting arcs, as it was for the intervals.) The arc-width of a graph G is the minimal possible width of such arc-representations, $aw(G)$ in notation.*

In a more formal way we use the following notations. Take the unit circle, which has points with coordinates $(\cos x, \sin x)$ where $x \in [0, 2\pi)$. The correspondence between the points of the unit circle and the central angles is a bijection. So henceforth we will refer to a point with its central angle. In this way, let us denote the clockwise arc on the unit circle from x_1 to x_2 as $arc(x_1, x_2)$.

Since the union of two arcs is an arc (we consider the whole circle as an arc), one can easily prove the minor-monotonicity of the newly introduced graph parameter.

Lemma 5.2 *Arc-width is minor-monotone.*

Proof: Assume we have a graph G with a given arc-representation ϱ.

(i) Deletion of an edge: After deleting an edge e, we can take the same representation for the graph $G \setminus e$. While the width remains the same.

(ii) Contraction of an edge: Let $xy = e$ be the contracted edge, and let u be the new vertex corresponding to the old x and y. Let $\varrho(x)$ and $\varrho(y)$ be the representing arcs of x and y. If we set $\varrho(u) := \varrho(x) \cup \varrho(y)$, delete $\varrho(x)$ and $\varrho(y)$, and leave all the other representing arcs unchanged, then we get an arc-representation of G/e without increasing the width. □

Definition 5.3 *We say that an arc-representation ϱ of G is a (u, w)-representation iff $u = w_{min}(\varrho)$ and $w = w_{max}(\varrho)$, where w_{min} and w_{max} denotes the minimum and maximum width of the representation respectively. $w_{max}(\varrho) \geq w_{min}(\varrho)$, moreover equality cannot hold. Let us call this minimum-maximum pair $wW(\varrho)$. Consider the natural order, so $wW(\varrho) < wW(\varrho')$ iff $w_{max}(\varrho) < w_{max}(\varrho')$ or $w_{max}(\varrho) = w_{max}(\varrho')$ and $w_{min}(\varrho) < w_{min}(\varrho')$. Finally $mM(G) := \min_\varrho wW(\varrho)$, where ϱ is an arc-representation of G. mM is a minimum-maximum pair as well, and is a graph parameter. We consider the natural order on this parameter. Namely $(u_1, w_1) < (u_2, w_2)$ iff $w_1 < w_2$ or $w_1 = w_2$ and $u_1 < u_2$.*

Arc-width is a natural modification of path-width. There is a quantitative connection, not just a formal one. The next lemma shows that the two measures are within a 2-factor. By tradition pw is one less than the maximum width in an optimal interval representation. But the maximum width is the natural parameter for us. By this reason let us use the following notation: $pw^* = pw + 1$.

Lemma 5.4 *Let ϱ be an arc-representation of G. Then*

(i) $pw^*(G) \leq w_{max}(\varrho) + w_{min}(\varrho)$

(ii) $\left\lceil \frac{1}{2}(pw^*(G) + 1) \right\rceil \leq aw(G) \leq pw^*(G)$

Proof: (i) Let $i := w_{min}(\varrho)$, $j := w_{max}(\varrho)$. The minimal width is i in ϱ. So there is point x of the base circle where the width is i. Cut the base circle at x to get a line. Delete the i arcs we cut. Replace each of them by an interval, which includes all non-cut arcs, which became now intervals. In this way we get an interval representation with width $i + j$.

(ii) The second inequality is trivial. The first one follows from (i). Because $pw^*(G) \leq w_{max}(\varrho) + w_{min}(\varrho)$ and $w_{max}(\varrho) > w_{min}(\varrho)$, hence $\frac{1}{2}(pw^*(G)) < w_{max}(\varrho)$ for any ϱ. So we obtain $\frac{1}{2}(pw^*(G)) < aw(G)$. □

Lemma 5.5 *$aw(T) = pw^*(T)$ where T is a tree. (The same statement holds for any graph which has no cycle.)*

Proof: We prove by induction that from any arc-representation of T, we can construct an interval-representation of T with the following properties:

- Every point p of the base circle has an image point p' on the real line s.t. this correspondence is continuous.
- The width of p' is at most the width of p for any point p of the base circle.

Let us call such an interval-representation 'good'.

If T is a single vertex, then we are easily done.

Assume now that the statement is true for any tree with at most $k-1$ vertices. Consider a tree T with k vertices. Delete a leaf v of T getting a graph T' with $k-1$ vertices. Let u be the only neighbor of v. Let ϱ be an arbitrary arc-representation of T. ϱ induces an arc-representation $\varrho|_{T'}$ of T'. It does not make any confusion if we write $\varrho(T')$ instead of $\varrho|_{T'}(T')$. Any arc-representation of T' implies a good interval-representation of T' by assumption. Hence ϱ induces a good interval-representation ϱ^* on T'. Consider the arc $\varrho(u)$ representing the vertex u of T'. $\varrho(u)$ induces an interval $\varrho^*(u)$ representing u in ϱ^*. Adding the vertex v (and the edge vu) to T' is represented by an arc $\varrho(v)$ intersecting $\varrho(u)$. ϱ^* was a good interval-representation, hence If $\varrho^*(v)$ intersects $\varrho^*(u)$, then we get a good extension of ϱ^* to the whole T.

This finishes the inductive argument. □

Lemma 5.6 $aw(K_n) = \left[\frac{n}{2}\right] + 1$

Proof: Let $V(K_n) = \{v_1, \ldots, v_n\}$. First we prove the upper bound by construction.

Consider the following arc which is slightly bigger than a half-circle:

$$a_1 = arc\left(0, c\frac{2\pi}{n}\right) \text{ where } c = \left[\frac{n}{2}\right] + 1$$

Let ϕ denote the clockwise rotation with $\frac{2\pi}{n}$, and ϕ^k that the rotation is repeated k times. Let $\varrho(v_i) = \phi^i a_1$. Then consider $\{\varrho(v_1), \ldots, \varrho(v_n)\}$. Clearly any two arcs in this set intersect, so ϱ is an arc-representation of K_n.

To see the lower bound, we refer to Lemma 5.4. □

5.2 Arc-width of the complete bipartite graph

Lemma 5.18 (see later) made us also curious about the bipartite graphs. First we give two easy constructions, showing the possible feature of this case.

Lemma 5.7 *$mM(K_{s,s}) \leq (s-1, s)$.*

Proof: Let the two color-classes of $K_{s,s}$ be $\{x_1, \dots, x_s\}$ and $\{y_1, \dots, y_s\}$. Consider the following arc-representation of $K_{s,s}$:

$$a_1 = arc\left(-\varepsilon, (s-2)\frac{2\pi}{s} + \varepsilon\right)$$

$$b_1 = arc\left(\varepsilon, \frac{2\pi}{s} - \varepsilon\right)$$

Let ϕ denote the clockwise rotation with $\frac{2\pi}{s}$, and ϕ^k that the rotation is repeated k times. Let $\varrho(x_i) = \phi^i a_1$, and $\varrho(y_i) = \phi^i b_1$. Clearly $\varrho(x_i) \cap \varrho(y_j) \neq \emptyset$, so ϱ is an arc-representation of $K_{s,s}$. The width of the points of the base circle is easy to count:
$width\left(\frac{2\pi}{s} \pm \varepsilon\right) = s$, $width(x) = s-1$ otherwise. □

Lemma 5.8 *$mM(K_{s,s}) \leq (s/2, s)$ if s is even.*

Proof: Take four arcs, say x_1, x_2, y_1, y_2 s.t. $x_i \cap y_j \neq \emptyset$, $x_1 \cap x_2 = \emptyset$, and $y_1 \cap y_2 = \emptyset$. For the moment this is an arc-representation of $K_{2,2}$ with color-classes $\{x_1, x_2\}$ and $\{y_1, y_2\}$. If we multiply all of these arcs $s/2$ times, then we get a proper representation of $K_{s,s}$ with x_1, x_2 and all of their multiples as one color-class. □

Combining these two ideas, we get the following slightly more general result.

Lemma 5.9 *$mM(K_{s,s}) \leq \left(\left(1 - \frac{1}{p}\right) \cdot s, s\right)$, where p is a prime factor of s.*

Proof: Consider the following arc-representation of $K_{s,s}$: Partition both color-classes into p equal groups. Let every group be represented by an arc with multiplicity $\frac{s}{p}$. Let the arcs corresponding to one color-class be disjoint, and let the other color-class behave like in the first construction. Namely they intersect every but two endpoint of the arcs representing the other color-class. This gives the result as before. □

Clearly $pw^*(K_{s,s}) \leq s+1$, see Figure 5.1. Hence the following is just a trivial observation:

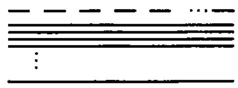


Figure 5.1:

Lemma 5.10 $aw(K_{s,t}) \leq \min(s+1, t+1)$.

In the case $s = t$ the previously given constructions are also best possible:

Lemma 5.11 $aw(K_{s,s}) = s$.

Proof: We have to prove $w_{max}(\varrho(K_{s,s})) \geq s$ for all arc-representations ϱ of $K_{s,s}$. Let the two color-classes — P and K — be called red and blue. $\mathcal{F} := \{\varrho(v) : v \in P\}$; this is a system not necessarily a set. We call the elements of $\mathcal{F}$ red arcs. Fix a direction of the base circle, clockwise say. Every arc $I \in \mathcal{F}$ has a left endpoint and a right endpoint in this direction. Let the set of left resp. right endpoints of the arcs in $\mathcal{F}$ be denoted by L and R.

1. W.m.a. the $2s$ endpoints to be different. Because this can be reached with a small movement of the endpoints.

2. W.m.a. $\cap_{I \in \mathcal{F}} I = \emptyset$. If this would be false, then the width is already at least s, so we are done.

3. W.m.a that if $I \neq J \in \mathcal{F}$, then $I \not\subset J$.

If there would be two arcs $I \subset J$, then the endpoints satisfy: $l(J) < l(I) < r(I) < r(J)$. Let now $I' := (l(J), r(I))$ and $J' := (l(I), r(J))$. Then clearly the width did not change, and now $I \not\subset J$. Also if any arc A intersects I (and hence J), then A intersects both I' and J' because $r(I) \in I'$ and $l(I) \in J'$. So we still have a representation of $K_{s,s}$.

It is easy to see that 3 is satisfied after a finite number of this kind of modifications. Because the length of the longer arc after the change is strictly less than the length of the longer arc before, and the possible length of the arcs are determined by the possible $2s$ endpoints (hence finite).

4. For every arc $I \in \mathcal{F}$ consider a candidate complementary arc I' which is defined as follows: $l(I') := r(I)$, and $r(I') := l(J)$, where $J \in \mathcal{F}$ is the arc s.t. if I' intersects J, then I' intersects every arc of $\mathcal{F}$. In this way we defined a set system of arcs, call it $\mathcal{F}' = \{I' : I \in \mathcal{F}\}$. The above described $I - I'$ correspondence is a bijection. Moreover the $2s$ different endpoints of the arcs of $\mathcal{F}'$ are exactly the same as the endpoints of the arcs of $\mathcal{F}$. Let us prove this. Clearly the left endpoints of the arcs in $\mathcal{F}'$ are different. Hence we only have to show that two right endpoints cannot coincide. Assume

that $l(I'_1) < l(I'_2)$ and $r(I'_1) = r(I'_2)$. Then by 3 and the definition of I'_2, $r(I'_2) > l(I'_1)$. But this contradicts the definition of $r(I'_1)$.

5. Consider all the arcs in $\mathcal{F}$ or in $\mathcal{F}'$ as disjoint arcs. Then glue them together at the common endpoints. Then this 'snake' covers the base circle 'exactly' $s-1$ times. More precisely the width of the system of these $2s$ arcs is $s-1$ everywhere, except at the endpoints of the arcs, where the width is s. To see this, consider a point, $l(I)$ say, $I \in \mathcal{F}$. Cut the base circle at $l(I)$ to get the non-negative real line with $l(I) = 0$, and the natural $<$ relation. $l(I) \in J'$ if $r(I) < r(J)$. $l(I) \in J$ if $l(I) < r(J) < r(I)$. And of course for every $J \in \mathcal{F}$, $J \neq I$, exactly one case occur.

Consider now an arc-representation of $K_{s,s}$, where the red vertices are represented by the elements of $\mathcal{F}$. Then we can assume that the blue vertices are represented by arcs from $\mathcal{F}'$. 'A priori' they could be 'bigger', but in any case an arc J representing a blue vertex must contain a $J' \in \mathcal{F}'$ as a subarc. So in this sense we can contract the 'blue' arcs into arcs from $\mathcal{F}'$. Of course here we can get an arc with multiplicity greater than one, and some elements of $\mathcal{F}'$ with multiplicity zero. And this is the point. Henceforth the arc-representation of $K_{s,s}$ is considered as the arcs of $\mathcal{F}$ with multiplicity one, and the arcs of $\mathcal{F}'$ with non-negative multiplicities.

6. The points of L and R divide the base circle into $2s$ open arcs. Let us call them elementary arcs. We can look at the width over an elementary arc, e say. There are some arcs representing red vertices which contain e. Moreover there are some candidate arcs $A_1, \ldots, A_k$ of $\mathcal{F}'$ containing e. There are associated multiplicities to these arcs, respectively $\mu(A_1), \ldots, \mu(A_k)$ say. Assume there are q red arcs containing e. Let $\mathcal{F}'_e$ denote the set of blue arcs containing e. From 5 we know that $q + |\mathcal{F}'_e| = s-1$. If $q + \sum_{A \in \mathcal{F}'_e} \mu(A) > s-1$, then we are done. Otherwise we get the following inequality:

$$\sum_{A \in \mathcal{F}'_e} \mu(A) \leq |\mathcal{F}'_e|$$

So the number on the right-hand side is the number of terms on the left-hand side.

7. For every elementary arc e, we define its successor e^* as follows. Cut in mind the base circle at $r(e)$. Consider the first arc $I_e \in \mathcal{F}$, which is completely after $r(e)$. (Observe that 'first' is well-defined. Because if an arc starts first, it also has to end first by 3.) There is an elementary arc beginning at $r(I_e)$. That is defined to be e^*. Formally $l(e^*) := \min_{I \in \mathcal{F}: r(e) \notin I} r(I)$.

Let us define a directed graph D. The elementary arcs are the vertices of D, and the edges are of form (e, e^*). More precisely we define the edges to be geometric objects, namely (e, e^*) is the arc of the base circle (in clockwise order) from the middle of e to the middle of e^*.

8. Every elementary arc has out-degree one in D, hence there is a directed circuit C in D.

9. The edges of C (glued together as in 5) cover the base circle homogeneously t times. By this we mean the following: If we consider an arbitrary point p of the base circle, then there exist exactly t edges going over p.

Let $I' \in \mathcal{F}'$ be the first arc which is completely after p. Every edge f over p has a tail e s.t. e is an elementary arc disjoint from I'. This is a consequence of the definition in 7. Also vice versa. If an edge $f \in D$ is not over p, then the tail e of f is an elementary arc which intersects I'. We can interpret this result in another way too. Namely, whenever we take an arbitrary arc $I' \in \mathcal{F}'$, then the number of elementary arcs intersecting I' is a constant positive number, c say.

10. Consider now the inequalities of 6 only for the vertices of C. Let $V(C) = \{e_1, \ldots, e_p\}$. Sum up all of these inequalities:

$$\sum_{I \in \mathcal{F}'_{e_1}} \mu(I) \leq |\mathcal{F}'_{e_1}|$$

$$\vdots$$

$$\sum_{I \in \mathcal{F}'_{e_p}} \mu(I) \leq |\mathcal{F}'_{e_p}|$$

$$\overline{\sum_{e \in C} \sum_{I \in \mathcal{F}'_e} \mu(I) \leq \sum_{e \in C} |\mathcal{F}'_e|}$$

By 9 the left-hand side is $c \sum_{I' \in \mathcal{F}} \mu(I')$. Hence by 6, the right-hand side is $c \cdot s$. We obtained the following:

$$c \sum_{I' \in \mathcal{F}} \mu(I') \leq c \cdot s$$

c is positive, so simplification gives:

$$\sum_{I' \in \mathcal{F}} \mu(I') \leq s$$

But we know that here equality holds. This is only possible if equality hold everywhere in the above inequalities. Hence the width of the representation over an elementary arc $e \in C$ is exactly $s - 1$. But then at the endpoint of an elementary arc the width is at least s. $\square$

Comparing the results of Lemma 5.6 and Lemma 5.11, one can see that $K_{s,s}$ has much less edges than K_{2s}, but its arc-width is almost the same.

Hence it is very natural to ask how many edges can be deleted from K_n without decreasing the arc-width?

Another natural question is to determine the least possible minimal width, when the arc-width of the representation of $K_{s,s}$ is s.

5.3 Arc-width of non-connected graphs; the mM parameter

In the excluded minor theorems for arc-width some disconnected excluded minors arise.

Remark 5.12 *Let G be the disjoint union of two graphs G_1 and G_2. In notation $G = G_1 \cup G_2$. Then the following hold:*

$$\max\{aw(G_1), aw(G_2)\} \leq aw(G) \leq aw(G_1) + aw(G_2)$$

It is natural and necessary to study the arc-width of the union of complete graphs.

Lemma 5.13 $aw(K_s \cup K_t) = \left\lceil \frac{s+t}{2} \right\rceil$.
Hence also $aw(K_s \cup K_s) = s = pw^*(K_s \cup K_s)$ (i.e. we cannot do 'better' with arcs than with intervals).

Proof: Consider a representation ϱ of $K_s \cup K_t$. ϱ induces a (u_1, w_1)-representation of K_s, and a (u_2, w_2)-representation of K_t, where $u_1 + w_1 \geq s$, and $u_2 + w_2 \geq t$ by Lemma 5.4. Then $aw(K_s \cup K_t) \geq \max(u_1 + w_2, u_2 + w_1) \geq \frac{(u_1+w_2)+(u_2+w_1)}{2} \geq \frac{s+t}{2}$

A representation satisfying the equality can be easily constructed. □

Clearly G has path-width at most k is the same as G has a $(0, k+1)$ arc-representation.

However now we have a minor-monotone graph parameter, so by the Graph Minor Theorem we can make some excluded minor theorems.

5.4 Excluded minor theorems for mM

$mM(G) = \min wW(\varrho)$, where ϱ is an arc-representation of G.

mM is a minimum-maximum pair, and is a graph parameter.

We can easily see that if G is a minor of H, then $mM(G) \leq mM(H)$.

Lemma 5.14 *mM is a minor-monotone graph parameter.*

Proof: Same as Lemma 5.2. □

If a graph G has an (i, j)-representation, we denote this fact by $G \in (i, j)$. Let us call the class of excluded minors for $mM = (i, j)$ by $obs(i, j)$. Hence for any graph $G \in obs(i, j)$, $G \notin (i, j)$, and for any minor H of G, $H \in (i, j)$.

So far the easiest non-trivial 'width-type' question was to ask the excluded minors for $pw1$. Now we have immediately two questions instead: $obs(0, 2) =$? and $obs(1, 2) =$?

Lemma 5.15
$obs(0, 2) = K_3, Y_1$, where Y_1 is the graph shown in Figure 3.1.
$obs(1, 2) = Y_1, K_3 \cup K_2, K_4^-$, where K_4^- arises from K_4 with one edge deleted.

Proof: $obs(0, 2) = obs(pw1)$, hence the first claim is well-known. See also Lemma 4.7.

The graphs in $obs(1, 2)$ are not representable in $(1, 2)$, but all of their minors do. The interesting fact here is that K_3 is in $(1, 2)$, see Figure 5.2. Every other minor is easy to check.

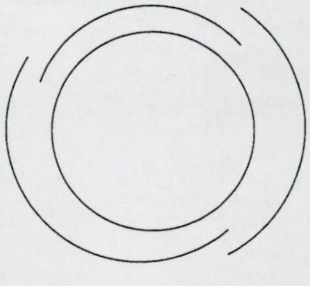


Figure 5.2:

What remaining is to prove that all graphs having no minor from the list is in fact representable in $(1, 2)$.

Let G be such a graph. If G is a tree, then the excluded Y_1-minor ensures us that G is a caterpillar, and we are done by Lemma 5.5. So assume that G has a cycle C. If there would be a cycle with a chord, then we find a K_4^- minor. So C is chordless. Consider a vertex $v \notin C$. Clearly v cannot be adjacent to more than one vertex of C. Can we have two adjacent vertices $v_1 \notin C$ and $v_2 \notin C$? No, because that gives us a $K_3 \cup K_2$ minor. So G must be a chordless cycle with some "hairs". More precisely if we contract C in G to a single point, we get a star. These graphs are clearly $(1, 2)$-representable. Because a cycle is in $(1, 2)$, and the remaining vertices can be added as points of the circle, where the width was one. □

Remark 5.16 $obs(0,3) = obs(pw2)$. *About this see Chapter 4.*

By looking at any graph G in $obs(0,3)$, we can observe that $mM(G) = (0,4)$ or $mM(G) = (1,3)$. We don't see a direct proof of this, a very wild conjecture would be the following: If $G \in obs(0,k)$, then either $mM(G) = (0,k+1)$ or $mM(G) = (1,k)$.

Lemma 5.17 *If $G \in obs(0,3)$, $H \in obs(0,2)$, and G is $(1,3)$-representable, then $G \cup H \in obs(1,3)$.*

Proof: $H \in obs(0,2)$ means that either $H \in (0,3)$ or $H \in (1,2)$.

If $H \in (0,3)$, then there is a point p of the base circle where the width is three. Cut the base circle at p, because a $(1,3)$-representation of $G \cup H$ cannot have other arcs including p. In this way we get the real line, and G should be represented by intervals. But $G \in obs(pw2)$, hence $G \cup H$ is not $(1,3)$-representable.

If $H \in (1,2)$, then the width is at least one at every point of the base circle. $G \notin (0,3)$ implies $G \notin (1,2)$. Hence $G \cup H \notin (1,3)$.

Assume that $G \cup H$ is not minor-minimal in $obs(1,3)$. If an edge of H is deleted or contracted getting a graph H', then H' is $(0,2)$-representable. Hence $G \cup H'$ is $(1,3)$-representable. If an edge of G is deleted or contracted getting a graph G', then G' is $(0,3)$-representable. Hence $G' \cup H$ is $(0,3)$-representable too. These contradictions prove the claim. □

Alright, but what kind of connected graphs are there in $obs(1,3)$?
We know that any star-composition of three disjoint (not necessarily different) graphs of $obs(0,2)$ yields a graph, which is in $obs(0,3)$, and all minors of these graphs are in $(0,3)$ (see [30]). Particularly in $(1,3)$ too. So these graphs are in $obs(1,3)$ too.
But the nice thing is the following:

Lemma 5.18 *K_5 and $K_{3,3}$ is in $obs(1,3)$.*

Proof: Because of symmetry it is easy to check that all minors of these two graphs are in $(1,3)$.

That K_5 is not in $(1,3)$ is a consequence of Lemma 5.4.

Consider now $K_{3,3}$. Assume this is in $(1,3)$. Let v be the vertex which produces the minimal width one. This means that $K_{3,3} \setminus v = K_{2,3}$ must be represented in $(0,3)$. This is possible essentially only in one way. Namely there are three disjoint intervals, a, b, c say, corresponding to one color-class, and the other two intervals, x, y say, intersect each of a, b, c. Now the interval corresponding to v must intersect a, b, c, which causes width four. This contradiction completes the proof. □

So the $(1,3)$-representable graphs form a subclass of planar graphs. Proper subclass, because we saw that even some trees are not $(1,3)$-representable.

Chapter 6

Open questions

In this Chapter we collect some problems which we think to be interesting. Some of them are really the 'results' of our research in some sense. A few are around for a while. While some questions are possibly nonsense. We try to motivate them, and describe what is already known.

Conjecture 6.1 *If you color the edges of the finite graphs with finitely many colors, then GMT remains true.*

Here the minor relation is refined s.t. the colors must match. This is said to be a consequence of a graph minor theorem for hypergraphs with labelled edges. But it would be nice to have a proof based on the GMT and a standard coloring trick.

$\star$

Conjecture 6.2 *If a graph G has minimum degree $k+1$, then G has a k-connected minor.*

The natural setting is of course to ask if minimum degree k implies a k-connected minor or not. This question is completely answered. For $k \leq 4$ it is true. But there is a counter-construction for $k \geq 5$ as follows. Take two copies of K_k minus a perfect matching. Connect every vertex to an extra vertex v. The resulting graph has minimum degree k, but has no K_k-minor.

So one should think of the $+1$ as $+c$, where c is a constant we would like to minimize.

Question 6.3 *What are the minor-minimal* 5*-connected graphs?*

What are the minor-minimal graphs with minimum degree 5*?*

If we replace the number 5 with 3 or 4 the two classes coincide. But they are really different for $k \geq 5$. See e.g. [12] for some details.

$\star$

Conjecture 6.4 *$cn(D) \leq 2cn^*(D)$, where $cn(D)$ is the cop-number for directed graphs in Definition 3.19, and $cn^*(D)$ is the cop-number in Definition 3.13.*

This is motivated in Section 3.2.

Conjecture 6.5 *Define a 'non-trivial' minor-monotone graph-parameter for directed graphs.*

This is a hard question.

Conjecture 6.6 *See [14]. For every integer k there is an integer N s.t. every directed graph D with $cn(D) \geq N$ has a minor isomorphic to J_k.*

Conversely we know that $cn(J_k) = k$ by Lemma 3.22. So a 'big' J_k implies 'big' cn parameter.

T.Johnson seems to have a proof if additionally D is planar. His proof is said to be technical, and a more natural proof is wanted.

$\star$

Conjecture 6.7 *See [15]. Among the excluded minors for pwk the trees have the most vertices.*

This seems to be very plausible. Even an inductive argument or so should work. But we do not see a proof of this.

Conjecture 6.8 *The 3-connected excluded minors for pw3 are: K_5, W, O, Q, P_4, $K_{3,3}^+$*

Here W = the Wagner graph, O = edge-graph of the octahedron, $Q = \Delta Y(O)$, P_4 = the cube or 4-prism, $K_{3,3}^+$ = complete bipartite graph on 3 vertices, plus an edge. See Figure 6.1.

The following was proved in [12].

Lemma 6.9 *Every 4-connected graph has K_5 or O as a minor.*

Moreover the condition 4-connected can be relaxed to minimum degree $\geq$ 4. So we only need to worry about those 3-connected graphs, which have minimum degree three.

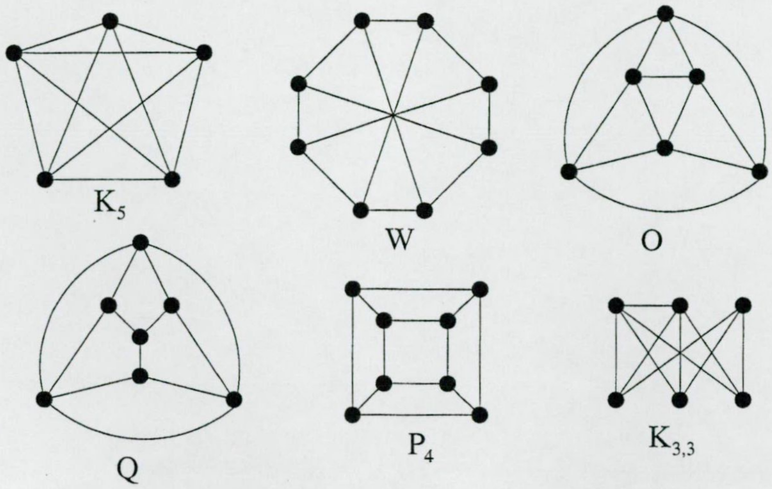


Figure 6.1: Minor-minimal $pw4$ graphs

⋆

The next two open questions are from Section 5.2. There is no strong reason to conjecture the answer, so we state them as questions.

Question 6.10 *How many edges can be deleted from K_n resp. $K_{s,s}$ without decreasing the arc-width?*

This kind of questions are very natural in graph theory. Hopefully this is not a difficult one.

Conjecture 6.11 *Determine the least possible minimal width m of an arc-representation of $K_{s,s}$, when the arc-width is s.*

There is a costruction in Lemma 5.9, showing that $m \leq \left(1 - \frac{1}{p}\right) s$, where p is the smallest prime factor of s. We wonder if this is best possible.

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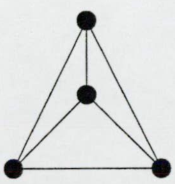
Appendix A

The excluded minors for path-width two



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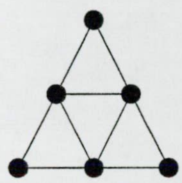

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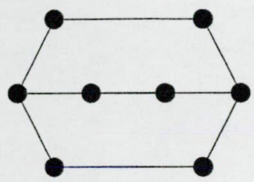

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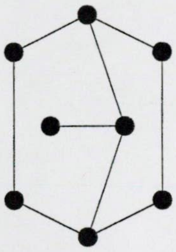
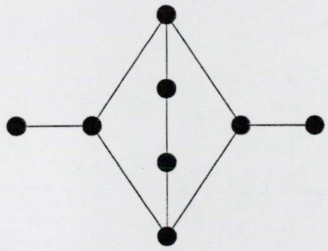


G_1



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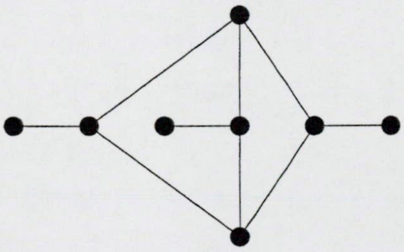

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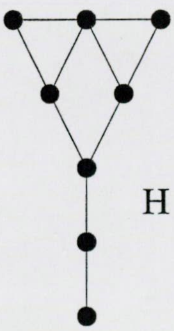

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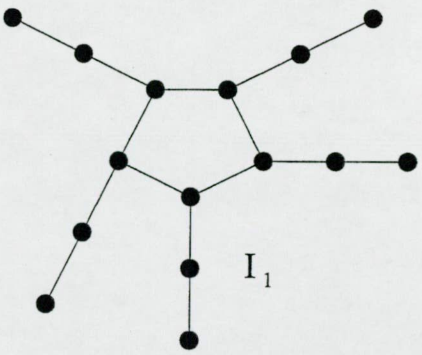

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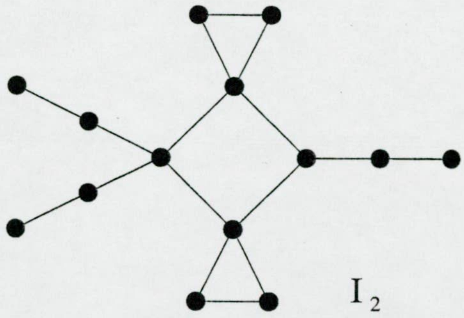

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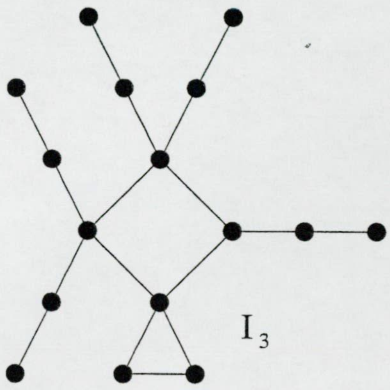

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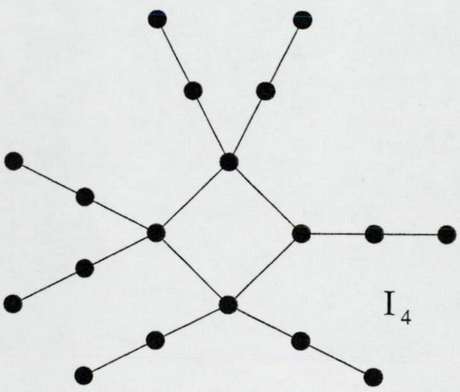

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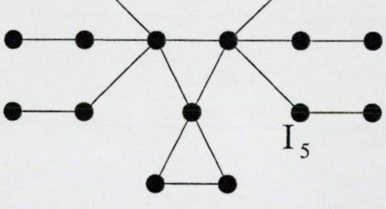

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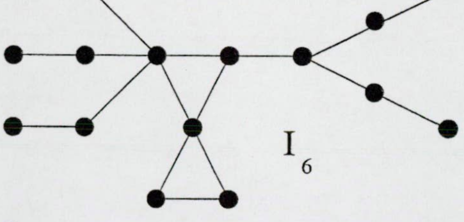

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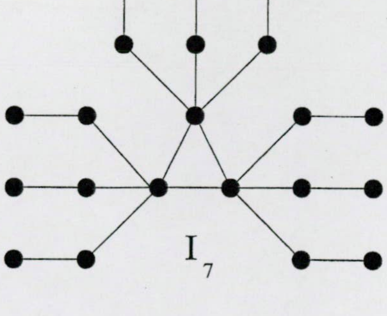

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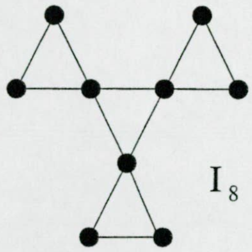

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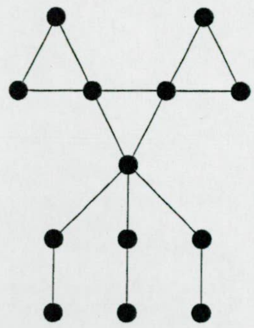

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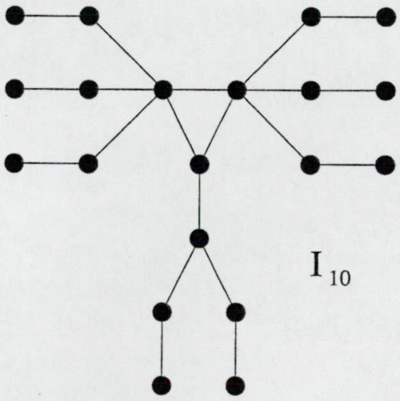

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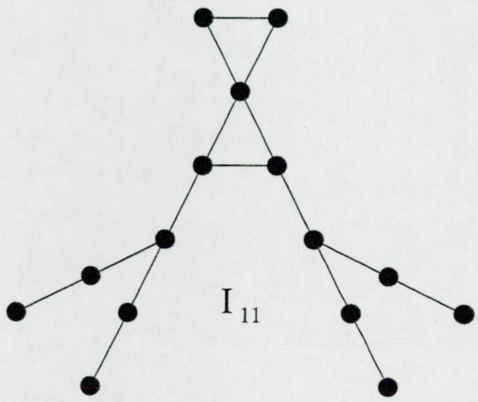

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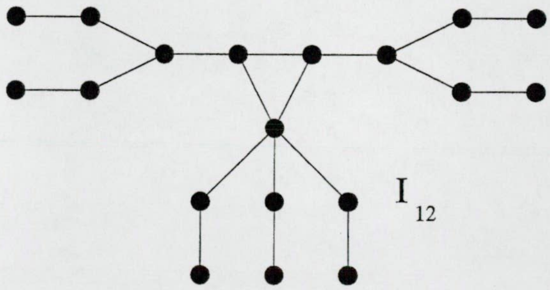

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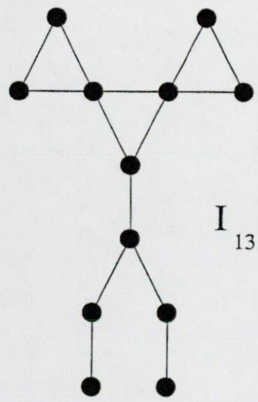

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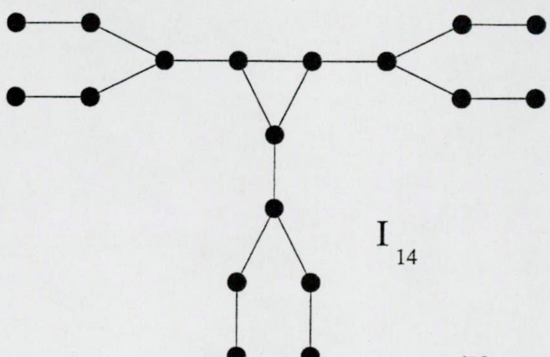

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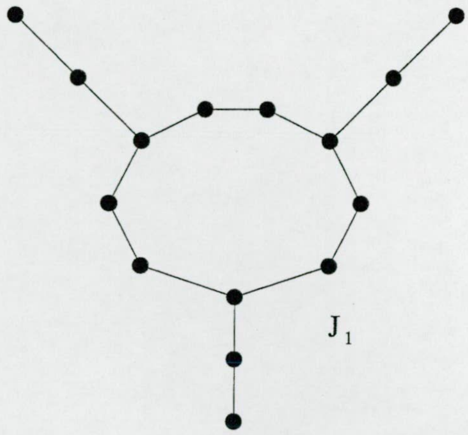

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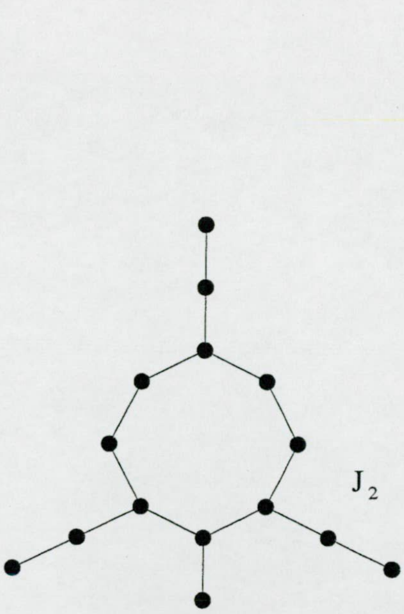

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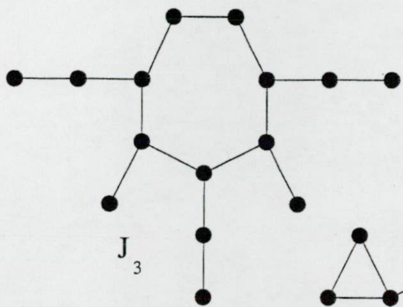

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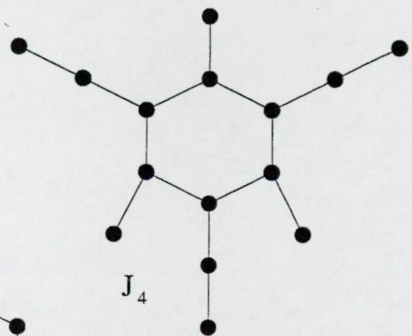

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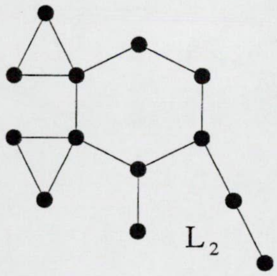

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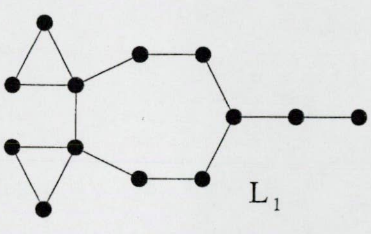

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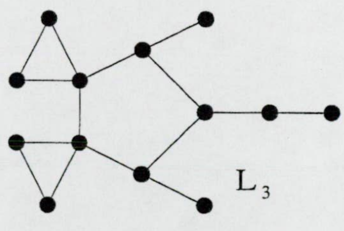

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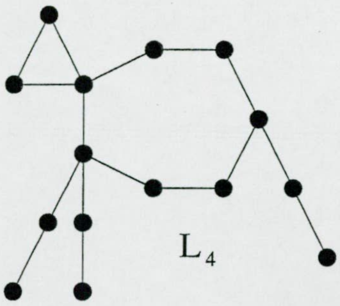

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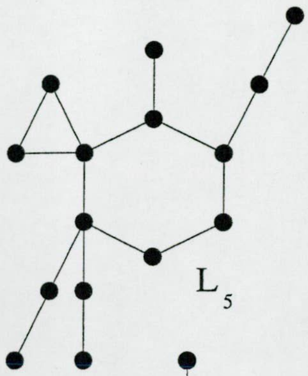

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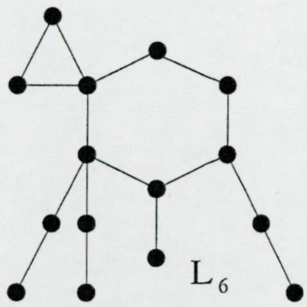

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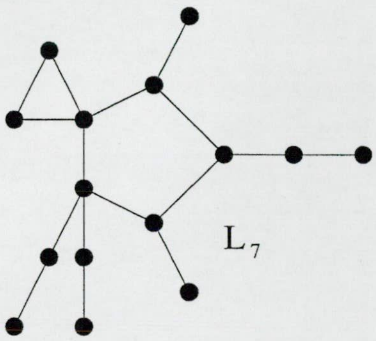

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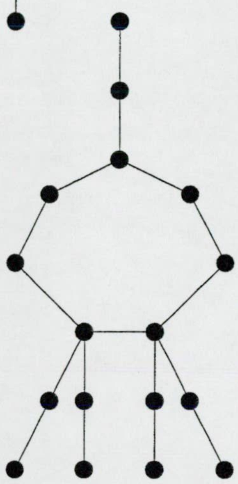

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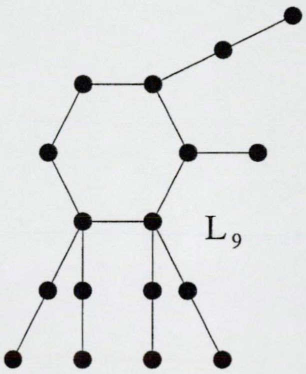

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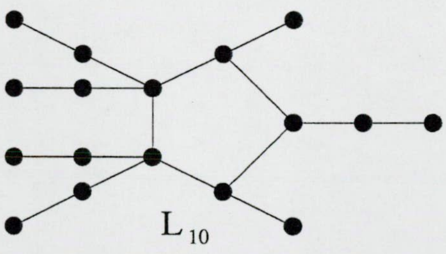

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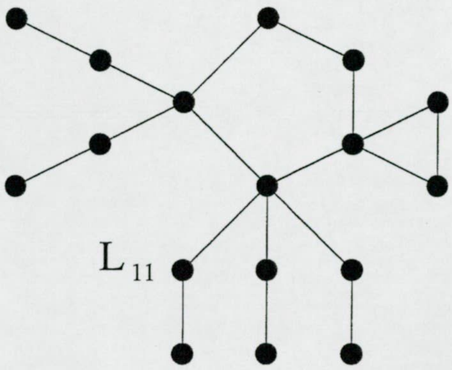

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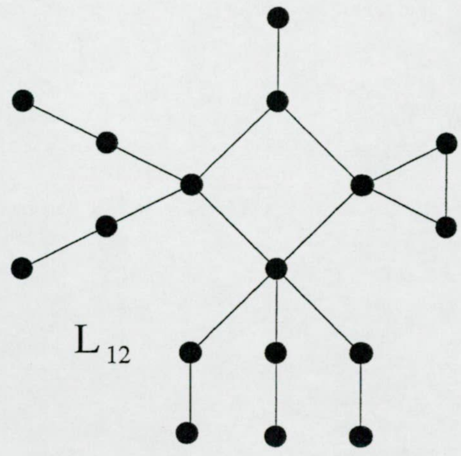

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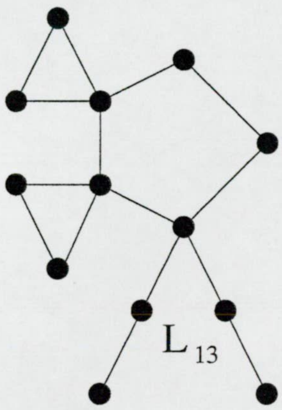

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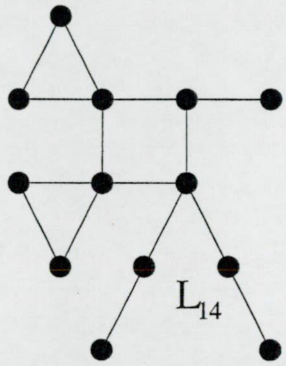

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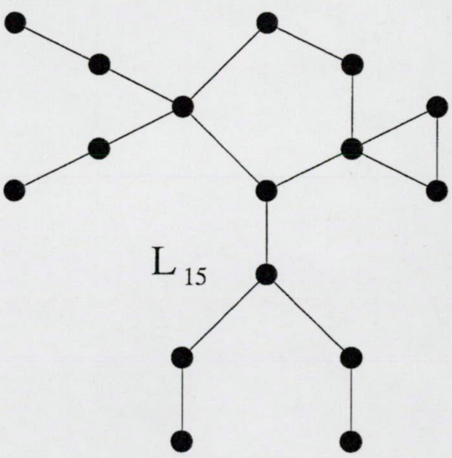

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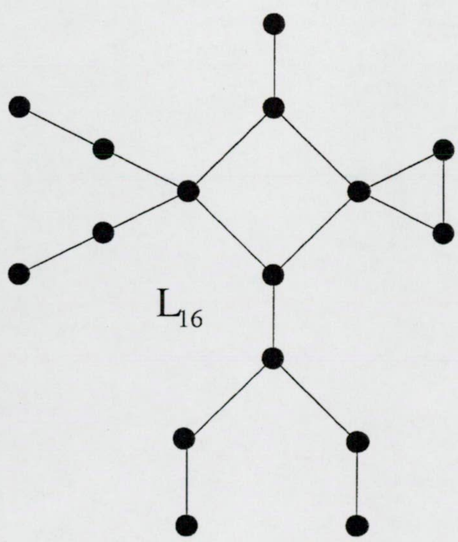

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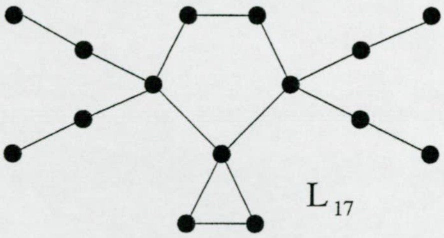

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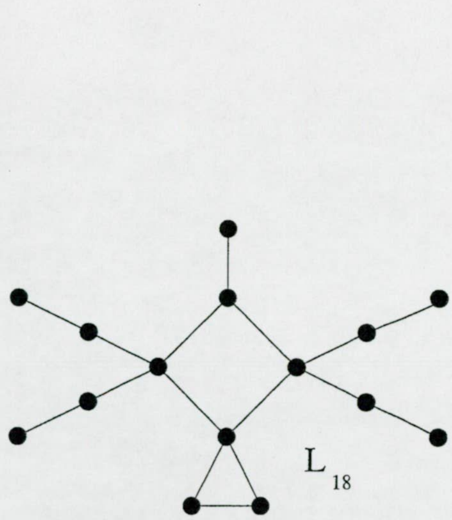

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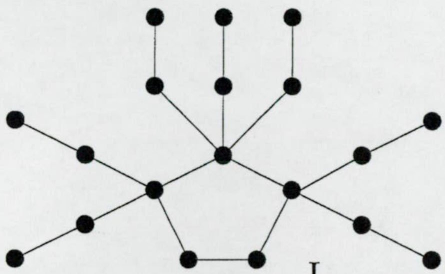

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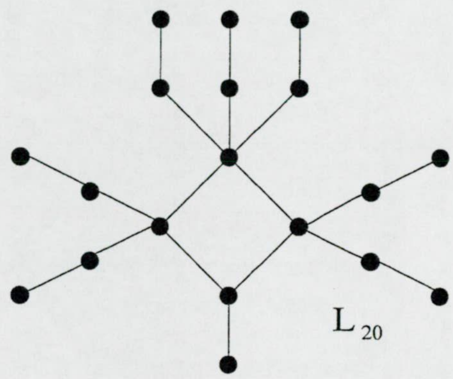

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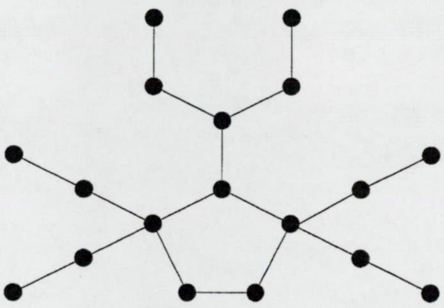

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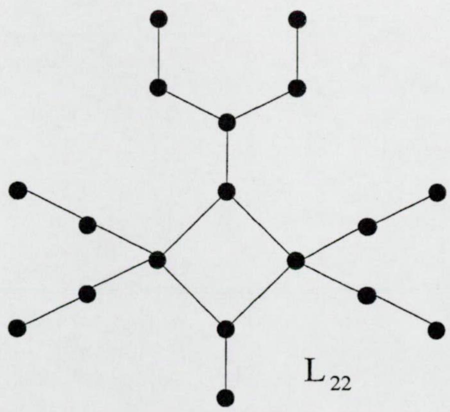

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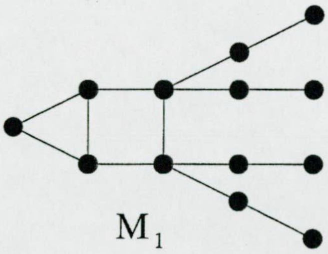

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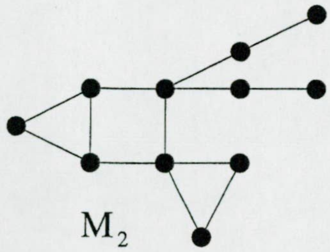

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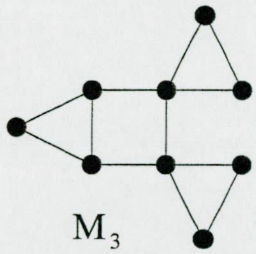

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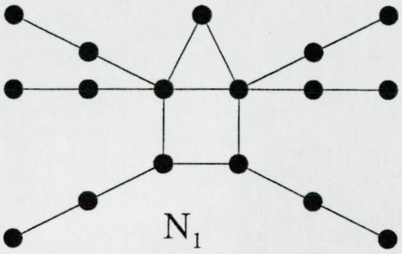

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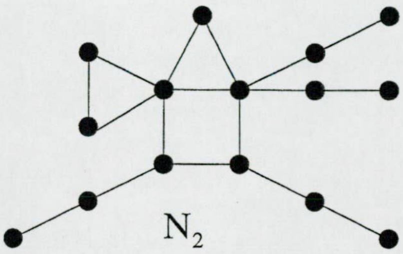

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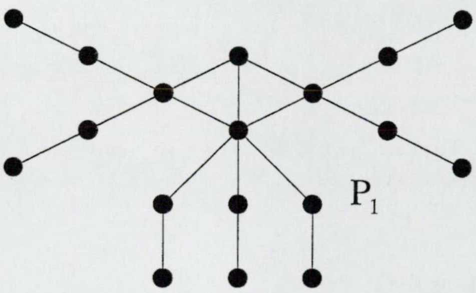

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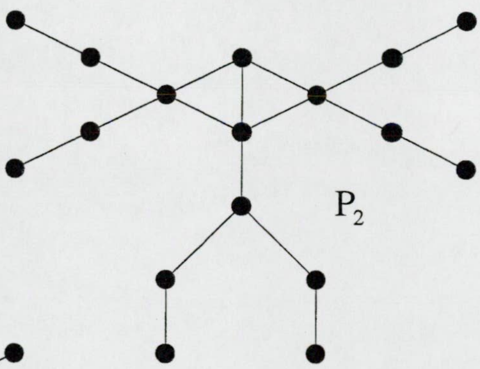

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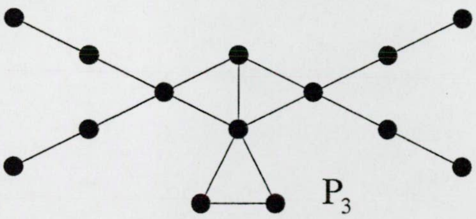

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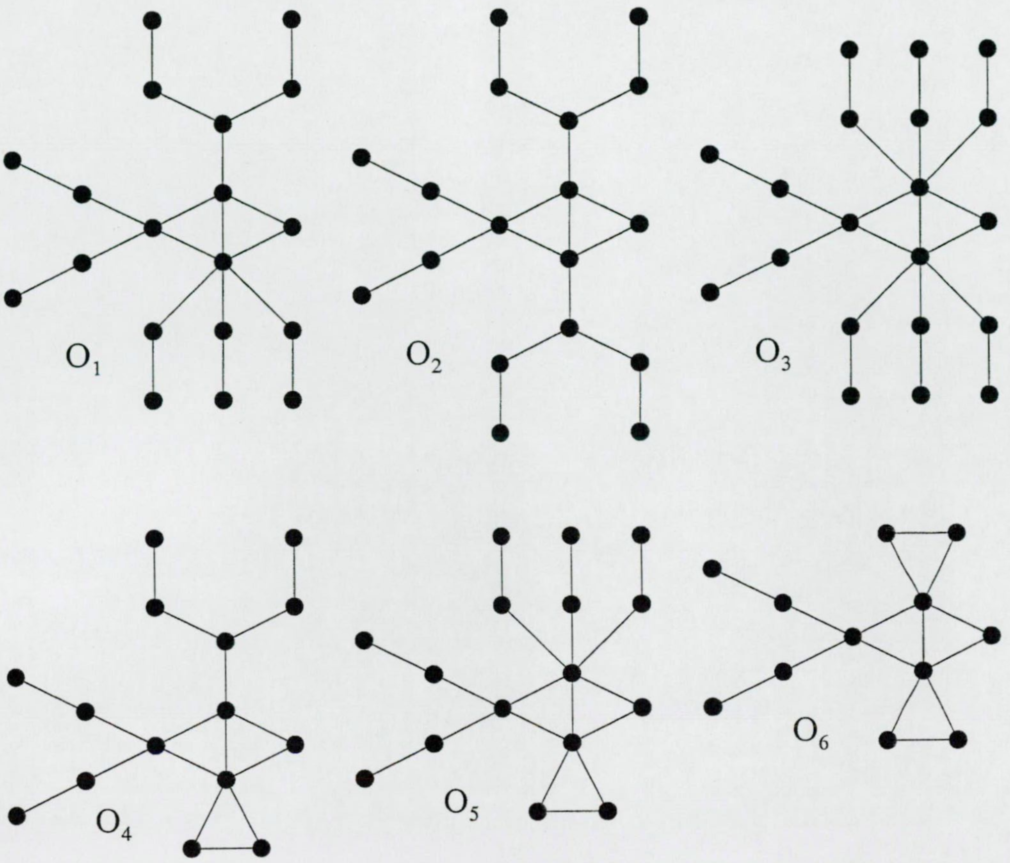
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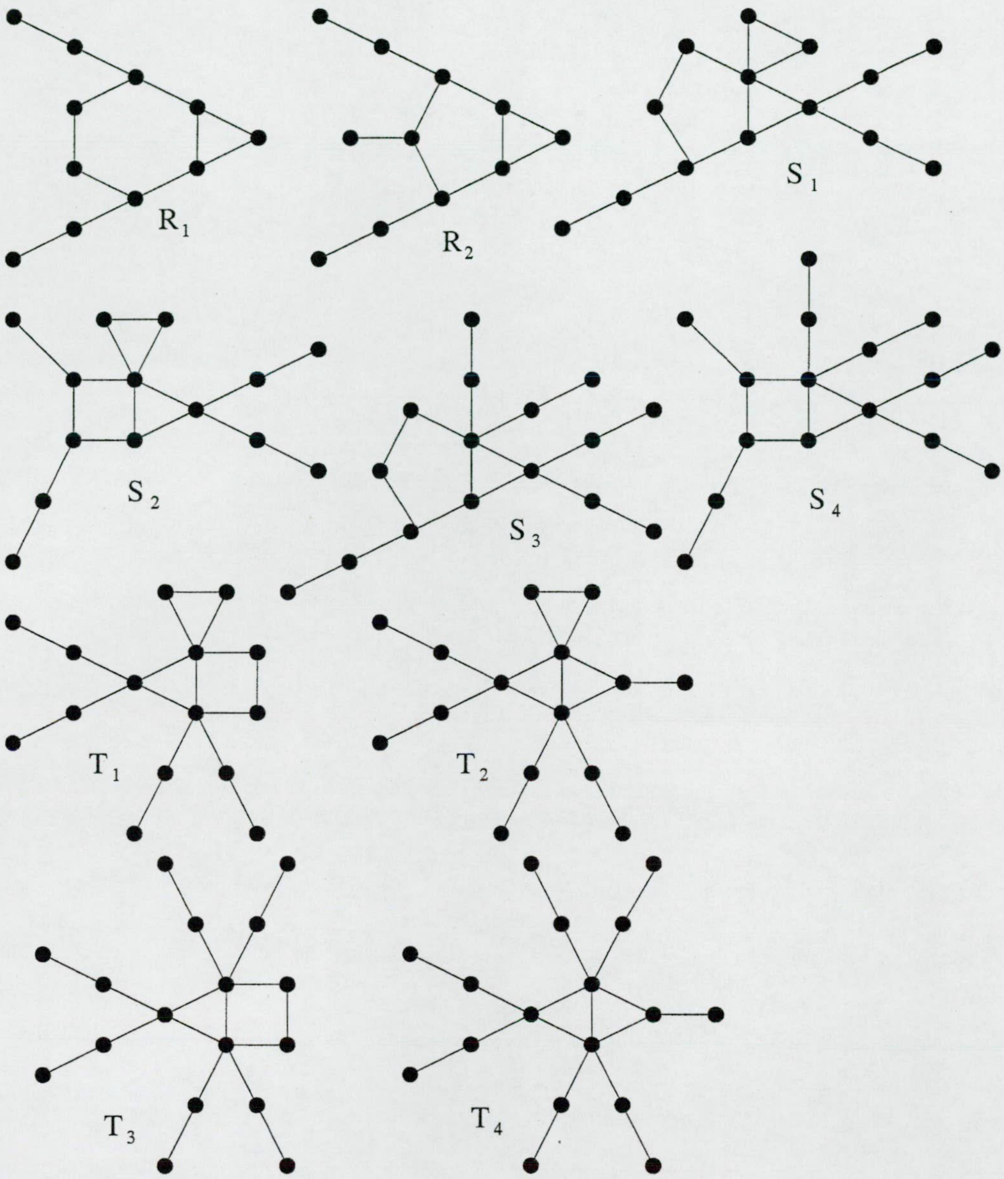
O_2
O_3
O_4
O_5
O_6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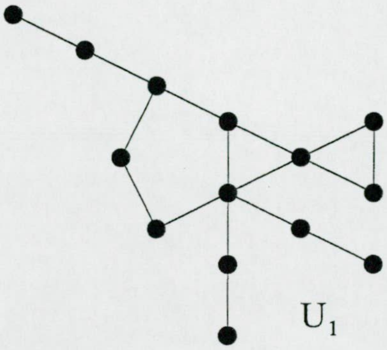
R_2
S_1
S_2
S_3
S_4
T_1
T_2
T_3
T_4



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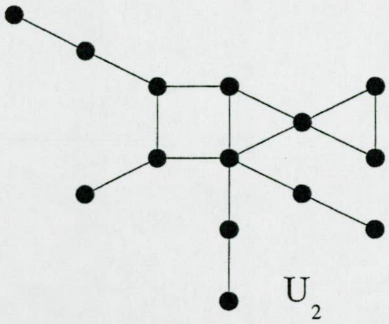

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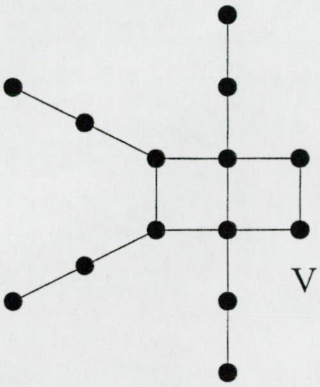

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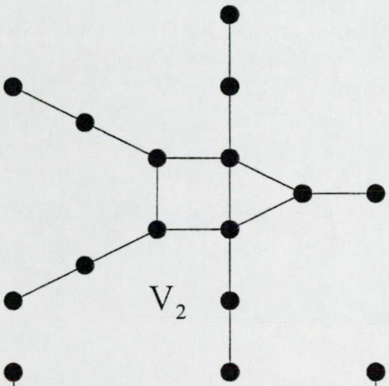

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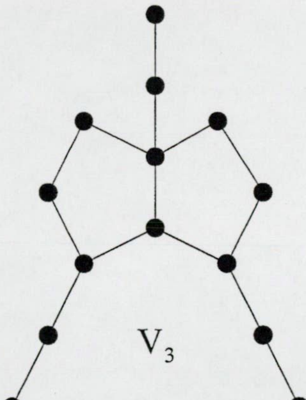

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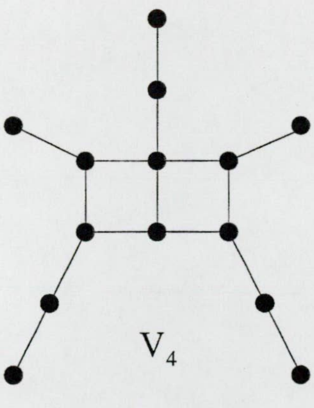

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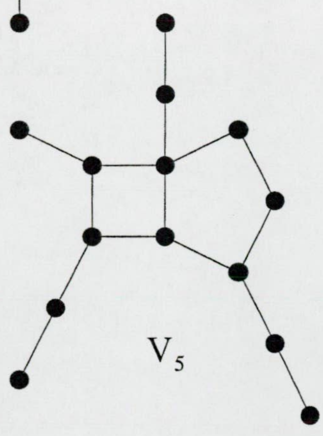

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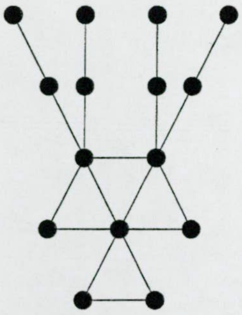

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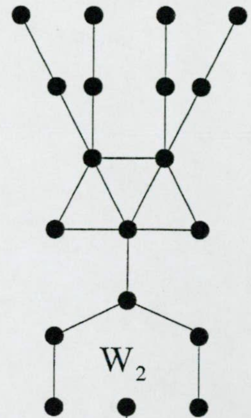

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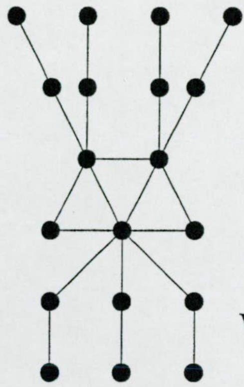

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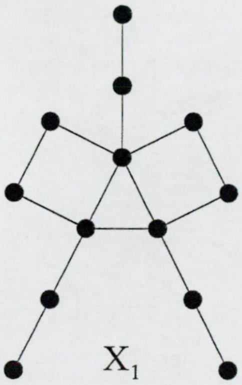

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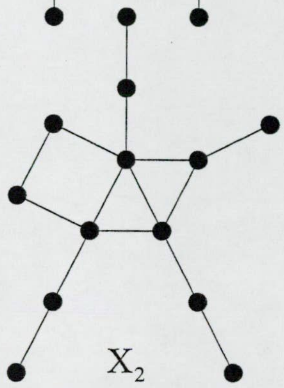

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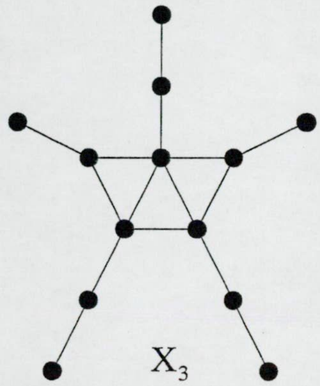

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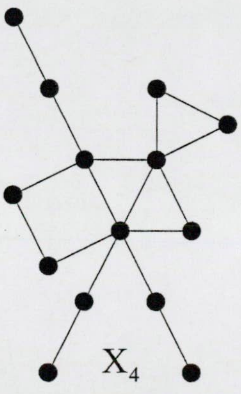

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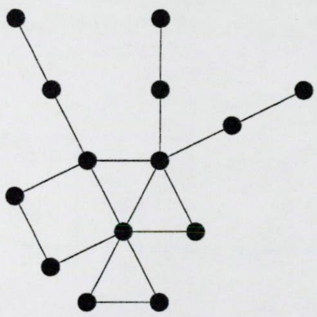

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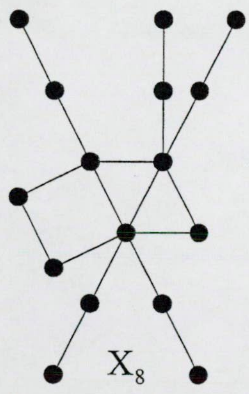

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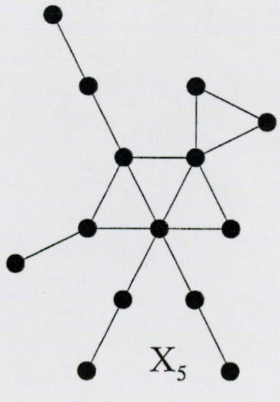

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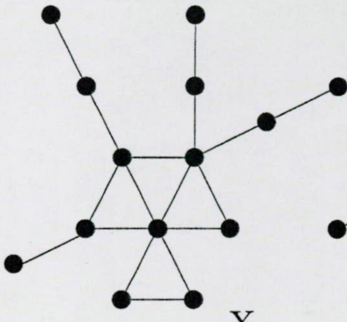

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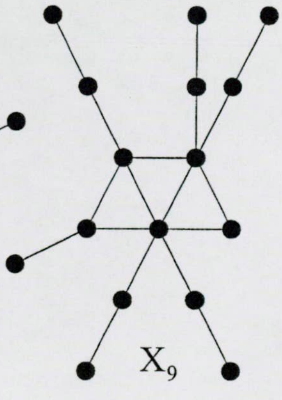

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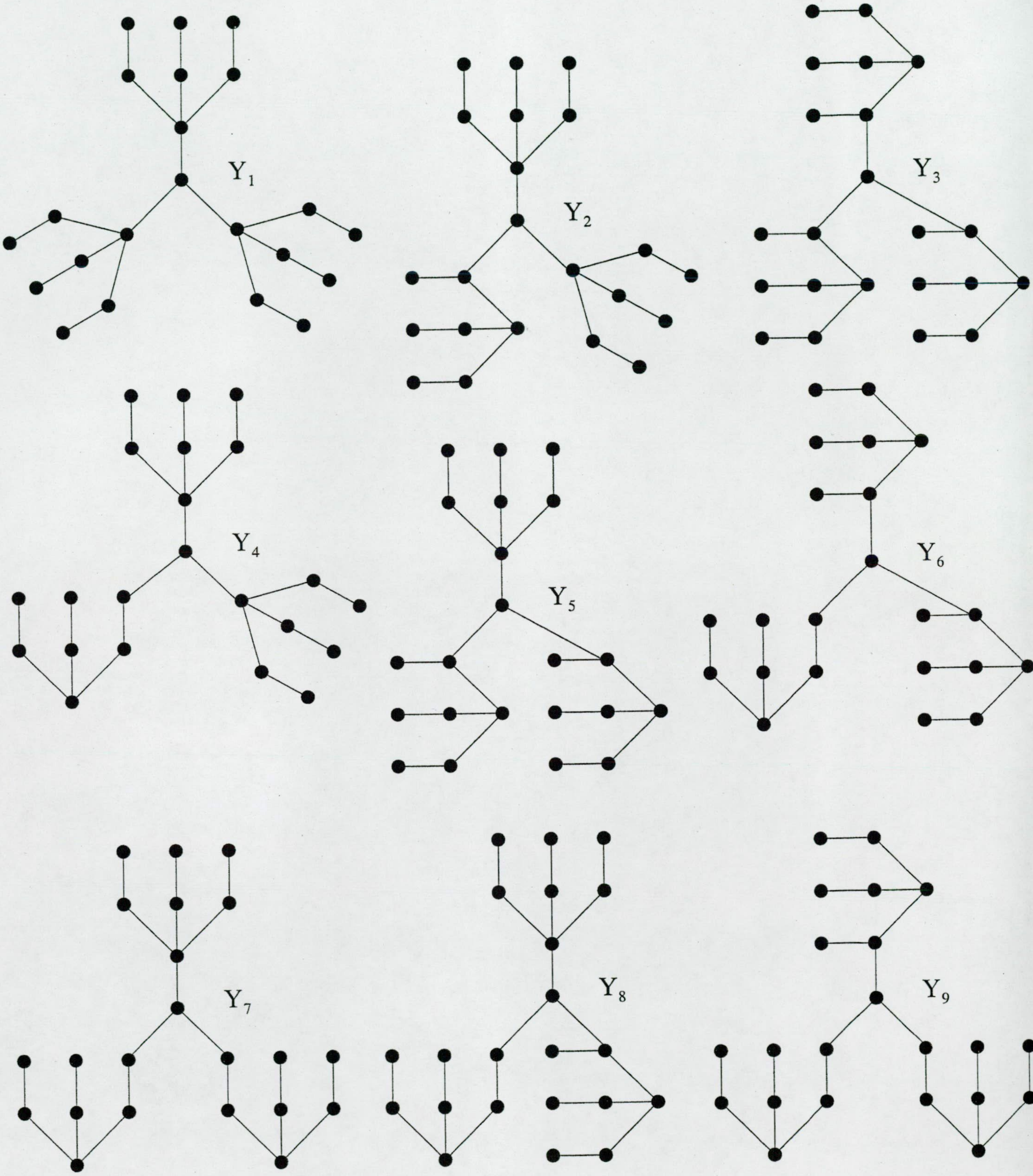
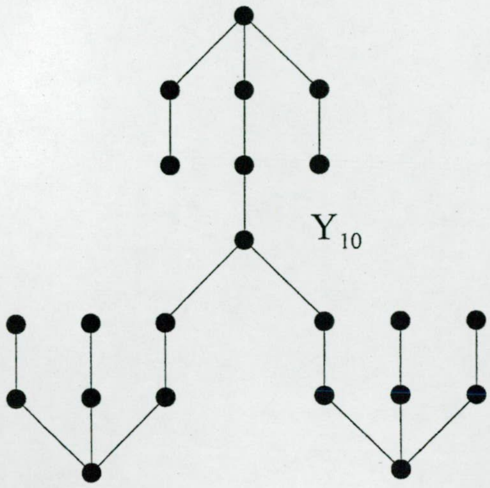
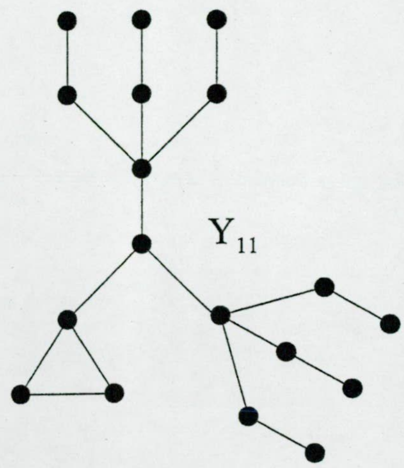
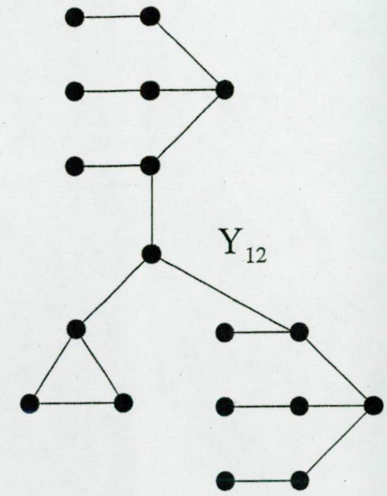

Y_1
Y_2
Y_3
Y_4
Y_5
Y_6
Y_7
Y_8
Y_9



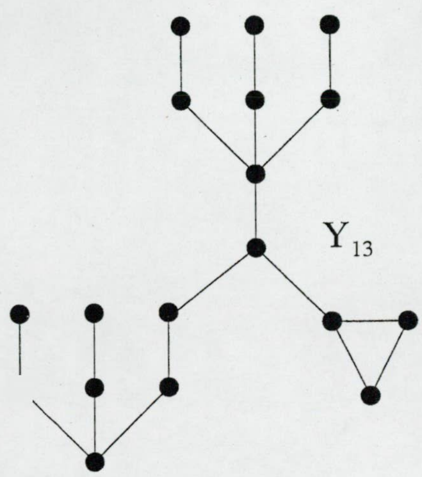
Y_{10}



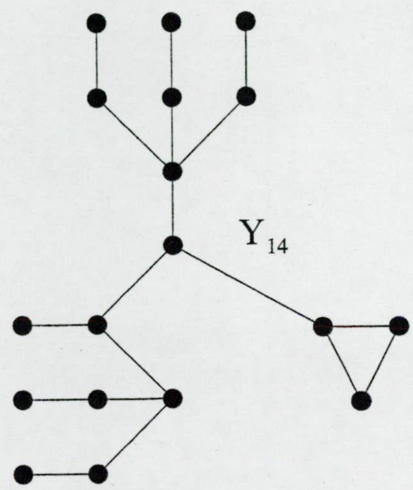
Y_{11}



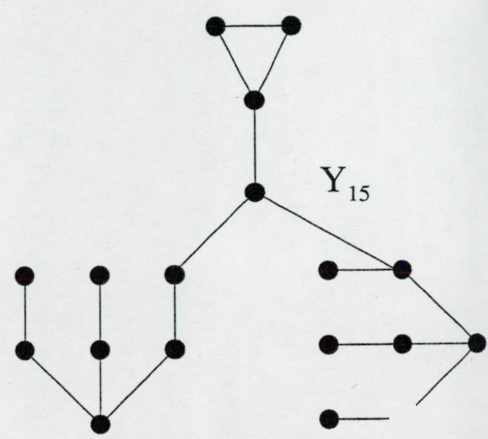
Y_{12}



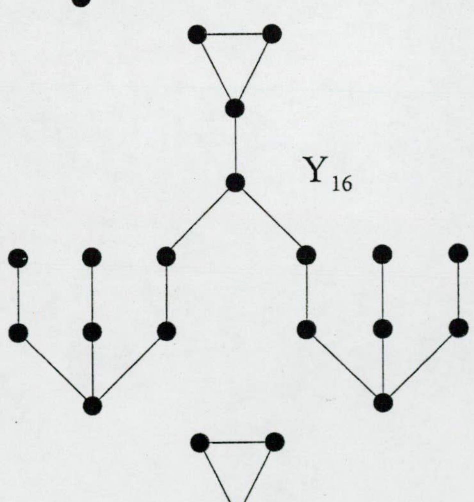
Y_{13}



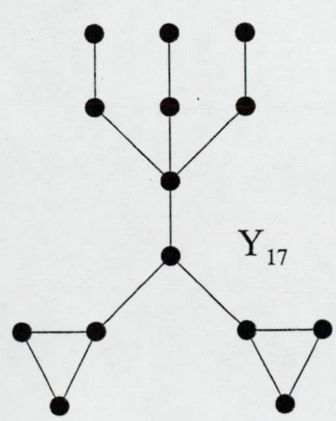
Y_{14}



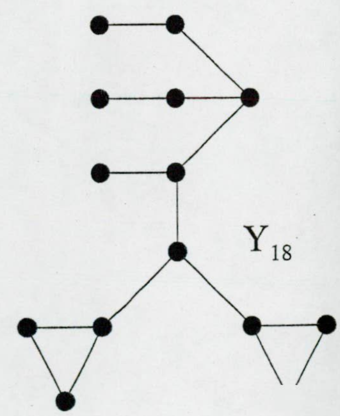
Y_{15}



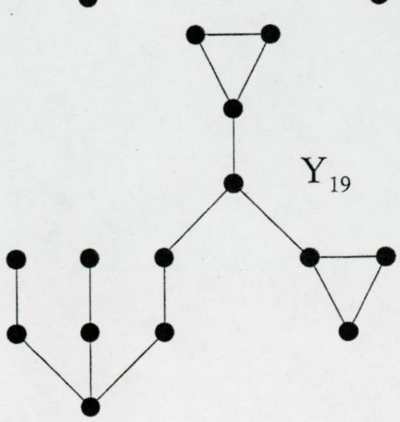
Y_{16}



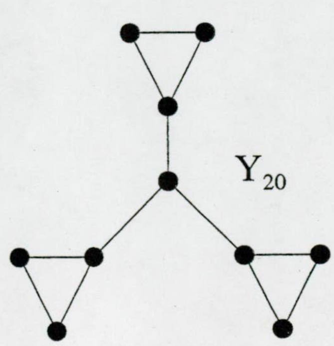
Y_{17}



Y_{18}



Y_{19}



Y_{20}