ALMA MATER STUDIORUM · UNIVERSITY OF BOLOGNA

School of Science

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Master Degree in Physics

Exact S-matrices for a class of 1+1-dimensional integrable factorized scattering theories with $\mathcal{U}_q(sl_2)$ symmetry and arbitrary spins

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Abstract

In this thesis we study the S-matrices associated to a new class of (1+1)-dimensional integrable models with $\mathcal{U}_q(sl_2)$ symmetry, whose asymptotic particle states organize into a $\frac{k}{2}$ isospin multiplet, with $k = 1, 2, \ldots$ Such S-matrices generalize the case study previously analyzed by S. R. Aladim and M. J. Martins, where the particular case of the non-deformed $q \to 1$ limit of pure SU(2) symmetry was investigated.
The formula for the two-particle S-matrix is obtained by multiplying the $\mathcal{U}_q(sl_2)$ R-matrix (where the spectral parameter is interpreted as rapidity) by a function of the rapidities difference that can be fixed with the requirements coming from crossing-symmetry and unitarity. The resulting S-matrix therefore defines a self-consistent integrable factorized scattering theory.

Abstract

In questa tesi studiamo la matrice S associata ad una nuova classe di modelli integrabili in (1+1)-dimensioni, con simmetria di quantum group $\mathcal{U}_q(sl_2)$, i cui stati asintotici si organizzano in multipletti di isospin $\frac{k}{2}$, con $k = 1, 2, \ldots$ Queste matrici S generalizzano il caso precedentemente analizzato da S. R. Aladim and M. J. Martins, in cui viene preso in esame il caso non deformato (limite $q \to 1$) in cui il gruppo di simmetria è SU(2).
La formula esatta per la matrice S è ottenuta moltiplicando alla matrice R del gruppo $\mathcal{U}_q(sl_2)$ (dove il parametro spettrale è interpretato come rapidità) una funzione della differenza delle rapidità. Tale funzione può essere fissata imponendo le condizioni dovute ad unitarietà e crossing-simmetria.
La matrice S risultante definisce pertanto una teoria di scattering che sia integrabile, fattorizzabile e autoconsistente.

Contents

Chapter 1

Introduction

Integrable models represent a class of systems that allows us to study non perturbative effects in field theories and statistical models. Integrability finds applications in many areas of physics, ranging from strings to condensed matter, from gauge/gravity dualities to the study of thermodynamics outside of equilibrium. In the last 40 years, many studies on factorizable S-matrices of integrable models in $(1+1)$-*dimensions* were performed and important results obtained.

One of the most relevant topics in these studies is indeed linked to factorizable S-matrices and their connection with the famous Yang-Baxter equation and with the mathematics of quantum groups [29].

A core theorem on which these theories are based on is the one proven by S. Parke [37], stating that "*In massive, (1 + 1)-dimensional, local, quantum field theories, the existence of two conserved charges is a sufficient condition for the absence of particle production and factorization of the S-matrix.*". It enabled all these theories and models to be studied and worked on, therefore becoming the stepping stone for this whole branch of physics.

Amongst the great results retrieved from these studies, one of the firsts and also one of the most important was obtained by A. B. Zamolodchikov and Al. B. Zamolodchikov[4] in their paper on factorizable S-matrices in 2-dimensions as solutions for quantum field theory models.

Between the years 1989 and 1995 other achievements of the parallelism between S-matrix and QFT piled up. Among them there are correlation functions obtained with the system of *form factors*, as well as physical quantities that can be computed in an exact, i.e. non-perturbative way.

Moreover, these integrable models are relevant in the studies of 4-dimensional theories as well: the Maldacena conjecture [35] is indeed a milestone in the studies on the gauge-gravity duality. At its core, the AdS/CFT was indeed originated from the statements presented in Maldacena's paper in 1997. The theory claims that a correspondence exists between a string theory (or supergravity) defined on a space product between an Anti de Sitter (AdS) space and a closed manifold, and a Conformal Field Theory (CFT) defined on the border of such space (immediately meaning that its description is going to be in a lower dimension). This is clearly a realization of the *Holographic principle*, first proposed by Gerard 't Hooft in 1993 [36].

A particularly relevant feature of these research works is that new integrable models were discovered: in fact, models obtained by adding a new deformation parameter to

previously known theories were introduced and studied (examples are the quantum sine-Gordon model[3] and the Sausage model as deformation of the $O(3)$ non-linear sigma model[6]). Their applications were, as anticipated, connected to many other topics in physics: one example of this is the fact that some calculations in AdS/CFT were found to be closely similar to those of the Sausage deformation[7].
The prototypical model for these deformed theories is the $O(3)$ non-linear sigma model treated in [6], showing the equivalence between the deformed S-matrices and the R-matrices emerging from the Hopf algebras[29] of the quantum groups.
An example of this connection can be found in the sine-Gordon model. Here, the system S-matrix reduces to a rational behaviour for the value of the parameter $\beta^2 = 8\pi$. When moving outside of that point, the model gets deformed and the associated S-matrix is a trigonometric one. Looking at the symmetry group underlying this model, the deformation means that the U(1) internal symmetry is sent to the spin-$\frac{1}{2}$ representation of the $\mathcal{U}_q(\mathrm{SU}(2))$ quantum group.
An analogous correspondence holds for the $O(3)$ non-linear sigma model, resulting in the symmetry group being related to the spin-1 representation of $\mathcal{U}_q(\mathrm{SU}(2))$.
It is legitimate to ask if it is possible to build consistent scattering theories (meaning that they satisfy the constraints imposed by the Yang-Baxter equation, Unitarity and Crossing-Symmetry relations) for higher spins. In their paper[8], S. R. Aladim and M. J. Martins proposed a generalization of the SU(2) symmetric models with spin $s = k/2$, $k = 1, 2, 3, \ldots$, i.e. the taking the generalization of the rational systems as sine-Gordon (at $\beta^2 = 8\pi$) ($k = 1$) and the $O(3)$ non-linear sigma model ($k = 2$) to higher spin representation. In their work they introduce an S-matrix for arbitrary k, distinguishing between two main branches that are even and odd values for k (i.e. integers and half-integers values for the spin $s = k/2$). After retrieving such S-matrices, they perform the TBA (Thermodynamic Bethe Ansatz) on that to compute physical relevant quantities. A final claim of the paper is to have calculated the central charge associated to those models in terms of the dilogarithm functions, which is a very delicate calculation but might lead to very relevant results.
After going through these studies, it is straightforward for one to ask himself what's lying ahead, waiting to be investigated. The first natural step would now be to try and take on the task of deforming such generalized models, in order to look for a complete description of this class of systems and eventually manage to connect them to their associated field theory, if it exists, and maybe even succeed in finding their application in useful tasks

involved in completely different topics, as already happened in the AdS/CFT instance. But is it possible to perform deformation on such a class of integrable models and obtain consistent scattering theories from that? How can one retrieve such S-matrices and what do they look like?
The focus of this thesis is to work towards building such deformed S-matrices for the scattering theories, respecting the consistency constraints and using them as tools to build them up. Following this work, new possible studies might take on the future tasks about these theories, namely performing the TBA on such models and eventually try to connect them to actual field theories to put the calculations to good use in that scope. As it is a common belief that these theories have an underlying field theory correspondence, it's important to point out that this connection is indeed not guaranteed, but only conjectured [26]. Even so, achieving such results would be amazingly powerful (in a physical sense), therefore it is only reasonable to try to look for these correlations.
We will now briefly go through the structure of this thesis.
In Chapter 2 there will be a review of S-matrix theory, starting from its historical development and going through the reasons why this approach to scattering problems is allowed and how does it describe the physics behind these processes, paying particular attention to describe two body scatterings and elasticity of those in (1+1)-dimensions, allowing the S-matrix to be factorized and therefore written as a series expansion in terms of the projectors on the spin states(which is going to be the main tool to actually build the matrix itself).
In Chapter 3, some already solved models, namely the sine-Gordon and the $O(3)$ non-linear sigma model are going to be introduced to present a review of their deformed versions, along with the S-matrices associated to them. The generalized rational case presented by Aladim and Martins will finally make its appearance and looked over.
Chapter 4 will be a description of the work done towards constructing the deformed S-matrices for such models, starting with the trigonometrization of the rational S-matrix and then introducing how to build the projectors over the spin states in the q-deformed case's Hilbert space[15]. Following these introductory elements, the consistence conditions will be imposed, namely verifying that the scattering theory build satisfies Yang-Baxter equation, Unitarity and Crossing-symmetry. In particular, in the process of working towards satisfying these constraints it will emerge that the S-matrix initially build does not immediately respect the crossing relation: it will be necessary to investigate the ratios between the non vanishing matrix elements of such S-matrix and its

crossing-symmetric. By doing so, we will find that these ratios are actually the same for each matrix element, resulting in a clear pre-factor that will need to be added to the S-matrix to compensate for this initial mismatch. We will finally present a finalized version for the q-deformed S-matrices that will envelop the contributes of Professors C. Ahn and F. Ravanini for the final calculations that will be developed in a yet to be published article.

The fifth and last chapter will be focused on conclusions, i.e. considerations over the results achieved, and on the outlook for what might be coming next, namely explicating the next natural tasks as continuations and complement of this work, such as looking for connections of the scattering theories developed with field theories.

Chapter 2

$\boldsymbol{S}$-Matrix

2.1 Historical development

With the purpose of dealing with the difficulties coming from the divergences of the perturbative series in quantum field theory, W. Heisenberg proposed S-matrix theory, which became relevant in the 1950s-60s, especially in studying the strong interaction of hadrons (i.e. protons, neutrons, pions...)[39].
Initial attempts to describe hadronic phenomena using quantum field theory were not successful at all. It was indeed an hard task to include in the theory unstable particles (the resonances) and the particles with spin higher than 1: the only quantum field theories that are consistent (renormalizable) are those of stable particles with spin 0, 1/2 and 1. Unusually large values for the effective coupling costants led to doubts about the actual validity of the possible perturbativie theories for such processes. All these reasons made it feel necessary to look for an alternative approach, one that would also be appropriate to extend to other interactions too. A set of principles and the analytical properties of the quantum amplitudes were the factors at the root of the *analitic theory of the S-matrix*. The first developers of the theory were part of a group of physicists in Berkeley, where they studied and proposed it. A fundamental part in that came from Chew and Mandelstam[40][41], with important contributions by Regge, Frautschi, Weisskopf and many others[42]. Scattering processes and their analysis are the closest point between theoretical and experimental aspects: results obtained with by the S-matrix were then expected to not depend on the fact that an underlying quantum field theory of the interactions would exist or not.
Another principle the theory was supposed to follow was that an S-matrix based description should be able to answer questions as such:

- How are stable and unstable particles different? Does a theoretical environment work for describing both exist? As known, the lagrangian formulation of quantum field theory only makes use of the stable asymptotic particles, thus an equal footing for both cases is not allowed.

- Is it possible to obtain coupling constants and the mass spectrum of the theory? For a lagrangian theory, one should recall that they are both free parameters of the model.

Studying the S-matrix as a function of energy, momentum, angular momentum etc., suggested that its structure was to be the simplest possible; this was then assumed as a

principle and formally submitted as the *principle of maximum analicity of the S-matrix*. Following this hypotesis, strong interaction's physics should not have arbitrary constants, except for the fundamental ones (lightspeed c, Plank's constant h and one parameter scale). As a consequence of this, all of the strong particles would then be composite particles and could be considered and studied in the same way. This was the basis of the *bootstrap principle*.
A strong influence came from the formalism introduced by Regge to study the scattering amplitudes as functions in the complex plane of the angular momentum. One thing in particular was that it was possible to analyze the asymptotic behavior of the amplitudes for large values of s and to give an estimate of the high-energy limit of the cross-sections in a very elegant way.
Some of the results achieved with Regge's formalism were the followings:

1. High-energy asymptotic behavior of the scattering processes dominated by the exchange of particles in the t-channel

$$\sigma_{tot} \simeq s^{\alpha_0 - 1}$$

2. Th relation between the total cross-section of a process with incoming particles $A + B$ and the cross-sections relative to the incoming particles $A + A$ and $B + B$:

$$\sigma_{tot}^{(A+B)} = \left[\sigma_{tot}^{(A+A)} * \sigma_{tot}^{(B+B)}\right]$$

The most important result conquered with the analytic S-matrix theory was the scattering amplitude discovered by Gabriele Veneziano, which can be used to exactly implement the duality between the s- and t-channels.
Going into specifics, when there are particles exchanged in the t-channel, having increasing values of mass and spin, the amplitude assumes the form

$$A(s,t) = -\sum_J \frac{g_J^2(-s)^J}{t - m_J^2}. \tag{2.1}$$

If the number of terms in the summation is finite, equation (2.1) defines an amplitude with no poles in the s-channel, which is easy to show since for every fixed value of t, the sum is an integer function of s. Nonetheless, if the series is infinite, it could diverge for different values of s, originating poles in the s-channel. In this eventuality, it is rather non-trivial how to implement crossing symmetry, since one should also include

the corresponding terms of the s-channel, knowing that they might be already included in (2.1).
If starting from the s-channel, an analogous conclusion might be reached, giving the formulae

$$\tilde{A}(s,t) = -\sum_J \frac{g_J^2(-t)^J}{s-m_J^2}. \tag{2.2}$$

One can now guess that, with an adequate choice of the coupling constants g_J and the masses m_J, the amplitudes $A(s,t)$ and $\tilde{A}(s,t)$ actually define the same function: with this statemente being true, the scattering amplitude could be equivalently written as a series on the infinite poles of the t-channel or the s-channel (with a clear remark on the duality of the two scenarios). Veneziano showed it with the amplitude

$$A(s,t) = \frac{\Gamma[-\alpha(s)]\Gamma[-\alpha(t)]}{\Gamma[-\alpha(s)]\Gamma[-\alpha(t)]}, \qquad \alpha(x) = \alpha_0 + \alpha' x. \tag{2.3}$$

Seeing the linear behaviour of $\alpha(x)$, it is possible to show that the singularities of the amplitude in (2.3) are simple poles, corresponding to the exchange of particle of mass $m^2 = (n-\alpha_0)/\alpha'$, with $n = 0, 1, 2, \ldots$ both in the t- and s-channels. A notable remark is the fact that the residue at the pole $\alpha(t) = n$ is a polynomial of order n in s, corresponding to a particle of spin n. The analogous happens in the s-channel. Finally, considering the asymptotic behaviour of $\Gamma(z)$, one can see that the amplitude from Veneziano presents a Regge behaviour in both vairables:

$$\begin{aligned} A(s,t) &\simeq s^{\alpha(t)}, \qquad s \to \infty, \qquad t \ \text{ fixed;} \\ A(s,t) &\simeq t^{\alpha(s)}, \qquad t \to \infty, \qquad s \ \text{ fixed.} \end{aligned}$$

This contribution from Veneziano had an important influence and impact on the development of strong interaction studies. It has also been the starting point for string theory. Despite the initial popularity, S-Matrix theory lost his appeal because it felt too complicated to handle properly and in years of studies, it had produced only modest progresses and results.
The actual check of basic principles of the S-Matrix theory comes from (1+1)-dimensional statistical and QFT models, with the achievement of solving important systems, like Ising model in an external magnetic field.

2.2 Scattering in QM

The purpose of this section is to recall some basic concepts from scattering theory in quantum mechanics and to show how the concept of S-Matrix can be included in such a context.
For simplicity, we are going to focus on 1-dimensional systems.
Considering a free particle, with mass m and momentum p, it is possible to associate to that the Hamiltonian

$$H_0 = \frac{p}{2m}, \tag{2.4}$$

where it is set $\hbar = 1$.
Making use of the fact that p and H_0 commute, it is immediate to write the well known eigenfunctions as plane waves:

$$\begin{aligned} \psi_k(x) &= e^{ikx} \\ p\psi_k(x) &= k\psi_k(x) \\ H_0\psi_k(x) &= \frac{k^2}{2m}\psi_k, \end{aligned}$$

the evolution of which is ruled by the following relation with time:

$$\psi_k(t,x) = e^{-iE_kt}\psi_k(x) = e^{-itk^2/2m}\psi_k(x). \tag{2.5}$$

This clearly shows the fact that the spectrum of energies has a double degeneration because of dependence from the square of the momentum k. An immediate consequence of this is that any eigenfunction obtained by linearly combining ψ_k and ψ_{-k} is still going to be an eigenfunction of H_0.H_0 also commutes with the parity operator P and therefore we can choose a basis with functions of a given parity.

$$\begin{aligned} \psi_{k1}(x) &= cos(kx), & P\psi_{k1}(x) &= \psi_{k1}(x) \\ \psi_{k2}(x) &= sin(kx), & P\psi_{k2}(x) &= -\psi_{k2}(x) \end{aligned} \tag{2.6}$$

One can now imagine to add to the hamiltonian a potential $V(x)$, finite and different from zero, only inside a region $|x| < x_0$, as in 2.1. For simplicity, let's assume that V is an even function, $V(x) = V(-x)$:

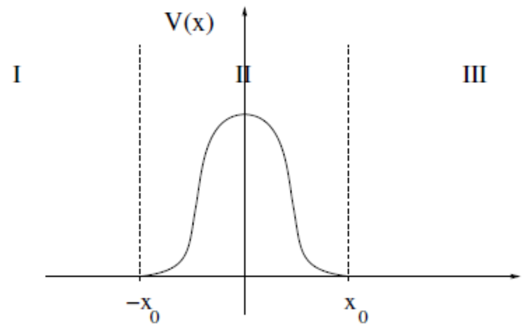

Figure 2.1: Potential $V(x)$ of the scattering process. The external regions (I and III) clearly identify a space where the particle moves without constraints.[1]

The resulting operators are then:

$$H = \frac{p^2}{2m} + v(x)$$
$$V(x) = 0 \quad for|x| > x_0. \tag{2.7}$$

The spectrum of the eigenvalues for $E \geq 0$ remains invariant, as well as the eigenfunctions in the free motion regions

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx}, & x < -x_0 \\ Ce^{ikx} + De^{-ikx}, & x > x_0. \end{cases} \tag{2.8}$$

Imposing boundary conditions and linking A and B with C and D leads to a relation that shapes the form of the potential $V(x)$.
Considering then the scattering solutions (i.e. the ones for which $D = 0$), of the Schrödinger problem, we get

$$\psi_+(x) = \begin{cases} Ae^{ikx} + Be^{-ikx}, & x < -x_0 \\ Ce^{ikx}, & x > x_0. \end{cases} \tag{2.9}$$

One can now very easily recognize A as the coefficient for the incoming wave, B as the amplitude for the reflected one and, finally, C as the amplitude for the transmitted wave.

[1]The figure was taken from [1]

Reflection and transmission coefficients are thus given by the formulas:

$$\mathcal{R} = \frac{B}{A}$$
$$\mathcal{T} = \frac{C}{A}. \tag{2.10}$$

To keep the same density for incoming and reflected plus transmitted wave, it is immediate to get to the relation

$$|\mathcal{R}|^2 + |\mathcal{T}|^2 = 1. \tag{2.11}$$

Considering now the phase shifts δ_0 and δ_1, defined by the stationary eigenfunctions of H, it is possible to use them to express the reflection and transmission coefficients

$$\begin{aligned} &\psi_0 = \cos(kx + \delta_0) \quad (x > x_0); && \psi_0 = \cos(kx - \delta_0) \quad (x < -x_0) \\ &\psi_1 = \sin(kx + \delta_1) \quad (x > x_0); && \psi_1 = \sin(kx - \delta_1) \quad (x < -x_0). \end{aligned} \tag{2.12}$$

The S-Matrix in channels of a given parity is then finally given by:

$$S_a = e^{2i\delta_a}, \quad a = 0, 1. \tag{2.13}$$

The linear combination of a given parity eigenstate that generates the scattering eigenfunction is then

$$\psi_+ = \begin{cases} e^{i\delta_0}\psi_0 + ie^{i\delta_1}\psi_1 = \frac{1}{2}\left(e^{2i\delta_0} + e^{2i\delta_1}\right)e^{ikx}, & (x > x_0) \\ e^{ikx}\frac{1}{2}\left(e^{2i\delta_0} - e^{2i\delta_1}\right)e^{-ikx}, & x < -x_0. \end{cases} \tag{2.14}$$

We then obtain

$$\begin{aligned} \mathcal{R} &= \frac{1}{2}\left(e^{2i\delta_0} - e^{2i\delta_1}\right) = \frac{1}{2}\left[\left(e^{2i\delta_0} - 1\right) - \left(e^{2i\delta_1} - 1\right)\right] \\ &= \sum_{l=0}^{1} i(-1)^l e^{i\delta_l} \sin\delta_l \end{aligned} \tag{2.15}$$

$$\tag{2.16}$$

$$\begin{aligned} \mathcal{T} &= \frac{1}{2}\left(e^{2i\delta_0} + e^{2i\delta_1}\right) = \frac{1}{2}\left[\left(e^{2i\delta_0} - 1\right) + e^{2i\delta_1} - 1\right] + 1 \\ &= 1 + \sum_{l=0}^{1} ie^{i\delta_l} \sin\delta_l, \end{aligned} \tag{2.17}$$

where it is very clear that the reflection and transmission coefficients are determined by the shifts.

An instructive example It is hereby shown an example to illustrate the traits of the object known as S-Matrix. Let us consider now a potential given by

$$V(x) = -2g\delta(x). \tag{2.18}$$

Imposing continuity of the wavefunction at the origin (x=0) and discontinuity of its derivative (because of $\delta(x)$) we get

$$\begin{aligned} \psi_0(0^+) &= \psi_0(0^-) \\ \frac{d\psi_0(0^+)}{dx} - \frac{d\psi_0(0^-)}{dx} &= -2k\sin\delta_0 = -2g\psi_0(0) = -g\cos\delta_0, \end{aligned} \tag{2.19}$$

from which is it straightforward to determine the even phase shift δ_0

$$\tan\delta_0 = \frac{g}{k}. \tag{2.20}$$

The S-Matrix in this channel is then

$$e^{2i\delta_0} = \frac{1 + i\tan\delta_0}{1 - i\tan\delta_0} = \frac{k + ig}{k - ig}. \tag{2.21}$$

The phase difference at infinity is then

$$\delta_0(+\infty) - \delta_0(-\infty) = -2\pi g/|g|, \tag{2.22}$$

showing the dependence on the sign of g. The odd solution vanishes for $x = 0$, meaning that the odd phase shift is identically zero. The corresponding S-Matrix is then equal to 1:

$$\begin{aligned} \delta_1 &= 0 \\ e^{2i\delta_1} &= 1. \end{aligned} \tag{2.23}$$

This expressions for $\delta_{0,1}$ allow us to get the reflection and transmission coefficients and define a solution for the Schrödinger equation for all k. It can be interesting to analyze the solution for complex values $k = k_1 + ik_2$. The real part can always be considered positive or zero since it corresponds to the physical momentum of the incoming particle. Substituting k in (2.14) one can see that the imaginary part k_2 enters the real part of the exponentials. Choosing now k as the value of the pole of the S-Matrix, i.e. $k = ig$, one can have a normalizable eigenfunction by imposing $A = 0$. This solution corresponds to a *bound state* of the system, whose energy is $E_b = -g^2/(2m)$. It is cleat that in this case we should have $g > 0$.

In general, one can show this properties of the S-Matrix for the non-relativistic case:

- The poles of the S-Matrix with positive imaginary values of the momentum, $k_n = ia_n.(an > 0)$ correspond to the energies $E_n = -a_n^2/(2m)$ of the bound states of the system.
- There are no poles in the complex plane of the variable $k = k_1 + ik_2$ with a non-vanishing real part k_1 in the half-plane $k_2 > 0$.
- The poles in the complex plane with negative imaginary part, $k_2 < 0$, correspond instead to the resonances.

Proofs of these properties are given by the following considerations:

- First property is consequence of properties of the potential $\delta(x)$.
- Let us suppose that the S-Matrix has a pole at a certain $k = k_1 + ik_2$, with positive k_2: substituting in (2.14) and setting A to zero, we recover a situation in which the eigenfunction is normalizable. In this situation, the problem lies in the time evolution of this eigenfunction, since it has an exponential growth for the limit $t \to +\infty$, which is a clear violation of conservation of probability.

A pole with negative imaginary part is nevertheless still totally admissible. It identifies a solution the probability of which decreases in a certain channel. It consequently implies that it grows in another channel, so that the global conservation of probability is recovered. Negative imaginary part poles correspond to *resonances*. A good description of k and E as complex variables is displayed in fig. 2.2.

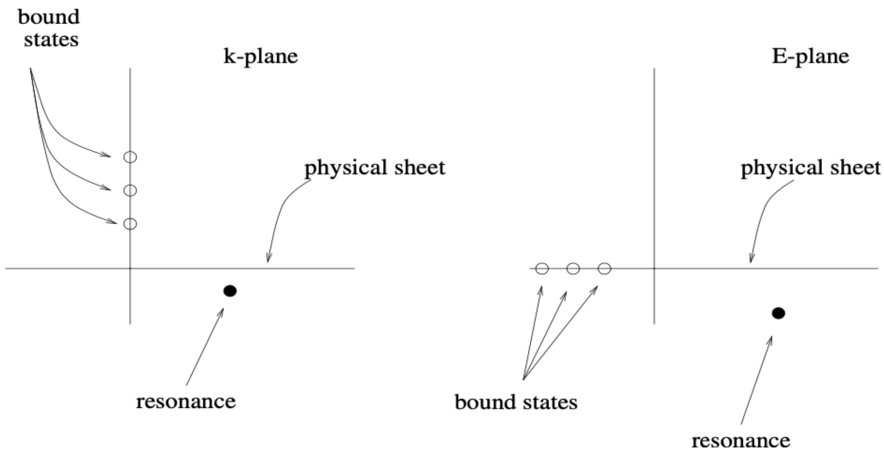

Figure 2.2: k and E complex planes, showing the S-Matrix structure related to those physical quantities.[2]

Being a unitary operator in any channel,in the vicinity of a pole k_0 we can describe the S-Matrix as

$$S = e^{2i\delta} = \frac{k - k_0^*}{k - k_0}. \tag{2.24}$$

Switching the formulation for $E = E_r - il/2$ (with $l > 0$, indicating the fact that there cannot be poles in the positive imaginary plane),we get

$$S = \frac{E - E_r - il/2}{E - E_r + il/2}. \tag{2.25}$$

We can notice that, getting closer to the resonance energy, the phase $\delta(E)$ has a discontinuity jump of 2π.
It is now possible to calculate the diffusion amplitude T, defined as $S = 1 + iT$:

$$T = -\frac{l}{E - E_r + il/2} \tag{2.26}$$

and the cross-section σ

$$\sigma \sim |T|^2 = \frac{l^2}{(E - E_r)^2 + l^2/4}. \tag{2.27}$$

As it can be seen in 2.3, the cross-section has the typical bell shape of a resonance phenomenon, with the width determined by the parameter l. It is quite simple to notice

[2]The figure was taken from [1]

that this is related to the life-time, which we call τ, of the resonance state given by the relation $\tau = 1/l$.

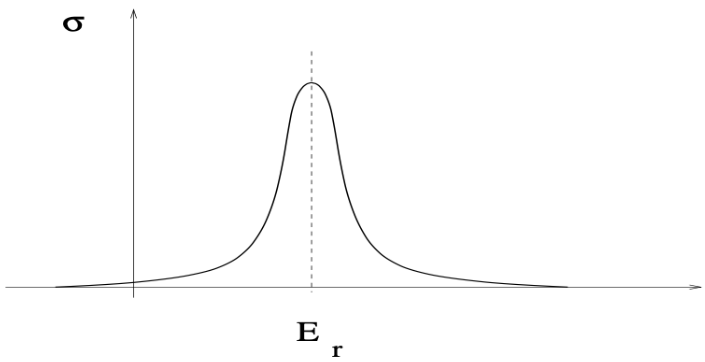

Figure 2.3: Representation of a cross-section σ for an S-Matrix with a resonance pole.[3]

[3]The figure was taken from [1]

2.3 From quantum scattering process to S-Matrix theory

The aim of S-Matrix theory is to allow the computation of scattering amplitudes without relying on an underlying lagrangian formalism associated to the model of one's studies (which is sometimes not easy to write).

S-Matrix properties and consequences It is appropriate to analyze key properties related to the S-Matrix, which are:

- short range interactions;
- superposition principle (from quantum mechanics);
- conservation of probability;
- Lorentz invariance;
- causality principle;
- analyticity principle.

Short range interaction In order to make use of the S-Matrix formalism in scattering processes, it is necessary to assume that the interactions are short range, so that the initial and final states, in which the particles are well separated one from another, consist of free particle states. These states can be identified assigning momenta and other quantum numbers. For simplicity, our focus will be on the scattering processes of scalar particles. Since this processes involve the physical particle states the components of their momenta satisfy the d-dimensional *on-shell condition*

$$p_\mu p^\mu = m^2, \tag{2.28}$$

in which m is obviously the mas of the particle. From (2.28) it is immediate to obtain the *dispersion relation*

$$E^2 - |\vec{p}|^2 = m^2 \tag{2.29}$$

where we adopted the usual notation $E = p_0$. We can now use the notation $|n\rangle$ to denote the states of the system, where n are the eigenvalues of the spatial momentum[4]. This states are a basis in the Hilbert space and they satisfy orthogonality and completeness relations:

$$\langle m|n\rangle = \delta_{m,n}, \qquad \sum_n |n\rangle \langle n| = 1. \tag{2.30}$$

Let's get started by considering a transition from an initial n-particle state to a final m-particle state, as it can be observed in 2.4.

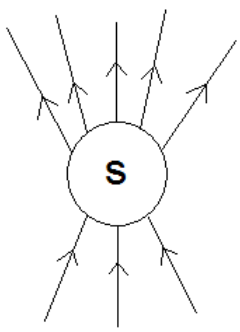

Figure 2.4: In this example, we can see the time evolution of the system with initial state consisting of three particles, leading to a final one with five.

As the picture shows, at $t = -\infty$ we can identify with $|i\rangle$ the initial state of the system, given by a certain number of free particles while for $t = +\infty$ the final state $|f'\rangle$ is still given by free particles but not in the same number as the initial one. Using the superposition principle from quantum mechanics, we can write $|f'\rangle$ as $|f'\rangle = S|i\rangle$, where S is a linear operator for now[5].
We can therefore calculate the probability of obtaining a state $|f\rangle$ as a result from

[4]Please note that the spectrum of the eigenvalues for spatial momentum is not discrete but it's obviously continuous. We will make use of the notation for a discrete spectrum to keep the reasoning behind it under the spotlight and for the sake of simplicity in explaining what's behind the theory of S-Matrix. One can, as usual, recover the correct continuum notation by applying the *formal* limit $\sum_n \to \int dn$

[5]S is the time evolution operator from $t = -\infty$ to $t = +\infty$. Given that the system admits a QFT formulation, it can be expressed as $S = \mathcal{T}_1 \exp\left[-i\int_{-\infty}^{+\infty} d^dx\mathcal{H}_i(x)\right]$, where $\mathcal{H}_i$ is the Hamiltonian density and $\mathcal{T}_1$ is used to point out the time-ordering of the expressions obtained by the series expansion of the exponential term.

measuring the final state, taking the square modulus of the matrix element

$$S_{fi} = \langle f| S |i\rangle . \tag{2.31}$$

If we now take into consideration an initial state which is normalizable, namely $|\psi\rangle$, that is a linear superposition of the basis

$$|\psi\rangle = \sum_n a_n |n\rangle , \quad where \quad \sum_n |a_n|^2 = 1, \tag{2.32}$$

we can compute the expression for the total probability that it evolves as a final state in any basis, which should obviously sum up to 1, therefore

$$\begin{aligned} 1 &= \sum_m |\langle m| S |\psi\rangle|^2 = \sum_m \langle\psi| S^\dagger |m\rangle \langle m| S |\psi\rangle \\ &= \langle\psi| S^\dagger S |\psi\rangle = \sum_{n,m} a_n^* a_m \langle n| S^\dagger S |m\rangle . \end{aligned} \tag{2.33}$$

Because of the fact that this should hold true for arbitrary values of the coefficients, we necessarily have

$$\langle n| S^\dagger S |m\rangle = \delta_{nm}, \tag{2.34}$$

which is equivalent to writing

$$S^\dagger S = 1 \tag{2.35}$$

in operator form. In an analogous way, imposing equal to 1 the total probability that an arbitrary final state comes from some initial state is, we immediately obtain the condition

$$S S^\dagger = 1. \tag{2.36}$$

It is then possible to say that probability conservation implies that S has to be a unitary operator. We can now analyze the Lorentz invariance for the scattering theory. Let L be an arbitrary proper Lorentz transformation and $L|m\rangle = |m'\rangle$. The relativistic invariance of the theory, which ensures the independence of the physical observables from the reference frames, is expressed by the identity

$$|\langle m'| S |n'\rangle|^2 = |\langle m| S |n\rangle|^2. \tag{2.37}$$

This relation cannot be used to fix the relative phase between the two matrix elements but, given the intrinsic arbitrariness of the overall phase of the S-Matrix, we can impose the stronger condition given by

$$\langle m'| S |n'\rangle = \langle m| S |n\rangle , \tag{2.38}$$

which implies that the S-Matrix, once we factorize a δ- function for the conservation of the total momentum, depends on the momenta of the particles only through their Lorentz invariant combinations i.e. their scalar products
Without interactions, the state of a system doesn't get changed and in this case the S-Matrix is simply the Identity. It is a common doing to separate the free time evolution, given by the identity operator, and write the S-Matrix as

$$S_{fi} = \delta_{fi} + i(2\pi)^d \delta^d(\mathbf{P}_f - \mathbf{P}_i) T_{fi}, \tag{2.39}$$

where $\mathbf{P}_f$ and $\mathbf{P}_i$ have been used to indicate the sum of the momenta of the final and initial particles. The term $\delta^d(\mathbf{P}_f - \mathbf{P}_i)$ explicitely express the conservation law of total momentum.
In the equation, the matrix elements T_{ij} indicate the *scattering amplitudes.*
For term outside of the diagonal, the identity part vanishes of course, leaving us with

$$S_{fi} = i(2\pi)^d \delta^d(\mathbf{P}_f - \mathbf{P}_i) T_{fi}. \tag{2.40}$$

The relative probability is obtained by the modulus squared of this amplitude. In computing such a modulus squared there is however a problem, whose origin is the interpretation to assign to the square of the delta function. This problem can be solved by using this representation of $\delta(x)$

$$\delta^d(\mathbf{P}_f - \mathbf{P}_i) = \frac{1}{(2\pi)^d} \int e^{i(\mathbf{P}_f - \mathbf{P}_i)x} d^d x. \tag{2.41}$$

If we now take up the task of computing another integral of this kind at $\mathbf{P}_f - \mathbf{P}_i$ (because of the δ-function) and taking the integral over a finite time interval t and on a $(d-1)$-dimenasional volume, namely V, which we will take as large but finite, the result is going to be $Vt/(2\pi)^d$. Taking the square modulus of the matrix elements S_{fi} we then obtain

$$|S_{fi}|^2 = (2\pi)^d \delta^d(\mathbf{P}_f - \mathbf{P}_i) |T_{fi}|^2 Vt. \tag{2.42}$$

Dividing for the factor Vt, we finally obtain the *transition probability* (per unit volume and unit time)

$$\mathbf{P}_{i \to f} = (2\pi)^d \delta^d(\mathbf{P}_f - \mathbf{P}_i) |T_{fi}|^2. \tag{2.43}$$

The most relevant cases, from both a theoretical and an experimental point of view, are those in which the initial state is describing either one or two particles. The first

situation concerns the so called *decay processes*, i.e. a heavy particle separating into a set of lighter ones, while the second is relative to the scattering of two particles, resulting in an elastic diffusion or in a production process.
We should now rapidly discuss the normalization of these states.
The convenient choice here is to use the covariant normalization of the 1-particle state:

$$\langle p'|p\rangle = 2E(2\pi)^{d-1}\delta^{d-1}(\vec{p'}-\vec{p}). \tag{2.44}$$

This normalization is Lorentz invariant and it's equivalent to integrating over the "mass-shell" state of a particle, i.e.

$$\int \frac{d^{d-1}p}{(2\pi)^{d-1}2E}\,|p\rangle\langle p|p'\rangle = \int \frac{d^d p}{(2\pi)^{d-1}}\delta(p^2-m^2)\,|p\rangle\langle p|p'\rangle = |p'\rangle \tag{2.45}$$

with $E > 0$, as one can easily see by using the completeness relation. The density of states associated to a "*on–shell* particle with momentum in the interval $(p, p+dp)$ is given by

$$\frac{d^{d-1}p}{(2\pi)^{d-1}2E}. \tag{2.46}$$

Decay and Scattering processes Considering the proper normalization of the states, the probability of a decay of a particle of energy E into an n-particle state is expressed by

$$d\Gamma = (2\pi)^d\delta^d(\mathbf{P}-p_1-p_2-\cdots-p_n)|T_{fi}|^2\frac{1}{2E}\prod_{i=1}^{n}\frac{d^{d-1}p_i}{(2\pi)^{d-1}2E_i}, \tag{2.47}$$

where $\mathbf{P}$ indicates the momentum of the decaying particle in this context. For the $2\to n$ scattering process, the probability that the scattering of two particles with respective momentum $p_1 = (E_1, \vec{p_1})$ and $p_2 = (E_2, \vec{p_2})$ produces an arbitrary number of particles with momentum $p'_j = (E'_j, \vec{p'_j})$ is

$$d\Gamma = (2\pi)^d\delta^d(\mathbf{P}-p'_1-p'_2-\cdots-p'_n)|T_{fi}|^2\frac{1}{4E_1E_2}\prod_{i=1}^{n}\frac{d^{d-1}p'_i}{(2\pi)^{d-1}2E'_i}. \tag{2.48}$$

In the second case, rather than the probability, it is often more interesting to compute the Lorentz invariant *cross-section* $d\sigma$ of the collision process. We can obtain it by dividing the probability $d\Gamma$ by the factor

$$j = \frac{I}{E_1E_2}, \tag{2.49}$$

where we used I to denote the (scalar) quantity

$$I = \sqrt{(p_1 \dot{p}_2)^2 - (m_1 m_2)^2}. \tag{2.50}$$

One can easily understand that j is the flux of the colliding particles. In fact, in the reference frame of the center of mass of the system ($\vec{p_1} = -\vec{p_2} = \vec{p}$), one has $I = |\vec{p}|(E_1 + E_2)$ and then

$$j = |\vec{p}| \left(\frac{1}{E_1} + \frac{1}{E_2} \right) = v_1 + v_2, \tag{2.51}$$

whit v_1 and v_2 being the velocities of the colliding particles. The cross-section is then the transition probability per unit of flux of the scattering particles.

It's interesting to notice that in both the decay and scattering processes, the following quantity appears

$$d\Phi_n = (2\pi)^d \delta^d(\mathbf{P} - p_1 - p_2 - \cdots - p_n) \prod_{i=1}^{n} \frac{d^{d-1} p_i}{(2\pi)^{d-1} 2E_i}. \tag{2.52}$$

This is the differential in the n-particle *phase space*. It allows us to express the density of states for an n-particle system with total momentum $\mathbf{P}$.

We will now briefly discuss the consequences of the unitarity of the S-Matrix.

Substituting (2.39) into (2.36) we obtain

$$T_{fi} - T^*_{if} = i(2\pi)^d \sum_n \delta^d(\mathbf{P}_f - \mathbf{P}_i) T_{fn} T^*_{in}, \tag{2.53}$$

where the sum over the index n here denotes, in compact notation, both a sum and an integral over all intermediate states allowed by the conservation of the total momentum of the process. We can also notice that the left-hand side of this equation is linear with respect to the matrix elements of T, whereas the right-hand side is quadratic. If the theory under investigation happens to have a coupling constant g that can be treated as a perturbative parameter, the first consequence of (2.53) is the hermiticity of the matrix T at the first perturbative order, which mathematically means

$$T_{fi} \simeq T^*_{if}. \tag{2.54}$$

In fact, one can easily see that the left-hand side of (2.53) is of the first order in g while the right-hand side is of second order.

The Optical Theorem We hereby show the Optical theorem relative to the scattering of two particles, which is an important consequence of (2.53).
To prove it, let's initially sandwich (2.53) with the states $|p_1, p_2\rangle$ and $|p_3, p_4\rangle$:

$$2\,\mathrm{Im}\{\langle p_3, p_4|\, T\, |p_1, p_2\rangle\} = (2\pi)^d \sum_n \delta^d(\mathbf{P}_f - \mathbf{P}_i)\, \langle p_3, p_4|\, T\, |n\rangle\, \langle n|\, T^*\, |p_1, p_2\rangle\,. \tag{2.55}$$

In the particular case in which the scattering process is purely elastic, the final state coincides with the initial one, which results in

$$2\,\mathrm{Im}\{T_{ii}\} = (2\pi)^d \sum_n \delta^d(\mathbf{P}_f - \mathbf{P}_i)|T_{in}|^2. \tag{2.56}$$

One can easily notice that the right-hand side of the last expression differs from the total cross-section, namely σ_t, of all the possible scatterings just by a multiplicative factor, in fact

$$\sigma_t = \left(\frac{(2\pi)^d}{j}\right) \sum_n \delta^d(\mathbf{P}_f - \mathbf{P}_i)|T_{in}|^2. \tag{2.57}$$

We can therefore state the *optical theorem* relation:

$$\sigma_t = \frac{2}{j}\,\mathrm{Im}\{T_{ii}\}. \tag{2.58}$$

The theorem allows us to compute the total cross-section of the theory (for all the inelastic processes also) in terms of the imaginary part of the purely elastic scattering amplitude of two particles.
We can eventually comment on the final principles on which S-matrix theory is based, namely causality and analyticity principles. One might think that these two aspects should be deeply related to each other, based on known examples [6]. Nevertheless, in relativistic quantum mechanics, it is in general a non-trivial task to identify the precise analytic structure of the S-matrix in terms of the causality principle. It is common, in fact, to conjecture the analytic properties of the S-matrix elements based on those obtained in the non-relativistic scattering processes or on the ones found from the perturbative diagrams of the associated quantum field theory, when they are known.

[6] One of these might be the dispersion relations satisfied by the Green functions of an ordinary quantum system

On (1+1)-dimensional systems and Integrability
From now on, our focus will be on the (1+1)-dimensional models.
This class of models is particularly relevant because of the properties that it envelops. In fact, the existence of one conserved charge other than the Hamiltonian of the system is a sufficient condition to guarantee the existence of an infinitely number of conserved quantities, which makes the model integrable.
A very notable theorem is in fact the one proven by S. Parke [37], which states the following:

> *"In massive, (1 + 1)-dimensional, local, quantum field theories, the existence of two conserved charges is a sufficient condition for the absence of particle production and factorization of the S-matrix".*

This is indeed a fundamental result in (1+1)-dimensional theories not only because it constraints the scattering to be of the type " *2 in 2*" but also because the outgoing particles are going to be of the same species of the incoming ones and have the same set of momenta (p_1,p_2) attached to them, meaning that it will therefore be elastic. These relevant results lead to the theory being integrable, and let us factorize the S-matrix built on these models.

Conserved Charges and Elasticity For the purpose of this work, this paragraph has the means of briefly review some implications on the scattering process deriving from having an infinite number of conserved charges $Q_{\pm s}$. The "*spin*" index can be used to label those charges. In particular, the local ones[7] can be expressed by means of the integral of their current densities:

$$Q_s = \int T_{s+1}(z,\bar{z})dz + \Theta_{s-1}(z,\bar{z})d\bar{z}, \qquad \text{with } s \geq 1. \tag{2.59}$$

In the previous equation, $T_{s+1}(z,\bar{z})$ and $\Theta_{s-1}(z,\bar{z})d\bar{z}$ are local fields that satisfy the conservation law

$$\partial_{\bar{z}} T_{s+1}(z,\bar{z}) = \partial_z \Theta_{s-1}(z,\bar{z}). \tag{2.60}$$

In a totally analogous way, it is possible to express the charges associated to negative spins with

$$\bar{Q}_s = \int \bar{T}_{s+1}(z,\bar{z})dz + \bar{\Theta}_{s-1}(z,\bar{z})d\bar{z}, \tag{2.61}$$

[7]In certain integrable theories, for example Sine-Gordon model and the $O(3)$ nonlinear sigma model, some non-local charges appear, but they're associated to fractional spin ($0 < s < 1$).

where the conservation law is now

$$\partial_z \bar{T}_{s+1}(z,\bar{z}) = \partial_{\bar{z}} \bar{\Theta}_{s-1}(z,\bar{z}). \tag{2.62}$$

An interesting observation can be made about $Q_{\pm 1}$, since it can be recognized that they're actually the momentum component[8] on the light-cone:

$$\begin{aligned} Q_1 &= P^{(0)} + P^{(1)} \\ Q_{-1} &= P^{(0)} - P^{(1)}. \end{aligned} \tag{2.63}$$

A very interesting feature comes from the fact that these charges commute with each other, namely

$$\left[Q_{s_i}, Q_{s_j}\right] = 0. \tag{2.64}$$

Knowing this, it is then also guaranteed that they can be diagonalized simultaneously. The conserved charges action on the one-particle state can be therefore written as:

$$Q_s \left|A_i(\theta)\right\rangle = \omega_s^i(\theta) \left|A_i(\theta)\right\rangle, \tag{2.65}$$

in which the function ω_s^i is determined by the tensorial properties of the Q_s. Under the Lorentz group transformations, the conserved charges with positive index $s > 0$ transform as s copies of the momentum operator P, while the ones with negative index, Q_{-s} do the same as they were s copies of $\bar{P}$; it is then straightforward to treat $Q_{\pm s}$ as an s-rank tensor.

With these information on the behaviour of Q_s, it is then possible to impose the separation

$$\omega_s^i(\theta) = \lambda_s^i e^{s\theta}, \tag{2.66}$$

where λ_s^i takes the name *eigenvalue of the charge Q_s with charge i*. Trying to resolve the spectrum of the λ_s^i is not an easy task in itself but it's not of our concern to deepen the calculation on this.

Other restrictions might come from discrete symmetries of the model. For instance, if the theory is invariant under the charge conjugation C, the conserved charges can be labelled as even or odd operators $Q_s^{\pm}$ with respect to C. In the eventuality that also the

[8] To avoid confusion, we recall that in this context, momentum is strictly intended as an operator, therefore acting on the Hilbert space.

parity P is a symmetry of the model, it is possible to derive the commutation relations:

$$\begin{aligned} CQ_s^+C &= Q_s^+ = (-1)^{s+1}Q_s^+ \\ CQ_s^-C &= -Q_s^- = (-1)^{s+1}Q_s^- . \end{aligned} \tag{2.67}$$

The above equations illustrates that the values that s can assume are only odd integer numbers for even charges (with respect to C), while they can only be even for odd charges.

We can now properly analyze how the infinite conserved charges constrain the scattering processes.

As illustrated by S. Coleman and J. Mandula[19], in their well known work, we have that in (3+1)-dimensional theories the existence of only one conserved charge of tensor rank larger than 2 implies a trivial S-matrix, i.e. $S = \mathbb{1}$. Of course the result does not apply to the (1+1)-dimensional theories but there is still a series of severe constraints that we can enumerate:

1. The number of particles with mass m_i does not change after the collision has taken place, meaning that we get the same number of particles in the initial and final asymptotic states.

2. The sets of particles' momenta before and after the scattering are the same, meaning that the process is purely elastic.

3. The scattering amplitude for the process in which n particles are involved can be completely separated (i.e. factorized) in terms of the $\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n-1} 1 = n(n-1)/2$ two-particles elastic scattering amplitudes.

We shall now provide proofs for these statements.

First, one should observe that the conserved charges act on the multi-particle states as

$$Q_s \left| A_{\alpha_1}(\theta_1) \dots A_{\alpha_n}(\theta_n) \right\rangle = \sum_{i=1}^{n} \lambda_s^{\alpha_i} e^{s\theta_i}(\theta) \left| A_{\alpha_1}(\theta_1) \dots A_{\alpha_n}(\theta_n) \right\rangle . \tag{2.68}$$

Since the Q_s are conserved quantities, the relation

$$\frac{d}{dt} Q_s = 0 \tag{2.69}$$

holds, meaning that there is an infinite sequence of constraints that involve the sum of the higher powers of the momenta of the initial and final particles, namely:

$$\sum_{j\in in}\lambda_s^{\alpha_j}e^{s\theta_j}(\theta)=\sum_{i\in fin}\lambda_s^{\alpha_i}e^{s\theta_i}(\theta), \tag{2.70}$$

where we used in to denote the initial state and fin to indicate the final one. The only possible solution to these infinite numbers of equations (apart from the trivial ones, i.e. permutations of particles with the same quantum numbers) corresponds to the situation in which the initial and final sets of rapidities are equal.
A very non-trivial and interesting consequence emerges from this: in theories having an infinite number of conserved charges, the annihilation and production processes are not happening, meaning that the scattering processes are all elastic.
Another key feature is the factorization of the processes. We shall first look at the action of the conserved charges Q_s on a localized wave packet. Denoting q_s as the space component for the two charges $Q_{\pm s}$, we have

$$e^{icq_s}\left|A_\alpha(p)\right\rangle=e^{icp^s}\left|A\alpha(p)\right\rangle, \tag{2.71}$$

where λ_s was put equal to 1 in order to have simplicity in this explanation of the feature. We can now take a look at the wavefunction ψ:

$$\psi(x)=\int_{-\infty}^{+\infty}dpe^{-\alpha(p-p_0)^2}e^{ip(x-x_0)} \tag{2.72}$$

describing a state that is well localized both in momentum space (around $(p=p_0)$ and in the coordinates' one (around $(x=x_0)$. By means of acting with e^{icq_s} on this state, we get

$$\tilde{\psi(x)}=\int_{-\infty}^{+\infty}dpe^{-\alpha(p-p_0)^2}e^{ip(x-x_0)}e^{icp^s}. \tag{2.73}$$

Clustering the exponential arguments, it can be seen that the new function is localized at $x=x_0-scp_0^{s-1}$, meaning that for $s>1$ the center of the wave packet is shifted by a quantity that depends on the $(s-1)$ power of its momentum (with $s=1$ we recover the case in which q_s coincides with the ordinary momentum, shifting all packets by the same quantity). This result clearly illustrates that wave packets with different momenta can be shifted differently by acting on them with the conserved charges e^{icqs} with higher spins [9].

[9]This result is a clarification of the Coleman-Mandula theorem. Indeed, in $(d+1)$-theories, with d being $d>1$, the possibility to translate differently particles that have momenta means that their

2.4 Two body scattering

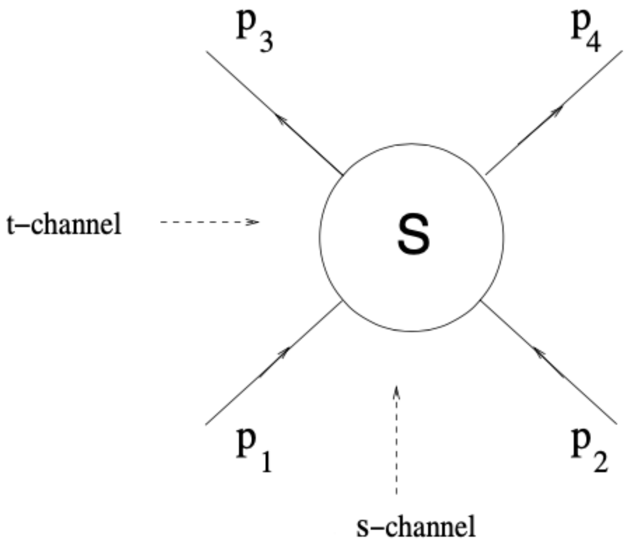

Figure 2.5: Scattering process for two particles.[10]

We can now consider in more details the scattering process of two initial scalar particles with momenta p_1 and p_2 which go into two scalar particles with momenta p_3 and p_4; a graphical representation of this process can be seen in 2.5.
We're going to denote with A_i the particle to which the momenta p_i is associated, resulting in the scattering being described by the following notation:

$$A_1 + A_2 \to A_3 + A_4. \tag{2.74}$$

If we now factorize the δ-function in the conservation of the total momentum, we obtain

$$\langle p_3, p_4| \, T \, |p_1, p_2\rangle = i(2\pi)^d \delta^d(p_1 + p_2 - p_3 - p_4)\mathcal{T}, \tag{2.75}$$

where we introduced $\mathcal{T}$, which is an analytic function of relativistic invariants for this scattering process that we can identify in term of the Mandelstam variables, namely s,

trajectories can never cross: their configuration is a free motion without collision and therefore S must be equal to $\mathbb{1}$

[10]The figure was taken from [1]

t and u.
Those quantities are described by the relations

$$\begin{aligned} s &= (p_1 + p_2)^2, \\ t &= (p_1 - p_3)^2, \\ u &= (p_1 - p_4)^2. \end{aligned} \tag{2.76}$$

The conservation law of momentum is clearly a condition related to these variable. One can in fact recast the condition

$$p_1 + p_2 = p_3 + p_4, \quad with \quad p_i^2 = m_i^2, (i = 1, \ldots, 4) \tag{2.77}$$

into the equation

$$s + t + u = \sum_{i=1}^{4} m_i^2. \tag{2.78}$$

It is straightforward to interpret the meaning of these variables in the appropriate situations.
By putting ourselves in the reference frame of the center of mass in the process showed in 2.5, which is the one described by the condition $\vec{p_1} + \vec{p_2} = 0$, We observe that $s = E^2$, with E being the total energy here, namely $E = E_1 + E_2$.
In an analogous context, we can see that the variable t has the same interpretation of being the square of the energy but in the channel described by the scattering

$$A_1 + \bar{A}_3 \to \bar{A}_2 + A_4. \tag{2.79}$$

Lastly, the same is also true for u in the channel

$$A_1 + \bar{A}_4 \to \bar{A}_2 + A_3. \tag{2.80}$$

In the equations above, use has been made of the notation $\bar{A}_i$, which indicates the antiparticle associated to A_i. In fact, by using the crossed-channels, we have to be careful of the way of the incoming particles and consider outgoing arrows as antiparticles whose nature gets reversed.

On Rapidities and Asymptotic States For the purpose of our work, we will specialize on (1+1)-dimensional scattering theories where there is an infinite number of conserved charges $\mathcal{Q}_i$, which is going to lead us to a very useful simplification of the S-Matrix structure and a possibility to calculate exact scattering amplitudes in some cases.

In particular, the momenta of the particles involved in these scattering processes are on-shell.

One particularly useful consequence of being in this dimensional situation, is the fact that a very handy parametrization of the energy and momentum of a particle is possible. We can in fact write them as a function of the rapidity θ_k, namely

$$p_k^{(0)} = m_k \cosh\theta_k, \qquad p_k^{(1)} = m_k \sinh\theta_k, \tag{2.81}$$

with m_k being the particle's mass. In this useful notation, the Lorentz transformation can be treated as a rotation of a hyperbolic angle θ_0 and therefore implemented as $\theta \to \theta + \theta_0$. Moreover, both components of the momentum can be changed by sign with the transformation $\theta_k \to i\pi - \theta_k$, meaning that in this way, the momentum of the original particle becomes that of its own antiparticle. This statement will become more clear once we will analyze Crossing-symmetry in two-particle scattering later in this chapter. Taking a quick look at the components on the light-cone, one can see that the relation they satisfy is quite interesting, indeed we have:

$$\begin{aligned} p_k &= p_k^{(0)} + p_k^{(1)} = m_k e^{\theta_k} \\ \bar{p_k} &= p_k^{(0)} - p_k^{(1)} = m_k e^{-\theta_k}, \end{aligned} \tag{2.82}$$

resulting in the on-shell condition being

$$p_k \bar{p_k} = m_k^2. \tag{2.83}$$

An interesting aspect might be the geometrical interpretation of the rapidity θ. In fact, the Italian mathematician Riccati pointed out that in the plane with axes given by E and p, the dispersion relation given by $E^2 = p^2 + m^2$ represents an hyperbola. The rapidity is proportional to the area Ω that is encompassed between the hyperbola and the straight line that joins the origin to the point of the hyperbola identified by the variable θ, in particular, the ratio is $\Omega = m^2\theta/2$. An similar result is achieved for the angle β that parameterizes the points of a circle $a^2 + b^2 = m^2$. Using polar coordinates $a = m\cos\beta$

and $a = m \sin\beta$ the area Ω between the x- axis and the line that indicates the point on the circles is in fact $\Omega = m^2\beta/2$. A very interesting fact is that the two geometrical representation are linked by the analytical continuation $\beta \to i\theta$.

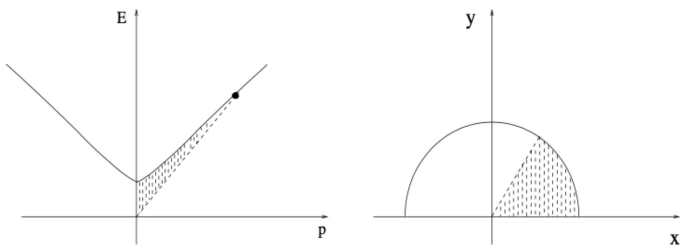

Figure 2.6: Geometrical interpretation of the rapidities as the parameter describing the colored area in the graphs, as described in this paragraph. [11]

We can now introduce a description for the n-particle state of the theory, which we will indicate as

$$|A_{\alpha_1}(\theta_1)A_{\alpha_2}(\theta_2)\dots A_{\alpha_n}(\theta_n)\rangle\,, \tag{2.84}$$

in which we used $A_{\alpha_j}(\theta_j)$ to indicate the particle of type α_j with associated rapidity θ_j. By means of a linear superposition of these states, we can construct wave packets that are localized both in momentum and coordinate space. In this way, we can imagine assigning a well-defined position to the particles in the state presented above. In massive theories, the interactions become short ranged, implying that a state like (2.84) represents a set of free particles except in the time intervals in which the wave packets overlap.
Being a fundamental point for our purposes, we will now discuss how to represent and introduce the initial and final states.
What we will call an *initial asymptotic state* is given by a set of free particles at $t \to -\infty$. Since in (1+1) dimensional theories the motion takes place on a line, this means that the fastest particle must be on the farthest left side of all the others, while the slowest must be on the right side. The remaining particles are ordered according to the value of their rapidities between those two. To properly formally describe this situation, we will therefore consider the symbols $A_{\alpha_j}(\theta_j)$ as non-commuting variables as well as interpreting their order as associated to the space ordering of the particles that they represent,

[11] The figure was taken from [1]

meaning that in (2.84) there is an underlying structure which puts the rapidities in a *decreasing* order:

$$\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n. \tag{2.85}$$

In a totally analogous way, we will describe the *final asymptotic state* as made of free particles at $t \to +\infty$. Following the reasoning pattern just discussed about motion on a line and ordering based on rapidities, we will have that each particle must be on the left-hand side of all the others that move faster than it. Accordingly, the final asymptotic state will be denoted with

$$\left|A'_{\alpha_1}(\theta_1)A'_{\alpha_2}(\theta_2)\ldots A'_{\alpha_n}(\theta_n)\right\rangle, \tag{2.86}$$

in a very similar way to the initial one, but with opposite ordering of the rapidities, meaning that we will have an *increasing* set:

$$\theta_1 \leq \theta_2 \leq \cdots \leq \theta_n. \tag{2.87}$$

An interesting feature is that we can always order (2.84) the way we like by means of a certain number of commutations of the $A_{\alpha_j}(\theta_j)$ between neighbours particles.
It can be shown that each commutation can be interpreted as a *scattering process of two particles.*
It is possible to normalize the states as

$$\left\langle A_{\alpha_i}(\theta_a)\middle|A_{\alpha_j}(\theta_b)\right\rangle = 2\pi\delta_{ij}\delta(\theta_a - \theta_b). \tag{2.88}$$

This normalization is going to come in handy as the density of states with rapidities ($\theta, \theta + d\theta$) is going to be $d\theta/2\pi$.

Unitarity and Crossing-Symmetry As the title of this paragraph anticipates, we will now take a look at the equations that we obtain and which constraint the S-Matrix after imposing unitarity and crossing-symmetric invariance. Those are the ones valid for the elastic scattering of two particles in (1+1)-dimensional integrable theories.
Keeping the previous notation for momenta, we will indicate with p_1 and p_2 the initial and final momentum for the incoming and outgoing particles. Since we are considering an elastic scattering, it is obvious that those momenta are going to keep the same modulus after the scattering process. We will also use A_i and A_j for incoming particles while A_k and A_l are going to be the outgoing ones. Since we are considering (1+1)-dimensional system, the conservation of total momentum becomes a 2-delta, meaning the term will appear as $\delta^2(p_1 + p_2 - p_3 - p_4)$. We also recall that asking the process to be Lorentz invariant, we need it to be such that the scattering amplitude depends on the particle momenta only by their invariant combinations, given by the Mandelstam variables s, t, and u, as defined in (2.76). One can notice that for (1+1)-dimensional processes, u vanishes, leaving us with only s an t to deal with.
It is also straightforward to realize that for elastic (1+1)-dimensional processes, we only have one independent Mandelstam variable because $p_1 = p_4$ and $p_2 = p_3$.
Recalling the parametrization of these quantities in terms of the rapidities of the particles θ_i, we have:

$$p_i^{(0)} = m_i \cosh\theta_i, \quad p_i^{(1)} = m_i \sinh\theta_i, \tag{2.89}$$

which leads us to writing the variable s scattering process

$$A_i + A_j = A_k + A_l \tag{2.90}$$

as[12]

$$s(\theta_{ij}) = (p_1 + p_2)^2 = m_i^2 + m_j^2 + 2m_i m_j \cosh\theta ij, \tag{2.91}$$

where we made use of the notation

$$\theta_{ij} = \theta_i - \theta_j. \tag{2.92}$$

[12]Since the reader might wonder about why the indexes 1 and 2 have been used, it's better to clarify that the elasticity of the scattering has been kept into consideration as well as the total momentum conservation. This means that we don't need to distinguish between 4 momentum (as we do for the 2 incoming plus 2 outgoing particles), but we can focus on the only two independent quantities.

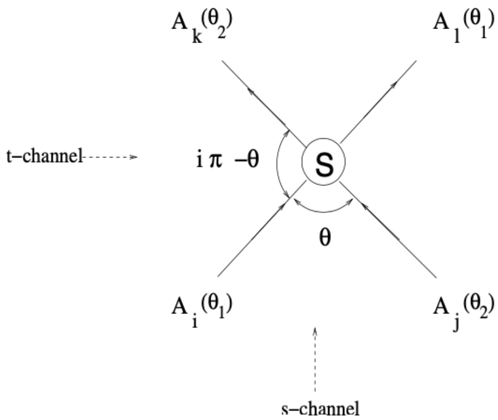

Figure 2.7: Elastic scattering of two particles in (1+1)-dimensions. [13]

For physical processes, the quantity θ_{ij} is real, therefore we can also say that s is going to be real as well as satisfying the relation $s \geq (m_i + m_j)^2$.
For the Mandelstam variable t, associated to the homonym channel, the relation that holds is instead the following one:

$$t(\theta_{ij}) = (p_1 - p_2)^2 = m_i^2 + m_j^2 - 2m_i m_j \cosh \theta ij. \tag{2.93}$$

As one can immediately guess from the similar function describing these two quantities, the straightforward step is to try to relate one to the other.
In particular, what we get is the following analytic continuation:

$$t(\theta) = s(i\pi - \theta), \tag{2.94}$$

which confirms the most natural way of interpreting θ as the angle between lines associated to the scattering particles, as the reader can see in 2.7. Using $|A_i(\theta_a)A_j(\theta_b)\rangle$ to identify the asymptotic state where we have particles A_i and A_j, with rapidities θ_a and θ_b respectively, we get the S-Matrix elements as defined by

$$|A_i(\theta_1)A_j(\theta_2)\rangle = S_{ij}^{kl}(\theta) \, |A_k(\theta_2)A_l(\theta_1)\rangle \,, \tag{2.95}$$

where $\theta = \theta_{12}$ and with $\theta_1 > \theta_2$, according to the definition of asymptotic states given in the previous paragraphs. The tensorial notation is in use in the equation (2.95), meaning

[13] The figure was taken from [1]

that a sum over the indexes k and l is in effect in the right-hand side term.
We can also notice that the dependence of the S-Matrix from the difference of the rapidities is a consequence of the relativistic invariance of the theory, since a Lorentz transformation changes the value of the rapidity of each particle by a constant.
Translating the relation (2.94) into one for the S-Matrix, the equation that we obtain is the following:

$$S_{ij}^{kl}(\theta) = S_{\bar{l}i}^{\bar{j}k}(i\pi - \theta), \tag{2.96}$$

while the unitarity condition can be written as

$$\sum_{n,m} S_{ij}^{nm}(\theta) S_{nm}^{kl}(-\theta) = \delta_i^k \delta_j^l. \tag{2.97}$$

It is important to stress on two fundamental characteristics of the formulation in term of rapidities for the S-Matrix. The former one is that the unitarity and crossing-symmetry equations can be analytically continued for arbitrary values of θ and therefore they hold true in all the complex plane of this variable. The latter is about the definition of the S-matrix itself, such that, as a function of θ, it can be written in an operatorial equation as

$$A_i(\theta_1) A_j(\theta_2) = S_{ij}^{kl}(\theta) A_k(\theta_2) A_l(\theta_1). \tag{2.98}$$

The equation above is very important since it defines an algebra for the symbols A_α, namely the one that goes by the name "*Faddev-Zamolodchikov algebra*". The scattering processes can be indeed interpreted in a totally equivalent way the commutation relations among the operators that create the particles. In this scenario, the unitarity equation (2.97) can be regarded as a simple consequence of this algebra. In particular, the Yang–Baxter equations are a straightforward consequence of the associativity condition of the Faddev–Zamolodchikov algebra.

Yang - Baxter equation We want now to consider the collision of three particles of momenta $p_3 > p_2 > p_1$, associated to wave packets that are well-localized both in the momentum and coordinate space. Depending on the initial positions of the three packets, we can have three types of collisions: we either get the simultaneous scattering of all three particles (which is, in general, unlikely) or, as shown in 2.8, we can get two different possibilities of a sequence of three two-particle scatterings. Between picture (a) and (b), it is very clear that the sequential order of the scatterings is different. In a general theory, the three processes have different amplitudes, but with the restriction of being in an integrable theory, this lets us say that the different scenarios can be obtained one from the other two by means of the application of the operators e^{icq_s}.
Being commutative with the Hamiltonian of the system (which is associated to $Q_{\pm 1}$), we need their action to lead to the same physical scenario. This immediately leads to having the same three scattering amplitudes in integrable theories.
Two very delicate results have therefore been achieved:

- In integrable theories, the S-matrix of a scattering process involving three particles can be factorized into the two-particle scattering amplitudes $S_2(p_i, p_j)$, which must satisfy the famous *Yang-Baxter Equation*:

$$S_2(p_2, p_3)S_2(p_3, p_1)S_2(p_1, p_2) = S_2(p_1, p_2)S_2(p_1, p_3)S_2(p_2 p_3). \tag{2.99}$$

- It is possible to show that this result can be generalized for n-particle processes; it is indeed easy to retrieve that the fulfilment of the Yang–Baxter equations (2.99) are sufficient and necessary conditions for the factorization of this amplitude in terms of the n(n − 1)/2 two-particle elastic amplitudes. As before, in these collisions a possible exchange of the momenta can occur only between particles with the same mass and the same quantum numbers.

The true strength of the Yang-Baxter equations is the fact that it is sufficient to find the two-particle scattering amplitudes to have full knowledge over any other scattering processes. In particular, the two-particle scattering amplitudes can be found as solutions of the Yang–Baxter equation, together with the general requirements of unitarity and crossing symmetry.

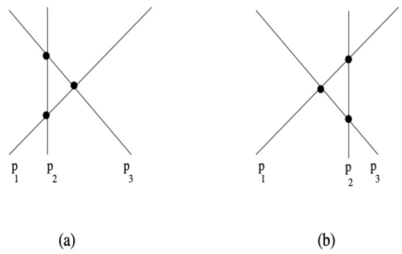

Figure 2.8: A graphical representation of elastic scattering of three particles, separated in sequential two-particles collision, showing a clear interpretation of the Yang-Baxter equation. The first picture (a) shows the scattering happening between particle 1 and 2, then between 1 and 3 and finally between 2 and 3, while the second one (b) illustrates a different elastic decomposition order for the scattering process, namely 2 and 3, followed by 1 and 3 and eventually 1 and 2. [14]

The explicit form of the Yang-Baxter equations is (with an understood sum over all the repeated indices, i.e. tensor notation)

$$S^{ab}_{jk}(\theta_{23})S^{ia}_{nc}(\theta_{31})S^{cb}_{ml}(\theta_{12}) = S^{ab}_{ij}(\theta_{12})S^{bk}_{cl}(\theta_{13})S^{ac}_{nm}(\theta_{23}). \tag{2.100}$$

It is important to bring to the reader's attention the fact that this is a set of κ^6 equations, which means that our problem is overdetermined, since we're only trying to retrieve the κ^4 amplitudes. A straightforward implication is the fact that this is only possible for some very particular functional forms of the functions $S^{kl}_{ij}(\theta)$.

[14] The figure was taken from [1]

Chapter 3

Exactly solved models

3.1 The Sine-Gordon Model

Following the considerations shown in the previous chapter, we will now show certain relativistic factorized S-matrices identified by the isotopic $O(n)$ symmetry. To introduce the symmetry, we shall consider the n-plet of particles A_i, with $i = 1, \dots, n$ having equal masses $m_1 = \dots = m_n = m$. We will then require the two-particle scattering to be $O(n)$ symmetric (this particular requirement guarantees the overall $O(n)$ symmetry of the S-matrix by means of the factorization). We then get the following form for the two-particle S-matrix

$$\begin{aligned} S_{ij}^{kl} &= \langle A_i(p_1')A_j(p_2')|A_k(p_1)A_l(p_2)\rangle \\ &= \delta(p_1 - p_1')\delta(p_2 - p_2')\left(\delta_{ij}\delta_{kl}S_t(s) + \delta_{ik}\delta_{jl}S_r(s) + \delta_{il}\delta_{kj}S_a(s)\right) \\ &\pm (i \leftrightarrow k, p_1 \leftrightarrow p_2), \end{aligned} \tag{3.1}$$

where s is the Mandelstam variable $s = (p_1^\mu + p_2^\mu)^2$, the $\pm$ sign refers to bosonic $(+)$ particles or fermionic $(-)$ particles and the primed momenta p_i' refer to outgoing particles, while the p_i identify the incoming ones. The quantities $S_t(s)$ and $S_r(s)$ are respectively the transition and reflection amplitudes while $S_a(s)$ is the one that describes the "annihilation" type process: $A_i + A_i \to A_j + A_j$ where $i \neq j$.
We can therefore write down the more general formulae in term of the rapidities θ_j of the particles that we will be using in the following paragraphs:

$$\begin{aligned} A_j(\theta_1)A_k(\theta_2) =& \delta_{jk}S_a(\theta)\sum_{l=1}^{n} A_l(\theta_2)A_l(\theta_1) \\ &+ S_t(\theta)A_k(\theta_2)A_j(\theta_1) + S_r(\theta)A_j(\theta_2)A_k(\theta_1), \end{aligned} \tag{3.2}$$

where we used θ to denote the difference between the particles' rapidities $\theta = \theta_1 - \theta_2$.

S-Matrix for the quantum sine-Gordon model The notorious sine-Gordon model is retrieved if we set $n = 2$ for the scattering theories of $O(n)$ models. In particular, its lagrangian can be written as:

$$\mathcal{L}_{SG}[\phi, \beta] = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{\beta^2} \cos(\beta\phi). \tag{3.3}$$

For the following analysis of the model, it's going to be useful to introduce the parameter ζ

$$\zeta = \frac{1}{1 - \frac{\beta^2}{8\pi}} \frac{\beta^2}{8}. \tag{3.4}$$

The variable ζ here takes the role of a renormalized coupling constant.
It is interesting to point out that it was shown[18] that the sine-Gordon model is equivalent to the massive Thirring model (MT) of a Dirac field, namely having the lagrangian

$$\mathcal{L}_{MT} = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m_0 \bar{\psi}\psi - \frac{g}{2}\left(\bar{\psi}\gamma^\mu\psi\right)^2, \tag{3.5}$$

where the mapping from sine-Gordon's to Thirring parameters is such that the coupling constant g takes the value

$$\frac{g}{\pi} = \frac{4\pi}{\beta^2} - 1, \tag{3.6}$$

for values of β^2 such that $0 \leq \frac{\beta^2}{8\pi} < 1$. In this instance, the Thirring fermion (i.e. the Dirac field ψ gets identified with the sine-Gordon soliton.
The interaction regimes can therefore be described by the sign of g

$$\begin{aligned} &\text{repulsive} \quad (g < 0) \quad \Leftrightarrow \quad \beta^2 > 4\pi \quad \Leftrightarrow \quad \zeta > \pi, \\ &\text{attractive} \quad (g > 0) \quad \Leftrightarrow \quad \beta^2 < 4\pi \quad \Leftrightarrow \quad \zeta < \pi. \end{aligned} \tag{3.7}$$

This will be relevant to keep in mind for discussing the sine-Gordon bound states that will come after.
The finest way to introduce the scattering theory for the model is to describe its excitations with the complex (linear) combination of the couples of degenerate particles of the initial $O(2)$ model, namely:

$$\begin{aligned} \mathcal{A}(\theta) &= A_1(\theta) + iA_2(\theta), \\ \bar{\mathcal{A}}(\theta) &= A_1(\theta) - iA_2(\theta), \end{aligned} \tag{3.8}$$

with $A_1(\theta)$ and $A_2(\theta)$ being the couple of degenerate particles mentioned. Making use of these renewed definition for the excitations it is possible to make use of the relation (3.2) in order to express the scattering amplitudes as

$$\begin{aligned}
\mathcal{A}(\theta_1)\bar{\mathcal{A}}(\theta_2) &= S_t(\theta)\bar{\mathcal{A}}(\theta_2)\mathcal{A}(\theta_1) + S_r(\theta)\mathcal{A}(\theta_2)\mathcal{A}(\bar{\theta}_1), \\
\mathcal{A}(\theta_1)\mathcal{A}(\theta_2) &= S_a(\theta)\mathcal{A}(\theta_2)\mathcal{A}(\theta_1), \\
\bar{\mathcal{A}}(\theta_1)\bar{\mathcal{A}}(\theta_2) &= S_a(\theta)\bar{\mathcal{A}}(\theta_2)\bar{\mathcal{A}}(\theta_1),
\end{aligned} \tag{3.9}$$

where we recall that we are making use of the notation $\theta = \theta_1 - \theta_2$ and $S_t(\theta), S_r(\theta), S_a(\theta)$ are the transition, reflection and annihilation amplitudes respectively.
It can be very handy to organize those amplitudes into a 4×4 matrix:

$$S_{SG} = \begin{pmatrix} S_a & & & \\ & S_t & S_r & \\ & S_r & S_t & \\ & & & S_a \end{pmatrix}, \tag{3.10}$$

where all the non-specified entries are vanishing elements. In this instance, S_a takes the role of the "*common transmission*" amplitude for scattering couples of solitons $\mathcal{A}(\theta_1)\mathcal{A}(\theta_2)$ or antisolitons $\bar{\mathcal{A}}(\theta_1)\bar{\mathcal{A}}(\theta_2)$. The unitarity condition becomes then the set of equations

$$\begin{aligned}
S_a(\theta)S_a(-\theta) &= 1 \\
S_t(\theta)S_t(-\theta) + S_r(\theta)S_r(-\theta) &= 1 \\
S_t(\theta)S_r(-\theta) + S_r(\theta)S_t(-\theta) &= 0,
\end{aligned} \tag{3.11}$$

while the crossing symmetry constraint untangle as

$$\begin{aligned}
S_a(\theta) &= S_t(i\pi - \theta) \\
S_r(\theta) &= S_r(i\pi - \theta),
\end{aligned} \tag{3.12}$$

where it is clearly shown that $S_a(\theta)$ and $S_t(\theta)$ are somehow reciprocally interacting while $S_r(\theta)$ does not blend with the other two amplitudes.
Making use of the Yang-Baxter equation, one can finally express the transition and

reflection amplitudes $S_t(\theta)$ and $S_r(\theta)$ in terms of $S_a(\theta)$

$$S_t(\theta) = \frac{\sinh \frac{\pi\theta}{\zeta}}{\sinh \frac{\pi(i\pi-\theta)}{\zeta}} S_a(\theta)$$
$$S_r(\theta) = i\frac{\sinh \frac{\pi^2}{\zeta}}{\sinh \frac{\pi(i\pi-\theta)}{\zeta}} S_a(\theta). \tag{3.13}$$

Inserting then (3.13) into the crossing-symmetry and unitarity conditions (i.e. (3.12) and (3.11)), one can finally retrieve the set of two equations that constraint the common transmission amplitude $S_a(\theta)$:

$$\begin{aligned} S_a(\theta)S_a(-\theta) &= 1 \\ S_a(i\pi-\theta) &= \frac{\sinh \frac{\pi\theta}{\zeta}}{\sinh \frac{\pi(i\pi-\theta)}{\zeta}} S_a(\theta), \end{aligned} \tag{3.14}$$

In particular, these relations enable us to retrieve an exact expression for it as an infinite product of Euler Γ functions:

$$\begin{aligned} S_a(\theta) = \prod_{l=0}^{\infty} & \frac{\Gamma\left(1+(2l+1)\frac{\pi}{\zeta}-i\frac{\theta}{\zeta}\right)\Gamma\left(1+2l\frac{\pi}{\zeta}+i\frac{\theta}{\zeta}\right)}{\Gamma\left(1+(2l+1)\frac{\pi}{\zeta}+i\frac{\theta}{\zeta}\right)\Gamma\left(1+2l\frac{\pi}{\zeta}-i\frac{\theta}{\zeta}\right)} \\ \times & \frac{\Gamma\left((2l+1)\frac{\pi}{\zeta}-i\frac{\theta}{\zeta}\right)\Gamma\left((2l+2)\frac{\pi}{\zeta}+i\frac{\theta}{\zeta}\right)}{\Gamma\left((2l+1)\frac{\pi}{\zeta}+i\frac{\theta}{\zeta}\right)\Gamma\left((2l+2)\frac{\pi}{\zeta}-i\frac{\theta}{\zeta}\right)}. \end{aligned} \tag{3.15}$$

It is interesting to also mention that an integral representation for $S_a(\theta)$ is also admitted. It can be retrieved by using the integral representation for the Euler Γ functions as the stepping stone and in fact we can express the scattering amplitude as

$$S_a(\theta) = -\exp\left\{-i\int_0^\infty \sin(\theta t)\frac{\sinh \frac{t(\pi-\zeta)}{2}}{\sinh\frac{\zeta t}{2}\cosh\frac{\pi t}{2}}\frac{dt}{t}\right\}. \tag{3.16}$$

Being able to write $S_a(\theta)$ in such integral form is an essential tool in taking the next step and perform the Thermodynamic Bethe Ansatz techniques on the model to pull out its physical quantities. Looking at the S-matrix now retrieved, one can observe the poles structure of it. It's quite important to focus on the physical region $0 < \mathrm{Im}\{\theta\} < \pi$. In particular, one can also rewrite the S_a scattering amplitude in a mixed representation, which is:

$$S_a(\theta) = -(-1)^n \prod_{l=1}^{\infty} \left(\frac{\theta + il\zeta}{\theta - il\zeta} \right)$$
$$\times \exp \left\{ -i \int_0^{\infty} \sin(\theta t) \frac{2 \sinh \frac{t(\pi-\zeta)}{2} e^{-n\zeta t} + \left(e^{-n\zeta t} - 1\right) \left(e^{(\zeta-\pi)t/2} + e^{-(\zeta+\pi)t/2}\right)}{2 \sinh \frac{\zeta t}{2} \cosh \frac{\pi t}{2}} \frac{dt}{t} \right\}. \tag{3.17}$$

One can pretty easily figure that $S_a(\theta)$ has poles in

$$\theta = i\kappa\zeta, \qquad \text{with } \kappa = 0, 1, 2, \ldots \tag{3.18}$$

while others emerge from (3.13) due to the denominators, leading to poles in

$$\theta = i(\pi - \kappa\zeta), \qquad \text{with } \kappa = 0, 1, 2, \ldots \tag{3.19}$$

The physical region allowing both poles to be in the $0 < \text{Im}\{\theta\} < \pi$ interval is then such that $\zeta < \pi$. If this requirement is satisfied, the poles described in (3.19) are connected to the bound states of the s-channel, while the poles (3.18) are linked to the crossed t-channel instead. It is also possible to compute the number of these poles, in fact one has

$$\mathcal{N}_{poles} = \left[\frac{\pi}{\zeta} \right], \tag{3.20}$$

where the notation $[n]$ has been used to indicate the *integer part* of n.
To utterly carry on this interpretation, one can introduce the amplitudes

$$\begin{aligned} S_+(\theta) &= (S_r(\theta) + S_t(\theta)) \\ S_-(\theta) &= (S_r(\theta) - S_t(\theta)) \,. \end{aligned} \tag{3.21}$$

Recalling (3.13), we can write down those amplitudes as

$$S_\pm(\theta) = -\frac{1}{\sinh \frac{\pi(\theta - i\pi)}{\zeta}} \left\{ i \sinh \frac{\pi^2}{\zeta} \pm \sinh \frac{\pi\theta}{\zeta} \right\} S_a(\theta). \tag{3.22}$$

The value of the residue in (3.19) takes then the final form

$$S_\pm(\theta) \simeq -\frac{i}{\theta - i\pi + i\kappa\zeta} (-1)^n \zeta \sin \frac{\pi^2}{\zeta} \left(1 \pm (-1)^\kappa\right) S_a(i\pi - i\kappa\zeta). \tag{3.23}$$

An interesting consequence of the way these scattering amplitudes were defined is that they correspond to processes where a quantum number under the charge conjugation operation is well defined. This unfolds assigning a charge $C_+ = 1$ to S_+ and a charge $C_- = 1$ to S_-.
This approach on the argument can also be checked by looking at the following scattering processes:

$$\begin{aligned} \mathcal{A}(\theta_1)\bar{\mathcal{A}}(\theta_2) + \bar{\mathcal{A}}(\theta_1)\mathcal{A}(\theta_2) &= S_+(\theta)\left[\mathcal{A}(\theta_2)\bar{\mathcal{A}}(\theta_1) + \bar{\mathcal{A}}(\theta_2)\mathcal{A}(\theta_1)\right] \\ \mathcal{A}(\theta_1)\bar{\mathcal{A}}(\theta_2) - \bar{\mathcal{A}}(\theta_1)\mathcal{A}(\theta_2) &= S_-(\theta)\left[\mathcal{A}(\theta_2)\bar{\mathcal{A}}(\theta_1) - \bar{\mathcal{A}}(\theta_2)\mathcal{A}(\theta_1)\right], \end{aligned} \tag{3.24}$$

directly confirming the statement presented above.
A final consideration can be made on the residue formulae (3.23) presented.
The element $(1 \pm (-1)^\kappa)$ in fact, prevents poles by making the whole expression vanish in certain instances. The immediate consequence of it, is that S_+ only has poles if κ has an *even* value, while the analogue is true for S_-, meaning that it can only get poles if κ is an *odd* number. Since both the sets of residue have positive values, they correspond to the s-channel poles and therefore its soliton-antisoliton bound states. These are scalar particles that take the name *breathers*, with the trait of having charge $C = (-1)^\kappa$ under the charge conjugation operation. Not being our goal, we will not further investigate these bound states for now. We report for completeness the masses associated to the soliton-antisoliton bound states, namely

$$m_\kappa = 2m \sin\left(\frac{\kappa\zeta}{2}\right), \qquad \text{with } \kappa = 0, 1, \dots < \frac{\pi}{\zeta}. \tag{3.25}$$

q-Deformation: Quantum group symmetry of the sine-Gordon The quantum group $\mathrm{SL}_q(2)$ is the deformation of the algebra of functions over SL(2). It is defined by the universal enveloping algebra $\mathcal{U}_q[sl(2)]$ with the commutation relations

$$[J_+, J_-] = \frac{q^H - q^{-H}}{q - q^{-}1}, \qquad [H, J_\pm] = \pm 2J_\pm. \tag{3.26}$$

In the limit $q \to 1$, (3.26) recover the usual commutation relations of SL(2) and the quantum group $\mathrm{SL}_q(2)$ reduces to SL(2).
$\mathcal{U}_q[sl(2)]$ actually constitutes an Hopf algebra where there is a *comultiplication* defined as

$$\begin{aligned}\Delta_q(H) &= 1 \otimes H + H \otimes 1, \\ \Delta_q(J_\pm) &= q^{H/2} \otimes J_\pm + J_\pm \otimes q^{-H/2}.\end{aligned} \tag{3.27}$$

Δ_q is actually an algebra homomorphism and constitutes the analogue of what is known as the addition of angular momentum in the SU(2) group environment (to which it reduces for the limit $q \to 1$). The comultiplication Δ_q dictates irreducible representations of $\mathrm{SL}_q(2)$.
As the similarity between the algebras of SL(2) and $\mathrm{SL}_q(2)$ suggests, the theory of representation for the quantum group is going to be very similar to the classical one. The irreducible representation for the quantum group can be indeed labelled by $j = 0, \frac{1}{2}, 1, \ldots$ and are acting on an Hilbert space where the basis vector are labelled as $|j, m\rangle$, with the expected constraint $-j \leq m \leq +j$.
In particular, we have:

$$J_3 |j, m\rangle = m |j, m\rangle, \quad J_\pm |j, m\rangle = \sqrt{[j \mp m]_q [j \pm m + 1]_q} |j, m\rangle, \tag{3.28}$$

where it's relevant the introduction of the *q-numbers*, defined as:

$$[\lambda]_q = \frac{q^{\frac{\lambda}{2}} - q^{-\frac{\lambda}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}. \tag{3.29}$$

As one can immediately notice, we recover the classical "numbers" for the $q \to 1$ limit:

$$[\lambda]_q \to \lambda \qquad \text{for } q \to 1. \tag{3.30}$$

All the j-labelled representations can be obtained from the fundamental representation, namely $j = 1/2$, by applying the comultiplication definition, as in (3.27), on the relation:

$$|J, M; j_1, j_2\rangle = \sum_{m_1, m_2} \begin{bmatrix} j_1 & j_2 & J \\ m_1 & m_2 & j \end{bmatrix}_q |j_1, m_1\rangle \otimes |j_2, m_2\rangle \tag{3.31}$$

where we see that the quantum counterpart of the Clebsh-Gordan coefficients appear. In particular, this quantity takes the name of $6-j$ coefficient or $6-j$ Wigner symbol.

3.2 The Sausage Model

The sausage model, initially introduced by Fateev, Onofri, and Zamolodchikov [6], is a deformation of the $O(3)$ sigma model which preserves integrability. The sphere that represents the target space is deformed to what is then called "*sausage*", parametrizing this deformation with a variable that we can call "ν", according to [7]'s notation. This model is retrieved by deforming the factorizable S-matrix of the $O(3)$ sigma model, using a parameter identified with λ.

Two-dimensional nonlinear sigma models(NLSM) constitute an interesting class of quantum field theories as they are conjectured to describe string theories on nontrivial target manifolds, continuum spin systems, quantum gravity and eventually black holes. Exactly solvable NLSM form an utterly interesting class since they provide consistent information on nonperturbative aspects of quantum fields.

An interesting development has seen these models also tangle with the AdS/CFT correspondence[1] which is strongly linked to the integrability of the target space because it guarantees factorization of the S-matrix which also implies the possibility to compute finite-size effects on the NLSM. To perform these calculations one usually resorts to TBA (Thermodynamic Bethe Ansatz) but it transcend the purpose of this work.

Another way of developing the studies on the NLSM is to "*extend*" the target spaces which preserve integrability: in this case in particular, the *sausage* model represents one of the first attempts at doing so. The paper [6] shows how the authors deformed the target space that was a sphere by "*stretching*" it into a sausage form (which gives the name to the model), calling ν the parameter which describes the "stretched" length. Despite the deformation, the model is still integrable, meaning that S-matrix can be factorized and TBA calculations can be performed.

By means of the assumption that the integrability holds, the authors of [7] have advanced an exact S-matrix obtained by deforming the one associated to the $O(3)$ NLSM, using the previously named λ parameter.

[1] An interested reader can look the articles [11] and [12] up for further and deeper knowledge on the argument.

S-Matrix As explained in the introductory section, the sausage model is based on the non-deformed $O(3)$ NLSM, which is described by the action functional:

$$\mathcal{S}_{O(3)} = \frac{1}{2g}\int\sum_{\alpha=1}^{3}\left(\partial_\mu n_\alpha\right)^2 d^2x + i\theta T, \tag{3.32}$$

where T is used to describe a Wess-Zumino topological term[2]. In the notation used, the fields n_α form a unit vector on the $O(3)$ spehre, meaning that $\sum_{\alpha=1}^{3} n_\alpha^2 = 1$. The model is integrable for the values for $\theta = 0, \pi$.

We will very briefly present both the S-matrices associated to those models.

For $\theta = 0$, the model is called $SSM_0^{(0)}$[7], in which the $O(3)$-invariant S-matrix for the model is

$$S(\theta) = S_0(\theta)\mathbf{P}_0 + S_1(\theta)\mathbf{P}_1 + S_2(\theta)\mathbf{P}_2 \tag{3.33}$$

in which the $S_k(\theta)$ are

$$\begin{aligned} S_0(\theta) &= \frac{\theta + 2i\pi}{\theta - 2i\pi}, \\ S_1(\theta) &= \frac{(\theta - i\pi)(\theta + 2i\pi)}{(\theta + i\pi)(\theta - 2i\pi)}, \\ S_2(\theta) &= \frac{\theta - i\pi}{\theta + i\pi}. \end{aligned} \tag{3.34}$$

and we used the symbols $\mathbf{P}_k$, with $k = 0, 1, 2$, to identify the projectors over the k-spin state spaces. It's interesting to notice that the S-matrix describes a triplet of $O(3)$ in this context. In the case $\theta = \pi$, the model emerging takes the name $SSM_0^{(\pi)}$ in a totally analogous way of the previous one. This sigma model describes instead a different scenario. The spectrum can be indeed identified with two doublets, which we're identifying with R (right moving) and L (left moving). Therefore, the scattering matrices are going to be all the same and in particular equal to:

$$S_{RR}(\theta) = S_{LL}(\theta) = S_{RL}(\theta) = \frac{\Gamma\left(\frac{1}{2} + \frac{\theta}{2i\pi}\right)\Gamma\left(-\frac{\theta}{2i\pi}\right)}{\Gamma\left(\frac{1}{2} - \frac{\theta}{2i\pi}\right)\Gamma\left(\frac{\theta}{2i\pi}\right)}\frac{\theta\mathbb{1} - i\pi\mathcal{P}}{\theta - i\pi}, \tag{3.35}$$

where $\mathcal{P}$ is hereby used to denote the permutation matrix.

The sausage model is obtained, by definition, through the deformation of the S-matrices

[2] One not familiar with topological terms, may intend those as terms that appear in action functional but don't depend on the metric $g_{\mu\nu}$. They can be regarded as characteristic of the model's topology and therefore intrinsic of the system

written above.
In particular, the scattering theories derived go under the name $SST_\lambda^{(\pm)}$, with the $\pm$ sign corresponding to the cases $\theta = 0, \pi$. Now, by identifying the non-vanishing terms of the S-matrix for $SST_\lambda^{(+)}$ and labelling them with $(-1, 0, +1)$, which, for a more contracted notation is going to be used as $(-, 0, +)$, we get:

$$\begin{aligned}
S_{++}^{++}(\theta) &= S_{+-}^{+-}(i\pi - \theta) = \frac{\sinh(\lambda(\theta - i\pi))}{\sinh(\lambda(\theta + i\pi))}, \\
S_{+0}^{0+}(\theta) &= S_{+-}^{00}(i\pi - \theta) = \frac{-i \sin(2\pi\lambda)}{\sinh(\lambda(\theta - 2i\pi))} S_{++}^{++}(\theta), \\
S_{+0}^{+0}(\theta) &= \frac{\sinh(\theta\lambda)}{\sinh(\lambda(\theta - 2i\pi))} S_{++}^{++}(\theta), \\
S_{+-}^{-+}(\theta) &= \frac{\sin(\pi\lambda)\sin(2\pi\lambda)}{\sinh(\lambda(\theta - 2i\pi))\sinh(\lambda(\theta + i\pi))}, \\
S_{00}^{00}(\theta) &= S_{+0}^{+0}(\theta) + S_{-+}^{+-}(\theta).
\end{aligned} \tag{3.36}$$

where we immediately see the crossing symmetry relation (2.96) holding. Referring to the triplet states with $m = -1, 0, +1$, one can see how the conjugation works in this scenario:

$$\begin{aligned}
-1 &\rightarrow +1, \\
0 &\rightarrow 0, \\
+1 &\rightarrow -1.
\end{aligned}$$

where in general, one can intend that as $\bar{m} \rightarrow (-m)$ if the spin states are properly defined to go along with this map. The S-matrix in (3.36) clearly reduces to (3.33), (3.34) in the limit $\lambda \rightarrow 0$ where the deformation does not actually happen.
Looking carefully at the expressions in (3.36), one can observe that all the matrix elements have no poles in the regime $0 \leq \mathrm{Im}\{\theta\} < \pi$ for values of λ such that $0 \leq \lambda < 1/2$. For $\lambda = 1/2$, the theory becomes free and the triplet configuration turns into a complex fermion and a boson with the same mass. The model becomes instead very hard to deal with in the region $\lambda \geq 1/2$; in particular, the S-matrix elements have bound-state poles (as opposed to the $0 \leq \lambda < 1/2$ region) which should be properly analyzed by complete bootstrap processes. We will not indulge in such calculations since they would stray from this work's goal.
Moving onto the $SST_\lambda^{(-)}$ model, we have the following form for the non-vanishing terms

of the S-matrix describing the interaction between two sets of massless doublets (L and R movers):

$$\begin{aligned} U_{++}^{++}(\theta) &= U_{--}^{--}(\theta) = U_0(\theta), \\ U_{+-}^{+-}(\theta) &= U_{-+}^{-+}(\theta) = -\frac{\sinh\left(\lambda\theta/(1-\lambda)\right)}{\sinh\left(\lambda(\theta-i\pi)/(1-\lambda)\right)}U_0(\theta), \\ U_{-+}^{+-}(\theta) &= U_{+-}^{-+}(\theta) = -i\frac{\sinh\left(\lambda\pi/(1-\lambda)\right)}{\sinh\left(\lambda(\theta-i\pi)/(1-\lambda)\right)}U_0(\theta), \end{aligned} \tag{3.37}$$

where we regrouped under the name $U_0(\theta)$ the quantity

$$U_0(\theta) = -\exp\left[i\int_0^\infty \frac{\sinh((1-2\lambda)\pi\omega/(2\lambda))\sin(\omega\theta)}{\cosh(\pi\omega/2)\sinh((1-\lambda)\pi\omega/(2\lambda))}\frac{d\omega}{\omega}\right]. \tag{3.38}$$

As one can check, the limit $\lambda \to 0$ recovers (3.35) as expected.
A fundamental claim from [6] is that the $SST_\lambda^{(\pm)}$ theories can be put in correspondence with deformed sigma models that can be described by an effective action of the type:

$$\mathcal{S}_{SSM_\nu^{(\theta)}} = \int \frac{(\partial_\mu Y)^2 + (\partial_\mu X)^2}{a(t) + b(t)\cosh(2Y)}\,(\partial_\mu n_\alpha)^2\, d^2x + i\theta T, \tag{3.39}$$

where θ can assume the values 0 or π to keep the model integrable and with $a(t)$ and $b(t)$ being RG flows in the leading order

$$a(t) = -\nu\coth\left(\frac{\nu(t-t_0)}{2\pi}\right), \qquad b(t) = -\nu/\sinh\left(\frac{\nu(t-t_0)}{2\pi}\right). \tag{3.40}$$

3.3 Non-deformed models: Aladim and Martins' $\mathbf{SU(2)}_k$ factorizable $\boldsymbol{S}$-Matrix

Interested in the study of the Bethe ansatz properties of factorizable S-matrices invariant under the SU(2) symmetry, the authors of [8] investigated the thermodynamics emerging from those scattering theories in their article. A reason for doing so consists in the fact that the finite size effects in certain quantum field theories can be investigated by an appropriate Bethe ansatz analysis of the SU(2) scattering S-matrices at some order k of its representation. The reader can intend the order k as related to the spin s of the particles involved in the scattering described by the S-matrix. In particular, the relation $k = 2s$ holds.

As an example of this, one can think of the kink excitations of the SU(2) Thirring model as described by the the SU(2) scattering amplitudes in the fundamental representation (which is $\mathrm{SU}(2)_k$ with $k = 1$). We hereby recall that the Thirring model can be interpreted as equivalent to the sine-Gordon model [18]. Moreover, as it is going to be shown later, the isomorphism $\mathrm{SU}(2)_2 \sim O(3)$ led to an analogous study on the $O(3)$ non-linear sigma model by using, in fact, the $\mathrm{SU}(2)_2$ scattering amplitudes. With the goal of generalizing the procedure, it was presented [8] a factorizable $\mathrm{SU}(2)_2$ invariant S-matrix for arbitrary order k of representation.

In order to obtain this general form for the $\mathrm{SU}(2)_2$ invariant S-matrix, the procedure followed consisted of looking for Boltzmann weights of the corresponding solvable lattice model: in this instance, the model corresponds to the Heisenberg chain with spin $k/2$ [22][23][3] . This Boltzmann weights have been computed through the fusion procedure by Kulish, Reshetikhin and Sklyanin in [24].

[3] from here one can immediately understand the correspondence anticipated between the order k or representation and the spin $s = k/2$ of the scattering particles in the theory described by the retrieved S-matrix.

SU(2)$_k$ invariant S-matrix Following the approach presented, we first consider the R-matrix (i.e. vertex operator) $R(\tilde{\theta}, \eta)$. It acts on the tensor space $\mathbb{C}^{k+1} \otimes \mathbb{C}^{k+1}$ and it can be written as a series expansion over the projectors $\mathbf{P}_j$ which are defined on the Hilbert space of the tensor product of the two spin $k/2$ particles[4].
The vertex operator reads then as:

$$R(\tilde{\theta}, \eta) = \mathbf{P}_0 + \sum_{j=1}^{2s} \prod_{l=1}^{j} \frac{\tilde{\theta} - il\eta}{\tilde{\theta} + il\eta} \mathbf{P}_j. \tag{3.41}$$

Here, the projectors $\mathbf{P}_j$ act on the states $|m\rangle$ belonging to the Hilbert space, selecting the spin-states subspaces, namely $\mathbf{P}_j |m\rangle = \delta_{j,m} |m\rangle$.
To recognize $R(\tilde{\theta}, \eta)$ as an S-matrix, one has to check (and impose) the unitarity condition and the crossing-symmetry constraint[5].
Properly dealing with the two conditions, one can finally obtain the S-matrix in the form

$$\begin{aligned} S(\theta) &= \prod_{l=even}^{2s} \frac{\theta + il\pi}{\theta - il\pi} R(\theta, \pi), \qquad \text{if } k \text{ is even} \\ &= S_0(\theta) \prod_{l=odd}^{2s} \frac{\theta + il\pi}{\theta - il\pi} R(\theta, \pi), \qquad \text{if } k \text{ is odd,} \end{aligned} \tag{3.42}$$

where we used $S_0(\theta)$ to indicate the quantity

$$S_0(\theta) = \frac{\Gamma\left(\frac{1}{2} - \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{i\theta}{2\pi}\right)}{\Gamma\left(\frac{1}{2} + \frac{i\theta}{2\pi}\right) \Gamma\left(-\frac{i\theta}{2\pi}\right)}. \tag{3.43}$$

As already anticipated, we can retrieve the S-matrix of two SU(2) invariant particles by setting $k = 1$ in (3.42), recovering the sine-Gordon (or the equivalent Thirring) model. For $k = 2$, it is possible to show, as we are about to do, that the scattering amplitudes are in correspondence with those of the $O(3)$ non-linear sigma model.
Analogously as what has been seen in the previous sections, it's possible to notice that the S-matrix has no poles in the region $0 \leq \mathrm{Im}\{\theta\} \leq \pi$ and envelops $2s + 1$ degenerate

[4]It's immediate here to separate the tensor product into the two $\mathbb{C}^{2s+1}$ subspaces, where each is assigned to one of the two scattering particles of spin $s = k/2$.

[5]As a vertex operator, $R(\tilde{\theta}, \eta)$ is already satisfying the Yang-Baxter equation. Therefore, there is no need to utterly check that constraint, even tho it's important to not lose focus on it since it's fundamental element of the scattering theory.

(spectrum wise) particles.
Finally, the crossing-symmetry is obtained by the means of the relation

$$S_{ij}^{kl}(\theta) = \mathcal{C}_{i\alpha} S_{\beta j}^{\alpha l}(i\pi - \theta) \mathcal{C}_{k\beta}^{\mathrm{t}}, \tag{3.44}$$

in which $\mathcal{C}$ is the charge-conjugation operator that in this case acts as a $(k+1) \times (k+1)$ matrix, namely:

$$\mathcal{C} = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & -1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & (-1)^{k-1} & \dots & 0 & 0 \\ (-1)^k & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{3.45}$$

As part of the work provided, the S-matrices introduced in (3.42) have been checked through the using of computers: a program was compiled to verify the unitarity and crossing-symmetry constraints, as well as the Yang-Baxter equation, on those quantities. The code used to perform this task is reported in Appendix B as a tool one can use to first-hand verify that those relations actually hold true.

SU(2)$_2 \sim O(3)$ isomorphism We will now very briefly show the connection between the S-matrices associated to the $O(3)$ non-linear sigma model and to the SU(2)$_2$ (i.e. spin 1) Heisenberg chain. From [25] we find that the S-matrix for the $O(3)$ NLSM can be written as

$$S'(\tilde{\theta}) = \frac{\tilde{\theta}+2}{\tilde{\theta}-2}\left(\mathbf{P}_0 + \frac{\tilde{\theta}-1}{\tilde{\theta}+1}\mathbf{P}_1 + \frac{\tilde{\theta}-1}{\tilde{\theta}+1}\frac{\tilde{\theta}-2}{\tilde{\theta}+2}\mathbf{P}_2\right) \tag{3.46}$$

with $\mathbf{P}_i$, $i = 0,1,2$, being the projectors on the subspace of spin i.
It's immediate to see that we recover (3.33) and (3.34) with the simple map $\tilde{\theta} = \theta/(i\pi)$. Taking the next step, by defining the matrices

$$S'^1 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S'^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S'^3 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \tag{3.47}$$

it is possible to retrieve the relations for the projectors

$$\begin{aligned} \mathbf{P}_0 &= \frac{x^2-1}{3}, \\ \mathbf{P}_1 &= 1 - \frac{(x+x^2)}{2}, \\ \mathbf{P}_2 &= \frac{x}{2} + \frac{x^2}{6} + \frac{1}{3}, \end{aligned} \tag{3.48}$$

where we have defined x as

$$x = \sum_{j=1}^{3} S'^j \otimes S'_j. \tag{3.49}$$

One can easily check that the $S'(\tilde{\theta})$ is equivalent to the $R(\tilde{\theta}, \pi)$ obtained in (3.41) after a unitary transformation with the structure $S_j = U S'_j U^\dagger$ is performed. In particular, U takes the form:

$$U = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \end{pmatrix}. \tag{3.50}$$

A very interesting feature of this equivalence is the fact that the excitations of the $O(3)$ spin chain in the fundamental representation are shown to be related to the ones of the SU(2)$_2$ Heisenberg chain. An immediate consequence is that the ground state of the $O(3)$ model has to be described in terms of two-*strings*, namely $\tilde{\theta}_{j_{1,2}} = \lambda_j \pm i/2$. This result also shows that the ground state of spin chains associated to the fundamental representation of non-simply laced Lie algebras is not described by real roots λ_j, meaning that the general belief that they clearly stray from the behaviour of the simply-laced ones.

Chapter 4

Exact $\boldsymbol{S}$-Matrix for $\mathcal{U}_q(sl_2)$ symmetric integrable models

In this final developing chapter, the work is going to focus on retrieving the deformed version for the S-matrices discussed in the final part of the previous one.
The task is going to start with finding the correct trigonometrization of the R-matrix presented in [8]. After retrieving such expression, we will then work on constructing the q-projectors, making use of the mathematical tools presented in [15].
After getting to the appropriate formulation for the core of the S-matrix, we will finally move on to build its complete expression, i.e. constructing the pre-factor. Such quantity is supposed to be introduced to adjust the physics: in fact, it is necessary to make the whole S-matrix simultaneously unitary and crossing-symmetric (while of course still satisfying the Yang-Baxter equation).
As it will be useful for the following section, we will briefly recall some notations. In particular, it is important to remember that we will use $s = k/2$ to indicate the particles' spin, while q is the parameter defining the algebra $\mathcal{U}_q(sl_2)$.
We will later on introduce the variable γ which is the one describing the deformation of the model and in particular will be q-dependent, i.e. $\gamma = \gamma(q)$.

4.1 Deformed space

The first step to take to work towards building the *q-deformation* of Martins' S-matrices is to find their trigonometrization. Recalling (3.41), one can in fact express the core of the deformed S-matrix as

$$\tilde{S}(\tilde{\theta}) = \mathcal{P} \sum_{\kappa=0}^{2s} \tilde{f}(\tilde{\theta}, q)_{\kappa} \, \mathbf{P}^{[\kappa]}. \tag{4.1}$$

We will now clarify the meaning of each element in the formula presented above.
First of all, the permutation operator $\mathcal{P}$ is needed in the S-matrix structure in order to satisfy the Yang-Baxter equation. It is indeed a necessary condition since its absence would imply the violation of such constraint. To define the operator, we have to preemptively introduce the quantities $e_{(a,b)}$, whose matrix elements are defined as [1].

$$e_{(a,b)_{ij}} = \delta_{i,a}\delta_{j,b}. \tag{4.2}$$

The permutation operator is then defined as

$$\mathcal{P} = \sum_{a,b=1}^{2s} e_{(a,b)} \otimes e_{(b,a)}. \tag{4.3}$$

The role of the permutation operator here is to swap the subspaces 1 and 2 where each of them identifies a single particle as it can be seen in fig. 4.1. The functions $\tilde{f}(\theta, q)_{\kappa}$ are simply the trigonometrization of the factors multiplying the projectors in the R-matrix shown in (3.41), namely

$$\tilde{f}(\theta, q)_{\kappa} = \prod_{l=0}^{\kappa} \frac{q^l - e^{\theta}}{q^l e^{\theta} - 1}. \tag{4.4}$$

We will now get a better look at the $\mathbf{P}^{[\kappa]}$. These operators are the projectors over the κ total spin state. To clarify, one should remember that thanks to the Parke theorem [37], the scattering theory that we are building is of the kind "*2 in 2*", sending two identical particles into two particle of the same species with the same set of rapidities.
In particular, when we consider the spins j_1 and j_2 of the incoming particles, it is immediate to realize the following:

[1] One can intuitively understand that $e_{(a,b)}$ is the matrix with all vanishing elements except the one in line a and column b. An example is $e_{(1,3)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

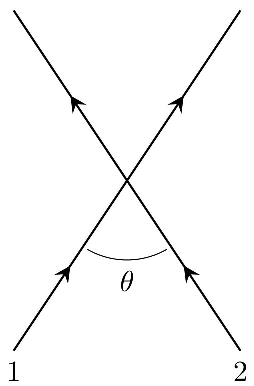

Figure 4.1: A graphical representation of the two particle scattering, each with an assigned number to identify its proper subspace.

- since the incoming particles are identical, we get $j_1 = j_2 = s$;
- furthermore, having the outgoing particles being of the same species of the incoming one means that their spins j'_1 and j'_2 respectively will also be equal to s, $j'_1 = j'_2 = s$;
- recalling the angular momentum addition rules, we get that the total momentum J ranges from the value $|j_1 - j_2|$ to $j_1 + j_2$, which immediately translates in the discrete interval $[0, 2s]$, therefore covering $2s + 1$ possible values;
- the eigenvalue M associated to the total spin J state is then ranging from $-J$ to $+J$.

Now, for a "classical" quantum mechanical system, the projectors P_J over the total spin J state would easily be expressed as

$$P_J = \sum_{M=-J}^{+J} |JM\rangle \langle JM| \, , \tag{4.5}$$

where the sum covers all the possible values of M for a certain J.
The matrix elements would then be

$$P_{J}{}^{m'_1 m'_2}_{m_1 m_2} = \sum_M \langle m_1 m_2 | JM \rangle \langle JM | m'_1 m'_2 \rangle \, , \tag{4.6}$$

where we have intuitively associated the quantum numbers m_1 and m_2 to the incoming particles and m'_1 and m'_2 to the outgoing ones [2]. As one may notice, the brakets in (4.6)

[2] One can already figure out that the projector is going to be a two dimensional matrix, meaning that the axis will have to represent the incoming and outgoing combinations for the m_i and m'_i. This

are the Clebsh-Gordan coefficients.
However, the framework in which we are now working is not the one of standard quantum mechanical systems anymore, but a q-deformed one. This deformation applies to the Hilbert space projectors as well. In fact, the quantity defined in (4.1) as $\mathbf{P}^{[J]}$ is not the projector we just described but it's the one built upon the deformed structure of our framework.
The core structure of these projectors is actually the same, namely

$$\mathbf{P}^{[J]} = \sum_{M=-J}^{+J} |JM\rangle_{q\,q}\langle JM|. \tag{4.7}$$

One needs now to be careful in handling these newly introduced quantities. In order to find the correct expression for these projectors, it is particularly useful the work of H. Ruegg [15]. This paper in fact, contains a complete description on how to retrieve the quantum Clebsh-Gordan coefficients. In fact, conveniently writing the matrix elements of the projector $\mathbf{P}^{[J]}$ we get

$$P^{[J]}{}^{m_1'm_2'}_{m_1m_2} = \sum_M \langle m_1m_2|JM\rangle_{q\,q}\langle JM|m_1'm_2'\rangle \tag{4.8}$$

where the quantum Clebsh-Gordan coefficient $\langle m_1m_2|JM\rangle_q$ is introduced.
Before introducing the proper way of calculating such quantity, some notations shall be introduced.
First of all, we recall from (3.29) the definition of q-number, which is given by

$$[\lambda]_q = \frac{q^{\frac{\lambda}{2}} - q^{-\frac{\lambda}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}. \tag{4.9}$$

Having defined that, one can also introduce the q-factorial, namely

$$[n]! = [n][n-1]\cdots[1], \qquad [0]! = 1, \qquad [-n]! = -\infty. \tag{4.10}$$

It is then convenient to introduce the notation

$$\begin{aligned} n &= j_1 + j_2 - J \\ n_2 &= j_1 - j_2 + J \\ n_1 &= -j_1 + j_2 + J \\ N &= j_1 + j_2 + J + 1, \end{aligned} \tag{4.11}$$

will result in $(2s+1)\times(2s+1)$ configurations for the incoming couple and an equal amount for the outgoing one. Projectors are then clearly going to be $(2s+1)^2\times(2s+1)^2$ matrices, meaning that $\tilde{S}$ is going to be of those dimensions as well.

so that we can now calculate the quantum Clebsh-Gordan coefficient $\langle m_1 m_2 | JM \rangle_q$ as

$$\begin{aligned} \langle j_1 m_1 j_2 m_2 | JM \rangle_q = f(j_1 j_2 J) q^{1/4nN + 1/2(j_1 m_2 - j_2 m_1)} \\ \times \left\{ \prod_{i=1}^{3} [j_i + m_i]! [j_i - m_i]! \right\}^{1/2} \\ \times \sum_{\nu \geq 0} (-1)^\nu q^{-1/2 \nu N} D^{-1}, \end{aligned} \tag{4.12}$$

where we used the notation

$$j_3 = J, \qquad m_3 = M \tag{4.13}$$

in order to better organize the expression's elements. Defining now the last elements for the formula (4.12), we have the quantity D being:

$$\begin{aligned} D = [\nu]! [j_1 + j_2 - J - \nu] | [j_1 - m_1 - \nu]! \\ \times [j_2 + m_2 - \nu]! [J - j_2 + m_1 + \nu]! \\ \times [J - j_1 - m_2 + \nu]!, \end{aligned} \tag{4.14}$$

while the normalization factor $f(j_1 j_2 J)$ is equal to

$$f(j_1 j_2 J) = \left\{ [2J+1][n_1]![n_2]![n]!([N]!)^{-1} \right\}^{1/2}. \tag{4.15}$$

As in standard quantum mechanics, we get the non-vanishing condition $M = m_1 + m_2$ for the quantum Clebsh-Gordan coefficients, meaning that all the quantum numbers' sets that do not satisfy such constraint lead to vanishing coefficients. We have finally cleared the way towards building the core of the S-matrix, namely $\tilde{S}(\theta)$. We shall now illustrate how to properly do so and how to check and fix the constraint coming from unitarity and crossing symmetry conditions along with Yang-Baxter equation.

4.2 q-deformed S-Matrix

Knowing how to build the matrix elements (4.8) one has to now be careful in the way they get organized in the code. A convenient structure for this might be progressively linking the index to all the couples (m_1, m_2): we start from both values at their minimum level (i.e. $m_1 = m_2 = -s$) and then run through all the values of m_2 for the first value of m_1. We then increase m_1 by one and we repeat the process until we have covered all the $(2s+1)^2$ combinations.
To give a better idea of the ordering, we present a very intuitive table showing how the combination of the quantum numbers for the incoming particles are mapped into the index.

j_1	j_2	index
$-s$	$-s$	0
$-s$	$-s+1$	1
$-s$	$-s+2$	2
$\vdots$	$\vdots$	$\vdots$
$-s$	$+s$	$2s$
$-s+1$	$-s$	$2s+1$
$\vdots$	$\vdots$	$\vdots$

The explicit relation linking m_1 and m_2 to the index j is then[3]

$$\begin{aligned} m_1 &= -s + [j/(2s+1)] \\ m_2 &= -s + (j \mod (2s+1)), \end{aligned} \tag{4.16}$$

with $[x]$ being the integer part of x. The inverse relation is then

$$j = (m_1 + s)(2s+1) + (m_2 + s). \tag{4.17}$$

Once a proper structure is set up, one can proceed with checking unitarity, crossing-symmetry and Yang-Baxter equation for $\tilde{S}(\tilde{\theta})$.
Unitarity is pretty much straightforward: it is indeed sufficient to check the relation

$$\tilde{S}(\tilde{\theta})\mathcal{P}\tilde{S}(-\tilde{\theta})\mathcal{P} = \mathbb{1}_{(2s+1)^2}, \tag{4.18}$$

[3]One can obviously choose another ordering system, as long as he deals with each element properly. A reasonable one might be the inverse order of the one presented, starting with the couple $(+s, +s)$ and ending with $(-s, -s)$ instead.

where we made use of the permutation operator to properly complement the adjoint $\tilde{S}$. The unitarity condition is immediately satisfied by the core $\tilde{S}(\tilde{\theta})$.
Moving on to Yang-Baxter equation, the check gets a bit tricky and it is necessary to pay attention to the role of the permutations.
Since we're dealing with three particles colliding, we use a three-particle S-matrix which we can write as a tensor product of a two-particle S-matrix and the identity for the third one , thanks to the factorizable property of the model. In fact, as we saw in the previous chapters, the Yang-Baxter equation allows us to consider the three identical particles scattering as a series of two particle scatterings. While these two particle scatterings happen, the particle not involved clearly stays unaltered, justifying the use of the identity to represent it.
We shall now denote with $\tilde{S}_{ab}$ the S-matrix related to the three particle scattering in which a and b represent the involved particles.
To properly define such matrices, we should first introduce the permutation operator for the three particle set: $\mathcal{P}_{ab}$ will be the permutation operator that swaps the particle a and b subspaces.
The reasoning behind the introduction of the tensor product with the identity is analogous to the one described above for the S-matrix. We can now write

$$\begin{aligned}\mathcal{P}_{12} &= \mathcal{P} \otimes \mathbb{1}_{(2s+1)} \\ \mathcal{P}_{23} &= \mathbb{1}_{(2s+1)} \otimes \mathcal{P} \\ \mathcal{P}_{13} &= \mathcal{P}_{23}\mathcal{P}_{12}\mathcal{P}_{23},\end{aligned} \tag{4.19}$$

where the definition of $\mathcal{P}_{13}$ is delicate (but still intuitive) and one should be careful in writing it as multiple swaps that will result in the permutation of the particles 1 and 3 in the end.
The matrices $\tilde{S}_{ab}$ are therefore:

$$\begin{aligned}\tilde{S}_{12}(\theta_{12}) &= \tilde{S}(\theta_{12}) \otimes \mathbb{1}_{(2s+1)} \\ \tilde{S}_{23}(\theta_{23}) &= \mathbb{1}_{(2s+1)} \otimes \tilde{S}(\theta_{23}) \\ \tilde{S}_{13}(\theta_{13}) &= \mathcal{P}_{23}\tilde{S}_{12}(\theta_{13})\mathcal{P}_{23},\end{aligned} \tag{4.20}$$

where the θ_{ab} are the difference of the rapidities associated to the scattering particles: $\theta_{ab} = \theta_a - \theta_b$.
The Yang-Baxter equation can be then written as

$$\tilde{S}_{12}(\theta_{12})\tilde{S}_{13}(\theta_{13})\tilde{S}_{23}(\theta_{23}) = \tilde{S}_{23}(\theta_{23})\tilde{S}_{13}(\theta_{13})\tilde{S}_{12}(\theta_{12}). \tag{4.21}$$

Once again, a positive check is obtained for $\tilde{S}(\theta)$ from the *Mathematica* code that was set up (in Appendix A).
The last constraint to check is the crossing-symmetry. This is indeed quite a challenging task since the $\tilde{S}(\theta)$ does not immediately satisfy the condition. It is convenient to rewrite

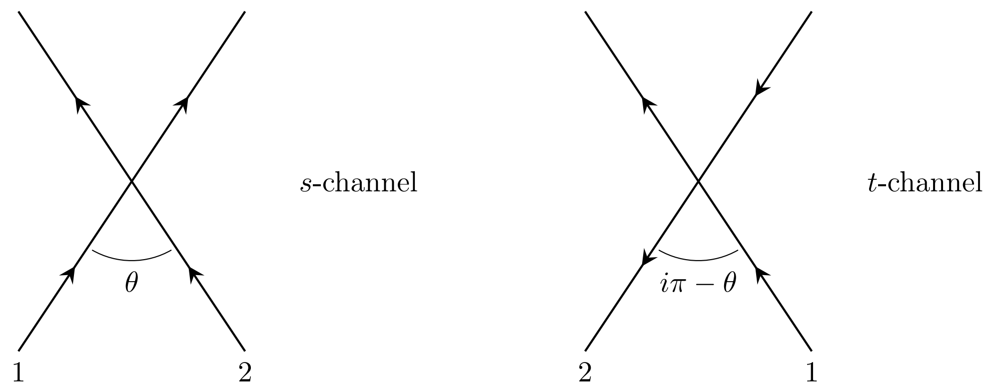

Figure 4.2: A graphical interpretation of the crossing-symmetry in a two particle scattering.

the crossing condition (2.96) with the newly acquired notation for the permutation operators.
Looking at the graphical representation in fig. 4.2, we have the condition:

$$
\begin{aligned}
S_{12}(\theta)_{ab}^{cd} &= \left(\mathcal{C}_2 S_{21}^{t_2}(i\pi-\theta)\mathcal{C}_2^{-1}\right)_{ab}^{cd} \\
&= \left((\mathcal{P}_{12}\mathcal{C}_1\mathcal{P}_{12})\left(\mathcal{P}_{12}S_{12}^{t_1}(i\pi-\theta)\mathcal{P}_{12}\right)\left(\mathcal{P}_{12}\mathcal{C}_1^{-1}\mathcal{P}_{12}\right)\right)_{ab}^{cd} \\
&= \left(\mathcal{P}_{12}\left(\mathcal{C}_1 S_{12}^{t_1}(i\pi-\theta)\mathcal{C}_1^{-1}\right)\mathcal{P}_{12}\right)_{ab}^{cd} \\
&= \left(\mathcal{C}_1 S_{12}^{t_1}(i\pi-\theta)\mathcal{C}_1^{-1}\right)_{ba}^{dc} \\
&= S_{12}(i\pi-\theta)_{\bar{d}a}^{\bar{b}c},
\end{aligned}
\tag{4.22}
$$

where we have made use of the fact that applying two permutation operator returns the identity, $\mathcal{P}_{12}^2 = \mathbb{1}$ and $\mathcal{C}_i$ is the charge conjugation operator on the particle space i, meaning that is the tensor product of the general charge conjugation $\mathcal{C}$ and the identity over the unaltered subspace:

$$
\mathcal{C}_1 = \mathcal{C} \otimes \mathbb{1}_{(2s+1)}. \tag{4.23}
$$

In (4.22), the apex t_i means that we are performing transposition only on the particle space i. The crossing symmetry is not satisfied by $\tilde{S}(\theta)$ but we can still retrieve the

charge conjugation which is going to be useful in our following arguments.
$\mathcal{C}$ is indeed a $(2s+1)\times(2s+1)$ matrix with the form

$$\mathcal{C} = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{4.24}$$

We should now work towards building an S-matrix which actually respect the crossing-symmetry constraint. In particular, one should check the ratios between the matrix elements of $\tilde{S}(\theta)$ and its crossed version.
A very interesting feature of this is the fact that we observe the same ratio for all the non-vanishing element of $\tilde{S}(\theta)$.

$$\frac{\tilde{S}(\tilde{\theta})_{ab}^{cd}}{\tilde{S}(i\pi-\tilde{\theta})_{\bar{d}a}^{\bar{b}c}} = \frac{1}{q^s}\prod_{a=1}^{2s}\frac{\left(e^{\tilde{\theta}}-q^{a+1}\right)}{\left(-1+e^{\tilde{\theta}}q^a\right)} = \prod_{a=1}^{2s}\frac{1}{q^{1/2}}\frac{\left(e^{\tilde{\theta}}-q^{a+1}\right)}{\left(-1+e^{\tilde{\theta}}q^a\right)} \tag{4.25}$$

We shall now simplify the ratio:

$$\begin{aligned}
\frac{1}{q^{1/2}}\frac{\left(e^{\tilde{\theta}}-q^{a+1}\right)}{\left(-1+e^{\tilde{\theta}}q^a\right)} &= \frac{1}{q^{1/2}}\frac{q^{(a+1)/2}}{q^{a/2}}\frac{\left(e^{\tilde{\theta}/2}q^{-(a+1)/2}-e^{-\tilde{\theta}/2}q^{(a+1)/2}\right)}{\left(-e^{-\tilde{\theta}/2}q^{-a/2}+e^{\tilde{\theta}/2}q^{a/2}\right)} \\
&= \frac{e^{(\tilde{\theta}/2-2i\pi\gamma(a+1)/2)}-e^{-(\tilde{\theta}/2-2i\pi\gamma(a+1)/2)}}{-e^{(-\tilde{\theta}/2-2i\pi\gamma(a+1)/2)}+e^{(\tilde{\theta}/2+2i\pi\gamma(a+1)/2)}} \\
&= \frac{\sinh\left(\frac{\tilde{\theta}}{2}-2i\pi\gamma\frac{(a+1)}{2}\right)}{\sinh\left(\frac{\tilde{\theta}}{2}+2i\pi\gamma\frac{a}{2}\right)} \\
&= \frac{\sinh\left(\gamma\left(\theta-i\pi(a+1)\right)\right)}{\sinh\left(\gamma\left(\theta+i\pi a\right)\right)}
\end{aligned} \tag{4.26}$$

In the simplification procedure, we made use of the map for the parameter q into the deformation variable γ such that

$$q = e^{2i\pi\gamma}, \tag{4.27}$$

and later redefined the rapidities such as

$$\tilde{\theta} = 2\gamma\theta. \tag{4.28}$$

We can finally write the ratio

$$\frac{\tilde{S}(\theta)_{ab}^{cd}}{\tilde{S}(i\pi-\theta)_{\bar{d}a}^{\bar{b}c}}=\prod_{a=1}^{2s}\frac{\sinh\left(\gamma\left(\theta-i\pi(a+1)\right)\right)}{\sinh\left(\gamma\left(\theta+i\pi a\right)\right)}. \tag{4.29}$$

To fully understand the value of retrieving this ratio, let us now consider the full S-matrix that we are trying to obtain. We have

$$S^{[s]}(\theta)=f(\theta)\tilde{S}(\theta), \tag{4.30}$$

where $f(\theta)$ is a pre-factor that we have to add in order to fix the crossing-symmetry for the S-matrix.

We recall that $\tilde{S}(\theta)$ has already been shown to be unitary. $f(\theta)$ will also need to be unitary itself, namely

$$f(\theta)f(-\theta)=1. \tag{4.31}$$

The crossing-symmetry condition is thus going to be

$$S^{[s]}(\theta)_{ab}^{cd}=S^{[s]}(i\pi-\theta)_{\bar{d}a}^{\bar{b}c}. \tag{4.32}$$

Inserting (4.30) into (4.32), we get

$$f(\theta)\tilde{S}(\theta)_{ab}^{cd}=f(i\pi-\theta)\tilde{S}(i\pi-\theta)_{\bar{d}a}^{\bar{b}c}, \tag{4.33}$$

that immediately leads to

$$\frac{f(\theta)}{f(i\pi-\theta)}=\frac{\tilde{S}(i\pi-\theta)_{\bar{d}a}^{\bar{b}c}}{\tilde{S}(\theta)_{ab}^{cd}}=\left[\frac{\tilde{S}(\theta)_{ab}^{cd}}{\tilde{S}(i\pi-\theta)_{\bar{d}a}^{\bar{b}c}}\right]^{-1}. \tag{4.34}$$

We shall now take a look at the constraint found for $f(\theta)$:

$$\frac{f(\theta)}{f(i\pi-\theta)}=\prod_{a=1}^{2s}\frac{\sinh\left(\gamma\left(\theta+i\pi a\right)\right)}{\sinh\left(\gamma\left(\theta-i\pi(a+1)\right)\right)}. \tag{4.35}$$

We will now define $U(\theta)$ as

$$f(\theta)=\left\{\prod_{a=1}^{2s}\sinh\left(\gamma\left(\theta+i\pi a\right)\right)\right\}\cdot U(\theta), \tag{4.36}$$

which is a function to be fixed which is supposed to satisfy crossing on its own

$$U(\theta)=U(i\pi-\theta). \tag{4.37}$$

From equation (4.31) we get

$$U(\theta)U(-\theta) = \frac{1}{\prod_{a=1}^{2s} \sinh\left(\gamma\left(\theta + i\pi a\right)\right) \sinh\left(\gamma\left(-\theta + i\pi a\right)\right)}. \tag{4.38}$$

Relations (4.37) and (4.38) constrain $U(\theta)$ strongly enough to be determined. We report the result of this quite cumbersome calculation, which will be published in a forthcoming article [5]. Since the expression is strongly dependent on the nature of s, we shall separate the two possibilities:[4]
Integer s:

$$\begin{aligned} U(\theta) &= \prod_{\substack{a=1 \\ \text{even}}}^{2s} \frac{1}{\sinh(\gamma(\theta - i\pi a))} \prod_{\substack{a=1 \\ \text{odd}}}^{2s-1} \frac{1}{\sinh(\gamma(\theta + i\pi a))} \\ &= \prod_{a=1}^{s} \frac{1}{\sinh(\gamma(2ai\pi - \theta))} \frac{1}{\sinh(\gamma((2a-1)i\pi - \theta))}, \end{aligned} \tag{4.39}$$

where the two right-hand side expressions can be easily shown to be equivalent.
Half-Integer s:

$$\begin{aligned} U(\theta) = \prod_{m=1}^{2s} \Bigg\{ &\left[\prod_{n=1}^{\infty} \frac{R_n^{[s,m]}(\theta) R_n^{[s,m]}(i\pi - \theta)}{R_n^{[s,m]}(0)) R_n^{[s,m]}(i\pi)} \right] \frac{\Gamma(m\gamma)}{\Gamma(1-(m-1)\gamma)} \\ &\times \Gamma\left(1 - \gamma(m-1) + \frac{i\gamma\theta}{\pi}\right) \Gamma\left(1 - \gamma m - \frac{i\gamma\theta}{\pi}\right) \Bigg\} \end{aligned} \tag{4.40}$$

where

$$\begin{aligned} R_n^{[s,m]} = &\frac{\Gamma\left(\frac{\gamma}{\pi}((4sn - 4s + 2m)\pi + i\theta)\right)}{\Gamma\left(\frac{\gamma}{\pi}((4sn - 2s + 2m)\pi + i\theta)\right)} \\ &\times \frac{\Gamma\left(1 + \frac{\gamma}{\pi}((4sn - 2m + 2)\pi + i\theta)\right)}{\Gamma\left(1 + \frac{\gamma}{\pi}((4sn - 2s - 2m + 2)\pi + i\theta)\right)}. \end{aligned} \tag{4.41}$$

Inserting now (4.36) into (4.30) we obtain

$$S^{[s]}(\theta) = \left\{ \prod_{a=1}^{2s} \sinh\left(\gamma\left(\theta + i\pi a\right)\right) \right\} \cdot U(\theta) \cdot \tilde{S}(\theta). \tag{4.42}$$

[4] $U(\theta)$ is defined up to a multiplicative constant, which can be omitted to preserve a more elegant notation.

The expression (4.42) can be utterly simplified by extracting the exact value of the (1,1) element of the core $\tilde{S}$-matrix. In fact, defining

$$S_0(\theta) = \tilde{S}^{ss}_{ss}(\theta) = \prod_{a=1}^{2s} \frac{\sinh\left(\gamma\left(\theta - i\pi a\right)\right)}{\sinh\left(\gamma\left(\theta + i\pi a\right)\right)}, \tag{4.43}$$

and consequently

$$\tilde{S}_{\mathcal{N}orm}(\theta) = \frac{1}{S_0(\theta)}\tilde{S}(\theta) = \begin{pmatrix} 1 & \cdots \\ \vdots & \ddots \end{pmatrix}, \tag{4.44}$$

where we defined as $\tilde{S}_{\mathcal{N}orm}(\theta)$ the core $\tilde{S}$-matrix with its elements divided by the value of the upper left one (which then gets normalized to 1).
The final expression for $S^{[s]}(\theta)$ becomes:
Integer s:

$$S^{[s]}(\theta) = \prod_{\substack{a=1\\ \text{odd}}}^{2s-1} \frac{\sinh\gamma(\theta - i\pi \cdot a))}{\sinh\gamma(\theta + i\pi \cdot a))}\tilde{S}_{\mathcal{N}orm}(\theta). \tag{4.45}$$

Half-Integer s:

$$\begin{aligned} S^{[s]}(\theta) = \prod_{m=1}^{2s} \Bigg\{ & \left[\prod_{n=1}^{\infty} \frac{R_n^{[s,m]}(\theta) R_n^{[s,m]}(i\pi - \theta)}{R_n^{[s,m]}(0)) R_n^{[s,m]}(i\pi)}\right] \frac{\Gamma(m\gamma)}{\Gamma(1-(m-1)\gamma)} \\ & \times \Gamma\left(1 - \gamma(m-1) + \frac{i\gamma\theta}{\pi}\right)\Gamma\left(1 - \gamma m - \frac{i\gamma\theta}{\pi}\right) \\ & \times \sinh(\gamma(\theta - im\pi))\Bigg\} \cdot \tilde{S}_{\mathcal{N}orm}(\theta), \end{aligned} \tag{4.46}$$

with $R_n^{[s,m]}(\theta)$ being defined as in (4.41).

Chapter 5

Conclusions and outlooks

Summary

We will now summarize the work done and look over the result.

The first step consisted in looking over the S. R. Aladim and M. J. Martins's studies, in order to both understand and check the process of building the S-matrix for the undeformed case. We then proceeded to apply such knowledge on the deformed models. In fact, the deformed and undeformed S-matrices do share the same fundamental structure. Even so, many differences had to be accounted for. Applying the deformation on the case study was quite a delicate task. We had indeed to properly introduce the q parameter in all the physical quantities definition, to adjust them to the new framework. In order to build the q-projectors we had in fact to go back to the Clebsh-Gordan coefficients and find the quantum Clebsh-Gordan[15] coefficients instead.

This was followed by a process of chained adjustments, until we got to properly define and build $\tilde{S}(\theta)$, the "core" for the final S-matrix, as written in eq. (4.1). To succeed in this task, it was fundamental to make proper use of the algebraic software system *Wolfram Mathematica* (see the codes in appendices A e B). It would not have been a fast duty to obtain such S-matrix with only calculation by hand. In fact, S being $k-$dependent meant that its dimensions are growing with k itself, making it impossible to retrieve such a large quantity of matrix elements without relying on an algebraic software.

$\tilde{S}(\theta)$ was then inspected and found to be satisfying both Yang-Baxter equation and unitarity condition. Stumbling upon the not satisfied crossing-symmetry, we had to further investigate the problem. By taking the ratios between the matrix elements of $\tilde{S}(\theta)$ and those of its crossed version, we were able to find out that these quantities do not depend on the indices but they have the same form for all the matrix elements.

It was then clear that a pre-factor, $f(\theta)$, was needed in front of $\tilde{S}(\theta)$ to complement the matrix part and obtain a consistent S-matrix. Such pre-factor was finally constrained by crossing-symmetry and unitarity (of the overall S-matrix). The general form of the full S-matrix is given in eqs. (4.45) and (4.46)

We shall now state a few observations on the final expression for $S(\theta)$. In particular, we can recover the known models from the general form shown: for $s = 1/2$ (i.e. $k = 1$), the S-matrix reduces to the one of sine-Gordon, while for $s = 1$ (i.e. $k = 2$) we retrieve the Sausage S-matrix. The limit $q \to 1$ also returns a positive match with the rational (not deformed cases) presented in Aladim and Martins work[8].

Taking a look at (4.45) and (4.46) we notice that the distinction between the integer and half-integer values for s is quite significant but it is in agreement with the initial expectations. As one would guess, integer values for s lead to a finite and quite elegant solution. Half-integer values instead get us to an infinite product of Euler Γ-functions, resulting into an expression for $S(\theta)$ way more complicated than the integer counterpart.

Outlooks

This result allows new scenarios to be explored thanks to the exact expression for the S-matrix obtained.

A first task one might take on would be to find the underlying quantum field theory associated to the scattering theory of S. To do so, the dynamic of the model would need to be investigated initially in order to check the properties of the bootstrap (which in general exist but are not guaranteed to be non-trivial). Looking back at the sine-Gordon model, one could think of the excitations of our case study as "generalized" solitons. It would only be natural for one to ask if it would be possible for them to generate bound states such as the *breathers* for sine-Gordon.

Another open aspect is the study of the TBA (Thermodynamic Bethe Ansatz) in order to get information on the UV limit. Performing the TBA could also lead to clues related to the underlying quantum field theory and eventually to its uncovering.

Moreover, the Wiener-Hopf approach is made possible by the knowledge of the S-matrix. This technique could be used to get an insight on which σ-model might possibly be identified with the scattering theory described by the S-matrix found.

Appendix A

Building the S-Matrix through *Mathematica* code

In this Appendix we report the *Mathematica* code used to reproduce the tools discussed in the previous chapters to work towards building the deformed S-matrix.

```
s = 1;
Num := 2 s + 1;
qN[q_, lam_] :=
   (q^(lam/2) - q^(-lam/2))/(q^(1/2) - q^(-1/2));
n[j1_, j2_, j_] := j1 + j2 - j;
n1[j1_, j2_, j_] := j2 - j1 + j;
n2[j1_, j2_, j_] := j1 - j2 + j;
nN[j1_, j2_, j_] := n[j1, j2, j] (j1 + j2 + j + 1);
QFact[n_, q_] := If[n < 0, Infinity, Product[qN[q, k],
    {k, 1, n}]];
f[q_, j1_, j2_, J_] :=
    (qN[q, 2 J + 1] x QFact[(n1[j1, j2, J]),q]
    x QFact[(n2[j1, j2, J]), q]xQFact[n[j1, j2, J], q]
    x (QFact[(j1 + j2 + J + 1), q])^(-1))^(1/2);
Di[q_, j1_, m1_, j2_, m2_, J_, M_, v_] :=
    QFact[v, q] x QFact[j1 + j2 - J - v, q] x
    QFact[j1 - m1 - v, q] x QFact[j2 + m2 - v, q] x
```

```
    QFact[J - j2 + m1 + v,q]xQFact[J - j1 - m2 + v,q];
PMQF[q_, j1_, m1_, j2_, m2_, J_, M_] :=
    QFact[j1 + m1, q] x QFact[j1 - m1, q] x
    QFact[j2 + m2, q] x QFact[j2 - m2, q] x
    QFact[J + M, q] x QFact[J - M, q];
InfSum[q_, j1_, m1_, j2_, m2_, J_, M_] :=
    Sum[(-1)^(v) q^((-1/2) v (j1 + j2 + J + 1))
    ((Di[q, j1, m1, j2, m2, J, M, v])^(-1)),
    {v, 0, Min[(j1 + j2 - J), (j1 - m1), (j2 + m2)]}];
qCG[q_, j1_, m1_, j2_, m2_, J_, M_] :=
    If[m1 + m2 == M, f[q, j1, j2, J] x
    q^(1/4 nN[j1, j2, J] + 1/2 (j1 m2 - j2 m1))
    Sqrt[PMQF[q, j1, m1, j2, m2, J, M]] x
    InfSum[q, j1, m1, j2, m2, J, M], 0];
DoubleqCG[q_,j1_,m1_,j2_,m2_,
    j1p_,m1p_,j2p_,m2p_,J_,M_] :=
    qCG[q, j1, m1, j2, m2, J, M] x
    qCG[q, j1p, m1p, j2p, m2p, J, M];
sDoubleCG[q_, s_, m1_, m2_, m1p_, m2p_, J_, M_] :=
    DoubleqCG[q, s, m1, s, m2, s, m1p, s, m2p, J, M];

qProjector[q_] =
    Table[Sum[sDoubleCG[q, s, -s + IntegerPart[(i/Num)],
     -s + ( Mod[i, Num]), -s + IntegerPart[(j/Num)],
     -s + ( Mod[j, Num]), J, k], {k, -J, +J}],
     {J, 0, Num - 1}, {i, 0, Num^2 - 1},
     {j, 0, Num^2 - 1}];
    (* i for lines (m1,m2); j for columns (m1p,m2p) *)

Simplify[ qProjector[q][[2]].qProjector[q][[2]] -
        qProjector[q][[2]]] ===
        IdentityMatrix[Num^2] - IdentityMatrix[Num^2]
    (* this statement returns true if
    idempotency is satisfied from the projectors. *)
```

```
e[a_, b_] :=
    Table[KroneckerDelta[a, i] x
    KroneckerDelta[b, j], {i, Num}, {j, Num}];
pp = Sum[KroneckerProduct[e[a, b], e[b, a]],
    {a, Num}, {b, Num}];
f[th_, k_, q_] :=
    Product[(q^l - E^th)/(q^l E^th - 1),
    {l, 0, k}];
qS[th_, q_] :=
    pp.Sum[f[th, k, q] x
    qProjector[q][[k + 1]], {k, 0, Num - 1}];

Simplify[qS[th, q], Assumptions -> q > 0] // MatrixForm

UNITARITY
Simplify[qS[th, q].pp.qS[-th, q].pp /. {q -> 2/3}] ==
     IdentityMatrix[Num^2]
(*One can numerically check, inserting whichever
value he needs to for the deformation parameter q*)

YANG-BAXTER Equation
Id[n_] := IdentityMatrix[Num^n];
S1[q_, th_, gam_] := pp.S[q, th, gam];

pp12 = KroneckerProduct[pp, Id[1]];
pp23 = KroneckerProduct[Id[1], pp];
pp13 = pp23.pp12.pp23;

qS12[th_, q_] :=
  KroneckerProduct[qS[th, q], IdentityMatrix[Num]];
qS23[th_, q_] :=
  KroneckerProduct[IdentityMatrix[Num], qS[th, q]];
qS13[th_, q_] := pp23.qS12[th, q].pp23;
```

```
Simplify[(qS12[lam1 - lam2, q].qS13[lam1 - lam3,q].
    qS23[lam2 - lam3, q] -
    qS23[lam2 - lam3, q].qS13[lam1 - lam3, q].
    qS12[lam1 - lam2, q]) /. {q -> 1/2}] ===
    IdentityMatrix[Num^3] - IdentityMatrix[Num^3]
    (*This statement returns true
    if Yang-Baxter equation is satisfied *)

CROSSING-SYMMETRY
Iindex[m1_, m2_] = (m1 + s) (2 s + 1) + (m2 + s) + 1;
Jindex[m1p_, m2p_] = (m1p + s) (2 s + 1) + (m2p + s) + 1;
CC[m_] = -m;

Scatt[m1_, m2_, m1p_, m2p_, th_, q_] =
    qS[th, q][[Iindex[m1, m2], Jindex[m1p, m2p]]];
CSRatio[m1_, m2_, m1p_, m2p_, th_, q_] :=
    Simplify[Scatt[m1, m2, m1p, m2p, th, q]/
    Scatt[CC[m2p], m1, CC[m2], m1p, (Log[q] - th), q]];
Do[If[(-s + IntegerPart[(i/Num) - s + ( Mod[i, Num])]) -
    (-s + IntegerPart[(j/Num) - s + ( Mod[j, Num])]) === 0,
    Print[Simplify[CSRatio[-s + IntegerPart[(i/Num)],
    -s + ( Mod[i, Num]), -s + IntegerPart[(j/Num)],
    -s + ( Mod[j, Num]), th, q]]]
    Print["(", i, ";", j, ")"]],
    {i, 0, Num^2 - 1}, {j, 0, Num^2 - 1}];
```

Appendix B

Check on Aladim and Martins' $\mathrm{SU}(2)_k$ invariant $\boldsymbol{S}$-Matrices

We hereby report a simple code developed in the Mathematica programming language that can be used to perform Yang-Baxter equation check, as well as unitarity and crossing-symmetry checks on the $\mathrm{SU}(2)_k$ invariant non-deformed matrices that appear in [8].
An interesting take at these S-matrices is that they are not immediately

```
DoubleCG[s_, m1_, m2_, m1p_, m2p_, J_, M_] =
  ClebschGordan[{s, m1}, {s, m2}, {J, M}]
  Conjugate[ClebschGordan[{s, m1p}, {s, m2p}, {J, M}]];
s = 1;
Num = 2s + 1;
Projector = Table[Sum[DoubleCG[(Num - 1)/2, -(Num - 1)/2 +
   IntegerPart[(i/Num)], -(Num - 1)/2 + ( Mod[i, Num]),
   -(Num - 1)/2 + IntegerPart[(j/Num)],
   -(Num - 1)/2 + ( Mod[j, Num]), J, k],
   {k, -J, +J}], {J, 0, Num - 1},
   {i, 0, Num^2 - 1}, {j, 0, Num^2 - 1}]
(* i for lines (m1,m2); j for columns (m1p,m2p) *)

f[l_, th_, eta_] :=
   (th - I l eta)/(th + I l eta);
```

```
(* Here, and in the following code,
        "I" is the imaginary unit *)

SMatrix[th_, eta_] =
   Sum[Product[f[l, th, Pi], {l, 1, J}] Projector[[J + 1]],
   {J, 0, Num - 1}];
S[th_] = SMatrix[th, Pi];

eta = Pi;
Simplify[SMatrix[th, eta]] // MatrixForm

Yang-Baxter Equation
Id[n_] := IdentityMatrix[Num^n];

e[a_, b_] =
   Table[KroneckerDelta[a, i] KroneckerDelta[j, b],
   {i, 1, Num}, {j, 1, Num}];
pp = Sum[KroneckerProduct[e[a, b], e[b, a]],
   {a, 1, Num}, {b, 1, Num}];
pp12 = KroneckerProduct[pp, Id[1]];
pp23 = KroneckerProduct[Id[1], pp];
pp13 = pp23.pp12.pp23;

S12[th_] = KroneckerProduct[S[th]], Id[1]];
S23[th_] = KroneckerProduct[Id[1], S[th]];
S13[th_] = pp23.S12[th].pp23;

Simplify[S12[lam1 - lam2].S13[lam1 - lam3].S23[lam2 - lam3]
   - S23[lam2 - lam3].S13[lam1 - lam3].S12[lam1 - lam2]]
   == Id[Num] - Id[Num]
(*This expression returns true if
   Yang-Baxter equation is satisfied *)
```

```
MARTINS'
Prefac[th_] = Gamma[(1/2) - th I /(2 Pi)]
   Gamma[th I /(2 Pi)] /
   (Gamma[(1/2) + th I /(2 Pi)] Gamma[-th I /(2 Pi)]);

Sp[th] =
   If[(Mod[2 s, 2] == 0),
   Product[(th + 2l I Pi)/(th - 2 l I Pi),
   {l, 0, s}] x S[th],
   Prefac[th] Product[(th +(2l + 1) I Pi)/
   (th - (2 l + 1) I Pi),
   {l, 0, (2 s - 1)/2}] x S[th]];

Iindex[m1_, m2_] = (m1 + s) (2 s + 1) + (m2 + s) + 1;
Jindex[m1p_, m2p_] = (m1p + s) (2 s + 1) + (m2p + s) + 1;
Scatt[m1_, m2_, m1p_, m2p_, th_] =
   Sp[th][[Iindex[m1, m2], Jindex[m1p, m2p]]];

CROSSING
CC[m_] = -m;
CS[m1_, m2_, m1p_, m2p_, th_] =
   Simplify[Scatt[m1, m2, m1p, m2p, th] -
   Scatt[CC[m2p], m1, CC[m2], m1p, (I Pi - th)]];

CCMatrix =
   Table[(-1) (-1)^a KroneckerDelta[a + b, 2 s + 2],
   {a, 1, 2 s + 1}, {b, 1, 2 s + 1}];
C1Matrix = KroneckerProduct[CCMatrix, Id[1]];
(* setup for charge-conjugation matrix*)

T1Projector =
   Table[Sum[DoubleCG[s, -s + IntegerPart[(i/Num)],
   -s + ( Mod[j, Num]), -s + IntegerPart[(j/Num)],
   -s + ( Mod[i, Num]), J, k], {k, -J, +J}],
```

```
    {J, 0, Num - 1}, {i, 0, Num^2 - 1}, {j, 0, Num^2 - 1}];

 T2SMatrix[th_, eta_] =
    Sum[Product[f[l, th, eta], {l, 1, J}]
    T1Projector[[J + 1]], {J, 0, Num - 1}];

 T2S[th_] = T2SMatrix[th, Pi];

 CrossedSMat[th_] :=
    C1Matrix.pp.T2S[I Pi - th].pp.Transpose[C1Matrix]

 Simplify[CrossedSMat[th]/(CrossedSMat[th][[1, 1]])]
     // MatrixForm
 (*Normalization of crossed S-matrix with its 1,1 elements*)

 Simplify[S[th]/(S[th][[1, 1]])] // MatrixForm
 (*Normalization of S-matrix with its 1,1 elements*)

 Simplify[CrossedSMat[th]/(CrossedSMat[th][[1, 1]]) -
S[th]/(S[th][[1, 1]])] // MatrixForm
 (* This returns , as expected, a matrix were
 all elements are vanishing, confirming crossing
 symmetry between the normalized matrices*)


 UNITARITY
 SNorm[th_] := S[th]/(S[th][[1, 1]]);
 Simplify[SNorm[th].SNorm[-th]] // MatrixForm
 (*Unitarity check*)
```

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