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Closed-form Solution for Secondary Perturbation Displacement in FE Implemented Koiter's Theory

Jiayi Yan and Shuguang Li

Faculty of Engineering, the University of Nottingham, Nottingham, NG7 2RD, UK

Abstract

The finite element implemented Koiter's initial post-buckling analysis has not managed to convince commercial FE code developers to incorporate it into their codes to benefit structural designers and analysts. Three key obstacles have been identified. With two of them being resolved recently, the present paper addressed the remaining one, *viz*. the efficient solution of the second order perturbation equation. A closed-form solution is obtained which is more computationally efficient than any approach adopted in the past. The closed-form solution is obtained mathematically rigorously and in the meantime it is computational efficient. The approach applies also to problems where buckling is of multiple modes. The solution procedure established in this paper has been verified and demonstrated through a structural application.

Keywords: Koiter's theory; The initial post-buckling analysis; Perturbation; Stability; FEM; Closed-form solution; Verification.

Nomenclature

a	First initial post-bucking coefficient
b	Second initial post-buckling coefficient
B	Breadth of plate
E	Young's modulus
$\mathbf{f}$ or f_i	Right-hand term of secondary perturbation equation
$\tilde{\mathbf{f}}$	The same as $\mathbf{f}$ but with k^{th} row removed
$\tilde{\mathbf{g}}$	The same as the k^{th} column of the $\mathbf{K}$ matrix but with k^{th} row removed
F_U^0, F_U^1 & F_U^2	In-plane stress resultant in the x direction for pre-buckling, buckling and secondary perturbation, respectively
F_V^0, F_V^1 & F_V^2	In-plane stress resultant in the y direction for pre-buckling, buckling and secondary perturbation, respectively
i and j	Degree of freedom (dof) numbers in a finite element model (i,j=1,2,…, n)
k	The number of dof at which the magnitude of buckling mode is maximum
$\mathbf{K}$ or K_{ij}	Tangential stiffness matrix, which is a singular, as the coefficient matrix of governing equation for buckling as well as the secondary perturbation
$\tilde{\mathbf{K}}$	The same as $\mathbf{K}$ but with k^{th} row and k^{th} column removed

$\mathbf{K}_0$	Stiffness matrix for linear problems
$\mathbf{K}_0^{\sigma}$	Initial stress or geometric stiffness matrix
L	Length of plate
m	Multiplicity of the buckling modes
n	Total number of dofs in the finite element model
$\mathbf{N}_1$ & $\mathbf{N}_2$	Nonlinear stiffness matrix as linear and quadratic functions of displacement
N_{xx}^0, N_{xx}^1 & N_{xx}^2	In-plane membrane in the x direction for pre-buckling, buckling and secondary perturbation
N_{yy}^0, N_{yy}^1 & N_{yy}^2	In-plane membrane in the y direction for pre-buckling, buckling and secondary perturbation
$\mathbf{q}^0$	Pre-buckling displacement as a function of loading parameter
$\mathbf{q}_c^0$	Pre-buckling displacement at the critical point of bifurcation
$\dot{\mathbf{q}}^c$	Derivative of the pre-buckling displacement with respect to the load parameter at the critical point of bifurcation
$\mathbf{q}^1$ or q_i^1	Buckling mode and also the first order of perturbation displacement
$\mathbf{q}^2$ or q_i^2	Secondary perturbation displacement (SPD)
$\mathbf{q}^c$	Complementary solution for SPD
$\mathbf{q}^{(k)}$	The k^{th} buckling mode in the SPD in a multiple mode buckling case
$\mathbf{q}^p$ or q_i^p	Particular solution for SPD
$\tilde{\mathbf{q}}^p$	The same as $\mathbf{q}^p$ but with k^{th} row removed
$\tilde{\mathbf{q}}_0^p$	A special case of $\tilde{\mathbf{q}}^p$ with q_k^p set to zero
q_k^p	The nodal displacement at the k^{th} dof in the particular solution
s	Magnitude of complementary solution, i.e. buckling mode, in the SPD
$s^{(r)}$	Magnitude of the r^{th} buckling mode in the SPD in a multiple mode buckling case (r=1,2,…, m)
t	Thickness of plate
$\mathbf{T}$	Arbitrary positive definite matrix to deliver the orthogonality condition
u & v	In-plane displacement in the x and y direction
u_i, v_i & w_i	Nodal displacements at the i^{th} node
u_i^a, v_i^a & w_i^a	Analytical displacements at the point of the coordinates of the i^{th} node
U^0, U^1 & U^2	Common values of displacement in the x direction on sides perpendicular to the x-axis for pre-buckling, buckling and secondary perturbation
V^0, V^1 & V^2	Common values of displacement in the y direction on sides perpendicular to the y-axis for pre-buckling, buckling and secondary perturbation
w	Lateral displacement in the z direction
λ	Loading parameter
λ_c	Buckling load
λ_s	The load at snap-through point on the equilibrium path of the structure with initial imperfection
μ	Amplitude of initial imperfection
ν	Poisson's ratio
ξ	Perturbation parameter and the amplitude of the buckling mode as a part of the post-buckling displacement

I. Introduction

The provision of buckling strength predictions from mainstream commercial finite element (FE) codes is generally unsatisfactory for practising structural designers and analysts, despite the ever-improving modelling capability. Nonlinear solvers available nowadays are in fact sufficient for the class of the so-called snap-through buckling problem, which features a maximum load point on the equilibrium path where snap-through takes place. The equilibrium path before snap-through is usually highly nonlinear due to finite deformation. There is another, practically larger, class of buckling problem characterised by a bifurcation preceded by a usually fairly linear pre-buckling equilibrium path, as will be the focus of the present paper. The prediction of such bifurcation point is straightforward as an eigenvalue problem. However, designers are well aware of the fact that it is simply not reliable to treat the predicted the bifurcation load as the buckling strength of the structure. Some structures can sustain much higher loads above the bifurcation load with increasing deformation such that the so-called secondary buckling becomes an issue, e.g. [1], while others could collapse at load levels much lower than the bifurcation load. As an example of the latter, cylinders under axial compression collapse over a wide range scattered between 20% and 50% of that as predicted by the classic buckling theory [2]. This phenomenon started troubling researchers in 1940s [3] which led to great efforts trying to obtain the post-buckling equilibrium paths, e.g. [4,5], in order to understand why. The campaign ended rather abruptly with the advent of Koiter's initial post-buckling theory in late 1960s [6], established in 1945 but unnoticed in the English world until 1963 [7], although the search for post-buckling equilibrium paths seems to find its way back in its computational form due to inavailability of convincingly FE implemented Koiter's initial post-buckling analysis in mainstream commercial FE codes. In essence, Koiter's theory revealed the true significance of the post-buckling behaviour on the buckling load. It was in fact not the whole post-buckling equilibrium path but the stability of the equilibrium state at the bifurcation point on the post-buckling path. In other words, being able to find the post-buckling equilibrium path is not enough, no matter how far one could follow the path beyond the bifurcation point. In order to assess whether the predicted bifurcation load can be used as the design load, one only needs the stability state of the equilibrium on the post-buckling path at the bifurcation point, which is not available from the existing mainstream commercial code in a straightforward manner. If one could assess the stability state, there would not be a need to go very far beyond the bifurcation point before one could conclude if there would be any risk of premature collapse. Koiter proved that the state of stability also dictated the sensitivity of the actual buckling load to the initial imperfection the presence of which is inevitable in reality due to manufacturing or testing practicality. Structures of a stable initial post-buckling state are insensitive to imperfection and hence able to sustain higher loads while those tend to collapse prematurely are due to their unstable initial post-buckling state and hence extreme sensitivity to imperfection. Koiter did not only prove the qualitative relationship as stated above, he also managed to express the actual collapsing load quantitatively and simply as a function of the magnitude of the initial imperfection in an asymptotic manner. Although Koiter's theory had the word of 'post-buckling' in its title, and it indeed generated the post-buckling equilibrium path in an asymptotic manner, it was more for effective and comprehensive understanding of the buckling behaviour but from the perspective of post-buckling. It should therefore not to be mixed with post-buckling analysis, especially those traced far into advanced post-buckling regime. In fact, to be able to trace the post-buckling path alone does not necessarily resolve the problem of lack of consistency in the buckling behaviour of structures.

According to Koiter's theory, the way in which a structure buckles can be fully described as sketched in Fig. 1, depending on two values, a and b, the initial post-buckling coefficients. In general, if $a \neq 0$ or $a=0$ and $b<0$ as in Fig. 1(a) and (b), respectively the post-buckling equilibrium is unstable at the bifurcation point and the structure is therefore sensitive to the initial imperfection. In presence of the initial deflection in the shape of the buckling mode at different amplitudes, μ, the equilibrium paths are shown in dashed curves and the actual buckling takes place as snap-through at λ^s as indicated in the figure. In fact, the sensitivity is infinite in an asymptotic sense as illustrated in Fig. 1(d) where the slope of the drop in buckling load is infinite at $\mu=0$. For structures in this category, with cylinders under axial compression as a typical example [2], for which new understanding has been achieved recently [8], wide scatters in the experimentally obtained buckling load are expected. On the other hand, in the case of $a=0$ and $b>0$ as in Fig. 1(c), the post-buckling equilibrium is stable at the bifurcation point and the structure can take loads higher than the predicted buckling load while undergoing increasing deformation. Flat plates and straight beams are typical structures in this category. The assessment of the buckling behaviour of a structure assisted with Koiter's theory is not only informative but also simple because one only needs to obtain two scalar values, a and b, in addition to the buckling load, although the underlying theory is sophisticated and the derivations of the expressions of a and b are demanding. However, once implemented through the finite element method (FEM), their numerical evaluation is almost computationally effortless.

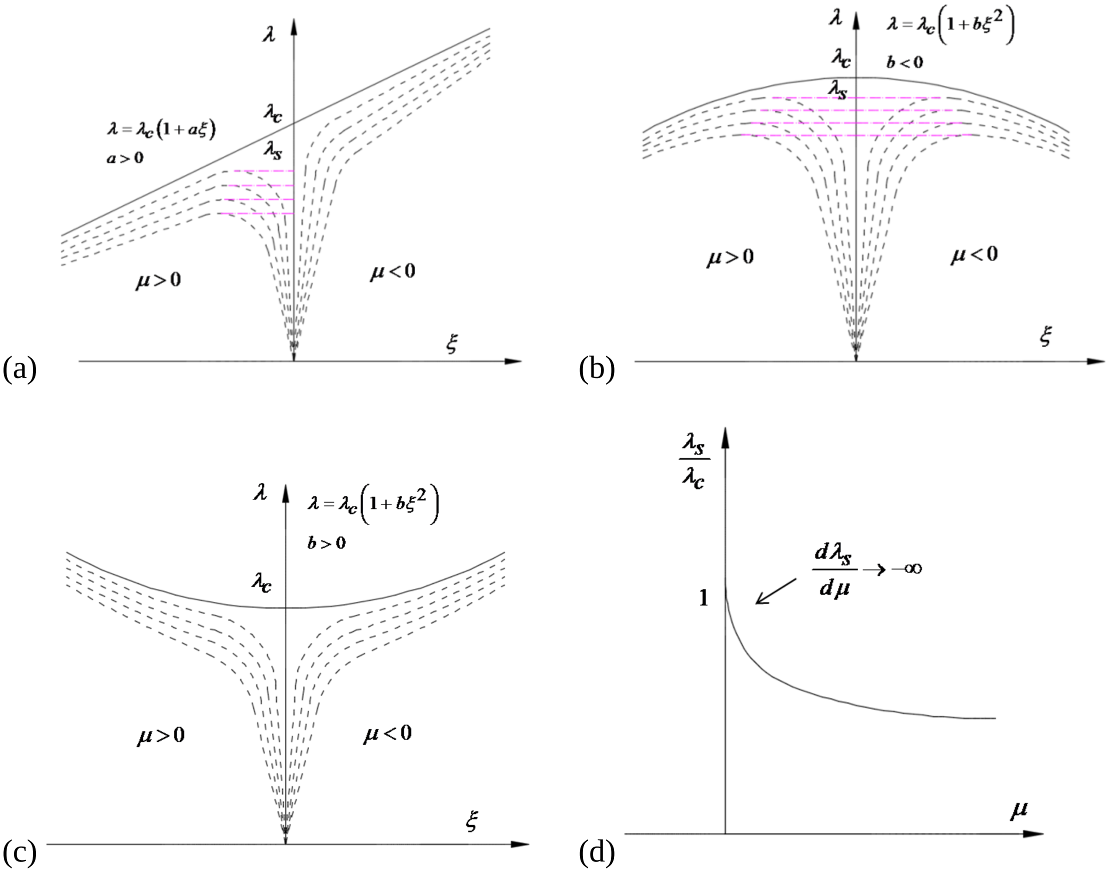


Fig. 1 A schematic presentation of the outcomes of Koiter's theory (a) asymmetric bifurcation, $a\neq0$; (b) symmetric bifurcation with a=0 and b<0; (c) symmetric bifurcation with a=0 and b>0; and (d) imperfection sensitivity, $a\neq0$ or a=0 and b<0

The 1970s as Koiter's theory was made widely known were also the era when modern commercial FE codes started being incubated. Half a century on, the development of FE capabilities of these codes has been phenomenal. However, given the practical significance of Koiter's theory in the problem of structural stability and given the attempts of its FE implementation since early 1970s [9], it is still unavailable from any mainstream commercial FE code, for which disappointment is an understatement. As a result, the research activities on Koiter's theory declined, given sparsely populated publications [10-23] as a representative spectrum of them from various perspectives, without occupying the length of this paper for a comprehensive literature review while a reasonable coverage can be found in relatively recent publications [24,25]. The reality is that for, most structural designers and analysts, Koiter's theory is becoming distant past instead of useful tool at their fingertips. A closer examination suggests that it has been in this position not without genuine reasons, owing much to the following obstacles.

1) A key output from Koiter's theory, coefficient b, has been found mesh sensitive. While several techniques had been proposed [10-13], an effective approach has been demonstrated very recently by the authors in [26], simply by adopting the N-notation in FEM.
2) To evaluate the same coefficient b, a set of special simultaneous linear equations as the governing equations has to be solved for the secondary perturbation displacement (SPD), where a convincing solution technique lacks. A more detailed description will be provided later as the subject of the present paper.
3) As far as plates and shells are concerned, FE predictions had not been verified by comparing with available analytical results until recently [24], although analytical solutions, e.g. in the case of a simply support plate under bi-axial compression [27], was available in early 1970s.

The present paper is devoted to the second obstacle as identified above as a final hurdle in fully establishing the FE implemented Koiter's theory. A crucial step of the initial post-buckling analysis according to Koiter's theory is to evaluate a and b, as they contain the full information about the state of stability of the post-buckling equilibrium path at the bifurcation point and hence the buckling behaviour, as depicted in Fig. 1. The evaluation of a requires little more information than the outcomes of conventional buckling analysis. If $a \neq 0$, the analysis can be concluded at this point. However, there is a large family of structures of which $a=0$ while $b \neq 0$. One requires the value of b in order to assess the stability of the post-buckling equilibrium path at the bifurcation point.

In order to evaluate b, one has to conduct the second order perturbation. This includes obtaining the SPD that plays an essential part in the expression of b. A set of simultaneous linear algebraic equations will have to be solved. The difficulty associated with this set of equations is that the coefficient matrix is singular and they are topped up with some orthogonality conditions. If the problem involved only a small number of equations, as it was often the case in analytical solutions, one or more redundant equations could be easily identified. However, in an FEM scenario, one often faces a large number of simultaneous equations. Which of them should be filtered out and replaced by the orthogonality conditions does not seem to be a straightforward decision to make. Even if one identified the redundant equation, simply replacing it with the orthogonality condition would spoil the symmetric and banded appearance of the coefficient matrix in FEM. An alternative is to solve the singular set of equations along with the orthogonality condition as a set of over determined simultaneous equations using the least square method, as was employed in [25]. Again, the solution demands a new solver since the banded numerical feature of the coefficient matrix have been spoiled. The efficiency of FEM counts heavily on the special numerical features of the coefficient matrix. The equation solvers employed in all FE codes are usually based on these features. The last thing commercial FE code developers want is to introduce a new solver specifically for this particular problem. Therefore, this presents an obstacle to the FE implementation of Koiter's theory.

Given the fact that every attempt to the FE implementation of the theory has to face this practical problem, there has not been much effort in order to resolve the problem in the literature, except those to be described in the next two paragraphs. In fact, most accounts in the literature tended to avoid mentioning it as a problem.

A complete description of the solution procedure could be found in a number of publications [11,13,28] in the literature where an iterative scheme was proposed. In addition to the increased computational cost and loss of numerical accuracy, a rigorous proof of the convergence of the iterative scheme is absent, although it was claimed to be rapid.

To the authors' best knowledge, there has been only one account in the literature, which described a non-iterative approach intuitively as 'the solution can therefore be found simply by adding a support that eliminates the singularity and afterwards the orthogonality condition can be applied' [12]. However, no instruction was given where the support should be added and why. While the intuition stood some truth, it was far from rigorous and it did not seem to have attracted much attention afterwards either. In the present paper, a systematic approach along this line will be established leading to a closed-form solution of mathematical rigor. The governing equations will be expressed in this paper using the N-notation finite element formulation as established in [26]. As it is a closed-form solution, no iteration will be required. Mathematically it is an exact solution if numerical errors involved in the eigenvalue extraction for the buckling load and the solution of a set of simultaneous linear equations are excluded. More importantly, this approach does not spoil the symmetric, positive definite and banded nature of the coefficient matrix in any way so that the same solver in the conventional FEM, such as the LDLT decomposition method [29], can be employed to solve the simultaneous equations while the remaining manipulations, though essential, are trivial in terms of computational cost.

After a brief recapitulation of FE implemented Koiter's theory associated with SPD in the next section, the approach to the closed-form solution will be presented in Section III based the mathematical principle of the construction of solutions for linear equations in general in terms of the sum of the complementary solution and a particular solution. The existence and uniqueness of the solution has also been discussed. The applicability of the solution technique for problems of multiple buckling modes has been delivered in Section IV before various examples as verifications or applications are presented in Section V.

II. FE initial post-buckling analysis and the secondary perturbation displacement

With the fully established N-notation [26], governing equation for buckling is given as

$$\left(\mathbf{K}_0+\mathbf{N}_1\left(\mathbf{q}_c^0\right)+\mathbf{N}_2\left(\mathbf{q}_c^0\right)\right)\mathbf{q}^1=\mathbf{0} \tag{1}$$

where $\mathbf{K}_0$ is the stiffness matrix for the linear problem, and $\mathbf{N}_1$ and $\mathbf{N}_2$ are the stiffness matrices due to geometrical nonlinearity, which are linear and quadratic functions of displacement, respectively, and $\mathbf{q}^1$ is the buckling mode. They can be obtained explicitly and uniquely [26]. $\mathbf{q}_c^0$ is defined as follows. The fundamental path is a function of load parameter λ, defined as

$$\mathbf{q}^0=\mathbf{q}^0\left(\lambda\right). \tag{2}$$

In general, it can be expressed into a series as Taylor's expansion at the point of buckling, i.e.

$$\mathbf{q}^0=\mathbf{q}_c^0+\left(\lambda-\lambda_c\right)\dot{\mathbf{q}}_c^0+\cdots. \tag{3}$$

where λ_c is the lowest eigenvalue of (1), i.e. the buckling load. Then $\mathbf{q}_c^0=\mathbf{q}^0\left(\lambda_c\right)$, and $\dot{\mathbf{q}}_c^0$ is the 1st order derivative with respect to λ at λ_c. The fundamental path expressed either in (2) or (3) is assumed to be known.

Equation (1) defines a nonlinear eigenvalue problem. In most practical problems, the fundamental path is linear, i.e. (2) is a linear function of λ. Even so, (1) is still a quadratic eigenvalue problem in general. However, if the pre-buckling deformation state on the fundamental path is also in a membrane state, (1) can be simplified to

$$\left(\mathbf{K}_0+\lambda_c\mathbf{K}_0^\sigma\right)\mathbf{q}^1=\mathbf{0} \tag{4}$$

where $\mathbf{K}_0^\sigma$ is the so-called initial stress or geometric stiffness matrix with detailed expression provided in [26] after the initial stresses have been normalised by the load parameter λ. Equation (4) is then a conventional generalised eigenvalue problem. In the examples later, the buckling problem will be solved using (4). However, to keep the generality of the presentation in the discussion to follow, (1) will be kept as the governing equations.

The 2nd order perturbation is required if the first initial post-buckling coefficient vanishes, i.e. a=0. In this case, the governing equations for the SPD, $\mathbf{q}^2$ can be given as follows [26].

$$\left(\mathbf{K}_0+\mathbf{N}_1\left(\mathbf{q}_c^0\right)+\mathbf{N}_2\left(\mathbf{q}_c^0\right)\right)\mathbf{q}^2=-\frac{1}{2}\mathbf{N}_1\left(\mathbf{q}^1\right)\mathbf{q}^1-\mathbf{N}_2\left(\mathbf{q}^1\right)\mathbf{q}_c^0 \tag{5}$$

where the coefficient matrix on the left hand side is identical to that of (1), which is therefore singular. Under the same conditions that helped reducing (1) to (4), this coefficient matrix can also be replaced by that in (4).

The orthogonality condition between buckling mode and SPD can be given as [26]

$$\left(\mathbf{q}^1\right)^T\mathbf{T}\mathbf{q}^2=0 \tag{6}$$

where $\mathbf{T}$ can be an arbitrary positive definite matrix and the superscript T on $\mathbf{q}^1$ indicates transposition. A conventional choice of $\mathbf{T}$ and its positive definitiveness will be justified in Appendix for the sake of completeness.

With $\mathbf{q}^1$ and $\mathbf{q}^2$ found, the initial post-buckling coefficients can be obtained as [26]

$$a=-\frac{1}{2\lambda_c}\frac{\left(\mathbf{q}^1\right)^T\mathbf{N}_1\left(\mathbf{q}^1\right)\mathbf{q}^1+2\left(\mathbf{q}^1\right)^T\mathbf{N}_2\left(\mathbf{q}^1\right)\mathbf{q}_c^0}{\left(\mathbf{q}^1\right)^T\mathbf{N}_1\left(\mathbf{q}^1\right)\dot{\mathbf{q}}_c^0+2\left(\mathbf{q}_c^0\right)^T\mathbf{N}_2\left(\mathbf{q}^1\right)\dot{\mathbf{q}}_c^0} \tag{7}$$

$$b=-\frac{1}{3\lambda_c}\frac{\left(\mathbf{q}^1\right)^T\mathbf{N}_2\left(\mathbf{q}^1\right)\mathbf{q}^1-6\left(\mathbf{q}^2\right)^T\left(\mathbf{K}_0+\mathbf{N}_1\left(\mathbf{q}_c^0\right)+\mathbf{N}_2\left(\mathbf{q}_c^0\right)\right)\mathbf{q}^2}{\left(\mathbf{q}^1\right)^T\mathbf{N}_1\left(\mathbf{q}^1\right)\dot{\mathbf{q}}_c^0+2\left(\mathbf{q}_c^0\right)^T\mathbf{N}_2\left(\mathbf{q}^1\right)\dot{\mathbf{q}}_c^0}\quad\text{(if } a\text{=0)} \tag{8}$$

Noticing the typographic errors in the above expressions as presented in [26] in the second term in the denominators of (7) and (8), which have been corrected here. The significance of a and b has been described in the previous section through Fig. 1. It is obvious that in order to evaluate b, it is essential to find $\mathbf{q}^2$. The problem of finding $\mathbf{q}^2$ can be presented mathematically as solving governing equations (5) subjected to orthogonality condition (6). It will be shown that a closed-form solution can be obtained in the next section.

III. The closed-form solution for the secondary perturbation displacement

A. The mathematical problem and the construction of the solution

For the derivation in this section, the buckling will be assumed to be of a single mode. For the convenience in mathematical manipulations, (5) is abbreviated as

$$\mathbf{K}\mathbf{q}^2 = \mathbf{f} \qquad \text{or} \qquad \begin{bmatrix} K_{1,1} & \cdots & K_{1,k} & \cdots & K_{1,n} \\ \vdots & \ddots & \vdots & & \vdots \\ K_{k,1} & \cdots & K_{k,k} & \cdots & K_{k,n} \\ \vdots & & \vdots & \ddots & \vdots \\ K_{n,1} & \cdots & K_{n,k} & \cdots & K_{n,n} \end{bmatrix} \begin{Bmatrix} q_1^2 \\ \vdots \\ q_k^2 \\ \vdots \\ q_n^2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ \vdots \\ f_k \\ \vdots \\ f_n \end{Bmatrix}. \tag{9}$$

Similarly, eigenvalue problem (1) or (4) is denoted as

$$\mathbf{K}\mathbf{q}^1 = \mathbf{0}. \tag{10}$$

The mathematical problem to be addressed is (9) as a set of nonhomogeneous simultaneous linear equations with a singular coefficient matrix as the governing equations which should be solved subject to condition (6). This is a linear problem mathematically.

In general, the solution to a linear problem can always be constructed from a complementary solution $\mathbf{q}^c$ and a particular solution $\mathbf{q}^p$. The complementary solution $\mathbf{q}^c$ satisfies the homogeneous counterpart of (9)

$$\mathbf{K}\mathbf{q}^c = \mathbf{0} \tag{11}$$

while the particular solution $\mathbf{q}^p$ can be anyone satisfying (9). Apparently, none of the complementary solution $\mathbf{q}^c$ and the particular solution $\mathbf{q}^p$ is unique, given the singularity of $\mathbf{K}$. In fact, for any constant s, one can easily verify that $s\mathbf{q}^c$ also gives a complementary solution while $s\mathbf{q}^c+\mathbf{q}^p$ is also a particular solution. According to linear algebra [30], if the buckling is of a single mode, the rank of $\mathbf{K}$ is (n-1), i.e. the solution to (11) is determined to a single arbitrary constant which can be s as introduced above. Thus, the complete solution to (9), if it exists, can be expressed as

$$\mathbf{q} = s\mathbf{q}^c + \mathbf{q}^p. \tag{12}$$

With the solution mathematically constructed as above, a closed-form solution will be sought for in the following subsections with appropriate mathematical considerations duly made.

B. The complementary solution and its implication

Given the buckling equations (10), $\mathbf{q}^1$ apparently satisfies (11). It is clear that the buckling mode $\mathbf{q}^1$ is a complementary solution. Thus

$$\mathbf{q}^c = \mathbf{q}^1 = \begin{bmatrix} q_1^1 & \cdots & q_{k-1}^1 & q_k^1 & q_{k+1}^1 & \cdots & q_n^1 \end{bmatrix}^T. \tag{13}$$

Equation (10) can be re-written into

$$\begin{Bmatrix} K_{1,1} \\ \vdots \\ K_{1,k} \\ \vdots \\ K_{1,n} \end{Bmatrix} q_1^1 + \cdots + \begin{Bmatrix} K_{k-1,1} \\ \vdots \\ K_{k-1,k} \\ \vdots \\ K_{k-1,n} \end{Bmatrix} q_{k-1}^1 + \begin{Bmatrix} K_{k,1} \\ \vdots \\ K_{k,k} \\ \vdots \\ K_{k,n} \end{Bmatrix} q_k^1 + \begin{Bmatrix} K_{k+1,1} \\ \vdots \\ K_{k+1,k} \\ \vdots \\ K_{k+1,n} \end{Bmatrix} q_{k+1}^1 + \cdots + \begin{Bmatrix} K_{n,1} \\ \vdots \\ K_{n,k} \\ \vdots \\ K_{n,n} \end{Bmatrix} q_n^1 = \begin{Bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{Bmatrix}. \tag{14}$$

Assuming that the k^{th} component of $\mathbf{q}^1$ is of the highest absolute magnitude, one is sure that

$$q_k^1 \neq 0\,. \tag{15}$$

Otherwise, the buckling mode would be a null vector that is not allowed as the buckling mode is supposed to be a non-trivial solution to (10). With (15), one can re-write (14) into

$$\begin{Bmatrix} K_{k,1} \\ \vdots \\ K_{k,k} \\ \vdots \\ K_{k,n} \end{Bmatrix} = -\frac{q_1^1}{q_k^1}\begin{Bmatrix} K_{1,1} \\ \vdots \\ K_{1,k} \\ \vdots \\ K_{1,n} \end{Bmatrix} - \cdots - \frac{q_{k-1}^1}{q_k^1}\begin{Bmatrix} K_{k-1,1} \\ \vdots \\ K_{k-1,k} \\ \vdots \\ K_{k-1,n} \end{Bmatrix} - \frac{q_{k+1}^1}{q_k^1}\begin{Bmatrix} K_{k+1,1} \\ \vdots \\ K_{k+1,k} \\ \vdots \\ K_{k+1,n} \end{Bmatrix} - \cdots - \frac{q_n^1}{q_k^1}\begin{Bmatrix} K_{n,1} \\ \vdots \\ K_{n,k} \\ \vdots \\ K_{n,n} \end{Bmatrix}. \tag{16}$$

The above equations can be stated as the k^{th} column of the $\mathbf{K}$ matrix can be linearly expressed in terms of other columns of the same matrix, with k corresponds to the row number of the buckling mode of the highest absolute magnitude. As $\mathbf{K}$ is a symmetric matrix, the above statement can be made equally for the i^{th} row of matrix $\mathbf{K}$, i.e. the i^{th} row of $\mathbf{K}$ can be linearly expressed in terms of the remaining rows of the same matrix.

C. Existence of a solution for the linear nonhomogeneous simultaneous equations

In Koiter's theory, there has never been rigorous proof of the existence of the solution for the secondary perturbation equations. In analytical approaches, usually the proof came with the fact that a solution had been actually found for a specific problem. This is not an ideal position. If a numerical algorithm fails to work, e.g. lack of convergence in the iterative scheme following [11,13,28], one would not be sure if it was because of the inexistence of a solution or the incapability of the iterative algorithm. A clear statement on the existence of a solution is therefore helpful from both mathematical and practical point of view. For (9) to have a solution, strictly according to linear algebra, one should have

$$\operatorname{rank}\left(\begin{bmatrix} K_{1,1} & \cdots & K_{1,k} & \cdots & K_{1,n} \\ \vdots & \ddots & \vdots & & \vdots \\ K_{k,1} & \cdots & K_{k,k} & \cdots & K_{k,n} \\ \vdots & & \vdots & \ddots & \vdots \\ K_{n,1} & \cdots & K_{n,k} & \cdots & K_{n,n} \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} K_{1,1} & \cdots & K_{1,k} & \cdots & K_{1,n} & f_1 \\ \vdots & \ddots & \vdots & & \vdots & \vdots \\ K_{k,1} & \cdots & K_{k,k} & \cdots & K_{k,n} & f_k \\ \vdots & & \vdots & \ddots & \vdots & \vdots \\ K_{n,1} & \cdots & K_{n,k} & \cdots & K_{n,n} & f_n \end{bmatrix}\right). \tag{17}$$

This will be satisfied if f_k can be linearly expressed by the rest of the components of $\mathbf{f}$ in the same way as the k^{th} row of $\mathbf{K}$ is expressed in terms of other rows of $\mathbf{K}$ as in (16), i.e.

$$f_k = -\frac{q_1^1}{q_k^1} f_1 - \cdots - \frac{q_{k-1}^1}{q_k^1} f_{k-1} - \frac{q_{k+1}^1}{q_k^1} f_{k+1} - \cdots - \frac{q_n^1}{q_k^1} f_n\,. \qquad \text{or} \qquad \mathbf{f}^T\mathbf{q}^1 = 0 \tag{18}$$

If so, condition (17) will be satisfied and existence of solution will be proven. The satisfaction (18) can be easily argued as follows. Given $\mathbf{f}$ as defined on the right hand side of (5), $\mathbf{f}^T\mathbf{q}^1$

reproduces the numerator of the expression of a as provided in (7). Condition (18) is therefore satisfied if a=0. The satisfaction of (18) will not be compromised even if $a\neq0$, as elaborated in Appendix. In other words, (18) is always satisfied and so is (17), whether a vanishes or not. The existence of a solution to (9) or (5) is thus proven. Although this is a simple argument mathematically, the explicit proof has been absent, whether for FE represented problem or for its analytical counterpart, e.g. from [6,27]. The proven existence of a solution lays down one of the cornerstones for the FE implementation of the Koiter's theory in a mathematical sense.

D. The particular solution

Equations (16) and (18) imply that the k^{th} equation in (9) can be linearly expressed by the remaining equations in (9) and therefore it is redundant. After removing the k^{th} equation and with the terms corresponding to the k^{th} column of the coefficient matrix moved to the right-hand side, the governing equation (9) can be re-written into

$$\begin{bmatrix} K_{1,1} & \cdots & K_{1,k-1} & K_{1,k+1} & \cdots & K_{1,n} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ K_{k-1,1} & \cdots & K_{k-1,k-1} & K_{k-1,k+1} & \cdots & K_{k-1,n} \\ K_{k+1,1} & \cdots & K_{k+1,k-1} & K_{k+1,k+1} & \cdots & K_{k+1,n} \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ K_{n,1} & \cdots & K_{n,k-1} & K_{n,k+1} & \cdots & K_{n,n} \end{bmatrix} \begin{bmatrix} q_1^p \\ \vdots \\ q_{k-1}^p \\ q_{k+1}^p \\ \vdots \\ q_n^p \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_{k-1} \\ f_{k+1} \\ \vdots \\ f_n \end{bmatrix} - q_k^p \begin{bmatrix} K_{1,k} \\ \vdots \\ K_{k-1,k} \\ K_{k+1,k} \\ \vdots \\ K_{n,k} \end{bmatrix} \tag{19a}$$

denoted as

$$\tilde{\mathbf{K}}\tilde{\mathbf{q}}^p = \tilde{\mathbf{f}} - q_k^p\tilde{\mathbf{g}} \tag{19b}$$

where $\mathbf{g}$ is the k^{th} column of the $\mathbf{K}$ and a tilde on top of any quantity means the same but with the k^{th} degree of freedom (dof) removed. The coefficient matrix $\tilde{\mathbf{K}}$ on the left hand side of (19) is now positive definite, given the single mode buckling assumption. Apparently, q_k^p is a free factor that can be set to any fixed value before solving (19) to deliver a particular solution. As stated previously, the particular solution is not unique and only one of them is required. Given the arbitrariness of q_k^p, one can choose $q_k^p = 0$, i.e.

$$\tilde{\mathbf{K}}\tilde{\mathbf{q}}_0^p = \tilde{\mathbf{f}} \tag{20}$$

which will be of exactly the same effect as constraining the dof q_k^p from (9), i.e.

$$\begin{bmatrix} K_{1,1} & \cdots & K_{1,k} & \cdots & K_{1,n} \\ \vdots & \ddots & \vdots & & \vdots \\ K_{k,1} & \cdots & \infty & \cdots & K_{k,n} \\ \vdots & & \vdots & \ddots & \vdots \\ K_{n,1} & \cdots & K_{n,k} & \cdots & K_{n,n} \end{bmatrix} \begin{Bmatrix} q_1^p \\ \vdots \\ q_k^p \\ \vdots \\ q_n^p \end{Bmatrix} = \begin{Bmatrix} f_1 \\ \vdots \\ f_k \\ \vdots \\ f_n \end{Bmatrix} \tag{21}$$

where ∞ is infinity but can be represented numerically by sufficiently high positive value computationally, as the higher it is, the closer will q_k^p be to zero and hence more accurately (21) be satisfied. With the dof corresponding to q_k^p constrained, the singularity of **K** has been eliminated. If the solution to (21) is denoted as $\mathbf{q}_0^p$, the difference between $\mathbf{q}_0^p$ and $\tilde{\mathbf{q}}_0^p$ is that the former is of *n* dofs with the k^{th}, i.e. q_k^p equal to zero, while the latter has (*n*-1) dofs with the k^{th} dof eliminated. Thus, $\mathbf{q}_0^p$ satisfies (9) if it satisfies (21), given (19), and therefore it is a particular solution to (9).

It is important to point out that the way of imposing constraints to displacement in (21) is also typical in FEM in general as it does not upset the appearance of the coefficient matrix that should remain symmetric and banded. The constraint as imposed in (21) turns the singular coefficient matrix into positive definite.

Although a particular solution could be found after constraining a dof as suggested in [12], it cannot be an arbitrary dof. If the dof had not been properly chosen, for instance, if the chosen dof of the buckling mode happened to have a zero value, (16) and (18) would not apply. Even if the component of the buckling mode associated with the dof was not exactly zero, the problem could become ill-conditioned if it was small relative to the amplitude of the buckling mode. Choosing the k^{th} component of $\mathbf{q}^1$ of the highest absolute magnitude is to ensure that a redundant equation is eliminated on one hand and that the problem is numerically well-posed on the other hand.

E. Orthogonality condition, the closed-form solution and its uniqueness

Having obtained the complementary solution and a particular solution as presented in the previous subsections, the complete solution can be constructed according to (12), which satisfies the governing equations (9) identically. However, it does not necessarily meet the requirement of the orthogonality condition (6). To impose it, given $\mathbf{q}^c$=$\mathbf{q}^1$, one has

$$\left(\mathbf{q}^1\right)^T \mathbf{T}\left(s\mathbf{q}^c + \mathbf{q}^p\right) = \left(\mathbf{q}^1\right)^T \mathbf{T}\left(s\mathbf{q}^1 + \mathbf{q}_0^p\right) = 0 \tag{22}$$

There is only one special value of *s* that will satisfy the orthogonality condition and it can be determined as follows.

$$s = -\left(\mathbf{q}^1\right)^T \mathbf{T}\mathbf{q}_0^p \Big/ \left(\mathbf{q}^1\right)^T \mathbf{T}\mathbf{q}^1 \tag{23}$$

The SPD can thus be obtained as

$$\mathbf{q}^2 = -\frac{\left(\mathbf{q}^1\right)^T \mathbf{T}\mathbf{q}_0^p}{\left(\mathbf{q}^1\right)^T \mathbf{T}\mathbf{q}^1}\mathbf{q}^1 + \mathbf{q}_0^p . \tag{24}$$

This uniquely determines the SPD in a closed form. Mathematically, after removing the redundant equation and topping up with the orthogonality condition (6), one has *n* independent simultaneous linear equations and they should uniquely determine a solution.

In terms of computational efforts in order to obtain the closed-form solution, one has to solve only a set of simultaneous linear equations as given in (21) to obtain a particular solution $\mathbf{q}_0^p$ while $\mathbf{q}^1$ has been made available already through the conventional buckling analysis. The coefficient matrix of (21) should also be readily available from the preceding buckling analysis without having to generate it again. Appropriate boundary conditions should be imposed as defined by the problem along with an additional constraint at the k^{th} dof as illustrated in (21). Then, the SPD can be obtained according to (24), which is computationally effortless.

Simultaneous linear equations (21) pose a well-presented problem as the coefficient matrix is symmetric, positive definite and, more importantly, banded in an identical form to the stiffness matrix of the structure concerned for its conventional linear analysis. Reflecting on the statement made in [12] as quoted in the Introduction, the present paper has turned it into a practical procedure users can follow with mathematical rigor, which can be extended to embrace broader generality as will be addressed in the next section.

IV. Secondary perturbation displacement in the case of multiple modes buckling

When the buckling is of multiple modes, the closed-form solution can be constructed in a similar way. Assume that, corresponding to the buckling load, the eigenvalue is repeated *m* times, there are *m* independent eigenvectors, denoted as $\mathbf{q}^{(r)}$ (r=1,2,…, m), and these eigenvectors as well as the buckling load have all been obtained already, where superscript *r* in brackets is to be distinguished from displacement perturbations $\mathbf{q}^2$, $\mathbf{q}^3$, etc. For each of eigenvectors, the row number of the component of the highest absolute magnitude can be identified and denoted as $i^{(r)}$. All eigenvectors are orthogonal to each other, i.e.

$$\left(\mathbf{q}^{(h)}\right)^T \mathbf{T}\mathbf{q}^{(r)} = 0 \qquad \text{if } h \neq r \qquad (h, r=1,2,\ldots, m) \tag{25}$$

The complementary solution can then be constructed from these eigenvectors as a linear combination of them. There are *m* redundant equations in (9) which can be eliminated by constraining the $k^{(r)}$-*th* dof (r=1,2,…, m) in the structure, where $k^{(r)}$ is defined as the row or dof number of the highest magnitude in the *r-th* eigenvector. After constraining these dofs, the positive definiteness of **K** is recovered and Equation (9) can be solved for a particular solution $\mathbf{q}^p$. Then the complete solution to the governing equations can be constructed as

$$\mathbf{q}^2 = \sum_{r=1}^{m} s_r \mathbf{q}^{(r)} + \mathbf{q}^p . \tag{26}$$

In the case of multiple eigenvectors, as opposed to (6), the orthogonality conditions become

$$\left(\mathbf{q}^{(r)}\right)^T \mathbf{T}\mathbf{q}^2 = 0 \qquad (r=1,2,\ldots,m) \tag{27}$$

Given the orthogonality conditions above, constants s_r can be determined as

$$s_r = -\left(\mathbf{q}^{(r)}\right)^T \mathbf{T}\mathbf{q}^p \Big/ \left(\mathbf{q}^{(r)}\right)^T \mathbf{T}\mathbf{q}^{(r)} \qquad (r = 1,2,\ldots m) \tag{28}$$

Substituting the above into (26), one obtains a closed-form solution again

$$\mathbf{q}^2 = \mathbf{q}^p - \sum_{r=1}^{m} \frac{\left(\mathbf{q}^{(r)}\right)^T \mathbf{T}\mathbf{q}^p}{\left(\mathbf{q}^{(r)}\right)^T \mathbf{T}\mathbf{q}^{(r)}} \mathbf{q}^{(r)} . \tag{29}$$

The derivation is equally rigorous as the single mode buckling scenario. Apparently, the single mode case can be considered as a special case of the present multiple mode formulation when m=1. The existence and uniqueness of $\mathbf{q}^2$ can be argued in the same way as that for the single mode buckling case as provided in the previous section.

V. Numerical examples

A. A verification through a manually solvable case

A simple example can be introduced as follows with

$$\mathbf{K}^0 = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}, \quad \mathbf{K}_\sigma^0 = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{f} = \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$$

Substituting them into Equation (4), a single eigenvalue and its corresponding eigenvector can be found as

$$\lambda_c = 1 \quad \text{and} \quad \mathbf{q}^1 = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}^T .$$

Given the magnitudes of the components of $\mathbf{q}^1$, one can easily identify that i=2. Apparently, **f** satisfies (18). The governing equations for $\mathbf{q}^2$ becomes

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} q_1^2 \\ q_2^2 \\ q_3^2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}.$$

Solving $\begin{bmatrix} 2 & -1 & 0 \\ -1 & \infty & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} q_1^p \\ q_2^p \\ q_3^p \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$ one obtains $\mathbf{q}^p = \begin{Bmatrix} 0.5 \\ 0 \\ 0.5 \end{Bmatrix}$.

After determining s according to (23), the complete solution can be obtained from (24) as

$$\mathbf{q}^2 = \frac{1}{3}\begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^T.$$

If the obtained result is substituted back into both the governing equations (5) and the orthogonality condition (6), they are satisfied exactly. The solution has therefore been verified.

In order to draw a comparison with Lanzo's iterative approach [11] to offer an indication of the rate of convergence, results are also obtained from such scheme and those after 6 and 12 iterations are shown as follows, respectively, along with the initial value

$$\mathbf{q}^2_{\text{0-iteration}} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \quad \mathbf{q}^2_{\text{6-iterations}} = \begin{Bmatrix} 0.33325 \\ -0.33325 \\ 0.33325 \end{Bmatrix} \quad \text{and} \quad \mathbf{q}^2_{\text{12-iterations}} = \begin{Bmatrix} 0.33333331 \\ -0.33333331 \\ 0.33333331 \end{Bmatrix}.$$

Each of the latter two could offer an approximation depending on the required accuracy. It is clear that the approximation, no matter how accurate it is, is unnecessary if a closed-form solution can be found, especially if the closed-form solution can be obtained with less computational effort and perfect mathematical rigor.

B. A manually solvable case involving multiple eigenvectors

If one replaces the $\mathbf{K}^0$ matrix as in the previous subsection by the one below

$$\mathbf{K}^0 = \begin{bmatrix} 1 & -2 & 2 \\ -2 & -2 & 4 \\ 2 & 4 & -2 \end{bmatrix}$$

while everything else remains the same, one finds a repeated eigenvalue at

$$\lambda_c^{(1)} = \lambda_c^{(2)} = 2.$$

The eigenvectors associated with the repeated eigenvalue are

$$\mathbf{q}^{(1)} = \begin{bmatrix} 2 & -1 & 0 \end{bmatrix}^T \quad \text{and} \quad \mathbf{q}^{(2)} = \begin{bmatrix} 2 & 4 & 5 \end{bmatrix}^T \quad \text{satisfying} \quad \left(\mathbf{q}^{(1)}\right)^T \mathbf{T}\,\mathbf{q}^{(2)} = 0.$$

Given the magnitudes of the components of the eigenvectors, one can easily identify that the first and the third equations can be eliminated from the governing equations for $\mathbf{q}^2$

$$\begin{bmatrix} -1 & -2 & 2 \\ -2 & -4 & 4 \\ 2 & 4 & -4 \end{bmatrix} \begin{Bmatrix} q_1^2 \\ q_2^2 \\ q_3^2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \\ -2 \end{Bmatrix} \quad \text{to give} \quad \begin{bmatrix} \infty & -2 & 2 \\ -2 & -4 & 4 \\ 2 & 4 & \infty \end{bmatrix} \begin{Bmatrix} q_1^p \\ q_2^p \\ q_3^p \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \\ -2 \end{Bmatrix}$$

leading to

$$\mathbf{q}^p = -\frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T .$$

From (28), one has $s_1 = -1/10$ and $s_2 = 2/45$, and (29) leads to

$$\mathbf{q}^2 = \frac{1}{9}\begin{bmatrix} -1 & -2 & 2 \end{bmatrix}^T$$

which satisfies both the governing equations (5) and the orthogonality conditions (27) and therefore is a solution and the only solution.

C. An examples of structural application

While verifying the derived solution obtained in this paper at a basic level, the simple examples as shown in previous subsections have illustrated the closed nature of the solution as well as the procedure of its application. As a proper verification of the FE implemented Koiter's theory, it will be applied to a structural problem with the results compared with available analytical solution. The in-house developed FE code to facilitate the Koiter's theory and applied in [26] was employed to conduct the FE analysis. As mentioned in the Introduction, such a meaningful comparison had remained one of the obstacles for the wide acceptance of FE implemented Koiter's theory for nearly half a century and it had not been achieved until the publication of [24] as the very first proper attempt of successful verification FE implemented Koiter's theory for its application to plates and shells (in modern FEM, shells have been unified with plates in terms of element formulation). The way of prescribing boundary conditions in the numerical models in [24] followed the approach proposed in [25] maintaining full consistence with those in analytical solution for a simply supported plate. These boundary conditions were also implied in the analytical solution as presented in [27].

<u>(1) The problem to be analysed and the boundary conditions</u>

The problem to be analysed is as sketched in Fig. 2 where a simply supported homogeneous and isotropic plate is subjected to biaxial compression for which an analytical solution is available [27]. Relevant parameters involved in the problem are as shown in Fig. 2. The boundary conditions consistent with the analytical solution [27] for pre-buckling, buckling (1st perturbation) and post-buckling (2nd perturbation) regimes are presented in Table 1. Uniaxial compression as analysed in [11, 26] can be considered as a special case when one of the prescribed membrane forces vanishes.

While the boundary conditions presented in Table 1 are in perfect consistence with analytical solution, why they should be in this form for each of the three phases of analyses for general applications of Koiter's theory remains a subtle problem to be addressed since insufficient details could be found in [24]. As will be established later through numerical result, the key to the reproduction of analytical results using FEM relies on the correct representation of the boundary conditions.

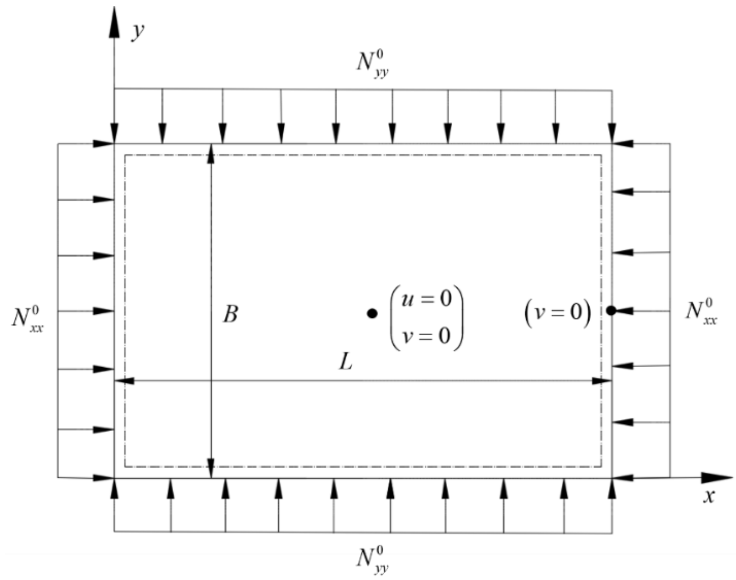


Fig. 2 A simply supported plate under biaxial compression

Table 1 Boundary conditions consistent with the analytical solution and also among all three phases involved

	$x=0\,\&\,L$		$y=0\,\&\,B$	
Nature of BC	Essential	Natural	Essential	Natural
In-plane boundary conditions				
Pre-buckling	$u^0\big\vert_{x=0\text{ or }L}=\pm U^0$	$F_U^0=N_{xx}^0B,\ N_{xy}^0=0$	$v^0\big\vert_{y=0\text{ or }B}=\pm V^0$	$F_V^0=N_{yy}^0L,\ N_{xy}^0=0$
Buckling	$u^1\big\vert_{x=0\text{ or }L}=\pm U^1$	$F_U^1=0,\ N_{xy}^1=0$	$v^1\big\vert_{y=0\text{ or }B}=\pm V^1$	$F_V^1=0,\ N_{xy}^1=0$
Post-buckling	$u^2\big\vert_{x=0\text{ or }L}=\pm U^2$	$F_U^2=0,\ N_{xy}^2=0$	$v^2\big\vert_{y=0\text{ or }B}=\pm V^2$	$F_V^2=0,\ N_{xy}^2=0$
Out-of-plane boundary conditions				
Pre-buckling	$w^0=0$	$M_{xx}^0=0,\ M_{xy}^0=0$	$w^0=0$	$M_{yy}^0=0,\ M_{xy}^0=0$
Buckling	$w^1=0$	$M_{xx}^1=0,\ M_{xy}^1=0$	$w^1=0$	$M_{yy}^1=0,\ M_{xy}^1=0$
Post-buckling	$w^2=0$	$M_{xx}^2=0,\ M_{xy}^2=0$	$w^2=0$	$M_{yy}^2=0,\ M_{xy}^2=0$

The simply supported boundary conditions for the out-of-plane deformation are uniquely defined and do not need any further discussion but included in Table 1 for completeness. The in-plane boundary conditions need some attention. In general, in order to reproduce the analytical result, the FE model should represent the boundary conditions in the analytical solution precisely. The trouble is that the in-plane part of analytical solutions, such as that in [27], were often obtained using the so-called semi-inversed method as in the theory of elasticity [32] based on Airy's stress function where in-plane boundary conditions were not provided explicitly. However, this does not mean that boundary conditions do not exist. One can always obtained the membrane forces from the Airy's stress function. The values of the normal and tangential components of the membrane forces at boundary conditions are the traction boundary conditions the solution satisfies. Alternatively, from the membrane forces, one can work out the strains using the generalised Hooke's law before integrating the strains through the kinematic equations to derive the displacement field. The in-plane displacement boundary conditions can then be obtained as the values of the displacement field at the boundary. Obviously, for the problem to be well-pose mathematically, one should not prescribe all the obtained boundary conditions at the same time, even though they are compatible with each other. Correct prescription of them can be in one of the three forms:

i) Displacement boundary conditions (normal and tangential to the boundary);
ii) Traction boundary conditions (normal and tangential membrane forces);
iii) A logical combination of selected components of the above two, e.g. displacement normal to the boundary and membrane force tangential to the boundary.

Even so, it is sometimes cumbersome to represent the boundary conditions as implied by the analytical solution in a way convenient for FE implementation of Koiter's theory. Some of the values to be prescribed, e.g. the magnitude of SPD, are unknown while the known condition is that the boundary should displace by a constant amount so that the corresponding membrane force resultant vanishes. The success of [25] was to introduce what is conventionally called 'kinematic coupling' [33] that allowed a displacement component along a part of the boundary to be kept the same as that at an additional dof artificially introduced. This was brought into the numerical analysis of Koiter's theory properly in [24]. The magnitude of the displacement at the additional dof can be prescribed as fixed value if it is so wished. Alternatively, it can be left for the concentrated force on that dof to decide. Among the three phases of analysis, the relevant concentrated forces associated with different sides of the boundary should be prescribed non-vanishing values only for pre-buckling analysis. In both buckling and initial post-buckling analyses, the 1st and 2nd order perturbation displacements are independent of load parameter λ and their contributions to the initial post-buckling equilibrium path is through the perturbation parameter ξ according to Koiter's theory. This is equivalent to prescribing zero loads at the additional dofs artificially introduced. In Table 1, U and V are such additional dofs, one for each set of edges in the direction perpendicular to the respective edges. F_U and F_V are the concentrated forces at U and V, respectively. The superscripts 0, 1 and 2 on them indicate the phases of the analysis they are associated with. It should be pointed out that these boundary conditions as shown in Table 1 are of different natures, *viz*. essential and natural, stemming to the variational calculus. Since FEM is based on variational principles, the nature of these boundary conditions matters, as essential boundary conditions have to be imposed and therefore satisfied by the solution strictly. On the other hand, natural boundary conditions in FEM are prescribed in terms of load application. In the case of zero load, no action is required from the user. As they are satisfied in the sense of energy minimisation, their satisfaction is meant to be approximate, as accurate as equilibrium conditions are satisfied in general. An important statement here is that homogeneous natural boundary conditions can be simply left alone while being perfectly correct in FEM in general. Therefore, in the column under 'Natural' boundary conditions, all conditions can be ignored, except for the pre-buckling loading condition.

For all three phases, the in-plane rigid body motions need to be constrained properly. In the present example, they are constrained by fixing both in-plane displacements at centre of the plate and the displacement in y direction on the $x=L$ side. The out-of-plane rigid motions are constrained naturally through the simple support conditions.

(2) Numerical results

The plate under consideration was assumed to be of a breadth of 100mm; the thickness of plate is 1mm; the material is assumed to be homogenous, isotropic and linearly elastic with E=210GPa and ν=0.3; and the 8-noded shell elements were employed with reduced integration. The buckling load and the second initial post-buckling coefficient b have been included in Table 2. Two types of boundary conditions have been prescribed, one consistent with those in the analytical solution as detailed in Table 1 and one denoted 'Free BC' as employed in [11] and confirm in [26], which is inconsistent with those in the analytical solution but obtained intuitively. Converged mesh as that in [26] has been employed. There was no mesh sensitivity as a result of the use of the N-notation [26]. The following measure has been adopted to assess the numerical errors in the obtained results as shown in Table 2.

$$\text{Error in } \mathbf{q}^2 = \underset{\text{all nodes}}{Max}\left\{\sqrt{\left(u_i-u_i^{\mathrm{a}}\right)^2+\left(v_i-v_i^{\mathrm{a}}\right)^2+\left(w_i-w_i^{\mathrm{a}}\right)^2}\right\}\Big/\underset{\text{all nodes}}{Max}\left\{\sqrt{\left(u_i^{\mathrm{a}}\right)^2+\left(v_i^{\mathrm{a}}\right)^2+\left(w_i^{\mathrm{a}}\right)^2}\right\} \quad (30)$$

where $\left(u_i, v_i, w_i\right)$ are nodal displacements of the secondary perturbation at the n^{th} node, and $\left(u_i^{\mathrm{a}}, v_i^{\mathrm{a}}, w_i^{\mathrm{a}}\right)$ the displacements from the analytical solution at the location of the n^{th} node. It is clear as shown in Table 2 that the errors in the present approach are indeed insignificant for all the cases dealt with. It should be noted that the sources of minor numerical errors in the solution obtain resulted from FEM, eigenvalue extraction and solving the simultaneous equations (21) for the particular solution but none from the closed-form solution for the SPD.

Table 2 Results for biaxial compression

Aspect ratio	Load ratio	Buckling load (N/mm)		b			Error in $\mathbf{q}^2$	
		Analytical	Present	Analytical	Present	Free BC	Present	Free BC
0.5	0.2	12.65	12.57	0.3413	0.3434	0.2060	0.316%	28.9%
	1	23.73	23.63	0.4641	0.4652	0.06169	0.243%	110%
	5	5.649	5.625		0.4654	0.06164	0.187%	88.9%
	10	2.893	2.881		0.4654	0.06165	0.184%	88.9%
1.0	0.2	12.65	12.55	0.3413	0.3440	0.1779	0.281%	41.8%
	1	37.96	37.64		0.3440	0.1779	0.258%	41.8%
	5	12.65	12.55		0.3440	0.1779	0.281%	41.8%
	10	6.902	6.843		0.3440	0.1779	0.281%	41.8%
2.0	0.2	5.649	5.625	0.4641	0.4654	0.06164	0.187%	88.9%
	1	23.73	23.62		0.4652	0.06159	0.243%	88.9%
	5	12.65	12.57	0.3413	0.3434	0.2060	0.316%	28.9%
	10	6.902	6.857		0.3434	0.2059	0.316%	28.9%
3.0	0.2	4.585	4.574	0.5597	0.5600	0.01778	0.189%	94.1%
	1	21.09	21.04		0.5596	0.01775	0.275%	94.1%
	5	12.29	12.23	0.3917	0.3932	0.1213	0.708%	51.0%
	10	6.902	6.861	0.3413	0.3413	0.2147	0.778%	20.1%

The results as shown in Table 2 show the reliance of the accuracy of the FEM predictions in comparison with those from the analytical solution on the correct interpretation of boundary conditions. The FEM results become close predictions to the analytical solution if the boundary conditions employed in the FE analysis are consistent with those in, or implied by, the analytical solution. Otherwise, as those presented in [11], they would be a completely different story, not necessarily wrong, but only correct to the boundary conditions employed as confirmed in [26] and one cannot draw any relevance between them and the verification of the FE implemented Koiter's theory against available analytical results. The inability to verify against available analytical solution is an impractical position for any FEM development to be considered as acceptable.

The obtained results and their accuracies in comparison with the analytical results are a convincing verification of the closed-form solution for the SPD, as well as the FE implementation of Koiter's theory for plates and shells.

(3) A structural example of dual mode buckling

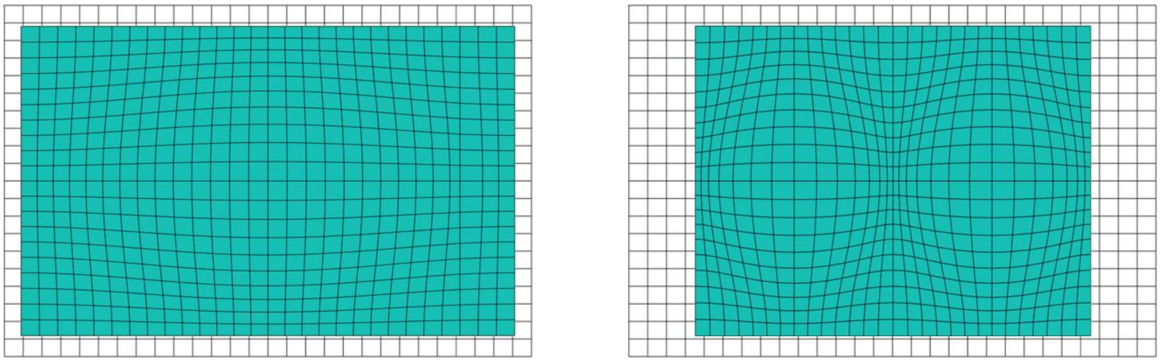
For the same plate as dealt with above, if the aspect ratio of it is set to $\sqrt{2}$, the buckling will be of dual modes. The obtained buckling load as a repeated eigenvalue and the buckling modes from the authors' FE analysis all agree well with the analytical results and hence are not shown here again. Given the two independent buckling modes, there are three sets of SPDs, $\mathbf{q}^{2(1,1)}$, $\mathbf{q}^{2(1,2)}$ and $\mathbf{q}^{2(2,2)}$, corresponding to different expressions on right hand side of (9), i.e.

$$\begin{aligned}
&\left(\mathbf{K}_0+\mathbf{N}_1\left(\mathbf{q}_c^0\right)+\mathbf{N}_2\left(\mathbf{q}_c^0\right)\right)\mathbf{q}^{2(1,1)}=-\frac{1}{2}\mathbf{N}_1\left(\mathbf{q}^{1(1)}\right)\mathbf{q}^{1(1)}-\mathbf{N}_2\left(\mathbf{q}^{1(1)}\right)\mathbf{q}_c^0\\
&\left(\mathbf{K}_0+\mathbf{N}_1\left(\mathbf{q}_c^0\right)+\mathbf{N}_2\left(\mathbf{q}_c^0\right)\right)\mathbf{q}^{2(1,2)}=-\frac{1}{2}\mathbf{N}_1\left(\mathbf{q}^{1(1)}\right)\mathbf{q}^{1(2)}-\frac{1}{2}\mathbf{N}_{11}\left(\mathbf{q}^{1(1)},\mathbf{q}^{1(2)}\right)\mathbf{q}_c^0\\
&\left(\mathbf{K}_0+\mathbf{N}_1\left(\mathbf{q}_c^0\right)+\mathbf{N}_2\left(\mathbf{q}_c^0\right)\right)\mathbf{q}^{2(2,2)}=-\frac{1}{2}\mathbf{N}_1\left(\mathbf{q}^{1(2)}\right)\mathbf{q}^{1(2)}-\mathbf{N}_2\left(\mathbf{q}^{1(2)}\right)\mathbf{q}_c^0
\end{aligned}\tag{31}$$

where $\mathbf{q}^{1(1)}$ and $\mathbf{q}^{1(2)}$ are the two independent buckling modes, respectively. Each of the three independent SPDs as the solutions to the above equations, respectively, is subject to orthogonality conditions

$$\left(\mathbf{q}^{2(i,j)}\right)^T\mathbf{q}^{1(k)}=0 \qquad i,j,k=1,2. \tag{32}$$

For this particular structure, the SPDs obtained happen to involve only in-plane deformation and two of them are plotted in Fig. 3 as demonstrators of the closed-form solution technique as presented in this paper. The undeformed mesh is shown as the regular grids while the shaded patches represent the deformed mesh with the magnitude of the deformation amplified by 1000 times in the plots, corresponding to the buckling modes so normalised that the amplitudes of their deflections are both unity. The obtained solutions agree so well with the analytical ones that they overlap with each other without visible differences while the actual errors are below 1% as the numerical accuracy of the FE mesh allows. To highlight the a characteristic feature of these plots, the two solutions share the common maximum in-plane displacement in the transverse direction. In the loading direction, $\mathbf{q}^{2(2,2)}$ is four times of $\mathbf{q}^{2(1,1)}$ precisely in terms of their maximum values. This is the case because the number of half-waves of $\mathbf{q}^{1(2)}$ doubles that of $\mathbf{q}^{1(1)}$ and these half-wave numbers appear in a bilinear form in the expressions for the right hand sides of the secondary perturbation equations in the analytical solution.



(a) (b)

Fig. 3 Patterns of the SPDs corresponding to (a) the first buckling mode $\mathbf{q}^{2(1,1)}$ alone, and (b) the second buckling mode $\mathbf{q}^{2(2,2)}$ alone

It should be pointed out that the results shown here are merely meant to demonstrate the applicability of the solution technique for SPDs to cases of multiple mode buckling. The third SPDs will be presented in a subsequent publication along with a complete initial post-buckling analysis which falls into a different category, *viz*. mode interaction, and hence beyond the scope of the present paper.

VI Concluding remarks

In this paper, the major obstacles in the FE implementation of Koiter's initial post-buckling theory have been identified. With two of them being resolved recently, the final one, *viz*. the solution for the secondary perturbation displacement (SPD), has been addressed as the subject of this paper, leading to a closed-form solution, whereas an iterative approach had been adopted previously. The difficulty associated with the SPD is the singularity of the coefficient matrix of the governing equations although the singularity can be eliminated by the supplemented orthogonality condition(s). The obtained closed-form solution is based firmly on mathematics and constructed rigorously from a complementary solution and a particular solution as an approach generally applicable to all linear systems. The buckling mode has been identified as a complementary solution. A particular solution can be obtained after constraining the dof corresponding to the component of the maximum magnitude in the buckling mode. The solution as a sum of the complementary and particular solutions can then be uniquely determined after imposing the orthogonality condition. The mathematical rigor has been partially demonstrated through the proof of the existence of a solution to the governing equations for SPD, which has never been made clear in the past, even in the analytical formulation of Koiter's theory. The procedure has been extended to the problem of multiple mode buckling. The solution is not only analytically elegant but also computationally efficient. The procedure has been verified through simple examples where exact solutions can be obtained manually. A structural application has been shown with results compared perfectly with analytical solutions.

As a most important outcome out of this paper, it has shown that all obstacles have been resolved and the FE implemented Koiter's theory is ready to be incorporated in mainstream commercial FE codes. While the outcomes from Koiter's theory have significant impact to the conventional buckling analyses by enhancing the much-needed confidence on their buckling predictions, as well as a useful and simple assessment on the sensitivity to initial imperfections, the computational efforts required after the conventional buckling analysis are almost negligible, requiring no additional resources other than what have been available in the conventional FEM. The required coding is most unlikely to upset any existing functionality in an available FE code.

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References

[1] Nakamura, T. and Uetani, K. 'The secondary buckling and post-secondary-buckling behaviours of rectangular plates', *International Journal of Mechanical Sciences*, vol. 21**,** 1979, pp. 265-286.

[2] Brush, D.O. and Almroth, B.O., *Buckling of bars, plates and shells*, McGraw-Hill Book Co., London, 1975 (Fig. 5.18 on page 186).

[3] Von Karman T. and Tsien, H.S., 'The buckling of thin cylindrical shells under axial compression', *Journal of Aeronautical Sciences*, vol. 8**,** 1941, pp. 303-312.

[4] Hoff, N.J., Madsen, W.A. and Mayers, J. 'Postbuckling equilibrium of axially compressed circular cylindrical shells', *AIAA Journal*, vol. 4, 1966, pp. 126-133.

[5] Almroth, B.O. 'Postbuckling behaviour of axially compressed circular cylinders', *AIAA Journal*, vol. 1, 1963, pp.630-633.

[6] Koiter, W.T. 'On the stability of elastic equilibrium', Thesis, Delft University, Amesterdam, 1945 (in Dutch). English translation: Stanford University 1967, NASA TT F-10, 833; also Dept Aero Astro Report AD 704124 1970, Airforce Flight Dynamics Lab Report TR-70-25, 1945.

[7] Koiter, W.T. 'Elastic stability and post-buckling behavior', *Proc. Symposium on Nonlinear Problems,* University of Wisconsin Press, Madison, 1963, pp. 257-275.

[8] Groh, R.M.J. and Pirrera, A., 'On the role of localizations in buckling of axially compressed cylinders', *Proc. R. Soc. A*, Vol. 475 (20190006), 2019. http://dx.doi.org/10.1098/rspa.2019.0006

[9] Haftka, R.T., Mallett, R.H. and Nachbar, W., 'Adaption of Koiter's method to finite element analysis of snap-through buckling behaviour', *International Journal of Solids and Structures*, vol.7, 1971, pp.1427-1445.

[10] Byskov, E. 'Smooth postbuckling stresses by a modified finite element method', *International Journal for Numerical Methods in Engineering*, vol. 28, 1989, pp.2877-2888.

[11] Lanzo, A.D., Garcea, G. and Casciaro, R. 'Asymptotic post-buckling analysis of rectangular plates by HC finite elements', *International Journal for Numerical Methods in Engineering*, vol.38, 1995, pp.2325-2345.

[12] Poulsen, P.N. and Damkilde, L. 'Direct determination of asymptotic structural postbuckling behaviour by the finite element method', *International Journal for Numerical Methods in Engineering*, vol.42, 1998, pp.685-702.

[13] Garcea, G., Madeo, A., Zagari, G. and Casciaro, R. 'Asymptotic post-buckling FEM analysis using corotational formulation', *International Journal of Solids and Structures*, vol.46, 2009, pp.377–397.

[14] Byskov, E. 'Applicability of an asymptotic expression for elastic buckling problem with mode interaction', *AIAA Journal*, vol.17, 1978, pp.630-633.

[15] Hilburger, M.W., Nemeth, M.P. and Starnes, Jr J.H., 'Shell buckling design criteria based on manufacturing imperfection signatures', *AIAA Journal*, vol.44, 2006, pp.654-663.

[16] Goldfeld, Y. 'Elastic buckling and imperfection sensitivity of generally stiffed conical shells', *AIAA Journal*, vol.45, 2007, pp.721-729.

[17] Khosravi, P. and Sedaghati, R 'Local buckling and mode switching in the optimum design of stiffened panels', *AIAA Journal.*, vol.46, 2008, pp.1542-1548.

[18] Casciaro, R, Salerno G and Lanzo, A. 'Finite element asymptotic analysis of slender elastic structures: a simple approach', *International Journal for Numerical Method in Engineering*, vol.35, 1992, pp.1397-1426.

[19] Garcea, G., Salerno, G., and Casciaro, R. 'Extrapolation locking and its sanitisation in Koiter's asymptotic analysis', *Computer Methods in Applied Mechanics and Engineering*, vol. 180, 1999, pp.137-167.

[20] Garcea, G., Liguori, F., Leonetti, L., Magisano, D. and Madeo, A. 'Accurate and efficient a-posteriori account of geometrical imperfections in Koiter finite element analysis', *International Journal for Numerical Method in Engineering*, vol.112, 2017, pp.1154-1174.

[21] Mania, R.J., Madeo, A., Zucco, G. and Kubiak, T. 'Imperfection sensitivity of post-buckling of FML channel section column', *Thin-walled Structures*, vol.114, 2017, pp.32-38.

[22] Madeo, A., Groh, R.M.J., Zucco, G., Weaver, P.M., Zagari, G. and Zinno, R. 'Post-buckling analysis of variable-angle tow composite plates using Koiter's approach and the finite element method', *Thin-walled Structures*, vol.110, 2017, pp.1-13.

[23] Rahman, T. and Jansen, E.L. 'Finite element based coupled mode initial post-buckling analysis of a composite cylindrical shell', *Thin-walled Structures*, vol.48, 2010, pp.25-32.

[24] Henrichsen, S.R., Weaver, P.M., Lindgaard, E. and Lund, E. 'Post-buckling optimization of composite structures using Koiter's method', *International Journal for Numerical Methods in Engineering*, vol.108, 2016, pp.902-940.

[25] White, S.C., Raju, G. and Weaver, P.M. 'Initial post-buckling of variable-stiffness curved panels', *Journal of the Mechanics and Physics of Solids*, vol.71, 2014, pp.132-155.

[26] Li, S., Yan, J., Zhang, G. and Duan, S. 'Commutativity of the strain energy density expression for the benefit of the FEM implementation of Koiter's initial post-buckling theory', *International Journal for Numerical Methods in Engineering*, vol. 114, 2018, pp.955–974.

[27] Budiansky, B. 'Theory of buckling and post-buckling behaviour of elastic structures', in *Advances in Applied Mechanics*, vol.14, 1974, pp.1-65, Academic Press, Cambridge, Massachusetts, USA.

[28] Kouhia, R. and Menken, C. 'On the solution of second-order post-buckling fields', *Communications in Numerical Methods in Engineering*, vol.11, 1995, pp.443-453.

[29] Bathe, K.-J. and Wilson, E.L. *Numerical Methods in Finite Element Analysis*, Prentice-Hall, New Jersey, USA, 1976.

[30] Lipschutz, S. and Lipson, M. *Linear algebra*, 5th edn., Schaum's outline series, London, 2013.

[31] Washizu, K. *Variational Methods in Elasticity and Plasticity,* Pergamon, Oxford, 1975.

[32] Timoshenko, S.P. and Goodier, J.N. *Theory of Elasticity,* 3rd edn, McGraw Hill, London, 1970.

[33] Abaqus, Abaqus Analysis User's Guide, Abaqus 2016 HTML Documentation, 2016.

Appendix A Matrix T for orthogonality condition (6) and its positive definiteness

The positive definite matrix **T** is conventionally choose as [10-13]

$$\mathbf{T} = -\left(\mathbf{N}_1\left(\dot{\mathbf{q}}_c^0\right) + \mathbf{N}_{11}\left(\mathbf{q}_c^0, \dot{\mathbf{q}}_c^0\right)\right). \tag{A1}$$

where $\mathbf{N}_{11}\left(\mathbf{q}_c^0, \dot{\mathbf{q}}_c^0\right)$ is defined through

$$\mathbf{N}_2\left(\mathbf{q}_c^0 + \dot{\mathbf{q}}_c^0\right) = \mathbf{N}_2\left(\mathbf{q}_c^0\right) + \mathbf{N}_{11}\left(\mathbf{q}_c^0, \dot{\mathbf{q}}_c^0\right) + \mathbf{N}_2\left(\dot{\mathbf{q}}_c^0\right). \tag{A2}$$

It is reasonable to assume that the equilibrium on the fundamental path is stable before reaching the buckling point, i.e.

$$\delta^2\Pi = \delta\mathbf{q}^{\mathrm{T}}\left(\mathbf{K}_0 + \mathbf{N}_1\left(\mathbf{q}^0\right) + \mathbf{N}_2\left(\mathbf{q}^0\right)\right)\delta\mathbf{q} > 0 \qquad \text{for } \lambda < \lambda_c \text{ and } \delta\mathbf{q} \neq \mathbf{0}. \tag{A3}$$

Taylor expand $\mathbf{q}^0$ at $\lambda = \lambda_c$ as in (3) and truncated at the linear term as

$$\begin{aligned} \delta^2\Pi \approx\ & \delta\mathbf{q}^{\mathrm{T}}\left(\mathbf{K}_0 + \mathbf{N}_1\left(\mathbf{q}_c^0\right) + \mathbf{N}_2\left(\mathbf{q}_c^0\right)\right)\delta\mathbf{q} \\ & + \left(\lambda - \lambda_c\right)\delta\mathbf{q}^{\mathrm{T}}\left(\mathbf{N}_1\left(\dot{\mathbf{q}}_c^0\right) + \mathbf{N}_{11}\left(\mathbf{q}_c^0, \dot{\mathbf{q}}_c^0\right)\right)\delta\mathbf{q} > 0. \end{aligned} \tag{A4}$$

Given the criterion for bucking, the first term on the right hand side of the above is semi-positive definite. For any $\lambda < \lambda_c$, the 2nd term on the right hand side of (A4) must be positive definite to keep (A3) satisfied, i.e.

$$-\delta\mathbf{q}^{\mathrm{T}}\left(\mathbf{N}_1\left(\dot{\mathbf{q}}_c^0\right) + \mathbf{N}_{11}\left(\mathbf{q}_c^0, \dot{\mathbf{q}}_c^0\right)\right)\delta\mathbf{q} > 0 \qquad \text{for } \delta\mathbf{q} \neq \mathbf{0}. \tag{A5}$$

Thus, matrix $-\left(\mathbf{N}_1\left(\dot{\mathbf{q}}_c^0\right) + \mathbf{N}_{11}\left(\mathbf{q}_c^0, \dot{\mathbf{q}}_c^0\right)\right)$ is positive definite.

Appendix B Satisfaction of (18) in the case of $a \neq 0$

In presence of non-vanishing *a*, there will be extra terms in the expression of **f** [9], given as in the present notations

$$\mathbf{f} = -a\lambda_c\left(\mathbf{N}_1\left(\dot{\mathbf{q}}_c^0\right) + \mathbf{N}_{11}\left(\mathbf{q}_c^0, \dot{\mathbf{q}}_c^0\right)\right)\mathbf{q}^1 - \frac{1}{2}\mathbf{N}_1\left(\mathbf{q}^1\right)\mathbf{q}^1 - \mathbf{N}_2\left(\mathbf{q}^1\right)\mathbf{q}_c^0 \, . \tag{B1}$$

Accordingly, the expression for *b* will also be slightly different from that given in (8) [9] without being further pursue in this paper. Given the expression of *a* as given in (7), one has

$$\mathbf{f} = \frac{1}{2}\frac{\left(\mathbf{q}^1\right)^T\mathbf{N}_1\left(\mathbf{q}^1\right)\mathbf{q}^1 + 2\left(\mathbf{q}^1\right)^T\mathbf{N}_2\left(\mathbf{q}^1\right)\mathbf{q}_c^0}{\left(\mathbf{q}^1\right)^T\mathbf{N}_1\left(\mathbf{q}^1\right)\dot{\mathbf{q}}_c^0 + 2\left(\mathbf{q}_c^0\right)^T\mathbf{N}_2\left(\mathbf{q}^1\right)\dot{\mathbf{q}}_c^0}\left(\mathbf{N}_1\left(\dot{\mathbf{q}}_c^0\right) + \mathbf{N}_{11}\left(\mathbf{q}_c^0, \dot{\mathbf{q}}_c^0\right)\right)\mathbf{q}^1 - \frac{1}{2}\mathbf{N}_1\left(\mathbf{q}^1\right)\mathbf{q}^1 - \mathbf{N}_2\left(\mathbf{q}^1\right)\mathbf{q}_c^0 . \tag{B2}$$

One can readily obtain

$$\mathbf{f}^T\mathbf{q}^1 = \frac{1}{2}\left(\left(\mathbf{q}^1\right)^T\mathbf{N}_1\left(\mathbf{q}^1\right)\mathbf{q}^1 + 2\left(\mathbf{q}^1\right)^T\mathbf{N}_2\left(\mathbf{q}^1\right)\mathbf{q}_c^0\right) - \frac{1}{2}\left(\mathbf{q}^1\right)^T\mathbf{N}_1\left(\mathbf{q}^1\right)\mathbf{q}^1 - \left(\mathbf{q}^1\right)^T\mathbf{N}_2\left(\mathbf{q}^1\right)\mathbf{q}_c^0 = 0 \tag{B3}$$

where use has been made of the commutativity of $\mathbf{N}_1$ and $\mathbf{N}_{11}$, i.e.

$$\left(\mathbf{q}^1\right)^T\mathbf{N}_1\left(\dot{\mathbf{q}}_c^0\right)\mathbf{q}^1 = \left(\mathbf{q}^1\right)^T\mathbf{N}_1\left(\mathbf{q}^1\right)\dot{\mathbf{q}}_c^0 \quad \text{and} \quad \left(\mathbf{q}^1\right)^T\mathbf{N}_{11}\left(\mathbf{q}_c^0, \dot{\mathbf{q}}_c^0\right)\mathbf{q}^1 = \left(\mathbf{q}_c^0\right)^T\mathbf{N}_{11}\left(\mathbf{q}^1, \mathbf{q}^1\right)\dot{\mathbf{q}}_c^0 \tag{B4}$$

and the relationship

$$\mathbf{N}_{11}\left(\mathbf{q}^1, \mathbf{q}^1\right) = 2\mathbf{N}_2\left(\mathbf{q}^1\right). \tag{B5}$$