Bifurcation Problems with Octahedral Symmetry

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Thesis presented for degree of Ph.D.

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July, 1987.

Contents

Acknowledgement

I should like to thank both my supervisor Ian Stewart for many helpful suggestions and comments, and for his encouragement throughout, and Mark Roberts for bringing to my attention several possible applications including the one in this thesis.

I am grateful to CTC Wall for pointing out a clearer proof of Theorem 2.3.4 in a more natural setting.

My thanks also to SERC for my grant.

Declaration

Except where explicitly stated to the contrary, the results in this thesis are the result of original work of the author.

It is intended to publish the contents of this thesis as three papers:

1. A singularity theory analysis of bifurcation problems with octahedral symmetry.

    (Accepted by: Dynamics and Stability of Systems.)

2. The recognition problem for equivariant singularities.

    (Submitted to: Nonlinearity.)

3. Classification of bifurcation problems with octahedral symmetry.

    (In preparation.)

Summary

We analyse local bifurcation problems with octahedral symmetry using results from singularity theory. The thesis is split up into three sections. §1 comprises the bifurcation theory, and §3 contains a full singularity theory classification up to topological codimension one. The classification relies heavily upon new results about the recognition problem. These results are presented in §2 together with several examples drawn from equivariant bifurcation theory. These examples illustrate the new methods more clearly than the work in §3.

In §1 we look at nondegenerate bifurcation problems equivariant with respect to the standard action of the octahedral group on $\mathbb{R}^3$. We find three branches of symmetry-breaking bifurcation corresponding to the three maximal isotropy subgroups of the symmetry group with one-dimensional fixed-point subspaces. Locally, one of these branches is never stable, but precisely one of the other branches is stable if and only if all three branches bifurcate supercritically.

In §2 we simplify the recognition problem by decomposing the group of equivalences into a unipotent group and a group of matrices. Building upon results of Bruce, du Plessis & Wall, we show that in many cases the unipotent problem can be solved by just using linear algebra. We give a necessary and sufficient condition for this, namely that the tangent space be invariant under unipotent equivalence. In addition we develop methods for checking whether the tangent space is invariant.

The classification theorem in §3 gives a list of seven normal forms together with recognition problem solutions and universal unfoldings. Certain anomalies arise when comparing these results with those in §1. We reconcile the anomalies by giving a qualitative classification in addition to the standard classification. An application to barium titanate crystals is considered briefly.

Introduction.

In this thesis, we apply the methods of singularity theory to study the local bifurcations of steady-state solutions to a three-dimensional system of equations in the presence of a group of symmetries, namely that of the cube. Many of the techniques required are those developed by Golubitsky & Schaeffer [1979a,b]. In these papers many explicit examples are considered: n-dimensional systems with no symmetry, the line with $\mathbb{Z}_2$ acting as reflections (see also Golubitsky & Langford [1981]), and the plane under actions of $\mathbb{Z}_2$ with one-dimensional fixed point set (see Armbruster, Dangelmayr & Güttinger [1985]), $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $O(2)$. Subsequently the actions of the family of symmetry groups of the n-gon D_n on $\mathbb{R}^2$ have been studied (see Buzano, Geymonat & Poston [1985]). These investigations essentially exhaust the possibilities for actions on $\mathbb{R}$ and $\mathbb{R}^2$. The natural next step is to look at the group of symmetries of a three-dimensional solid, giving an irreducible action on $\mathbb{R}^3$. We have selected the cube, for which there is a particularly easy choice of coordinates. In addition, the action is *absolutely irreducible* (the only linear maps commuting with the action are real multiples of the identity) and this simplifies the analysis.

Apart from the mathematical naturality of the cube, there is also the question of applications. Many physical phenomena may be modelled by an idealisation with cubic symmetry. Then the results in this paper would go some way towards predicting the qualitative behaviour of steady state solutions. For example, it should be possible to apply our results to elastic deformations of a cube. Indeed, Ball & Schaeffer [1983] have already looked at a problem in elasticity where cubic symmetry is assumed. However, they make further assumptions which first factor out the reflectional symmetry of the cube and then reduce the system of equations from three dimensions to two dimensions. They are then left with D_3 acting on $\mathbb{R}^2$ and are able to call on the results of Golubitsky & Schaeffer [1982].

Our aim here is to set up the mathematical generalities of bifurcation with octahedral symmetry, and applications are not emphasised. However, in §3 we indicate how the results might be applied to model the phenomenological changes in the crystal form of barium titanate as temperature is varied, Devonshire [1949].

The traditional name for the symmetry group of the cube is the *octahedral group* because it is also the symmetry group of the octahedron, the dual of the cube. We shall be interested throughout in the standard absolutely irreducible representation of $\mathbb{O}$ as the symmetry group of the cube acting on $\mathbb{R}^3$ by orthogonal transformations and acting trivially on all other variables (for example, λ in (2) below). The group $\mathbb{O}$ has 48 elements and is generated by

$$\kappa_{x_1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; R_{x_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \; R_{x_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

These elements represent reflection in the $x_1=0$ plane and $\pi/2$ rotations about the x_1- and x_2-axes respectively. The symmetric group S_3 is a subgroup of $\mathbb{O}$ and other elements include $\kappa_{x_2}, \kappa_{x_3}, R_{x_3}$ with the obvious matrix representations. Note that we are including reflectional symmetries of the cube. As mentioned above, Ball & Schaeffer [1983] use a different representation of $\mathbb{O}$, on $\mathbb{R}^2$, with kernel $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Consider the steady state solutions of the system of ODEs

$$\dot{x} + g(x,\lambda) = 0, \qquad (1)$$

where λ is a distinguished real bifurcation parameter, $x=(x_1, x_2, x_3) \in \mathbb{R}^3$ and $g : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ is a smooth map-germ at 0 commuting with the symmetry group of the cube $\mathbb{O}$, that is, satisfying

$$g(\gamma x,\lambda) = \gamma g(x,\lambda) \text{ for all } \gamma \in \mathbb{O}. \qquad (2)$$

Such g are said to be $\mathbb{O}$-*equivariant*. We denote by $\vec{\mathcal{E}}_{x,\lambda}(\mathbb{O})$ the set of all $\mathbb{O}$-equivariant mappings g. The set $\vec{\mathcal{E}}_{x,\lambda}(\mathbb{O})$ is a module over $\mathcal{E}_{x,\lambda}(\mathbb{O})$, the ring of all $\mathbb{O}$-*invariant* C^∞ function-germs f at 0; that is, those f satisfying $f(\gamma x,\lambda) = f(x,\lambda)$ for all $\gamma \in \mathbb{O}$. (See Golubitsky, Stewart & Schaeffer [1988].)

A *bifurcation problem with* $\mathbb{O}$*-symmetry* is an equation $f(x,\lambda) = 0$ where $f \in \vec{\mathcal{E}}_{x,\lambda}(\mathbb{O})$ and $(d_x f)_0 = 0$. Clearly, if g in (1) satisfies $(d_x g)_0 = 0$ or in other words has a *singularity* at the origin, then the steady state solutions of (1) define a bifurcation problem with $\mathbb{O}$-symmetry. Note that since we work with germs, the entire analysis is local.

Following the ideas introduced in Golubitsky & Schaeffer [1979a,b] we apply singularity-theoretic methods to analyse the qualitative nature of bifurcation problems with $\mathbb{O}$-symmetry. Two germs g and h in $\vec{\mathcal{E}}_{x,\lambda}(\mathbb{O})$ are said to be $\mathbb{O}$*-equivalent* if there exist smooth germs at 0

$$S : (\mathbb{R}^4,0) \to \mathcal{L}(\mathbb{R}^3,\mathbb{R}^3),\ X : (\mathbb{R}^4,0) \to \mathbb{R}^3,\ \Lambda : (\mathbb{R},0) \to \mathbb{R}$$

such that

$$h(x,\lambda) = S(x,\lambda)g(X(x,\lambda),\Lambda(\lambda)), \qquad (3)$$

$$X(0) = 0,\ \Lambda(0) = 0, \qquad (4)$$

$$S(0) = \mu I,\ (d_x X)_0 = \nu I,\ \mu,\nu > 0,\ \Lambda'(0) > 0, \qquad (5)$$

$$X(\gamma x,\lambda) = \gamma(x,\lambda),\ \gamma^{-1} S(\gamma x,\lambda)\gamma = S(x,\lambda) \text{ for all } \gamma \in \mathbb{O}. \qquad (6)$$

Here $\mathcal{L}(\mathbb{R}^3,\mathbb{R}^3)$ is the space of linear maps $\mathbb{R}^3 \to \mathbb{R}^3$. This definition is analogous to that of contact equivalence in singularity theory, but the purely λ dependence of Λ in (3) preserves the special nature of the distinguished parameter, whilst (6) ensures that h is $\mathbb{O}$-equivariant precisely when g is. The sign conditions (5) are a special case of a refinement of the original definition put forward by Golubitsky & Schaeffer; see Golubitsky, Stewart & Schaeffer [1988]. They are imposed in order to preserve the asymptotic stabilities of solutions. In general this is not always possible, but for a group acting absolutely irreducibly the conditions reduce to those in (5) and the asymptotic stabilities of many solutions are preserved. We say that x_0, λ_0 is *linearly stable* if every eigenvalue of $(dg)_{x_0,\lambda_0}$ has positive real part, and *linearly unstable* if at least one eigenvalue of $(dg)_{x_0,\lambda_0}$ has a negative real part. Provided none of the eigenvalues lie on the imaginary axis, linear stability is a necessary and sufficient condition for asymptotic stability.

Notice that the condition on X in (6) just says that $X \in \vec{\mathcal{E}}_{x,\lambda}(\mathbf{O})$. We denote by $\overleftrightarrow{\mathcal{E}}_{x,\lambda}(\mathbf{O})$ the $\mathcal{E}_{x,\lambda}(\mathbf{O})$-module of all smooth matrix-valued germs at 0 satisfying the condition on S in (6). Finally $\mathcal{E}_\alpha$ is just the ring of C^∞ function-germs at 0 in the variable(s) α.

In §1 we obtain nondegeneracy conditions under which we can predict the directions and stabilities of branching from the trivial solution $x = 0$ for a bifurcation problem $g \in \vec{\mathcal{E}}_{x,\lambda}(\mathbf{O})$. Our results are consistent with a theorem of Vanderbauwhede [1982] and Cicogna [1981], the Equivariant Branching Lemma. For $x \in \mathbb{R}^3$ we define the *isotropy subgroup* Σ_x of $\mathbf{O}$ to be

$$\Sigma_x = \{\gamma \in \mathbb{O} \mid \gamma x = x\},$$

and its *fixed-point subspace* $Fix(\Sigma_x)$ to be

$$Fix(\Sigma_x) = \{y \in \mathbb{R}^3 \mid \gamma y = y \text{ for all } \gamma \in \Sigma_x\}.$$

Then subject to certain hypotheses, the Equivariant Branching Lemma states that corresponding to each isotropy subgroup with one-dimensional fixed-point subspace there exists locally a unique branch of solutions with the symmetry of that subgroup. It turns out that there are three conjugacy classes of isotropy subgroups of $\mathbb{O}$ with one-dimensional fixed-point subspaces. The sign conditions in (5) ensure that the stabilities of the three corresponding branches and the trivial solution are indeed preserved by $\mathbb{O}$-equivalences. Assuming the trivial solution to be stable subcritically (in order to normalize signs), we show that of the three guaranteed branches, one is never stable and one of the others is stable only if all three branches bifurcate supercritically. Further, in the situation where all three branches are supercritical, it is one non-vanishing coefficient in the Taylor expansion that determines stabilities. This same coefficient ensures that no eigenvalue of dg evaluated on a branch has a vanishing real part so that linearised stability is a necessary and sufficient condition for asymptotic stability. Finally we use the condition that this coefficient is non-zero to prove that no other branches are possible.

In §2 we widen our field of study to the situation where Γ is any compact Lie group acting on $\mathbb{R}^n$. Definitions (1) and (2) are the same but with $\mathbb{R}^3$ replaced by $\mathbb{R}^n$ and $\mathbb{O}$ replaced by Γ. Similarly we define Γ-equivalences in a way analogous to (3) - (6). The Γ-equivalences (S, X, Λ) form a group $\mathfrak{D}(\Gamma)$. Since Golubitsky & Schaeffer [1979a,b] introduced the idea of applying singularity-theoretic methods to the study of equivariant bifurcation problems, many authors have produced classifications up to some codimension in a given context. These classifications include the following three components:

(i) A list of normal forms, with the property that all bifurcation problems up to the given codimension are equivalent to precisely one normal form.

(ii) The universal unfolding of each normal form.

(iii) The solution to the recognition problem for each normal form.

The *recognition problem* is one of the least explored facets of singularity theory and it is with this third component that we deal in this thesis. We are interested in knowing precisely when a bifurcation problem is equivalent to a given normal form. Hence we must find a characterisation of the orbit of the normal form under the group of equivalences $\mathfrak{D}(\Gamma)$. This problem can often be reduced to one of finite dimensions via a key idea from singularity theory; that of *finite* *determinacy*. Many smooth map-germs are determined up to Γ-equivalence by finitely many coefficients in their Taylor expansion. Modulo other *high order terms* $\mathfrak{D}(\Gamma)$ acts as a Lie group. It is well known that the orbits under the resulting Lie group are semialgebraic sets, so we can characterise the orbit as comprising those germs whose Taylor coefficients satisfy a finite number of polynomial constraints in the form of equalities and inequalities. This characterisation is the solution to the recognition problem.

We will always assume that the bifurcation problems under discussion are finitely determined. Indeed, finite codimension implies finite determinacy, and so for the purpose of classifying bifurcation problems up to low codimension, this assumption is always valid. The next step is to discover precisely which terms are high order terms. Gaffney [1986] uses results from Bruce, du Plessis & Wall [1985] in providing the answer to this problem. However an additional assumption is required, namely that $\mathfrak{D}(\Gamma)$ acts linearly. The group of (contact) equivalences used in studying bifurcation problems does indeed act linearly and the results in this thesis require the same assumption. In fact, the linearity of the group action is the key hypothesis in our results which hold equally well for the recognition problem under right equivalence and contact equivalence in classical singularity theory.

Because of the Lie group structure of $\mathfrak{D}(\Gamma)$, we can speak of the *tangent space* to the orbit of a bifurcation problem f, or the *Lie algebra* at f

$$T(f,\mathfrak{D}(\Gamma)) = L\mathfrak{D}(\Gamma).f = \{ \tfrac{d}{dt}(\delta_t f)|_{t=0} \mid \delta_t \in \mathfrak{D}(\Gamma),\ \delta_0 = 1 \}. \qquad (7)$$

Most of the low codimension classifications in the literature have been performed in the presence of a group of symmetries Γ acting absolutely irreducibly. Such classifications include bifurcation problems in one state variable with no symmetry up to codimension seven (Keyfitz [1986]) and with $\mathbb{Z}_2$-symmetry up to codimension three (Golubitsky & Schaeffer [1984]),and in two state variables with D_4-symmetry up to topological codimension two (Golubitsky & Roberts [1986]). We tackle the case of three state variables with $\mathbb{O}$-symmetry up to topological codimension one in §3 of this thesis. Apart from these, the most exhaustive classification in the literature is that performed by Dangelmayr & Armbruster [1983] who consider an action of $\mathbb{Z}_2$ on $\mathbb{R}^2$ which is not irreducible. They go up to codimension four.

It is shown in §2 that provided Γ acts absolutely irreducibly, then the group of equivalences $\mathfrak{D}(\Gamma)$ can be decomposed into a group $\mathcal{U}(\Gamma)$ of equivalences whose linear parts are the identity and a group $\mathcal{S}(\Gamma)$ of linear equivalences (which hence must be scalar multiples of the identity). We refer to these as the group of *unipotent* equivalences and the group of *scalings* and define the *unipotent tangent space* $T(f, \mathcal{U}(\Gamma))$ in an analogous way to $T(f,\mathcal{D}(\Gamma))$ in (7).

Examination of the solutions of the recognition problem in the aforementioned classifications leads to the following observations:

(1) Calculating the effect of the scalings alone is easy, although the results look complicated and are often very nonlinear.

(2) If we consider the recognition problems with respect to unipotent equivalences alone, the solutions consist only of equalities.

(3) In many cases, these equalities are linear.

(4) The linearity of these equalities is usually disguised when the effect of the scalings are included.

The following remarks on these observations are in order:

(1) If Γ does not act absolutely irreducibly then it is possible for the effect of the linear equivalences to be rather complicated (for example, two state variable problems with no symmetry, Golubitsky & Schaeffer [1984]). This complexity does not occur provided linear equivalences are forced by the action of Γ to be diagonal matrices. In this thesis we study only such examples.

(2) This property is in fact always true and is stated algebraically in Proposition 3.3 of Bruco, du Plessis & Wall [1985] and Theorem 2.2.2(a) of this thesis.

(3) The main result of §2, Theorem 2.3.4, gives a necessary and sufficient

condition for this property of *linear determinacy* to hold. The condition is that $T(f, U(\Gamma))$ should be invariant under $U(\Gamma)$. In this case the orbit of f under $U(\Gamma)$ is simply the affine space $f + T(f, U(\Gamma))$.

(4) In the light of the examples in this thesis, it seems reasonable to solve the unipotent part of a recognition problem separately, whether or not the bifurcation problem is linearly determined.

In §3 we return to the setting of §1 and, with the aid of §2, perform the classification of bifurcation problems with octahedral symmetry up to topological codimension 1. The classification consists of seven normal forms together with their recognition problem solutions and universal unfoldings. We would expect that one normal form would encapsulate the nondegenerate bifurcation problems of §1 and that the remaining normal forms would reflect each of the possible degeneracies. It turns out, however, that we need two infinite families of germs in order to represent all $\mathbb{O}$-orbits of nondegenerate bifurcation problems. These families are

$$g_m = (\delta m(x^2+y^2+z^2)+\varepsilon\lambda+\sigma(x^2+y^2+z^2)^2)\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \delta \begin{pmatrix} x^3 \\ y^3 \\ z^3 \end{pmatrix},$$

$$h_m = (\delta m(x^2+y^2+z^2)+\varepsilon\lambda)\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \delta \begin{pmatrix} x^3 \\ y^3 \\ z^3 \end{pmatrix},$$

where $\delta, \varepsilon, \sigma = \pm 1$, $m \neq -1, -\frac{1}{2}, -\frac{1}{3}$.

Now, the results of §1 show that the different assignments for δ and ε

do indeed lead to different branching directions and stabilities. This is also true of the interval

$$(-\infty,-1),\ (-1,-\tfrac{1}{2}),\ (-\tfrac{1}{2},-\tfrac{1}{3}),\ (-\tfrac{1}{3},+\infty),$$

in which m lies. According to §1, these possibilities should give rise to all distinct *qualitative* behaviour. The local branching and stability does not depend on the precise value of m. However m is a *modal parameter* and as such is invariant under $\mathbb{D}$-equivalence.

We also have the parameter σ whose sign (positive, negative or zero) is invariant under $\mathbb{D}$, and yet which again has no qualitative relevance. The smooth singularity theory suggests that h_m is more degenerate than g_m but for the purposes of bifurcation theory they are the same. Unfortunately, there is at present no good mathematical theory for studying qualitative equivalence of bifurcation problems; even a slight weakening of the smoothness properties of the equivalences throws away large amounts of structure. Nevertheless, within the bounds of our low codimension classification we are able to deal with qualitative considerations simply by inspection of the bifurcation diagrams.

Hence §3 consists of both a smooth and a qualitative classification. Under the latter, the modal families g_m and h_m collapse into one family. The recognition problems are correspondingly more straightforward since much of the fine detail can now be omitted.

The calculations are similarly simplified in the application to the changes in structure of barium titanate crystals with temperature. Devonshire [1949], noted that as temperature is decreased from above 120°C, the structure of a barium titanate crystal undergoes successive changes from one having the full group of symmetries of the cube to three structures with less symmetry. These states are referred to in the Physics literature as *cubic*, *tetragonal*, *orthorhombic* and *rhombohedral* respectively, the last three corresponding to the three conjugacy classes of isotropy subgroups of $\mathbb{O}$ with one-dimensional fixed-point subspaces. Our results in §1 say that the orthorhombic state cannot be stable locally, but this is not a contradiction since we do not preclude the possibility of stability away from the origin. By analysing the unfolding of a suitably degenerate normal form, we are able to reproduce exactly the scenario described above.

§1. Bifurcations in the Presence of Octahedral Symmetry.

In this section we consider the general bifurcation problem with the symmetry group of the cube, $g \in \vec{\mathcal{E}}_{x,y,z,\lambda}(\mathbb{O})$ and show that generically there are three different types of solution to $g = 0$ branching from the trivial solution $x = y = z = 0$ at the origin. The symmetry of a solution (x, y, z) is defined in terms of its *isotropy subgroup*

$$\Sigma_{x,y,z} = \{\gamma \in \mathbb{O} \mid \gamma(x,y,z) = (x,y,z)\},$$

which is a subgroup of $\mathbb{O}$. An isotropy subgroup H is called *maximal* if H is a proper subgroup of $\mathbb{O}$ and the only isotropy subgroups of $\mathbb{O}$ containing H are $\mathbb{O}$ and H themselves.

The trivial solution has the full symmetry group of the cube so that $\Sigma_{0,0,0} = \mathbb{O}$. However, in accordance with a general phenomenon called *spontaneous symmetry breaking*, each branch of solutions corresponds to a proper isotropy subgroup of $\mathbb{O}$, and so has less symmetry. Furthermore the three isotropy subgroups of these solutions turn out to be the three maximal isotropy subgroups. Thus the losses of symmetry are in some sense the least possible. This situation is fairly general though examples of submaximal isotropy subgroups with generic branches of solutions can be found in Chossat [1983] and Lauterbach [1986].

We follow the standard procedure (see Golubitsky [1983], Golubitsky, Stewart & Schaeffer [1988]) of finding the lattice of isotropy subgroups. and seeing whether branches of solutions exist for each isotropy subgroup. In fact a result of Vanderbauwhede [1982] and Cicogna [1981], the Equivariant Branching Lemma, guarantees under certain hypotheses the existence of a unique branch corresponding to each maximal isotropy subgroup. We impose nondegeneracy conditions on g enabling us to decide whether each branch bifurcates subcritically or supercritically (that is whether the branch of solutions exists for λ less than or greater than zero). A further nondegeneracy condition allows us to determine stabilities, and we use this condition to show that no nondegenerate branching other than that guaranteed by the Equivariant Branching Lemma, is possible locally.

In §1.1 we give the lattice of conjugacy classes of isotropy subgroups of $\mathbb{O}$ together with their fixed-point subspaces. Then a simplified form for an $\mathbb{O}$-equivariant bifurcation problem is found in §1.2. This simplifies further on fixed-point subspaces and we are able to solve the branching equations in §1.3. We also compute the stabilities.

Fig. 1.3.1 illustrates eight of the possible bifurcation diagrams for a nondegenerate bifurcation problem. There are in fact sixteen distinct diagrams in all but we draw only those in which the trivial solution is stable subcritically and unstable supercritically. These are the diagrams of interest in applications. We see that there exists a stable branch if and only if all three branches bifurcate supercritically. However, there is one maximal isotropy subgroup for which the corresponding branch is never stable.

§1.1. The Octahedral Group and Lattice of Isotropy Subgroups.

In this section we obtain the lattice of isotropy subgroups of $\mathbb{O}$. This is a standard part of the procedure for analysing equivariant bifurcation problems, see Golubitsky [1983] and Golubitsky, Stewart & Schaeffer [1988]. First, we give a brief review of the approach.

Suppose we have an equivariant map-germ $g \in \vec{\mathcal{E}}_{x,y,z,\lambda}(\mathbb{O})$. Then

$$g(\gamma(x,y,z),\lambda) = \gamma g(x,y,z,\lambda).$$

Hence, given g at (x,y,z,λ), we know the value of g at $(\gamma(x,y,z),\lambda)$ for all $\gamma \in \mathbb{O}$. In other words, the $\mathbb{O}$-orbit of g is determined by the value of g on a representative of that orbit. Furthermore, solutions to $g = 0$ come in orbits: if the value of g on an orbit representative is zero then g is zero on the whole orbit.

The *isotropy subgroup* $\Sigma_{x,y,z,\lambda}$ of a solution (x,y,z,λ) is given by

$$\Sigma_{x,y,z,\lambda} = \{\gamma \in \mathbb{O} \mid \gamma(x,y,z) = (x,y,z)\}.$$

It is an easy calculation to see that

$$\Sigma_{\gamma(x,y,z),\lambda} = \gamma \Sigma_{x,y,z,\lambda} \gamma^{-1}. \qquad (1)$$

We have seen that solutions to $g = 0$ come in orbits. It follows from (1) that each solution has isotropy subgroup conjugate to that of its orbit representative.

The *fixed-point subspace* of an isotropy subgroup Σ is given by

$$\mathrm{Fix}(\Sigma) = \{(x,y,z) \in \mathbb{R}^3 \mid \sigma(x,y,z)=(x,y,z) \text{ for all } \sigma \in \Sigma\}.$$

An easy but fundamental fact is that

$$g(\mathrm{Fix}(\Sigma)\times\mathbb{R}) \subset \mathrm{Fix}(\Sigma), \tag{2}$$

since

$$\sigma g(x,y,z,\lambda) = g(\sigma(x,y,z),\lambda) = g(x,y,z,\lambda)$$

for all $(x,y,z) \in \mathrm{Fix}(\Sigma)$, $\sigma \in \Sigma$.

Our strategy is to obtain a list of orbit representatives and to find the isotropy subgroup of this representative and the fixed-point subspace of the isotropy subgroup. We can then find zeroes of g by restricting g to orbit representatives. Simultaneously we know the symmetry of the solution.

Table 1.1.1 lists the different orbit representatives, the isotropy subgroup of $\mathbb{O}$ fixing that representative, and the subspace of $\mathbb{R}^3$ fixed by the isotropy subgroup. The entries in Table 1.1.1 are easy to verify. Elements of $\mathbb{O}$ can only permute and/or change signs of the x,y,z variables. Thus we have ordered the variables so that the nonzero elements come first and elements of the same magnitude are grouped together and equal. We could have insisted that all elements were nonnegative and in descending order of magnitude but this is no simpler and actually leads to an extra case to consider: case (f) would split up into (x,x,z) and (x,y,y).

Table 1.1.1. Isotropy Subgroups for $\mathbb{O}$ and their Fixed-Point Subspaces.

	Orbit Rep	Isotropy Subgroup	Fixed-Point Space
(a)	(0,0,0)	$\mathbb{O}$	(0,0,0)
(b)	$(x,0,0)$	D_4 (generated by R_x, κ_z)	$(x,0,0)$
(c)	$(x,x,0)$	$\mathbf{Z}_2^r \oplus \mathbf{Z}_2^t$	$(x,x,0)$
(d)	(x,x,x)	S_3	(x,x,x)
(e)	$(x,y,0)$	$\mathbf{Z}_2^r = \{1,\kappa_z\}$	$(x,y,0)$
(f)	(x,x,z)	$\mathbf{Z}_2^t = \{1,(12)\}$	(x,x,z)
(g)	(x,y,z) $\lvert x\rvert,\lvert y\rvert,\lvert z\rvert$ distinct and non-zero	$\mathbb{1}$	(x,y,z) $x,y,z \in \mathbb{R}$

We have used r and t in $\mathbf{Z}_2^r$ and $\mathbf{Z}_2^t$ to denote reflection and transposition respectively. In Fig. 1.1.1, we sketch the isotropy subgroups with one-dimensional fixed-point subspaces, to show that they are geometrically very natural. Fig. 1.1.2 illustrates the lattice of isotropy subgroups up to conjugacy. In this lattice, $A \subset B$ if a member of the conjugacy class of A is a subgroup of B. The inclusions are all trivial. The most difficult thing to check in Fig. 1.1.2 is that $\mathbf{Z}_2^r$ is not included in S_3. However, let $K = \{\kappa_x, \kappa_y, \kappa_z\}$. Then it is easy enough to check that

$$\gamma K \gamma^{-1} \subset K, \tag{3}$$

for $\gamma = \kappa_x$, R_x and R_y. But these generate $\mathbb{O}$ and so (3) holds for all $\gamma \in \mathbb{O}$. In particular, $\gamma \kappa_z \gamma^{-1} \notin S_3$ for any $\gamma \in \mathbb{O}$. It follows that $\mathbf{Z}_2^r$ is not included in S_3.

Note that the three maximal isotropy subgroups D_4, $\mathbb{Z}_2^r \oplus \mathbb{Z}_2^t$, S_3, are precisely those instances (b),(c),(d) in Table 1.1.1 where the dimension of the fixed-point subspace is minimal, that is one. In general the latter condition implies maximality of the isotropy subgroup (see Golubitsky [1983]) but not vice versa (Ihrig & Golubitsky [1984]).

Fig 1.1.1 The maximal isotropy subgroups of $\mathbb{O}$.

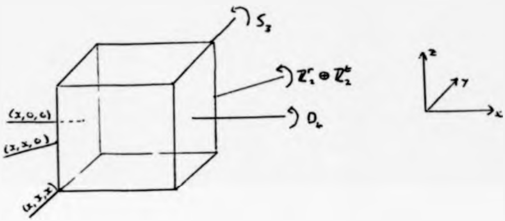
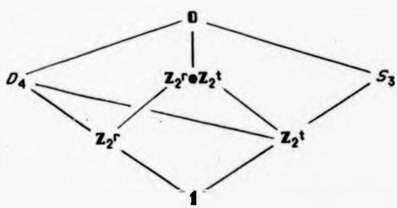


Fig. 1.1.2. The lattice of isotropy subgroups of $\mathbb{O}$.



§1.2. Calculation of Invariants and Equivariants.

Our aim in this section is to arrive at a simplified expression for a general $\mathbb{D}$-equivariant bifurcation problem $g \in \vec{\mathcal{E}}_{x,y,z,\lambda}(\mathbb{D})$. In particular, we will prove the following result.

Theorem 1.2.1 *Let* $g \in \vec{\mathcal{E}}_{x,y,z,\lambda}(\mathbb{D})$. *Then there exist* $P, Q, R \in \mathcal{E}_{u,v,w,\lambda}$ *such that*

$$g(x,y,z,\lambda) = P(u,v,w,\lambda)X_1 + Q(u,v,w,\lambda)X_2 + R(u,v,w,\lambda)X_3 \qquad (1)$$

where

$$u = x^2+y^2+z^2, \quad v = x^2y^2+y^2z^2+z^2x^2, \quad w = x^2y^2z^2$$

and

$$X_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad X_2 = \begin{pmatrix} x^3 \\ y^3 \\ z^3 \end{pmatrix}, \quad X_3 = \begin{pmatrix} y^2z^2x \\ z^2x^2y \\ x^2y^2z \end{pmatrix}.$$

Further, all coefficients in the Taylor expansions of P, Q, R *at* 0 *are uniquely determined.*

Remark 1.2.2 As a consequence of Theorem 1.2.1 we can adopt the *invariant coordinate* notation

$$g = [P, Q, R], \qquad (2)$$

for g given as in (1). This representation is essentially unique for our purposes: up to any given order in the Taylor expansion of g we have P, Q and R given uniquely. Note that an equivariant germ g is automatically zero at the origin. The other condition that g must satisfy in order to be a bifurcation problem becomes $P(0) = 0$.

The remainder of this section is devoted to proving Theorem 1.2.1. In Lemma 1.2.3 we show that the ring of invariant function germs $\mathcal{E}_{x,y,z}(\mathbf{O})$ is generated in some sense by u, v and w. Then Lemma 1.2.4 demonstrates that X_1, X_2 and X_3 generate the $\mathcal{E}_{x,y,z}(\mathbf{O})$-module, $\vec{\mathcal{E}}_{x,y,z}(\mathbf{O})$, of equivariant map germs. Finally we prove uniqueness.

Lemma 1.2.3 *Suppose* $f \in \mathcal{E}_{x,y,z}(\mathbf{O})$ *is an invariant germ. Then there exists* $P \in \mathcal{E}_{u,v,w}$ *such that*

$$f(x,y,z) = P(u,v,w),$$

where

$$u = x^2+y^2+z^2,\quad v = x^2y^2+y^2z^2+z^2x^2,\quad w = x^2y^2z^2.$$

Proof Using a result of Schwarz [1975] it is sufficient to show that u, v and w generate the ring of $\mathbf{O}$-invariant polynomials and that there is no relation between these generators. Now the κ_x, κ_x and κ_x invariance tells us that f is even in x, y and z. Apart from this f is just S_3 invariant. The result follows as it is well known that the ring of S_3-invariant polynomials is generated by the elementary symmetric polynomials. (For example, see Theorem 268, page 442 of Redei [1967].) □

In the following, let $a = x^2$, $b = y^2$, $c = z^2$ and let

$$\langle\varphi(a,b,c)\rangle = \begin{pmatrix} \varphi(a,b,c)x \\ \varphi(b,c,a)y \\ \varphi(c,a,b)z \end{pmatrix}. \qquad (3)$$

Lemma 1.2.4 *The module of equivariant maps* $\vec{\mathcal{E}}_{x,y,z}(0)$ *is generated over* $\mathcal{E}_{x,y,z}(0)$ *by* X_1, X_2 and X_3 where

$$X_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \langle 1\rangle,\ X_2 = \begin{pmatrix} x^3 \\ y^3 \\ z^3 \end{pmatrix} = \langle x^2\rangle,\ X_3 = \begin{pmatrix} y^2z^2x \\ z^2x^2y \\ x^2y^2z \end{pmatrix} = \langle y^2z^2\rangle.$$

Furthermore,

$$\vec{\mathcal{E}}_{x,y,z}(0) = \{\langle\varphi\rangle \mid \varphi \in \mathcal{E}_{a,b,c},\ \varphi(a,b,c) = \varphi(a,c,b)\}. \qquad (4)$$

Proof Applying Lemma 1.4.1, page 106 of Poénaru [1976], we restrict attention to polynomials. We start by verifying expression (4). In other words we show that an equivariant polynomial g is characterised by being of the form

$$g(x,y,z) = \sum_{ijk} g_{ijk} \langle a^i(b^j c^k + c^j b^k)\rangle, \tag{5}$$

where the g_{ijk} are real numbers. Observe that such a map satisfies

$$g(\gamma(x,y,z)) = \gamma g(x,y,z), \tag{6}$$

for $\gamma = \kappa_x$, R_x and R_y and hence for all $\gamma \in \mathbf{O}$, so a map of the form (5) is indeed $\mathbf{O}$-equivariant. Now suppose that $g = (g^1, g^2, g^3)$ is an equivariant polynomial map satisfying (6) for all γ. Setting $\gamma = \kappa_x, \kappa_y, \kappa_z$, we find that g^1 is odd in x, even in y and z, g^2 is odd in y, even in z and x, and g^3 is odd in z, even in x and y. Hence we can write

$$g(x,y,z) = \sum_{ijk} a^i b^j c^k \begin{bmatrix} g^1_{ijk} x \\ g^2_{ijk} y \\ g^3_{ijk} z \end{bmatrix}. \tag{7}$$

Now set γ to be the transpositions (12), (23), (31) to find

$$g^1_{ijk} = g^2_{jik}, \quad g^1_{ijk} = g^1_{ikj}, \quad g^2_{ijk} = g^2_{kji}.$$

Hence (7) becomes

$$g(x,y,z) = \sum_{ijk} \begin{bmatrix} g^1_{ijk} a^i(b^j c^k + c^j b^k)x \\ g^2_{ijk} b^i(c^j a^k + a^j c^k)y \\ g^3_{ijk} c^i(a^j b^k + b^j a^k)z \end{bmatrix}. \tag{8}$$

Using $\gamma = (12)$ and (13) in (8) yields the required form (5).

Now we show that the general term $\langle a^i(b^j c^k + c^j b^k)\rangle$ of (5) can be written in the form

$$\langle a^i(b^jc^k+c^jb^k)\rangle = P(a,b,c)\langle 1\rangle + Q(a,b,c)\langle a\rangle + R(a,b,c)\langle bc\rangle, \qquad (9)$$

where P, Q and R are symmetric polynomials in a, b and c. First note that

$$\langle a^2\rangle = u\langle a\rangle - v\langle 1\rangle + \langle bc\rangle,$$

$$\langle a^n\rangle = u\langle a^{n-1}\rangle - v\langle a^{n-2}\rangle + w\langle a^{n-3}\rangle;\ n \ge 3,$$

so $\langle a^n\rangle$ can be written in the form (9) for $n \ge 0$. Also

$$\langle a^m(b^n+c^n)\rangle = (a^n+b^n+c^n)\langle a^m\rangle - \langle a^{n+m}\rangle;\ m,n \ge 0.$$

Furthermore,

$$\langle b^nc^n\rangle = (a^nb^n+b^nc^n+c^na^n)\langle 1\rangle - \langle a^n(b^n+c^n)\rangle,$$

and for $m \ge n \ge 0$, setting $k = m-n$, we have

$$\begin{aligned}\langle b^mc^n+c^mb^n\rangle &= \langle b^nc^n(b^k+c^k)\rangle \\ &= (a^k+b^k+c^k)\langle b^nc^n\rangle - \begin{cases} w^k\langle b^pc^p\rangle; & n \ge k,\ p = n-k, \\ w^n\langle a^q\rangle; & n \le k,\ q = k-n. \end{cases}\end{aligned}$$

Finally

$$\begin{aligned}\langle a^k(b^lc^m+c^lb^m)\rangle &= w^k\langle b^pc^q+c^pb^q\rangle; && k \text{ is least, } p = l-k,\ q = m-k, \\ &= w^m\langle a^p(b^q+c^q)\rangle; && k \text{ is not least, say } m \le k,l \\ & && p = k-m,\ q = l-m.\end{aligned}$$

□

Proof of Theorem 1.2.1 We have to show that the uniqueness condition holds. Theorem 1.2.1 then follows immediately from Lemmas 1.2.3 and 1.2.4 by the triviality of the D action on the λ variable. Now Theorem 268, page 442 of Rédei [1967] shows that at the level of polynomials, there is no nontrivial relation between u, v and w. This is not enough to guarantee that there is no relation at the level of germs, but it does give uniqueness up to arbitrarily high order in the Taylor expansion.

It only remains to show that X_1, X_2 and X_3 generate $\vec{\mathcal{E}}_{x,y,z}(0)$ freely over $\mathcal{E}_{x,y,z}(0)$. Suppose that $P, Q, R \in \mathcal{E}_{u,v,w}$ satisfy

$$PX_1 + QX_2 + RX_3 = 0. \tag{10}$$

We must show that each of P, Q and R is identically zero. Now for x, y, z nonzero, (10) reduces to

$$\left.\begin{aligned} P(u,v,w) + x^2 Q(u,v,w) + y^2 z^2 R(u,v,w) = 0, \\ P(u,v,w) + y^2 Q(u,v,w) + z^2 x^2 R(u,v,w) = 0, \\ P(u,v,w) + z^2 Q(u,v,w) + x^2 y^2 R(u,v,w) = 0, \end{aligned}\right\} \tag{11}$$

and by continuity, the identities (11) hold for all x, y, z. Eliminating P from these identities, we obtain

$$\left.\begin{aligned} Q - z^2 R = 0, \\ Q - z^2 R = 0, \end{aligned}\right\} \tag{12}$$

holding everywhere by continuity. Eliminating Q from (12) and appealing once again to continuity yields $R = 0$, and so by (12) and (11) we have $Q = 0$, $P = 0$, as required. □

§1.3. Branching and Stability.

In §1.1 we obtained the lattice if isotropy subgroups of $\mathbb{O}$ together with a list of orbit representatives and fixed-point subspaces. In this section we look for zeroes of the general $\mathbb{O}$-equivariant bifurcation problem restricted to each fixed-point subspace. The analysis is greatly simplified due to the special form that an equivariant germ must take, see Theorem 1.2.1.

Our main result of §1, Theorem 1.3.1, is in accordance with the Equivariant Branching Lemma. This result due to Vanderbauwhede [1982] and Cicogna [1981] predicts, under certain hypotheses, the existence locally of a unique branch corresponding to each isotropy group with one-dimensional fixed-point subspace. The first hypothesis is that the group of symmetries should act absolutely irreducibly. Then for a bifurcation problem

$$g = [P, Q, R], \; P(0) = 0,$$

the only other hypothesis is that the nondegeneracy condition $P_\lambda(0) \neq 0$ holds.

Theorem 1.3.1. *Suppose that g is as in* Remark 1.2.2 *and that* $P(0) = 0$, $P_\lambda(0) \neq 0$. *Suppose further that the following nondegeneracy* *conditions are satisfied:*

$$Q(0) \neq 0, \quad P_u(0)/Q(0) \neq -1, -\tfrac{1}{2}, -\tfrac{1}{3}. \tag{1}$$

Then (i) *the branches of solutions corresponding to the three maximal isotropy subgroups satisfy the following equations:*

$$D_4: \qquad \lambda = -\frac{P_u(0) + Q(0)}{P_\lambda(0)} x^2 + o(x^4), \tag{2}$$

$$\mathbf{Z}_2^r \bullet \mathbf{Z}_2^t: \qquad \lambda = -\frac{2P_u(0) + Q(0)}{P_\lambda(0)} x^2 + o(x^4). \tag{3}$$

$$S_3: \qquad \lambda = -\frac{3P_u(0) + Q(0)}{P_\lambda(0)} x^2 + o(x^4). \tag{4}$$

(ii) *There are no other branches of solutions locally.*

(iii) *The* D_4 *branch is stable if and only if* $P_u(0) + Q(0) > 0$, $Q(0) < 0$.

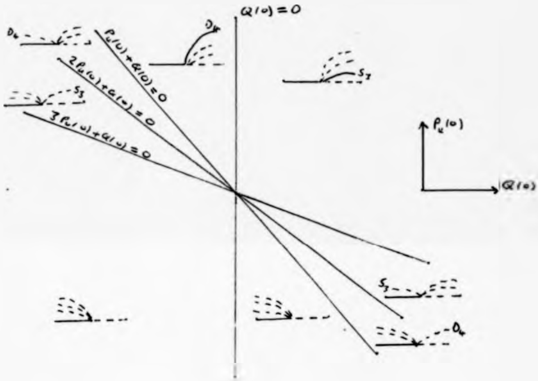
The S_3 *branch is stable if and only if* $3P_u(0) + Q(0) > 0$, $Q(0) > 0$.

The $\mathbf{Z}_2^r \bullet \mathbf{Z}_2^t$ *branch is never stable.*

Remarks 1.3.2 (a) The results of Theorem 1.3.1 are summarised in Figure 1.3.1. The branches here represent **O**-orbits. We consider the case $\varepsilon < 0$ where the trivial solution is stable subcritically and unstable supercritically.

(b) One of the bifurcating solutions can be stable if and only if all three branches bifurcate supercritically. The others are then unstable. The sign

Fig.1.3.1. Branching and stability for the different types of solution branch in the D-symmetric context. The ($Q(0), P_u(0)$) plane divides into 8 regions: for values interior to these the schematic bifurcation diagrams are as shown for $\varepsilon<0$. (Solid lines correspond to stable branches, dotted ones to unstable branches.)



of $Q(0)$ determines which of the D_4 and S_3 branches is stable whilst a stable $\mathbf{Z}_2^r \oplus \mathbf{Z}_2^t$ branch would require $Q(0)$ to be simultaneously positive and negative, and hence cannot occur.

Proof of Theorem 1.3.1 (i) We only need look at g evaluated at points $(x,0,0)$, $x>0$ when looking for D_4-solutions to $g = 0$, since g vanishes on $\mathbf{O}$-orbits by the equivariance of g. The equation $g = 0$ becomes

$$P(x^2,0,0,\lambda) + x^2 Q(x^2,0,0,\lambda) = 0. \qquad (5)$$

Now $P(0) = 0$ and $P_\lambda(0) \neq 0$ so by the Implicit Function Theorem,

$$P(x^2,0,0,\lambda(x^2)) + x^2 Q(x^2,0,0,\lambda(x^2)) = 0,$$

where $\lambda(0) = 0$ and $\lambda(x^2) = \lambda_2 x^2 + o(x^4)$. Therefore

$$P_u(0) + P_\lambda(0)\lambda_2 + Q(0) = 0$$

yielding equation (2). Evaluated on $(x,x,0)$ and (x,x,x), the equation $g = 0$ becomes

$$P(2x^2,x^4,0,\lambda) + x^2 Q(2x^2,x^4,0,\lambda) = 0, \qquad (6)$$

and

$$P(3x^2,3x^4,x^6,\lambda) + x^2 Q(3x^2,3x^4,x^6,\lambda) + x^4 R(3x^2,3x^4,x^6,\lambda) = 0, \qquad (7)$$

respectively. These lead by the Implicit Function Theorem to (3) and (4) as required.

(ii) Case (a) in Table 1.1.1 is just the trivial solution. In case (e) $g = 0$

reduces to

$$P + x^2 Q = 0,\ P + y^2 Q = 0,\ |x| \neq |y|,\ x, y \neq 0.$$

Subtracting one equation from the other and dividing by $x^2 - y^2$ gives $Q = 0$ on the supposed branch which contradicts the nondegeneracy condition $Q(0) \neq 0$. Hence there are no solution branches with $\mathbf{Z}_2^r$ symmetry that do not have at least D_4 or $\mathbf{Z}_2^r \bullet \mathbf{Z}_2^t$ symmetry. Cases (f) and (g) offer similar contradictions.

(iii) A solution branch of g is stable if all the real parts of the eigenvalues of (dg) evaluated at points on the branch are positive, and is unstable if one of the real parts is negative. Now $g = (A, B, C)$ where

$$A = Px + Qx^3 + Ry^2 z^2 x, \quad B = Py + Qy^3 + Rz^2 x^2 y, \quad C = Pz + Qz^3 + Rx^2 y^2 z.$$

We consider the three cases in turn.

<u>Case 1</u>: D_4. When evaluated at $(x, 0, 0, \lambda)$, $\partial B/\partial x$, $\partial C/\partial x$ and $\partial C/\partial y$ all vanish. Hence $(dg)_{x,0,0,\lambda}$ is an upper triangular matrix with eigenvalues

$$\frac{\partial A}{\partial x}(x,0,0,\lambda), \quad \frac{\partial B}{\partial y}(x,0,0,\lambda), \quad \frac{\partial C}{\partial z}(x,0,0,\lambda).$$

Now

$$\frac{\partial B}{\partial y}(x,0,0,\lambda) = \frac{\partial C}{\partial z}(x,0,0,\lambda) = P(x^2,0,0,\lambda)$$
$$= -x^2 Q(x^2,0,0,\lambda)$$

by (5). Also

$$\frac{\partial A}{\partial x}(x,0,0,\lambda) = P(x^2,0,0,\lambda)+2x^2P_u(x^2,0,0,\lambda)+3x^2Q(x^2,0,0,\lambda)+o(x^4)$$

$$= 2x^2(P_u(x^2,0,0,\lambda)+Q(x^2,0,0,\lambda))+o(x^4).$$

To use the characterisation of stability stated in the introduction, we ensure that the eigenvalues do not vanish near the origin by demanding that $Q(0) \neq 0$ and $P_u(0)+Q(0) \neq 0$. Note that the latter condition corresponds to that needed to predict direction of branching.

<u>Case 2</u>: $\mathbb{Z}_2^r \oplus \mathbb{Z}_2^t$. Along $(x,x,0,\lambda)$,

$$(dg) = \begin{pmatrix} \frac{\partial A}{\partial x} & \frac{\partial A}{\partial y} & * \\ \frac{\partial A}{\partial y} & \frac{\partial A}{\partial x} & * \\ 0 & 0 & P+x^4R \end{pmatrix}.$$

Therefore, one eigenvalue is positive if $Q(0) < 0$. The other eigenvalues are the eigenvalues of a 2×2 matrix and so the signs of their real parts are determined by the signs of the trace and determinant of the matrix. We have stability if

$$Q(0) < 0, \quad \frac{\partial A}{\partial x} > 0 \text{ and } \left(\frac{\partial A}{\partial x}\right)^2 - \left(\frac{\partial A}{\partial y}\right)^2 > 0,$$

and instability if at least one of these expressions is negative. Now

$$\frac{\partial A}{\partial x} = 2x^2(P_u+Q) + o(x^4)$$

using (6), and

$$\frac{\partial A}{\partial y} = 2x^2 P_u + o(x^4).$$

Hence

$$\left(\frac{\partial A}{\partial x}\right)^2 - \left(\frac{\partial A}{\partial y}\right)^2 = 4x^4 Q(2P_u + Q) + o(x^6).$$

The conditions

$$Q(0) < 0,\ P_u(0) + Q(0) > 0,\ 2P_u(0) + Q(0) < 0$$

cannot hold simultaneously, and so the $\mathbb{Z}_2^r \oplus \mathbb{Z}_2^t$ branch is always unstable.

Case 3: S_3. The eigenvalues of $(dg)_{x,x,x,\lambda}$ satisfy the equation

$$\begin{vmatrix} E-\lambda & F & F \\ F & E-\lambda & F \\ F & F & E-\lambda \end{vmatrix} = 0 \qquad (8)$$

where, using (7),

$$E = \frac{\partial A}{\partial x} = 2x^2(P_u + Q) + o(x^4)$$

$$F = \frac{\partial A}{\partial y} = 2x^2 P_u + o(x^4).$$

By performing column and row operations, (8) can be reduced to

$$\begin{vmatrix} 0 & 0 & E-F-\lambda \\ 0 & E-F-\lambda & 0 \\ E+2F-\lambda & F & F \end{vmatrix} = 0\,.$$

Therefore we require $E-F$ and $E+2F$ to be non-zero for nondegeneracy and positive for stability. This yields the required results. □

§2. The Equivariant Recognition Problem.

In this section we explore more efficient ways of solving the recognition problem. Recall that we wish to characterise the orbit of a bifurcation problem g under the group of equivalences $\mathfrak{D}(\Gamma)$ in terms of the Taylor coefficients of g. We show that this problem can be simplified by decomposing $\mathfrak{D}(\Gamma)$ into a group $\mathcal{U}(\Gamma)$ of equivalences whose linear parts are the identity and a group $\mathcal{S}(\Gamma)$ of linear equivalences. Then the $\mathfrak{D}(\Gamma)$-recognition problem can be solved by combining the solutions of the $\mathcal{U}(\Gamma)$- and $\mathcal{S}(\Gamma)$-recognition problems.

For many Γ-actions the $\mathcal{S}(\Gamma)$-recognition problem is trivial and so we concentrate on the $\mathcal{U}(\Gamma)$-recognition problem. In particular, we give a criterion for this problem to reduce to linear algebra, namely that the unipotent tangent space $T(f, \mathcal{U}(\Gamma))$ of the bifurcation problem f should be invariant under $\mathcal{U}(\Gamma)$. In this case the orbit of f under $\mathcal{U}(\Gamma)$ is simply the affine space

$$f + T(f, \mathcal{U}(\Gamma)),$$

and we say that f is *linearly determined*.

The organisation of this section is as follows. §2.1 sets up the necessary singularity theory background. In §2.2 we show that $\mathfrak{D}(\Gamma)$ can be decomposed into $\mathcal{U}(\Gamma)$ and $\mathcal{S}(\Gamma)$, and that the recognition problem can be similarly decomposed. We then give a theory for $\mathcal{U}(\Gamma)$-equivalence that is almost identical to that developed by Gaffney [1986] for $\mathcal{D}(\Gamma)$-equivalence. In particular, results by Bruce, du Plessis & Wall [1985] lead to a characterisation of a module of high order terms. §2.3 contains our main result which gives the criterion for a bifurcation problem to be linearly determined. In §2.4 we give results which make it easier to check whether or not this criterion holds. Even if the bifurcation problem in question is not linearly determined, the calculations discussed in §2.4 are still necessary in order to determine the module of high order terms.

In §2.5 we solve the recognition problem for many linearly determined bifurcation problems. A common link between these examples is that Γ acts absolutely irreducibly. We conclude by discussing briefly in §2.6 the complications that can be introduced into both the $\mathcal{U}(\Gamma)$ and the $\mathcal{S}(\Gamma)$ recognition problems when Γ does not act absolutely irreducibly.

§2.1. Background Singularity Theory.

We summarise the main concepts that will be needed, and establish notation. This notation is the same as that used in Golubitsky & Schaeffer [1984], Golubitsky, Stewart & Schaeffer [1988], Golubitsky & Roberts [1986] and Stewart [1987], and generalises that used in §1.

Let Γ be a compact Lie group acting on $\mathbb{R}^n$. A smooth map-germ at 0, $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is said to be Γ-*equivariant* if

$$g(\gamma x, \lambda) = \gamma g(x, \lambda) \text{ for all } \gamma \in \Gamma,\ x \in \mathbb{R}^n,\ \lambda \in \mathbb{R}.$$

We denote the space of all such mappings by $\vec{\mathcal{E}}_{x,\lambda}(\Gamma)$. The variable $x = (x_1, \ldots, x_n)$ is called the *state variable* and λ is the *bifurcation parameter*. Let $\mathcal{E}_{x,\lambda}(\Gamma)$ be the ring of all Γ-*invariant* smooth function-germs at 0, $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$; that is, those f satisfying

$$f(\gamma x, \lambda) = f(x, \lambda) \text{ for all } \gamma \in \Gamma,\ x \in \mathbb{R}^n,\ \lambda \in \mathbb{R}.$$

Then $\vec{\mathcal{E}}_{x,\lambda}(\Gamma)$ is a module over $\mathcal{E}_{x,\lambda}(\Gamma)$. We must also consider the $\mathcal{E}_{x,\lambda}(\Gamma)$-module $\overset{\leftrightarrow}{\mathcal{E}}_{x,\lambda}(\Gamma)$, which consists of the germs at 0 of all smooth matrix valued maps $S : \mathbb{R}^n \times \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ satisfying the condition

$$\gamma^{-1} S(\gamma x, \lambda) \gamma = S(x, \lambda) \text{ for all } \gamma \in \Gamma,\ x \in \mathbb{R}^n,\ \lambda \in \mathbb{R}.$$

A result of Schwarz [1975] ensures that there exists a finite set of invariant generators $u_1, \ldots, u_k \in \mathcal{E}_{x,\lambda}(\Gamma)$ such that any element $f \in \mathcal{E}_{x,\lambda}(\Gamma)$

can be written as a function of $u_1,\dots,u_k$. In other words $\mathcal{E}_{x,\lambda}(\Gamma) = \mathcal{E}_{u,\lambda}$. The ring $\mathcal{E}_{u,\lambda}$ has a unique maximal ideal $\mathcal{M}_{u,\lambda} = \langle u_1,\dots,u_k,\lambda\rangle$ comprising all invariant functions that vanish at the origin. The kth power of the maximal ideal $\mathcal{M}^k_{u,\lambda}$ consists of all invariant functions whose derivatives in u and λ up to any degree less than k vanish at the origin. Similarly we can define $\vec{\mathcal{M}}^k_{x,\lambda}(\Gamma)$ to be the space of equivariant maps whose derivatives in x and λ of degree less than k vanish at the origin. A *bifurcation problem with* Γ *symmetry* is an equation $g(x,\lambda) = 0$ where $g \in \vec{\mathcal{M}}_{x,\lambda}(\Gamma)$ and $(d_x g)_0 = 0$.

The group of Γ-equivalences acting on $\vec{\mathcal{M}}_{x,\lambda}(\Gamma)$ is defined in the following way. Let $\mathcal{L}(\Gamma)^\circ$ denote the connected component of $\mathrm{Hom}_\Gamma(\mathbb{R}^n) \cap GL(\mathbb{R}^n)$ containing the identity, where $\mathrm{Hom}_\Gamma(\mathbb{R}^n)$ is the vector space of all Γ-equivariant linear mappings on $\mathbb{R}^n$. Then $g, h \in \vec{\mathcal{M}}_{x,\lambda}(\Gamma)$ are Γ*-equivalent* if there exists a triple $(S, X, \Lambda) \in \overleftrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma) \times \vec{\mathcal{M}}_{x,\lambda}(\Gamma) \times \mathcal{M}_\lambda$ such that

$$h(x,\lambda) = S(x,\lambda)\, g(X(x,\lambda), \Lambda(\lambda)),$$

$$S(0),\ (d_x X)_0 \in \mathcal{L}(\Gamma)^\circ,\ \Lambda'(0) > 0.$$

Let

$$\mathfrak{D}(\Gamma) = \{(S, X, \Lambda) \in \vec{\mathcal{E}}_{x,\lambda}(\Gamma) \times \vec{\mathcal{M}}_{x,\lambda}(\Gamma) \times \mathcal{M}_{\lambda} \mid S(0), (d_x X)_0 \in \mathcal{L}(\Gamma)^{\circ}, \Lambda'(0) > 0\}.$$

Then under a suitable multiplication, the group action of $\mathfrak{D}(\Gamma)$ on $\vec{\mathcal{M}}_{x,\lambda}(\Gamma)$ induces the required equivalence relation. If we write $\varphi_i = (X_i, \Lambda_i)$ $i = 1,2$, then the multiplication is given by

$$(S_2, \varphi_2) \cdot (S_1, \varphi_1) = (S_2 \cdot (S_1 \circ \varphi_2), \varphi_1 \circ \varphi_2)$$

where

$$S_2 \cdot (S_1 \circ \varphi_2)(x, \lambda) = S_2(x, \lambda) \cdot S_1(\varphi_2(x, \lambda)),$$

$$\varphi_1 \circ \varphi_2(x, \lambda) = (X_1 \circ \varphi_2(x, \lambda), \Lambda_1 \circ \Lambda_2(\lambda)).$$

Recall that the tangent space $T(f, \mathfrak{D}(\Gamma))$ is given by

$$T(f, \mathfrak{D}(\Gamma)) = \{ \tfrac{d}{dt}(\delta_t f)|_{t=0} \mid \delta_t \in \mathfrak{D}(\Gamma), \delta_0 = 1 \}. \qquad (1)$$

A calculation shows that

$$T(f, \mathfrak{D}(\Gamma)) = \tilde{T}(f, \mathfrak{D}(\Gamma)) + \mathcal{E}_{\lambda}\{\lambda f_{\lambda}\}, \qquad (2a)$$

where

$$\tilde{T}(f, \mathfrak{D}(\Gamma)) = \{ Sf + (df)X \mid (S, X) \in \vec{\mathcal{E}}_{x,\lambda}(\Gamma) \times \vec{\mathcal{M}}_{x,\lambda}(\Gamma) \}. \qquad (2b)$$

Note that $\tilde{T}(f, \mathfrak{D}(\Gamma))$ is an $\mathcal{E}_{x,\lambda}(\Gamma)$-module, but this is not necessarily so for $T(f, \mathfrak{D}(\Gamma))$. (2) gives an alternative 'formal' definition for $T(f, \mathfrak{D}(\Gamma))$. Unlike in (1) we do not require $\mathfrak{D}(\Gamma)$ to be a Lie group. The following result is a fundamental lemma from singularity theory relating the concepts of finite determinacy and finite codimension.

Lemma 2.1.1 *The following are equivalent:*

(a) *$T(f,\mathfrak{D}(\Gamma))$ has finite codimension in $\vec{\mathcal{E}}_{x,\lambda}(\Gamma)$, that is*

$$T(f,\mathfrak{D}(\Gamma)) \oplus V = \vec{\mathcal{E}}_{x,\lambda}(\Gamma),$$

for some finite dimensional vector space V.

(b) *f is finitely determined, that is there is some $k > 0$ such that*

$$f + p \in \mathfrak{D}(\Gamma).f \text{ for all } p \in \vec{\mathcal{M}}^k_{x,\lambda}(\Gamma).$$

If (a) *and* (b) *hold then $\mathfrak{D}(\Gamma)$ can be considered as acting modulo $\vec{\mathcal{M}}^k_{x,\lambda}(\Gamma)$. The induced action is that of a Lie group acting algebraically. The tangent space definitions in* (1) *and* (2) *coincide.* □

Definition 2.1.2 A bifurcation problem $f \in \vec{\mathcal{M}}_{x,\lambda}(\Gamma)$ has *finite Γ-codimension* if *$T(f,\mathfrak{D}(\Gamma))$ has finite codimension in $\vec{\mathcal{E}}_{x,\lambda}(\Gamma)$.*

§2.2 Unipotent Actions and the Recognition Problem.

Let $\mathfrak{D}(\Gamma)$ be the following group of Γ-equivalences acting on $\vec{\mathcal{M}}_{x,\lambda}(\Gamma)$:

$$\mathfrak{D}(\Gamma) = \{ (S, X, \Lambda) \in \vec{\mathcal{E}}_{x,\lambda}(\Gamma)\times\vec{\mathcal{M}}_{x,\lambda}(\Gamma)\times\mathcal{M}_{\lambda} \mid S(0), (d_x X)_0 \in \mathcal{Z}(\Gamma)^*, \Lambda'(0)>0\}.$$

Consider the map projecting equivalences onto their linear parts

$$\pi : \vec{\mathcal{E}}_{x,\lambda}(\Gamma)\times\vec{\mathcal{M}}_{x,\lambda}(\Gamma)\times\mathcal{M}_{\lambda} \to \vec{\mathcal{E}}_{x,\lambda}(\Gamma)\times\vec{\mathcal{M}}_{x,\lambda}(\Gamma)\times\mathcal{M}_{\lambda},$$

$$\pi(S, X, \Lambda) = (S(0), (d_x X)_0, \Lambda'(0)).$$

Let $\mathcal{S}(\Gamma) = \mathcal{Z}(\Gamma)^*\times\mathcal{Z}(\Gamma)^*\times\mathbf{R}^{>0}$ where $\mathbf{R}^{>0}$ is the set of positive real numbers.

It is easy to check that

$$\pi|_{\mathfrak{D}(\Gamma)} : \mathfrak{D}(\Gamma) \to \mathcal{S}(\Gamma)$$

is a group epimorphism. Its kernel

$$\mathcal{U}(\Gamma) = \{ (S, X, \Lambda) \in \mathfrak{D}(\Gamma) \mid S(0)=I, (d_x X)_0=I, \Lambda'(0)=1 \}, \qquad (1)$$

is therefore a normal subgroup of $\mathfrak{D}(\Gamma)$. We can decompose $\delta \in \mathfrak{D}(\Gamma)$ as

$$\delta = s u_1 = u_2 s$$

where $s \in \mathcal{S}(\Gamma)$, $u_1, u_2 \in \mathcal{U}(\Gamma)$. To do this set

$$s = \pi(\delta),\ u_1 = \pi(\delta)^{-1}\delta,\ u_2 = \delta\pi(\delta)^{-1}.$$

Furthermore the decomposition is unique since

$$\pi(\delta) = \pi(s)\pi(u_1) = s.$$

Note however that in general $u_1 \neq u_2$.

The group $\mathcal{U}(\Gamma)$ consists of unipotent diffeomorphisms, whose linear parts are unipotent matrices. (A unipotent matrix is one that in some coordinate system can be written as an uppertriangular matrix with ones on the diagonal). In consequence we can use the methods of Bruce, du Plessis & Wall [1985], from algebraic geometry.

Remark 2.2.1 (a) The decomposition described above allows us to solve a $\mathcal{D}(\Gamma)$-recognition problem by combining the solutions of the corresponding $\mathcal{U}(\Gamma)$- and $\mathcal{S}(\Gamma)$-recognition problems in the following way. Our method is to compute $\mathcal{S}(\Gamma).n$ for a given normal form n, and then to calculate $\mathcal{U}.f$ for all $f \in \mathcal{S}(\Gamma).n$. Since

$$\mathcal{D}(\Gamma).n = \mathcal{U}(\Gamma).\mathcal{S}(\Gamma).n,$$

we have $g \in \mathcal{D}(\Gamma).n$ if and only if $g \in \mathcal{U}(\Gamma).f$ for some $f \in \mathcal{S}(\Gamma).n$.

The elements of $\mathcal{S}(\Gamma)$ are linear, hence we might hope to solve the $\mathcal{S}(\Gamma)$-recognition problem without too much difficulty. This hope is not always realised; see Chapter IX of Golubitsky & Schaeffer [1984] for the case of two state variables without symmetry. However, in the examples which we consider in this paper, Γ acts in such a way that $\mathcal{S}(\Gamma)$ is *scalar*, that is $\mathcal{L}(\Gamma)^\circ$ contains only diagonal matrices (in some coordinate system). In §2.6 we give a criterion for $\mathcal{S}(\Gamma)$ to be scalar in terms of the action of Γ. In these cases solving $\mathcal{S}(\Gamma)$-recognition problems is a trivial matter. In the remainder of this section we concentrate on the $\mathcal{U}(\Gamma)$-recognition problem. From now on we usually suppress the Γ dependence.

(b) Our results require bifurcation problems $f \in \vec{\mathcal{M}}_{x,\lambda}$ to have finite codimension. It is not necessary to specify whether this is finite codimension with respect to $\mathfrak{D}$ or $\mathcal{U}$. A calculation shows that

$$T(f,\mathcal{U}) = \tilde{T}(f,\mathcal{U}) + \mathcal{E}_\lambda\{\lambda^2 f_\lambda\}, \qquad (2a)$$

where

$$\tilde{T}(f,\mathcal{U}) = \{\, Sf + (df)X \mid (S,X) \in \overleftrightarrow{\mathcal{E}}_{x,\lambda} \times \vec{\mathcal{M}}_{x,\lambda}\ ,\ S(0) = (dX)_0 = 0\,\}. \qquad (2b)$$

Comparing the definitions of $T(f,\mathfrak{D})$ and $T(f,\mathcal{U})$, we see that

$$T(f,\mathfrak{D}) = T(f,\mathcal{U}) + W, \qquad (3)$$

where

$$W = \mathbb{R}\{\, Sf + (d_x f)X + \lambda f_\lambda \mid S,\ d_x X \in \mathrm{Hom}(\mathbb{R}^n)\}.$$

Now $\vec{\mathcal{M}}_{x,\lambda}$ and $\overleftrightarrow{\mathcal{E}}_{x,\lambda}$ are finitely generated as modules over $\mathcal{E}_{x,\lambda}$, say by

$$X_1, \ldots, X_r,\ S_1, \ldots, S_s\,,$$

(Theorems XII,5.2 and XII,5.3, and Exercise XIV,1.3 of Golubitsky, Stewart & Schaeffer [1988]) and so $\mathrm{Hom}(\mathbb{R}^n)$ is spanned by

$$(d_x X_1)_0, \ldots, (d_x X_r)_0\ ;\ S_1(0), \ldots, S_s(0)\,.$$

Therefore W is a finite dimensional vector space and hence, by (3), it follows that the two tangent spaces have finite or infinite codimension in $\vec{\mathcal{E}}_{x,\lambda}$ together.

(c) The results in §2.2 and §2.3 hold in a more general setting. In particular $\mathcal{U}$ and $\mathcal{S}$ can be any subgroups of $\mathfrak{D}$ satisfying the following three properties:

for all $\delta \in \mathfrak{D}$, $\delta = su$ for some $u \in \mathcal{U}$, $s \in \mathcal{S}$,

$\mathcal{U}$ acts unipotently,

the codimension property (3) holds with $\mathcal{W}$ finite-dimensional.

We will require the following two results from algebraic geometry. They deal with actions of unipotent groups and are Proposition 3.3 and Corollary 3.5 respectively of Bruce, du Plessis & Wall [1985].

Theorem 2.2.2 *Let U be a unipotent affine algebraic group over* $\mathbb{R}$ *acting algebraically on an affine variety V. Then*

(a) *The orbits of U are Zariski-closed in V.*

(b) *If* $x \in V$ *and W is a U-invariant subspace of V then* $x + W$ *is contained in an orbit of U if and only if* $LU.x \supset W$. □

Theorem 2.2.2 is restated in our particular context in Corollary 2.2.6.

Definition 2.2.3 For $f \in \vec{\mathcal{M}}_{x,\lambda}$,

$$M(f,U) = \{\, p \in \vec{\mathcal{M}}_{x,\lambda} \mid f+p \in Uf \,\}$$
$$= \{\, uf - f \mid u \in U \,\}.$$

Remark 2.2.4 Notice that $g \in Uf$ if and only if $g - f \in M(f,U)$. Hence, solving the U-recognition problem amounts to computing $M(f,U)$.

Definition 2.2.5 A subspace of $\vec{\mathcal{M}}_{x,\lambda}$ is *U-intrinsic* if it is invariant under the action of U. If a subset M of $\vec{\mathcal{M}}_{x,\lambda}$ contains a unique maximal U-intrinsic subspace, then this subspace is called the *U-intrinsic part* of M and is denoted $\mathrm{Itr}_{\mathcal{U}} M$.

Note that a U-intrinsic subspace of $\vec{\mathcal{M}}_{x,\lambda}$ is automatically an $\mathcal{E}_{x,\lambda}$-submodule of $\vec{\mathcal{M}}_{x,\lambda}$ since it is closed under multiplication on the left by $S = hI$ for any $h \in \mathcal{E}_{x,\lambda}$.

Clearly $\mathrm{Itr}_{\mathcal{U}} M$ exists for any subspace M. In Proposition 2.2.8 we see that $\mathrm{Itr}_{\mathcal{U}} M(f,U)$ always exists provided f has finite codimension.

Corollary 2.2.6 *Suppose $f \in \vec{\mathfrak{M}}_{x,\lambda}$ is of finite codimension. Then*

(a) *The orbit $\mathcal{U}.f$ is determined by a finite system of polynomial equations.*

(b) *Suppose M is a $\mathcal{U}$-intrinsic subspace of $\vec{\mathfrak{M}}_{x,\lambda}$. Then*

$$M \subset M(f,\mathcal{U}) \text{ if and only if } M \subset T(f,\mathcal{U}).$$

Proof By Lemma 2.1.1 we can work modulo $\vec{\mathfrak{M}}^k_{x,\lambda}$, some $k > 0$, and so regard $\mathcal{U}$ as an algebraic group acting algebraically. Now (a) and (b) are then just rewordings of Theorem 2.2.2 (a) and (b) respectively. □

We now define the analogue to the module $\mathcal{P}$ of high order terms in the $\mathfrak{D}$ context (see Gaffney [1986]).

Definition 2.2.7 $\mathcal{P}(f,\mathcal{U}) = \{\, p \in \vec{\mathfrak{M}}_{x,\lambda} \mid g+p \in \mathcal{U}f$ for all $g \in \mathcal{U}f \,\}$.

Proposition 2.2.8 *If f has finite codimension then*

$$\mathcal{P}(f,\mathcal{U}) = \mathrm{Itr}_{\mathfrak{u}} M(f,\mathcal{U}).$$

Proof We have to show that $\mathcal{P}(f,\mathcal{U})$ is the unique maximal $\mathcal{U}$-intrinsic subspace contained in $M(f,\mathcal{U})$. The proof is identical to that of Proposition 1.7 in Gaffney [1986] with one exception. Closure under addition is still straightforward: if $p_1, p_2 \in \mathcal{P}(f, U)$ and $g \in Uf$ then $g+p_1 \in Uf$ and so $(g+p_1)+p_2 \in Uf$ by definition. The problem is closure under scalar multiplication. However, consider the set

$$T = \{ t \in \mathbb{R} \mid g+tp \in Uf \},$$

where $p \in \mathcal{P}(f, U)$, $g \in Uf$. By the property of closure under addition, we have $\mathbb{N} \subset T$. But by Corollary 2.2.6(a), Uf is determined by finitely many polynomials. Therefore $t \in T$ if and only if t is a simultaneous zero of a finite set of polynomials. But T contains $\mathbb{N}$, an infinite set, and so $T = \mathbb{R}$ as required. Therefore $\mathcal{P}(f, U)$ is a subspace.

The rest of the proof proceeds as expected. Suppose $g \in \mathcal{P}(f, U)$, $u \in U$. Then $g+up = u(u^{-1}g+p) \in Uf$, so $up \in \mathcal{P}(f, U)$. Therefore $\mathcal{P}(f, U)$ is a U-intrinsic subspace. Clearly $\mathcal{P}(f, U) \subset M(f, U)$. Suppose $P \subset M(f, U)$ where P is U-intrinsic. Let $p \in P$ and $g = uf$, $u \in U$. Then

$$g + p = uf + p = u(f+u^{-1}p) \in Uf.$$

Thus $P \in \mathcal{P}(f, U)$ and $\mathcal{P}(f, U)$ is maximal and unique. □

Corollary 2.2.9 *If f has finite codimension then*

$$\mathcal{P}(f, U) = \operatorname{Itr}_U T(f, U).$$

Proof Taking U-intrinsic parts in Corollary 2.2.6(b) and applying Proposition 2.2.8 yields

$$M \subset \mathcal{P}(f, U) \text{ if and only if } M \subset \operatorname{Itr}_{\mathcal{U}} T(f, U),$$

for any U-intrinsic subspace M. Setting $M = \mathcal{P}(f, U)$ and $M = \operatorname{Itr}_{\mathcal{U}} T(f, U)$ in turn gives the result. □

§2.3. Linearly Determined Bifurcation Problems.

In Remark 2.2.4, we observed that the computation of $M(f,U)$ would solve the U-recognition problem. By Corollary 2.2.6(a), $M(f,U)$ is determined by a finite set of polynomial equations. We concentrate on the simplest case when these equations are linear, so that $M(f,U)$ is a vector subspace of finite codimension. Note that this codimension is the same as that of $T(f,U)$, because

$$\begin{aligned} \operatorname{codim} T(f,U) &= \text{number of defining equations for } Uf \\ &= \operatorname{codim} M(f,U). \end{aligned}$$

Definition 2.3.1 A bifurcation problem $f \in \vec{\mathbf{M}}_{x,\lambda}$ of finite codimension is *linearly determined* if $M(f,U)$ is a vector subspace of $\vec{\mathbf{M}}_{x,\lambda}$.

Remark 2.3.2 Linearly determined bifurcation problems are by no means rare. Indeed in examples that have been studied up to now, the majority of bifurcation problems are linearly determined. In the context of one state variable with no symmetry, nine out of the thirteen bifurcation problems of codimension ≤ 4 are linearly determined, whilst if $\Gamma = \mathbf{Z}_2$ all problems up to at least codimension 3 are linearly determined. In this section we give a simple criterion for linear determinacy. If this is satisfied, then

$M(f,U)$ is immediately known.

Proposition 2.3.3 *f is linearly determined if and only if*

$$M(f,U) = \mathcal{P}(f,U).$$

Proof We have to show that $M(f,U)$ is a subspace if and only if it is a U-intrinsic subspace. One implication is trivial. To prove the converse suppose $p \in M(f,U)$ and $g \in Uf$, so that there exist $u, u' \in U$ such that

$$f + p = uf,\ g = u'f.$$

Then

$$(g + p) - f = (u'f - f) + (uf - f) \in M(f,U).$$

Therefore $g + p \in Uf$ and so $p \in \mathcal{P}(f,U)$. □

Theorem 2.3.4 *f is linearly determined if and only if $T(f,U)$ is U-intrinsic, in which case*

$$M(f,U) = T(f,U).$$

Proof Suppose that f is linearly determined. Then by Proposition 2.3.3,

$$M(f,U) = \mathcal{P}(f,U) \subset T(f,U).$$

But $M(f,U)$ is a subspace with the same codimension as $T(f,U)$.

Therefore

$$T(f,U) = M(f,U) = \mathcal{P}(f,U),$$

the latter being a U-intrinsic subspace. The converse can be proved directly in the case when $U(\Gamma)$ is defined as in (2.1). However the proof is quite unwieldy. C.T.C. Wall found a more natural setting for the result in Lemma 2.3.5. The upshot of this Lemma is that $T(f,U) = M(f,U)$. But $T(f,U) = \mathcal{P}(f,U)$ and so f is linearly determined by Proposition 2.3.3. □

In the remainder of this section we revert to the notation of Theorem 2.2.2. Recall that the Lie algebra LU at f and the tangent space T(f,U) are the same object

Lemma 2.3.5 *Let U be a unipotent group acting linearly on a vector space V, and let $v \in V$ such that $LU.v$ is a U-invariant subspace of V. Then $U.v$ is the affine subspace $v + LU.v$.*

Proof (C.T.C Wall, private communication.) Let $N_1,\ldots,N_k$ be a basis of the Lie algebra LU. Since this is nilpotent, there is an integer r such that any product of more than r of the N_i is zero. The tangent space $LU.v$ is spanned by the $N_i v$. Since it is invariant, any $N_i N_j v$ also belongs to $LU.v$ (see Proposition 2.4.1).

It suffices to show that $U.v \subset v + LU.v$ for these have the same dimension. As $U.v$ is closed, it follows that it is the whole space. Because the exponential map for U is surjective, it is enough to show that for any $N = \Sigma\lambda_i N_i$ in LU, $e^N v$ belongs to $v + LU.v$. But

$$e^N v = v + \Sigma\lambda_i N_i v + \tfrac{1}{2}(\Sigma\lambda_i N_i)^2 v + \ldots + \tfrac{1}{n!}(\Sigma\lambda_i N_i)^r v$$

and since any N_iN_jv is a linear combination of the N_jv it follows by induction that each term except the first lies in $L\mathcal{U}.v$. □

Corollary 2.3.6 *Let U be a unipotent group acting linearly on a vector space V and let $v \in V$. then $L\mathcal{U}.v$ is a U-invariant subspace of V if and only if*

$$\mathcal{U}v = v + L\mathcal{U}v \ .$$

Proof It remains to prove that if $\mathcal{U}v = v + L\mathcal{U}v$ then $L\mathcal{U}v$ is U-invariant. Suppose that $M \in L\mathcal{U}$, $u \in U$. We must show that $uMv \in L\mathcal{U}v$. The hypothesis implies that $v + L\mathcal{U}v$ is invariant under U and so

$$u(v + Mv) \in v + L\mathcal{U}v.$$

Therefore

$$uv + uMv - v \in L\mathcal{U}v.$$

But $uv \in \mathcal{U}v$ and so $uv - v \in L\mathcal{U}v$. Hence we have

$$uMv \in L\mathcal{U}v$$

as required. □

§2.4. Tools for Calculating Maximal U-Intrinsic Subspaces.

In order to calculate $\mathcal{P}(f, U)$ we need an efficient method for calculating the U-intrinsic part of a subspace. The first result gives a necessary and sufficient condition for a subspace to be U-intrinsic.

Proposition 2.4.1 *If $M \subset \vec{\mathcal{M}}_{x,\lambda}$ is a subspace of finite codimension then M is U-intrinsic if and only if $LU.M \subset M$.*

Proof By the finite codimension of M we can work modulo $\vec{\mathcal{M}}^k_{x,\lambda}$, $k > 0$, and so regard U as a Lie group or as an algebraic group acting algebraically. For a unipotent group U, the exponential map

$$\exp : LU \to U$$

is continuous and surjective (Lemma 3.1 of Bruce, du Plessis & Wall [1985]), so U is the continuous image of a connected space. Therefore U is a connected Lie group acting smoothly on $\vec{\mathcal{M}}_{x,\lambda}$. Hence by Lemma 2.2 of Bruce, du Plessis & Wall [1985] we obtain the required result. □

In general verifying the condition in Proposition 2.4.1 is a laborious task. A better method is to recognise that a 'large part' of a subspace is U-intrinsic and then apply Proposition 2.4.1 as a last resort on whatever is remaining.

It is clear that applying a Γ-equivalence to a monomial $p \in \vec{\mathcal{M}}_{x,\lambda}(\Gamma)$ cannot reduce the overall degree of p. Furthermore, because the Λ part of a Γ-equivalence is only allowed to depend on λ, the degree of p in λ alone can also not be reduced. Hence for all $k, l > 0$, the subspace

$$\vec{\mathcal{M}}^{k}_{x,\lambda}(\Gamma)\langle\lambda^{l}\rangle \tag{1}$$

is both $\mathfrak{D}$-intrinsic and $\mathcal{U}$-intrinsic. By the linearity of the action of $\mathfrak{D}$, sums of subspaces such as in (1) are also intrinsic.

In the examples considered in §2.5, the action of Γ is irreducible. Suppose further that the action is nontrivial. The fixed point subspace

$$V^{\Gamma} = \{\, v \in \mathbb{R} \mid \gamma v = v \text{ for all } \gamma \in \Gamma \,\}$$

is a Γ-invariant subspace of $\mathbb{R}$ and so is just $\{0\}$. Now suppose $X \in \vec{\mathcal{E}}_{x,\lambda}(\Gamma)$. Then

$$\gamma X(0,\lambda) = X(\gamma.0,\lambda) = X(0,\lambda) \text{ for all } \gamma \in \Gamma.$$

Hence $X(0,\lambda) \in V^{\Gamma}$ and so $X(0,\lambda) = 0$. Thus the following useful hypothesis is often satisfied.

$$X(0,\lambda) = 0 \text{ for all } X \in \vec{\mathcal{E}}_{x,\lambda}(\Gamma). \tag{2}$$

Condition (2) implies that the degree in x is preserved by Γ-equivalence in the same way as the degree in λ is preserved. Therefore it is useful to define a space of germs vanishing up to some specified degree in x. For

$k \geq 1$, we define

$$\vec{\mathcal{M}}_k(\Gamma) = \left\{ f \in \vec{\mathcal{E}}_x(\Gamma) \mid \frac{d^\alpha f}{dx^\alpha}(0) = 0 \quad \text{for all multi-indices } |\alpha| < k \right\}.$$

The following result is elementary.

Proposition 2.4.2 *Suppose* (2) *holds. Then sums of subspaces of the form*

$$\vec{\mathcal{M}}_k(\Gamma)\langle \lambda^l \rangle, \ k \geq 1, \ l \geq 0,$$

are $\mathfrak{D}$*-intrinsic and U-intrinsic.* □

Note that

$$\vec{\mathcal{E}}_x(\Gamma) = \vec{\mathcal{M}}_1(\Gamma) \supset \vec{\mathcal{M}}_2(\Gamma) \supset \vec{\mathcal{M}}_3(\Gamma) \supset \dots .$$

These inclusions need not be strict. For example, consider $\Gamma = \mathbf{Z}_2$ acting on $\mathbf{R}$. Then $\vec{\mathcal{E}}_x(\mathbf{Z}_2)$ consists only of odd functions and so

$$\vec{\mathcal{M}}_{2k}(\mathbf{Z}_2) = \vec{\mathcal{M}}_{2k+1}(\mathbf{Z}_2) \text{ for all } k \geq 1.$$

For $k > 1$, let k^- denote the largest integer less than k such that $\vec{\mathcal{M}}_k(\Gamma)$ is strictly contained in $\vec{\mathcal{M}}_{k^-}(\Gamma)$.

Remark 2.4.3 (a) k^- is either $k-1$ or $k-2$. This is due to the fact that Γ is a compact Lie group acting on $\mathbf{R}^n$ and so is a subgroup of $O(n)$. Hence there is always an invariant of degree two, the norm $\|x\|$. In consequence, there is an equivariant of degree r for any odd number r. Furthermore, the existence of an equivariant of degree two would guarantee the existence of an equivariant of any given degree. Hence we have the following.

$$\text{Either } k^- = k-1 \text{ for all } k > 1, \text{ or } 3^- = 1. \qquad (3)$$

(b) Both cases in (3) can obtain for $V^\Gamma = \{0\}$. The examples in §2.5 and §3 all satisfy $3^- = 1$, but if $\Gamma = S_3$ acting on $\mathbb{C}$ as the symmetries of an equilateral triangle, then $\bar{z}^2$ is an equivariant of degree two. (See Golubitsky & Schaeffer [1983].)

Theorem 2.4.4 *Suppose* (2) *holds. Let V be a subspace of*

$$\vec{M}_{k_1^-}(\Gamma)\langle\lambda^{l_1-1}\rangle + \dots + \vec{M}_{k_s^-}(\Gamma)\langle\lambda^{l_s-1}\rangle, \; k_i > 1, \; l_i > 0, \; i = 1,\dots,s.$$

Then

$$\vec{M}_{k_1}(\Gamma)\langle\lambda^{l_1-1}\rangle + \vec{M}_{k_1^-}(\Gamma)\langle\lambda^{l_1}\rangle + \dots + \vec{M}_{k_s}(\Gamma)\langle\lambda^{l_s-1}\rangle + \vec{M}_{k_s^-}(\Gamma)\langle\lambda^{l_s}\rangle + V$$

is U-intrinsic.

Proof By Proposition 2.4.2

$$H = \vec{M}_{k_1}(\Gamma)\langle\lambda^{l_1-1}\rangle + \vec{M}_{k_1^-}(\Gamma)\langle\lambda^{l_1}\rangle + \dots + \vec{M}_{k_s}(\Gamma)\langle\lambda^{l_s-1}\rangle + \vec{M}_{k_s^-}(\Gamma)\langle\lambda^{l_s}\rangle$$

is U-intrinsic. Hence by Proposition 2.4.1 it suffices to show that

$LU.V \subset H.$

We show that if $p \in \vec{\mathcal{M}}_{k^-}(\Gamma)\langle\lambda^{l-1}\rangle$ then

$$T(p,U) \subset H_0 = \vec{\mathcal{M}}_k(\Gamma)\langle\lambda^{l-1}\rangle + \vec{\mathcal{M}}_{k^-}(\Gamma)\langle\lambda^l\rangle.$$

The result follows by linearity of the $\mathfrak{D}$-action. Now

$$T(p,U) = \left\{ \begin{array}{l} Sp + (dp)X + \Lambda p_\lambda \mid (S,X,\Lambda) \in \mathcal{E}_{x,\lambda}(\Gamma) \times \vec{\mathcal{M}}_{x,\lambda}(\Gamma) \times \mathcal{M}_\lambda \\ S(0) = 0, (dX)_0 = 0, \Lambda'(0) = 0 \end{array} \right\}.$$

It is easy enough to see that

$$Sp \in H_0, \quad \Lambda p_\lambda \in \vec{\mathcal{M}}_{k^-}(\Gamma)\langle\lambda^l\rangle \subset H_0.$$

To show that $(dp)X \in H_0$ we have to use Remark 2.4.3(a). By (3) we have two cases to consider.

<u>Case 1.</u> $k^- = k-1$ for all $k > 1$.

Now p is of degree at least $k-1$ in x and at least $l-1$ in λ, and so dp is of degree at least $k-2$ in x and at least $l-1$ in λ. Also we have $X \in \vec{\mathcal{M}}_2(\Gamma)\mathcal{E}_\lambda + \vec{\mathcal{M}}_1(\Gamma)\langle\lambda\rangle$ since $X(0,\lambda) = 0$ and $(dX)_0 = 0$. Thus

$$(dp)X \in \vec{\mathcal{M}}_k(\Gamma)\langle\lambda^{l-1}\rangle + \vec{\mathcal{M}}_{k-1}(\Gamma)\langle\lambda^l\rangle = H_0$$

as required.

<u>Case 2.</u> $3^- = 1$.

This time $X \in \vec{\mathcal{M}}_3(\Gamma)\mathcal{E}_\lambda + \vec{\mathcal{M}}_1(\Gamma)\langle\lambda\rangle$. Hence

$$(dp)X \in \vec{\mathcal{M}}_{k^-+2}(\Gamma)\langle\lambda^{l-1}\rangle + \vec{\mathcal{M}}_{k^-}(\Gamma)\langle\lambda^l\rangle.$$

By Remark 2.4.2(a), $k^{-}+2 \geq k$ and so the result is proved. □

If (2) does not hold then the property of 'preservation of degree in x' does not stand. However we can prove a weak analogue of Theorem 2.4.4 which holds true for all compact Lie group actions. Note that

$$\vec{\mathcal{E}}_{x,\lambda}(\Gamma) = \vec{\mathcal{M}}^{0}_{x,\lambda}(\Gamma) \supset \vec{\mathcal{M}}^{1}_{x,\lambda}(\Gamma) \supset \vec{\mathcal{M}}^{2}_{x,\lambda}(\Gamma) \supset \ldots \ .$$

This time each inclusion is strict.

Theorem 2.4.5 *Let W be a subspace of*

$$\vec{\mathcal{E}}_{x,\lambda}(\Gamma)\langle\lambda^{l_0}\rangle + \vec{\mathcal{M}}^{k_1}_{x,\lambda}(\Gamma)\langle\lambda^{l_1-1}\rangle + \ldots + \vec{\mathcal{M}}^{k_s}_{x,\lambda}(\Gamma)\langle\lambda^{l_s-1}\rangle\ ,\ k_i > 0,\ l_i > 0.$$

Then

$$\vec{\mathcal{M}}_{x,\lambda}(\Gamma)\langle\lambda^{l_0}\rangle + \vec{\mathcal{M}}^{k_1+1}_{x,\lambda}(\Gamma)\langle\lambda^{l_1-1}\rangle + \vec{\mathcal{M}}^{k_1-1}_{x,\lambda}(\Gamma)\langle\lambda^{l_1}\rangle$$
$$+ \ldots + \vec{\mathcal{M}}^{k_s+1}_{x,\lambda}(\Gamma)\langle\lambda^{l_s-1}\rangle + \vec{\mathcal{M}}^{k_s-1}_{x,\lambda}(\Gamma)\langle\lambda^{l_s}\rangle + W$$

is U-intrinsic.

Proof This is similar to that of Theorem 2.4.4. However we have only

$$X \in \vec{\mathcal{M}}_{x,\lambda}(\Gamma) + \vec{\mathcal{E}}_{x,\lambda}(\Gamma)\langle\lambda\rangle$$

rather than $\vec{\mathcal{M}}_{x,\lambda}(\Gamma) + \vec{\mathcal{M}}_{x,\lambda}(\Gamma)\langle\lambda\rangle$ as in case 1 of the proof of Theorem 2.4.4. In particular

$$\mathcal{X}(x,\lambda) = a\lambda,\ a \in \mathbb{R}^n,$$

is a possibility now that the restriction $\mathcal{X}(0,\lambda) = 0$ no longer holds in general. This accounts for the slightly weaker result. □

§2.5. Examples with Γ acting Absolutely Irreducibly.

<u>1. One state variable. No symmetry.</u> (Keyfitz [1986].)

Up to codimension ≤ 4, all bifurcation problems fall into one of the following families:

$$\varepsilon x^k + \delta\lambda, \quad k \geq 2; \quad \text{codim} = k-2,$$
$$\varepsilon x^k + \delta x\lambda, \quad k \geq 3; \quad \text{codim} = k-1,$$
$$\varepsilon x^2 + \delta\lambda^k, \quad k \geq 2; \quad \text{codim} = k-1,$$
$$\varepsilon x^3 + \delta\lambda^2, \quad \text{codim} = 3.$$

(See Table IV2.2 and Exercise IV2.1 of Golubitsky & Schaeffer [1984].) Our methods apply to all the above germs except those in the third family. Indeed even the solutions to the full recognition problems consist of linear defining and nondegeneracy conditions. Furthermore, in these cases the unipotent tangent spaces are invariant not only under unipotent equivalences but under the full group of equivalences. For this reason, the solution of these recognition problems is almost trivial even without making use of the results in this paper. Therefore it is necessary to go up to higher codimension to find instructive examples. First however we must calculate the unipotent tangent space $T(f, U)$. By definition

$$T(f,U) = \{ \tfrac{d}{dt} u_t|_{t=0} \mid u_t \in U, u_0 = 1 \}$$
$$= \{ Sf + (d_x f)X + \Lambda f_\lambda \mid (S, X, \Lambda) \in \mathcal{E}_{x,\lambda} \times \vec{\mathcal{M}}_{x,\lambda} \times \mathcal{M}_\lambda,\ S(0) = (d_x X)_0 = \Lambda'(0) = 0 \}.$$

Therefore

$$T(f,U) = \tilde{T}(f,U) + \mathcal{E}_{\lambda}\{\lambda^2 f_{\lambda}\}, \tag{1a}$$

where

$$\tilde{T}(f,U) = \mathcal{E}_{x,\lambda}\{xf, \lambda f, x^2 f_x, \lambda f_x\}. \tag{1b}$$

The tangent space $T(f,U)$ is the same as L_{MAX} in Corollary 1.9 of Gaffney [1986].

Example 2.5.1(1) $\varepsilon x^k + \delta x\lambda^2$, $k \geq 4$; codim $= 2k-1$.

This is the family II.5,2 in Table 1 of Keyfitz[1986]. The lowest codimension in the family is 7. First we calculate the orbit of $\varepsilon x^k + \delta x\lambda^2$ under scaling equivalences (S, X, Λ) where

$$S(x,\lambda) = \mu,\ X(x,\lambda) = \nu x,\ \Lambda(\lambda) = l\lambda;\ \mu,\nu,l \geq 0.$$

It is easy to ascertain that the orbit is

$$\{\mu\nu^k\varepsilon x^k + \mu\nu l^2\delta x\lambda^2 \mid \mu,\nu,l \geq 0\},$$

and that f is contained in this orbit if and only if

$$f = ax^k + bx\lambda^2,\ \text{sign}\, a = \varepsilon,\ \text{sign}\, b = \delta. \tag{2}$$

Now consider the unscaled germ

$$f(x,\lambda) = ax^k + bx\lambda^2,\ a,b \neq 0,\ k \geq 4.$$

By (1) we have

$$T(f,U) = \tilde{T}(f,U) + \mathcal{E}_{\lambda}\{\lambda^2 f_{\lambda}\},$$

where

$$\tilde{T}(f,U) = \mathcal{E}_{x,\lambda}\{x^2 f_x, \lambda f_x, xf, \lambda f\}$$

$$= \mathcal{E}_{x,\lambda}\{\ kax^{k+1} + bx^2\lambda^2,\ kax^{k-1}\lambda + b\lambda^3,$$
$$ax^{k+1} + bx^2\lambda^2,\ ax^k\lambda + bx\lambda^3\ \}\ .$$

The first and third generators simplify to x^{k+1} and $x^2\lambda^2$ and then it is easy to obtain

$$T(f,U) = \mathcal{M}^{k+1} + \mathcal{M}^2\langle\lambda^2\rangle + \mathbb{R}\{\ kax^{k-1}\lambda + b\lambda^3\ \},$$

where $\mathcal{M} = \vec{\mathcal{M}}_{x,\lambda} = \langle x,\lambda\rangle$ is the maximal ideal in $\mathcal{E}_{x,\lambda}$.

Note that

$$\mathcal{P}(f,\mathcal{D}) = \mathcal{M}^{k+1} + \mathcal{M}^2\langle\lambda^2\rangle.$$

Now $kax^{k-1}\lambda + b\lambda^3 \notin \mathcal{P}(f,\mathcal{D})$, for if we apply the scaling

$$\lambda \mapsto 2\lambda,$$

then

$$kax^{k-1}\lambda + b\lambda^3 \mapsto 2(kax^{k-1}\lambda + 4b\lambda^3) \notin T(f,U).$$

Hence $T(f,U)$ is not $\mathcal{D}$-intrinsic. However

$$\mathbb{R}\{\ kax^{k-1}\lambda + b\lambda^3\ \} \subset \mathcal{M}^{k-1}\langle\lambda\rangle + \langle\lambda^3\rangle,$$

and

$$T(f,U) \supset \mathcal{M}^k\langle\lambda\rangle + \mathcal{M}^{k-2}\langle\lambda^2\rangle + \mathcal{M}\langle\lambda^3\rangle,$$

and so by Theorem 2.4.5

$$\mathcal{P}(f,U) = T(f,U).$$

Hence by Theorem 2.3.4, f is linearly determined and

$$Uf = f + T(f,U)$$
$$= ax^k + bx\lambda^2 + A(kax^{k-1}\lambda + b\lambda^3) + \mathcal{M}^{k+1} + \mathcal{M}^2\langle\lambda^2\rangle\,.$$

Further, $g \in \mathcal{U}.f$ if and only if

$$g = g_x = \cdots = g_{x^{k-1}} = 0, \; g_\lambda = g_{x\lambda} = \cdots = g_{x^{k-2}\lambda} = 0, \; g_{\lambda\lambda} = 0, \tag{3a}$$

$$g_{x^k} = k!\,a, \; g_{x\lambda\lambda} = 2b, \tag{3b}$$

$$g_{x^{k-1}\lambda} = k!\,Aa, \; g_{\lambda\lambda\lambda} = 6Ab. \tag{3c}$$

The equations in (3c) are equivalent to the condition

$$k!\,a g_{\lambda\lambda\lambda} - 6 b g_{x^{k-1}\lambda} = 0. \tag{3d}$$

We have now solved both the unipotent recognition problem (3a,b,d) and the scaling recognition problem (2). Combining the two solutions gives the solution to the full recognition problem. Hence we see that g is $\mathfrak{D}$-equivalent to $\varepsilon x^k + \delta x \lambda^2$ if and only if

$$\left.\begin{array}{l} g = g_x = \cdots = g_{x^{k-1}} = 0, \; g_\lambda = g_{x\lambda} = \cdots = g_{x^{k-2}\lambda} = 0, \; g_{\lambda\lambda} = 0, \\ \operatorname{sign} g_{x^k} = \varepsilon, \; \operatorname{sign} g_{x\lambda\lambda} = \delta, \\ g_{x^k} g_{\lambda\lambda\lambda} - 3 g_{x^{k-1}\lambda} g_{x\lambda\lambda} = 0. \end{array}\right\}$$

Although the defining conditions for the unipotent problem are linear, the defining and nondegeneracy conditions for the corresponding full problem are not linear.

Example 2.5.1(ii) $\varepsilon(x^2+\delta\lambda)^2 + \sigma x^5$, codim = 5.

(See Table 3.5 of Keyfitz [1986] and Example 1.13 of Gaffney [1986].)

It is easy to check that f is equivalent by scalings to $\varepsilon(x^2+\delta\lambda)^2 + \sigma x^5$ if and only if

$$f = a(x^2+b\lambda)^2 + cx^5, \ \text{sign}\, a = \varepsilon, \ \text{sign}\, b = \delta, \ \text{sign}\, c = \sigma. \tag{4}$$

Consider

$$f(x,\lambda) = a(x^2+b\lambda)^2 + cx^5, \ a,b,c \neq 0.$$

Computations show that

$$T(f,U) = \tilde{T}(f,U) = H + \mathbb{R}\{\ x^5+bx^3\lambda,\ x^3\lambda+bx\lambda^2\ \},$$

where

$$H = \mathcal{M}^6 + \mathcal{M}^4\langle\lambda\rangle + \mathcal{M}^2\langle\lambda^2\rangle + \langle\lambda^3\rangle,$$

and that

$$\mathcal{P}(f,U) = T(f,U).$$

However $\mathcal{P}(f,\mathfrak{D})$ is only H. Gaffney shows that in this case a sufficient condition for g to be $\mathfrak{D}$-equivalent to f is that $g \equiv f \bmod T(f,U)$. In fact Theorem 2.3.4 shows that this condition is necessary and sufficient for U-equivalence. Hence

$$\begin{aligned} U.f = {} & ax^4 + 2abx^2\lambda + ab^2\lambda^2 + cx^5 \\ & + A(x^5+bx^3\lambda) + B(x^3\lambda+bx\lambda^2) + H \end{aligned}$$

and $g \in U.f$ if and only if

$$\left.\begin{aligned} & g = g_x = g_{xx} = g_{xxx} = 0,\ g_\lambda = g_{x\lambda} = 0, \\ & g_{xxxx} = 24a,\ g_{xx\lambda} = 4ab,\ g_{\lambda\lambda} = 2ab^2, \\ & g_{xxxxx} = 120(c+A),\ g_{xxx\lambda} = 6(Ab+B),\ g_{x\lambda\lambda} = 2Bb. \end{aligned}\right\} \tag{5}$$

Conditions (5) are equivalent to

$$\left.\begin{aligned}&g = g_x = g_{xx} = g_{xxx} = 0,\ g_\lambda = g_{x\lambda} = 0,\\ &g_{xxxx} = 24a,\ 6g_{xx\lambda} = b\,g_{xxxx},\ g_{xxxx}g_{\lambda\lambda} - 3g_{xx\lambda}^2 = 0,\\ &\frac{g_{xxxxx}}{120} - \frac{g_{xxx\lambda}}{6b} + \frac{g_{x\lambda\lambda}}{2b^2} = c\ .\end{aligned}\right\}\quad (6)$$

Combining this with (4) yields the required result: $g \in \mathfrak{D}f$ if and only if

$$\left.\begin{aligned}&g = g_x = g_{xx} = g_{xxx} = 0,\ g_\lambda = g_{x\lambda} = 0,\\ &\operatorname{sign} g_{xxxx} = \varepsilon,\ \operatorname{sign} g_{xx\lambda} = \varepsilon\delta,\ g_{xxxx}g_{\lambda\lambda} - 3g_{xx\lambda}^2 = 0,\\ &\operatorname{sign}\left(g_{xxxxx} - 10\,\frac{g_{xxx\lambda}g_{xx\lambda}}{g_{\lambda\lambda}} + 15\,\frac{g_{x\lambda\lambda}g_{xx\lambda}^2}{g_{\lambda\lambda}^2}\right) = \sigma.\end{aligned}\right\}$$

Note that example 2.5.1(ii) is the first member of the infinite family

$$\varepsilon(x^2+\delta\lambda)^2 + \sigma x^l,\ l \geq 5,\ \text{codim} = l,$$

in Keyfitz [1986]. In fact it is the only member of the family that is linearly determined.

<u>2. One state variable $\Gamma = \mathbf{Z}_2$.</u> (Golubitsky & Schaeffer [1984], VI.)

Here Γ acts on $\mathbb{R}$ as multiplication by -1. The ring of Γ-invariant *polynomials* in x is merely the ring of even polynomials, while the module of Γ-equivariant polynomials just consists of odd polynomials. Every odd polynomial can be written as an even polynomial multiplied by x, and so the module of Γ-equivariant polynomials is generated over the ring of Γ-invariant polynomials by the single element x. Results of Schwarz [1975] and Poénaru [1976] state that these properties are shared by smooth germs. Thus if we let $u = x^2$, then

$$\mathcal{E}_{x,\lambda}(\mathbf{Z}_2) = \mathcal{E}_{u,\lambda},$$

$$\vec{\mathcal{E}}_{x,\lambda}(\mathbf{Z}_2) = \mathcal{E}_{u,\lambda}.x.$$

Suppose $f \in \vec{\mathcal{E}}_{x,\lambda}(\mathbf{Z}_2)$, $f(x,\lambda) = r(u,\lambda).x$, $r \in \mathcal{E}_{u,\lambda}$. The unipotent tangent space is given by

$$T(f, U, \mathbf{Z}_2) = \tilde{T}(f, U, \mathbf{Z}_2) + \mathcal{E}_\lambda(\lambda^2 r_\lambda.x), \qquad (7a)$$

where

$$\tilde{T}(f, U, \mathbf{Z}_2) = \mathcal{E}_{u,\lambda}\{\, ur, \lambda r, u^2 r_u, u\lambda r_u \,\}.x. \qquad (7b)$$

A list of $\mathbf{Z}_2$-equivariant germs up to codimension 3 is given in Table VI,5.1 of Golubitsky & Schaeffer [1984]. It turns out that all but one of the eleven bifurcation problems satisfy

$$\mathcal{P}(f, \mathcal{D}, \mathbf{Z}_2) = \mathcal{P}(f, U, \mathbf{Z}_2) = T(f, U, \mathbf{Z}_2).$$

The missing problem is linearly determined but $\mathcal{P}(f,\mathfrak{D},\mathbb{Z}_2)$ is strictly contained in $T(f,U,\mathbb{Z}_2)$. This means that there is a distinct advantage in considering the unipotent recognition problem separately and we choose this as our next example:

Example 2.5.2 $(\varepsilon(u+\delta\lambda)^2 + \sigma u^3)x$, $\mathrm{codim}_{\mathbb{Z}_2} = 3$.

Now f is equivalent by scalings to $(\varepsilon(u+\delta\lambda)^2 + \sigma u^3)x$ if and only if

$$f = (a(u+b\lambda)^2 + cu^3)x,\ \mathrm{sign}\, a = \varepsilon,\ \mathrm{sign}\, b = \delta,\ \mathrm{sign}\, c = \sigma. \tag{8}$$

Consider the germ

$$f(x,\lambda) = r(u,\lambda)x,$$

where

$$r(u,\lambda) = a(u+b\lambda)^2 + cu^3,\ a,b,c \neq 0.$$

A computation using (7) shows that

$$T(f,U,\mathbb{Z}_2) = \tilde{T}(f,U,\mathbb{Z}_2) = H + V,$$

where

$$H = \mathcal{E}_{u,\lambda}\{u^4,\ u^3\lambda,\ u^2\lambda^2,\ u\lambda^3,\ \lambda^4\}.x,$$

and

$$V = \mathbb{R}\{u^3 + bu^2\lambda,\ u^2\lambda + bu\lambda^2,\ u\lambda^2 + b\lambda^3\}.x.$$

Notice that $u^4.x \in H$ and hence H contains any monomial of order ≥ 9 in x. Therefore $H \supset \vec{\mathcal{M}}_9(\mathbb{Z}_2)\mathcal{E}_\lambda$. In this way we see that

$$H = \vec{\mathcal{M}}_9(\mathbb{Z}_2)\mathcal{E}_\lambda + \vec{\mathcal{M}}_7(\mathbb{Z}_2)\langle\lambda\rangle + \vec{\mathcal{M}}_5(\mathbb{Z}_2)\langle\lambda^2\rangle + \vec{\mathcal{M}}_3(\mathbb{Z}_2)\langle\lambda^3\rangle + \vec{\mathcal{M}}_1(\mathbb{Z}_2)\langle\lambda^4\rangle$$

and so by Proposition 2.4.2

$$\mathcal{P}(f,\mathfrak{D},\mathbb{Z}_2) = H.$$

Now

$$V \subset \vec{\mathcal{M}}_7(\mathbb{Z}_2)\mathcal{E}_\lambda + \vec{\mathcal{M}}_5(\mathbb{Z}_2)\langle\lambda\rangle + \vec{\mathcal{M}}_3(\mathbb{Z}_2)\langle\lambda^2\rangle + \vec{\mathcal{M}}_1(\mathbb{Z}_2)\langle\lambda^3\rangle,$$

and since there are no equivariants of even order

$$9^- = 7,\ 7^- = 5,\ 5^- = 3 \text{ and } 3^- = 1.$$

Thus, by Theorem 2.4.4

$$T(f,U,\mathbb{Z}_2) = H + V$$

is U-intrinsic and so f is linearly determined. Therefore

$$Uf = (au^2 + 2abu\lambda + ab^2\lambda^2 + cu^3).x$$

$$+ (A(u^3 + bu^2\lambda) + B(u^2\lambda + bu\lambda^2) + C(u\lambda^2 + b\lambda^3)).x + H.$$

Hence $g(x,\lambda) = s(u,\lambda)x$ is U-equivalent to f if and only if

$$\left.\begin{aligned} & s = s_u = s_\lambda = 0, \\ & s_{uu} = 2a,\ s_{u\lambda} = 2ab,\ s_{\lambda\lambda} = 2ab^2, \\ & s_{uuu} = 6(c+A),\ \ s_{uu\lambda} = 2(Ab+B),\ s_{u\lambda\lambda} = 2(Bb+C),\ s_{\lambda\lambda\lambda} = 6Cb. \end{aligned}\right\} \quad (9)$$

Equations (9) can be replaced by

$$\left.\begin{aligned} & s = s_u = s_\lambda = 0, \\ & s_{uu} = 2a,\ \ s_{u\lambda} = b s_{uu},\ \ s_{uu}s_{\lambda\lambda} - s_{u\lambda}^2 = 0, \\ & s_{uuu} - 3\frac{s_{uu\lambda}}{b} + 3\frac{s_{u\lambda\lambda}}{b^2} - \frac{s_{\lambda\lambda\lambda}}{b^3} = 6c. \end{aligned}\right\} \quad (10)$$

Together with (8) this gives the necessary and sufficient conditions for g

to be $\mathbb{Z}_2$-equivalent to $\varepsilon(u+\delta\lambda)^2+\sigma u^3$, namely

$$\left.\begin{array}{l} s = s_u = s_\lambda = 0, \\ \operatorname{sign} s_{uu} = \varepsilon,\ \operatorname{sign} s_{u\lambda} = \varepsilon\delta,\ \ s_{uu}s_{\lambda\lambda} - s_{u\lambda}^2 = 0, \\ \operatorname{sign}\ s_{uuu} - 3\dfrac{s_{uu\lambda}s_{uu}}{s_{u\lambda}} + 3\dfrac{s_{u\lambda\lambda}s_{uu}^2}{s_{u\lambda}^2} - \dfrac{s_{\lambda\lambda\lambda}s_{uu}^3}{s_{u\lambda}^3} = \sigma. \end{array}\right\}$$

<u>3. Two state variables, $\Gamma = D_4$.</u> (Golubitsky & Roberts [1986].)

Here D_4 is taken to be acting on $\mathbb{R}^2$ as the symmetry group of the square and is generated by the symmetries

$$(x,y) \mapsto (x,-y),\ (x,y) \mapsto (y,x).$$

The ring of D_4-invariant germs is given by

$$\mathcal{E}_{x,y,\lambda}(D_4) = \mathcal{E}_{N,\Delta,\lambda},$$

where

$$N = x^2+y^2 \text{ and } \Delta = (x^2-y^2)^2.$$

$\vec{\mathcal{E}}_{x,y,\lambda}(D_4)$ is generated as a module over $\mathcal{E}_{x,y,\lambda}(D_4)$ by

$$\begin{pmatrix} x \\ y \end{pmatrix},\ (y^2-x^2)\begin{pmatrix} x \\ -y \end{pmatrix}.$$

Hence every D_4-equivariant map germ can be written as

$$f(x,y,\lambda) = p(N,\Delta,\lambda)\begin{pmatrix} x \\ y \end{pmatrix} + r(N,\Delta,\lambda)(y^2-x^2)\begin{pmatrix} x \\ -y \end{pmatrix}.$$

We adopt the 'invariant coordinate' notation

$$f = [p,r].$$

Table 2.1 of Golubitsky & Roberts [1986] gives a list of the fifteen bifurcation problems with D_4 symmetry of topological codimension ≤ 2. Of these, ten are linearly determined. We remark that these are precisely those bifurcation problems satisfying the nondegeneracy condition $r(0) \neq 0$. An analogous situation exists in the O-symmetric context; see §3. Of the linearly determined germs, $\mathcal{P}(f, U, D_4)$ is strictly larger than $\mathcal{P}(f, U, D_4)$ for all but cases I and II. We treat problem XII:

Example 2.5.3 $[\varepsilon N+\delta\lambda^2+\sigma\Delta+mN\lambda, \varepsilon]$, $m^2 \neq 4\delta\sigma$, top. codim$_{D_4}$ = 2.

The scaling problem is not quite as trivial as in the previous examples. f is equivalent by scalings to $[\varepsilon N+\delta\lambda^2+\sigma\Delta+mN\lambda, \varepsilon]$ if and only if

$$f = [aN+b\lambda^2+c\Delta+dN\lambda, a], \tag{11a}$$

and there are positive numbers μ, ν, l such that

$$\varepsilon\mu\nu^3 = a,\ \delta\mu\nu l^2 = b,\ \sigma\mu\nu^5 = c,\ m\mu\nu^3 l = d.$$

Clearly we require

$$\text{sign}\, a = \varepsilon,\ \text{sign}\, b = \delta,\ \text{sign}\, c = \sigma. \tag{11b}$$

A short computation shows that in addition we require

$$m = \frac{d}{\sqrt{|bc|}}\ . \tag{11c}$$

As usual we now consider the unscaled germ

$$f = [aN + b\lambda^2 + c\Delta + dN\lambda, a],\ d^2 \neq 4bc\ .$$

In Example 9.2 of Golubitsky & Roberts [1986] it is shown that

$$T(f, U, D_4) \supset H,$$

where

$$H = [\mathcal{M}^3 + \mathcal{M}\langle\Delta\rangle, \mathcal{M}^2 + \langle\Delta\rangle],$$

$\mathcal{M}$ being the maximal ideal $\langle N, \Delta, \lambda\rangle$ in $\mathcal{E}_{N,\Delta,\lambda}$. In fact

$$T(f, U, D_4) = H + \mathbb{R}\{\,[N^2, N], [\Delta, N], [N\lambda, \lambda]\,\}. \qquad (12)$$

In order to translate (12) into the notation of §2.4, we first note that H is generated as an $\mathcal{E}_{N,\Delta,\lambda}$-module by

$$[N^3, 0], [\Delta^2, 0], [\lambda^3, 0], [N^2\lambda, 0], [N\Delta, 0], [\Delta\lambda, 0], \qquad (13)$$
$$[0, N^2], [0, \Delta], [0, \lambda^2], [0, N\lambda].$$

Ignoring factors of λ we start to list monomials in $\vec{\mathcal{E}}_{x,y,\lambda}(D_4)$ in order of degree in (x, y). Note that N and Δ have degrees 2 and 4 and that $[1,0]$ and $[0,1]$ have degrees 1 and 3.

Order	[*,0]	[0,*]	
1	1		
3	N	1	
5	N^2, Δ	N	
7	$N^3, N\Delta$	N^2, Δ	etc.

Glancing at (13) we note that the only monomials in (x, y) which are missing are

$$[1,0], [N,0], [N^2,0], [\Delta,0], [0,1], [0,N].$$

These are all terms of degree ≤ 5 in (x, y) and hence

$$H \supset \vec{\mathcal{M}}_7(D_4)\mathcal{E}_\lambda .$$

In this way it is easily seen that

$$H = \vec{\mathcal{M}}_7(D_4)\mathcal{E}_\lambda + \vec{\mathcal{M}}_5(D_4)\langle\lambda\rangle + \vec{\mathcal{M}}_3(D_4)\langle\lambda^2\rangle + \vec{\mathcal{M}}_1(D_4)\langle\lambda^3\rangle.$$

Thus by Proposition 2.4.2 H is U-intrinsic and so is contained in $\mathcal{P}(f, U, D_4)$. Furthermore

$$\mathbb{R}\{\,[N^2, N], [\Delta, N], [N\lambda, \lambda]\,\} \subset \vec{\mathcal{M}}_5(D_4)\mathcal{E}_\lambda + \vec{\mathcal{M}}_3(D_4)\langle\lambda\rangle,$$

and so by Theorem 2.4.4, $\mathcal{P}(f, U, D_4) = T(f, U, D_4)$. Therefore by Theorem 2.3.4 we have

$$U.f = [aN + b\lambda^2 + c\Delta + dN\lambda, a]$$
$$+ A[N^2, N] + B[\Delta, N] + C[N\lambda, \lambda] + H.$$

Hence $[p, r] \in U.f$ if and only if

$$\left.\begin{array}{l} p = p_\lambda = 0,\ p_N = a,\ p_{\lambda\lambda} = 2b,\ p_\Delta = c + B, \\ p_{N\lambda} = d + C,\ p_{NN} = 2A,\ r = a,\ r_N = A + B,\ r_\lambda = C, \end{array}\right\}$$

that is if and only if

$$\left.\begin{array}{l} p = p_\lambda = 0,\ p_N = a,\ p_{\lambda\lambda} = 2b,\ p_N - r = 0, \\ p_{N\lambda} - r_\lambda = d,\ p_{NN} + 2p_\Delta - 2r_N = 2c. \end{array}\right\} \quad (14)$$

Combining (14) with (11) we see that $[p, r]$ is D_4-equivalent to $[\varepsilon N + \delta\lambda^2 + \sigma\Delta + mN\lambda, \varepsilon]$ if and only if

$$\rho = \rho_\lambda = 0,\ \mathrm{sign}\,\rho_N = \varepsilon,\ \mathrm{sign}\,\rho_{\lambda\lambda} = \delta,\ \rho_N - r = 0,$$

$$\mathrm{sign}(\rho_{NN} + 2\rho_\Delta - 2r_N) = \sigma,$$

$$m = \frac{2(\rho_{N\lambda} - r_\lambda)}{\sqrt{|\rho_{\lambda\lambda}(\rho_{NN} + 2\rho_\Delta - 2r_N)|}} .$$

§2.6. Examples with Γ not acting Absolutely Irreducibly.

In §2.5 we considered examples where Γ acts irreducibly. Using Theorem 2.4.4 or Theorem 2.4.5, we were able to show that the unipotent tangent spaces of certain bifurcation problems are $\mathcal{U}$-intrinsic. Then by Theorem 2.3.4 the unipotent recognition problems can be solved using only linear algebra. Furthermore it is then trivial to recover the solution to the full recognition problem because the group $\mathcal{S}(\Gamma)$ of linear Γ-equivalences just consists of scalar multiples of the identity. In other words, the triviality of the $\mathcal{S}(\Gamma)$-recognition problems in §2.5 relies on the absolute irreducibility rather than the irreducibility of the Γ action.

Schur's Lemma (Theorem 2, p.119 of Kirillov [1976]) states that if Γ acts irreducibly on V and $\mathrm{Hom}_{\Gamma}(V)$ denotes the space of linear maps on V that commute with Γ, then

$$\mathrm{Hom}_{\Gamma}(V) \cong \mathbb{R}, \mathbb{C} \text{ or } \mathbb{H}.$$

If $\mathrm{Hom}_{\Gamma}(V) \cong \mathbb{R}$, then Γ acts absolutely irreducibly, whereas if $\mathrm{Hom}_{\Gamma}(V) \cong \mathbb{C}$ or $\mathbb{H}$, then there is no coordinate system in which $\mathrm{Hom}_{\Gamma}(V)$ consists only of diagonal matrices.

Definition 2.6.1 Supose Γ is a compact Lie group acting on $\mathbb{R}^n$. We say that $\mathcal{S}(\Gamma)$ is *scalar* if in some coordinate system

$$\mathrm{Hom}_\Gamma(\mathbb{R}^n) \subset \{\text{diagonal matrices}\}.$$

Proposition 2.6.2 *Suppose Γ acts irreducibly on $\mathbb{R}^n$. Then $\mathcal{S}(\Gamma)$ is scalar if and only if Γ acts absolutely irreducibly.* □

Suppose now that Γ does not act irreducibly. By Theorem 3.20 of Adams [1969], $\mathbb{R}^n$ can be decomposed into irreducible subspaces

$$\mathbb{R}^n = V_1 \oplus \dots \oplus V_k.$$

Lemma 2.6.3 $\mathcal{S}(\Gamma)$ *is scalar if*

(i) *The actions of Γ on V_i and V_j are not isomorphic for $i \neq j$.*

(ii) $\mathrm{Hom}_\Gamma(V_i) \cong \mathbb{R}$, $i = 1,\dots,k$.

Proof Supose $L \in \mathcal{Z}(\Gamma)^* \subset \mathrm{Hom}_\Gamma(\mathbb{R}^n)$. Then, as in Proposition 4.2 of Stewart [1987],

$$L(V_i) \subset V_i, \; i = 1,\dots,k,$$

and so L has the block matrix structure

$$\begin{matrix} V_1 \\ \vdots \\ V_k \end{matrix} \begin{pmatrix} L_1 & & 0 \\ & \ddots & \\ 0 & & L_k \end{pmatrix}$$

where each $L_i \in \mathrm{Hom}_\Gamma(V_i)$. Furthermore, since each $\mathrm{Hom}_\Gamma(V_i) \cong \mathbf{R}$ we have

$$L_i = \mu_i I, \quad \mu_i \in \mathbf{R}, \ i = 1,\ldots,k. \qquad \square$$

In this paper we consider only examples where $\mathcal{S}(\Gamma)$ is scalar. A nonscalar problem is studied by Golubitsky & Schaeffer [1984], Chapter IX. They look at the nondegenerate bifurcation problems in two state variables with no symmetry. Their result for high order terms is easily recovered using Corollary 2.2.9; indeed the problems are linearly determined. However it is in the $\mathcal{S}$-recognition problem that all the difficulties lie.

In the remainder of this section we look at a straightforward example where Γ does not act irreducibly but where $\mathcal{S}(\Gamma)$ is scalar.

<u>1. $\Gamma = \mathbf{Z}_2$ acting on $\mathbf{R}^2$, by reflection on one copy of $\mathbf{R}$, trivially on the other.</u> (Dangelmayr & Armbruster [1983].)

The $\mathbf{Z}_2$ action is generated by

$$(x,y) \mapsto (x,-y).$$

Every $\mathbf{Z}_2$-equivariant germ can be written in the form

$$f(x,y,\lambda) = \begin{pmatrix} f_1(x,y,\lambda) \\ f_2(x,y,\lambda) \end{pmatrix},$$

where

$$f_1(x,y,\lambda) = p(u,v,\lambda), \quad f_2(x,y,\lambda) = r(u,v,\lambda)y,$$

$$u = x, \quad v = y^2.$$

In the invariant coordinate notation

$$f = [p,r].$$

In this notation the unipotent tangent space

$$T(f,U,\mathbb{Z}_2) = \tilde{T}(f,U,\mathbb{Z}_2) + \mathcal{E}_\lambda\{\lambda^2[p_\lambda,r_\lambda]\},$$

where $\tilde{T}(f,U,\mathbb{Z}_2)$ is generated as a $\mathcal{E}_{u,v,\lambda}$-module by

$$z[p,0],\ z[0,r],\ z[vp_v,vr_v],\quad z = u, v \text{ or } \lambda,$$

$$[0,p],\ [vr,0],\ u^2[up_u,ur_u],\ v[up_u,ur_u],\ \lambda[up_u,ur_u].$$

Let $\mathcal{M} = \langle u,v,\lambda\rangle$ denote the maximal ideal in $\mathcal{E}_{u,v,\lambda}$. Let I and J consist of sums and products of ideals of the form

$$\mathcal{M},\ \langle v\rangle \text{ and } \langle\lambda\rangle.$$

Then it is easily seen from the tangent space generators that (I,J) is an intrinsic module if and only if

$$vJ \subset I \subset J.$$

This characterisation of 'obvious' intrinsic modules proves more useful in this particular case than the more general Theorem 2.4.5.

It turns out that the methods of this paper simplify calculations for relatively few of the bifurcation problems. Linear determinacy holds for three out of the five problems of topological codimension ≤ 1, but for only three of a further twelve problems of topological codimension 2. There are two types of equivalence that restrict the number of intrinsic subspaces:

$$x \mapsto x + \lambda \text{ and } [p,0] \mapsto [0,p].$$

The first of these types also occurs when there is one state variable without symmetry and causes bifurcation problems of low codimension to fail to be linearly determined. This does not happen when there is reflectional symmetry present. For example in our present context we do not have equivalences of the form

$$y \mapsto y + \lambda \text{ or } [0,q] \mapsto [q,0].$$

We would expect the action of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ on $\mathbb{R}^2$

$$(x,y) \mapsto (-x,y),\ (x,y) \mapsto (x,-y)$$

to behave far better, in much the same way that $\mathbf{Z}_2$ behaves better than $\mathbf{1}$ when acting on $\mathbb{R}$.

Example 2.6.4 $[u^m + \varepsilon_1\lambda + \varepsilon_3 v, \varepsilon_2 u^2 + v]$, $m \geq 3$, top. $\text{codim}_{\mathbf{Z}_2} = m-1$.

This is family $(3)_{m2}$ of Dangelmayr & Armbruster [1983]. First we solve the $\mathcal{S}(\mathbf{Z}_2)$-recognition problem. Note that $\mathcal{S}(\mathbf{Z}_2)$ is scalar:

$$\mathrm{Hom}_{\mathbb{Z}_2}(\mathbb{R}^2) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{R} \right\}.$$

We usually require that

$$S(0),\ (dX)_0 \in \mathcal{L}(\mathbb{Z}_2)^{\circ},\ \Lambda'(0) > 0,$$

$\mathcal{L}(\mathbb{Z}_2)^{\circ}$ being the connected component of $\mathrm{Hom}_{\mathbb{Z}_2}(\mathbb{R}^2) \cap GL(\mathbb{R}^2)$ containing the identity (see Chapter XIV, §1 of Golubitsky, Stewart & Schaeffer [1988]). Then

$$\mathcal{L}(\mathbb{Z}_2)^{\circ} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b > 0 \right\}.$$

Dangelmayr & Armbruster [1983] impose the alternative restrictions

$$\det S(0) \neq 0,\ (dX)_0 > 0,\ \Lambda'(0) > 0.$$

In other words $(S, X, \Lambda) \in \mathcal{S}(\mathbb{Z}_2)$ must satisfy

$$S(x,y,\lambda) = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix},\quad X(x,y,\lambda) = \begin{pmatrix} \nu_1 x & 0 \\ 0 & \nu_2 y \end{pmatrix},\ \Lambda(\lambda) = l\lambda,$$

where $\mu_1, \mu_2 \neq 0$, $\nu_1, \nu_2, l > 0$. It can be shown that f is $\mathcal{S}(\mathbb{Z}_2)$-equivalent to $[u^m + \varepsilon_1\lambda + \varepsilon_3 v, \varepsilon_2 u^2 + v]$ subject to the following conditions:

$$f = [au^m + b\lambda + cv, du^2 + ev], \tag{1a}$$

$$\mathrm{sign}(de) = \varepsilon_2,\ \mathrm{sign}(ac) = \varepsilon_3,\ \mathrm{sign}(ab) = \varepsilon_1, \text{ if } m \text{ even}, \tag{1b}$$

$$\mathrm{sign}(de) = \varepsilon_2,\ \mathrm{sign}(bc) = \varepsilon_1\varepsilon_3, \text{ if } m \text{ odd}. \tag{1c}$$

As always we now consider the unscaled bifurcation problem

$$f = [au^m + b\lambda + cv, du^2 + ev],\ m \geq 3,\ a, b, c, d, e \neq 0.$$

A simple calculation reveals that

$$T(f,U,\mathbb{Z}_2) = [I,J] + \mathbb{R}\{[0, b\lambda + cv]\},$$

where

$$I = \mathcal{M}^{m+1} + \mathcal{M}\langle v,\lambda\rangle, \quad J = \mathcal{M}^3 + \mathcal{M}\langle v,\lambda\rangle.$$

Clearly $vJ \subset I \subset J$ and so $\mathcal{P}(f,U,\mathbb{Z}_2) \supset [I,J]$. Furthermore it is easily checked from the tangent space generators that if $p \in \mathbb{R}\{[0, b\lambda + cv]\}$ then

$$T(p,U,\mathbb{Z}_2) \subset [I,J].$$

By Theorem 2.3.4

$$U.f = f + A[0, b\lambda + cv] + [I,J].$$

Thus $[p,r] \in U.f$ if and only if

$$\left.\begin{array}{l} p = p_u = \dots = p_{u^{m-1}} = 0,\ r = r_u = 0, \\ p_{u^m} = m!a,\ p_\lambda = b,\ p_v = c,\ r_{uu} = 2d, \\ r_v = e + Ac,\ r_\lambda = Ab. \end{array}\right\}$$

These conditions are equivalent to

$$\left.\begin{array}{l} p = p_u = \dots = p_{u^{m-1}} = 0,\ r = r_u = 0, \\ p_{u^m} = m!a,\ p_\lambda = b,\ p_v = c,\ r_{uu} = 2d, \\ p_\lambda r_v - r_\lambda p_v = p_\lambda e. \end{array}\right\}$$

Hence by (1) $[p,r]$ is $\mathbb{Z}_2$-equivalent to $[u^m + \varepsilon_1\lambda + \varepsilon_3 v, \varepsilon_2 u^2 + v]$ if and only if

$$\left.\begin{array}{l} \rho = \rho_u = \ldots = \rho_{u^{m-1}} = 0, \quad r = r_u = 0, \\ \operatorname{sign}(r_{uu}\rho_\lambda(\rho_\lambda r_v - r_\lambda \rho_v)) = \varepsilon_2 , \\ \text{and} \quad \operatorname{sign}(\rho_{u^m}\rho_v) = \varepsilon_3, \ \operatorname{sign}(\rho_{u^m}\rho_\lambda) = \varepsilon_1 , \text{ if } m \text{ even,} \\ \operatorname{sign}(\rho_\lambda \rho_v) = \varepsilon_1\varepsilon_3 , \text{ if } m \text{ odd.} \end{array}\right\}$$

§3. Singularity Theory Classification of Bifurcation Problems with Octahedral Symmetry.

We give a singularity theory classification of bifurcation problems with octahedral symmetry up to topological codimension one. This consists of a list of seven normal forms with the property that any $\mathbb{O}$-equivariant bifurcation problem on $\mathbb{R}^3$ with topological codimension ≤ 1 is $\mathbb{O}$-equivalent to precisely one of the germs represented by the normal forms. In addition we give the universal unfolding of each normal form and solve the recognition problems where possible. The results are displayed in Table 3.1.1.

The recognition problem was discussed in §2. We now explain briefly the concepts of universal unfolding and codimension. For a more detailed discussion see Golubitsky & Schaeffer [1984] and Golubitsky, Stewart & Schaeffer [1988].

Roughly speaking, a universal unfolding of a germ gives us all possible local behaviour in the bifurcation diagram under small perturbation. More rigorously, we say that a germ $F \in \vec{\mathcal{E}}_{x,y,z,\lambda,\alpha}(\mathbb{O})$ is an *unfolding* (or $\mathbb{O}$ *-unfolding* to emphasise the role of the group $\mathbb{O}$) of $f \in \vec{\mathcal{E}}_{x,y,z,\lambda}(\mathbb{O})$ if $F(x,y,z,\lambda,0) = f(x,y,z,\lambda)$. The unfolding F *factors through* another unfolding $G \in \vec{\mathcal{E}}_{x,y,z,\lambda,\beta}(\mathbb{O})$ if there exist $S \in \mathcal{E}_{x,y,z,\lambda,\beta}(\mathbb{O})$,

$X \in \vec{\mathcal{E}}_{x,y,z,\lambda,\alpha}(\mathbf{D})$, $\Lambda \in \mathcal{E}_{\lambda,\alpha}$ and a smooth germ at 0 $A : \mathbb{R}^k \to \mathbb{R}^l$ such that

$$F(x,y,z,\lambda,\alpha) = S(x,y,z,\lambda,\alpha)\, G(X(x,y,z,\lambda,\alpha), \Lambda(\lambda,\alpha), A(\alpha)),$$

$$S(x,y,z,\lambda,0) = 1, \quad X(x,y,z,\lambda,0) = (x,y,z), \quad \Lambda(\lambda,0) = \lambda, \quad A(0) = 0.$$

An $\mathbf{D}$-unfolding F of f is *versal* if all other $\mathbf{D}$-unfoldings of f factor through F. A necessary and sufficient condition for versality is given in terms of the *extended tangent space* $T_e(f, \mathcal{D}(\mathbf{D}))$ defined as follows.

$$T_e(f, \mathcal{D}(\mathbf{D})) = \{ Sf + (df)X + \Lambda f_\lambda \mid S \in \vec{\mathcal{E}}_{x,y,z,\lambda}(\mathbf{D}), X \in \vec{\mathcal{E}}_{x,y,z,\lambda}(\mathbf{D}), \Lambda \in \mathcal{E}_\lambda \}.$$

There is a simple relation between $T_e(f, \mathcal{D}(\mathbf{D}))$ and the tangent space $T(f, \mathcal{D}(\mathbf{D}))$ defined in the introduction, namely

$$T_e(f, \mathcal{D}(\mathbf{D})) = T(f, \mathcal{D}(\mathbf{D})) + \mathbb{R}\{ f_\lambda \}.$$

Then the Equivariant Universal Unfolding Theorem (Golubitsky & Schaeffer [1979b], Golubitsky, Stewart & Schaeffer [1988]) states that F is a versal $\mathbf{D}$-unfolding of f if and only if

$$\vec{\mathcal{E}}_{x,y,z,\lambda}(\mathbf{D}) = T_e(f, \mathcal{D}(\mathbf{D})) + \mathbb{R}\{ \partial F / \partial \alpha_i (x,y,z,\lambda,0) \}. \qquad (1)$$

A *universal unfolding* is a versal unfolding with the minimum number of *unfolding parameters* $\alpha_1, \ldots, \alpha_k$. It follows from (1) that k is the codimension of $T_e(f, \mathcal{D}(\mathbf{D}))$ in $\vec{\mathcal{E}}_{x,y,z,\lambda}(\mathbf{D})$. This same number we call the $\mathbf{D}$*-codimension* of the germ f.

The codimension of a germ gives a rough measure of the complexity of the bifurcation diagram. The higher the codimension the greater the number of degeneracies that can be unfolded. However we saw in the introduction that some degeneracies are irrelevant from the *qualitative* point of view. Even in the nondegenerate situation of §1, we have a *modal parameter* that is invariant under $\mathbb{D}$-equivalence. This parameter must therefore be an unfolding parameter and yet it does not change the qualitative behaviour. To deal with this we slightly alter our definition of codimension by considering the *topological codimension* :

$$\text{top. codim}_{\mathbb{D}} f = \text{codim}_{\mathbb{D}} f - \#(\text{modal parameters}).$$

This definition is not totally satisfactory; we still have two modal families corresponding to nondegenerate bifurcation problems, one of topological codimension zero and the other of topological codimension one. Hence in addition to our standard classification we produce a qualitative classification. This gives the right answers but only because it is defined to do so! In particular, the two modal families described above collapse into one family of codimension zero.

Thus §3.1 comprises both a standard (smooth) classification and a qualitative classification up to (topological) codimension one. §3.2 deals briefly with an application to barium titanate crystals. The calculations for the smooth classification are presented in §§3.3 to 3.5, and the qualitative classification is discussed in §3.6. Many of the tangent space calculations are reserved for the Appendix.

§3.1. Tabulation of Results.

In this section we present the results of our classification. There is one normal form of topological codimension zero, normal form 1(i). Associated with this normal form are six nondegeneracy conditions. Breaking each of these conditions in the least degenerate manner leads to the six topological codimension one normal forms 1(ii), 2 to 6. Normal forms 1(i) and 1(ii) correspond to the nondegenerate bifurcation problems of §1. The reason for the strange numbering is that the qualitative behaviour of these two normal forms is the same. The other five normal forms all lead to distinct qualitative behaviour.

Fig. 3.1.1 gives a flow diagram for the classification up to topological codimension one. Then in Table 3.1.1 the recognition problem is solved for the first six normal forms. In other words, polynomial restraints are imposed on Taylor coefficients at the origin in order to determine whether a bifurcation problem is $\mathfrak{D}$-equivalent to a normal form. We also give the additional terms required to obtain a universal unfolding of each normal form. Note however that for example normal form 3 has codimension three but that we only give one unfolding term corresponding to the topological codimension. The other unfolding terms correspond to the two modal parameters and are omitted.

The full solution to the recognition problem for normal form 6 is not given as the calculations involved are far more difficult. However, we are able to solve a slightly different recognition problem.Here we look only for qualitative differences between germs; that is topological differences between bifurcation diagrams and furthermore differences between unfolded diagrams. We say that germs belonging to distinct smooth equivalence classes are *qualitatively equivalent* if they are the same in the above sense.

Table 3.1.2 gives the same information as Table 3.1.1 but for qualitative equivalence. Note that the entries are far simpler than those in Table 3.1.1. For a start, normal forms 1(i) and 1(ii) coincide as promised. Also we see that the modal parameters denoted by p and q have no qualitative bearing. Hence (usually complicated) expressions that had to be evaluated precisely for smooth equivalence can be ignored completely for qualitative equivalence. In practical terms, this simplification does not make a great deal of difference when recognising to which normal form a given bifurcation problem corresponds. However, actually solving the simplified recognition problem is much easier. In particular the recognition problem for normal form 6 can be solved with relatively little trouble. (In fact, a good mathematical theory for qualitative equivalence would make the solution quite easy.)

Some notation in Table 3.1.2 has to be explained. For example, in the case of normal form 6, one of the nondegeneracy conditions is

$$\mathcal{J} \in (m). \qquad (1)$$

The modal parameter m has four intervals of possible values

$$(-\infty,0),\ (0,\tfrac{1}{4}),\ (\tfrac{1}{4},\tfrac{1}{3}) \text{ and } (\tfrac{1}{3},\infty).$$

Each range of values gives rise to distinct qualitative behaviour. However it is not necessary to specify the precise value of $\mathcal{J}$ but merely the interval into which the value falls. Thus if m had the value $\tfrac{1}{8}$, then condition (1) would read $\mathcal{J} \in (0,\tfrac{1}{4})$.

§§3.3–3.5 are concerned with obtaining the information in Table 3.1.1; a sample of the calculations for normal forms 1 to 5 are given in §§3.3 and 3.4 and the details for normal form 6 are presented in §3.5. Then we deal with Table 3.1.2 in §3.6.

Theorem 3.1.1 (Classification Theorem)

Suppose $f = [P, Q, R] \in \vec{\mathcal{E}}_{x,y,z,\lambda}$ *of topological codimension* ≤ 1. *Then* f *is equivalent to precisely one of the normal forms in* Table 3.1.1 *in which case it satisfies the defining equations and nondegeneracy conditions* (*where given*). *The universal unfolding is obtained by adding the unfolding terms and replacing* m, n, p, q *by* $\bar{m}, \bar{n}, \bar{p}, \bar{q}$ *respectively, where*

$$\alpha,\ \bar{m}-m,\ \bar{n}-n,\ \bar{p}-p,\ \bar{q}-q,$$

are arbitrarily close to zero.

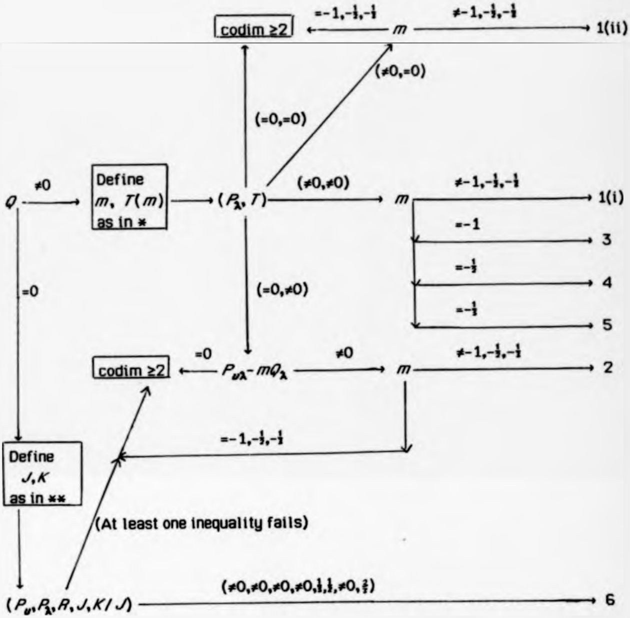
Remark 3.1.2 An analogous result to Theorem 3.1.1 exists for qualitative equivalence. The codimension in Table 3.1.2 is the least topological codimension of germs in the same qualitative equivalence class.

Proof of Theorem 3.1.1 Most of the work is done in verifying the entries in Tables 3.1.1 and 3.1.2. Then the flow chart in Fig. 3.1.1 all but constitutes a proof of the theorem. It only remains to show that the 'top. codim ≥ 2' boxes are accurate. Now

$$\text{top. codim } f = \#(\text{topological defining equations for } f) - 1.$$

Hence we must show that each box corresponds to at least three independent topological defining equations for f. Our strategy rests on the following observation: if we go along an arrow that says '$\varphi = a$' and if '$\varphi = a$' is an invariant of equivalence by that stage of the flow diagram, then '$\varphi = a$' must be a defining condition from then on. For example, Q is invariant whilst in general R is not. However, once we have $P = Q = 0$, then R is invariant. Therefore if we go along the arrows which say '$Q = 0$' and '$R = 0$' then we have already reached germs of codimension at least two. Special caution must be paid to expressions involving moduli. For example, if $Q \neq 0$, then we define $m = P_u / Q$. This is a smooth defining condition but not necessarily a topological defining condition. Indeed if we have $m \neq -1, -\frac{1}{2}, -\frac{1}{3}$, then $m = P_*/Q$ is a topological nondegeneracy condition. Of course, if we have $m = -1, -\frac{1}{2}, -\frac{1}{3}$, then $m = P_*/Q$ is both a smooth and a topological defining condition.

Hence it remains to prove that all expressions in the flow chart are invariant (once they appear). Now if $Q \neq 0$, then it is clear from the calculations in §§3.3 and 3.4 that all subsequent expressions such as $T(m)$ are invariant. Also, if $Q = 0$, then it is clear that P_*, P_1, and R are all invariant. This leaves J and K. This time, the calculations in §3.6 lead us to the required conclusion. □



* $m = P_v/Q, \quad T(m) = P_{vv} + (m+1)P_v - 2mQ_v + (m+1)(2m+1)R$

** $J = (P_\lambda Q_v - P_v Q_\lambda)/P_\lambda R, \quad K = (P_\nu Q_v - P_v Q_\nu)/R^2$

Fig 3.1.1. Flow Chart for Classification Theorem.

	Normal Form	(Top.) Codim.	Defining Equations	Nondegeneracy Conditions	Unfolding Terms
1(i)	$[\varepsilon mu+\delta\lambda+\sigma u^2,\varepsilon,0]$ $m \neq -1, -\tfrac{1}{2}, -\tfrac{1}{3}$	(0) 1	$P = T_1 = 0$	$\operatorname{sign} Q = \varepsilon$, $\operatorname{sign} P_\lambda = \delta$ $\operatorname{sign} T_2 = \sigma$	$\emptyset$
1(ii)	$[\varepsilon mu+\delta\lambda,\varepsilon,0]$ $m \neq -1, -\tfrac{1}{2}, -\tfrac{1}{3}$	(1) 2	$P = T_1 = T_2 = 0$	$\operatorname{sign} Q = \varepsilon$, $\operatorname{sign} P_\lambda = \delta$	$[\alpha u^2,0,0]$
2	$[\varepsilon mu+\delta\lambda^2+\sigma u^2+\rho u\lambda,\varepsilon,0]$ $m \neq -1, -\tfrac{1}{2}, -\tfrac{1}{3}$	(1) 3	$P = P_\lambda = T_1 = 0$ $\rho = \dfrac{2T_{1,\lambda}}{\sqrt{\lvert P_{\lambda\lambda} T_2\rvert}}$	$\operatorname{sign} Q = \varepsilon$, $\operatorname{sign} P_{\lambda\lambda} = \delta$ $\operatorname{sign} T_2 = \sigma$	$[\alpha\lambda,0,0]$
3 4 5	$[-\varepsilon u+\delta\lambda+\sigma u^2+\rho u^3,\varepsilon,0]$ $[-\tfrac{1}{2}\varepsilon u+\delta\lambda+\sigma u^2+\rho u^3,\varepsilon,0]$ $[-\tfrac{1}{3}\varepsilon u+\delta\lambda+\sigma u^2+\rho u^3,\varepsilon,0]$	(1) 2	$P = T_1(m) = 0$ $\rho = \dfrac{2\lvert Q\rvert}{T_2(m)}\left(\dfrac{1}{3}\dfrac{T_3(m)}{T_2(m)} + \dfrac{H(m)}{Q} + \dfrac{T_{1,\lambda}(m)}{P_\lambda}\right)$	$\operatorname{sign} Q = \varepsilon$, $\operatorname{sign} P_\lambda = \delta$ $\operatorname{sign} T_2(m) = \sigma$ $m = -1, -\tfrac{1}{2}, -\tfrac{1}{3}$ resp.	$[\alpha u,0,0]$
6	$[\varepsilon u+\delta\lambda+\sigma nv,\sigma mu+\rho u^2+qu^3,\sigma]$ $m \neq 0, \tfrac{1}{2}, \tfrac{1}{3}$ $n \neq 0, \tfrac{3}{4}$	(1) 5	$P = Q = 0$ $J = m$, $K = mn$ and two further equations	$\operatorname{sign} P_v = \varepsilon$, $\operatorname{sign} P_\lambda = \delta$ $\operatorname{sign} R = \sigma$	$[0,\alpha,0]$

$\varepsilon,\delta,\sigma = \pm 1$

$T_1(m) = P_v - mQ \qquad T_{1,\lambda}(m) = P_{v\lambda} - mQ_\lambda \qquad H(m) = P_v - 2Q_u + (4m+3)R$

$T_2(m) = P_{uv} + (m+1)P_v - 2mQ_u + (m+1)(2m+1)R$

$T_3(m) = P_{uuv} + 3(m+1)P_{uv} + (m+1)(2m+1)P_u - 3mQ_u - 3m(m+1)Q_v + 3(m+1)(2m+1)R_v$

$J = (P_\lambda Q_v - P_v Q_\lambda)/P_\lambda R \qquad K = (P_v Q_u - P_u Q_v)/R^2$

Table 3.1.1. Solution of the Smooth Recognition Problem for $\mathbb{Z}_2$-Equivariant Singularities of Top. Codimension ≤ 1.

	Normal Form	Codim.	Defining Equations	Nondegeneracy Conditions	Unfolding Terms
1	$[\varepsilon m u+\delta\lambda, \varepsilon, 0]$ $m \neq -1, -\frac{1}{2}, -\frac{1}{3}$	0	$P = 0$	$\operatorname{sign} Q = \varepsilon$, $\operatorname{sign} P_\lambda = \delta$ $P_u / Q \in (m)$	$\varnothing$
2	$[\varepsilon m u+\delta\lambda^2+\sigma u^2, \varepsilon, 0]$ $m \neq -1, -\frac{1}{2}, -\frac{1}{3}$	1	$P = P_\lambda = 0$	$\operatorname{sign} Q = \varepsilon$, $\operatorname{sign} P_{\lambda\lambda} = \delta$ $\operatorname{sign} T_2 = \sigma$, $P_u / Q \in (m)$	$[\alpha\lambda, 0, 0]$
3 4 5	$[-\varepsilon u+\delta\lambda+\sigma u^2, \varepsilon, 0]$ $[-\frac{1}{2}\varepsilon u+\delta\lambda+\sigma u^2, \varepsilon, 0]$ $[-\frac{1}{3}\varepsilon u+\delta\lambda+\sigma u^2, \varepsilon, 0]$	1	$P = T_1(m) = 0$	$\operatorname{sign} Q = \varepsilon$, $\operatorname{sign} P_\lambda = \delta$ $\operatorname{sign} T_2(m) = \sigma$ $m = -1, -\frac{1}{2}, -\frac{1}{3}$ resp.	$[\alpha u, 0, 0]$
6	$[\varepsilon u+\delta\lambda+\sigma n v, \sigma m u, \sigma]$ $m \neq 0, \frac{1}{3}, \frac{1}{2}$ $n \neq 0, \frac{2}{3}$	1	$P = Q = 0$	$\operatorname{sign} P_v = \varepsilon$, $\operatorname{sign} P_\lambda = \delta$ $\operatorname{sign} R = \sigma$, $J \in (m)$ $K / J \in (n)$	$[0, \alpha, 0]$

$\varepsilon, \delta, \sigma = \pm 1$

$T_1(m) = P_u - mQ$

$T_2(m) = P_{uu} + (m+1)P_v - 2mQ_u + (m+1)(2m+1)R$

$J = \dfrac{P_\lambda Q_v - P_v Q_\lambda}{P_\lambda R} \qquad K = \dfrac{P_v Q_u - P_u Q_v}{R^2}$

Table 3.1.2. Solution of the Qualitative Recognition Problem for D-Equivariant Singularities of Codimension ≤ 1.

§3.2. Application to Barium Titanate Crystals.

One possible application for equivariant bifurcation theory is the phenomenological theory of crystals. The theory deals with the change in structure of a crystal with temperature and the resulting polarisation along an axis. This behaviour corresponds to *spontaneous symmetry breaking* from a trivial solution with full symmetry to a bifurcation with a smaller isotropy subgroup of symmetries. An example of a crystal with cubic symmetry is the barium titanate crystal. This has a barium ion at each vertex, an oxygen ion at each face centre, and a titanium ion at the centre of the crystal. Our following analysis should work equally well for other crystals with cubic symmetry. However different parameter values corresponding to differing properties of crystals could lead to strikingly different bifurcation diagrams.

Devonshire [1949] noted that as temperature is decreased from above 120°C, the structure of a barium titanate crystal undergoes successive changes from one having the full group of symmetries of the cube to three structures with less symmetry. These states are referred to in the Physics literature as cubic, tetragonal, orthorhombic and rhombohedral respectively, the last three corresponding to $\mathbf{Z}_4$, $\mathbf{Z}_2 \oplus \mathbf{Z}_2$, and S_3, the three maximal isotropy subgroups of $\mathbf{O}$. Furthermore, the axes of polarisation are the corresponding one-dimensional fixed-point subspaces.

Our results say that in the nongeneric situation the orthorhombic state cannot be stable locally but this is not a contradiction since we do not preclude the possibility of stability away from the origin. In fact by considering the universal unfolding of a suitably degenerate normal form we are able to produce precisely the scenario described above. (Our unfolding corresponds to the derivative of a free energy polynomial. Employing Landau theory, Devonshire [1949] minimised sucn a polynomial in order to explain the transitions.) Our choice of normal form must permit local submaximal branching and according to Theorem 1.3.1 (ii) this is impossible with $Q(0)$ nonzero. Hence the normal form in question is the sixth in our list:

$$[\varepsilon u + \delta\lambda + \sigma n v, \sigma m u + p u^2 + q u^3, \sigma],$$

$$\varepsilon, \delta, \sigma = \pm 1, \ m \neq 0, 1, \tfrac{1}{3}, \ n \neq 0, \tfrac{1}{3}.$$

Our analysis is simplified by choosing suitable values and ranges of values for the various parameters. As mentioned in §3.1, the qualitative nature of the bifurcation diagrams is not affected by the values of p and q. Accordingly, we set $p = q = 0$. Then, as is standard for physical applications, we set $\varepsilon = +1$, $\delta = -1$. These values are necessary to ensure that the trivial solution is stable subcritically and unstable supercritically, thus allowing spontaneous symmetry breaking as λ passes through zero. The qualitative effect of the modal parameter n is almost negligible and in order to simplify calculations we will set $n = 0$. This is a critical value for n and will thus lead to degeneracies in the bifurcation diagrams. However we will keep track of the rare occasions when we should not have taken n to be zero and will indicate the true picture at these 'degeneracies'. Finally, we will postpone a choice of value for σ and range of values for m. These choices are far more sensitive and lead to a wide range of interesting scenarios including the one required for the intended application.

Thus we analyse the unfolding

$$G(\alpha) = [\,u-\lambda,\alpha+\sigma mu,\sigma\,],$$
$$\sigma = \pm 1,\ m \neq 0, \tfrac{1}{2}, \tfrac{1}{3}.$$

The branching and stability data are summarised in Table 3.2.1. We notice that there are four λ values at which an eigenvalue on a maximal branch changes sign giving rise to a submaximal secondary bifurcation. These intersections values are

$$\lambda_1 = -\frac{1}{\sigma m}\alpha, \qquad (x,0,0),\ (x,y,0) \text{ and } (x,y,y) \text{ branches},$$

$$\lambda_2 = -\frac{1}{\sigma m}\alpha, \qquad (x,x,0) \text{ and } (x,y,0) \text{ branches},$$

$$\lambda_3 = \frac{2}{\sigma(1-2m)}\alpha + \frac{\sigma}{(1-2m)^2}\alpha^2, \qquad (x,x,0) \text{ and } (x,x,z) \text{ branches},$$

$$\lambda_4 = \frac{3}{\sigma(1-3m)}\alpha + \frac{2\sigma}{(1-3m)^2}\alpha^2, \qquad (x,x,x),\ (x,x,z) \text{ and } (x,y,y) \text{ branches}.$$

It would appear that $\lambda_1 = \lambda_2$. In fact, the coefficient of α^2 in $\lambda_1-\lambda_2$ is not identically zero. This is the artificial degeneracy introduced by setting $n = 0$. We shall draw the bifurcation diagrams as if $\lambda_1 = \lambda_2$ but it should be remembered that one of the λ-values occurs slightly before the other (α^2 is small compared with α). Which value occurs first depends on the sign of n.

The order of occurrence of λ_1, λ_3, λ_4 is indicated in Table 3.2.2 together with the stability assignments on the maximal branches and the existence of submaximal branching. The + and - signs indicate the signs of the three eigenvalues on a branch. The branch is stable if and only if the signs are +++.

It is now possible to pick out the sequence of events corresponding to the results in Devonshire [1949]. In particular, the $(x, x, 0)$ branch must start unstable, stabilise, and then become unstable again. Hence we must have either

$$\alpha < 0,\ \sigma m > 0,\ \sigma(2m-1) > 0,\ \sigma = +1, \qquad (1)$$

or

$$\alpha > 0,\ \sigma m < 0,\ \sigma(2m-1) < 0,\ \sigma = -1. \qquad (2)$$

However, if condition (2) holds then it is impossible for the $(x, 0, 0)$ branch to be stable from the origin. Hence we must have condition (1). Finally, we need

$$\sigma(3m-1) > 0,$$

in order for the (x,x,x) branch to restabilise. Henco we have the following independent requirements:

$$\alpha < 0,\ \sigma = 1,\ m > \tfrac{1}{2}. \qquad (3)$$

The parameter values in (3) lead to Fig. 3.2.1. We have included the submaximal branches in the diagram even though it is not clear that they are relevant to the application. What appears to happen in practice is that a mixture of say $(x,0,0)$ and $(x,x,0)$ states occurs for a short time between the two pure states. For this reason we have not performed the complicated calculations that would give us the stabilities of the submaximal branches. In any case, these stabilities are not necessarily invariant under the equivalence relation.

Fig. 3.2.1 could equally well model the behaviour of other crystals with cubic symmetry and similar characteristics to barium titanate, or even totally different physical systems. (Note that for certain ranges of parameter values, it might be possible for the $(x,x,0)$ state to be bypassed via the (x,y,y) secondary bifurcation). On the other hand, a better model might be given by Figs. 3.2.2-3.2.8.

We have only drawn the bifurcation diagrams that arise from the unfolding of normal form 6. This is because the diagrams associated to the other normal forms are not very interesting. In particular, no mode interactions are possible.

	Maximal Branching Equations	Stabilities
$(x,0,0)$	$\lambda = x^2+\sigma m x^4+\alpha x^2$	$\mu_1 > 0$ $\text{sign}\ \mu_2 = \text{sign}\ \mu_3 = -\text{sign}(\alpha+\sigma m x^2)$
$(x,x,0)$	$\lambda = 2x^2+2\sigma m x^4+\alpha x^2$	$\text{sign}\ \mu_1 = \text{sign}(\alpha+2\sigma m x^2)$ $\mu_2 > 0$ $\text{sign}\ \mu_3 = -\text{sign}(\alpha+\sigma(2m-1)x^2)$
(x,x,x)	$\lambda = 3x^2+\sigma(3m+1)x^4+\alpha x^2$	$\text{sign}\ \mu_1 = \text{sign}\ \mu_2 = \text{sign}(\alpha+\sigma(3m-1)x^2)$ $\mu_3 > 0$

	Submaximal Branching Equations	Intersection with Maximals
$(x,y,0)$	$\sigma m\lambda = -\alpha$ $\sigma m y^2 = -\sigma m x^2-\alpha$	$y=0 \quad \sigma m\lambda = \sigma m x^2 = -\alpha$ $y=x \quad \sigma m\lambda = 2\sigma m x^2 = -\alpha$
(x,x,z)	$\sigma m\lambda = \sigma x^2-\alpha+(1-m)x^4-\sigma\alpha x^2$ $\sigma m z^2 = \sigma(1-2m)x^2-\alpha$	$z=0 \quad \sigma(1-2m)\lambda = 2\sigma(1-2m)x^2+o(x^4)$ $= 2\alpha+o(\alpha^2)$ $z=x \quad \sigma(1-3m)\lambda = 3\sigma(1-3m)x^2+o(x^4)$ $= 3\alpha+o(\alpha^2)$
(x,y,y)	$\lambda = x^2+2y^2+\sigma y^2(x^2+y^2)$ $\sigma(2m-1)y^2 = -\sigma m x^2-\alpha$	$y=0 \quad \sigma m\lambda = \sigma m x^2 = -\alpha$ $y=x \quad \sigma(1-3m)\lambda = 3\sigma(1-3m)x^2+o(x^4)$ $= 3\alpha+o(\alpha^2)$
(x,y,z)	No solution	

Table 3.2.1. Branching Equations and Eigenvalues of Normal Form 6.

	Parameter Values	$\alpha < 0$	$\alpha > 0$
$(x,0,0)$	$\sigma m<0$	$+++$	$+--,\lambda_1,+++$
	$\sigma m>0$	$+++,\lambda_1,+--$	$+--$
$(x,x,0)$	$\sigma m<0$, $\sigma(2m-1)<0$, $\sigma=-1$	$-++$	$++-,\lambda_1,-+-,\lambda_3,-++$
	$\sigma=+1$		$++-,\lambda_3,+++,\lambda_1,-++$
	$\sigma m<0$, $\sigma(2m-1)>0$	$-++,\lambda_3,-+-$	$++-,\lambda_1,-+-$
	$\sigma m>0$, $\sigma(2m-1)<0$	$-++,\lambda_1,+++$	$++-,\lambda_3,+++$
	$\sigma m>0$, $\sigma(2m-1)>0$, $\sigma=-1$	$-++,\lambda_3,-+-,\lambda_1,++-$	$++-$
	$\sigma=+1$	$-++,\lambda_1,+++,\lambda_3,++-$	
(x,x,x)	$\sigma(3m-1)<0$	$--+$	$+++,\lambda_4,--+$
	$\sigma(3m-1)>0$	$--+,\lambda_4,+++$	$+++$
$(x,y,0)$	$\sigma m<0$	×	$(x,0,0,\lambda_1)\rightarrow(x,x,0,\lambda_1)$
	$\sigma m>0$	$(x,0,0,\lambda_1)\rightarrow(x,x,0,\lambda_1)$	×
(x,x,z)	$\sigma(2m-1)<0$, $\sigma(3m-1)<0$, $\sigma=-1$	×	$(x,x,x,\lambda_4)\rightarrow(x,x,0,\lambda_3)$
	$\sigma=+1$		$(x,x,0,\lambda_3)\rightarrow(x,x,x,\lambda_4)$
	$\sigma(2m-1)<0$, $\sigma(3m-1)>0$	(x,x,x,λ_4)	$(x,x,0,\lambda_3)$
	$\sigma(2m-1)>0$, $\sigma(3m-1)<0$	$(x,x,0,\lambda_3)$	(x,x,x,λ_4)
	$\sigma(2m-1)>0$, $\sigma(3m-1)>0$, $\sigma=-1$	$(x,x,0,\lambda_3)\rightarrow(x,x,x,\lambda_4)$	×
	$\sigma=+1$	$(x,x,x,\lambda_4)\rightarrow(x,x,0,\lambda_3)$	
(x,y,y)	$\sigma m<0$, $\sigma(3m-1)<0$, $\sigma=-1$	×	$(x,0,0,\lambda_1)\rightarrow(x,x,x,\lambda_4)$
	$\sigma=+1$		$(x,x,x,\lambda_4)\rightarrow(x,0,0,\lambda_1)$
	$\sigma m<0$, $\sigma(3m-1)>0$	(x,x,x,λ_4)	$(x,0,0,\lambda_1)$
	$\sigma m>0$, $\sigma(3m-1)<0$	$(x,0,0,\lambda_1)$	(x,x,x,λ_4)
	$\sigma m>0$, $\sigma(3m-1)>0$, $\sigma=-1$	$(x,0,0,\lambda_1)\rightarrow(x,x,x,\lambda_4)$	×
	$\sigma=+1$	$(x,x,x,\lambda_4)\rightarrow(x,0,0,\lambda_1)$	

Table 3.2.2. Order of Branching and Stability Assignments for Normal Form 6.

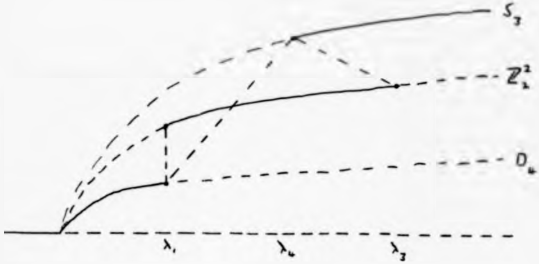


Fig 3.2.1. $\sigma > 0$, $m > \frac{1}{2}$, $\alpha < 0$.

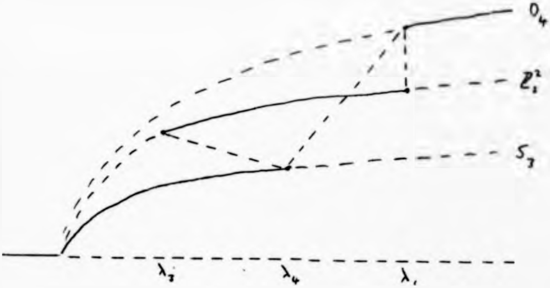


Fig. 3.2.2. $\sigma > 0$, $m < 0$, $\alpha > 0$.

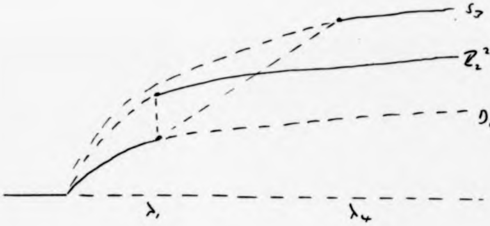


Fig. 3.2.3. $\sigma > 0, \frac{1}{3} < m < \frac{1}{2}, \alpha < 0.$

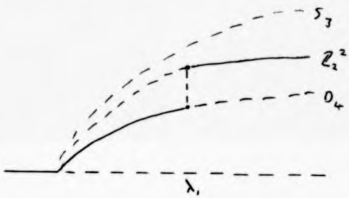


Fig. 3.2.4. $\sigma > 0, 0 < m < \frac{1}{3}, \alpha < 0.$

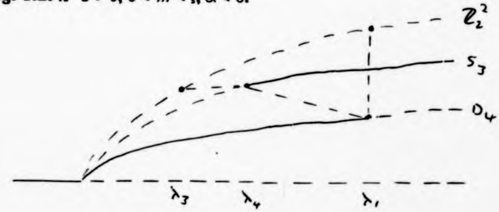


Fig. 3.2.5. $\sigma < 0, m < 0, \alpha < 0.$

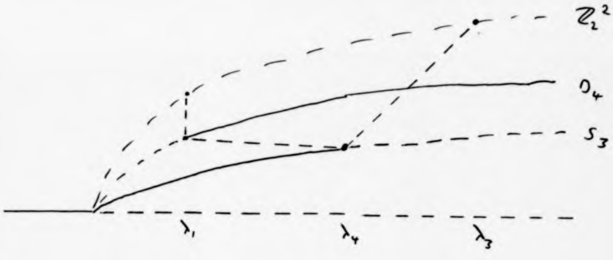


Fig. 3.2.6. $\sigma < 0$, $m > \frac{1}{2}$, $\alpha > 0$.

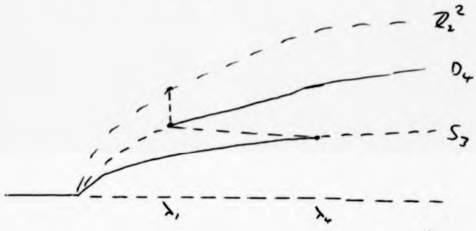


Fig. 3.2.7. $\sigma < 0$, $\frac{1}{3} < m < \frac{1}{2}$, $\alpha > 0$.

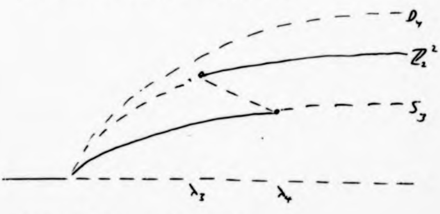


Fig. 3.2.8. $\sigma > 0$, $0 < m < \frac{1}{3}$, $\alpha > 0$.

§3.3. Tangent Space Calculations for the Linearly Determined Bifurcation Problems.

In this section, we obtain the information displayed in Table 3.3.1. In particular, we show that all but one of the low codimension bifurcation problems are linearly determined; that is their unipotent tangent spaces are invariant under the group of unipotent equivalences. It is convenient to work throughout under the coordinate change in the Appendix. We split the calculations up into three stages.

Step 1 Calculate $\tilde{T}(f,U)$, the $\mathcal{E}_{u,v,w,\lambda}$-module part of $T(f,U)$. As we have a $\mathcal{E}_{u,v,w,\lambda}$-module we can use Nakayama's Lemma (Lemma 3.10 of Golubitsky & Schaeffer [1979a]). In order to verify that $\tilde{T}(f,U)$ contains a finitely generated $\mathcal{E}_{u,v,w,\lambda}$-submodule I, we need only show

$$I \subset \tilde{T}(f,U) + \mathcal{M}I,$$

where $\mathcal{M}$ is the maximal ideal in $\mathcal{E}_{u,v,w,\lambda}$.

Step 2 Calculate the unipotent tangent space,

$$T(f,U) = \tilde{T}(f,U) + \mathcal{E}_\lambda\{\lambda^2 f_\lambda\}.$$

Step 3 Find the high order term module,

$$\mathcal{P}(f,U) = \operatorname{Itr}_U T(f,U).$$

If $\mathcal{P}(f,U) = T(f,U)$, then f is linearly determined.

Step 1 This is just an application of Nakayama's Lemma. As an example, we give the details for the two infinite families

$$f = (b\lambda + cu^k + du^{k+1}, a, 0)_m;\ m = -1 \text{ or } -\tfrac{1}{2};\ k \geq 2. \qquad (1)$$

We have to prove that

$$\tilde{T}(f, U) = I + W,$$

where, suppressing subscript m's, I is generated as an $\mathcal{E}_{u,v,w,\lambda}$-module by

$$\begin{array}{lllllll} (u^{k+2},0,0) & (v^2,0,0) & (w,0,0) & (\lambda^2,0,0) & (uv,0,0) & (u^2\lambda,0,0) & (v\lambda,0,0) \\ (0,u^2,0) & (0,v,0) & (0,w,0) & (0,\lambda,0) & & & \\ (0,0,u) & (0,0,v) & (0,0,w) & (0,0,\lambda) & & & \end{array}$$

and

$$W = \mathbb{R}\left\{\begin{array}{ll} ((k-1)cu^{k+1} - bu\lambda, 0, 0) & (kcu^{k+1} - 2av, 0, 0) \\ (2v, u, 0) & ((4m+3)v, 0, -1) \end{array}\right\}.$$

To prove this it suffices by Nakayama's Lemma to work modulo $\mathcal{M}I$. Modulo $\mathcal{M}I$ the generators in Corollary A.4 reduce to

$(v^2,0,0)$ $(0,v,0)$ $(0,0,v)$ $(v\lambda,0,0)$ $(w,0,0)$ $(0,w,0)$ $(0,0,w)$ $(0,0,\lambda)$ $(0,0,u)$

$(bu\lambda + cu^{k+1} + du^{k+2}, au, 0)$ $(kcu^{k+1} + (k+1)du^{k+2}, au, 0)$

$(b\lambda^2 + cu^k\lambda, a\lambda, 0)$ $(kcu^k\lambda, a\lambda, 0)$

$(bu\lambda + cu^{k+1} + du^{k+2} - 2av, 0, 0)$

$(-av - bmu\lambda - cmu^{k+1} - dmu^{k+2}, a(m+1)u + b\lambda + cu^k, a)$

$(-bmu\lambda + (k-1)cmu^{k+1} + kdmu^{k+1} + 2amv + 2kcu^{k-1}v, amu + b\lambda + cu^k, 0)$

$(bu^2\lambda + cu^{k+2}, au^2, 0)$ $(kcu^{k+2}, au^2, 0)$

$(bu^2\lambda + cu^{k+2} - 2auv, 0, 0)$ $(-auv - bmu^2\lambda - cmu^{k+2}, a(m+1)u^2, 0)$

$(-bu^2\lambda + (k-1)cu^{k+2} + 2auv, au^2, 0)$.

The last five generators yield $(u^{k+2},0,0)$, $(u^2\lambda,0,0)$, $(uv,0,0)$ and

$(0, u^2, 0)$ since the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ k & 0 & 0 & 1 \\ 1 & 1 & -2 & 0 \\ m & m & 1 & -(m+1) \\ (k-1) & -1 & 2 & 1 \end{pmatrix}$$

has full rank. The required result now follows easily.

Step 2 This step is trivial. Indeed, for the normal forms 1(ii), 3, 4 and 5 $T(f,U) = \tilde{T}(f,U)$. In the cases of the normal forms 1(i) and 2, $\tilde{T}(f,U)$ contains $\mathcal{E}_\lambda\{\lambda^3 f_\lambda\}$, but the introduction of the term $\lambda^2 f_\lambda$ simplifies the form of the tangent space quite considerably.

Step 3 Our method is to find as big a U-intrinsic part as possible by Theorem 2.4.4 and then to use Proposition 2.4.1 on what is left over. For example, consider normal form 1(ii) in Table 3.3.1. In this case $T(f,U) = \tilde{T}(f,U)$. We claim firstly that $T(f,U)$ has the alternative characterisation given in the third column of the table. The following table gives all monomials in (x,y,z) of a given degree. (Recall that u, v, w have degree 2,4,6 and the equivariant generators X_1, X_2, X_3 have degrees 1,3,5. Also note that our coordinate changes in the Appendix preserve these degrees.)

Order	(*,0,0)	(0,*,0)	(0,0,*)	
1	1			
3	u	1		
5	u^2, v	u	1	
7	u^3, uv, w	u^2, v	u	etc.

We see from Table 3.3.1 that ignoring terms with λ we are only missing

$$(1,0,0),\ (u,0,0),\ (u^2,0,0),\ (0,1,0).$$

Hence we have all terms of order ≥ 7 and so

$$T(f,U) \supset \vec{M}_7 E_\lambda .$$

Similarly

$$T(f,U) \supset \vec{M}_3\langle\lambda^k\rangle,\ \vec{M}_1\langle\lambda^{k+1}\rangle.$$

We are left with

$$(v\lambda^r,0,0),\ (0,u\lambda^r,0),\ (0,0,\lambda^r);\quad r = 0,\dots,k-1$$
$$(0,\lambda^r,0);\qquad\qquad r = 1,\dots,k-1.$$

Hence we have verified the entry in Table 3.3.1. It remains to show that

$T(f,U)$ is U-intrinsic. By Proposition 2.4.2 we have

$$\mathcal{P}(f,U) \supset \vec{M}_7 E_\lambda + \vec{M}_3\langle\lambda^k\rangle + \vec{M}_1\langle\lambda^{k+1}\rangle.$$

Now consider the set

$$V = \{(v,0,0)\ (0,u,0)\ (0,0,1)\}$$
$$\subset \vec{M}_5 E_\lambda \cap T(f,U).$$

It is a simple application of Lemma 2.4.1 to see that

$$\mathcal{P}(f,U) \supset (\vec{\mathcal{M}}_7 + V)\mathcal{E}_\lambda + \vec{\mathcal{M}}_3\langle\lambda^k\rangle + \vec{\mathcal{M}}_1\langle\lambda^{k+1}\rangle.$$

Hence it remains only to show that

$$\mathcal{P}(f,U) \supset W,$$

where

$$W = \mathbb{R}\{(0,\lambda,0)_\#, \ldots, (0,\lambda^{k-1},0)_\#\}.$$

Using Corollary A.4, it is easy to check that

$$\tilde{T}(w,U) \subset \mathcal{P}(f,U) \text{ for all } w \in W.$$

Also $\lambda^2 w_\lambda$ is clearly contained in $\mathcal{P}(f,U) + W$ for all $w \in W$. Hence by Proposition 2.4.1 we have

$$\mathcal{P}(f,U) = T(f,U).$$

The calculations for normal forms 1(i) and 2 are even more straightforward. $T(f,U)$ can be written as in Table 3.3.1 and it is then immediate by Proposition 2.4.2 that it is U-intrinsic. However normal forms 3,4 and 5 present more difficulties. Again there are no problems in obtaining the results in Table 3.3.1. Now suppose $k = 2$. By Theorem 2.4.4 we have

$$\mathcal{P}(f,U) \supset (\vec{\mathcal{M}}_9 + \mathbb{R}\{(uv,0,0), (w,0,0), (0,u^2,0), (0,v,0), (0,0,u)\})\mathcal{E}_\lambda$$
$$+ (\vec{\mathcal{M}}_5 + \mathbb{R}(0,1,0))\langle\lambda\rangle + \vec{\mathcal{M}}_1\langle\lambda^2\rangle + \mathbb{R}(cu^3 - bu\lambda,0,0).$$

It remains to show that $\mathcal{P}(f,U)$ contains W where

$$W = \mathbb{R}\{(cu^3 - av,0,0), (2v,u,0), ((4m+3)v,0,-1)\}.$$

It suffices by Proposition 2.4.1 to show that

$$T(\rho, U) \subset \mathcal{P}(f, U)$$

for $\rho = (v,0,0), (0,u,0), (0,0,1)$. This is an awkward calculation. We cannot use the tangent space generators in Corollary A.4 since ρ does not satisfy the conditions $P_v, Q_*, R = 0$. We have to write ρ in the original coordinates, work out the tangent space generators using Theorem A.3, and then change coordinates again to verify that the generators are contained in $\mathcal{P}(f, U)$. Great care must be taken with these calculations. For example, consider $\rho = (v_*,0,0)_*$ and T_{12}. In the original coordinates

$$\rho = [v + \tfrac{1}{2}(m+1)u^2, 0, 0].$$

We have

$$\begin{aligned}
T_{12} &= [-(u^2-2v)(m+1)u + uv - 3w, \tfrac{1}{2}(m+1)u^2 - v, 0] \\
&= [-(m+1)u^3 + (2m+3)uv - 3w, \tfrac{1}{2}(m+1)u^2 - v, 0] \\
&= [-(m+1)u^3 + (2m+3)u(v_* + \tfrac{1}{2}(m+1)u^2) - 3(w_* + \tfrac{1}{6}(m+1)(2m+1)u^3), -v_*, 0] \\
&= [\tfrac{1}{2}(m+1)(-2+2m+3-2m-1)u^3 + (2m+3)uv_* - 3w_*, -v_*, 0] \\
&= [(2m+3)uv_* - 3w_*, -v_*, 0] \\
&= ((2m+3)uv_* - 3w_*, -v_*, 0)_* \subset \mathcal{P}(f, U).
\end{aligned}$$

The third and sixth equalities are the second and first coordinate changes respectively. T_{12} is the most difficult generator for $(v,0,0)$ and eventually we do obtain the required result when $k = 2$.

When $k > 2$ even more work is required, although the fact that we exclude the case $m = -\tfrac{1}{2}$ simplifies things slightly. In particular, $w_* = w$

and $(0,0,R)_{\#} = [0,0,R]$. We can see immediately that

$$\mathcal{P}(f,U) \supset \vec{M}_{2k+5}\mathcal{E}_{\lambda} + \vec{M}_5\langle\lambda\rangle + \vec{M}_1\langle\lambda^2\rangle$$
$$+ \mathbb{R}\{((k-1)cu^{k+1} - bu\lambda, 0, 0)\}.$$

Also, a glance at the tangent space generators in Theorem A.3 reveals that

factors of w cannot be removed and so

$$\mathcal{P}(f,U) \supset \mathcal{E}_{u,v,w,\lambda}\{(w,0,0)\,(0,w,0)\,(0,0,w)\}.$$

Furthermore, if $p = (0,0,R)$, then

$$T(p,U) \subset \mathcal{E}_{u,v,w,\lambda}\{(wR,0,0)\,(0,wR,0)\,(0,0,R)\}.$$

Hence all the hard work lies in checking that

$$T(p,U) \subset \mathcal{P}(f,U)$$

for $p \in \mathcal{E}_{u,v}\{(v,0,0)\,(0,u,0)\,(0,v,0)\}$. We omit these details.

	Normal Form f	$\tilde{F}(f,U)$	Tangent Space $T(f,U) = \mathcal{P}(f,U)$
1(i)	$(b\lambda^k + cu^2, a, 0)_m$ $k \geq 1;\ a,b,c \neq 0$ $m \neq -1, -\frac{1}{2}, -\frac{1}{3}$	$(u^3,0,0)\ (v,0,0)\ (w,0,0)\ (\lambda^{2k},0,0)\ (u\lambda^k,0,0)$ $(0,u,0)\ \ (0,v,0)\ (0,w,0)\ (0,\lambda^k,0)$ $(0,0,1)$ $(b\lambda^{k+1} - cu^2\lambda,0,0)\ (2b\lambda^{k+1}, a\lambda, 0)$	$(\vec{M}_7 + \mathbb{R}\{(v,0,0),(0,u,0),(0,0,1)\})E_\lambda$ $+ (\vec{M}_5 + \mathbb{R}(0,1,0))\langle\lambda\rangle$ $+ \vec{M}_3\langle\lambda^k\rangle + \vec{M}_1\langle\lambda^{k+1}\rangle$
1(ii)	$(b\lambda^k, a, 0)_m$ $k \geq 1;\ a,b \neq 0$ $m \neq -1, -\frac{1}{2}, -\frac{1}{3}$	$(u^3,0,0)\ (v,0,0)\ (w,0,0)\ (\lambda^{k+1},0,0)\ (u\lambda^k,0,0)$ $(0,u,0)\ \ (0,v,0)\ (0,w,0)\ (0,\lambda,0)$ $(0,0,1)$	$(\vec{M}_7 + \mathbb{R}\{(v,0,0),(0,u,0),(0,0,1)\})E_\lambda$ $+ \vec{M}_3\langle\lambda^k\rangle + \vec{M}_1\langle\lambda^{k+1}\rangle$ $+ \mathbb{R}\{(0,\lambda,0),\ldots,(0,\lambda^{k-1},0)\}$
2	$(b\lambda^2 + cu^2 + du\lambda, a, 0)_m$ $a,b,c \neq 0$ $m \neq -1, -\frac{1}{2}, -\frac{1}{3}$	$(u^3,0,0)\ (v^2,0,0)\ (w,0,0)\ (\lambda^4,0,0)\ (uv,0,0)\ (u\lambda^2,0,0)\ (v\lambda,0,0)$ $(0,u^2,0)\ (0,v,0)\ \ (0,w,0)\ (0,\lambda^2,0)\ \ \ \ \ (0,u\lambda,0)$ $(0,0,u)\ \ (0,0,v)\ \ (0,0,w)\ (0,0,\lambda)$ $(b\lambda^3 - cu^2\lambda,0,0)\ (2b\lambda^3, a\lambda, 0)$ $(2v,u,0)\ (-v,(2m+1)u,1)\ (du^2\lambda - 2av,0,0)$	$(\vec{M}_7 + \mathbb{R}\{(v,0,0),(0,u,0),(0,0,1)\})E_\lambda$ $+ (\vec{M}_5 + \mathbb{R}(0,1,0))\langle\lambda\rangle$ $+ \vec{M}_3\langle\lambda^2\rangle + \vec{M}_1\langle\lambda^3\rangle$
3-5	$(b\lambda + cu^k + du^{k+1}, a, 0)_m$ $m = -1$ or $-\frac{1}{2}$, $k \geq 2$ or $m = -\frac{1}{3}$, $k = 2$	$(u^{k+2},0,0)\ (v^2,0,0)\ (w,0,0)\ (\lambda^2,0,0)\ (uv,0,0)\ (u^2\lambda,0,0)\ (v\lambda,0,0)$ $(0,u^2,0)\ \ \ (0,v,0)\ \ (0,w,0)\ (0,\lambda,0)$ $(0,0,u)\ \ \ \ (0,0,v)\ \ (0,0,w)\ (0,0,\lambda)$ $((k-1)cu^{k+1} - bu\lambda,0,0)\ (kcu^{k+1} - 2av,0,0)$ $(2v,u,0)\ ((4m+3)v,0,-1)$	$\vec{M}_{2\leq k\leq 5}E_\lambda + (\vec{M}_5 + \mathbb{R}(0,1,0))\langle\lambda\rangle)$ $+ \vec{M}_1\langle\lambda^2\rangle + \mathbb{R}((k-1)cu^{k+1} - bu\lambda,0,0)$ $+ E_{u,v,w,\lambda}\{(w,0,0),(0,w,0),(0,0,u),(0,0,v),(0,0,w),(0,0,\lambda)\}$ $+ E_{u,v}\{(v,0,0),(0,u,0),(0,v,0)\}$ $+ \mathbb{R}\{(kcu^{k+1} - 2av,0,0),(2v,u,0),((4m+3)v,0,-1)\}$

Table 3.3.1. Algebraic Data for the Unipotent Recognition Problem.

§3.4. The Recognition Problem for the Linearly Determined Bifurcation Problems.

In this section, we verify the entries in Tables 3.1.1, 3.4.1 and 3.4.2 for the normal forms 1 to 5. We showed in §3.3 that the unipotent tangent space was U-intrinsic for each of these normal forms and hence the solution of the unipotent recognition problems only requires linear algebra. The solution of the full recognition problem falls naturally into three stages.

Step 1 Solution of the unipotent recognition problem in the preferred coordinates (Table 3.4.1).

Step 2 Solution of the full recognition problem in the preferred coordinates (Table 3.4.2).

Step 3 Solution of the full recognition problem in the original coordinates (Table 3.1.1).

Step 1 Suppose f is one of the normal forms 1 to 5. Then $T(f,U)$ is U-intrinsic, and hence by Theorem 2.3.4 we have

$$g \text{ is } U\text{-equivalent to } f \text{ if and only if } g \equiv f \bmod T(f,U).$$

Now, for the normal forms 1(i), 1(ii) and 2, $T(f,U)$ is generated by monomials and so the linear algebra is very straightforward. For example, consider normal form 1(i),

$$f = (b\lambda^k + cu^2, a, 0);\ k \geq 1.$$

Notice that the unipotent tangent space contains all monomials except

$$(1,0,0), (\lambda,0,0), \ldots, (\lambda^{k-1},0,0),$$
$$(u,0,0), (u\lambda,0,0), \ldots, (u\lambda^{k-1},0,0),$$
$$(u^2,0,0), (\lambda^k,0,0), (0,1,0).$$

Hence (P, Q, R) is U-equivalent to f if and only if

$$P = P_\lambda = \ldots = P_{\lambda^{k-1}} = 0,$$

$$P_u = P_{u\lambda} = \ldots = P_{u\lambda^{k-1}} = 0,$$

$$P_{uu} = 2c\,,\ P_{\lambda^k} = k!\,b,\ Q = a.$$

Normal forms 3, 4 and 5 do not cause many more problems. In the preferred coordinates, each normal form is represented by

$$f = (b\lambda + cu^k + du^{k+1}, a, 0)_m\ ; m = -1 \text{ or } -\tfrac{1}{2},\ k \geq 2; \text{ or } m = -\tfrac{1}{2},\ k = 2.$$

The unipotent tangent space contains all monomials except for

$$(1,0,0), (u,0,0), \ldots, (u^k,0,0), (\lambda,0,0), (0,1,0),$$
$$(u^{k+1},0,0), (u\lambda,0,0), (v,0,0), (0,u,0), (0,0,1).$$

In addition $T(f, U) \supset W$ where

$$W = \mathbb{R}\left\{\begin{matrix}((k-1)cu^{k+1} - bu\lambda,0,0)\ (kcu^{k+1} - 2av,0,0) \\ (2v,u,0)\ ((4m+3)v,0,-1)\end{matrix}\right\}.$$

Hence (P, Q, R) is U-equivalent to f if and only if

$$P = P_u = \ldots = P_{u^{k-1}} = 0\,,\quad P_{u^k} = k!\,c\,,\ P_\lambda = b\,,\ Q = a$$

and there exist $A, B, C, D \in \mathbb{R}$ such that

$$P_{uk+1} = (k+1)\{d+(k-1)cA+kcB\},$$

$$P_{u\lambda} = -bA,$$

$$P_v = -2aB+2C+(4m+3)D,$$

$$Q_u = C,$$

$$R = -D.$$

These equations yield the required relation between P_{uk+1}, $P_{u\lambda}$, P_v, Q_u and R.

Step 2 Suppose n is a one of the normal forms 1 to 5 in Table 3.4.2. By Remark 2.2.1(a), a germ $f \in \vec{\mathcal{E}}_{x,y,z,\lambda}$ is D-equivalent to n if and only if f is U-equivalent to the corresponding normal form in Table 3.4.1 and if that normal form is equivalent by scalings to n.

For example, (P,Q,R) is U-equivalent to normal form 2 in Table 3.4.1 if and only if

$$Q = a,\ P_{\lambda\lambda} = 2b,\ P_{uu} = 2c,\ P_{u\lambda} = d,$$

$$P = P_\lambda = P_u = 0.$$

This normal form is equivalent by scalings to

$$(\delta\lambda^2+\sigma u^2+\rho u\lambda,\varepsilon,0)$$

if and only if there exist positive numbers μ, ν and l such that the equivalence

$$S(x,y,z,\lambda) = \mu,\ X(x,y,z,\lambda) = \nu(x,y,z),\ \Lambda(\lambda) = l\lambda,$$

transforms one normal form into the other. In other words μ, ν and l must satisfy

$$(\delta\mu\nu l^2\lambda^2+\sigma\mu\nu^5 u^2+\rho\mu\nu^3 lu\lambda, \epsilon\mu\nu^3, 0) = (b\lambda^2+cu^2+du\lambda, a, 0).$$

The equations

$$\begin{aligned} \epsilon\mu\nu^3 &= a, \\ \delta\mu\nu l^2 &= b, \\ \sigma\mu\nu^5 &= c, \\ \rho\mu\nu^3 l &= d, \end{aligned}$$

can be solved for $\mu, \nu, l > 0$ if and only if

$$\operatorname{sign} a = \epsilon,\ \operatorname{sign} b = \delta,\ \operatorname{sign} c = \sigma \text{ and } \frac{d}{\sqrt{|bc|}} = \rho. \tag{1}$$

But from the unipotent recognition problem we have

$$a = Q,\quad b = \tfrac{1}{2}P_{\lambda\lambda},\quad c = \tfrac{1}{2}P_{uu},\quad d = P_{u\lambda}$$

so (1) becomes

$$\operatorname{sign} Q = \epsilon,\ \operatorname{sign} P_{\lambda\lambda} = \delta,\ \operatorname{sign} P_{uu} = \sigma \text{ and}$$

$$\frac{P_{u\lambda}}{\sqrt{|P_{\lambda\lambda}P_{uu}|}} = d.$$

In addition, the unipotent recognition problem gives

$$P = P_\lambda = P_u = 0$$

as required.

Step 3 It remains to recover the necessary and sufficient conditions of Table 3.4.2 in the original coordinates. Note that in Table 3.1.1 we have given these results only up to topological codimension one.The results needed to do this are summarised in Proposition A.7. In fact we could recover the results for the infinite families 1(i) and 1(ii) since high partial derivatives with respect to λ are allowed for in Proposition A.7. Also we could deal with the infinite family 3. Here $m = -1$ and the second coordinate change is the identity, so we need only use Proposition A.5 which allows for all partial derivatives. It is the second coordinate change which causes more problems and our results in the Appendix do not suffice for the infinite family 4.

	Normal Form	Defining Equations
1(i)	$(b\lambda^k + cu^2, a, 0)_m$ $k \geq 1;\ a, b, c \neq 0$ $m \neq -1, -\frac{1}{2}, -\frac{1}{4}$	$Q = a,\ P_{\lambda^k} = k!b,\ P_{uu} = 2c$ $P = P_\lambda = \dots = P_{\lambda^{k-1}} = 0$ $P_u = P_{u\lambda} = \dots = P_{u\lambda^{k-1}} = 0$
1(ii)	$(b\lambda^k, a, 0)_m$ $k \geq 1;\ a, b \neq 0$ $m \neq -1, -\frac{1}{2}, -\frac{1}{4}$	$Q = a,\ P_{\lambda^k} = k!b$ $P = P_\lambda = \dots = P_{\lambda^{k-1}} = 0$ $P_u = P_{u\lambda} = \dots = P_{u\lambda^{k-1}} = 0$ $P_{uu} = P_{uu\lambda} = \dots = P_{uu\lambda^{k-1}} = 0$
2	$(b\lambda^2 + cu^2 + du\lambda, a, 0)_m$ $a, b, c \neq 0$ $m \neq -1, -\frac{1}{2}, -\frac{1}{4}$	$Q = a,\ P_{\lambda\lambda} = 2b,\ P_{uu} = 2c$ $P = P_\lambda = P_u = 0$ $P_{u\lambda} = d$
3-5	$(b\lambda + cu^k + du^{k+1}, a, 0)_m$ $a, b, c \neq 0$ $m = -1$ or $-\frac{1}{2}$, $k \geq 2$ or $m = -\frac{1}{4}$, $k = 2$	$Q = a,\ P_\lambda = b,\ P_{u^k} = k!c$ $P = P_u = \dots = P_{u^{k-1}} = 0$ $\frac{P_{u^{k+1}}}{(k+1)!} + c\left(\frac{kH}{2a} + (k-1)\frac{P_{u\lambda}}{b}\right) = d$

$$H(m) = P_v - 2Q_u + (4m+3)R$$

Table 3.4.1. Unipotent Equivalence.

	Normal Form	(Top.) Codim.	Defining Equations	Nondegeneracy Conditions
1(i)	$(\delta\lambda^k+\sigma u^2,\varepsilon,0)_m ; k \geq 1$ $m \neq -1, -\frac{1}{2}, -\frac{1}{4}$	$(2k-2)$ $2k-1$	$P = P_\lambda = \ldots = P_{\lambda^{k-1}} = 0$ $P_u = P_{u\lambda} = \ldots = P_{u\lambda^{k-1}} = 0$	$\text{sign}\, Q = \varepsilon$, $\text{sign}\, P_{\lambda^k} = \delta$ $\text{sign}\, P_{uu} = \sigma$
1(ii)	$(\delta\lambda^k,\varepsilon,0)_m ; k \geq 1$ $m \neq -1, -\frac{1}{2}, -\frac{1}{4}$	$(3k-2)$ $3k-1$	$P = P_\lambda = \ldots = P_{\lambda^{k-1}} = 0$ $P_u = P_{u\lambda} = \ldots = P_{u\lambda^{k-1}} = 0$ $P_{uu} = P_{uu\lambda} = \ldots = P_{uu\lambda^{k-1}} = 0$	$\text{sign}\, Q = \varepsilon$, $\text{sign}\, P_{\lambda^k} = \delta$
2	$(\delta\lambda^2+\sigma u^2+pu\lambda,\varepsilon,0)_m$ $m \neq -1, -\frac{1}{2}, -\frac{1}{4}$	(1) 3	$P = P_\lambda = P_u = 0$ $p = \dfrac{2P_{u\lambda}}{\sqrt{\lvert P_{\lambda\lambda}P_{uu}\rvert}}$	$\text{sign}\, Q = \varepsilon$, $\text{sign}\, P_{\lambda\lambda} = \delta$ $\text{sign}\, P_{uu} = \sigma$
3-5	$(\delta\lambda+\sigma u^k+pu^{k+1},\varepsilon,0)_m$ $m = -1$ or $-\frac{1}{2}$, $k \geq 2$ or $m = -\frac{1}{4}$, $k = 2$	$(k-1)$ k	$P = P_u = \ldots = P_{u^{k-1}} = 0$ $p = G(k,m)$	$\text{sign}\, Q = \varepsilon$, $\text{sign}\, P_\lambda = \delta$ $\text{sign}\, P_{u^k} = \sigma$

$\varepsilon,\delta,\sigma = \pm 1$

$$G(k,m) = \sigma\left(\frac{k\lvert Q\rvert}{\lvert P_{u^k}\rvert}\right)^{\frac{1}{k-1}}\left(\frac{P_{u^{k+1}}}{(k+1)P_{u^k}} + \frac{k}{2}\,\frac{P_v - 2Q_u + (4m+3)R}{Q} + (k-1)\frac{P_{u\lambda}}{P_\lambda}\right)$$

Table 3.4.2. Smooth Equivalence in Preferred Coordinates.

§3.5. The Degeneracy $Q(0)=0$.

From Table 3.1.1 we see that a bifurcation problem $[P, Q, R]$ has topological codimension zero provided six nondegeneracy conditions are satisfied, namely

$$Q(0) \neq 0,\ P_\lambda(0) \neq 0,\ T_2(0) \neq 0,\ m \neq -1, -\tfrac{1}{2}, -\tfrac{1}{3},$$

where

$$m = P_u(0)/Q(0),$$

$$T_2(0) = P_{uu}(0) + (m+1)P_v(0) - 2mQ_u(0) + (m+1)(2m+1)R(0)\,.$$

Normal forms 1(ii), 2, 3, 4 and 5 correspond to the degeneracies

$$T_2(0) = 0,\ P_\lambda(0) = 0,\ m = -1, -\tfrac{1}{2}, -\tfrac{1}{3},$$

respectively. Further nondegeneracy conditions are imposed where necessary in order to ensure the only degeneracy is the required one.

In this section we show that normal form 6

$$[\varepsilon u + \delta\lambda + \sigma nv, \sigma mu + pu^2 + qu^3, \sigma],$$

$$m \neq 0, \tfrac{1}{2}, \tfrac{1}{3},\ n \neq 0, \tfrac{2}{3},$$

possesses only the degeneracy $Q(0) = 0$ and so has topological codimension one.

First we consider the germ

$$f = [au + b\lambda + cnv, cmu + du^2 + eu^3, c],$$

$$a, b, c \neq 0,\ m \neq 0, \tfrac{1}{2}, \tfrac{1}{3},\ n \neq 0, \tfrac{2}{3}\,.$$

Lemma 3.5.1 $\tilde{T}(f,U) \supseteq I + J$, *where*

$$I = [\mathfrak{M}\langle w\rangle, \mathfrak{M}\langle w\rangle, \langle w\rangle],$$

$$J = [\mathfrak{M}^3, \mathfrak{M}^4 + \mathfrak{M}^2\langle v\rangle, \mathfrak{M}^3 + \mathfrak{M}\langle v\rangle].$$

Proof By Theorem A.3, $\tilde{T}(f,U)$ is generated by

$$zT_1 = z[au + b\lambda + cnv, cmu + du^2 + eu^3, c]$$

$$zT_{13} = z[au + 2b\lambda, -du^2 - 2eu^3, 0] \qquad z = u, v, w \text{ and } \lambda$$

$$T_3 = [c(m-1)uw + du^2w + eu^3w, cw, au + b\lambda + c(n+1)v]$$

$$T_{14} = [c(m+2n+1)uw + du^2w + eu^3w + 3aw, c(3m-1)w + 6duw + 9eu^2w, 0]$$

$$T_5 = [au^2 + bu\lambda + c(n-2m)uv + cmu^3 + du^4 + eu^5 - 2du^2v - 2eu^3v + 3cw, 0, 0]$$

$$T_6 = [0, au^2 + bu\lambda + c(n-2m)uv + cmu^3 + du^4 + eu^5 - 2du^2v - 2eu^3v + 3cw, 0]$$

$$T_7 = [0, 0, au^2 + bu\lambda + c(n-2m)uv + cmu^3 + du^4 + eu^5 - 2du^2v - 2eu^3v + 3cw]$$

$$T_8 = [3auw + 3bw\lambda + c(3n+1)vw + cmu^2w + du^3w + eu^4w, 0, 0]$$

$$T_9 = [0, 3auw + 3bw\lambda + c(3n+1)vw + cmu^2w + du^3w + eu^4w, 0]$$

$$T_{10} = [0, 0, 3au + 3b\lambda + c(3n+1)v + cmu^2 + du^3 + eu^4]$$

$$T_{11} = [-cmuv - du^2v - eu^3v + cw, au + b\lambda + cnv + cmu^2 + du^3 + eu^4, cmu + du^2 + eu^3]$$

$$T_{15} = [c(n-m)uv - du^2v - eu^3v + au^2 - 2av - c(3n+1)w, -2cmv + 2cmu^2 + 3du^3 + 4eu^4 - 4duv - 6eu^2v, c(m+1)u + du^2 + eu^3]$$

where $T_{13} = \frac{1}{2}(5T_1 - T_2)$, $T_{14} = \frac{1}{2}(T_4 - T_3)$, $T_{16} = \frac{1}{2}(T_{12} - T_{11})$.

First we show that $\tilde{T}(f,U)$ contains I. By Nakayama's Lemma we may work modulo $\mathfrak{M}I$. Now

$$wT_5 \equiv [3cw^2, 0, 0], \quad wT_6 \equiv [0, 3cw^2, 0] \bmod \mathfrak{M}I,$$

and then wT_{11}, T_9 yield $[0, vw, 0]$, $[0, auw + bw\lambda, 0]$. vT_{14} gives $[vw, 0, 0]$ and so T_8 and wT_{13} give $[uw, 0, 0]$ and $[w\lambda, 0, 0]$. Multiplying T_{14} by u

and λ gives $[0, uw, 0]$ and $[0, w\lambda, 0]$ respectively since $m \neq \frac{1}{2}$. Finally wT_1 yields $[0,0,w]$. Similarly $\tilde{T}(f,U)$ contains J provided $m \neq 0, \frac{1}{2}$. □

We can now write

$$\tilde{T}(f,U) = (I + J) \oplus V,$$

where V is a finite dimensional vector space. In fact

$$\lambda^2 f_\lambda = [a\lambda^2,0,0],$$

and so

$$T(f,U) = (I + J) \oplus W \oplus \mathbb{R}\{[\lambda^2,0,0]\},$$

where W is the real vector space generated by

$A_1 = [au^2+2bu\lambda, -du^3, 0]$ $\quad A_2 = [au\lambda, -du^2\lambda, 0]$

$A_3 = [auv+2bv\lambda, 0, 0]$ $\quad A_4 = [au^2+bu\lambda+cnuv, cmu^2+du^3, cu]$

$A_5 = [au\lambda+cnv\lambda, cmu\lambda+du^2\lambda, c\lambda]$ $\quad A_6 = [auv+bv\lambda+cnv^2, cmuv, cv]$

$A_7 = [0, mu^3, u^2]$ $\quad A_8 = [0, mu^2\lambda, u\lambda]$

$A_9 = [0, mu\lambda^2, \lambda^2]$

$A_{10} = [-cmuv+cw, au+b\lambda+cnv+cmu^2+du^3, cmu+du^2]$

$A_{11} = [0, au^2+bu\lambda+cnuv+cmu^3, cmu^2]$

$A_{12} = [0, au\lambda+b\lambda^2+cnv\lambda+cmu^2\lambda, cmu\lambda]$

$A_{13} = [0, auv+bv\lambda+cnv^2, 0]$ $\quad A_{14} = [0, au^3+bu^2\lambda, 0]$

$A_{15} = [0, au^2\lambda+bu\lambda^2, 0]$ $\quad A_{16} = [0, au\lambda^2+b\lambda^3, 0]$

$$A_{17} = [c(n-m)uv+au^2-2av-c(3n+1)w, -2cmv+2cmu^2+3du^3-4duv, c(m+1)u+du^2]$$

$A_{18} = [-2auv, -2cmuv+2cmu^3, c(m+1)u^2]$

$A_{19} = [-2av\lambda, -2cmv\lambda+2cmu^2\lambda, c(m+1)u\lambda]$

$A_{20} = [av^2, cmv^2, 0]$ $\qquad$ $A_{21} = [0, cw, au + b\lambda + c(n+1)v]$

$A_{22} = [0, 0, au^2 + bu\lambda]$ $\qquad$ $A_{23} = [0, 0, au\lambda + b\lambda^2]$

$A_{24} = [3aw, c(3m-1)w, 0]$

$A_{25} = [au^2 + bu\lambda + c(n-2m)uv + 3cw, 0, 0]$

$A_{26} = [0, au^2 + bu\lambda + c(n-2m)uv + cmu^3 + 3cw, 0]$

$A_{27} = [0, 0, 3au + 3b\lambda + c(3n+1)v + cmu^2]$.

Our first task is to cast out redundancies. Inspecting the generators

$$A_7 \text{ to } A_9,\ A_{14} \text{ to } A_{16},\ A_{22},\ A_{23}$$

we see that A_8 and A_9 are redundant. Now replace A_{19} and A_6 by

$$A_{28} = a\{-\tfrac{1}{2}(aA_{18} + bA_{19}) + cm A_{14} + \tfrac{1}{2}c(m+1)A_{22} - cm A_{13} + cn A_{20}\}$$

$$= [auv + bv\lambda + cnv^2, 0, 0],$$

$$A_{29} = A_6 - A_{28} = [0, muv, v].$$

Also replace A_{26} by

$$A_{30} = A_{11} - A_{26} = [0, 2muv - 3w, mu^2].$$

We can now see that A_{27} is a linear combination of A_{21}, A_{29} and A_{30}. It is also possible to check that A_{11} can be written in terms of A_1 to A_5, A_{14}, A_{18}, A_{21}, A_{24}, A_{25}, A_{29} and A_{30}!

By Theorem A.3 the extended tangent space

$$T_e(f, \mathfrak{D}) = T(f, U) + \mathbb{R}\{T_1, T_{13}, f_\lambda, \lambda f_\lambda\}$$

$$= (I + J) \oplus W_1 \oplus \mathbb{R}\{[1,0,0], [\lambda,0,0], [\lambda^2,0,0]\},$$

where

$$W_1 = W + \mathbb{R}\{[au + cnv, cmu + du^2 + eu^3, c], [au, -du^2 - 2eu^3, 0]\}.$$

We have cast out seven redundancies from W and so have at most 25 independent tangent space generators in 30 variables. Hence the codimension of $T_e(f,\mathcal{D})$ is at least 5. We claim that

$$U = \mathbb{R}\{\, [0,1,0], [\, v,0,0], [0, u,0], [0, u^2,0], [0, u^3,0] \,\}$$

is the required tangent space complement and so

$$\operatorname{codim} f = 5.$$

The claim is easy to verify. We must show that

$$T_e(f,\mathcal{D}) + U = \vec{\mathcal{E}}_{x,y,z,\lambda}\,. \tag{1}$$

It is immediate that the left hand side of (1) contains

$$[\, u,0,0], [\, u\lambda,0,0], [\, u^2,0,0], [0, u^2\lambda,0], [0, u\lambda^2,0], [0,\lambda^3,0]$$

$$[0,0,1], [0,0, u^2], [0,0, u\lambda], [0,0,\lambda^2]$$

in addition to the generators of U. Then the four simplified generators

$$A_{18}\,,\ A_{24}\,,\ A_{25}\,,\ A_{30}$$

give

$$[\, uv,0,0], [\, w,0,0], [0, uv,0], [0, w,0]$$

since $m \neq 0$ and $n \neq \frac{1}{2}$. The rest is easy with $n \neq 0$. Hence we have proved the following.

Theorem 3.5.2 *Let $f \in \vec{\mathcal{M}}_{x,y,z,\lambda}$ be the germ*

$$f = [\varepsilon u + \delta\lambda + \sigma n v, \sigma m u + p u^2 + q u^3, \sigma],$$

where

$$\delta, \varepsilon, \sigma = \pm 1, \quad m \neq 0, \tfrac{1}{2}, \tfrac{1}{3}, \quad n \neq 0, \tfrac{3}{2}.$$

Then m,n,p,q are modal parameters and f has codimension 5, *but topological codimension* 1. *A universal unfolding of f is*

$$F = [\varepsilon u + \delta\lambda + \sigma \tilde{n} v, \alpha + \sigma \tilde{m} u + \tilde{p} u^2 + \tilde{q} u^3, \sigma]$$

where $\alpha, \tilde{m}, \tilde{n}, \tilde{p}, \tilde{q}$ *are close to* $0, m, n, p, q$. □

§3.6. Qualitative Equivalence.

It is generally recognised that the equivalence relation used in this thesis is too strong. Two bifurcation problems can exhibit the same qualitative behaviour and yet not be related by a smooth change of coordinates. On the other hand if we replaced 'smooth' by 'continuous' then the resulting equivalence relation would be too weak. For example, the germs

$$f_1(x,\lambda) = x^2 - \lambda,$$
$$f_2(x,\lambda) = x^4 - \lambda,$$

are related by a continuous change of coordinates, yet behave quite differently under small perturbations. A more subtle example is the $\mathbb{Z}_2$-equivariant 'nondegenerate quadratic' discussed in VI §7 of Golubitsky & Schaeffer [1984]:

$$g_m(x,\lambda) = \varepsilon x^5 + 2m\lambda x^3 + \delta\lambda^2 x, \qquad (1)$$
$$\varepsilon,\delta = \pm 1, \; m^2 \neq \varepsilon\delta.$$

Here m is a modal parameter determining a distinct smooth equivalence class for each value of m satisfying $m^2 \neq \varepsilon\delta$. However if $\varepsilon\delta = -1$, g_m only represents one topological equivalence class, while if $\varepsilon\delta = +1$, g_m represents three equivalence classes depending on whether m falls in the range $(-\infty,-1)$, $(-1,+1)$, $(+1,+\infty)$. Moreover on studying the universal unfolding of g_m it is revealed that $m = 0$ is also a special value of the modal parameter. See Fig. VI 7.3 of Golubitsky & Schaeffer [1984]. We will say that two germs are *qualitatively equivalent* if their zero sets exhibit the same topological behaviour under small perturbations. We have not attempted to make this definition precise. However it is clear that under the general definition, the germ in (1) should represent precisely two equivalence classes when $\varepsilon\delta = -1$ and four equivalence classes when $\varepsilon\delta = +1$.

We use barehand techniques to deal with the bifurcation problems considered in this paper. For example, consider the normal forms

1(i) $\quad g_m = [\varepsilon mu + \delta\lambda + \sigma u^2, \varepsilon, 0],$

1(ii) $\quad h_m = [\varepsilon mu + \delta\lambda, \varepsilon, 0],$

$$\varepsilon, \delta, \sigma = \pm 1, \quad m \neq -1, -\tfrac{1}{2}, -\tfrac{1}{3}.$$

These are the nondegenerate bifurcation problems of §1. We saw in Fig. 1.3.1 that the only qualitatively important factors are

$$\text{sign}\,\phi(0),\ \text{sign}\,P(0) \text{ and } m.$$

Furthermore the importance of m only lay in the question of whether its value was in the range $(-\infty,-1)$, $(-1,-\tfrac{1}{2})$, $(-\tfrac{1}{2},-\tfrac{1}{3})$ or $(-\tfrac{1}{3},+\infty)$. In other words the two modal families g_m and h_m collapse into precisely 16 different bifurcation problems, the different problems corresponding to the various permutations of

$$\varepsilon = +1 \text{ or } -1,\ \delta = +1 \text{ or } -1,\ m \in (-\infty,-1), (-1,-\tfrac{1}{2}), (-\tfrac{1}{2},-\tfrac{1}{3}) \text{ or } (-\tfrac{1}{3},+\infty).$$

Note that the value of the expression T_2 in Table 3.1.1 has no bearing on the qualitative question. Similarly the expressions relating to the modal parameters ρ in normal forms 2 to 5 have no qualitative relevance. Hence the simplified recognition problem solutions in Table 3.1.2. The arguments are quite simple for normal forms 1 to 5. Since $Q(0)$ is bounded away from zero we have no submaximal branching (see Theorem 1.3.1(ii)). Hence we need only consider the subspaces

$$(0,0,0),\ (x,0,0),\ (x,x,0),\ (x,x,x).$$

On (0,0,0) we just have the trivial solution, while on each of the other subspaces we have the zeroes of one polynomial in x and λ. One nonvanishing term in each of x and λ is sufficient to give all local behaviour. Hence in normal form 3 the term

$$[\sigma u^2,0,0] = \sigma(x^2+y^2+z^2)\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

gives a nonvanishing x^5 term and so ρu^3 is unnecessary. However the term $[\sigma u^2,0,0]$ is needed because the cubic terms vanish on the $(x,0,0)$ subspace:

$$[-\varepsilon u,\varepsilon,0] = 0 \text{ when } y = z = 0.$$

As usual, the arguments for normal form 6 are much harder. However on inspection of branching and stability equations it becomes clear that the values of the modal parameters p and q are irrelevant. Hence the normal form given in Table 3.1.2. Now since the value of q does not matter $[0,u^3,0]$ is qualitatively a high order term. Instead of choosing $[0,u^3,0]$ in the complement to the extended tangent space, we could just as well have chosen any of

$$[0,u^2\lambda,0],\ [0,u\lambda^2,0],\ [0,\lambda^3,0],\ [0,0,u^2],\ [0,0,u\lambda],\ [0,0,\lambda^2].$$

Hence to solve the unipotent qualitative recognition problem, we can work modulo the U-intrinsic space

$$M = [\mathcal{M}^3 + \mathcal{M}\langle w\rangle, \mathcal{M}^3 + \mathcal{M}\langle w\rangle, \mathcal{M}^2 + \langle v, w\rangle],$$

which is contained in the U-intrinsic part of

$$I + J + [0,\mathcal{M}^3,\mathcal{M}^2].$$

Define $T_N(f,U) = T(f,U) + M$, $\mathcal{P}_N(f,U) = \mathrm{Itr}_U T_N(f,U)$. Then

$$T_N(f,U) = [\mathcal{M}^2 + \langle w\rangle, \mathcal{M}^3 + \mathcal{M}\langle v\rangle + \langle w\rangle, \mathcal{M}^2 + \langle v, w\rangle]$$

$$+ \mathbb{R}\left\{\begin{matrix}[0,mu^2,u]\,[0,mu\lambda,\lambda]\,[0,au\lambda+b\lambda^2,0]\,[0,0,au+b\lambda]\\ [0,au+b\lambda+cnv+cm(1-m)u^2,0]\,[2av,2cmv+cm(m-1)u^2,0]\end{matrix}\right\}.$$

Repeating the argument with the modal parameter p we can work modulo

$$N = [\mathcal{M}^2 + \langle w\rangle, \mathcal{M}^2 + \langle w\rangle, \mathcal{M}].$$

We might hope that

$$\mathcal{P}_N(f,U) = T_N(f,U)$$

working modulo N, but unfortunately

$$T_N(f,U) = N + \mathbb{R}\{[0, au + b\lambda + cnv, 0][av, cmv, 0]\}.$$

Applying T_{12} of Theorem A.3 to $[av, cmv, 0]$ gives

$$[0, -av, 0] \bmod \mathcal{P}_N(f,U),$$

and clearly $[0, v, 0] \notin \mathcal{P}_N(f,U)$. Hence by Proposition 2.4.1,

$$\mathcal{P}_N(f,U) \neq T_N(f,U),$$

and so f is not qualitatively linearly determined.

However, we can still solve the unipotent recognition problem working modulo N. First note that $\mathcal{E}_\lambda\{\lambda^2\} \subset N$, and so without loss of generality, the Λ part of an $\mathbb{O}$-equivalence can be taken to be the identity. Now consider the X part, $X \in \vec{\mathcal{M}}_{x,y,z,\lambda}$, $(dX)_0 = I$, so

$$X = [X_1, X_2, X_3], \quad X_1(0) = 1, \; X_2(0) = A.$$

Lemma 3.6.1 *Modulo N, X has the following effect:*

$$\lambda \mapsto \lambda, \quad u \mapsto u - 4Av, \quad v \mapsto v,$$

$$[z_1, z_2, k] \mapsto [z_1, z_2, k]; \; z_1, z_2 = u, v, w \text{ or } \lambda, \; k \in \mathbb{R}.$$

Proof As an example, we show that $u \mapsto u - 4Av$. Notice that N contains all invariants except for u and v. In particular, we can work modulo terms of order 6. Now X sends x onto

$$x(X_1+x^2X_2+y^2z^2X_3),$$

similarly y and z. Therefore modulo N

$$u = x^2+y^2+z^2 \mapsto x^2(X_1+x^2X_2)^2+y^2(X_1+y^2X_2)^2+z^2(X_1+z^2X_2)^2$$
$$= uX_1^2+2X_1(0)X_2(0)(x^4+y^4+z^4)$$
$$= u+2A(u^2-2v)$$
$$= u-4Av. \qquad \square$$

The effect of general S with $S(0) = 1$ is easier to compute. The generators of $\bar{E}_{x,y,z,\lambda}$ are given in Lemma A.1. From Theorem A.3 it can be seen that

$$Sf = \Sigma^* S_i T_i$$

where $S_i \in E_{u,v,w,\lambda}$ with $S(0) = 1$, and Σ^* denotes summation over the set

$$\{1,\dots,12\} \setminus \{2,4,6\}.$$

It turns out in the same way that under general S,

$$[z_1,z_2,k] \mapsto [z_1, Bz_1+z_2, k], \bmod N,$$

where $S_{11}(0) = B$. We can now prove the following.

Corollary 3.6.2 *$[P,Q,R]$ is unipotently* $\mathbb{D}$ *-equivalent to $[au+b\lambda+cnv, cmu, c]$ modulo N if and only if*

$$P = Q = 0,\ P_u = a,\ P_\lambda = b,\ R = c,$$

$$bQ_u - aQ_\lambda = bcm,\ \ Q_uP_v - aQ_v = c^2mn.$$

Proof Using the results of Lemma 3.6.1, we see that under a general equivalence (S, X, Λ) we have

$$[au+b\lambda+cnv, cmu, c] \mapsto_\Lambda [au+b\lambda+cnv, cmu, c] \bmod N$$

$$\mapsto_X [a(u-4Av)+b\lambda+cnv, cm(u-4Av), c] \bmod N$$

$$\mapsto_S [a(u-4Av)+b\lambda+cnv, B(a(u-4Av)+b\lambda+cnv)+cm(u-4Av), c] \bmod N$$

$$= [au+b\lambda+(cn-4aA)v, (aB+cm)u+bB\lambda+(cBn-4aAB-4cAm)v, c].$$

Hence $[P, Q, R]$ is contained in the orbit if and only if there exist $A, B \in \mathbb{R}$ such that

$$P_u = a,\ P_\lambda = b,\ R = c,\ P_v = cn-4aA,$$

$$Q_u = aB+cm,\ Q_\lambda = bB,\ Q_v = cBn-4aAB-4cAm.$$

Rearrangement gives the required conditions. □

Corollary 3.6.3 $[P, Q, R]$ *is* $\mathbb{R}$ *-equivalent to* $[\varepsilon u+\delta\lambda+\sigma nv, \sigma mu, \sigma]$ *modulo* N *if and only if*

$$P = Q = 0,\ \operatorname{sign} P_u = \varepsilon,\ \operatorname{sign} P_\lambda = \delta,\ \operatorname{sign} R = \sigma,$$

$$\frac{P_\lambda Q_u - P_u Q_\lambda}{P_\lambda R} = m,\quad \frac{P_v Q_u - P_u Q_v}{R^2} = mn.$$

Proof Combine the effect of scalings with Corollary 3.6.2. □

Appendix to §3. Tangent Space Generators and Change of Coordinates.

<u>Tangent space generators.</u>

In order to calculate the tangent space of a germ we first need to find generators for the $\mathcal{E}_{x,y,z}$-module $\check{\mathcal{E}}_{x,y,z}$ of all matrix-valued germs

$$S : (\mathbb{R}^3,0) \to L(\mathbb{R}^3,\mathbb{R}^3)$$

satisfying

$$\gamma^{-1} S(\gamma(x,y,z))\gamma = S(x,y,z), \qquad (1)$$

for all $\gamma \in \mathbb{D}$.

Lemma A.1 *$\check{\mathcal{E}}_{x,y,z}$ is generated over $\mathcal{E}_{x,y,z}$ by $S_1,\dots,S_9$ where*

$$S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_4 = \begin{pmatrix} 0 & xy & xz \\ yx & 0 & yz \\ zx & zy & 0 \end{pmatrix}, \quad S_7 = \begin{pmatrix} 0 & x^3yz^2 & x^3yz^2 \\ x^3yz^2 & 0 & x^3yz^2 \\ x^3yz^2 & x^3yz^2 & 0 \end{pmatrix},$$

$$S_2 = \begin{pmatrix} x^2 & 0 & 0 \\ 0 & y^2 & 0 \\ 0 & 0 & z^2 \end{pmatrix}, \quad S_5 = \begin{pmatrix} 0 & xyz^2 & xzy^2 \\ yxz^2 & 0 & yzx^2 \\ zxy^2 & zyx^2 & 0 \end{pmatrix}, \quad S_8 = \begin{pmatrix} 0 & xy^3z^2 & xz^3y^2 \\ yx^3z^2 & 0 & yz^3x^2 \\ zx^3y^2 & zy^3x^2 & 0 \end{pmatrix},$$

$$S_3 = \begin{pmatrix} y^2z^2 & 0 & 0 \\ 0 & z^2x^2 & 0 \\ 0 & 0 & x^2y^2 \end{pmatrix}, \quad S_6 = \begin{pmatrix} 0 & x^3y & x^3z \\ y^3x & 0 & y^3z \\ z^3x & z^3y & 0 \end{pmatrix}, \quad S_9 = \begin{pmatrix} 0 & x^5yz^2 & x^5zy^2 \\ y^5xz^2 & 0 & y^5zx^2 \\ z^5xy^2 & z^5yx^2 & 0 \end{pmatrix}.$$

Proof As in the proof of Lemma 1.2.4, we exploit Lemma 1.4.1 of Poènaru [1976] to restrict attention to matrix maps with polynomial entries satisfying (1). Again we set

$a = x^2,\ b = y^2,\ c = z^2.$

It is easy to check that the following two spaces contain such maps.

$$D = \left\{ \sum_{ijk} d_{ijk} \begin{bmatrix} a^i(b^jc^k + c^jb^k) & 0 & 0 \\ 0 & b^i(c^ja^k + a^jc^k) & 0 \\ 0 & 0 & c^i(a^jb^k + b^ja^k) \end{bmatrix} \right\}, \quad (2)$$

$$A = \left\{ \sum_{ijk} e_{ijk} \begin{bmatrix} 0 & xya^ib^jc^k & xza^ic^jb^k \\ yxb^ia^jc^k & 0 & yzb^ic^ja^k \\ zxc^ia^jb^k & zyc^ib^ja^k & 0 \end{bmatrix} \right\}. \quad (3)$$

We show that $D \oplus A$ contains all such maps. Suppose that S is a diagonal matrix with polynomial entries satisfying (1), with diagonal entries S^1, S^2, S^3. Then setting $\gamma = \kappa_x, \kappa_y, \kappa_z$ shows that each S^r is even in x, y, z. Hence we can write S in the form

$$S(x,y,z) = \sum_{ijk} a^ib^jc^k \begin{bmatrix} s^1_{ijk} & 0 & 0 \\ 0 & s^2_{ijk} & 0 \\ 0 & 0 & s^3_{ijk} \end{bmatrix}. \quad (4)$$

As in the proof of Lemma 1.2.4, taking γ to be the transpositions (12), (23) and (31) shows first that

$$s^1_{ijk} = s^1_{ikj},\ s^2_{ijk} = s^2_{kji},\ s^3_{ijk} = s^3_{jik},$$

and then that

$$s^1_{ijk} = s^2_{ijk} = s^3_{ijk}.$$

Now suppose that S is an antidiagonal matrix satisfying (1) with polynomial entries S^{qr}; $q, r = 1,2,3$. Letting $\gamma = \kappa_x, \kappa_y, \kappa_z$ we find that

S_{qr} is odd in the qth and rth variables and even in the other. Thus we can write

$$S(x,y,z) = \sum_{ijk} a^i b^j c^k \begin{bmatrix} 0 & s^{12}_{ijk}xy & s^{13}_{ijk}xz \\ s^{21}_{ijk}yx & 0 & s^{23}_{ijk}yz \\ s^{31}_{ijk}zx & s^{32}_{ijk}zy & 0 \end{bmatrix}. \qquad (5)$$

By choosing $\chi = (12)$ we obtain the relations

$$s^{12}_{ijk} = s^{21}_{jik},\ s^{13}_{ijk} = s^{23}_{jik},\ s^{31}_{ijk} = s^{32}_{jik},$$

and (5) becomes

$$S(x,y,z) = \sum_{ijk} \begin{bmatrix} 0 & s^{12}_{ijk}xya^ib^jc^k & s^{13}_{ijk}xza^ib^jc^k \\ s^{12}_{ijk}yxb^ia^jc^k & 0 & s^{13}_{ijk}yzb^ia^jc^k \\ s^{31}_{ijk}zxa^ib^jc^k & s^{31}_{ijk}zyb^ia^jc^k & 0 \end{bmatrix}.$$

Now choosing $\chi = (13)$ yields

$$s^{12}_{ijk} = s^{31}_{kij} = s^{13}_{kji},$$

as required.

It is clear by analogy with Lemma 1.2.4 that S_1, S_2, S_3 generate D and so it remains to show that $S_4, \ldots, S_9$ generate A. We use the notation

$$\langle a^ib^jc^k \rangle = \begin{bmatrix} 0 & xya^ib^jc^k & xza^ic^jb^k \\ yxb^ia^jc^k & 0 & yzb^ic^ja^k \\ zxc^ia^jb^k & zyc^ib^ja^k & 0 \end{bmatrix}. \qquad (6)$$

Then

$$\langle a^2 \rangle = u\langle a \rangle - v\langle 1 \rangle + \langle bc \rangle,$$

$$\langle a^n \rangle = u\langle a^{n-1} \rangle - v\langle a^{n-2} \rangle + w\langle a^{n-3} \rangle;\ n \geq 3,$$

and

$$\langle\!\langle b^n c^n \rangle\!\rangle = (a^{n-1}b^{n-1} + b^{n-1}c^{n-1} + c^{n-1}a^{n-1})\langle\!\langle bc \rangle\!\rangle$$
$$- w(a^{n-1} + b^{n-1} + c^{n-1})\langle\!\langle a^{n-2} \rangle\!\rangle + w\langle\!\langle a^{2n-3} \rangle\!\rangle;\ n \geq 2.$$

Hence $\langle\!\langle a^n \rangle\!\rangle$ and $\langle\!\langle b^n c^n \rangle\!\rangle$, and similarly $\langle\!\langle b^n \rangle\!\rangle$, $\langle\!\langle c^n \rangle\!\rangle$, $\langle\!\langle a^n b^n \rangle\!\rangle$ and $\langle\!\langle a^n c^n \rangle\!\rangle$, are generated by $S_4, \ldots, S_8$. We claim that $\langle\!\langle a^{n+1} b^n \rangle\!\rangle$ and $\langle\!\langle a^{n+2} b^n \rangle\!\rangle$ can be generated. When $n = 1$

$$\langle\!\langle a^2 b \rangle\!\rangle = S_9 \text{ and } \langle\!\langle a^3 b \rangle\!\rangle = u\langle\!\langle a^2 b \rangle\!\rangle - \langle\!\langle a^2 b^2 \rangle\!\rangle - w\langle\!\langle b \rangle\!\rangle,$$

while for $n \geq 2$,

$$\langle\!\langle a^{n+1} b^n \rangle\!\rangle = v\langle\!\langle a^n b^{n-1} \rangle\!\rangle - w\langle\!\langle a^n b^{n-2} \rangle\!\rangle - w\langle\!\langle a^{n-1} b^{n-1} \rangle\!\rangle,$$
$$\langle\!\langle a^{n+2} b^n \rangle\!\rangle = u\langle\!\langle a^{n+1} b^n \rangle\!\rangle - \langle\!\langle a^{n+1} b^{n+1} \rangle\!\rangle - w\langle\!\langle a^n b^{n-1} \rangle\!\rangle,$$

thus verifying the claim. It is now easy to see that we have $\langle\!\langle a^m b^n \rangle\!\rangle$: we know this for $\langle\!\langle a^{n+r} b^n \rangle\!\rangle$, $n = 0$ or $r = 0,1,2$. For $n \geq 1$, $r \geq 3$ we have

$$\langle\!\langle a^{n+r} b^n \rangle\!\rangle = u\langle\!\langle a^{n+r-1} b^n \rangle\!\rangle - \langle\!\langle a^{n+r-1} b^{n+1} \rangle\!\rangle - w\langle\!\langle a^{n+r-2} b^{n-1} \rangle\!\rangle.$$

Finally, the general term is $\langle\!\langle a^i b^j c^k \rangle\!\rangle$. Without loss of generality $i \leq j \leq k$, and so

$$\langle\!\langle a^i b^j c^k \rangle\!\rangle = w^i\langle\!\langle b^l c^m \rangle\!\rangle;\ l = j - i,\ n = k - i. \qquad \square$$

We recall some notation from §1.2. In Theorem 1.2.1 it was shown that every equivariant germ $f \in \vec{\mathcal{E}}_{x,y,z,\lambda}$ can be written in invariant coordinates as

$$f = [P, Q, R], \tag{7}$$

where $P, Q, R \in \mathcal{E}_{x,y,z,\lambda}$. It was also shown (Theorem 1.2.4) that f can be written as

$$f = \langle\varphi(a,b,c)\rangle = \begin{pmatrix}\varphi(a,b,c)x \\ \varphi(b,c,a)y \\ \varphi(c,a,b)z\end{pmatrix}. \tag{8}$$

for some function-germ φ. In the notations (7) and (8) we have the following useful identities.

Lemma A.2 (a) $x^4+y^4+z^4 = u^2-2v$,

$x^6+y^6+z^6 = u^3-2uv+3w$,

(b) $\langle x^4\rangle = [-v, u, 1]$,

$\langle x^6\rangle = [-uv+w, u^2-v, u]$,

$\langle y^4z^4\rangle = [-uw, w, v]$,

$\langle y^2z^4+z^2y^4\rangle = [-w, 0, u]$,

$\langle y^2z^6+z^2y^6\rangle = [0, -w, u^2-2v]$.

Proof (a) Can be checked directly.

(b) These are straightforward consequences of the inductive arguments used in the proof of Lemma 1.2.4. □

Theorem A.3 *Suppose* $f=[P,Q,R]\in\vec{\mathcal{E}}_{x,y,z,\lambda}$. *Then the unipotent tangent space is given by*

$$T(f,U)=\tilde{T}(f,U)+\mathcal{E}_\lambda\{\lambda^2 f_\lambda\}.$$

Here $\tilde{T}(f,U)$ *is generated as an* $\mathcal{E}_{u,v,w,\lambda}$ *module by*

$zT_1,\ zT_2,\ z=u,v,w$ or λ, $T_3,\dots,T_{12}$, *where*

$$T_1=[P,Q,R],$$

$$T_2=[P+2uP_u+4vP_v+6wP_w,\ 3Q+2uQ_u+4vQ_v+6wQ_w,\ 5R+2uR_u+4vR_v+6wR_w],$$

$$T_3=[wQ-uwR,\ wR,\ P+vR],$$

$$T_4=[wQ+uwR+w(3P_u+2uP_v+vP_w),\ -wR+w(3Q_u+2uQ_v+vQ_w),\ w(3R_u+2uR_v+vR_w)],$$

$$T_5=[uP+(u^2-2v)Q+3wR,0,0],$$

$$T_6=[0,uP+(u^2-2v)Q+3wR,0],$$

$$T_7=[0,0,uP+(u^2-2v)Q+3wR],$$

$$T_8=[3wP+uwQ+vwR,0,0],$$

$$T_9=[0,3wP+uwQ+vwR,0],$$

$$T_{10}=[0,0,3P+uQ+vR],$$

$$T_{11}=[-vQ+wR,\ P+uQ,\ Q],$$

$$T_{12}=[-2wR+u(uP_u+vP_v+wP_w)-2vP_u-3wP_v,\ -P+u(uQ_u+vQ_v+wQ_w)-2vQ_u-3wQ_v,\ uR+u(uR_u+vR_v+wR_w)-2vR_u-3wR_v].$$

The extended tangent space is given by

$$T_e(f,\mathcal{D})=T(f,U)+\mathbb{R}\{T_1,T_2,f_\lambda,\lambda f_\lambda\}.$$

Proof Recall from §2.2 that $\tilde{T}(f,U)$ is given by

$$\tilde{T}(f,U) = \{\, Sf + (df)X \mid (S,X) \in \vec{\mathcal{E}}_{x,y,z,\lambda} \times \vec{\mathcal{M}}_{x,y,z,\lambda},\ S(0) = (dX)_0 = 0 \,\}.$$

Thus $\tilde{T}(f,U)$ is generated as an $\mathcal{E}_{x,y,z,\lambda}$-module by

$$zS_1 f,\ z(df)X_1,\ z = u, v, w \text{ or } \lambda,$$
$$S_2 f, \ldots, S_9 f, (df)X_2, (df)X_3.$$

We obtain the set of generators in the statement of this Theorem by the following rules:

$$T_1 = S_1 f,\ T_2 = (df)X_1,\ T_3 = S_3 f,\ T_4 = (df)X_2,$$
$$T_5 = S_4 f + S_2 f,\ T_6 = S_6 f + uS_2 f + S_3 f - vS_1 f,$$
$$T_7 = S_8 f + wS_1 f,\ T_8 = S_7 f + wS_1 f,$$
$$T_9 = S_9 f + wS_2 f,\ T_{10} = S_5 f + S_3 f,$$
$$T_{11} = S_2 f,\ T_{12} = (df)X_2.$$

Since $\tilde{T}(f,U)$ is a submodule of $\vec{\mathcal{E}}_{x,y,z,\lambda}$, each generator is equivariant and so must be of the form (8) for some $\varphi \in \mathcal{E}_{x^2,y^2,z^2}$. Hence we need only find φ. Then using the results in Lemma A.3 we write the generators in the invariant coordinates (7). Suppose for example we wish to calculate T_{11}. Then

$$T_{11} = S_2 f = \begin{pmatrix} x^2 & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} \begin{pmatrix} x(P + x^2 Q + y^2 z^2 R) \\ * \\ * \end{pmatrix}$$

$$= \begin{pmatrix} x(x^2 P + x^4 Q + x^2 y^2 z^2 R) \\ * \\ * \end{pmatrix}$$

$$= \langle x^2 P + x^4 Q + wR \rangle$$

$$= [0,1,0]P + [-v,u,1]Q + [w,0,0]R$$

$$= [-vQ + wR, P + uQ, Q].$$

Similarly, to find the $(df)X_i$ it suffices to consider the first row of df, namely

$$(A, xB(y,z), xB(z,y)),$$

where

$$A = \frac{\partial f_1}{\partial x} = x(2xP_u + 2x(y^2+z^2)P_v + 2xy^2z^2P_w)$$

$$+ x^3(2xQ_u + 2x(y^2+z^2)Q_v + 2xy^2z^2Q_w)$$

$$+ xy^2z^2(2xR_u + 2x(y^2+z^2)R_v + 2xy^2z^2R_w)$$

$$+ P + 3x^2Q + y^2z^2R,$$

and

$$xB(y,z) = \frac{\partial f_1}{\partial y} = x(2yP_u + 2y(x^2+z^2)P_v + 2yx^2z^2P_w)$$

$$+ x^3(2yQ_u + 2y(x^2+z^2)Q_v + 2yx^2z^2Q_w)$$

$$+ xy^2z^2(2yR_u + 2y(x^2+z^2)R_v + 2yx^2z^2R_w)$$

$$+ 2xyz^2R.$$

We exploit the symmetry of the partial derivatives of P, Q, R in A and xB by splitting up the calculations as follows. We have

$$(df)X_1 = \langle A + yB(y,z) + zB(z,y) \rangle,$$

and so P_u, P_v and P_w contribute

$$2\langle x^2 P_u + x^2(y^2+z^2)P_v + x^2y^2z^2P_w + (y^2+z^2)P_u$$

$$+ (y^2(x^2+z^2)+z^2(x^2+y^2))P_v + 2x^2y^2z^2P_w\rangle$$

$$= 2[uP_u+2vP_v+3wP_w,0,0]. \qquad (9)$$

To compute the Q_u, Q_v and Q_w terms, multiply (9) by x^2 and change P's to Q's. Similarly, multiply (9) by y^2z^2 and change P's to R's to yield the R_u, R_v and R_w terms. Ignoring the partial derivative terms, $\langle A\rangle$ is $[P,Q,R]$ and $\langle yB(y,z)+zB(z,y)\rangle$ is $[0,0,4R]$. The calculation of $(df)X_2$ and $(df)X_3$ is similar.

Finally the relation between the unipotent tangent space and the extended tangent space follows immediately from their respective definitions in §2.2 and §3. □

<u>Change of coordinates.</u>

It turns out that many of the calculations in this thesis are best performed in a different set of coordinates. Six out of the seven normal forms in our classification have $Q(0) \neq 0$. It is the calculations for these germs that are simplified. These germs all have the modal parameter

$$m = P_u(0)/Q(0)$$

and the coordinate change is also parametrised by m. In the remainder of the Appendix, we describe the coordinate change and give results enabling us to recover some solutions in the original coordinates.

The change of coordinates is performed in two stages. Firstly we set

$$(P',Q',R')_m = [P,Q,R],$$

where

$$P' = P - muQ + \tfrac{1}{2}(m+1)(2m+1)u^2R, \quad Q' = Q, \quad R' = R, \tag{10}$$

The subscript m shows the dependence of the change of coordinates on m. Secondly, we set

$$P''(u, v_m, w_m, \lambda) = P'(u, v, w, \lambda), \text{ similarly } Q'' \text{ and } R'',$$

where

$$v_m = v - \tfrac{1}{2}(m+1)u^2, \quad w_m = w - \tfrac{1}{6}(m+1)(2m+1)u^3, \tag{11}$$

For the bifurcation problems that we consider in this paper, the partial derivatives P_v, P_w, Q_u, Q_v, Q_w and R are identically zero. Using this simplification we write the generators $T_1, \dots, T_{12}$ in the new coordinates.

Corollary A.4 *Let* $f = (P, Q, 0)_{\blacksquare}$ *where* P_v, P_w, Q_u, Q_v, Q_w *are* 0. *Then in the coordinates of* (10) *and* (11), $\bar{T}(f, U)$ *is generated by*

$$zT_1 = z(P, Q, 0)_{\blacksquare}$$

$$zT_2 = z(P + 2uP_u, 3Q, 0)_{\blacksquare} \qquad z = u, v_{\blacksquare}, w_{\blacksquare} \text{ or } \lambda$$

$$T_3 = (\tfrac{1}{2}(m+1)(2m+1)u^2 P + (w_{\blacksquare} + \tfrac{1}{6}(m+1)(2m+1)(3m+1)u^3)Q,\ 0, P + muQ)_{\blacksquare}$$

$$T_4 = ((w_{\blacksquare} + \tfrac{1}{6}(m+1)(2m+1)u^3)(3P_u + (3m+1)Q), 0, 0)_{\blacksquare}$$

$$T_5 = (uP - 2v_{\blacksquare}Q, 0, 0)_{\blacksquare}$$

$$T_6 = (0, uP - 2v_{\blacksquare}Q, 0)_{\blacksquare}$$

$$T_7 = (0, 0, uP - 2v_{\blacksquare}Q)_{\blacksquare}$$

$$T_8 = ((w_{\blacksquare} + \tfrac{1}{6}(m+1)(2m+1)u^3)(3P + (3m+1)uQ), 0, 0)_{\blacksquare}$$

$$T_9 = (0, (w_{\blacksquare} + \tfrac{1}{6}(m+1)(2m+1)u^3)(3P + (3m+1)uQ), 0)_{\blacksquare}$$

$$T_{10} = (\tfrac{1}{2}(m+1)(2m+1)u^2(3P + (3m+1)uQ), 0, 3P + (3m+1)uQ)_{\blacksquare}$$

$$T_{11} = (-muP - v_{\blacksquare}Q, P + (m+1)uQ, Q)_{\blacksquare}$$

$$T_{12} = (mu(uP_u - P) + 2v_{\blacksquare}(P_u + mQ), P + muQ, 0)_{\blacksquare} \qquad \square$$

The following three results tell us how partial derivatives in the original coordinates can be recovered. The first of these results deals with the coordinate change (10).

Proposition A.5 *Let*

$$\frac{\partial^l}{\partial u^l} = \begin{cases} \dfrac{\partial^{l+l_1+l_2+l_3}}{\partial u^l \partial v^{l_1} \partial w^{l_2} \partial \lambda^{l_3}} & ; l \geq 0, \\ 0 & ; l < 0. \end{cases}$$

Then, at the origin, for $l \geq 0$, *we have*

$$\frac{\partial^l P'}{\partial u^l} = \frac{\partial^l P}{\partial u^l} - lm \frac{\partial^{l-1} Q}{\partial u^{l-1}} + \tfrac{1}{2}(l-1)l(m+1)(2m+1) \frac{\partial^{l-2} R}{\partial u^{l-2}},$$

$$\frac{\partial^l Q'}{\partial u^l} = \frac{\partial^l Q}{\partial u^l}, \qquad \frac{\partial^l R'}{\partial u^l} = \frac{\partial^l R}{\partial u^l}.$$

Proof By induction we have

$$\frac{\partial^l P'}{\partial u^l} = \frac{\partial^l P}{\partial u^l} - lm \frac{\partial^{l-1} Q}{\partial u^{l-1}} + \tfrac{1}{2}(l-1)l(m+1)(2m+1) \frac{\partial^{l-2} R}{\partial u^{l-2}}$$

$$- mu \frac{\partial^l Q}{\partial u^l} + l(m+1)(2m+1)u \frac{\partial^{l-1} R}{\partial u^{l-1}} + \tfrac{1}{2}(m+1)(2m+1)u^2 \frac{\partial^l R}{\partial u^l},$$

$$\frac{\partial^l Q'}{\partial u^l} = \frac{\partial^l Q}{\partial u^l}, \qquad \frac{\partial^l R'}{\partial u^l} = \frac{\partial^l R}{\partial u^l}.$$

The result at the origin follows. □

The next result gives a series of identities for obtaining partial derivatives of P' in terms of partial derivatives of P. Similar results exist for Q' and R'.

Proposition A.6 *At the origin, we have, suppressing v, w, λ partials*

$$P'' = P', \quad P_u = P_u ,$$

and, suppressing λ partials

$$P_{uu} = P_{uu} + (m+1)P_v ,$$

$$P_{uuu} = P_{uuu} + 3(m+1)P_{uv} + (m+1)(2m+1)P_u ,$$

and so on.

Proof Straightforward. □

Finally we combine the two results.

Proposition A.7 *At the origin, we have, suppressing v, w, λ partials*

$$P'' = P, \quad Q'' = Q, \quad R'' = R,$$

$$P_u = P_u - mQ, \quad Q_u = Q_u , \quad R_u = R_u ,$$

and, suppressing λ partials,

$$P_{uu} = P_{uu} + (m+1)P_v - 2mQ_u + (m+1)(2m+1)R,$$

$$Q_{uu} = Q_{uu} + (m+1)Q_v ,$$

$$P_{uuu} = P_{uuu} + 3(m+1)P_{uv} + (m+1)(2m+1)P_u - 3mQ_u$$
$$- 3m(m+1)Q_v + 3(m+1)(2m+1)R_u ,$$

and so on. □

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