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THE EXACT MINIMUM NUMBER OF TRIANGLES IN GRAPHS WITH GIVEN ORDER AND SIZE

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Abstract

What is the minimum number of triangles in a graph of given order and size? Motivated by earlier results of Mantel and Turán, Rademacher solved the first nontrivial case of this problem in 1941. The problem was revived by Erdős in 1955; it is now known as the Erdős–Rademacher problem. After attracting much attention, it was solved asymptotically in a major breakthrough by Razborov in 2008. In this paper, we provide an exact solution for all large graphs whose edge density is bounded away from 1, which in this range confirms a conjecture of Lovász and Simonovits from 1975. Furthermore, we give a description of the extremal graphs.

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1. Introduction

The celebrated theorem of Turán [42] (with the case $r = 3$ proved earlier by Mantel [27]) states that, among all K_r-free graphs with $n \geqslant r$ vertices, the *Turán graph* $T_{r-1}(n)$, the complete balanced $(r-1)$-partite graph, is the unique graph maximizing the number of edges. Here, the *r-clique* K_r is the complete graph with r vertices (and $\binom{r}{2}$ edges).

Let $t_r(n) := e(T_r(n))$ denote the number of edges in $T_r(n)$ and let an (n, e)-*graph* mean a graph with n vertices and e edges. Thus the above result implies that every $(n, t_2(n) + 1)$-graph H contains at least one triangle. Rademacher in 1941 (unpublished; see [6]) showed that H must have at least $\lfloor n/2 \rfloor$ triangles. This naturally leads to the following general question that first appeared in print in a paper of Erdős [6] and is now called the *Erdős–Rademacher problem*: determine

$$g_r(n,e) := \min\{K_r(H) : (n,e)\text{-graph } H\}, \quad n, e \in \mathbb{N},\ e \leqslant \binom{n}{2},$$

where $K_r(H)$ denotes the number of K_r-subgraphs in a graph H and $\mathbb{N} := \{1, 2, \ldots\}$ consists of natural numbers.

Before discussing the history of this problem in some detail, let us present the general upper bound $h^*(n,e)$ on $g_3(n,e)$, which, as far as the authors know, may actually equal $g_3(n,e)$ for all pairs (n,e). In fact, one of the main results of this paper (stated in a stronger form in Theorem 1.6) is that $g_3(n,e) = h^*(n,e)$ if n is large and $e/\binom{n}{2}$ is bounded away from 1. In order to define h^*, we need to introduce some auxiliary parameters.

DEFINITION 1 (Parameters k, m^* and h^*, vector $\boldsymbol{a}^*$ and graph H^*). Let $n, e \in \mathbb{N}$ satisfy $e \leqslant \binom{n}{2}$. Define

$$k = k(n,e) := \min\{s \in \mathbb{N} : e \leqslant t_s(n)\}, \tag{1.1}$$

that is, k is the unique positive integer with $t_{k-1}(n) < e \leqslant t_k(n)$.

Next, let $\boldsymbol{a}^* = \boldsymbol{a}^*(n,e)$ be the unique integer vector $(a_1^*, \ldots, a_k^*)$ such that

- $a_k^* := \min\{a \in \mathbb{N} : a(n-a) + t_{k-1}(n-a) \geqslant e\}$;
- $a_1^* + \cdots + a_{k-1}^* = n - a_k^*$ and $a_1^* \geqslant \cdots \geqslant a_{k-1}^* \geqslant a_1^* - 1$.

Further, define

$$m^* = m^*(n,e) := \sum_{1 \leqslant i < j \leqslant k} a_i^* a_j^* - e, \tag{1.2}$$

$$h^*(n,e) := \sum_{1 \leqslant h < i < j \leqslant k} a_h^* a_i^* a_j^* - m^* \sum_{i=1}^{k-2} a_i^*.$$

Also, let the graph $H^* = H^*(n,e)$ be obtained from $K^k_{a_1^*,\ldots,a_k^*}$, the complete k-partite graph with part sizes $a_1^*, \ldots, a_k^*$, by removing m^* edges between the last two parts (say, for definiteness, all incident to a vertex in the last part).

Let us rephrase the above definitions and also argue that H^* is well defined. We look for an upper bound on $g_3(n,e)$, where we take a complete partite graph, say with parts $A_1^*, \ldots, A_k^*$, and remove a star incident to a vertex of A_k^*. First, we choose the smallest k for which such an (n,e)-graph exists and then the smallest possible size a_k^* of A_k^*. Then we let the first $k-1$ parts form the Turán graph $T_{k-1}(n-a_k^*)$, that is, their sizes are $a_1^*, \ldots, a_{k-1}^*$. Since $T_{k-1}(n-a_k^*)$ has at least as many edges as any other $(k-1)$-partite graph of order $n-a_k^*$, it holds that $m^* := e(K^k_{a_1^*,\ldots,a_k^*}) - e$ is nonnegative. Furthermore, we have that

$$0 \leqslant m^* \leqslant a_{k-1}^* - a_k^* \tag{1.3}$$

because, if the upper bound fails, then

$$e(K^k_{a_1^*,\ldots,a_{k-2}^*,a_{k-1}^*+1,a_k^*-1}) = e(K^k_{a_1^*,\ldots,a_k^*}) - (a_{k-1}^* - a_k^* + 1) \geqslant e,$$

contradicting the minimality of a_k^* (or the minimality of k if $a_k^* = 1$). In particular, we have $m^* \leqslant a_{k-1}^*$, so H^* is well defined. Thus H^* is an (n, e)-graph and

$$h^*(n, e) := K_3(H^*) \geqslant g_3(n, e)$$

is indeed an upper bound on $g_3(n, e)$.

For example, if $e \leqslant t_2(n)$, then $H^*(n, e)$ is bipartite and $h^*(n, e) = 0$ (here $k = 2$). Also, $H^*(n, t_r(n)) = T_r(n)$. If $1 \leqslant \ell < \lceil n/r \rceil$, then $H^*(n, t_r(n) + \ell)$ is obtained from the Turán graph $T_r(n)$ by adding the ℓ-star $K_{1,\ell}$ into a largest part (here, $k = r + 1$ and $a_k^* = 1$) and so on.

Let us return to the history of the triangle-minimization problem. The problem was revived by Erdős [6] in 1955, who in particular conjectured that for $1 \leqslant \ell < \lfloor n/2 \rfloor$, it holds that $g_3(n, t_2(n)+\ell) = \ell \lfloor n/2 \rfloor$. This is exactly the h^*-bound; also, note that if n is even and $\ell = n/2$, then $h^*(n, t_2(n) + \ell)$ is strictly smaller than $\ell n/2$ (here, $k = 3$ and $a_3^* = 2$). So the Erdős conjecture cannot be extended here. In the same paper, Erdős [6] proved the conjecture when $\ell \leqslant 3$; the same result also appears in Nikiforov [31]. Erdős in [7] was able to prove his conjecture when $\ell < \gamma n$ for some positive constant γ. The conjecture was eventually proved in totality for large n by Lovász and Simonovits [25] in 1975, with the proof of the conjecture also announced by Nikiforov and Khadzhiivanov [32].

Moon and Moser [28, page 285] and, independently, Nordhaus and Stewart [33, Equation (5)] proved that

$$g_3(n, e) \geqslant \frac{e(4e - n^2)}{3n}, \tag{1.4}$$

with equality achieved if and only if $e = t_k(n)$ with k dividing n. The bound in (1.4) can be derived by using the triangle counting method from an earlier paper by Goodman [13] and is often referred to as the *Goodman bound*.

In order to state some of the following results, it will be convenient to define the asymptotic version of the problem. Namely, given $\lambda \in [0, 1]$, take any integervalued function $0 \leqslant e(n) \leqslant \binom{n}{2}$ with $e(n)/\binom{n}{2} \to \lambda$ as $n \to \infty$ and define

$$g_r(\lambda) := \lim_{n\to\infty} \frac{g_r(n, e(n))}{\binom{n}{r}}.$$

It is easy to see from basic principles that the limit exists and does not depend on the choice of the function $e(n)$.

The upper bound on the function $g_3(\lambda)$ given by the graphs H^* from Definition 1 is as follows. Let $n \to \infty$ and $e = \lambda n^2/2 + o(n^2)$. It always holds that, for example, $m^* \leqslant n$ and $a_1^* - a_{k-1}^* \leqslant 1$. So these have negligible effect on the limit and one can consider only complete partite graphs with all parts equal, except at most one part of smaller size. Therefore, for $\lambda \in [0, 1)$, let us define

$$k(\lambda) := \min\{k \in \mathbb{N} : \lambda \leqslant 1 - 1/k\}. \tag{1.5}$$

Thus if $\lambda \in (0, 1)$, then $k(\lambda)$ is the unique integer $k \geqslant 2$ satisfying $1 - \frac{1}{k-1} < \lambda \leqslant 1 - \frac{1}{k}$, while $k(0) = 1$. Let $k = k(\lambda)$ and let $c = c(\lambda)$ be the unique root with $c \geqslant 1/k$ of the quadratic equation

$$\binom{k-1}{2}c^2 + (1 - c')c' = \lambda/2, \tag{1.6}$$

where $c' := 1 - (k-1)c$. The above equation is the limit version of the desired equality $e(K^k_{cn,\dots,cn,c'n}) = \lambda\binom{n}{2} + o(n^2)$. Explicitly,

$$c(\lambda) = \frac{1}{k}\left(1 + \sqrt{1 - \frac{k}{k-1}\cdot\lambda}\right), \quad \lambda \in (0, 1), \quad \text{while } c(0) = 1. \tag{1.7}$$

Thus

$$g_3(\lambda) \leqslant h^*(\lambda) := 3!\left(\binom{k-1}{3}c^3 + \binom{k-1}{2}c^2c'\right), \quad \lambda \in [0, 1). \tag{1.8}$$

(For $\lambda = 1$, we just let $h^*(1) := 1$.)

The upper bound in (1.8) coincides with the lower bound on $g_3(\lambda)$ given by (1.4) when $\lambda = 1 - 1/k$ for all integers $k \geqslant 1$. Thus

$$g_3(1 - 1/k) = \frac{(k-1)(k-2)}{k^2}, \quad k \in \mathbb{N}. \tag{1.9}$$

Some of the early results on $g_3(\lambda)$ concentrated on finding good convex lower bounds. McKay (unpublished; see [33, page 35]) showed that $g_3(\lambda) \geqslant \lambda - \frac{1}{2}$. Nordhaus and Stewart [33] conjectured that $g_3(\lambda) \geqslant \frac{4}{3}(\lambda - \frac{1}{2})$ and presented some partial results in this direction. This conjecture was proved by Bollobás [1], who in fact established the best possible convex lower bound on g_3, namely, the piecewise linear function that coincides with g_3 at all values in (1.9).

However, the upper bound $h^*(\lambda)$ is a strictly concave function between any two consecutive values in (1.9) for $\lambda \geqslant 1/2$. This is one of the reasons why the triangle-minimization problem is so difficult.

After Bollobás [1], the first improvement 'visible in the limit' was achieved by Fisher [10], who showed that $g_3(\lambda) = h^*(\lambda)$ for all $1/2 \leqslant \lambda \leqslant 2/3$. (There was a hole in Fisher's proof, which can be fixed using the results of Goldwurm and Santini [12]; see [4, Remark 3.3].) Then Razborov used his newly developed theory of *flag algebras* first to give a different proof of Fisher's result in [36] and then to determine the whole function $g_3(\lambda)$ in [37] (see Figure 1 for a plot of the function).

THEOREM 1.1 [37]. *For all* $\lambda \in [0, 1]$, *we have that* $g_3(\lambda) = h^*(\lambda)$.

Nikiforov [30] presented a new proof of Razborov's result and also determined $g_4(\lambda)$ for all $\lambda \in [0, 1]$. More recently, Reiher [38] determined $g_r(\lambda)$ for all $\lambda \in [0, 1]$ and $r \geqslant 5$ (also reproving the case $r \in \{3, 4\}$).

Another property that makes this problem difficult is that in general there are many asymptotically extremal (n, e)-graphs, as the following family demonstrates.

DEFINITION 2 (Family $\mathcal{H}^*(n, e)$). Given $n, e \in \mathbb{N}$ with $e \leqslant \binom{n}{2}$, let $k = k(n, e)$, $\boldsymbol{a}^* = (a_1^*, \ldots, a_k^*)$ and m^* be as in Definition 1. The family $\mathcal{H}^*(n, e) := \bigcup_{i=0}^{2} \mathcal{H}_i^*(n, e)$ is defined as the union of the following three families. Let $T := K[A_1^*, \ldots, A_k^*]$ be the complete partite graph with part sizes $a_1^* \geqslant \cdots \geqslant a_k^*$, respectively.

$\mathcal{H}_1^*(n, e)$: If $m^* = 0$, then take all graphs obtained from T by replacing, for some $i \in [k-1]$, $T[A_i^* \cup A_k^*]$ with an arbitrary triangle-free graph with $a_i^* a_k^*$ edges. If $m^* > 0$, take all graphs obtained from T by replacing $T[A_{k-1}^* \cup A_k^*]$ with an arbitrary triangle-free graph with $a_{k-1}^* a_k^* - m^*$ edges.

$\mathcal{H}_0^*(n, e)$: Take the family $\mathcal{H}_1^*(n, e)$ and, if $a_k^* = 1$, add all graphs obtained from $K_{a_1^*,\ldots,a_{k-2}^*,a_{k-1}^*+1}$ by adding a triangle-free graph with $a_{k-1}^* - m^*$ edges such that each added edge lies inside some part of size $a_{k-1}^* + 1$.

$\mathcal{H}_2^*(n, e)$: Take those graphs in $\mathcal{H}_1^*(n, e)$ that are k-partite, along with the following family. Take disjoint sets $A_1, \ldots, A_k$ of sizes $a_1^*, \ldots, a_k^*$,

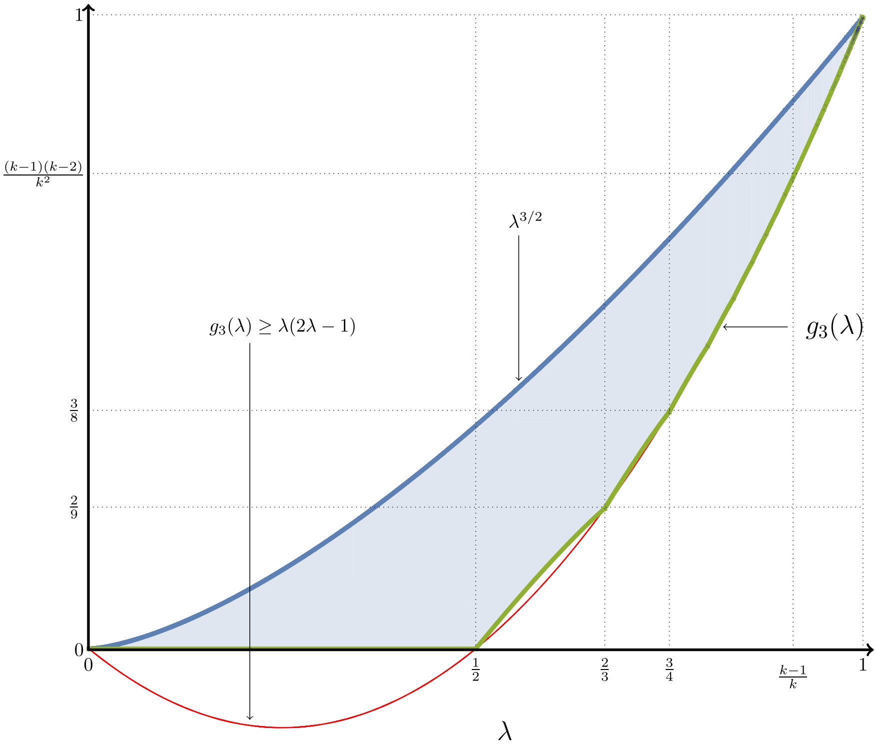

Figure 1. The green function is $g_3(\lambda)$, as determined by Theorem 1.1. The red curve is Goodman's bound (1.4). The blue curve $\lambda^{3/2}$ is asymptotically the maximum triangle density in a graph of edge density λ. This follows easily from the Kruskal–Katona theorem [20, 23].

respectively, and let $m := m^*$. If $m^* = 0$ and $a_1^* \geqslant a_k^* + 2$, then we also allow $(|A_1|, \ldots, |A_k|) = (a_2^*, \ldots, a_{k-1}^*, a_1^* - 1, a_k^* + 1)$ and let $m := a_1^* - a_k^* - 1$. Take all graphs obtained from $K[A_1, \ldots, A_k]$ by removing m edges, each connecting B_i to A_i for some $i \in I$, where $I := \{i \in [k-1] : |A_i| = |A_{k-1}|\}$ and $(B_i)_{i \in I}$ are some disjoint subsets of A_k.

One can check by the definition that every graph in $\mathcal{H}^*(n, e)$ has e edges and $h^*(n, e)$ triangles. Also, the graph $H^*(n, e)$ belongs to $\mathcal{H}_i^*(n, e)$ for each $i \in \{0, 1, 2\}$. Proposition 1.5 and Conjecture 1.8, to be stated shortly, will motivate the above definitions.

Note that every graph in $\mathcal{H}_0^*(n,e)\setminus\mathcal{H}_1^*(n,e)$ has at most $a_{k-1}^*-m^*\leqslant\frac{n-1}{k-1}$ more edges than the Turán graph $T_{k-1}(n)$. In other words,

$$\mathcal{H}_0^*(n,e)=\mathcal{H}_1^*(n,e),\quad \text{for } t_{k-1}(n)+\tfrac{n-1}{k-1}<e\leqslant t_k(n). \tag{1.10}$$

In general, $\mathcal{H}^*(n,e)$ contains many nonisomorphic graphs. Nonetheless, a 'stability' result was established by Pikhurko and Razborov [34], who showed that every almost extremal (n,e)-graph is within edit distance $o(n^2)$ from $\mathcal{H}_1^*(n,e)$ (or, equivalently, from $\mathcal{H}^*(n,e)$).

THEOREM 1.2 [34]. *For every $\varepsilon>0$, there are $\delta,n_0>0$ such that, for every (n,e)-graph G with $n\geqslant n_0$ vertices and at most $g_3(n,e)+\delta\binom{n}{3}$ triangles, there exists $H\in\mathcal{H}_1^*(n,e)$ such that $|E(G)\bigtriangleup E(H)|\leqslant\varepsilon\binom{n}{2}$.*

Although Theorems 1.1 and 1.2 deal only with the asymptotic values, they can also be used to derive some exact results. Namely, if $n=(k-1)a+b$, where $k,a,b\in\mathbb{N}$ with $a\geqslant b$ and $e=\binom{k-1}{2}a^2+(k-1)ab=e(K^k_{a,\ldots,a,b})$, then

$$g_3(n,e)=K_3(K^k_{a,\ldots,a,b})=\binom{k-1}{3}a^3+\binom{k-1}{2}a^2b. \tag{1.11}$$

Indeed, if some (n,e)-graph H violates the lower bound, then the uniform blow-ups of H violate Theorem 1.1; furthermore, every extremal (n,e)-graph contains the complete $(k-1)$-partite graph $K^{k-1}_{a,\ldots,a,a+b}$ as a spanning subgraph, as otherwise its blow-ups violate Theorem 1.2.

The above blow-up trick also shows that $g_3(n,e)\geqslant(n^3/6)\,g_3(2e/n^2)$ for every (n,e). Although, for $e>t_2(n)$, one can show that this bound is tight only when the pair (n,e) is as in (1.11), it gives a rather good approximation to $g_3(n,e)$. Namely, calculations based on the explicit formula for $g_3(\lambda)=h^*(\lambda)$ (see, for example, [30, Theorem 1.3]) give that

$$0\leqslant g_3(n,e)-\frac{n^3}{6}\,g_3\!\left(\frac{2e}{n^2}\right)\leqslant\frac{n^3}{n^2-2e},\qquad n,e\in\mathbb{N},\ e\leqslant\binom{n}{2}. \tag{1.12}$$

In a long and difficult paper, Lovász and Simonovits [26] established the exact result for a large range of parameters. In order to state their main result, we have to define some graph families (which will also appear in our results and proofs).

DEFINITION 3 (Families $\mathcal{H}_0$, $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}$). Given positive integers e,n with $e\leqslant\binom{n}{2}$, let $k=k(n,e)$ be as in (1.1) and define the following families.

$\mathcal{H}_0(n,e)$: the family of (n,e)-graphs H obtained from adding a triangle-free graph J to a complete $(k-1)$-partite graph on n vertices.

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$\mathcal{H}_1(n, e)$: the family of (n, e)-graphs H with a partition $V(H) = A_1 \cup \cdots \cup A_{k-2} \cup B$ such that $|A_1| \geqslant \cdots \geqslant |A_{k-2}|$; $H[A_1 \cup \cdots \cup A_{k-2}]$ is the complete partite graph $K[A_1, \ldots, A_{k-2}]$; $H[B, V(H) \setminus B]$ is complete; and $H[B]$ is a triangle-free graph.

$\mathcal{H}_2(n, e)$: the family of k-partite (n, e)-graphs H with a partition $A_1, \ldots, A_k$ of $V(H)$ such that $|A_1| \geqslant \cdots \geqslant |A_k|$; $H[A_1 \cup \cdots \cup A_{k-1}] = K[A_1, \ldots, A_{k-1}]$, and for every vertex $x \in A_k$ there is at most one $j \in [k-1]$ such that x is not complete to A_j.

Also, let $\mathcal{H}(n, e) := \mathcal{H}_1(n, e) \cup \mathcal{H}_2(n, e)$ and define

$$h(n, e) := \min\{K_3(H) : H \in \mathcal{H}(n, e)\}. \tag{1.13}$$

Note that $\mathcal{H}_1(n, e) \subseteq \mathcal{H}_0(n, e)$; this inclusion is in general strict as the added edges in the definition of $\mathcal{H}_0(n, e)$ can lie inside different parts.

The main result proved by Lovász and Simonovits [26] (first announced in their 1975 paper [25]) is the following.

THEOREM 1.3 [25, 26]. *For all integers $k \geqslant 3$ and $r \geqslant 3$, there exist $\alpha = \alpha(r, k) > 0$ and $n_0 = n_0(r, k) > 0$ such that, for all positive integers (n, e) with $n \geqslant n_0$ and $t_{k-1}(n) < e \leqslant t_{k-1}(n) + \alpha n^2$, we have that*

$$g_r(n, e) = h_r(n, e) := \min\{K_r(H) : H \in \mathcal{H}(n, e)\}.$$

If $r = 3$, then every extremal graph lies in $\mathcal{H}_0(n, e) \cup \mathcal{H}_2(n, e)$, and there is at least one extremal graph in $\mathcal{H}_1(n, e)$. If $r \geqslant 4$, then every extremal graph lies in $\mathcal{H}_1(n, e) \cup \mathcal{H}_2(n, e)$.

Although the proof of Theorem 1.3 does not use the regularity lemma, the constant $\alpha(r, k)$ given by it is nonetheless so small that Lovász and Simonovits [26, page 465] write that they 'did not even dare to estimate' $\alpha(3, 3)$. In the same papers [25, 26], the following bold conjecture was stated.

CONJECTURE 1.4 [25, 26]. *For all integers $r \geqslant 3$, there exists $n_0 = n_0(r) > 0$ such that $g_r(n, e) = h_r(n, e)$ for all positive integers $n \geqslant n_0$ and $e \leqslant \binom{n}{2}$.*

Of course, the triangle-minimization problem for such a restricted class as any of $\mathcal{H}_i(n, e)$ is much easier than the unrestricted function $g_3(n, e)$. In fact, we can solve it exactly.

PROPOSITION 1.5. *For $i \in \{0, 1, 2\}$ and all $n, e \in \mathbb{N}$ with $e \leqslant \binom{n}{2}$, we have that $\min\{K_3(H) : H \in \mathcal{H}_i(n, e)\} = h^*(n, e)$ and $\mathcal{H}_i^*(n, e)$ is the set of graphs in $\mathcal{H}_i(n, e)$ that attain this bound.*

In particular, we have that $h(n, e) = h^*(n, e)$.

An interesting consequence of Proposition 1.5 that has not been observed before is that, for $r = 3$, if Conjecture 1.4 is true, then its conclusion is in fact true for **all** $n \geqslant 1$; see Lemma 10.1.

Apart from some cases when e is very close to $\binom{n}{2}$, to the best of the authors' knowledge, all established cases of the conjecture are confined to the direct consequences of Theorem 1.1 via the blow-up trick and to Theorem 1.3 (the latter superseding, as $n \to \infty$, all remaining exact results that we mentioned). The main contribution of this paper is to prove the conjecture when $r = 3$ and $e/\binom{n}{2}$ is bounded away from 1, and to characterize the extremal graphs in this range.

THEOREM 1.6. *For all $\varepsilon > 0$, there exists $n_0 > 0$ such that for all positive integers $n \geqslant n_0$ and $e \leqslant \binom{n}{2} - \varepsilon n^2$, we have that $g_3(n, e) = h(n, e)$. Furthermore, the family of extremal (n, e)-graphs is precisely $\mathcal{H}_0^*(n, e) \cup \mathcal{H}_2^*(n, e)$.*

By Theorem 1.3 and Proposition 1.5, it is enough to prove Theorem 1.6 when $e \geqslant t_{k-1}(n) + \Omega(n^2)$, where $k = k(n, e)$. This is done in the next theorem. (Note that $\mathcal{H}_0$ is irrelevant in this range by (1.10).)

THEOREM 1.7. *For all $\varepsilon, \alpha > 0$ and every integer $3 \leqslant k \leqslant 1/\varepsilon$, there exists $n_0 > 0$ such that the following holds. For all integers n, e with $n \geqslant n_0$ and $t_{k-1}(n) + \alpha n^2 \leqslant e < t_k(n)$, we have $g_3(n, e) = h(n, e)$ and every extremal graph lies in $\mathcal{H}(n, e) = \mathcal{H}_1(n, e) \cup \mathcal{H}_2(n, e)$.*

We believe that the following strengthening of the case $r = 3$ of Conjecture 1.4 holds where, additionally, the exact structure of all extremal graphs is described.

CONJECTURE 1.8. *For all positive integers n and $e \leqslant \binom{n}{2}$, an (n, e)-graph G satisfies $K_3(G) = g_3(n, e)$ if and only if $G \in \mathcal{H}_0^*(n, e) \cup \mathcal{H}_2^*(n, e)$.*

1.1. Organization of the paper. We collect some frequently used notation in Section 2 (and there is a symbolic glossary at the end of the paper). Theorem 1.6 is formally derived from Theorem 1.7 in Section 5.1. Since the proof of Theorem 1.7 is very involved and long, we provide a sketch in Section 3 and also try to provide all details in calculations. In Section 4, we investigate the function $h(n, e)$ and provide some preliminary tools that will be used later on; in particular, we prove Proposition 1.5. The proof of Theorem 1.7 begins in Section 5. Sections 6–8 continue the proof in the 'intermediate' case, which, roughly speaking, is when e is bounded away from any Turán density. The remaining 'boundary' case is dealt with in Section 9. Some concluding remarks can be found in Section 10.

2. Notation

Given a set X and $k \in \mathbb{N}$, let $\binom{X}{k}$ denote the set of k-subsets of X. Also, $[k] := \{1, \ldots, k\}$. We may abbreviate $\{a, b\}$ to ab. We write $x = y \pm \varepsilon$ if $y - \varepsilon \leqslant x \leqslant y + \varepsilon$.

We use standard graph theoretic notation. Given a graph G and $A \subseteq V(G)$, we write $\overline{A} := V(G) \setminus A$ for the *complement of A in G* and $\overline{G}$ for the graph with vertex set $V(G)$ and edge set $\binom{V(G)}{2} \setminus E(G)$, which we call the *complement of G*. Further, we write $G[A]$ for the graph induced by G on A. Given disjoint $A, B \subseteq V(G)$, we write $G[A, B]$ for the graph with vertex set $A \cup B$ and edge set $\{ab \in E(G) : a \in A, b \in B\}$. For $x \in V(G)$ and $A \subseteq V(G)$, we set $N_G(x, A) := \{y \in A : xy \in E(G)\}$ and $d_G(x, A) := |N_G(x, A)|$. Additionally, we write $N_G(x) := N_G(x, V(G))$ and $d_G(x) := |N_G(x)|$. Given pairwise-disjoint vertex sets $A_1, \ldots, A_\ell$, we write $K[A_1, \ldots, A_\ell]$ for the complete partite graph with parts $A_1, \ldots, A_\ell$. When $a_1, \ldots, a_\ell$ are integers, we write $K^\ell_{a_1,\ldots,a_\ell}$ (or $K_{a_1,\ldots,a_\ell}$) for the complete ℓ-partite graph with parts of sizes $a_1, \ldots, a_\ell$.

A partition of $V(H)$ witnessing that $H \in \mathcal{H}_i(n, e)$ in Definition 3 will be called *$\mathcal{H}_i$-canonical* (or just *canonical*).

Given $x \in V(G)$, we write $K_3(x, G)$ for the number of triangles in G that contain x. That is,

$$K_3(x, G) := e(G[N_G(x)]).$$

Given $A_1, A_2 \subseteq V(G) \setminus \{x\}$, we write $K_3(x, G; A_1, A_2)$ for the number of triples $\{x, a_1, a_2\}$ that span a triangle in G, where $a_i \in A_i$ for $i \in [2]$. (Note that we do not double count when both a_1, a_2 lie in $A_1 \cap A_2$.) If $A_1 = A_2 = A$, we let $K_3(x, G; A) := K_3(x, G; A, A)$. Similarly, given $\{x, y\} \in \binom{V(G)}{2}$, let $P_3(xy, G)$ be the number of 3-vertex paths with endpoints x and y; that is,

$$P_3(xy, G) := |N_G(x) \cap N_G(y)|.$$

Let $P_3(xy, G; A) := |N_G(x, A) \cap N_G(y, A)|$. Given a graph G with vertex partition $A_1, \ldots, A_k$, a *cross-edge* is any edge that lies between parts. Given two graphs G, H on the same vertex set V and $U \subseteq V$, we say that G and H *only differ at U* if $E(G) \triangle E(H) \subseteq \binom{U}{2}$.

Given a family $\mathcal{G}(n, e)$ of (n, e)-graphs, we write $\mathcal{G}^{\min}(n, e) \subseteq \mathcal{G}(n, e)$ for the subfamily consisting of all graphs with the minimum number of triangles.

Since we are interested in the case $r = 3$, we will say that a pair (n, e) is *valid* if $n, e \in \mathbb{N}$ are such that $\lfloor \frac{n^2}{4} \rfloor < e \leqslant \binom{n}{2}$ (that is, there exist graphs with n vertices and e edges, and every such graph contains at least one triangle).

Given $\ell \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_\ell \in \mathbb{R}$, for convenience, we write

$$e(K^{\ell}_{\alpha_1,\ldots,\alpha_\ell}) := \sum_{ij \in \binom{[\ell]}{2}} \alpha_i \alpha_j \quad \text{and} \quad K_3(K^{\ell}_{\alpha_1,\ldots,\alpha_\ell}) := \sum_{hij \in \binom{[\ell]}{3}} \alpha_h \alpha_i \alpha_j$$

in analogy with the number of edges and triangles in the complete ℓ-partite graph $K^{\ell}_{n_1,\ldots,n_\ell}$, which is defined when the n_i's are positive integers.

The *edit distance* between two graphs G and H on the same vertex set is $|E(G) \triangle E(H)|$, and these graphs are said to be d*-close* if $|E(G) \triangle E(H)| \leqslant d$.

3. Sketch of the proof of Theorem 1.7

The asymptotic results of Fisher [10], Razborov [37], Nikiforov [30], Pikhurko-Razborov [34] and Reiher [38] all use spectral or analytic methods. Such techniques do not seem to be helpful for the exact problem, and indeed our proof of Theorem 1.7 uses purely combinatorial methods. At its heart, our proof uses the well-known stability method: Theorem 1.2 implies that any extremal graph G is structurally close to *some* H in $\mathcal{H}^*(n, e)$ and hence some graph in $\mathcal{H}_1(n, e)$. Then the goal would be to analyse G and show that it cannot contain any imperfections and must in fact lie in $\mathcal{H}_1(n, e)$. The stability approach stems from the work of Erdős [8] and Simonovits [40] and has been used to solve many major problems in extremal combinatorics.

However, a major obstacle here is the fact that there is a large family of conjectured extremal graphs. Given any $H \in \mathcal{H}_1(n, e)$ with canonical partition $A_1, \ldots, A_{k-2}, B$ as in the definition, one can obtain a different $H' \in \mathcal{H}_1(n, e)$ such that $K_3(H') = K_3(H)$ simply by replacing $H[B]$ with another triangle-free graph containing the same number of edges. In general, there are many choices for this triangle-free graph.

An additional difficulty is that $\mathcal{H}_1(n, e)$ does not in fact contain every extremal graph, as in Theorem 1.3. So our goal as stated above must be modified.

Let us present a brief outline of the proof of Theorem 1.7. Suppose that Theorem 1.7 is false. Let k be the minimum integer for which there is an arbitrarily large integer n and some e with $t_{k-1}(n) < e \leqslant t_k(n)$ such that $\mathcal{H}(n, e)$ does not contain every extremal graph. Choose a fixed large n and then e as above such that $g_3(n, e) - h(n, e) \leqslant 0$ is minimal, and let $G \notin \mathcal{H}(n, e)$ be an (n, e)-graph with $K_3(G) = g_3(n, e)$. We call such a G a *worst counterexample*. One consequence of the choice of G is, for example, that no edge can lie in too many triangles, and the endpoints of every nonedge have many common neighbours.

I: The intermediate case $t_k(n) - e = \Omega(n^2)$.

1. Approximate structure (Section 6)

Theorem 1.2 implies that G is close in edit distance to some graph $H \in \mathcal{H}^*(n, e)$. Note that $H \in \mathcal{H}_1(n, e')$ for some e', which is close to e. The first step is to show that actually G is close to the specific graph $H^*(n, e)$ (namely, the edit distance is $o(n^2)$; see Lemma 6.4). The ith part of $H^*(n, e)$ has size a_i^*, which is roughly cn for all $i \in [k-1]$ (Lemma 4.16). Since e is bounded away from $t_k(n)$, it is not hard to see that $n - (k-1)cn < cn - \Omega(n)$. So G is close to a complete partite graph with one small part and the other parts equally sized. In fact, we can show (Lemma 6.1) that every max-cut partition $A_1, \ldots, A_k$ of G is such that $||A_i| - cn| = o(n)$ for $i \in [k-1]$ (and $||A_k| - (n - (k-1)cn)| = o(n)$) and $m + h = o(n^2)$, where

$$m := \sum_{ij \in \binom{[k]}{2}} e(\overline{G}[A_i, A_j]) \quad \text{and} \quad h := \sum_{i \in [k]} e(G[A_i]).$$

Following [26], we say that any pair of vertices in different parts that does not span an edge is a *missing edge*, and any edge inside a part is *bad*. As usual, we now identify some vertices that are atypical in the sense that they are incident to many missing edges. Let Z be the set of vertices incident with $\Omega(n)$ missing edges. Thus

$$|Z| = O(m/n) = o(n). \tag{3.1}$$

It turns out that every bad edge is incident to a vertex in Z. Thus, if $Z = \emptyset$, then G is k-partite and it is not hard to show (see Corollary 4.4(i)) that every extremal k-partite (n, e)-graph lies in $\mathcal{H}_2(n, e)$, a contradiction.

2. Transformations (Section 7)

Now we would like to make a series of local changes to G to obtain a new n-vertex e-edge graph G' such that $K_3(G') - K_3(G) = 0$, but the structure of G' is much simpler. Here, 'simpler' means 'no bad edges', so G' would be k-partite, and we would obtain our desired contradiction. From the property of Z above, these local changes would then only have to be made at Z. Unfortunately, this is too ambitious as we do not have fine enough control on the structure of the graph. Therefore we reduce our expectations and aim to find G' such that $K_3(G') - K_3(G)$ is *small* (Lemma 7.1). That is, we simplify the structure (and thus it is easier to count triangles) at the expense of a few additional triangles. To be more precise, *small* means $o(m^2/n)$. Although the transformations themselves are easy to describe, this is the longest and most technical part of the proof.

- Transformation 1 (Figure 3, Lemmas 7.3 and 7.4): Removing bad edges in the large parts $A_1, \ldots, A_{k-1}$.
- Transformation 2 (Figure 4, Lemmas 7.5 and 7.6): Reassigning those vertices in $Z \cap A_k$ incident to many missing edges to a large part.
- Transformations 3–6 (Figures 5–7, Lemmas 7.7–7.10 and the proof of Lemma 7.1): Dealing with those vertices in $Z \cap A_k$ incident to few missing edges.

3. Finishing the proof in this case (Section 8)

i. Suppose that $m > Cn$ for some large constant C (Section 8.1). Write $A_1'', \ldots, A_k''$ for the parts of G'. Keeping track of the transformation $G \to G'$ allows us to use G' to obtain additional structural information about G. To do this, we apply Lemma 4.19, which measures the difference in the numbers of triangles between a k-partite (n, e)-graph (such as G') and an 'ideal' k-partite graph (which is essentially $H^*(n, e)$). Because the same is true in G in the intermediate case, the difference in size between the smallest part of G' and the other parts is $\Omega(n)$. In Lemma 8.2, this fact and $K_3(G') - K_3(H^*(n, e)) \leqslant K_3(G') - K_3(G) = o(m^2/n)$ imply via Lemma 4.19 that $e(\overline{G'}[A_i'', A_k'']) = \Omega(m)$ for exactly one $i \in [k-1]$, and the other A_j'' satisfy $||A_j''| - cn| = o(m/n)$ and $|Z \cap A_j''| = o(m/n)$ (which is much stronger than (3.1)).

Since we had fine control on the transformation $G \to G'$, similar statements hold in G (Lemma 8.4): $e(\overline{G}[A_i, A_k]) = \Omega(m)$ for exactly one $i \in [k-1]$, and the other A_j satisfy $||A_j| - cn| = o(m/n)$ and $|Z \cap A_j| = o(m/n)$. This new information about G is substantial enough to show that most of the local changes we did earlier actually *decrease* the number of triangles. This applies, for example, to Transformation 1, and we conclude that $Z \cap A_j = \emptyset$ for all $j \in [k-1] \setminus \{i\}$. So A_j contains no bad edges (Lemma 8.6). This analysis requires tight 'step-by-step' control on the effect of the transformations, which is what makes the proofs more technical than they would otherwise have to be. Then a final global change (see Figure 8) brings us to a graph $H \in \mathcal{H}_1(n, e)$, which, if $Z \neq \emptyset$, satisfies $K_3(H) - K_3(G) < 0$, a contradiction.

ii. Suppose that $m < Cn$ (Section 8.2). This case is different as the errors stemming from G' are too large to allow us to glean any extra information. Instead, we show directly that most of the transformations we did earlier *do not increase* the number of triangles. This is possible since we now know that, for example, Z has constant size (see (3.1)).

This case has a different flavour because we may enter the situation where, for example, after performing Transformation 1 to obtain G_1, we have $K_3(G_1) =$

$K_3(G)$ and $G_1 \in \mathcal{H}(n, e)$. Then we have to argue that in fact this must imply $G \in \mathcal{H}(n, e)$, a contradiction. This is the only part of the proof where we are not able to obtain a contradiction by strictly decreasing the number of triangles, and must actually analyse the extremal family $\mathcal{H}(n, e)$ (Section 8.2.1).

II: The boundary case $t_k(n) - e = r$, where $r = o(n^2)$ (*Section* 9).

The proof in this case turns out to be much shorter than the intermediate case. We now have that $cn = n/k + O(\sqrt{r})$. A different argument is required to determine the approximate structure of G as we need better bounds in terms of r: we use an averaging argument (Lemma 9.2), which is very similar to [**26**, Theorem 2]. Thus we obtain a rather strong structure property (Lemma 9.1): every max-cut partition $A_1, \ldots, A_k$ of G is such that $||A_i| - n/k| = O(\sqrt{r})$ for *all* $i \in [k]$, and $\sum_{ij \in \binom{[k]}{2}} e(\overline{G}[A_i, A_j]) + \sum_{i \in [k]} e(G[A_i]) = O(r)$.

Again, we let Z be the set of vertices with $\Omega(n)$ missing edges, and show that $|Z| = o(n)$ and every bad edge is incident to a vertex in Z. In the intermediate case, the most troublesome vertices were those in $Z \cap A_k$ dealt with in Transformations 3–6. Now, A_k is not substantially smaller than the other parts, so this is no longer the case and some difficulties from the intermediate case disappear.

We show that, for every $i \in [k]$, the set $A_i \setminus Z$ is 'significantly smaller' than cn. This then implies that $G[A_1 \setminus Z, \ldots, A_k \setminus Z]$ is complete partite (Lemma 9.9). Finally, we show that $Z = \emptyset$, completing the proof as before. For these final steps, we again build up a repository of structural information by performing (much simpler) transformations that strictly decrease the number of triangles unless a desired property holds.

4. Extremal families and preliminary tools

One of the main results of this section is to prove Proposition 1.5 that for all $i = 0, 1, 2$, we have $\mathcal{H}_i^{\min}(n, e) = \mathcal{H}_i^*(n, e)$, and $h(n, e) = h^*(n, e)$ for all valid pairs (n, e). In order to do this, we present some auxiliary definitions and results first.

4.1. Extremal $k(n, e)$-partite graphs. The main conclusion of this section will be Corollary 4.4, which states that all extremal $k(n, e)$-partite (n, e)-graphs lie in $\mathcal{H}_2(n, e)$ and at least one such graph is in $\mathcal{H}_1(n, e)$.

In order to prove it, we need to define a somewhat related family $\mathcal{H}_2'(n, e)$. Given a valid pair (n, e), let $k := k(n, e)$. Define $\mathcal{H}_2'(n, e)$ to be the family of k-partite (n, e)-graphs H with parts $A_1, \ldots, A_k$ of sizes $|A_1| \geqslant \cdots \geqslant |A_k|$ such that

(1) for all $i \in [k]$ and $x \in A_i$, there is at most one $j \in [k] \setminus \{i\}$ such that $d_{\overline{H}}(x, A_j) > 0$;

(2) if $|A_i| + |A_j| > |A_{k-1}| + |A_k|$, then $H[A_i, A_j]$ is complete.

We say that $A_1, \ldots, A_k$ is an $\mathcal{H}_2'$-*canonical partition*. The above definition is motivated by the following easy lemma.

LEMMA 4.1. *Let (n, e) be valid and let $k = k(n, e)$. Let $\mathcal{G}(n, e)$ be the set of k-partite (n, e)-graphs. Then $\mathcal{G}^{\min}(n, e) \subseteq \mathcal{H}_2'(n, e)$.*

Proof. Let $G \in \mathcal{G}^{\min}(n, e)$. Let $A_1, \ldots, A_k$ be the parts of G, where $a_i := |A_i|$ for all $i \in [k]$ and $a_1 \geqslant \cdots \geqslant a_k$. Let $m := \sum_{ij \in \binom{[k]}{2}} e(\overline{G}[A_i, A_j])$.

We have that $m \leqslant a_{k-1}a_k$, for otherwise

$$e < e(K_{a_1,\ldots,a_{k-2},a_{k-1}+a_k}) \leqslant t_{k-1}(n)$$

and so $k(n, e) \leqslant k - 1$, a contradiction. Consider $G^* := K[A_1, \ldots, A_k] \setminus E^*$, where E^* consists of some m edges of $K[A_{k-1}, A_k]$. Clearly, $G^* \in \mathcal{G}(n, e)$. Thus, by the minimality of $G \in \mathcal{G}(n, e)$, we have $K_3(G^*) \geqslant K_3(G)$. On the other hand, since each pair of E^* is in exactly $a_1 + \cdots + a_{k-2}$ triangles of $K[A_1, \ldots, A_k]$ and no such triangle is counted more than once, we have

$$\begin{aligned} K_3(G^*) - K_3(G) &= (K_3(K[A_1, \ldots, A_k]) - K_3(G)) \\ &\quad - (K_3(K[A_1, \ldots, A_k]) - K_3(G^*)) \\ &\leqslant \sum_{ij \in \binom{[k]}{2}} e(\overline{G}[A_i, A_j]) \left(\sum_{h \in [k] \setminus \{i,j\}} a_h \right) - |E^*|(a_1 + \cdots + a_{k-2}) \\ &= \sum_{ij \in \binom{[k]}{2}} e(\overline{G}[A_i, A_j]) \left(\sum_{h \in [k] \setminus \{i,j\}} a_h - (a_1 + \cdots + a_{k-2}) \right) \\ &= \sum_{ij \in \binom{[k]}{2}} e(\overline{G}[A_i, A_j]) \big((a_{k-1} + a_k) - (a_i + a_j) \big) \leqslant 0, \end{aligned} \tag{4.1}$$

so we have equality throughout. The sharpness of the first (respectively, second) inequality in (4.1) implies the first (respectively, second) property from the definition of $\mathcal{H}_2'(n, e)$. Thus $G \in \mathcal{H}_2'(n, e)$, as required. □

We also need the following result concerning extremal graphs in $\mathcal{H}_2'(n, e)$.

LEMMA 4.2. *Let (n, e) be valid with $k = k(n, e)$. Let $H \in (\mathcal{H}_2')^{\min}(n, e)$ with an $\mathcal{H}_2'$-canonical partition $A_1, \ldots, A_k$ having part sizes $a_1 \geqslant \cdots \geqslant a_k$, respectively. Let $m := \sum_{ij \in \binom{[k]}{2}} e(\overline{H}[A_i, A_j])$. Then the following statements hold.*

(i) *There exists* $G \in \mathcal{H}_1(n, e) \cap \mathcal{H}_2'(n, e) \cap \mathcal{H}_2(n, e)$ *with* $K_3(G) = K_3(H)$.

(ii) *If* $a_{k-2} = a_{k-1}$, *then* $m \leqslant a_{k-1} - a_k + 1$.

Proof. If $m > a_{k-1}a_k$, then $e < t_{k-1}(n)$, a contradiction. Thus there exists $G := K[A_1, \dots, A_k] \backslash E^*$, where $E^* \subseteq K[A_{k-1}, A_k]$ and $|E^*| = m$. Clearly, $G \in \mathcal{H}_1(n, e) \cap \mathcal{H}_2'(n, e) \cap \mathcal{H}_2(n, e)$. Also, the calculation as in (4.1) shows that $K_3(G) \leqslant K_3(H)$. This is equality by the minimality of H, proving the first part of the lemma.

Now, let us show (ii). Let $a := a_{k-2} = a_{k-1}$. Suppose on the contrary that $s := m - a + a_k - 1$ is at least 1. Then $(a + 1)(a_k - 1) - (aa_k - m) = s \geqslant 1$. If $s > a(a_k - 1)$, then

$$\begin{aligned} e &= e(K_{a_1,\dots,a_k}) - m = e(K_{a_1,\dots,a_{k-3},a+1,a,a_k-1}) - s < e(K_{a_1,\dots,a_{k-3},a+1,a+a_k-1}) \\ &\leqslant t_{k-1}(n), \end{aligned}$$

a contradiction to the definition of k. Thus there is an (n, e)-graph J obtained from the complete k-partite graph $K_{a_1,\dots,a_{k-3},a+1,a,a_k-1}$ by removing s edges between the last two classes (that have sizes a and $a_k - 1$). Note that $J \in \mathcal{H}_2'(n, e)$. But then we have

$$\begin{aligned} K_3(H) - K_3(J) &\geqslant (a^2 a_k - (s + a - a_k + 1)a) - (a(a + 1)(a_k - 1) - s(a + 1)) \\ &= s > 0. \end{aligned}$$

This contradiction completes the proof of the second part. □

LEMMA 4.3. *Let* (n, e) *be valid with* $k = k(n, e)$. *Then* $(\mathcal{H}_2')^{\min}(n, e) = \mathcal{H}_2^{\min}(n, e)$. *Moreover, for all graphs in this family, an* $\mathcal{H}_2'$*-canonical partition is an* $\mathcal{H}_2$*-canonical partition up to relabelling parts, and vice versa.*

Proof. Throughout this proof, we omit (n, e) for brevity.

We first show that $(\mathcal{H}_2')^{\min} \subseteq \mathcal{H}_2^{\min}$. Take any $H \in (\mathcal{H}_2')^{\min}$ with an $\mathcal{H}_2'$-canonical partition $A_1, \dots, A_k$. We claim that $H \in \mathcal{H}_2$, and some ordering of $\{A_1, \dots, A_k\}$ is an $\mathcal{H}_2$-canonical partition. Assume that $|A_{k-2}| = |A_{k-1}| = |A_k|$ for otherwise $e(\overline{H}[A_i, A_j]) > 0$ only if $k \in \{i, j\}$ in which case $H \in \mathcal{H}_2$, as desired. Lemma 4.2(ii) gives that

$$\sum_{ij \in \binom{[k]}{2}} e(\overline{H}[A_i, A_j]) \leqslant |A_{k-1}| - |A_k| + 1 = 1.$$

Thus H has at most one missing edge, which (if exists) is incident to some part A_i with $|A_i| = |A_k|$. In any case, $H \in \mathcal{H}_2$ with the same canonical partition, up to relabelling, as claimed. If H is not in $\mathcal{H}_2^{\min}$, then any $H' \in \mathcal{H}_2^{\min}$ has fewer triangles than H. However, by Lemma 4.1 there is $G \in \mathcal{H}_2'$ with $K_3(G) \leqslant K_3(H') < K_3(H)$, contradicting the extremality of H. In particular, writing $h_2 := K_3(F)$ and $h_2' := K_3(F')$, where $F \in \mathcal{H}_2^{\min}$ and $F' \in (\mathcal{H}_2')^{\min}$, we see that $h_2 = h_2'$.

We now show the other direction, that is, $(\mathcal{H}_2')^{\min} \supseteq \mathcal{H}_2^{\min}$. Let $\mathcal{G}(n, e)$ be the set of k-partite (n, e)-graphs. By definition, $\mathcal{H}_2 \subseteq \mathcal{G}$. As $\mathcal{G}^{\min} \subseteq \mathcal{H}_2'$ due to Lemma 4.1 and $h_2 = h_2'$, we have that $\mathcal{H}_2^{\min} \subseteq \mathcal{G}^{\min} \subseteq (\mathcal{H}_2')^{\min}$ as desired. Furthermore, if $A_1, \ldots, A_k$ is an $\mathcal{H}_2$-canonical partition of $G \in \mathcal{H}_2^{\min}$, some ordering of it is an $\mathcal{H}_2'$-canonical partition. □

For ease of reference, let us summarize some facts that we will need later.

COROLLARY 4.4. *Let (n, e) be valid with $k = k(n, e)$. Then the following statements hold.*

(i) *Every extremal k-partite (n, e)-graph lies in $\mathcal{H}_2(n, e)$.*

(ii) *At least one extremal k-partite (n, e)-graph lies in $\mathcal{H}_1(n, e)$.*

(iii) *Let $H \in \mathcal{H}_2^{\min}(n, e) \setminus \mathcal{H}_1(n, e)$ with an $\mathcal{H}_2$-canonical partition $A_1^*, \ldots, A_k^*$. Then*
$$\sum_{ij \in \binom{[k]}{2}} e(\overline{H}[A_i^*, A_j^*]) \leqslant |A_{k-1}^*| - |A_k^*| + 1 \leqslant n.$$

Proof. Part (i) (respectively, (ii)) is a direct consequence of Lemma 4.1 when combined with Lemma 4.3 (respectively, with Lemma 4.2(i)). To see (iii), let H and $A_1^*, \ldots, A_k^*$ be as stated. We claim that $|A_{k-2}^*| = |A_{k-1}^*|$. Indeed, if $|A_{k-2}^*| \geqslant |A_{k-1}^*| + 1$, then all the missing edges in H should lie in $[A_{k-1}^*, A_k^*]$ as otherwise moving all missing edges to $[A_{k-1}^*, A_k^*]$ would result in a graph still in $\mathcal{H}_2(n, e)$ having strictly fewer triangles than H, contradicting the choice of H. But then if all missing edges lie in $[A_{k-1}^*, A_k^*]$, we have $H \in \mathcal{H}_1(n, e)$, a contradiction. This together with Lemma 4.2(ii) and Lemma 4.3 implies (iii). □

For future reference, let us state here the following auxiliary lemma, which implies that if the condition on a that defines a_k^* in Definition 1 fails for some $a \leqslant n/k$, then it fails for all smaller values of $a \in \mathbb{N}$.

LEMMA 4.5. *For any integers $a \geqslant 1$, $k \geqslant 2$ and $n \geqslant ak$, we have*
$$a(n - a) + t_{k-1}(n - a) > (a - 1)(n - a + 1) + t_{k-1}(n - a + 1).$$

Proof. Let $a_1 \geqslant \cdots \geqslant a_{k-1}$ be the part sizes of $T_{k-1}(n-a)$. If we increase its order by 1, then the part sizes of the new Turán graph, up to a reordering, can be obtained by increasing a_{k-1} by 1. Thus we need to estimate the following difference:

$$e(K_{a_1,\dots,a_{k-1},a}) - e(K_{a_1,\dots,a_{k-2},a_{k-1}+1,a-1}) = a_{k-1}a - (a_{k-1}+1)(a-1) = a_{k-1} - a + 1, \tag{4.2}$$

which is positive since $a_{k-1} \geqslant \lfloor (n-a)/(k-1) \rfloor$ is at least a by our assumption $a \leqslant \lfloor n/k \rfloor$. □

4.2. Proof of Proposition 1.5. First, we describe a transformation that converts an arbitrary $\mathcal{H}_0(n,e)$-extremal graph G into another extremal graph H' of a rather simple structure. Then, we argue in Lemma 4.6 that H' is in fact isomorphic to the special graph $H^*(n,e)$ from Definition 1. Since $H^*(n, e) \in \mathcal{H}_1(n,e) \subseteq \mathcal{H}_0(n,e)$, this determines the minimum number of triangles for graphs in these two families. From here, it is relatively easy to derive all remaining claims of Proposition 1.5.

Let (n,e) be valid and set $k = k(n,e)$. Take an arbitrary graph $G \in \mathcal{H}_0^{\min}(n,e)$ with a vertex partition $B_1, \dots, B_{k-1}$ such that G consists of the union of $K[B_1, \dots, B_{k-1}]$ and an edge-disjoint triangle-free graph J. We say that a part B_j, $j \in [k-1]$, is *partially full (in G)* if $0 < e(G[B_j]) < t_2(b_j)$, where $b_j := |B_j|$. Since we can move edges in both directions between such parts (keeping the parts triangle-free and thus staying within the family $\mathcal{H}_0(n,e)$), we have by the minimality of G that

$$b_i = b_j, \quad \text{for all } i, j \in [k-1] \text{ such that } B_i \text{ and } B_j \text{ are partially full.} \tag{4.3}$$

We construct another graph $H' = H'(G)$ in $\mathcal{H}_0^{\min}(n,e)$ using the following steps.

Step 1 For each partially full part B_j, replace $G[B_j]$ by a balanced bipartite graph of the same size (which is possible by Mantel's theorem).

Step 2 Move edges between partially full parts (keeping them balanced bipartite), until at most one remains. By (4.3), the current graph (denote it by G_1) is still in $\mathcal{H}_0^{\min}(n,e)$.

Step 3 If there is a part B_i which is partially full in G_1, then let $B := B_i$; otherwise, let $B := B_i$ for some $i \in [k-1]$ with $e(G_1[B_i]) = t_2(b_i)$ (such i exists since $e(G_1) = e > t_{k-1}(n)$).

Step 4 As $V(G) \setminus B$ induces a complete partite graph in G_1, let $A_1, \dots, A_{t-2}$ be its parts of sizes $a_1 \geqslant \cdots \geqslant a_{t-2}$, respectively. Thus each part B_i of G is equal to either B, or some A_j, or the union of some two parts $A_j \cup A_\ell$.

Step 5 Choose integers $a_{t-1} \geqslant a_t \geqslant 1$ such that $a_{t-1} + a_t = |B|$ and $(a_{t-1} + 1)(a_t - 1) < e(G_1[B]) \leqslant a_{t-1}a_t$, which is possible since $G_1[B]$ is bipartite. Let A_{t-1}, A_t be a partition of B with $|A_i| = a_i$ for $i \in \{t-1, t\}$. If $e(G_1[B]) = t_2(|B|)$, then we additionally require that the parts A_{t-1} and A_t are given by the bipartition of $G_1[B] \cong T_2(|B|)$.

Step 6 Let H' be obtained from $K[A_1, \dots, A_t]$ by removing a star centred at A_t with m' leaves all of which lie in A_{t-1}, where $m' := \sum_{ij \in \binom{[t]}{2}} a_i a_j - e = a_{t-1}a_t - e(G_1[B])$. This is possible because, like in (1.3), we have

$$0 \leqslant m' \leqslant a_{t-1} - a_t. \tag{4.4}$$

LEMMA 4.6. *For every valid (n, e) and $G \in \mathcal{H}_0^{\min}(n, e)$, the graph H' produced by Steps 1–6 above is isomorphic to $H^*(n, e)$.*

Proof. We will use the notation defined in Steps 1–6 (such as the sets B_i and A_i and so on). As H' is obtained from $G_1 \in \mathcal{H}_0^{\min}(n, e)$ by replacing a bipartite graph on B with another bipartite graph of the same size (while B is complete to the rest in both graphs), we have that $K_3(H') = K_3(G_1)$ and thus $H' \in \mathcal{H}_0^{\min}(n, e)$.

CLAIM 4.7. *If $m' = 0$, then $e(H'[A_h \cup A_i \cup A_j]) > t_2(|A_h| + |A_i| + |A_j|)$ for all $hij \in \binom{[t]}{3}$. If $m' > 0$, then the stated inequality holds for every triple $\{h, t-1, t\}$ with $h \in [t-2]$.*

Proof of Claim. Let $W := A_h \cup A_i \cup A_j$. Suppose on the contrary that $e(H'[W]) \leqslant t_2(|W|)$. Then one can obtain a new graph G_2 from H' by replacing $H'[W]$ with a bipartite graph of the same size. Note that H' is complete between W and $\overline{W}$. (Indeed, this is trivially true if $m' = 0$ as then $H' = K[A_1, \dots, A_t]$; otherwise, the only noncomplete pair is $[A_{t-1}, A_t]$, but both of these sets lie inside W.)

As H' is t-partite, the graph G_2 is $(t-1)$-partite (with at most one noncomplete pair of parts). By Steps 4–5, we have $t \leqslant 2(k-1)$. So we can represent G_2 as the union of complete $(k-1)$-partite and triangle-free graphs, that is, $G_2 \in \mathcal{H}_0(n, e)$. We have that $K_3(G_2[W]) = 0$ and W is complete to $\overline{W}$ in both H' and G_2. Thus the fact that $H' \in \mathcal{H}_0^{\min}(n, e)$ implies that $K_3(H'[W]) = 0$. However, if $\{t-1, t\} \not\subseteq \{h, i, j\}$, then $H'[W]$ is complete tripartite and so clearly contains at least one triangle. Otherwise, if, say, $\{i, j\} = \{t-1, t\}$, then H' spans at least one edge between A_{t-1} and A_t (since there are $m' \leqslant a_{t-1} - a_t < a_{t-1}a_t$ missing edges by (4.4)). Each such edge lies in $|A_h| > 0$ triangles in $H'[W]$. So in either case, we obtain a contradiction. □

CLAIM 4.8. *If $m' > 0$, then $a_{t-2} \geqslant a_{t-1}$.*

Proof of Claim. Suppose the claim is false. Now, make a new graph G_3 from H' by replacing $[A_{t-2}, A_t]$-edges with $[A_{t-1}, A_t]$-edges until this is no longer possible. Let $W := A_{t-2} \cup A_{t-1} \cup A_t$. If $A_{t-2} \cup A_t$ is an independent set in G_3 (that is, if $m' \geqslant a_{t-2}a_t$), then $e(H'[W]) = e(G_3[W]) \leqslant t_2(|W|)$, contradicting Claim 4.7 for the triple $\{t-2, t-1, t\}$. Thus $G_3[W]$ is obtained from $K[A_{t-2}, A_{t-1}, A_t]$ by removing m' edges from $K[A_{t-2}, A_t]$. So $G_3 \in \mathcal{H}_0(n, e)$, and

$$K_3(G_3)-K_3(H') = m'((n-a_{t-1}-a_t)-(n-a_{t-2}-a_t)) = m'(a_{t-2}-a_{t-1}) \leqslant -1,$$

a contradiction proving the claim. □

If $m' > 0$, let $C_i := A_i$ for $i \in [t]$. If $m' = 0$, then let $C_1, \ldots, C_t$ be a relabelling of $A_1, \ldots, A_t$ so that the sizes of the sets do not increase. Regardless of the value of m', the following statements hold. First, $c_1 \geqslant \cdots \geqslant c_t$, where $c_i := |C_i|$ for $i \in [t]$. (Indeed, if $m' > 0$, this follows from Steps 4–5 and Claim 4.8.) Also, we have

$$0 \leqslant m' \leqslant c_{t-1} - c_t. \tag{4.5}$$

(Indeed, if $m' > 0$, this is the same as (4.4); otherwise, this is a trivial consequence of $m' = 0$ and $c_{t-1} \geqslant c_t$.) Also, Claim 4.7 applies to any triple C_i, C_{t-1}, C_t.

The rest of the proof is written so that it works for both $m' = 0$ and $m' > 0$.

CLAIM 4.9. *We have that $c_1 \leqslant c_{t-1} + 1$.*

Proof of Claim. Suppose that this is false. Let $W := C_1 \cup C_{t-1} \cup C_t$. Note that

$$e(K_{c_1-1,c_{t-1}+1,c_t}) - e(H'[W]) = m' - c_{t-1} + c_1 - 1 =: m''.$$

Now, $m'' \geqslant m' + 1$. We claim that additionally $m'' < (c_{t-1}+1)c_t$. Suppose that this is not true. Then $e(H'[W]) \leqslant (c_1 - 1)(c_{t-1} + c_t + 1) \leqslant t_2(|W|)$, contradicting Claim 4.7. Take a partition C'_1, C'_{t-1}, C'_t of W of sizes $c_1 - 1$, $c_{t-1}+1, c_t$, respectively, and let a graph H_W be obtained from $K[C'_1, C'_{t-1}, C'_t]$ by removing m'' edges between C'_{t-1} and C'_t. Then $e(H_W) = e(H'[W])$. Obtain H'' from H' by replacing $H'[W]$ with H_W. Note that $H'' \in \mathcal{H}_0(n, e)$. By (4.5), we have that

$$\begin{aligned} K_3(H') - K_3(H'') &= K_3(H'[W]) - K_3(H_W) \\ &= \big(c_1c_{t-1}c_t - m'c_1\big) - \big((c_1-1)(c_{t-1}+1)c_t \\ &\quad - (m' - c_{t-1} + c_1 - 1)(c_1 - 1)\big) \\ &\geqslant (c_1 - c_t)(c_1 - c_{t-1} - 2) + 1 \geqslant 1, \end{aligned}$$

a contradiction proving $c_{t-1} + 1 \geqslant c_1$. □

It follows that $C_1, \ldots, C_{t-1}$ induce a Turán graph in H'. (Indeed, the sizes of these independent sets are almost equal by Claim 4.9; furthermore, if $m' > 0$, then all missing edges in H' are between $C_{t-1} = A_{t-1}$ and $C_t = A_t$ while otherwise there are no missing edges at all.)

Now, we can argue that $t = k$. By the definition of k, we have to show that $t_{t-1}(n) < e \leqslant t_t(n)$. Clearly, H' is t-partite, so $e \leqslant t_t(n)$. So it remains to show $t_{t-1}(n) < e$. Let $T := H'[C_1 \cup \cdots \cup C_{t-1}] \cong T_{t-1}(n - c_t)$. We can obtain both H' and $T_{t-1}(n)$ from T by adding c_t vertices one by one. First, let us make H' from T. The number of additional edges is $e - e(T) = c_t(n - c_t) - m'$. If we instead add vertices one by one to T to make $T_{t-1}(n)$, each vertex must miss an entire part of the current graph, so its degree is at most $n - c_{t-1} - 1$. Thus $t_{t-1}(n) - e(T) \leqslant c_t(n - c_{t-1} - 1)$. By (4.5), we have

$$e - t_{t-1}(n) \geqslant c_t(c_{t-1} + 1 - c_t) - m' \geqslant (c_t - 1)(c_{t-1} - c_t) + c_t > 0.$$

Thus $t = k$, as stated.

Now we can show that H' has part sizes given by the vector $\boldsymbol{a}^* = \boldsymbol{a}^*(n, e)$ from Definition 1, finishing the proof of the lemma. By Claim 4.9, we have that $\sum_{ij \in \binom{[k-1]}{2}} c_i c_j = t_{k-1}(n - c_k)$. Note that $m' = c_{k-1}c_k - e(H'[C_{k-1} \cup C_k])$. Thus we have by (4.5) that $e - t_{k-1}(n - c_k) = c_k(n - c_k) - m' \leqslant c_k(n - c_k)$.

So it remains only to show that c_k is the *smallest* natural number a with $f(a) := a(n - a) + t_{k-1}(n - a) \geqslant e$. Note that $c_k \leqslant n/k$ as it is the smallest among $c_1 + \cdots + c_k = n$. Thus, by Lemma 4.5, it is enough to check that $c_k - 1$ violates this condition. The calculation in (4.2), the estimates that we stated in the previous paragraph and (4.5) give that

$$f(c_k - 1) = f(c_k) - (c_{k-1} - c_k + 1) \leqslant e + m' - (m' + 1) < e,$$

as desired. This finishes the proof of the lemma. $\square$

Proof of Proposition 1.5. Let $n, e \in \mathbb{N}$ with $e \leqslant \binom{n}{2}$ and let $k := k(n, e)$. Corollary 4.4 and Lemma 4.6 show that, for each $i \in \{0, 1, 2\}$, the minimum number of triangles over the graphs in $\mathcal{H}_i(n, e) \ni H^*(n, e)$ is $K_3(H^*(n, e)) = h^*(n, e)$. Thus it remains to describe the extremal graphs. Assume that $k \geqslant 3$ as otherwise $h(n, e) = h^*(n, e) = 0$ and trivially $\mathcal{H}_i^{\min}(n, e) = \mathcal{H}_i^*(n, e)$ for $i = 0, 1, 2$.

First, we will prove that $\mathcal{H}_i^{\min}(n, e) = \mathcal{H}_i^*(n, e)$ for $i = 0, 1$. Let $G \in \mathcal{H}_0^{\min}(n, e)$ be arbitrary. Let G have vertex partition $B_1, \ldots, B_{k-1}$ such that G consists of the union of $K[B_1, \ldots, B_{k-1}]$ and an edge-disjoint triangle-free graph J. Write $b_i := |B_i|$ for all $i \in [k - 1]$. Apply Steps 1–6 to G to obtain a t-partite graph H' with parts $A_1, \ldots, A_t$. By Lemma 4.6, H' is isomorphic to $H^* := H^*(n, e)$.

Thus $t = k$ and, by relabelling parts, we can assume that $|A_i| = a_i^*$ for all $i \in [k]$ and that all missing edges, if any exist, are in $H'[A_{k-1}, A_k]$.

We will also need the following claim.

CLAIM 4.10. *If a part B_i is not partially full in G (that is, if $e(G[B_i])$ is 0 or $t_2(b_i)$), then $G[B_i] = H'[B_i]$ (that is, no adjacency inside B_i is modified).*

Proof of Claim. If $e(G[B_i]) = 0$, then $B_i = A_j$ for some $j \in [k-2]$ and so $e(H'[B_i]) = 0 = e(G[B_i])$, giving the required. If $e(G[B_i]) = t_2(b_i)$, then by construction, $G_1[B_i] = G[B_i]$ are maximum bipartite graphs and so $H'[B_i] = G[B_i]$, as required. □

Since $t = k$, exactly one part B_p of G is subdivided as $A_q \cup A_r$ in Steps 4–5 (that is, $B_p = A_q \cup A_r$), while the remaining parts of G correspond to the remaining parts of H'. In particular, $b_p = a_q^* + a_r^*$, where, say, $1 \leqslant q < r \leqslant k$.

Let us show that $e(G[B_p]) > 0$. Indeed, if this is not true, then, by (1.3), $H'[B_p]$ contains $a_q^* a_r^* - m^* \geqslant a_q^* a_r^* - (a_{k-1}^* - a_k^*) > 0$ edges, and so is different from the edgeless graph $G[B_p]$. Then Claim 4.10 implies that B_p is partially full, a contradiction.

Case 1. *There exists $h \in [k-1] \setminus \{p\}$ such that $e(G[B_h]) > 0$. In other words, $G \in \mathcal{H}_0^{\min}(n, e) \setminus \mathcal{H}_1(n, e)$.*

We claim that $b_h = b_p$. This follows from (4.3) if B_h and B_p are both partially full. Note that B_h is an independent set in H' and so $G[B_h] \neq H'[B_h]$, and Claim 4.10 implies that B_h is partially full. So it suffices to show that B_p is partially full. If not, then $e(G[B_p]) = t_2(b_p)$ (as $e(G[B_p]) = 0$ is already excluded). Since $G[B_i, B_j] = H'[B_i, B_j]$ for all $ij \in \binom{[k-1]}{2}$ and $e(H'[B_h]) = 0 < e(G[B_h])$, there is some $\ell \in [k-1] \setminus \{h\}$ such that $e(H'[B_\ell]) > e(G[B_\ell])$. Since $H'[B_p]$ is bipartite and $e(G[B_p]) = t_2(b_p) \geqslant e(H'[B_p])$, we have that $\ell \neq p$. But then $B_\ell = A_j$ for some $j \in [k]$, and so B_ℓ is an independent set in H', a contradiction. This proves that $b_h = b_p$.

Since B_p is the only part that was subdivided, there is $s \in [k-1]$ such that $A_s = B_h$ and thus $a_s^* = b_h = b_p = a_q^* + a_r^*$. Since $a_1^* \geqslant \cdots \geqslant a_{k-1}^* \geqslant \max\{a_1^* - 1, a_k^*\}$, we have $a_s^* - a_q^* = 1$ and $a_r^* = 1$. So $a_k^* = 1$ and $a_q^* = a_{k-1}^*$. Since h was arbitrary, we conclude that for all $i \in [k-1]$ such that $e(G[B_i]) > 0$, we have $b_i = a_{k-1}^* + 1$. So $G \in \mathcal{H}_0^*(n, e)$, as required.

Case 2. *For all $h \in [k-1] \setminus \{p\}$, we have $e(G[B_h]) = 0$. In other words, $G \in \mathcal{H}_1^{\min}(n, e)$.*

Suppose first that $m^* = 0$. Then $H' = K[A_1, \ldots, A_k]$, and G can be obtained from it by replacing $H'[A_q \cup A_r]$ with $G[B_p]$. Moreover, $G[B_p]$ is a triangle-free graph on $a_q^* + a_r^*$ vertices with $a_q^* a_r^*$ edges. If $a_r^* = a_k^*$, then $G \in \mathcal{H}_1^*(n, e)$; otherwise $|a_q^* - a_r^*| \leqslant 1$, so $G[B_p] \cong T_2(a_q^* + a_r^*)$ and thus $G \cong H' \in \mathcal{H}_1^*(n, e)$, getting the required in either case.

Suppose instead that $m^* > 0$. Since $G[A_i, A_j]$ is complete for all $\{i, j\} \neq \{q, r\}$, and $H'[A_i, A_j]$ is complete if and only if $\{i, j\} \neq \{k-1, k\}$, we have $\{q, r\} = \{k-1, k\}$. Thus G can be obtained from $K[A_1, \ldots, A_k]$ by replacing $K[A_{k-1} \cup A_k]$ with a triangle-free graph with $a_{k-1}^* a_k^* - m^*$ edges. This gives that $G \in \mathcal{H}_1^*(n, e)$, as required.

Note that if $G \in \mathcal{H}_1^{\min}(n, e)$, then the above argument always concludes that $G \in \mathcal{H}_1^*(n, e)$, apart from Case 1 (that does not apply here). Thus we have proved that $\mathcal{H}_i^{\min}(n, e) = \mathcal{H}_i^*(n, e)$ for $i = 0, 1$.

Now let $G \in \mathcal{H}_2^{\min}(n, e)$ be arbitrary. If $G \in \mathcal{H}_1(n, e)$, then, as we have just established, $G \in \mathcal{H}_1^*(n, e)$ (and also G is k-partite). So $G \in \mathcal{H}_2^*(n, e)$, and thus we may assume that $G \in \mathcal{H}_2^{\min}(n, e) \setminus \mathcal{H}_1(n, e)$.

Let G have an $\mathcal{H}_2$-canonical partition $A_1, \ldots, A_k$ with part sizes $a_1 \geqslant \cdots \geqslant a_k$, respectively. By Lemma 4.3, we have that $G \in (\mathcal{H}_2')^{\min}(n, e)$, and $A_1, \ldots, A_k$ is an $\mathcal{H}_2'$-canonical partition. Since $G \notin \mathcal{H}_1(n, e)$, Corollary 4.4(iii) gives that

$$m := \sum_{1 \leqslant i < j \leqslant k} e(\overline{G}[A_i, A_j]) \leqslant a_{k-1} - a_k + 1. \tag{4.6}$$

Since $\mathcal{H}_2^{\min}(n, e) = (\mathcal{H}_2')^{\min}(n, e)$ by Lemma 4.3, we see that if, for i in $I := \{j \in [k-1] : a_j = a_{k-1}\}$, we let B_i consist of those $x \in A_k$ that have at least one nonneighbour in A_i, then these subsets of A_k are disjoint and every missing edge in G intersects one of them. So to prove that $G \in \mathcal{H}_2^*(n, e)$, it suffices to show that

(i) $(a_1, \ldots, a_k) = (a_1^*, \ldots, a_k^*)$; or

(ii) $m^* = 0$, $a_1^* \geqslant a_k^* + 2$ and $(a_1, \ldots, a_k) = (a_2^*, \ldots, a_{k-1}^*, a_1^* - 1, a_k^* + 1)$.

By (4.6), we can obtain a graph G' from G by moving all m missing edges between parts A_{k-1} and A_k. Then $G' \in \mathcal{H}_1^{\min}(n, e)$, which equals $\mathcal{H}_1^*(n, e)$ as we have already shown. So G' has a partition $A_1^*, \ldots, A_k^*$, where $|A_i^*| = a_i^*$, and there is some $i \in [k-1]$ such that G' can be obtained from $K[A_1^*, \ldots, A_k^*]$ by replacing $K[A_i^* \cup A_k^*]$ with a triangle-free graph with $a_i^* a_k^* - m^*$ edges. Thus there is a bijection $\sigma : [k-1] \setminus \{i\} \to [k-2]$ such that

$$A_{\sigma(j)} = A_j^*, \quad \text{for all } j \in [k-1] \setminus \{i\}, \tag{4.7}$$

while $A_{k-1} \cup A_k = A_i^* \cup A_k^*$ and $a_{k-1} + a_k = a_i^* + a_k^*$. Thus, by the monotonicity of the involved sequences, if we remove the ith and kth entries from $\boldsymbol{a}^*$, then we obtain $(a_1, \ldots, a_{k-2})$.

By the minimality of a_k^*, we have $a_k^* \leqslant a_k$. Suppose that $a := a_k - a_k^* \geqslant 1$ as otherwise $(a_{k-1}, a_k) = (a_i^*, a_k^*)$ and the desired property (i) follows from (4.7). Since $a_k^* + a = a_k \leqslant a_{k-1} = a_i^* - a$, we have

$$m - m^* = a_{k-1}a_k - a_i^* a_k^* = a_{k-1}a_k - (a_{k-1}+a)(a_k - a) \geqslant a_{k-1} - a_k + a^2. \quad (4.8)$$

By (4.6) and (4.8), we have $a = 1$, $m^* = 0$, $a_k = a_k^* + 1$ and $a_{k-1} = a_i^* - 1$. Also, $a_1^* - 1 \geqslant a_i^* - 1 = a_{k-1} \geqslant a_k = a_k^* + 1$. Recall that $0 \leqslant a_1^* - a_i^* \leqslant 1$ by the definition of $\boldsymbol{a}^*$. If $a_i^* = a_1^* - 1$, then for all $j \in [k-1] \setminus \{i\}$, by (4.7), we have $a_j = a^*_{\sigma^{-1}(j)} \geqslant a_1^* - 1 = a_i^* = a_{k-1} + 1$. But then the set I of indices of parts that are not complete to A_k consists only of $k-1$, so $G \in \mathcal{H}_1(n, e)$, a contradiction. Thus $a_i^* = a_1^*$. This gives all the statements from (ii) by (4.7), finishing the proof of the proposition. □

4.3. Approximating the increment of the function $h^*(n, \cdot)$.

Let a pair (n, e) be valid and let $k = k(2e/n^2)$, where the single-variable function k is defined in (1.5). Also, define $c(n, e) := c(2e/n^2)$ to be the larger root of (1.6) for $\lambda = 2e/n^2$; this root can be explicitly written as

$$c(n, e) := c(2e/n^2) = \frac{1}{k}\left(1 + \sqrt{1 - \frac{k}{k-1} \cdot \frac{2e}{n^2}}\right). \quad (4.9)$$

Let $c := c(n, e)$. By definition,

$$\binom{k-1}{2}c^2 + (k-1)c(1-(k-1)c) = (k-1)c - \binom{k}{2}c^2 = \frac{e}{n^2} \quad (4.10)$$

and so

$$\begin{aligned} e(K^k_{cn,\ldots,cn,n-(k-1)cn}) &= e \quad \text{and} \\ K_3(K^k_{cn,\ldots,cn,n-(k-1)cn}) &= \binom{k-1}{3}c^3n^3 + \binom{k-1}{2}c^2(1-(k-1)c)n^3 \\ &= \binom{k-1}{2}c^2n^3 - 2\binom{k}{3}c^3n^3. \end{aligned} \quad (4.11)$$

In this section, we show that the increment of the function $h^*(n, \cdot)$ at e is very closely approximated by $(k-2)cn$.

First, we need the following standard estimate of the Turán number.

LEMMA 4.11. *Let s, n be integers such that $2 \leqslant s \leqslant n$. Then*

$$\left(1 - \frac{1}{s}\right)\frac{n^2}{2} - \frac{s}{8} \leqslant t_s(n) \leqslant \left(1 - \frac{1}{s}\right)\frac{n^2}{2}. \quad (4.12)$$

Proof. Divide n by s with remainder: $n = s\ell + r$ with $r \in \{0, \dots, s-1\}$. Then the Turán graph $T_s(n)$ has r parts of size $\ell + 1$ and $s - r$ parts of size ℓ. Routine calculations show that

$$t_s(n) = \binom{r}{2}(\ell+1)^2 + \binom{s-r}{2}\ell^2 + r(s-r)(\ell+1)\ell = \left(1 - \frac{1}{s}\right)\frac{(s\ell+r)^2}{2} + \frac{r^2 - sr}{2s}.$$

For real $r \in [0, s-1]$, the quadratic function $r^2 - rs$ has its minimum at $r = s/2$ and its maximum at $r = 0$, giving the required bounds on $t_s(n)$. □

Because of the gap in (4.12), the values of $k(2e/n^2)$ and $k(n, e)$ may be different when e is slightly above a Turán number. The following lemma implies that this never occurs inside the proof of Theorem 1.7, where $t_{k-1}(n) + \Omega(n^2) < e \leqslant t_k(n)$; in particular, (4.9) holds then with $k(2e/n^2)$ replaced by $k(n, e)$.

LEMMA 4.12. *Let a pair (n, e) be valid. Then*

(i) *$k(2e/n^2) \leqslant k(n, e)$;*

(ii) *if $t_{k-1}(n) + (k-1)/8 \leqslant e \leqslant t_k(n)$, then $k(2e/n^2) = k = k(n, e)$.*

Proof. Clearly, each of the functions $k(n, e)$ and $k(2e/n^2)$ is nondecreasing in e. Let $s \in \mathbb{N}$. Recall that $k(\lambda)$ jumps from s to $s + 1$ when λ becomes larger than $(s-1)/s$ while $k(n, e)$ jumps from s to $s + 1$ when e becomes larger than $t_s(n)$. Now, both of the stated claims follow from Lemma 4.11. □

LEMMA 4.13. *For every $\lambda \in [0, 1)$, we have $(k(\lambda) - 1)c(\lambda) < 1$.*

Proof. Assume that $s := k(\lambda) \geqslant 2$, as otherwise there is nothing to prove. The formula in (1.7) shows that $c(x)$ is a strictly decreasing continuous function for $x \in (\frac{s-2}{s-1}, \frac{s-1}{s}]$ and the limit of $c(x)$ as x tends to $\frac{s-2}{s-1}$ from above is $1/(s-1)$. Thus $c(x) < 1/(s-1)$ in this half-open interval, as required. □

LEMMA 4.14. *For all valid (n, e), if $c = c(n, e)$ is such that $cn \in \mathbb{N}$, then $k(n, e) = k(2e/n^2) =: k$, and $\boldsymbol{a}^* = \boldsymbol{a}^*(n, e)$ is equal to $(cn, \dots, cn, n - (k-1)cn)$.*

Proof. Let $k := k(2e/n^2)$ and $a := n - (k-1)cn$. Since $c \geqslant 1/k$ by definition, we have that $a \leqslant cn$. Also, $c < 1/(k-1)$ by Lemma 4.13. Thus a is positive. From $e(K_{cn,\dots,cn,a}) = e$, we conclude that $k(n, e) \leqslant k$. This must be equality by the first part of Lemma 4.12.

Recall by Definition 1 that a_k^* is the minimum $s \in \mathbb{N}$ with $s(n-s) + t_{k-1}(n-s) \geqslant e$, which is satisfied (with equality) for $s = a$. Thus $a_k^* \leqslant a$. Now, Lemma 4.5 implies by the induction on $a-s$ that for every $s=a-1, a-2, \ldots, 1$, we have $s(n-s)+t_{k-1}(n-s)<e$. Thus indeed $a_k^*=a$. This clearly implies that $a_i^*=cn$ for each $i\in[k-1]$. □

The following simple lemma describes the change in $H^*(n,e)$ when we increase e by 1. Informally speaking, (i) one missing edge is added, (ii) the smallest part increases by 1, or (iii) the number of parts increases by 1.

LEMMA 4.15. *Let $e,n\in\mathbb{N}$ with $e<\binom{n}{2}$. Let $k=k(n,e)$, $\boldsymbol{a}^*=\boldsymbol{a}^*(n,e)$, $m^*=m^*(n,e)$, $k^+=k(n,e+1)$ and $\boldsymbol{a}^+=\boldsymbol{a}^*(n,e+1)$ be as in Definition 1. Then the following statements hold.*

(i) *If $m^*>0$, then $k^+=k$ and $\boldsymbol{a}^+=\boldsymbol{a}^*$.*

(ii) *If $m^*=0$ and $a_1^*\geqslant a_k^*+2$, then $k^+=k$, $a_k^+=a_k^*+1$ and $(a_1^+,\ldots,a_{k-1}^+)$ is obtained from $(a_1^*-1, a_2^*,\ldots,a_{k-1}^*)$ by ordering it nonincreasingly.*

(iii) *If $m^*=0$ and $a_1^*\leqslant a_k^*+1$, then $k^+=k^*+1$.*

Proof. Let us consider Cases (i) and (ii) together. We can increase the size of $H^*(n,e)$ without increasing the number of parts: namely, let $H^{(\mathrm{i})}$ and $H^{(\mathrm{ii})}$ be obtained from H^* by, respectively, adding a missing edge or moving a vertex from the first part to the last. Since $k(n,\cdot)$ is a nondecreasing function, we have that $k^+=k$ in both cases. Furthermore, $a_k^*\leqslant n/k$ by (1.3). This and the equality $k^+=k$ imply by Lemma 4.5 that $a_k^+\geqslant a_k^*$ if $m^*>0$ and $a_k^+\geqslant a_k^*+1$ if $m^*=0$, with the matching upper bounds on a_k^+ witnessed by (the part sizes of) $H^{(\mathrm{i})}$ and $H^{(\mathrm{ii})}$, giving the required.

The third case is also easy: $k^+>k$ since $H^*(n,e)$ is the Turán graph $T_k(n)$ while $k^+\leqslant k+1$ since $k<n$ and $t_{k+1}(n)\geqslant t_k(n)+1$. □

LEMMA 4.16. *For all valid (n,e), if $e\in[t_{k-1}(n)+k, t_k(n)-1]$, then with $c=c(n,e)$ we have*

$$|(h^*(n,e+1)-h^*(n,e))-(k-2)cn|\leqslant k \quad \textit{and}$$
$$|(h^*(n,e)-h^*(n,e-1))-(k-2)cn|\leqslant k.$$

Moreover, $|a_i^*-cn|\leqslant 2$ for all $i\in[k-1]$, where $\boldsymbol{a}^*=\boldsymbol{a}^*(n,e)$ is defined in Definition 1.

Proof. For valid (n,f) with $k(n,f)$ equal to $k=k(n,e)$, let

$$L(n,f):=\sum_{i\in[k-2]}a_i^*(n,f)=n-a_{k-1}^*(n,f)-a_k^*(n,f),$$

where $\boldsymbol{a}^*(n,f)=(a_1^*(n,f),\ldots,a_k^*(n,f))$ is as in Definition 1.

Note that if $f + 1 \leqslant t_k(n)$ (that is, $k(n, f + 1) = k(n, f) = k$), then

$$L(n, f + 1) - L(n, f) \in \{-1, 0\}. \tag{4.13}$$

Indeed, consider how the vector $\boldsymbol{a}^*$ changes when we increase f by 1. Suppose that $m^*(n, f) = 0$ as otherwise the vector stays the same by Lemma 4.15(i). Note that $a_1^*(n, f) \geqslant a_k^*(n, f) + 2$ since $f < t_k(n)$, so Lemma 4.15(ii) applies. Here the kth entry of $\boldsymbol{a}^*$ increases by 1 while one of the other entries decreases by 1. In any case, $a_{k-1}^* + a_k^*$ stays the same or increases exactly by 1, giving (4.13).

CLAIM 4.17. *There exist integers e^-, e^+ such that*

(i) *$e^- \leqslant e \leqslant e^+$ and $k(n, e^-) = k(n, e^+) = k$;*

(ii) *$L(n, e^-) \leqslant (k-2)\lceil cn \rceil$ and $L(n, e^+) \geqslant (k-2)\lfloor cn \rfloor$.*

Proof of Claim. Given some e^- and e^+ satisfying (i), we will write $\boldsymbol{a}^*(n, e^-) = (a_1^-, \ldots, a_k^-)$ and similarly $\boldsymbol{a}^*(n, e^+) = (a_1^+, \ldots, a_k^+)$.

Let us consider e^-. Suppose first that $\lceil cn \rceil \geqslant n/(k-1)$. Then we let $e^- := t_{k-1}(n)+1$. Now $k(n, e^-) = k$ by definition, and $a_k^- = 1$, so $a_{k-1}^- = \lfloor (n-1)/(k-1) \rfloor$. Thus

$$L(n, e^-) = n - \left(\left\lfloor \frac{n-1}{k-1} \right\rfloor + 1 \right) \leqslant \frac{k-2}{k-1} \cdot n \leqslant (k-2)\lceil cn \rceil,$$

as desired.

So suppose that $a := \lceil cn \rceil < n/(k-1)$. Let e^- satisfy $c(n, e^-) = a/n$, that is, e^- is the size of the complete k-partite graph $K^k_{a,\ldots,a,n-(k-1)a}$. Clearly, $e^- \leqslant t_k(n)$. Since $a < n/(k-1)$, we have that $e^- > t_{k-1}(n)$. Thus $k(n, e^-) = k$. The explicit formula in (4.9) shows that $c(n, x)$ is a decreasing function of x, even when $k(n, x)$ jumps. Since $c(n, e^-) = a/n$ is at least $c = c(n, e)$, it holds that $e^- \leqslant e$. For this e^- we have that $a_i^- = \lceil cn \rceil$ for all $i \in [k-1]$, so Lemma 4.14 implies that $L(n, e^-) = (k-2)\lceil cn \rceil$, as required.

It remains to obtain e^+. Suppose first that $b := \lfloor cn \rfloor < n/k$. Let $e^+ := t_k(n)$. Then $k(n, e^+) = k$, $a_k^+ = \lfloor n/k \rfloor$ and $a_{k-1}^+ = \lfloor (n - a_k^+)/(k-1) \rfloor$. Since $b < n/k \leqslant cn$ by definition, we have that $\lfloor n/k \rfloor = b$. Thus

$$L(n, e^+) = n - \left\lfloor \frac{n}{k} \right\rfloor - \left\lfloor \frac{n - \lfloor n/k \rfloor}{k-1} \right\rfloor \geqslant (n-b)\left(1 - \frac{1}{k-1}\right) > (k-2)b,$$

as required.

So suppose that $b \geqslant n/k$. By our assumption $e \geqslant t_{k-1}(n)+k$ and Lemma 4.12, we have that $k(n, e) = k(2e/n^2)$. By Lemma 4.13, we have that $(k-1)b \leqslant$

$(k-1)cn < n$. Thus, similarly as above, if we define $e^+ = e(K_{b,\ldots,b,n-(k-1)b})$, then $k(n, e^+) = k$, $c(n, e^+) = b/n$ is at most $c = c(n, e)$ and thus $e^+ \geqslant e$. In this case, $a_i^+ = \lfloor cn \rfloor$ for all $i \in [k-1]$, so Lemma 4.14 implies that $L(n, e^+) = (k-2)\lfloor cn \rfloor$, as required. □

By (4.13), $L(n, \cdot)$ is a nonincreasing function in the range between $t_{k-1}(n) + k$ and $t_k(n)$. Together with the second part of Claim 4.17, this then implies that

$$(k-2)\lfloor cn \rfloor \leqslant L(n, e^+) \leqslant L(n, e) \leqslant L(n, e^-) \leqslant (k-2)\lceil cn \rceil. \qquad (4.14)$$

From this we have that $\lfloor cn \rfloor \leqslant a_i^* \leqslant \lceil cn \rceil$ for all $i \in [k-2]$. Since $a_{k-1}^* \geqslant a_{k-2}^* - 1$, the second part of the lemma is proved.

Now, we claim that

$$L(n, e) - 1 \leqslant L(n, e+1) \leqslant h^*(n, e+1) - h^*(n, e) \leqslant L(n, e). \qquad (4.15)$$

If this holds, then

$$\begin{aligned} &|h^*(n, e+1) - h^*(n, e) - (k-2)cn| \\ &\quad \leqslant |h^*(n, e+1) - h^*(n, e) - L(n, e)| + |L(n, e) - (k-2)cn| \\ &\quad \overset{(4.14),(4.15)}{\leqslant} 1 + (k-2)\max\{cn - \lfloor cn \rfloor, \lceil cn \rceil - cn\} \leqslant k - 1, \end{aligned}$$

proving the first inequality. Similarly, noting that $k(n, e-1) = k(n, e) = k$ by Lemma 4.12 and the fact that $e \geqslant t_{k-1}(n) + k$, we have that

$$\begin{aligned} &|h^*(n, e) - h^*(n, e-1) - (k-2)cn| \\ &\quad \leqslant |h^*(n, e) - h^*(n, e-1) - L(n, e-1)| \\ &\qquad + |L(n, e-1) - L(n, e)| + |L(n, e) - (k-2)cn| \\ &\quad \leqslant 1 + 1 + (k-2) = k, \end{aligned}$$

where the last inequality follows from (4.13)–(4.15), proving the second.

So it suffices to prove (4.15). The first inequality follows from (4.13). If $m^* > 0$, then by Lemma 4.15(i), the difference $h^*(n, e+1) - h^*(n, e)$ is the number of triangles created by adding one missing edge to $H^*(n, e)$, which is exactly $L(n, e)$. If $m^* = 0$, then we are in the second case of Lemma 4.15, where we add one more edge into the union of two parts of sizes a_1^* and a_k^*, keeping this graph bipartite. Clearly, this new edge creates $n - a_1^* - a_k^*$ triangles. This is $L(n, e)$ if $a_1^* = a_{k-1}^*$ and $L(n, e+1)$ otherwise (that is, if $a_1^* = a_{k-1}^* + 1$). □

Lemma 4.16 will imply that if there is a counterexample to Theorem 1.7, then in an appropriately defined 'worst counterexample', no edge lies in more than $(k-2)cn + k$ triangles and no nonedge lies in less than $(k-2)cn - k$ copies of P_3. This fact will be extremely useful in our proof of Theorem 1.7.

COROLLARY 4.18. *Let $n \in \mathbb{N}$ and $e \in [t_{k-1}(n) + k, t_k(n) - 1]$ and let $p > 0$ and $c = c(n, e)$. Suppose that $g_3(n, e) - h^*(n, e) \leqslant g_3(n, e^*) - h^*(n, e^*)$ for all e^* with $k(n, e^*) = k$. Let G and G' be (n, e)-graphs such that $K_3(G) = g_3(n, e) \geqslant K_3(G') - p$. Then, for every $\overline{f} \in E(\overline{G})$, $\overline{f'} \in E(\overline{G'})$, $f \in E(G)$ and $f' \in E(G')$, we have that*

(i) *$P_3(\overline{f}, G) \geqslant (k-2)cn - k$ and $P_3(\overline{f'}, G') \geqslant (k-2)cn - k - p$;*

(ii) *$P_3(f, G) \leqslant (k-2)cn + k$ and $P_3(f', G') \leqslant (k-2)cn + k + p$.*

Proof. Let $\overline{f} \in E(\overline{G})$. Then $k(n, e+1) = k$ and by the assumption on G, for any $(n, e+1)$-graph G'', we have that

$$K_3(G) - h^*(n, e) \leqslant g_3(n, e+1) - h^*(n, e+1) \leqslant K_3(G'') - h^*(n, e+1).$$

Thus, by Lemma 4.16,

$$P_3(\overline{f}, G) = K_3(G \cup \{\overline{f}\}) - K_3(G) \geqslant h^*(n, e+1) - h^*(n, e) \geqslant (k-2)cn - k,$$

where $G \cup \{\overline{f}\}$ denotes the graph G with the pair $\overline{f}$ added as an edge. Similarly, for $\overline{f'} \in E(\overline{G'})$, we have

$$P_3(\overline{f'}, G') = K_3(G' \cup \{\overline{f'}\}) - K_3(G') \geqslant K_3(G' \cup \{\overline{f'}\}) - K_3(G) - p \geqslant (k-2)cn - k - p.$$

The second part can be proved similarly via the inequality $|h^*(n, e) - h^*(n, e-1) - (k-2)cn| \leqslant k$ from Lemma 4.16. □

4.4. Comparing k-partite graphs. The next lemma will be used to compare the number of triangles in two k-partite (n, e)-graphs G and F, in terms of their part sizes and the number of edges missing between parts. It will later be applied with $\ell := \lfloor cn \rfloor$ and F a graph in $\mathcal{H}_1(n, e)$; and G a graph obtained by switching a small number of adjacencies in a hypothetical counterexample to Theorem 1.7. Informally speaking, the lemma can be used to derive a quantitative conclusion of the form that, if the part sizes of G deviate from the almost optimal vector $(\ell, \ldots, \ell, n - (k-1)\ell)$, then $K_3(G)$ is larger than $K_3(F)$.

LEMMA 4.19. *Let $n \geqslant k \geqslant 3$ and $d > 0$ be integers. Suppose that G and F are n-vertex k-partite graphs with $e(G) = e(F)$ such that the following hold.*

(i) *G has parts $A_1, \ldots, A_k$.*

(ii) *$G[A_i, A_j]$ is complete whenever $ij \in \binom{[k-1]}{2}$.*

(iii) *F has parts $B_1, \dots, B_k$ with $\ell_i := |B_i|$ for $i \in [k]$ satisfying $\ell_1 = \cdots = \ell_{k-1} =: \ell > \ell_k > 0$.*

(iv) *$F[B_i, B_j]$ is complete for all $ij \in \binom{[k]}{2} \setminus \{\{k-1, k\}\}$; also, $e(\overline{F}[B_{k-1}, B_k]) \leqslant d$.*

(v) *For all $i \in [k]$, we have that $|d_i| \leqslant \frac{\ell-\ell_k}{12k^3}$, where $d_i := s_i - \ell_i$ and $s_i := |A_i|$. Moreover, $d_k \geqslant 0$.*

Let $m_i := |A_i||A_k| - e(G[A_i, A_k])$ for all $i \in [k-1]$ and $m := m_1 + \cdots + m_{k-1}$. Then

$$K_3(G) - K_3(F) \geqslant \sum_{t \in [k-1]} \frac{m_t}{m} \cdot \frac{\ell - \ell_k}{4} \left((d_t + d_k)^2 + \sum_{\substack{i \in [k-1] \\ i \neq t}} d_i^2 \right) - \frac{12d^2}{\ell - \ell_k}.$$

Proof. Define $d_0 := e(\overline{F}[B_{k-1}, B_k]) \leqslant d$. Let H be the complete k-partite graph with parts $B_1, \dots, B_k$. As $\sum_{ij \in \binom{[k]}{2}} s_i s_j - m = e(G) = e(F) = \sum_{ij \in \binom{[k]}{2}} \ell_i \ell_j - d_0$, we have

$$m' := m - (e(H) - e(F)) = m - d_0 = \sum_{ij \in \binom{[k]}{2}} s_i s_j - \sum_{ij \in \binom{[k]}{2}} \ell_i \ell_j.$$

CLAIM 4.20. *For all $t \in [k-1]$, we have*

$$\sum_{ijh \in \binom{[k]}{3}} s_i s_j s_h - m' \sum_{\substack{i \in [k-1] \\ i \neq t}} s_i - \sum_{ijh \in \binom{[k]}{3}} \ell_i \ell_j \ell_h \geqslant \frac{\ell - \ell_k}{3} \left((d_t + d_k)^2 + \sum_{\substack{i \in [k-1] \\ i \neq t}} d_i^2 \right). \tag{4.16}$$

Proof. For notational convenience, we prove this for $t = k-1$ and observe that the proof uses only properties (i)–(iii) and (v), which are all symmetric in $t \in [k-1]$.

We have that the left-hand side of (4.16) (with $t = k-1$) is equal to

$$\begin{aligned} &\sum_{ijh \in \binom{[k]}{3}} d_i d_j d_h + \sum_{ij \in \binom{[k]}{2}} \ell_i \ell_j \sum_{\substack{h \in [k] \\ h \neq i,j}} d_h + \sum_{ij \in \binom{[k]}{2}} d_i d_j \sum_{\substack{h \in [k] \\ h \neq i,j}} \ell_h \\ &- \left(\sum_{i \in [k]} \ell_i \sum_{\substack{j \in [k] \\ j \neq i}} d_j + \sum_{ij \in \binom{[k]}{2}} d_i d_j \right) \sum_{h \in [k-2]} (\ell_h + d_h). \end{aligned} \tag{4.17}$$

This is a cubic polynomial in $d_1, \ldots, d_k$. For each $0 \leqslant t \leqslant 3$ and $1 \leqslant i_1 \leqslant \cdots \leqslant i_t \leqslant k$, let $C_{i_1 \ldots i_t}$ denote the coefficient of $d_{i_1} \ldots d_{i_t}$. By a slight abuse of notation, we assume a pair $ij \in \binom{[k]}{2}$ satisfies $i < j$ (and similarly for triples). Note that $C_\emptyset = 0$. Now, for all $i \in [k]$,

$$C_i = \sum_{hj \in \binom{[k] \setminus \{i\}}{2}} \ell_h \ell_j - \sum_{\substack{j \in [k] \\ j \neq i}} \ell_j \sum_{h \in [k-2]} \ell_h.$$

So $C_1 = \cdots = C_{k-1}$ since $\ell_1 = \cdots = \ell_{k-1}$. Also

$$\begin{aligned} C_k &= \binom{k-1}{2} \ell^2 - (n - \ell_k)(k-2)\ell \\ &= \binom{k-2}{2} \ell^2 + (k-2)\ell\ell_k - (n-\ell)(k-2)\ell = C_1. \end{aligned}$$

But

$$\sum_{i \in [k]} d_i = 0 \tag{4.18}$$

and hence

$$\sum_{i \in [k]} C_i d_i = 0, \tag{4.19}$$

that is, the linear part of (4.17) is zero.

Next, we simplify the quadratic part. Suppose that $ij \in \binom{[k-2]}{2}$. Then

$$C_{ij} = \sum_{\substack{h \in [k] \\ h \neq i,j}} \ell_h - \sum_{\substack{h \in [k] \\ h \neq i}} \ell_h - \sum_{\substack{h \in [k] \\ h \neq j}} \ell_h - \sum_{h \in [k-2]} \ell_h = \ell + \ell_k - 2n. \tag{4.20}$$

Suppose that $i \in [k-2]$. Then

$$C_{ii} = -\sum_{\substack{h \in [k] \\ h \neq i}} \ell_h = \ell - n.$$

Suppose that $i \in [k-2]$ and $j \in \{k-1, k\}$. Then

$$C_{ij} = \sum_{\substack{h \in [k] \\ h \neq i,j}} \ell_h - \sum_{\substack{h \in [k] \\ h \neq j}} \ell_h - \sum_{h \in [k-2]} \ell_h = \ell_k - n. \tag{4.21}$$

This implies that

$$\sum_{\substack{i \in [k-2] \\ j \in \{k-1,k\}}} C_{ij} d_i d_j = \sum_{i \in [k-2]} (\ell_k - n)(d_{k-1} + d_k) d_i$$

$$\overset{(4.18)}{=} -(\ell_k - n)\left(\sum_{i\in[k-2]} d_i^2 + 2\sum_{ij\in\binom{[k-2]}{2}} d_i d_j\right).$$

Note that if $i, j \in \{k-1, k\}$, then $C_{ij} = 0$. So

$$\sum_{i\in[k]} C_{ii} d_i^2 = \sum_{i\in[k-2]} (\ell - n) d_i^2. \tag{4.22}$$

Thus the quadratic terms in (4.17) give

$$\begin{aligned}
\sum_{1\leqslant i\leqslant j\leqslant k} C_{ij} d_i d_j &= \sum_{i\in[k]} C_{ii} d_i^2 + \sum_{ij\in\binom{[k-2]}{2}} C_{ij} d_{ij} + \sum_{\substack{i\in[k-2]\\ j\in\{k-1,k\}}} C_{ij} d_i d_j \\
&= \sum_{i\in[k-2]} (\ell - n) d_i^2 + \sum_{ij\in\binom{[k-2]}{2}} d_i d_j (\ell + \ell_k - 2n) \\
&\quad - (\ell_k - n)\left(\sum_{i\in[k-2]} d_i^2 + 2\sum_{ij\in\binom{[k-2]}{2}} d_i d_j\right) \\
&= (\ell - \ell_k)\left(\sum_{ij\in\binom{[k-2]}{2}} d_i d_j + \sum_{i\in[k-2]} d_i^2\right).
\end{aligned} \tag{4.23}$$

Now let us consider the cubic terms in (4.17). We have

$$\begin{aligned}
\sum_{\substack{ijh\in[k]^3\\ i\leqslant j\leqslant h}} C_{ijh} d_i d_j d_h &= \sum_{ijh\in\binom{[k]}{3}} d_i d_j d_h - \sum_{ij\in\binom{[k]}{2}} d_i d_j \cdot \sum_{h\in[k-2]} d_h \\
&= d_{k-1} d_k \sum_{i\in[k-2]} d_i + (d_{k-1} + d_k) \sum_{ij\in\binom{[k-2]}{2}} d_i d_j \\
&\quad + \sum_{ijh\in\binom{[k-2]}{3}} d_i d_j d_h - \sum_{ij\in\binom{[k]}{2}} d_i d_j \sum_{h\in[k-2]} d_h \\
&= d_{k-1} d_k \sum_{i\in[k-2]} d_i - \sum_{h\in[k-2]} d_h \cdot \sum_{ij\in\binom{[k-2]}{2}} d_i d_j \\
&\quad + \sum_{ijh\in\binom{[k-2]}{3}} d_i d_j d_h - \sum_{ij\in\binom{[k]}{2}} d_i d_j \sum_{h\in[k-2]} d_h.
\end{aligned}$$

Note that, adding the first and the last terms, we get

$$d_{k-1} d_k \sum_{i\in[k-2]} d_i - \sum_{ij\in\binom{[k]}{2}} d_i d_j \cdot \sum_{h\in[k-2]} d_h$$

$$\overset{(4.18)}{=} \left(\sum_{i\in[k-2]} d_i\right)\left(\sum_{i\in[k-2]} d_i^2 + \sum_{jh\in\binom{[k-2]}{2}} d_j d_h\right),$$

which gives some cancellations when combined with the second term. Also, for every $\{i, j, h\} \in \binom{[k-2]}{3}$,

$$|d_i d_j d_h| \leqslant \max_{s\in[k-2]} |d_s| \cdot \frac{1}{2}(d_j^2 + d_h^2) < \max_{s\in[k-2]} |d_s| \cdot \sum_{t\in[k-2]} d_t^2.$$

These, together with $\max_i |d_i| \leqslant \frac{\ell-\ell_k}{12k^3}$, imply that

$$\left|\sum_{\substack{ijh\in[k]^3\\ i\leqslant j\leqslant h}} C_{ijh} d_i d_j d_h\right| \leqslant \left|\left(\sum_{h\in[k-2]} d_h\right)\sum_{i\in[k-2]} d_i^2\right| + \left|\sum_{ijh\in\binom{[k-2]}{3}} d_i d_j d_h\right|$$
$$\leqslant \frac{\ell-\ell_k}{6} \cdot \sum_{i\in[k-2]} d_i^2. \tag{4.24}$$

Thus, combining (4.19), (4.23) and (4.24), we have that (4.17) is equal to

$$\sum_{i\in[k]} C_i d_i + \sum_{1\leqslant i\leqslant j\leqslant k} C_{ij} d_i d_j + \sum_{ijh\in[k]^3} C_{ijh} d_i d_j d_h$$
$$\geqslant \frac{\ell-\ell_k}{2}\left((d_{k-1}+d_k)^2 + \sum_{i\in[k-2]} d_i^2\right) - \frac{\ell-\ell_k}{6} \cdot \sum_{i\in[k-2]} d_i^2$$
$$\geqslant \frac{\ell-\ell_k}{3}\left((d_{k-1}+d_k)^2 + \sum_{i\in[k-2]} d_i^2\right).$$

This completes the proof of the claim. □

Now,

$$K_3(G) - K_3(F) = K_3(G) - K_3(H) + d_0(\ell_1 + \cdots + \ell_{k-2})$$
$$\geqslant \sum_{ijh\in\binom{[k]}{3}} s_i s_j s_h - \sum_{h\in[k-1]} m_h \sum_{\substack{i\in[k-1]\\ i\neq h}} s_i - \sum_{ijh\in\binom{[k]}{3}} \ell_i \ell_j \ell_h + (k-2)d_0\ell$$
$$= \sum_{t\in[k-1]} \frac{m_t}{m}\left(\sum_{ijh\in\binom{[k]}{3}} s_i s_j s_h - m' \sum_{\substack{i\in[k-1]\\ i\neq t}} s_i - \sum_{ijh\in\binom{[k]}{3}} \ell_i \ell_j \ell_h\right.$$

$$
\left. - d_0 \sum_{\substack{i\in[k-1]\\ i\neq t}} s_i + (k-2)d_0\ell \right)
$$

$$
= \sum_{t\in[k-1]} \frac{m_t}{m}\left(\sum_{ijh\in\binom{[k]}{3}} s_i s_j s_h - m' \sum_{\substack{i\in[k-1]\\ i\neq t}} s_i - \sum_{ijh\in\binom{[k]}{3}} \ell_i\ell_j\ell_h \right)
$$

$$
- \sum_{t\in[k-1]} \frac{d_0 m_t}{m} \sum_{\substack{i\in[k-1]\\ i\neq t}} d_i
$$

$$
\overset{(4.16)}{\geqslant} \sum_{t\in[k-1]} \frac{m_t}{m}\cdot\frac{\ell-\ell_k}{3}\left((d_t+d_k)^2 + \sum_{\substack{i\in[k-1]\\ i\neq t}} d_i^2 \right) + \sum_{t\in[k-1]} \frac{d_0 m_t}{m}(d_t+d_k).
$$

Let $\mathcal{I}\subseteq[k-1]$ be such that $t\in\mathcal{I}$ if and only if

$$
\frac{\ell-\ell_k}{3}(d_t+d_k)^2 + d_0(d_t+d_k) > \frac{\ell-\ell_k}{4}(d_t+d_k)^2.
$$

If $s\in[k-1]\setminus\mathcal{I}$, then $|d_s+d_k|\leqslant 12d_0/(\ell-\ell_k)$. Thus

$$
\sum_{t\in[k-1]} \frac{m_t}{m}\left(\frac{\ell-\ell_k}{3}(d_t+d_k)^2 + d_0(d_t+d_k) \right)
$$

$$
\geqslant \sum_{t\in\mathcal{I}} \frac{m_t}{m}\cdot\frac{\ell-\ell_k}{4}(d_t+d_k)^2 + \sum_{s\in[k-1]\setminus\mathcal{I}} \frac{m_t}{m}\cdot\frac{\ell-\ell_k}{3}(d_t+d_k)^2 - \frac{12d_0^2}{\ell-\ell_k}
$$

$$
\geqslant \sum_{t\in[k-1]} \frac{m_t}{m}\cdot\frac{\ell-\ell_k}{4}(d_t+d_k)^2 - \frac{12d^2}{\ell-\ell_k}.
$$

Thus

$$
K_3(G)-K_3(F) \geqslant \sum_{t\in[k-1]} \frac{m_t}{m}\cdot\frac{\ell-\ell_k}{4}\left((d_t+d_k)^2 + \sum_{\substack{i\in[k-1]\\ i\neq t}} d_i^2 \right) - \frac{12d^2}{\ell-\ell_k},
$$

as required. □

4.5. Partitions. The structure of the graphs G we will be working with is somewhat complicated, and for much of the proof, we make a sequence of local changes to G to obtain a collection of new graphs. Therefore it is useful to define some types of partition to record all the relevant structural information about these graphs.

Let $k, n, e \in \mathbb{N}$ and $\beta > 0$ and let $c = c(n, e)$. We say that an (n, e)-graph H *has a* $(V_1, \ldots, V_k; \beta)$*-partition* if both of the following hold:

P1(H): $V_1 \cup \cdots \cup V_k$ is a partition of $V(H)$ and

$$\big||V_i| - cn\big|, \big||V_k| - (1-(k-1)c)n\big| \leqslant \beta n$$

for all $i \in [k-1]$;

P2(H): $H[V_i, V_j]$ is complete for all $ij \in \binom{[k-1]}{2}$.

Let $\delta > 0$. We say that H *has a* $(V_1, \ldots, V_k; U, \beta, \delta)$*-partition* if, in addition to P1(H) and P2(H), U is a subset of $V(H)$ such that the following properties hold:

P3(H): $|U| \leqslant \delta n$ and every edge in $\bigcup_{i\in[k]} E(H[V_i])$ is incident with a vertex of U; also, $\Delta(H[V_i]) \leqslant \delta n$ for all $i \in [k]$;

P4(H): $U \cap V_k$ has a partition $U_k^1 \cup \cdots \cup U_k^{k-1}$ such that for all $ij \in \binom{[k-1]}{2}$, we have that $G[U_k^i, V_j]$ is complete.

If $\gamma_1, \gamma_2 > 0$ and in addition to P1(H)–P4(H), the following property holds, then we say that H *has a* $(V_1, \ldots, V_k; U, \beta, \gamma_1, \gamma_2, \delta)$*-partition.*

P5(H): If $y \in V_i \setminus U$ then $d_H^m(y) := e_{\overline{H}}(y, \overline{V_i}) < \gamma_2 n$ and if $y \in V_i \cap U$ then $d_H^m(y) \geqslant \gamma_1 n$, for all $i \in [k]$.

If P1(H), P3(H) and P5(H) hold, then we say that H *has a weak* $(V_1, \ldots, V_k; U, \beta, \gamma_1, \gamma_2, \delta)$*-partition*. Observe that if $\beta^+ \geqslant \beta$; $\gamma_1^- \leqslant \gamma_1$; $\gamma_2^+ \geqslant \gamma_2$ and $\delta^+ \geqslant \delta$, then a $(V_1, \ldots, V_k; U, \beta, \gamma_1, \gamma_2, \delta)$-partition is also a $(V_1, \ldots, V_k; U, \beta^+, \gamma_1^-, \gamma_2^+, \delta^+)$-partition. We call $d_H^m(y)$ the *missing degree* of a vertex $y \in V(H)$ with respect to the partition $V_1, \ldots, V_k$. Let $\underline{m} = (m_1, \ldots, m_{k-1})$, where for all $i \in [k-1]$, we have $m_i := e(\overline{H}[V_i, V_k])$. We say that $\underline{m}$ is the *missing vector* of H with respect to $(V_1, \ldots, V_k)$. Observe that, by P2(H),

$$m_i = \sum_{v \in V_i} d_H^m(v).$$

An edge is *bad* if both of its endpoints lie in the same V_i. Let $h := \sum_{i\in[k]} e(H[V_i])$ be the total number of bad edges.

5. Initial steps in the proof of Theorem 1.7

We start by deriving Theorem 1.6 from Theorems 1.3 and 1.7 and Proposition 1.5. The rest of the paper will concentrate on proving Theorem 1.7.

5.1. The proof of Theorem 1.6 given Theorem 1.7. Let $\varepsilon > 0$. Assume $\varepsilon < 1/2$. Theorem 1.3 gives $\alpha(3, k) > 0$ and $n_0(3, k)$ for each integer $3 \leqslant k \leqslant 1/\varepsilon$. Let $\alpha > 0$ be the minimum of the above constants $\alpha(3, k)$. Apply Theorem 1.7 with parameters ε, α to obtain $n_0(\alpha, k)$ for each integer $3 \leqslant k \leqslant 1/\varepsilon$. Let n_0 be the maximum of $n_0(3, k)$ and $n_0(\alpha, k)$ over such k.

Now let $n \geqslant n_0$ and $e \leqslant \binom{n}{2} - \varepsilon n^2$ be positive integers. Let $k = k(n, e)$, so $t_{k-1}(n) < e \leqslant t_k(n)$. If $k \leqslant 2$, then $g_3(n, e) = 0$, and we are done as then

$$\mathcal{H}_2^*(n, e) \subseteq \mathcal{H}_0^*(n, e) = \{K_3\text{-free } (n, e)\text{-graphs}\}.$$

So we may assume that $k \geqslant 3$. Further, Lemma 4.11 implies that $k \leqslant 1/(2\varepsilon) + 1 \leqslant 1/\varepsilon$. Suppose first that $t_{k-1}(n) < e \leqslant t_{k-1}(n) + \alpha n^2$. Then, since $\alpha \leqslant \alpha(3, k)$, Theorem 1.3 applied with $r := 3$ implies that $g_3(n, e) = h(n, e)$ and every extremal graph lies in $\mathcal{H}_0(n, e) \cup \mathcal{H}_2(n, e)$. Proposition 1.5 then implies that the extremal value is $h^*(n, e) = h(n, e)$ and the family of extremal graphs is precisely $\mathcal{H}_0^*(n, e) \cup \mathcal{H}_2^*(n, e)$.

Suppose instead that $t_{k-1}(n) + \alpha n^2 \leqslant e \leqslant t_k(n)$. Then Theorem 1.7 implies that every extremal graph lies in $\mathcal{H}(n, e)$. Proposition 1.5 then implies that the family of extremal graph is precisely $\mathcal{H}_1^*(n, e) \cup \mathcal{H}_2^*(n, e)$ (and note that $\mathcal{H}_0^*(n, e) = \mathcal{H}_1^*(n, e)$ for this e by (1.10)). So certainly $g_3(n, e) = h(n, e)$.

5.2. Beginning the proof of Theorem 1.7. Let $\varepsilon > 0$. Suppose that Theorem 1.7 does not hold for this ε. Then take the minimal integer $k \leqslant 1/\varepsilon$ such that the conclusion is not true at this k for some α, and then choose such an α. By decreasing α, we can assume that $\alpha \ll \varepsilon$ and that $\alpha \leqslant (\alpha_{1.3})^5$, where $\alpha_{1.3}$ is the minimum constant $\alpha(3, k)$ obtained by applying Theorem 1.3 with parameters k and $r = 3$, for all $3 \leqslant k \leqslant 1/\varepsilon$.

By the minimality of k, we have that, for all $\ell \in [k-1]$ and all $\alpha' > 0$, there exists $n_0(\ell, \alpha') > 0$ such that every extremal (n, e)-graph with $n \geqslant n_0(\ell, \alpha')$ and $t_{\ell-1}(n) + \alpha' n^2 \leqslant e \leqslant t_\ell(n)$ lies in $\mathcal{H}(n, e)$.

Note that $k \geqslant 3$ as when $k(n, e) = 2$, the family $\mathcal{H}(n, e)$ is the family of n-vertex e-edge triangle-free graphs, and $g_3(n, e) = 0$. (So we can set $n_0(2, \alpha) = 1$ for every $\alpha > 0$.)

Choose $n_0 = n_0(k) \in \mathbb{N}$ and additional constants such that the dependencies between them are as follows:

$$0 < \frac{1}{n_0} \ll \rho_4 \ll \cdots \ll \rho_0 \ll \eta \ll \delta \ll \beta \ll \xi \ll \gamma \ll \alpha \leqslant (\alpha_{1.3})^5$$
$$\ll \delta' \ll \xi' \ll \varepsilon \leqslant \frac{1}{k}. \tag{5.1}$$

In particular, we assume that Theorem 1.2 holds for n_0 with ρ_4 playing the role of ε and that

$$n_0 \geqslant \max\{2 \cdot n_0(k-1, \alpha/3),\ \ n_{1.2}(\rho_4),\ \ 2 \cdot n_{1.3}(k)\}, \tag{5.2}$$

where $n_{1.2}(\rho_4)$ is the output of Theorem 1.2 applied with parameter ρ_4; and $n_{1.3}(k)$ is (along with $\alpha_{1.3}$) the output of Theorem 1.3 applied with $k-1$ and $r = 3$. For the reader's convenience, the glossary at the end of the paper gives an informal overview of the roles of the constants in (5.1). We may ignore floors and ceilings where they do not affect our argument.

Now, suppose that Theorem 1.7 fails for this n_0, k and α. Pick the smallest $n \geqslant n_0$ such that there is e with

$$t_{k-1}(n) + \alpha n^2 \leqslant e \leqslant t_k(n) \tag{5.3}$$

for which at least one extremal (n, e)-graph is not in $\mathcal{H}(n, e)$. If there is more than one choice for e then choose one with $g_3(n, e) - h(n, e)$ being smallest possible. By Theorem 1.3, the inequality

$$g_3(n, e) - h(n, e) \leqslant g_3(n, e') - h(n, e') \tag{5.4}$$

holds in fact for every e' with $k(n, e') = k$. (Indeed, if $t_{k-1}(n) \leqslant e' < t_{k-1}(n) + \alpha n^2$, then (5.4) holds as its right-hand side is zero.)

Next, choose an (n, e)-graph G according to the following criteria in the given order:

(C1) $G \notin \mathcal{H}(n, e)$ and G has the minimum number of triangles: $K_3(G) = g_3(n, e)$;

(C2) G has a maximum max-cut k-partition: If $A_1^G, \ldots, A_k^G$ is a max-cut partition of $V(G)$, then for every (n, e)-graph $J \notin \mathcal{H}(n, e)$ with $K_3(J) = g_3(n, e)$ and every (equivalently, some) max-cut partition $A_1^J, \ldots, A_k^J$ of $V(J)$, we have that

$$\sum_{ij \in \binom{[k]}{2}} e\left(G[A_i^G, A_j^G]\right) \geqslant \sum_{ij \in \binom{[k]}{2}} e\left(J[A_i^J, A_j^J]\right).$$

(C3) There exists a max-cut k-partition $A_1^G, \ldots, A_k^G$ of $V(G)$ such that for every (n, e)-graph J satisfying (C1) and (C2) and every max-cut partition $A_1^J, \ldots, A_k^J$ of $V(J)$, we have

$$\min_{i \in [k]} \left|A_i^G\right| \leqslant \min_{i \in [k]} \left|A_i^J\right|.$$

We say that such a graph G is a *worst counterexample*. From now on, G, n, e and all the constants in (5.1) are fixed. Define $c = c(n, e)$. Corollary 4.18, Proposition 1.5 and (5.4) imply that

$$P_3(wx, G) \geqslant (k-2)cn - k \quad \text{and} \quad P_3(yz, G) \leqslant (k-2)cn + k \tag{5.5}$$

for all $wx \in E(\overline{G})$ and $yz \in E(G)$. Since n and e satisfy (5.3), we have by (4.9) and Lemma 4.11 that

$$\frac{1}{k} \leqslant c \leqslant \frac{1}{k} + \frac{\sqrt{1 - 2\alpha k(k-1)}}{k(k-1)} + O(1/n) < \frac{1}{k-1} - \alpha. \tag{5.6}$$

(Here we used $\sqrt{1-x} < 1 - x/2$ for $x \in (0, 1]$.) Thus

$$0 \leqslant kc - 1 < c - (k-1)\alpha. \tag{5.7}$$

Further, using Theorem 1.1 and the fact that $e \leqslant \binom{n}{2} - \varepsilon n^2$, we have

$$\left|K_3(K^k_{cn,\ldots,cn,n-(k-1)cn}) - K_3(G)\right| \overset{(1.8),(4.11)}{=} \left|\frac{n^3}{6} g_3\left(\frac{2e}{n^2}\right) - g_3(n, e)\right| \overset{(1.12)}{\leqslant} \frac{n}{2\varepsilon}. \tag{5.8}$$

Before splitting into cases depending on the size of the difference $t_k(n) - e$, we prove the following useful statement about some structural properties of G.

LEMMA 5.1. *Let $0 < 1/n \ll \rho \ll 1/k$, and let $p, d > 0$ be such that*

$$p^2 \leqslant d \leqslant \rho n^2 \quad \textit{and} \quad 2\rho^{1/6} \leqslant 1 - (k-1)c. \tag{5.9}$$

Suppose that there is a partition $V_1, \ldots, V_k$ of $V(G)$ for which P1(G) holds with parameter p/n and

$$|E(G) \bigtriangleup E(K[V_1, \ldots, V_k])| \leqslant d. \tag{5.10}$$

Let $A_1, \ldots, A_k$ be a max-cut partition of G, where $|A_k| \leqslant |A_i|$ for all $i \in [k-1]$. Then

(i) *P1(G) holds with respect to $A_1, \ldots, A_k$ with parameter $2k^2\sqrt{d}/n$;*

(ii) *we have*

$$m := \sum_{ij\in\binom{[k]}{2}} e(\overline{G}[A_i, A_j]) \leqslant 2k^2\sqrt{d}(kc-1)n + d \leqslant 3k^2\sqrt{\rho}n^2. \qquad (5.11)$$

Moreover, for all $i \in [k]$,

(iii) *if* $xy \in E(G[A_i])$, *then* $d_{\overline{G}}(x, \overline{A_i}) + d_{\overline{G}}(y, \overline{A_i}) \geqslant (1-(k-1)c)n - 3k^2\sqrt{\rho}n \geqslant \rho^{1/6}n$;

(iv) $\Delta(G[A_i]) \leqslant \rho^{1/5}n$;

(v) $e(G[A_i]) \leqslant \rho^{1/30}m$.

Proof. By (5.10), there is a partition $V_1, \dots, V_k$ of $V(G)$ such that, defining $n_i := |V_i|$ for $i \in [k]$, we have

$$|n_i - cn| \leqslant p \quad \text{for all } i \in [k-1] \quad \text{and} \quad |n_k - (n-(k-1)cn)| \leqslant p; \quad (5.12)$$

and

$$\sum_{i\in[k]} e(G[V_i]) + \sum_{ij\in\binom{[k]}{2}} e(\overline{G}[V_i, V_j]) \leqslant d.$$

The max-cut property implies that

$$\sum_{ij\in\binom{[k]}{2}} e(G[A_i, A_j]) \geqslant \sum_{ij\in\binom{[k]}{2}} e(G[V_i, V_j]) \geqslant e - d$$

and so

$$h := \sum_{i\in[k]} e(G[A_i]) = e - \sum_{ij\in\binom{[k]}{2}} e(G[A_i, A_j]) \leqslant d. \qquad (5.13)$$

For $i \in [k]$, choose $j = j(i) \in [k]$ such that $|A_i \cap V_j|$ is maximal. Suppose that there exists $h \in [k] \setminus \{j\}$ such that $|A_i \cap V_h| > \sqrt{2d}$. Then

$$e(G[A_i]) \geqslant |A_i \cap V_j||A_i \cap V_h| - |E(G) \triangle E(K[V_1, \dots, V_k])| > (\sqrt{2d})^2 - d = d,$$

a contradiction to (5.13). Thus for each $i \in [k]$, there exists at most one $h \in [k]$ such that $|A_i \cap V_h| > \sqrt{2d}$. Suppose that there is some $j \in [k]$ for which no $i \in [k]$ satisfies $j(i) = j$. Then, using (5.9), we get

$$2k\sqrt{2d} + p \leqslant 3k\sqrt{2d} \leqslant 3k\sqrt{2\rho}\, n < n - (k-1)cn,$$

and so

$$n_j = \sum_{i\in[k]} |A_i \cap V_j| < k\sqrt{2d} < \frac{n-(k-1)cn-p}{2}.$$

Recall from (5.7) that $c \geqslant 1-(k-1)c$, so this is a contradiction to (5.12). Thus, the function $j : [k] \to [k]$ is a bijection and, for each $i \in [k]$,

$$|A_i| \geqslant |V_{j(i)}| - \sum_{i'\in[k]\setminus\{i\}} |A_{i'} \cap V_{j(i)}| \geqslant n_{j(i)} - k\sqrt{2d},$$

and similarly $|A_i| \leqslant n_{j(i)} + k\sqrt{2d}$. Suppose first that $j(k) = k$. Then

$$\big|\,|A_k| - (n-(k-1)cn)\,\big| \leqslant |n_k - (n-(k-1)cn)| + k\sqrt{2d} \leqslant p + k\sqrt{2d} \leqslant 2k\sqrt{d}$$

and similarly $||A_i| - cn| \leqslant 2k\sqrt{d}$ for all $i \in [k-1]$. Suppose instead that $j(k) \neq k$. Then $||A_k| - cn| \leqslant k\sqrt{2d}$, and since A_k is the smallest part, we have that $n = \sum_{i\in[k]} |A_i| \geqslant k(cn - k\sqrt{2d})$. Thus $cn - k^2\sqrt{2d} \leqslant n-(k-1)cn \leqslant cn$, where the last inequality follows from (5.7). So

$$\begin{aligned}\big|\,|A_k| - (n-(k-1)cn)\,\big| &\leqslant \big|\,|A_k| - n_{j(k)}\big| \\ &\quad + |n_{j(k)} - cn| + |cn - (n-(k-1)cn)| \\ &\leqslant k\sqrt{2d} + p + k^2\sqrt{2d} \leqslant 2k^2\sqrt{d},\end{aligned}$$

and similarly $||A_i| - cn| \leqslant 2k^2\sqrt{d}$ for all $i \in [k-1]$. Hence P1(G) holds with parameter $2k^2\sqrt{d}/n$, proving (i). So it also holds with parameter $2k^2\sqrt{\rho} \geqslant 2k^2\sqrt{d}/n$.

We now prove (ii). Write $p_i := cn$ for $i \in [k-1]$ and $p_k := n-(k-1)cn$; and $d_i := p_i - |A_i|$ for all $i \in [k]$. Then $\sum_{i\in[k]} d_i = 0$, and we have

$$\begin{aligned} m &\overset{(5.13)}{=} \sum_{ij\in\binom{[k]}{2}} |A_i||A_j| - e + h \\ &= \frac{1}{2}\left(n^2 - \sum_{i\in[k]} p_i^2 + 2\sum_{i\in[k]} p_i d_i - \sum_{i\in[k]} d_i^2\right) - e + h \\ &\overset{(5.13)}{\leqslant} \frac{1}{2}\left(n^2 - (k-1)c^2n^2 - (n-(k-1)cn)^2\right) \\ &\quad + cn \sum_{i\in[k-1]} d_i + (n-(k-1)cn)d_k - e + d \\ &\overset{(4.10)}{=} -d_k(kc-1)n + d \overset{(i)}{\leqslant} 2k^2\sqrt{d}(kc-1)n + d\end{aligned}$$

$$\overset{(5.9)}{\leqslant} \quad 3k^2\sqrt{\rho}n^2, \tag{5.14}$$

as required.

Next we prove (iii). For any $i \in [k]$, and $xy \in E(G[A_i])$,

$$(k-2)cn + k \overset{(5.5)}{\geqslant} P_3(xy, G) \geqslant n - |A_i| - (d_{\overline{G}}(x, \overline{A_i}) + d_{\overline{G}}(y, \overline{A_i}))$$

and so

$$\begin{aligned} d_{\overline{G}}(x, \overline{A_i}) + d_{\overline{G}}(y, \overline{A_i}) &\overset{(i),(5.7)}{\geqslant} n - (k-2)cn - k - cn - 2k^2\sqrt{\rho}n \\ &\geqslant (1-(k-1)c)n - 3k^2\sqrt{\rho}n \overset{(5.9)}{\geqslant} \rho^{1/6}n, \end{aligned}$$

as required.

For (iv), suppose on the contrary that there exist $i \in [k]$ and $x \in A_i$ with $d_G(x, A_i) > \rho^{1/5}n$. Suppose first that $d_{\overline{G}}(x, \overline{A_i}) \geqslant k\rho^{1/5}n$. By averaging, there is some $\ell \in [k] \setminus \{i\}$ such that $d_{\overline{G}}(x, A_\ell) \geqslant \rho^{1/5}n$. For each $j \in [k]$, let $X_j := N_G(x, A_j)$ and $x_j := |X_j|$. By the max-cut property, for any $j \neq i$, we have $x_j \geqslant x_i \geqslant \rho^{1/5}n$. Let L be the number of triangles containing x and no other vertices from $A_i \cup A_\ell$. Part (ii) implies that

$$K_3(x, G) \geqslant L + x_\ell x_i + (x_i + x_\ell)(n - x_i - x_\ell) - 3k^2\sqrt{\rho}n^2.$$

Obtain a new graph G' by choosing $A'_i \subseteq X_i$ and $A'_\ell \subseteq A_\ell \setminus X_\ell$ with $|A'_i| = |A'_\ell| = \rho^{1/5}n$ and letting $E(G') := (E(G) \cup \{xy : y \in A'_\ell\}) \setminus \{xz : z \in A'_i\}$. Now

$$K_3(x, G') \leqslant L + (x_\ell + \rho^{1/5}n)(x_i - \rho^{1/5}n) + (x_i + x_\ell)(n - x_i - x_\ell).$$

Thus

$$K_3(G') - K_3(G) \leqslant \rho^{1/5}n(x_i - x_\ell) - \rho^{2/5}n^2 + 3k^2\sqrt{\rho}n^2 < -\rho^{2/5}n^2/2,$$

a contradiction. Thus $d_{\overline{G}}(x, \overline{A_i}) < k\rho^{1/5}n$. But (ii) also implies that

$$\sum_{y \in X_i} d_{\overline{G}}(y, \overline{A_i}) \leqslant e(\overline{G}[A_i, \overline{A_i}]) \leqslant 3k^2\sqrt{\rho}n^2,$$

so there exists $y \in X_i$ with $d_{\overline{G}}(y, \overline{A_i}) \leqslant 3k^2\sqrt{\rho}n^2/x_i \leqslant 3k^2\rho^{3/10}n$. But then

$$d_{\overline{G}}(x, \overline{A_i}) + d_{\overline{G}}(y, \overline{A_i}) \leqslant (k\rho^{1/5} + 3k^2\rho^{3/10})n < \rho^{1/6}n,$$

contradicting (iii).

Finally, we prove (v). Using the previous parts, we have for all $i \in [k]$ that

$$\rho^{1/5}nm \geqslant \rho^{1/5}n \cdot e(\overline{G}[A_i, \overline{A_i}]) \overset{(iv)}{\geqslant} \sum_{\substack{xy \in E(\overline{G}[A_i, \overline{A_i}]) \\ x \in A_i}} d_G(x, A_i)$$

$$= \sum_{uv \in E(G[A_i])} (d_{\overline{G}}(u, \overline{A_i}) + d_{\overline{G}}(v, \overline{A_i})) \overset{(iii)}{\geqslant} e(G[A_i])\rho^{1/6}n,$$

giving the required. □

6. The intermediate case: approximate structure

We will assume in this section and the succeeding two sections that

$$t_{k-1}(n) + \alpha n^2 < e < t_k(n) - \alpha n^2 \tag{6.1}$$

and say that we are in the *intermediate case*. (The remaining *boundary* case is treated in Section 9.) Equations (1.7) and (6.1) imply that

$$c \geqslant \frac{1}{k} + \sqrt{\frac{2\alpha}{k(k-1)}} > \frac{1+\sqrt{2\alpha}}{k}. \tag{6.2}$$

Thus we can improve one inequality in (5.7):

$$\sqrt{2\alpha} < kc - 1 \leqslant c - (k-1)\alpha. \tag{6.3}$$

The aim of this section is to prove the forthcoming lemma about the approximate structure of G in the intermediate case. One consequence of the statement is that, when $A_1, \ldots, A_k$ is a max-cut partition of G, then actually G is close to the complete partite graph $K[A_1, \ldots, A_k]$. Note that this is not true for an arbitrary extremal graph H, so here we crucially use the fact that G is a worst counterexample, that is, it satisfies (C1)–(C3).

LEMMA 6.1 (Approximate structure). *Suppose that (6.1) holds. Let $A_1, \ldots, A_k$ be a max-cut partition of $V(G)$ such that $|A_k| \leqslant |A_i|$ for all $i \in [k-1]$. Then there exists $Z \subseteq V(G)$ such that G has an $(A_1, \ldots, A_k; Z, \beta, \xi, \xi, \delta)$-partition with missing vector $\underline{m} =: (m_1, \ldots, m_{k-1})$ such that $m \leqslant \eta n^2$ and $h \leqslant \delta m$, where $m := m_1 + \cdots + m_{k-1}$ and h is defined in (5.13).*

To prove the lemma, we will use Theorem 1.2 together with a somewhat involved series of deductions. Define a function $f : V(G) \to \mathbb{R}$ by setting

$$f(x) := (d_G(x) - (k-2)cn)(k-2)cn + \binom{k-2}{2}c^2n^2 - K_3(x, G), \quad x \in V(G). \tag{6.4}$$

The intuition behind this formula is that it becomes the zero function if we apply it to $H := K^k_{cn,\ldots,cn,(1-(k-1)c)n}$ with $c = c(n, e)$:

$$(d_H(x)-(k-2)cn)(k-2)cn+\binom{k-2}{2}c^2n^2-K_3(x,H)=0 \quad \text{for all } x \in V(H). \tag{6.5}$$

It turns out that $f(x)$ is small in absolute value for every $x \in V(G)$.

LEMMA 6.2. *$|f(x)| \leqslant 6n/\sqrt{\alpha}$ for all $x \in V(G)$.*

Proof. We first give a bound on the gradient of the function $c(n, \cdot)$ that was defined in (4.9). We will write $c := c(n, e)$ as usual. Note that $k(2e/n^2) = k(n, e)$ by Lemma 4.12. Setting $s := 1/\sqrt{\alpha}$, we have

$$\begin{aligned} e(K^k_{cn,\ldots,cn,cn-s,(1-(k-1)c)n+s}) - e = s(kc-1)n - s^2 &\overset{(6.3)}{\geqslant} \sqrt{2\alpha}sn - 1/\alpha \\ &> \sqrt{\alpha}sn = n. \end{aligned} \tag{6.6}$$

Let $p := e(K^k_{cn-\frac{s}{k-1},\ldots,cn-\frac{s}{k-1},(1-(k-1)c)n+s})$ and $c' := c(n, e + n)$. Then

$$p > e(K^k_{cn,\ldots,cn,cn-s,(1-(k-1)c)n+s}) \overset{(6.6)}{\geqslant} e + n = e(K^k_{c'n,\ldots,c'n,(1-(k-1)c')n}).$$

This, together with the fact that $c(n, \cdot)$ is a nonincreasing function, implies that $c \geqslant c' \geqslant c - \frac{s}{(k-1)n}$, so

$$(k-2)c'n \geqslant (k-2)\left(cn - \frac{s}{k-1}\right) \geqslant (k-2)cn - \frac{1}{\sqrt{\alpha}}. \tag{6.7}$$

Next, (6.5) (or a direct calculation using (1.6), (1.8) and (6.4)) shows that

$$\sum_{v \in V(G)} f(v) = 3\left(K_3(K^k_{cn,\ldots,cn,n-(k-1)cn}) - K_3(G)\right). \tag{6.8}$$

Now let $x, y \in V(G)$ be two arbitrary distinct vertices. Let G' be the graph obtained from G by deleting y and cloning x. (By cloning, we mean adding a new vertex x' whose neighbourhood is identical to $N_G(x) \setminus \{y\}$; so, in particular, $xx' \notin E(G')$.) Then, letting $e' := e(G') - e(G)$, we have that

$$e' = \begin{cases} d(x) - d(y) & \text{if } xy \notin E(G), \\ d(x) - d(y) - 1 & \text{otherwise.} \end{cases}$$

Clearly, $|e'| \leqslant n$ and so $k(n, e + e') = k(n, e)$.

Suppose first that $e' \geqslant 0$. Using Lemma 4.16, (6.1) and the facts that G is a worst counterexample and that $c(n,\cdot)$ is a nonincreasing function, we have

$$\begin{aligned}
K_3(G') - K_3(G) &\overset{(5.4)}{\geqslant} h(n, e+e') - h(n,e) \\
&= \sum_{i=1}^{e'} (h(n, e+i) - h(n, e+i-1)) \\
&\geqslant \sum_{i=1}^{e'} ((k-2)\cdot c(n, e+i-1)\cdot n - k) \geqslant e'(k-2)c'n - kn \\
&\overset{(6.7)}{\geqslant} e'(k-2)cn - \frac{2n}{\sqrt{\alpha}}.
\end{aligned}$$

On the other hand, $K_3(G') - K_3(G) \leqslant K_3(x, G) - K_3(y, G) + (n-2)$. Thus

$$\begin{aligned}
K_3(x, G) - K_3(y, G) &\geqslant (k-2)cn(d(x) - d(y) - 1) - \frac{2n}{\sqrt{\alpha}} \\
&\geqslant (k-2)cn(d(x) - d(y)) - \frac{3n}{\sqrt{\alpha}}.
\end{aligned}$$

This implies that

$$f(x) - f(y) = (d(x) - d(y))(k-2)cn - (K_3(x, G) - K_3(y, G)) \leqslant \frac{3n}{\sqrt{\alpha}}.$$

Using an analogous argument assuming $e' < 0$ and the fact that x, y were arbitrary, we derive that for any $x, y \in V(G)$,

$$|f(x) - f(y)| \leqslant \frac{3n}{\sqrt{\alpha}}. \tag{6.9}$$

Suppose now for some $x \in V(G)$, we have $|f(x)| \geqslant 6n/\sqrt{\alpha}$. Then

$$\frac{3n^2}{\sqrt{\alpha}} \overset{(6.9)}{\leqslant} \left| \sum_{v \in V(G)} f(v) \right| \overset{(6.8)}{=} 3\left| K_3(K^k_{cn,\ldots,cn,n-(k-1)cn}) - K_3(G) \right| \overset{(5.8)}{\leqslant} \frac{3n}{2\varepsilon},$$

so $1/n_0 \geqslant 1/n \geqslant 2\varepsilon/\sqrt{\alpha} \geqslant \sqrt{\varepsilon}$, a contradiction to (5.1). □

COROLLARY 6.3.

$$\Delta(G) \leqslant (k-1)cn + \frac{42}{\sqrt{\alpha}} \quad \textit{and} \quad \delta(G) \geqslant (k-2)cn - k.$$

Proof. Let $x \in V(G)$ be arbitrary. By Lemma 6.2,

$$
\begin{aligned}
(d_G(x) - (k-2)cn)(k-2)cn + \binom{k-2}{2}c^2n^2 &= K_3(x,G) + f(x) \\
&\leqslant \frac{1}{2}\sum_{y \in N_G(x)} P_3(xy,G) + \frac{6n}{\sqrt{\alpha}} \overset{(5.5)}{\leqslant} \frac{1}{2}d_G(x)((k-2)cn+k) + \frac{6n}{\sqrt{\alpha}} \\
&\leqslant \frac{1}{2}d_G(x)(k-2)cn + \frac{7n}{\sqrt{\alpha}}.
\end{aligned}
$$

Solving for $d_G(x)$, we have, using $c \geqslant 1/k$, that

$$
d_G(x) \leqslant (k-1)cn + \frac{14}{\sqrt{\alpha}(k-2)c} \leqslant (k-1)cn + \frac{14k}{\sqrt{\alpha}(k-2)} \leqslant (k-1)cn + \frac{42}{\sqrt{\alpha}}.
$$

The claim about minimum degree trivially follows from (5.5). □

6.1. *G* is almost complete *k*-partite. Theorem 1.2 implies that our worst counterexample G is close in edit distance to *some* graph in $\mathcal{H}^*(n,e)$. In this subsection, we prove that in fact G is close in edit distance to the specific graph $H^*(n,e)$ in $\mathcal{H}^*(n,e)$. Recall from Definition 1 and (1.3) that the edit distance between $H^*(n,e)$ and $K_{a_1^*,\ldots,a_k^*}$ is at most n. But Lemma 4.16 implies that additionally $|a_i^* - cn| \leqslant 2$ for all $i \in [k-1]$, so we will in fact show that the edit distance between G and the complete k-partite graph with $k-1$ parts of size $\lfloor cn \rfloor$ is $o(n^2)$.

LEMMA 6.4. $|E(G) \mathbin{\triangle} E(K^k_{\lfloor cn\rfloor,\ldots,\lfloor cn\rfloor,n-(k-1)\lfloor cn\rfloor})| \leqslant \rho_0 n^2$.

Proof. Suppose that the statement is not true. We will first derive some structural properties of G under this assumption.

Let $\mathcal{H}_1(n)$ be the set of n-vertex graphs H with vertex partition $A \cup B$ such that $H[A]$ is complete $(k-2)$-partite, $H[A,B]$ is complete and $H[B]$ is triangle-free. Pick $H \in \mathcal{H}_1(n)$ with the minimal edit distance to G. Theorem 1.2 and (5.2) imply that

$$
|E(H) \mathbin{\triangle} E(G)| \leqslant \rho_4 n^2. \tag{6.10}
$$

(Note that H need not have e edges although we do have $|e - e(H)| \leqslant \rho_4 n^2$.) By definition, H comes with a *canonical partition* $A_1,\ldots,A_{k-2},B$ such that each A_i is an independent set and $H[B]$ is triangle-free, and $H[A_1,\ldots,A_{k-2},B]$ is complete $(k-1)$-partite. Now, G is $\rho_4 n^2$-close to some graph $H' \in \mathcal{H}_1^*(n,e)$ in which for $i \in [k-2]$, the ith part has size $a_i^* = cn \pm 2$ (by Lemma 4.16). Thus

H is $2\rho_4 n^2$-close to H' and consequently,

$$||A_i| - cn| < \rho_3 n \quad \text{for all } i \in [k-2]. \tag{6.11}$$

Let $A := \bigcup_{i \in [k-2]} A_i$.

CLAIM 6.5. *The following hold in G:*

(i) *for every $x \in A$, $d_G(x, B) > (c + \rho_0)n$ or $d_G(x, A) < ((k-2)c - \rho_0)n$;*

(ii) *for any $y \in V(G)$ and $ij \in \binom{[k-2]}{2}$ such that* $\min\{d_{\overline{G}}(y, A_i), d_{\overline{G}}(y, A_j)\} \geqslant \rho_3 n$, *we have* $\min\{d_G(y, A_i), d_G(y, A_j)\} \leqslant \rho_3 n$;

(iii) *for every $y \in B$, $d_G(y, A) > (k-3)cn + \rho_0 n$ or $d_G(y, B) < cn - \rho_0 n$.*

Proof of Claim. To prove (i), suppose that there is a vertex $x \in A$ with $d_G(x, B) \leqslant cn + \rho_0 n$ and $d_G(x, A) \geqslant ((k-2)c - \rho_0)n$. Without loss of generality, we may suppose that $x \in A_1$. Now modify H to obtain $H' \in \mathcal{H}_1(n)$ by replacing the neighbourhood of x with $A \setminus \{x\}$. Then H' has a canonical partition $A_1 \setminus \{x\}$, $A_2, \ldots, A_{k-2}, B \cup \{x\}$. We have that

$$\begin{aligned} d_{G \setminus H}(x) + d_{H \setminus G}(x) &\geqslant d_G(x, A) - |A \setminus A_1| + |B| - d_G(x, B) \\ &\overset{(6.11)}{\geqslant} ((k-2)c - \rho_0)n - (k-3)(c + \rho_3)n \\ &\quad + (1 - (k-2)(c + \rho_3))n - (c + \rho_0)n \\ &\geqslant (1 - (k-2)c - 3\rho_0)n, \end{aligned}$$

while

$$\begin{aligned} d_{G \setminus H'}(x) + d_{H' \setminus G}(x) &= d_G(x, B) + |A| - d_G(x, A) \\ &\leqslant (c + \rho_0)n + (k-2)(c + \rho_3)n - ((k-2)c - \rho_0)n \\ &\leqslant cn + 3\rho_0 n. \end{aligned}$$

Thus

$$\begin{aligned} &|E(H') \triangle E(G)| - |E(H) \triangle E(G)| \\ &\quad = d_{G \setminus H'}(x) + d_{H' \setminus G}(x) - d_{G \setminus H}(x) - d_{H \setminus G}(x) \\ &\quad \leqslant (kc - 1 - c)n + 6\rho_0 n \overset{(6.3)}{\leqslant} -((k-1)\alpha - 6\rho_0)n < -\alpha n, \end{aligned} \tag{6.12}$$

contradicting the choice of H.

To prove (ii), suppose that there exist $y \in V(G)$ and $ij \in \binom{[k-2]}{2}$ such that $d_{\overline{G}}(y, A_i), d_{\overline{G}}(y, A_j) \geqslant \rho_3 n$ and $d_G(y, A_j) \geqslant d_G(y, A_i) > \rho_3 n$. Then we can obtain a new graph G' by replacing $\rho_3 n$ neighbours of y in A_i with $\rho_3 n$ new neighbours in A_j. There are at most $\rho_4 n^2$ edges missing between A_i and A_j in G, so

$$\begin{aligned}K_3(G) - K_3(G') &= K_3(y, G) - K_3(y, G') \\ &\geqslant (d_G(y, A_i)d_G(y, A_j) - \rho_4 n^2) \\ &\quad - (d_G(y, A_i) - \rho_3 n)(d_G(y, A_j) + \rho_3 n) \\ &\geqslant \rho_3^2 n^2 - \rho_4 n^2 \geqslant \rho_4 n^2.\end{aligned}$$

This contradicts the fact that G is a worst counterexample (namely, (C1)).

For (iii), suppose there is some $y \in B$ with $d_G(y, A) \leqslant (k-3)cn + \rho_0 n$ and $d_G(y, B) \geqslant cn - \rho_0 n$. Suppose without loss of generality that $d_G(y, A_1) = \min_{j\in[k-2]}\{d_G(y, A_j)\}$. We claim that

$$d_G(y, A_1) \leqslant 2\rho_0 n. \tag{6.13}$$

Indeed, when $k = 3$, we have $A_1 = A$ and so $d_G(y, A_1) = d_G(y, A) \leqslant \rho_0 n$. Suppose now that $k \geqslant 4$. If $d_G(y, A_1) \geqslant 2\rho_0 n$, then

$$\begin{aligned}d_{\overline{G}}(y, A \setminus A_1) &= |A \setminus A_1| - d_G(y, A) + d_G(y, A_1) \\ &\overset{(6.11)}{\geqslant} (k-3)(c-\rho_3)n - (k-3)cn - \rho_0 n + 2\rho_0 n \geqslant \frac{\rho_0 n}{2}.\end{aligned}$$

Thus there is some $j \in [k-2]\setminus\{1\}$ for which $d_{\overline{G}}(y, A_j) \geqslant \rho_0 n/(2k) \geqslant \rho_3 n$. On the other hand, as $d_G(y, A_1) = \min_{j\in[k-2]}\{d_G(y, A_j)\}$, we have that

$$d_{\overline{G}}(y, A_1) = |A_1| - d_G(y, A_1) \geqslant |A_1| - d_G(y, A)/(k-2) \geqslant \rho_3 n.$$

Then (ii) implies that $d_G(y, A_1) \leqslant \rho_3 n < 2\rho_0 n$, a contradiction. Thus (6.13) holds.

Obtain H' from H by replacing $N_H(y)$ with $V(H) \setminus A_1$. Then $H' \in \mathcal{H}_1(n)$ has a canonical partition $A_1 \cup \{y\}, A_2, \ldots, A_{k-2}, B \setminus \{y\}$. We have $d_{G\setminus H}(y) + d_{H\setminus G}(y) \geqslant d_{\overline{G}}(y, A)$, while

$$\begin{aligned}d_{G\setminus H'}(y) + d_{H'\setminus G}(y) &\leqslant d_G(y, A_1) + d_{\overline{G}}(y, A \setminus A_1) + d_{\overline{G}}(y, B) \\ &\leqslant 2d_G(y, A_1) + d_{\overline{G}}(y, A) - |A_1| + |B| - d_G(y, B) \\ &\leqslant 4\rho_0 n + d_{\overline{G}}(y, A) - (c-\rho_3)n + (1-(k-2)(c-\rho_3))n \\ &\quad - (c-\rho_0)n \\ &\leqslant d_{\overline{G}}(y, A) + (1-kc)n + 6\rho_0 n \\ &\overset{(6.3)}{\leqslant} d_{\overline{G}}(y, A) - (\sqrt{2\alpha} - 6\rho_0)n.\end{aligned}$$

Again, this implies that $|E(H') \triangle E(G)| < |E(H) \triangle E(G)|$, contradicting the choice of H. This completes the proof of the claim. □

The next claim shows that every large enough subset of B must contain many edges.

CLAIM 6.6. *For all $X \subseteq B$ with $|X| \geqslant (c - \rho_1)n$, we have $E(G[X]) \geqslant \rho_1 n^2$.*

Proof of Claim. Suppose that some X violates the claim. By taking a subset, we can assume that $|X| = (c - \rho_1)n$. Now (6.2) implies that $c \geqslant 1/k$, and so $|X| \geqslant n/(2k)$. Let $\tilde{d}(X, \overline{X}) := \frac{1}{|X|} \sum_{x \in X} d_G(x, \overline{X})$ denote the average degree of vertices in X into $\overline{X}$ in G. Then the average degree of vertices in X in G is

$$\frac{1}{|X|} \sum_{x \in X} d_G(x) = \tilde{d}(X, \overline{X}) + \frac{2e(G[X])}{|X|} \leqslant \tilde{d}(X, \overline{X}) + 4k\rho_1 n.$$

Let $Y := B \setminus X$. By Corollary 6.3, the average degree of vertices in Y is certainly at most

$$\Delta(G) \leqslant (k-1)cn + 42/\sqrt{\alpha} \leqslant (k-1)cn + \rho_3 n. \tag{6.14}$$

The average degree of vertices in A in $G[A]$ is

$$\begin{aligned} \frac{1}{|A|} \sum_{a \in A} d_G(a, A) &\overset{(6.10)}{\leqslant} \frac{1}{|A|} \left(\sum_{a \in A} d_H(a, A) + 2\rho_4 n^2 \right) \\ &\overset{(6.11)}{\leqslant} (k-3)(c + \rho_3)n + \rho_3 n \leqslant (k-3)cn + k\rho_3 n. \end{aligned}$$

Thus the average degree of vertices of A in G is

$$\begin{aligned} \frac{1}{|A|} \sum_{a \in A} d_G(a) &\leqslant |B| + (k-3)cn + k\rho_3 n \\ &\overset{(6.11)}{\leqslant} (1 - (k-2)(c - \rho_3))n + (k-3)cn + k\rho_3 n \\ &\leqslant (1 - c + 2k\rho_3)n. \end{aligned}$$

Hence, by taking the weighted average of these average degrees to obtain the average degree of G, we have

$$\begin{aligned} 2\left((k-1)c - \binom{k}{2}c^2\right) &\overset{(4.10)}{=} \frac{2e}{n^2} \\ &\leqslant \frac{1}{n^2}((\tilde{d}(X, \overline{X}) + 4k\rho_1 n)|X| + ((k-1)cn + \rho_3 n)|Y| + (1 - c + 2k\rho_3)n|A|) \\ &\overset{(6.11)}{\leqslant} \left(\frac{\tilde{d}(X, \overline{X})}{n} + 4k\rho_1 \right) c + ((k-1)c + \rho_3)(1 - (k-1)c + 2\rho_1) \end{aligned}$$

$$+ (1 - c + 2k\rho_3)(k-2)(c+\rho_3)$$
$$\leqslant 2\left((k-1)c - \binom{k}{2}c^2\right) + c\left(\frac{\tilde{d}(X,\overline{X})}{n} - (1-c)\right) + 6k\rho_1.$$

Thus

$$\tilde{d}(X,\overline{X}) \geqslant \left((1-c) - \frac{6k\rho_1}{c}\right)n \geqslant |\overline{X}| - \sqrt{\rho_1}n.$$

In particular, the number of missing edges in G between X and Y is $e(\overline{G}[X, Y]) \leqslant (c-\rho_1)\sqrt{\rho_1}n^2 \leqslant \sqrt{\rho_1}n^2$. This further implies that

$$\begin{aligned} e(G[Y]) &\leqslant |Y|\cdot\Delta(G) - e(G[A,Y]) - e(G[X,Y]) \\ &\overset{(6.10)}{\leqslant} |Y|\Delta(G) - (|A||Y| - \rho_4 n^2) - (|X||Y| - \sqrt{\rho_1}n^2) \\ &\overset{(6.11),(6.14)}{\leqslant} |Y|((k-1)cn + 42/\sqrt{\alpha} - (k-2)(c-\rho_3)n - (c-\rho_1)n) \\ &\qquad + \rho_4 n^2 + \sqrt{\rho_1}n^2 \\ &\leqslant 2\sqrt{\rho_1}n^2. \end{aligned}$$

Let $H' \in \mathcal{H}_1(n)$ be the n-vertex complete k-partite graph with partition $A_1, \ldots, A_{k-2}, X, Y$. Then

$$\begin{aligned} |E(G) \triangle E(H')| &\leqslant |E(G) \triangle E(H)| + e(G[Y]) + e(G[X]) + e(\overline{G}[X,Y]) \\ &\overset{(6.10)}{\leqslant} (\rho_4 + 2\sqrt{\rho_1} + \rho_1 + \sqrt{\rho_1})n^2 < 4\sqrt{\rho_1}n^2. \end{aligned}$$

But there is a one-to-one mapping of parts of H' to parts of $K^k_{\lfloor cn\rfloor,\ldots,\lfloor cn\rfloor,n-(k-1)\lfloor cn\rfloor}$ such that two corresponding parts have size within $2\rho_1$ of one another. Therefore

$$\left|E(H') \triangle E(K^k_{\lfloor cn\rfloor,\ldots,\lfloor cn\rfloor,n-(k-1)\lfloor cn\rfloor})\right| \leqslant \frac{\rho_0 n^2}{2}.$$

Then $|E(G) \triangle E(K^k_{\lfloor cn\rfloor,\ldots,\lfloor cn\rfloor,n-(k-1)\lfloor cn\rfloor})| < \rho_0 n^2$, a contradiction to our initial assumption on G. □

We are now able to show that vertices in every A_i have small degree in their own part, and further that for distinct i, j, the bipartite graph $G[A_i, A_j]$ is complete.

CLAIM 6.7. *For all $i \in [k-2]$, we have $\Delta(G[A_i]) < \rho_2 n$. Moreover, $G[A] \supseteq K[A_1, \ldots, A_{k-2}]$.*

Proof of Claim. Suppose on the contrary that for some $i \in [k-2]$, there is an $x \in A_i$ with $d_G(x, A_i) \geqslant \rho_2 n$. Let $Z := N_G(x, A_i)$ and $X := N_G(x, B)$. We claim that

$$d_{\overline{G}}(x, A \setminus A_i) < 6k\rho_3 n. \tag{6.15}$$

This is vacuously true if $k = 3$. So suppose that $k \geqslant 4$. We will first show that for any $j \in [k-2] \setminus \{i\}$, we have

$$d_G(x, A_j) \geqslant d_G(x, A_i) - \rho_3 n. \tag{6.16}$$

Indeed, let $H' \in \mathcal{H}_1(n)$ have a canonical partition obtained from $A_1, \ldots, A_{k-2}$, B by moving x from A_i to A_j. We have that

$$\begin{aligned} 0 &\leqslant |E(G) \triangle E(H')| - |E(G) \triangle E(H)| \\ &\leqslant d_G(x, A_j) + |A_i| - d_G(x, A_i) - (d_G(x, A_i) + |A_j| - d_G(x, A_j)) \\ &\overset{(6.11)}{\leqslant} 2(d_G(x, A_j) - d_G(x, A_i)) + 2\rho_3, \end{aligned}$$

giving (6.16). So $d_G(x, A_j) \geqslant |Z| - \rho_3 n \geqslant (\rho_2 - \rho_3)n \geqslant \rho_3 n$. If $d_{\overline{G}}(x, A \setminus A_i) \geqslant 6k\rho_3 n$, then there exists some $j \in [k-2] \setminus \{i\}$ such that $d_{\overline{G}}(x, A_j) \geqslant 6\rho_3 n$. Then (6.16) implies that

$$|A_i| - 1 - d_{\overline{G}}(x, A_i) = d_G(x, A_i) \leqslant d_G(x, A_j) + \rho_3 n = |A_j| - d_{\overline{G}}(x, A_j) + \rho_3 n$$

and so

$$\begin{aligned} d_{\overline{G}}(x, A_i) &\geqslant d_{\overline{G}}(x, A_j) + |A_i| - 1 - |A_j| - 2\rho_3 n \\ &\overset{(6.11)}{\geqslant} 6\rho_3 n + (c - \rho_3)n - 1 - (c + \rho_3)n - 2\rho_3 n > \rho_3 n. \end{aligned}$$

Then Claim 6.5(ii) implies $d_G(x, A_i) < \rho_3 n < \rho_2 n$, a contradiction. Thus (6.15) holds.

We have

$$\begin{aligned} \sum_{z \in Z} (d_{\overline{G}}(z, X) + d_{\overline{G}}(z, A \setminus A_i)) &= e(\overline{G}[Z, X]) + e(\overline{G}[Z, A \setminus A_i]) \\ &\leqslant |E(G) \triangle E(H)| \overset{(6.10)}{\leqslant} \rho_4 n^2. \end{aligned}$$

Thus, by averaging, there is some $z \in Z$ such that

$$d_{\overline{G}}(z, X) + d_{\overline{G}}(z, A \setminus A_i) \leqslant \rho_4 n / \rho_2 \leqslant \rho_3 n.$$

Then

$$\begin{aligned}
(k-2)cn+k &\overset{(5.5)}{\geqslant} P_3(xz,G)\\
&\geqslant |X|+|A\setminus A_i| - (d_{\overline{G}}(z,X)+d_{\overline{G}}(z,A\setminus A_i))\\
&\quad - d_{\overline{G}}(x,A\setminus A_i)\\
&\overset{(6.11),(6.15)}{\geqslant} |X|+(k-3)(c-\rho_3)n-\rho_3 n-6k\rho_3 n\\
&\geqslant |X|+(k-3)cn-7k\rho_3 n.
\end{aligned}$$

Consequently,

$$|X| \leqslant cn + 8k\rho_3 n. \tag{6.17}$$

We now bound $d_G(x)$ and $K_3(x,G)$ as follows. We have

$$d_G(x) \leqslant |X|+|Z|+|A\setminus A_i| \overset{(6.11)}{\leqslant} |X|+|Z|+(k-3)cn+k\rho_3 n. \tag{6.18}$$

We wish to bound $K_3(x,G)$ from below. Let $Y := N_G(x, A\setminus A_i)$. We will need the following lower bound on $|Y|$:

$$|Y| = |A\setminus A_i| - d_{\overline{G}}(x,A\setminus A_i) \overset{(6.11),(6.15)}{\geqslant} (k-3)cn-7k\rho_3 n \geqslant |A\setminus A_i|-8k\rho_3 n. \tag{6.19}$$

Note also that

$$\begin{aligned}
K_3(x,G;A\setminus A_i) &= e(G[Y]) \geqslant e(G[A\setminus A_i]) - (|A\setminus A_i|-|Y|)n\\
&\overset{(6.10),(6.19)}{\geqslant} e(H[A\setminus A_i]) - \rho_4 n^2 - 8k\rho_3 n^2\\
&\geqslant \left(\binom{k-3}{2}(c-\rho_3)^2-\rho_4-8k\rho_3\right)n^2\\
&\geqslant \binom{k-3}{2}c^2n^2 - \frac{\sqrt{\rho_3}n^2}{2}.
\end{aligned}$$

Thus

$$\begin{aligned}
K_3(x,G) &\overset{(6.10)}{\geqslant} |X||Y|+|Y||Z|+|Z||X|-\rho_4 n^2+e(G[X])\\
&\quad + K_3(x,G;A\setminus A_i)\\
&\overset{(6.11),(6.19)}{\geqslant} |X||Z|+(|X|+|Z|)(k-3)cn+e(G[X])\\
&\quad + \binom{k-3}{2}c^2n^2-\sqrt{\rho_3}n^2.
\end{aligned}$$

This together with Lemma 6.2 implies that

$$-\frac{6n}{\sqrt{\alpha}} \leqslant f(x) \tag{6.20}$$

$$
\begin{aligned}
&= && (d_G(x)-(k-2)cn)(k-2)cn+\binom{k-2}{2}c^2n^2-K_3(x,G)\\
&\overset{(6.11),\ (6.18)}{\leqslant} && (|X|+|Z|-cn+k\rho_3 n)(k-2)cn+\binom{k-2}{2}c^2n^2\\
& && -\Bigg(|X||Z|+(|X|+|Z|)(k-3)cn+e(G[X])\\
& && \quad+\binom{k-3}{2}c^2n^2-\sqrt{\rho_3}n^2\Bigg)\\
&\leqslant && (|Z|-cn)(cn-|X|)-e(G[X])+\rho_2 n^2.
\end{aligned}
$$

Then, by considering two cases where the coefficient $cn-|X|$ of $|Z|$ is negative or nonnegative and recalling that $\rho_2 n \leqslant |Z| \leqslant |A_i|$, we have

$$
\begin{aligned}
e(G[X]) &\overset{(6.17)}{\leqslant} \frac{6n}{\sqrt{\alpha}}+\rho_2 n^2+\max\{(\rho_2 n-cn)(-8k\rho_3 n),(|A_i|-cn)cn\}\\
&\overset{(6.11)}{\leqslant} 2\rho_2 n^2+8k\rho_3 cn^2 \leqslant 3\rho_2 n^2.
\end{aligned}
$$

Thus, by Claim 6.6, we have $d_G(x,B)=|X|<(c-\rho_1)n$. Claim 6.5(i) now implies that

$$
((k-2)c-\rho_0)n > d_G(x,A)=|Z|+|Y| \overset{(6.19)}{\geqslant} |Z|+(k-3)cn-7k\rho_3 n,
$$

implying that $|Z|\leqslant cn-\rho_0 n/2$. We look again at (6.20) to see that

$$
e(G[X]) \leqslant \frac{6n}{\sqrt{\alpha}}+\rho_2 n^2-\frac{\rho_0\rho_1 n^2}{2}<0,
$$

a contradiction. This proves the first part of the claim.

For the second part, let $x\in A_i$ and $y\in A_j$ with $ij\in\binom{[k-2]}{2}$. Then, using the first part,

$$
\begin{aligned}
P_3(xy,G) &\leqslant (n-|A_i|-|A_j|)+\Delta(G[A_i])+\Delta(G[A_j])\\
&\overset{(6.11)}{\leqslant} (1-2c+2\rho_3)n+2\rho_2 n\\
&\overset{(6.3)}{<} (k-2)cn-(\sqrt{2\alpha}-2\rho_3-2\rho_2)n<(k-2)cn-\sqrt{\alpha}n.
\end{aligned}
$$

Then (5.5) implies that $xy\in E(G)$. Since ij was arbitrary, we have shown that $K[A_1,\dots,A_{k-2}]\subseteq G[A]$, as required. □

We now prove some useful properties of vertices in B.

CLAIM 6.8. *For every $y \in B$, the following holds:*

(i) *If $d_G(y, B) \leqslant cn + \rho_2 n$, then $A \subseteq N_G(y)$.*

(ii) *If $d_G(y, B) > (c - \rho_1/2)n$, then there exists $i \in [k-2]$ such that $d_{\overline{G}}(y, A \setminus A_i) < k\rho_3 n$.*

Proof of Claim. Let $y \in B$ be arbitrary, and let $Y := N_G(y, B)$. We will first prove (ii). Note that (ii) is vacuously true when $k = 3$, so assume $k \geqslant 4$. Suppose that $d_G(y, B) > (c - \rho_1/2)n$. Claim 6.5(iii) implies that

$$d_G(y, A) > (k-3)cn + \rho_0 n. \tag{6.21}$$

Let $i \in [k-2]$ be such that $d_{\overline{G}}(y, A_i) = \max_{j \in [k-2]} d_{\overline{G}}(y, A_j)$.

Let us show that this i satisfies (ii). Suppose on the contrary that $d_{\overline{G}}(y, A \setminus A_i) \geqslant k\rho_3 n$. Then there exists $j \in [k-2] \setminus \{i\}$ such that $\rho_3 n \leqslant d_{\overline{G}}(y, A_j) \leqslant d_{\overline{G}}(y, A_i)$. Claim 6.5(ii) and (6.11) imply that $d_G(y, A_i \cup A_j) \leqslant \rho_3 n + (c + \rho_3)n = (c + 2\rho_3)n$. But then

$$\begin{aligned} d_G(y, A) &\leqslant d_G(y, A_i \cup A_j) + |A \setminus (A_i \cup A_j)| \overset{(6.11)}{\leqslant} (c + 2\rho_3)n + (k-4)(c+\rho_3)n \\ &\leqslant (k-3)cn + \rho_2 n, \end{aligned}$$

contradicting (6.21). Thus $d_{\overline{G}}(y, A \setminus A_i) < k\rho_3 n$. This completes the proof of (ii).

For (i), suppose now that $|Y| \leqslant cn + \rho_2 n$. First, consider the case when additionally $|Y| \leqslant (c - \rho_1/2)n$. Let $x \in A$ be arbitrary, and let $i \in [k-2]$ be such that $x \in A_i$. Then Claim 6.7 implies that

$$\begin{aligned} P_3(xy, G) &\leqslant \Delta(G[A_i]) + |Y| + |A \setminus A_i| \overset{(6.11)}{\leqslant} \rho_2 n + (c - \rho_1/2)n + (k-3)(c+\rho_3)n \\ &\leqslant (k-2)cn - \rho_1 n/3. \end{aligned}$$

Then (5.5) implies that $xy \in E(G)$. Since x was arbitrary, we have proved that $A \subseteq N_G(y)$. So (i) holds in this case.

Consider the other case when $(c - \rho_1/2)n < |Y| \leqslant (c + \rho_2)n$. Part (ii) implies that there exists $i \in [k-2]$ such that $d_{\overline{G}}(y, A \setminus A_i) < k\rho_3 n$.

Let $Z := N_G(y, A \setminus A_i)$. Then

$$\begin{aligned} |Z| = |A \setminus A_i| - d_{\overline{G}}(y, A \setminus A_i) &\geqslant (k-3)(c - \rho_3)n - k\rho_3 n \\ &\geqslant (k-3)cn - 2k\rho_3 n. \end{aligned} \tag{6.22}$$

Let also $X := N_G(y, A_i)$. Note that $d_G(y) \leqslant |X| + |Y| + |A \setminus A_i| \leqslant |X| + |Y| + (k-3)(c + \rho_3)n$ by (6.11). Then Lemma 6.2 implies that

$$K_3(y, G) \leqslant (d_G(y) - (k-2)cn)(k-2)cn + \binom{k-2}{2} c^2 n^2 + \frac{6n}{\sqrt{\alpha}}$$

$$\leqslant (|X| + |Y| - cn)(k-2)cn + \binom{k-2}{2} c^2n^2 + \rho_2 n^2. \tag{6.23}$$

Recall that every pair among X, Y, Z spans a complete bipartite graph in H. Moreover, (ii) implies that

$$e(G[Z]) \geqslant e(G[A \setminus A_i]) - d_{\overline{G}}(y, A \setminus A_i)n \geqslant e(G[A \setminus A_i]) - k\rho_3 n^2.$$

Thus we can use Claim 6.7 to lower bound $K_3(y, G)$:

$$\begin{aligned} K_3(y,G) &\geqslant e(G[X,Y]) + e(G[Y,Z]) + e(G[Z,X]) \\ &\quad + e(G[Z]) + e(G[Y]) \\ &\overset{(6.10)}{\geqslant} |X||Y| + |Y||Z| + |Z||X| - \rho_4 n^2 + \sum_{hj \in \binom{[k-2]\setminus\{i\}}{2}} |A_h||A_j| \\ &\quad -k\rho_3 n^2 + e(G[Y]) \\ &\overset{(6.11),(6.22)}{\geqslant} |X||Y| + (k-3)cn(|X|+|Y|) + \binom{k-3}{2} c^2n^2 + e(G[Y]) \\ &\quad -\sqrt{\rho_3}n^2. \end{aligned}$$

This together with (6.23) implies that

$$e(G[Y]) \leqslant (cn - |X|)(|Y| - cn) + 2\rho_2 n^2.$$

As before, considering the two cases when $cn - |X|$ is positive and nonpositive and recalling that $(c - \rho_1/2)n < |Y| \leqslant (c + \rho_2)n$, we have

$$\begin{aligned} e(G[Y]) &\leqslant \max\left\{cn \cdot \rho_2 n,\ (|A_i| - cn) \cdot \rho_1 n/2\right\} + 2\rho_2 n^2 \\ &\overset{(6.11)}{\leqslant} \max\left\{c\rho_2 n^2,\ \rho_1\rho_3 n^2/2\right\} + 2\rho_2 n^2 < \rho_1 n^2. \end{aligned}$$

This is a contradiction to Claim 6.6. □

CLAIM 6.9. *For every $i \in [k-2]$ and $y \in B$ with $d_{\overline{G}}(y, A \setminus A_i) \leqslant \rho_2 n/2$, we have that $A_i \subseteq N_G(y)$.*

Proof of Claim. Choose $i \in [k-2]$ and $y \in B$ with $d_{\overline{G}}(y, A \setminus A_i) \leqslant \rho_2 n/2$. Let $X := N_G(y, A_i)$ and $Y := N_G(B, y)$. Suppose that there exists $x' \in A_i$ such that $x'y \notin E(G)$. Then Claim 6.8(i) implies that $|Y| > (c + \rho_2)n$. Claim 6.5(iii) implies that $d_G(y, A) > (k-3)cn + \rho_0 n$. Therefore

$$|X| \geqslant d_G(y, A) - |A \setminus A_i| \overset{(6.11)}{>} (k-3)cn + \rho_0 n - (k-3)(c + \rho_3)n \geqslant \rho_0 n/2.$$

Furthermore,

$$\sum_{x\in X}\left(d_{\overline{G}}(x,Y)+d_{\overline{G}}(x,A\setminus A_i)\right)=e(\overline{G}[X,Y])+e(\overline{G}[X,A\setminus A_i])\overset{(6.10)}{\leqslant}\rho_4 n^2,$$

so there exists $x\in X$ with

$$d_{\overline{G}}(x,Y)+d_{\overline{G}}(x,A\setminus A_i)\leqslant\frac{\rho_4 n^2}{|X|}\leqslant\frac{2\rho_4 n}{\rho_0}<\rho_3 n.$$

Since $d_{\overline{G}}(y,A\setminus A_i)\leqslant\rho_2 n/2$, we have that

$$\begin{aligned}P_3(xy,G)&\geqslant(|A\setminus A_i|+|Y|)-d_{\overline{G}}(x,Y)-d_{\overline{G}}(x,A\setminus A_i)-d_{\overline{G}}(y,A\setminus A_i)\\&\overset{(6.11)}{\geqslant}(k-3)(c-\rho_3)n+(c+\rho_2)n-\rho_3 n-\rho_2 n/2\\&\geqslant(k-2)cn+\rho_2 n/3,\end{aligned}$$

a contradiction to (5.5). □

We are now able to show that G consists of the complete $(k-1)$-partite graph with parts $A_1,\ldots,A_{k-2},B$, together with some additional edges in B.

CLAIM 6.10. $G\setminus G[B]\cong K[A_1,\ldots,A_{k-2},B]$.

Proof of Claim. We will first show that $G[A,B]$ is a complete bipartite graph. Let $y\in B$ be arbitrary. It suffices to show that $A\subseteq N_G(y)$. By Claim 6.9, we may assume that $k\geqslant 4$. Let $Y:=N_G(y,B)$. By Claim 6.8(i), we may assume that $|Y|\geqslant(c+\rho_2)n$, and Claim 6.5(iii) implies that $d_G(y,A)\geqslant(k-3)cn+\rho_0 n$. Claim 6.8(ii) implies that there exists $i\in[k-2]$ such that $d_{\overline{G}}(y,A\setminus A_i)<k\rho_3 n<\rho_2 n/2$. Then, by Claim 6.9, we have that $A_i\subseteq N_G(y)$. Thus, for all $j\in[k-2]$, we have $d_{\overline{G}}(y,A\setminus A_j)\leqslant d_{\overline{G}}(y,A)=d_{\overline{G}}(y,A\setminus A_i)<\rho_2 n/2$. But Claim 6.9 now implies that $A_j\subseteq N_G(y)$ for all $j\in[k-2]$. Thus $A\subseteq N_G(y)$, proving the first part of the claim.

To complete the proof, it suffices by the second assertion of Claim 6.7 to show that $e(G[A_i])=0$ for all $i\in[k-2]$. So let $i\in[k-2]$ and let $x,z\in A_i$ be distinct. Claim 6.7 implies that $A_j\subseteq N_G(x)\cap N_G(z)$ for all $j\in[k-2]$, and since $G[A,B]$ is complete, we also have $B\subseteq N_G(x)\cap N_G(z)$. Thus

$$P_3(xz,G)\geqslant n-|A_i|\overset{(6.11)}{\geqslant}n-(c+\rho_3)n\overset{(6.3)}{\geqslant}(k-2)cn+((k-1)\alpha-\rho_3)n.$$

So (5.5) implies that $xz\notin E(G)$. This completes the proof of the claim. □

The rigid structural information provided by the last claim allows us to finish the proof by deriving a contradiction to our assumption that G is far in edit distance from $K^k_{\lfloor cn\rfloor,\ldots,\lfloor cn\rfloor,n-(k-1)\lfloor cn\rfloor}$.

Suppose first that $k=3$. Claim 6.10 implies that $G[A,B]$ is complete bipartite and $G[A]$ contains no edges. Thus $G[B]$ exactly minimizes the number of triangles given its size, that is, $K_3(G[B]) = g_3(n, e(G[B]))$ (otherwise, we could replace $G[B]$ in G to obtain an (n,e)-graph with fewer triangles). Now, $K_3(G[B]) > 0$, otherwise $G \in \mathcal{H}_1(n,e)$, a contradiction. Therefore

$$e(G[B]) > t_2(|B|) \overset{(6.11)}{\geqslant} \left\lfloor \frac{(1-(c+\rho_3))^2 n^2}{4} \right\rfloor \geqslant \frac{(1-c)^2n^2}{4} - \rho_2 n^2. \quad (6.24)$$

Recalling the definition of c (that is, (4.10)) in the case $k=3$ and the fact that $c < 1/2$ (that is, (5.6)), we have

$$\begin{aligned} e(G[B]) &= e - |A||B| \leqslant e - (c-\rho_3)(1-(c+\rho_3))n^2 \\ &\leqslant e - c(1-c)n^2 + \rho_2 n^2 \\ &\overset{(4.10)}{=} c(1-2c)n^2 + \rho_2 n^2. \end{aligned}$$

This together with (6.24) implies that $(3c-1)^2 \leqslant 8\rho_2$ and so

$$c < \frac{1}{3} + \rho_0 < \frac{1+\sqrt{2\alpha}}{3},$$

contradicting (6.2).

Therefore we may suppose that $k \geqslant 4$. Now, by Claim 6.10, for each $i \in [k-2]$, we have that A_i is an independent set in G and $G[A_i, \overline{A_i}]$ is a complete bipartite graph. Let $n_i := |\overline{A_i}|$ and $e_i := e(G[\overline{A_i}]) = e - n_i(n-n_i)$ and $G_i := G[\overline{A_i}]$. Then $g_3(n,e) = K_3(G) = K_3(G_i) + (n-n_i)e_i$. Thus $K_3(G_i) = g_3(n_i, e_i)$. Recall the definition of the function $k(\cdot,\cdot)$ given in (1.1).

CLAIM 6.11. $t_{k-2}(n_i) + \alpha n_i^2/3 \leqslant e_i \leqslant t_{k-1}(n_i) - \alpha n_i^2/3$.

Proof of Claim. By (6.11), $|n_i - (1-c)n| \leqslant \rho_3 n$. We then have

$$\frac{e_i}{n_i^2} - \frac{1}{2}\left(1 - \frac{1}{k-2}\right) \geqslant \frac{(1-kc+c)((kc-1)(k-2)+(1-c))}{2(1-c)^2(k-2)} - \rho_2,$$

where the first term follows by routine calculations with n_i approximated by $(1-c)n$ while the second term $-\rho_2$ absorbs all errors. By (6.3), the left-hand side is at least

$$\frac{(k-1)\alpha\cdot(1-c)}{2(1-c)^2(k-2)} - \rho_2 > \frac{\alpha}{3}$$

and thus $e_i \geqslant t_{k-2}(n_i) + \alpha n_i^2/3$. The other inequality is similar:

$$\frac{e_i}{n_i^2} - \frac{1}{2}\left(1 - \frac{1}{k-1}\right) \leqslant -\frac{(k-2)\cdot(kc-1)^2}{2(k-1)} + \rho_2 \overset{(6.3)}{\leqslant} -\frac{(k-2)\cdot 2\alpha}{k-1} + \rho_2 < -\frac{\alpha}{2}$$

and so $e_i \leqslant t_{k-1}(n_i) - \alpha n_i^2/3$. □

But

$$n_i = n - |A_i| \overset{(6.11)}{\geqslant} (1 - c - \rho_3)n \overset{(6.3)}{\geqslant} n/2 \geqslant n_0/2 \overset{(5.2)}{\geqslant} n_0(k-1, \alpha/3)$$

and so the minimality of k implies that $G_i \in \mathcal{H}(n_i, e_i)$. Suppose first that $G_i \in \mathcal{H}_1(n_i, e_i)$. Since G is an (n, e)-graph obtained by adding every edge between the independent set A_i and $V(G_i)$, we have that $G \in \mathcal{H}_1(n, e)$, a contradiction to (C1). Suppose instead that $G_i \in \mathcal{H}_2(n_i, e_i)$. Then G_i is $(k-1)$-partite and so G is k-partite. Corollary 4.4(i) then implies that $G \in \mathcal{H}_2(n, e)$, again contradicting (C1). Thus our original assumption was false, and we have shown that $|E(G) \triangle E(K^k_{\lfloor cn \rfloor, \ldots, \lfloor cn \rfloor, n-(k-1)\lfloor cn \rfloor})| \leqslant \rho_0 n^2$. This completes the proof of Lemma 6.4. □

6.2. Proof of Lemma 6.1. Now we are ready to show that every max-cut partition $A_1, \ldots, A_k$ of our worst counterexample G has the required approximate structure.

Proof of Lemma 6.1. Choose a max-cut k-partition $V(G) = A_1 \cup \cdots \cup A_k$. Assume that $|A_k| \leqslant |A_i|$ for all $i \in [k-1]$. Define

$$Z_i := \{z \in A_i : d_{\overline{G}}(z, \overline{A_i}) \geqslant \xi n\} \quad \text{for } i \in [k],$$
$$Z := Z_1 \cup \cdots \cup Z_k.$$

We need to show that G has an $(A_1, \ldots, A_k; Z, \beta, \xi, \xi, \delta)$-partition, that is, that P1(G)–P5(G) hold with the appropriate parameters.

Let $p := k$; $d := \rho_0 n^2$ and $\rho := \rho_0$. Then $p^2 \leqslant d \leqslant \rho n^2$ and, using (6.3), $2\rho^{1/6} \leqslant (k-1)\alpha \leqslant 1 - (k-1)c$. We can apply Lemma 5.1 with parameters d, p and ρ, using the k-partition returned by Lemma 6.4 that has $k-1$ parts of size $\lfloor cn \rfloor$. Lemma 5.1 implies that P1(G) holds for $(A_1, \ldots, A_k)$ with parameter $2k^2\sqrt{d}/n \leqslant 2k^2\sqrt{\rho_0}$ and hence with parameter β.

For P2(G), let $ij \in \binom{[k-1]}{2}$ and let $x \in A_i$ and $y \in A_j$. Then Lemma 5.1(iv) implies that

$$P_3(xy, G) \leqslant n - |A_i| - |A_j| + d_G(x, A_i) + d_G(y, A_j)$$

$$\overset{P1(G)}{\leqslant} n - 2(c-\beta)n + 2\rho_0^{1/5}n$$
$$\overset{(6.3)}{\leqslant} (k-2)cn - (\sqrt{2\alpha} - 2\beta - 2\rho_0^{1/5})n < (k-2)cn - \sqrt{\alpha}n.$$

Thus (5.5) implies that $xy \in E(G)$. So P2(G) holds. Lemma 5.1(ii) implies that

$$m = \sum_{ij \in \binom{[k]}{2}} e(\overline{G}[A_i, A_j]) \leqslant 3k^2\sqrt{\rho_0}n^2 < \eta n^2. \tag{6.25}$$

For P3(G), note that $|Z| \leqslant 2m/(\xi n) \leqslant 2\eta n/\xi \leqslant \delta n$. Furthermore, Lemma 5.1(iii) implies that for every $i \in [k]$ and $e \in E(G[A_i])$, there is at least one endpoint x of e with

$$d_{\overline{G}}(x, \overline{A_i}) \geqslant \frac{1}{2}\left(n - (k-1)cn - 3k^2\sqrt{\rho_0}n\right) \overset{(6.3)}{\geqslant} \frac{(k-1)\alpha n}{3} > \xi n.$$

Thus $x \in Z$. The final part of P3(G) follows from Lemma 5.1(iv) and the fact that $\rho_0 \ll \delta$.

We now prove P4(G). Let $z \in Z \cap A_k$ be arbitrary. By the definition of Z, there is some $i \in [k-1]$ such that $d_{\overline{G}}(z, A_i) \geqslant \xi n/k$. Let $j \in [k-1] \setminus \{i\}$ and $y \in A_j$ be arbitrary. We have

$$\begin{aligned}
P_3(zy, G) &\leqslant d_G(y, A_j) + d_G(z, A_k) \\
&\qquad + d_G(z, A_i) + (n - |A_i| - |A_j| - |A_k|) \\
&\overset{P1(G),P3(G)}{\leqslant} 2\delta n + (c+\beta)n - \xi n/k + ((k-3)c + 3\beta)n \\
&\leqslant (k-2)cn - \xi n/(2k).
\end{aligned}$$

Thus (5.5) implies that $xy \in E(G)$. This proves P4(G).

The property P5(G) holds immediately from the definition of Z.

The bound on m claimed in the lemma was established in (6.25). Finally, Lemma 5.1(v) implies that $h \leqslant k\rho_0^{1/30}m \leqslant \delta m$. □

6.3. Applying Lemma 6.1. Let G be a worst counterexample, that is, G satisfies (C1)–(C3). Let $A_1, \dots, A_k$ be a max-cut partition of G satisfying (C3). Assume that $|A_k| = \min_{i \in [k]} |A_i|$. Until the end of Section 8, we fix the $(A_1, \dots, A_k; Z, \beta, \xi, \xi, \delta)$-partition of G obtained from applying Lemma 6.1 to G and $A_1, \dots, A_k$ using the parameters in (5.1). Let $\underline{m} = (m_1, \dots, m_{k-1})$ be the missing vector of this partition and let

$$m := m_1 + \cdots + m_{k-1} \leqslant \eta n^2. \tag{6.26}$$

By permuting $A_1, \dots, A_{k-1}$ if necessary, we may assume that $m_{k-1} = \max_{i\in[k-1]} m_i$. (This assumption will not be used until the proof of Lemma 8.2.) Further,

$$h := \sum_{i\in[k]} e(G[A_i]) \leqslant \delta m. \tag{6.27}$$

Define

$$t := \frac{m}{(kc-1)n} \overset{(6.3)}{\geqslant} \frac{m}{cn}. \quad \text{Then} \quad t^2 \overset{(6.3)}{\leqslant} \frac{m^2}{2\alpha n^2} \overset{(6.26)}{\leqslant} \frac{\eta m}{2\alpha} \overset{(5.1)}{\leqslant} \sqrt{\eta}m. \tag{6.28}$$

Since P5(G) holds with both γ_1 and γ_2 set to the same value ξ, this uniquely determines the set Z as

$$Z = \bigcup_{i\in[k]} \left\{z \in A_i : d_{\overline{G}}(z, \overline{A_i}) \geqslant \xi n\right\}. \tag{6.29}$$

For all $i \in [k]$, let

$$Z_i := A_i \cap Z \quad \text{and} \quad R_i := A_i \setminus Z. \tag{6.30}$$

By P3(G), R_i is an independent set for all $i \in [k]$. By P2(G) and P5(G), for each $i \in [k-1]$, every $z \in Z_i$ has $d_{\overline{G}}(z, A_k) \geqslant \xi n$. Note that, by P4$(G)$, the set Z_k has a partition $Z_k^1 \cup \cdots \cup Z_k^{k-1}$ such that, for all $ij \in \binom{[k-1]}{2}$ we have that $G[Z_k^i, A_j]$ is complete. In particular, each vertex in Z_k^i sends at least ξn missing edges to A_i. Thus we have for all $i \in [k-1]$

$$|Z_i \cup Z_k^i| \leqslant \frac{2m_i}{\xi n} \quad \text{and} \quad |Z| \leqslant \frac{2(m_1 + \cdots + m_{k-1})}{\xi n} = \frac{2m}{\xi n} \overset{(6.26)}{\leqslant} \sqrt{\eta}n. \tag{6.31}$$

For each $i \in [k-1]$, let

$$Y_i := \{y \in Z_k^i : d_G(y, A_i) \leqslant \gamma n\}, \quad Y := \bigcup_{i\in[k-1]} Y_i \tag{6.32}$$

$$X_i := Z_k^i \setminus Y_i, \quad \text{and} \quad X := \bigcup_{i\in[k-1]} X_i.$$

See Figure 2 for an illustration. In the proof, we will perform various transformations on G, which will mainly involve changing adjacencies at vertices in Y and X. It turns out that vertices in X are much harder to deal with than those in Y, and much of the proof is devoted to these troublesome vertices.

We need a simple proposition before we start with the first main ingredient of the proof in Section 7.

PROPOSITION 6.12. *The following hold in G:*

(i) *Suppose that $xy \in E(G[A_k])$ and $x \in R_k$. Then $y \in Y$.*

(ii) *For all $ij \in \binom{[k-1]}{2}$, we have that $G[Y_i, Y_j]$ is complete.*

Proof. For (i), first note that $d_{\overline{G}}(x, \overline{A_k}) < \xi n$ by P5(G) since x is in $R_k = A_k \setminus Z$. Next, P3(G) implies that $y \in Z_k$. By P4(G), there is $i \in [k-1]$ such that $y \in Z_k^i$. Using (5.5) and that $G[Z_k^i, A_j]$ is complete for every $j \in [k-1] \setminus \{i\}$, we have that

$$\begin{aligned}(k-2)cn + k \geqslant P_3(xy, G) &\overset{P1(G), P5(G)}{\geqslant} \sum_{j \in [k-1] \setminus \{i\}} |A_j| + d_G(y, A_i) - \xi n \\ &\overset{P1(G)}{\geqslant} (k-2)(c-\beta)n + d_G(y, A_i) - \xi n\end{aligned}$$

and so $d_G(y, A_i) \leqslant (k\beta + \xi)n < \gamma n$. Thus $y \in Y$.

To prove (ii), let $y \in Y_i$ and $x \in Y_j$. Then

$$\begin{aligned}P_3(xy, G) &\leqslant \sum_{\substack{t \in [k-1] \\ t \neq i,j}} |A_t| + d_G(y, A_i) + d_G(x, A_j) + \max_{z \in Y} d_G(z, A_k) \\ &\overset{P1, P3(G)}{\leqslant} (k-3)(c+\beta)n + 2\gamma n + \delta n \leqslant (k-2)cn - cn/2.\end{aligned}$$

Thus (5.5) implies that $xy \in E(G)$. □

7. The intermediate case: transformations

The aim of this section is to prove the following lemma, which enables us to find a k-partite (n, e)-graph G' that inherits many of the useful properties of G but does not contain many more triangles than G (see Figure 7 for an illustration of G'). Let

$$C := \frac{1}{\sqrt{\delta}}. \tag{7.1}$$

LEMMA 7.1. *Suppose that $m \geqslant Cn$. Then there exists an (n, e)-graph G' with $V(G') = V(G)$, which has the following properties.*

(i) *For all $i \in [k-1]$, there exists $U_i \subseteq X_i$ such that, letting $A_i'' := A_i \cup Y_i \cup U_i$ and $A_k'' := V(G) \setminus \bigcup_{i \in [k-1]} A_i''$, the graph G' is k-partite with partition A_1'', $\ldots$, A_k'', and further has an $(A_1'', \ldots, A_k''; 3\beta)$-partition.*

(ii) *The missing vector* $\underline{m}' := (m'_1, \ldots, m'_{k-1})$ *of* G' *with respect to this partition satisfies* $\alpha^2 m_i - 2\sqrt{\delta}m \leqslant m'_i \leqslant 2m_i + 2\sqrt{\delta}m$ *for all* $i \in [k-1]$.

(iii) $K_3(G') \leqslant K_3(G) + \delta^{1/4}m^2/(2n)$.

It is important to note that we do *not* assume $m \geqslant Cn$ in any of the lemmas that precede the proof of Lemma 7.1 in Section 7.7. Indeed, we will require some of these lemmas in both cases $m \geqslant Cn$ and $m < Cn$.

We will obtain a sequence of (n, e)-graphs $G =: G_0, G_1, \ldots, G_6 =: G'$ via a series of transformations such that Transformation i is applied to G_{i-1} to obtain G_i and it preserves the number of edges and vertices: $e(G_{i-1}) = e(G_i)$. For each i, G_i has at most as many bad edges as G_{i-1}, and $K_3(G_i)$ is not much larger than $K_3(G_{i-1})$. The final graph G' is required to have a special partition and a missing vector with the property that each entry is within a constant multiplicative factor of the corresponding entry in G. So each G_i must also have these properties.

Transformation i for $i \in \{1, 2, 3\}$ consists of a 'local' transformation applied to each of a given set of vertices U in turn, producing graphs $G_{i-1} =: G^0_{i-1}, G^1_{i-1}, \ldots, G^{|U|}_{i-1} =: G_i$. We first derive some fairly precise properties of the graph G^j_{i-1}, and then after that we derive the required less precise properties of the graph G_i obtained after the final step. The reason for this is that a single step (that is, obtaining G^1_{i-1} only) is also needed at a later stage in the proof to derive a contradiction.

For all $i \in [k-1]$, we will let

$$a_i := \sum_{j \in [k-1]\setminus\{i\}} |A_j| = n - |A_i| - |A_k|. \tag{7.2}$$

7.1. Vertices with small missing degree. In the sequence of transformations described, we will often want to 'fill in' some missing edges, and thus we must remove some edges from another part of the graph to compensate. It will be useful if we have a fairly large stockpile of such edges that somehow exhibit average behaviour, and this property is preserved even after removing many of these well-behaved edges. For this reason, we define $Q_1, \ldots, Q_{k-1}$ and $R'_k \subseteq R_k$ below.

PROPOSITION 7.2. *Let* A_i, R_i, m_i *for* $i \in [k]$ *and* Z *be as in Section 6.3. Let* J *be an* n*-vertex graph with an* $(A_1, \ldots, A_k; Z, 2\beta, \xi/4, 2\xi, \delta)$*-partition and missing vector* $\underline{m}^* = (m^*_1, \ldots, m^*_{k-1})$*, where* $m^*_i \leqslant m_i$ *for all* $i \in [k-1]$*. Then, for all* $i \in [k-1]$*, there exists* $Q_i \subseteq J[R_i, R_k]$ *such that* Q_i *is a collection of* $2\delta n$ *edge-disjoint stars, each with a distinct centre in* A_k *and with* δn *leaves; and*

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the centre of each star has missing degree at most $2\sqrt{\eta}n$. (In particular, for all $e \in Q_i$, we have $P_3(e, J) \geqslant \sum_{j\in[k-1]\setminus\{i\}} |A_j| - 2\sqrt{\eta}n$.)

Proof. Let $R_k^* \subseteq R_k$ consist of vertices with missing degree at least $2\sqrt{\eta}n$ in J. Then

$$|R_k^*| \leqslant \frac{\sum_{i\in[k-1]} m_i^*}{2\sqrt{\eta}n} \leqslant \frac{m}{2\sqrt{\eta}n} \overset{(6.26)}{\leqslant} \frac{\sqrt{\eta}n}{2}.$$

By P1,P3(J), we have that $|R_i| \geqslant (c - 2\beta)n - |Z| \geqslant (c - 3\beta)n$ for every $i \in [k-1]$ and $|R_k \setminus R_k^*| \geqslant (1 - (k-1)c - 4\beta)n \geqslant 2\delta n \cdot (k-1)$. Thus, each Q_i can be chosen by picking a distinct set of $2\delta n$ vertices in $R_k \setminus R_k^*$ along with δn of each one's R_i-neighbours (of which there are at least $(c - \beta - 2\xi)n$ by P1,P3(J)). □

Let $R_k' \subseteq R_k$ be such that $|R_k'| = |R_k| - \xi n/2$ and $d_G(x', Z_k) \leqslant d_G(x, Z_k)$ for all $x' \in R_k'$ and $x \in R_k \setminus R_k'$. Let also

$$\Delta := \max_{x\in R_k'} d_G(x, Z_k) = \max_{x\in R_k'} d_G(x, A_k), \tag{7.3}$$

where the second inequality follows from P3(G). By P3(G) and (6.27),

$$2\delta m \geqslant 2e(G[A_k]) \geqslant \sum_{x\in R_k\setminus R_k'} d_G(x, A_k) \geqslant (|R_k| - |R_k'|)\Delta = \frac{\xi n}{2} \cdot \Delta.$$

Therefore every $x \in R_k'$ is such that

$$d_G(x, A_k) \leqslant \Delta \leqslant \frac{4\delta m}{\xi n} \leqslant \frac{\delta^{1/3} m}{n}. \tag{7.4}$$

7.2. Transformation 1: removing bad edges in $A_1, \ldots, A_{k-1}$. Our first goal is to obtain a graph G_1 from G, which has the property that $G_1[A_i]$ is independent for all $i \in [k-1]$ and G_1 does not contain many more triangles than G. The following lemma concerns the local transformation of removing all bad edges incident to a single $z \in Z \setminus Z_k$ and replacing them with certain missing edges incident to z (see the left-hand image in Figure 3).

LEMMA 7.3. *Let $p := |Z \setminus Z_k|$ and let $z_1, \ldots, z_p$ be any ordering of $Z \setminus Z_k$. For each $r \in [p]$, let $s(r)$ be such that $z_r \in A_{s(r)}$. Then there exists a sequence $G =: G^0, G^1, \ldots, G^p =: G_1$ of graphs such that for all $j \in [p]$, we have the following:*

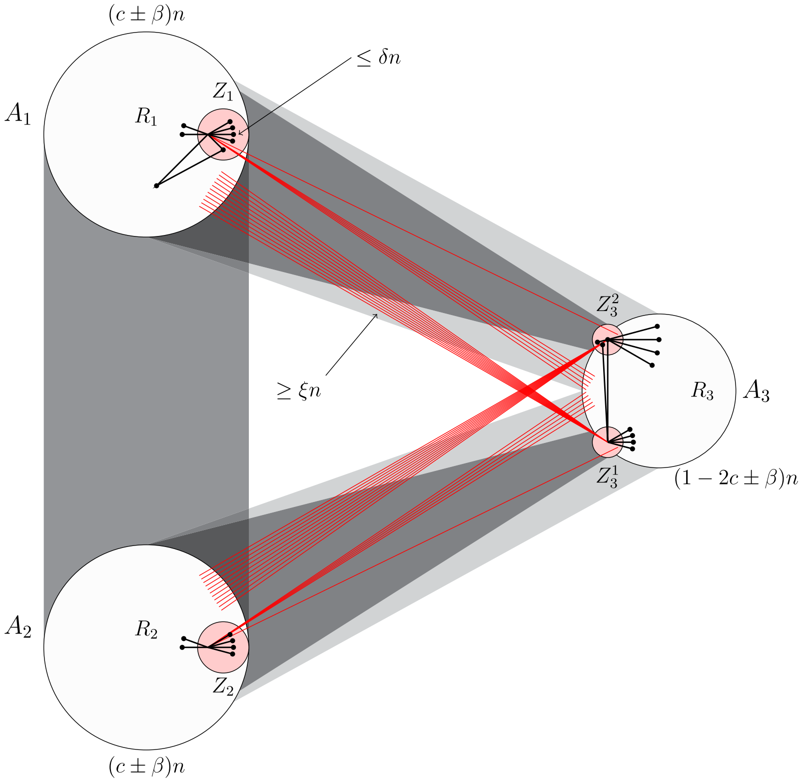

Figure 2. An $(A_1, A_2, A_3; Z, \beta, \xi, \xi, \delta)$-partition of G (here $k = 3$). Here and in the other figures, dark grey represents a complete bipartite pair, and light grey represents an 'almost complete' bipartite pair, in which each vertex has a small missing degree. The red edges are missing edges, and Z is also coloured (light) red.

J(1,j): *G^j is an (n, e)-graph and has an $(A_1, \ldots, A_k; Z, \beta, \xi/2, \xi, \delta)$-partition.*

J(2,j): *$E(G^j) \setminus E(G^{j-1}) = \{z_j x : x \in R(z_j)\}$ for some $R(z_j) \subseteq R'_k$, and $E(G^{j-1}) \setminus E(G^j)$ is the set of $xz_j \in E(G)$ with $x \in A_{s(j)} \setminus \{z_1, \ldots, z_{j-1}\}$.*

J(3,j): *$K_3(G^j) - K_3(G^{j-1}) \leqslant \sum_{y \in N_{G^{j-1}}(z_j, A_{s(j)})} (\Delta - |Z_k \setminus Z_k^{s(j)}| - P_3(yz_j, G^{j-1}; R_k))$. Furthermore, equality holds only if $G^{j-1}[N_{G^j \setminus G^{j-1}}(z_j, R_k), \bigcup_{i \in [k-1] \setminus \{s(j)\}} A_i]$ is complete.*

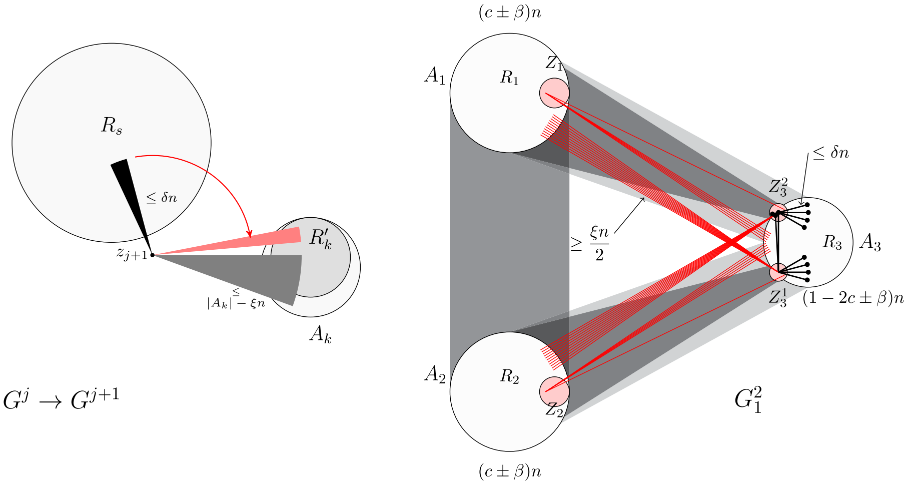

Figure 3. Transformation 1: $G \to G_1^2$ (here $k = 3$). Left: A single step $G^j \to G^{j+1}$ as in Lemma 7.3, in which the black edges are replaced by the pink edges. Right: The final graph G_1^2 obtained in Lemma 7.4, in which A_1 and A_2 are now independent sets.

Remark. The combined properties of Lemma 7.3 state that each G^j is obtained from the previous graph G^{j-1} by replacing all current edges connecting z_j to its part with the same number of new edges between z_j and R'_k. Thus $d_{G^j}(z_t, A_{s(t)}) = 0$ for all $t \in [j]$; $e(\overline{G^j}[A_i, A_k]) = e(\overline{G^{j-1}}[A_i, A_k])$ for all $i \neq s(j)$, and $e(\overline{G^j}[A_{s(j)}, A_k]) = e(\overline{G^{j-1}}[A_{s(j)}, A_k]) - d_{G^{j-1}}(z_j, A_{s(j)})$.

Proof of Lemma 7.3. Let $G^0 := G$. Suppose we have obtained $G^0, \ldots, G^j$ for some $j < p$ such that, for all $r \leqslant j$, properties J$(1, r)$–J$(3, r)$ hold. For $g \in [3]$, let J(g) denote the conjunction of J$(g, 1), \ldots,$J(g, j). We obtain G^{j+1} as follows. Let $s := s(j+1)$. Choose $R(z_{j+1}) \subseteq R'_k \setminus N_{G^j}(z_{j+1})$ such that $|R(z_{j+1})| = d_{G^j}(z_{j+1}, A_s)$. Let us first see why this is possible. One consequence of J(2) is that the neighbourhood of z_{j+1} in G^j is obtained from its neighbourhood in G by removing its G-neighbours among $\{z_1, \ldots, z_j\} \cap A_s$. Thus, as $|R'_k| = |R_k| - \xi n/2$, we have

$$\begin{aligned} d_{\overline{G^j}}(z_{j+1}, R'_k) &\overset{J(2)}{=} d_{\overline{G}}(z_{j+1}, R'_k) \geqslant d_{\overline{G}}(z_{j+1}, A_k) - |Z_k| - \xi n/2 \\ &\overset{P5(G)}{\geqslant} \xi n/2 - \delta n \geqslant \delta n \\ &\overset{P3(G)}{\geqslant} d_G(z_{j+1}, A_s) \overset{J(2)}{\geqslant} d_{G^j}(z_{j+1}, A_s). \end{aligned}$$

So $R(z_{j+1})$ exists. Now define G^{j+1} by setting $V(G^{j+1}) := V(G^j)$ and

$$E(G^{j+1}) := \big(E(G^j) \cup \{z_{j+1}x : x \in R(z_{j+1})\}\big) \setminus E(G^j[z_{j+1}, A_s]).$$

Thus G^{j+1} is obtained by replacing all bad edges of G^j that are incident with z_{j+1} by the same number of missing edges of G^j that are incident to z_{j+1}. The endpoints x of these new edges are chosen in R'_k to ensure that the number of new triangles created is not too large.

We will now show that G^{j+1} satisfies $J(1, j+1), \ldots, J(3, j+1)$, beginning with $J(1, j+1)$. By construction, G^{j+1} is an (n, e)-graph. To show that G^{j+1} has an $(A_1, \ldots, A_k; Z, \beta, \xi/2, \xi, \delta)$-partition, we need to show that P1(G^{j+1})–P5(G^{j+1}) hold with the appropriate parameters. All properties except P5(G^{j+1}) are immediate. For P5, let $i \in [k]$ and let $y \in A_i$ be arbitrary. We have that

$$d^m_{G^{j+1}}(y) = \begin{cases} d^m_{G^j}(y) - 1 & \text{if } y \in R(z_{j+1}), \\ d^m_{G^j}(y) - d_{G^j}(z_{j+1}, A_s) & \text{if } y = z_{j+1}, \\ d^m_{G^j}(y) & \text{otherwise.} \end{cases} \tag{7.5}$$

Thus if $y \in A_i \setminus Z$, we have $d^m_{G^{j+1}}(y) \leqslant d^m_{G^j}(y) \leqslant \xi n$ since G^j has an $(A_1, \ldots, A_k; Z, \beta, \xi/2, \xi, \delta)$-partition. It remains to consider the case $y = z_{j+1}$ (since missing degree is unchanged for all other vertices in Z). By the consequence of $J(2)$ stated above,

$$d^m_{G^j}(z_{j+1}) = d^m_G(z_{j+1}) \text{ and } d_{G^j}(z_{j+1}, A_s) = d_G(z_{j+1}, A_s \setminus \{z_1, \ldots, z_j\}). \tag{7.6}$$

Thus, as G has an $(A_1, \ldots, A_k; Z, \beta; \xi, \xi, \delta)$-partition,

$$d^m_{G^j}(z_{j+1}) \geqslant \xi n - d_G(z_{j+1}, A_s \setminus \{z_1, \ldots, z_j\}) \overset{P3(G)}{\geqslant} (\xi - \delta)n \geqslant \xi n/2.$$

Thus P5(G^{j+1}) holds. We have shown that $J(1, j+1)$ holds. That $J(2, j+1)$ holds is clear from $J(2)$ and the construction of G^{j+1}.

For J(3, $j+1$), observe that a triangle is in G^{j+1} but not G^j if and only if it contains an edge xz_{j+1}, where $x \in R(z_{j+1})$; furthermore, no triangle contains two such edges; and a triangle is in G^j but not G^{j+1} if and only if it contains an edge yz_{j+1}, where $y \in N_{G^j}(z_{j+1}, A_s)$. Thus

$$\begin{aligned} K_3(G^{j+1}) = K_3(G^j) &+ \sum_{x \in R(z_{j+1})} P_3(xz_{j+1}, G^{j+1}) \\ &- \sum_{y \in N_{G^j}(z_{j+1}, A_s)} P_3(yz_{j+1}, G^j; \overline{A_s}) - K_3(z_{j+1}, G^j; A_s). \end{aligned} \tag{7.7}$$

Fix $y \in N_{G^j}(z_{j+1}, A_s)$. By J(1,j), P2($G^j$) holds and, since $y, z_{j+1} \in A_s$, both of these vertices are incident to all of $A_t \cup Z_k^t$ for $t \in [k-1] \setminus \{s\}$. Recall the definition of a_s from (7.2). So

$$\begin{aligned} P_3(yz_{j+1}, G^j; \overline{A_s}) &= a_s + |Z_k \setminus Z_k^s| + P_3(yz_{j+1}, G^j; R_k \cup Z_k^s) \\ &\geqslant a_s + |Z_k \setminus Z_k^s| + P_3(yz_{j+1}, G^j; R_k). \end{aligned}$$

Now fix $x \in R(z_{j+1}) \subseteq R'_k$. Then, by $J(2, j+1)$, we have $d_{G^{j+1}}(z_{j+1}, A_s) = 0$ and $d_{G^{j+1}}(x, R_k) = d_G(x, R_k) = 0$. So

$$\begin{aligned} P_3(xz_{j+1}, G^{j+1}) &= a_s - d_{\overline{G^j}}\Big(x, \bigcup_{i \in [k-1] \setminus \{s\}} A_i\Big) + P_3(xz_{j+1}, G^{j+1}; Z_k) \\ &\leqslant a_s + d_{G^{j+1}}(x, Z_k) \overset{J(2)}{=} a_s + d_G(x, Z_k) \overset{(7.3)}{\leqslant} a_s + \Delta. \end{aligned} \tag{7.8}$$

Therefore,

$$K_3(G^{j+1}) - K_3(G^j) \overset{(7.7),(7.8)}{\leqslant} \sum_{y \in N_{G^j}(z_{j+1}, A_s)} \left(\Delta - |Z_k \setminus Z_k^s| - P_3(yz_{j+1}, G^j; R_k)\right),$$

where equality holds only when equality in (7.8) holds for every $x \in R(z_{j+1})$. This happens only if $d_{\overline{G^j}}(x, \bigcup_{i \in [k-1] \setminus \{s\}} A_i) = 0$ for every $x \in R(z_{j+1})$; in other words, $G^j[R(z_{j+1}), \bigcup_{i \in [k-1] \setminus \{s\}} A_i]$ is complete. Recall that $R(z_{j+1}) = N_{G^{j+1} \setminus G^j}(z_{j+1}, R_k)$. This completes the proof of $J(3, j+1)$. □

We can now derive some properties of $G_1 := G^p$ obtained in Lemma 7.3, namely that its only bad edges have endpoints in A_k and G_1 does not have many more triangles than G. In fact, we consider the graph G_1^ℓ, which is obtained by applying Lemma 7.3 for only vertices $z_j \in Z_1 \cup \cdots \cup Z_\ell$. See the right-hand side of Figure 3 for an illustration of G_1^2 in the case $k = 3$.

LEMMA 7.4. *Let $\ell \in [k-1]$. There exists an (n, e)-graph G_1^ℓ on the same vertex set as G such that we have the following:*

(i) *G_1^ℓ has an $(A_1, \ldots, A_k; Z, \beta, \xi/2, \xi, \delta)$-partition with missing vector $\underline{m}^{(1,\ell)} := (m_1^{(1,\ell)}, \ldots, m_{k-1}^{(1,\ell)})$, where $m_i/2 \leqslant m_i^{(1,\ell)} \leqslant m_i$ for all $i \in [k-1]$.*

(ii) *$E(G_1^\ell[A_i]) = \emptyset$ for all $i \in [\ell]$, and $E(G_1^\ell[A_i]) = E(G[A_i])$ otherwise.*

(iii) *$K_3(G_1^\ell) \leqslant K_3(G) + \delta^{7/8} m^2/n$.*

(iv) *$N_{G_1^\ell}(z) = N_G(z)$ for all $z \in Z_k$ and $N_{G_1^\ell}(x, A_k) = N_G(x, A_k)$ for all $x \in A_k$.*

Proof. Let $p := |Z \setminus Z_k|$ and let $p' := |Z_1 \cup \cdots \cup Z_\ell| \leqslant p$. Let $z_1, \ldots, z_p$ be an ordering of $Z \setminus Z_k$ such that for $1 \leqslant i < i' \leqslant k-1$, every vertex in Z_i appears before any vertex in $Z_{i'}$. Apply Lemma 7.3 to obtain $G_1^\ell := G^{p'}$ satisfying $J(1, p'), \ldots, J(3, p')$. By J(1,$p'$), G_1^ℓ has an $(A_1, \ldots, A_k; Z, \beta, \xi/2, \xi, \delta)$-partition. Further, J(2) (defined at the beginning of the proof of Lemma 7.3) implies that, for $i \in [\ell]$,

$$\sum_{\substack{j \in [p'] \\ s(j)=i}} d_{G^{j-1}}(z_j, A_i) = \sum_{\substack{j \in [p'] \\ s(j)=i}} d_G(z_j, A_i \setminus \{z_1, \ldots, z_{j-1}\}) = e(G[A_i]). \tag{7.9}$$

If $i \in [k-1] \setminus [\ell]$, then $m_i^{(1,\ell)} = m_i$. If $i \in [\ell]$, then

$$\begin{aligned} m_i^{(1,\ell)} &= e(\overline{G^{p'}}[A_i, A_k]) \overset{J(2,p')}{=} e(\overline{G}[A_i, A_k]) - \sum_{\substack{j \in [p'] \\ s(j)=i}} d_{G^{j-1}}(z_j, A_i) \\ &\overset{(7.9)}{=} m_i - e(G[A_i]) \overset{P3(G)}{\geqslant} m_i - |Z_i| \cdot \delta n \geqslant m_i - |Z_i| \cdot \frac{\xi n}{4} \overset{P5(G)}{\geqslant} \frac{m_i}{2} \end{aligned}$$

while clearly $m_i^{(1,\ell)} \leqslant m_i$, proving (i). Part (ii) follows immediately from $J(2)$.

Equation (6.27) states that $\sum_{i \in [k]} e(G[A_i]) \leqslant \delta m$. Therefore

$$\begin{aligned} K_3(G_1^\ell) - K_3(G) &= \sum_{j \in [p']} \left(K_3(G^j) - K_3(G^{j-1}) \right) \overset{J(3)}{\leqslant} \sum_{j \in [p']} d_{G^{j-1}}(z_j, A_{s(j)}) \cdot \varDelta \\ &\overset{(7.9)}{=} \sum_{i \in [\ell]} e(G[A_i]) \cdot \varDelta \overset{(7.4)}{\leqslant} \delta m \cdot \frac{4\delta m}{\xi n} \leqslant \frac{\delta^{7/8} m^2}{n}. \end{aligned}$$

Finally, part (iv) follows from J(2). □

7.3. Transformation 2: removing Y_i-A_i edges. The next transformation is applied to G_1^ℓ to obtain a graph that inherits the properties of G_1^ℓ whilst also reassigning Y_i to A_i and removing any edges that are bad relative to this new partition. The only bad edges that remain are incident to X in A_k. Observe that the $(A_1, \ldots, A_k; Z, \beta, \xi/2, \xi, \delta)$-partition of G_1^ℓ is also an $(A_1, \ldots, A_k; Z, 2\beta, \xi/4, 2\xi, \delta)$-partition.

LEMMA 7.5. *Let $\ell \in [k-1]$ and let G_1^ℓ be any graph satisfying the conclusion of Lemma 7.4 applied with ℓ. Let $q = q(\ell) := |Y_1 \cup \cdots \cup Y_\ell|$ and let $y_1, \ldots, y_q$ be an arbitrary ordering of $Y_1 \cup \cdots \cup Y_\ell$. For all $j \in [q]$, let $s(j) \in [k-1]$ be such that $y \in Y_{s(j)}$. Let $A_i^0 := A_i$ for $i \in [k]$. Let $Q_i^0 := Q_i$ be obtained by*

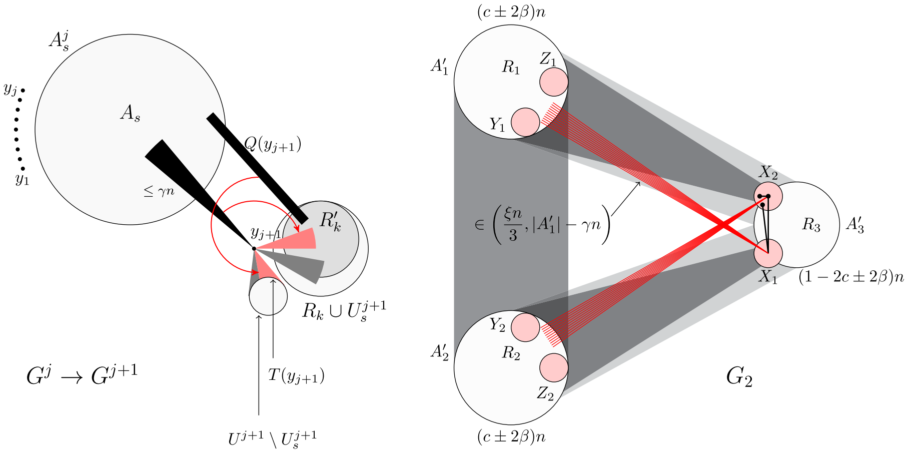

Figure 4. Transformation 2: $G_1 \to G_2$ (here $k = 3$). Left: A single step $G^j \to G^{j+1}$ as in Lemma 7.5, in which the two sets of black edges are replaced by the corresponding sets of pink edges. Right: The final graph G_2^2 obtained in Lemma 7.6, with the updated partition A'_1, A'_2, A'_3.

applying Proposition 7.2 to the graph $J := G_1^\ell$ and the partition $(A_1^0, \dots, A_k^0)$, for all $i \in [k-1]$. For all $j \in [q]$, let

$$A_t^j := \begin{cases} A_t^{j-1} \cup \{y_j\} & \text{if } t = s(j), \\ A_t^{j-1} \setminus \{y_j\} & \text{if } t = k, \\ A_t^{j-1} & \text{otherwise,} \end{cases} \tag{7.10}$$

and $U^j := Z_k \cap A_k^j$ and $U^{j,i} := Z_k^i \cap A_k^j$ for every $i \in [k-1]$. Then there exists a sequence $G_1^\ell =: G^0, G^1, \dots, G^q =: G_2^\ell$ of graphs such that for all $j \in [q]$, we have the following:

K(1, j):
- *$E(G^j) \setminus E(G^{j-1})$ is a star with centre y_j, where the set of leaves consists of $T(y_j)$ together with some vertices in R'_k, where $T(y_j)$ is the set of non-G^{j-1}-neighbours of y_j in $U^{j-1} \setminus U^{j-1,s(j)}$.*
- *$E(G^{j-1}) \setminus E(G^j) = \{y_j v \in E(G) : v \in A_{s(j)}^{j-1}\} \cup Q(y_j)$, where $Q(y_j) \subseteq Q_{s(j)}^{j-1}$ and $|Q(y_j)| \leqslant \delta n$.*
- *If $Z_k = X_{s(j)} \cup Y_{s(j)}$, then $T(y_j) = Q(y_j) = \emptyset$.*
- *The total number of cross-edges in G^j is at least that in G^0, that is,*

$$\sum_{ip \in \binom{[k]}{2}} e(G^j[A_i^j, A_p^j]) \geqslant \sum_{ip \in \binom{[k]}{2}} e(G^0[A_i^0, A_p^0]).$$

Define $Q_i^j := Q_i^{j-1} \setminus Q(y_j)$ *for all* $i \in [k-1]$.

K(2, j): G^j *is an* (n, e)*-graph and has an* $(A_1^j, \ldots, A_k^j; Z, \beta + \frac{j}{n}, \frac{\xi}{2} - \frac{j}{n}, \xi + 2\delta + \frac{j}{n}, \delta)$*-partition, where* $U^{j,1}, \ldots, U^{j,k-1}$ *is the partition of* $U^j := Z \cap A_k^j$ *given by P4(*G^j*).*

K(3, j):

$$K_3(G^j) - K_3(G^{j-1}) \leqslant \sum_{y \in N_{G^{j-1}}(y_j, A_{s(j)}^{j-1})} \left(\Delta - \frac{\xi}{6\gamma} |U^{j-1} \setminus U^{j-1,s(j)}| - P_3(yy_j, G^{j-1}; R_k) \right).$$

Furthermore, equality holds only if $G^{j-1}[N_{G^j \setminus G^{j-1}}(y_j, R_k), \bigcup_{i \in [k-1] \setminus \{s(j)\}} A_i^{j-1}]$ *is complete.*

Proof. Let $G^0 := G_1^\ell$. Note that $s(r) \leqslant \ell$ for every $r \in [q]$. Suppose that we have obtained $G^0, \ldots, G^j$ for some $j < q$ such that, for all $r \leqslant j$, properties $K(1, r)$–$K(3, r)$ hold. For $g \in [3]$, let $K(g)$ denote the conjunction of properties $K(g, 1), \ldots, K(g, j)$. Let $s := s(j+1)$. By definition, $U^j \setminus U^{j,s} = (Z_k \setminus Z_k^s) \setminus \{y_1, \ldots, y_j\}$. Recall that

$$T(y_{j+1}) = N_{\overline{G^j}}(y_{j+1}, U^j \setminus U^{j,s}).$$

We obtain G^{j+1} as follows. Choose a set $R(y_{j+1})$ of $d_{G^j}(y_{j+1}, A_s^j)$ vertices in $R_k' \setminus N_{G^j}(y_{j+1})$. Note that $R_i \subseteq A_i^r$ for all $0 \leqslant r \leqslant j$ and $i \in [k]$ by (7.10). Choose a set $Q(y_{j+1}) \subseteq Q_s^j$ of size $|T(y_{j+1})|$ with

$$V(Q(y_{j+1})) \cap R_k \subseteq N_{\overline{G^j}}(y_{j+1}). \tag{7.11}$$

Note that if $Z_k = X_s \cup Y_s$, then by definition $U^j \setminus U^{j,s} = \emptyset$. Therefore, $T(y_{j+1}) = Q(y_{j+1}) = \emptyset$. Now define G^{j+1} by setting $V(G^{j+1}) := V(G^j)$ and

$$E(G^{j+1}) := \big(E(G^j) \cup \{y_{j+1}x : x \in R(y_{j+1})\} \cup \{y_{j+1}z : z \in T(y_{j+1})\} \big) \setminus \big(E(G^j[y_{j+1}, A_s^j]) \cup Q(y_{j+1}) \big).$$

So G^{j+1} is obtained from G^j by replacing every neighbour of y_{j+1} in A_s^j with a nonneighbour in R_k'; and moving some previously unused edges from Q_s to lie between y_{j+1} and those nonneighbours in $Z_k \setminus Z_k^s$ that lie in A_k^j (see the left-hand side of Figure 4 for an illustration of the transformation $G^j \to G^{j+1}$).

Let us check that G^{j+1} exists, that is, one can choose the sets $R(y_{j+1})$ and $Q(y_{j+1})$ with the stated properties. Recall that G and G_1^ℓ agree on Y due to

Lemma 7.4(iv). Thus by Proposition 6.12(ii), $G_1^\ell[Y_i, Y_j]$ is complete for all $ij \in \binom{[k-1]}{2}$. Consequently, $T(y_r) \cap Y = \emptyset$ for all $1 \leqslant r \leqslant j$; in other words, no edge incident to $\{y_{j+1}, \ldots, y_q\}$ was modified when we passed from G^0 to G^j. This implies that

$$N_{G^j}(y_{j+1}) = N_{G_1^\ell}(y_{j+1}) \supseteq \bigcup_{i \in [k-1]\setminus\{s\}} (A_i \cup Y_i). \tag{7.12}$$

As $A_s^j = A_s \cup \{y_r : r \leqslant j; s(r) = s\}$, together with (7.12), this implies that $N_{G^j}(y_{j+1}, A_s^j) \subseteq N_{G_1^\ell}(y_{j+1}, A_s) \cup Y$. Since $|Y| \leqslant |Z| \leqslant \delta n$ by P3(G), we have from Lemma 7.4(iv) that

$$d_{G^j}(y_{j+1}, A_s^j) \leqslant d_{G_1^\ell}(y_{j+1}, A_s) + \delta n = d_G(y_{j+1}, A_s) + \delta n \leqslant (\gamma + \delta)n \leqslant 2\gamma n. \tag{7.13}$$

Thus

$$\begin{aligned} d_{\overline{G^j}}(y_{j+1}, R_k') &\overset{K(1)}{=} d_{\overline{G_1^\ell}}(y_{j+1}, R_k') \geqslant |A_k| - |Z_k| - \xi n/2 - d_{G_1^\ell}(y_{j+1}, A_k) \\ &\overset{P3(G_1^\ell), P5(G_1^\ell)}{\geqslant} |A_k| - (\xi/2 + 2\delta)n \overset{P1(G_1^\ell), (6.3)}{\geqslant} 2\gamma n \geqslant d_{G^j}(y_{j+1}, A_s^j). \end{aligned}$$

So we can choose $R(y_{j+1})$ as required. Also, by $K(1)$ and Lemma 7.4(iv), $N_{G^j}(y_{j+1}, R_k) = N_G(y_{j+1}, R_k)$, which is of size at most δn by P3(G). Thus

$$|V(Q_s) \cap N_{\overline{G^j}}(y_{j+1}, R_k)| \geqslant |V(Q_s) \cap R_k| - |N_{G^j}(y_{j+1}, R_k)| \geqslant 2\delta n - \delta n = \delta n.$$

Recall that $|Y| \leqslant |Z| \leqslant \sqrt{\eta} n$ by (6.31), and Q_s consists of $2\delta n$ stars each with δn leaves centred at R_k. Thus the number of available edges in Q_s^j (that is, all edges in $Q_s \setminus \bigcup_{\ell \in [j]} Q(y_\ell)$ whose endpoints in R_k are not adjacent to y_{j+1}) is at least

$$\delta n(\delta n - |Y|) \geqslant \delta n \geqslant |Z| \geqslant |U^j| \geqslant d_{\overline{G^j}}(y_{j+1}, U^j \setminus U^{j,s}) = |T(y_{j+1})| = |Q(y_{j+1})|,$$

so we can choose the desired $Q(y_{j+1}) \subseteq Q_s^j$. Hence G^{j+1} exists.

Recall that the sets A_t^{j+1}, $t \in [k]$, were defined in (7.10). It remains to check that $K(1, j+1)$–$K(3, j+1)$ hold. The first three points in Property $K(1, j+1)$ follow immediately from the construction. To see the last point, note that from G^0 to G^{j+1}, the cross-edges that are no longer present are precisely those in $Q(y_r)$ and $E(G^{r-1}[y_r, A_{s(r)}^{r-1}])$, which are compensated by $\{xy_r : x \in T(y_r)\}$ and $\{xy_r : x \in R(y_r)\}$, respectively, for every $1 \leqslant r \leqslant j+1$. In fact, G^{j+1} will have more cross-edges than G^0 if there are $G[A_k]$-edges incident to $\{y_1, \ldots, y_{j+1}\}$.

To check that G^{j+1} has an

$$(A_1^{j+1}, \ldots, A_k^{j+1}; Z, \beta + (j+1)/n, \xi/2 - (j+1)/n, \xi + 2\delta + (j+1)/n, \delta)$$

-partition, we need to show that P1(G^{j+1})–P5(G^{j+1}) hold with the required parameters. For P1(G^{j+1}), the part sizes $|A_t^{j+1}|$, $|A_t^j|$ differ by at most one. So for $t \in [k-1]$ we have

$$\left||A_t^{j+1}|-cn\right| \leqslant \left||A_t^{j+1}|-|A_t^j|\right|+\left||A_t^j|-cn\right| \leqslant \left(\beta+\frac{j}{n}\right)n+1=\left(\beta+\frac{j+1}{n}\right)n,$$

as required. The case $t=k$ is similar.

By P2(G^j) we have that $G^j[A_i^j, A_p^j]$ is complete for all $ip \in \binom{[k-1]}{2}$. Thus, for P2(G^{j+1}), we need only check that $xy_{j+1} \in E(G^{j+1})$ for all $x \in A_i^{j+1}$ with $i \in [k-1] \setminus \{s\}$. Indeed, if $i \in [k-1] \setminus \{s\}$, then $A_i^{j+1} = A_i^j = A_i \cup \{y_r : r \leqslant j; s(r) = i\}$ and, by (7.12) and Lemma 7.4(iv), $N_{G^j}(y_{j+1}) \supseteq A_i^{j+1}$. Finally, note that by construction, $N_{G^{j+1}}(y_{j+1}, A_i^{j+1}) = N_{G^j}(y_{j+1}, A_i^{j+1})$.

Note that P3(G^{j+1}) holds by P3(G_1^ℓ) and K(1). For P4(G^{j+1}), it suffices to show that, for all $ip \in \binom{[k-1]}{2}$, the bipartite graph $G^{j+1}[U^{j+1,i}, A_p^{j+1}]$ is complete. By P4(G^j) and $K(2, j)$, we have that $G^j[U^{j,i}, A_p^j]$ is complete. For $i, p \neq s$, this means that $G^j[U^{j+1,i}, A_p^{j+1}]$ is complete. But G^j and G^{j+1} are identical between these two sets by construction, so we are done in this case. Suppose instead that $i = s$. Then note that $U^{j+1,s} = U^{j,s} \setminus \{y_{j+1}\}$ and $A_p^{j+1} = A_p^j$, so we are done as $G^j[U^{j,s}, A_p^j]$ is complete and G^{j+1} is identical in this part. Suppose finally that $p = s$. Then $U^{j+1,i} = U^{j,i}$ and $G^j[U^{j,i}, A_s^{j+1} \setminus \{y_{j+1}\}]$ is complete. Thus, it suffices to show that $U^{j+1,i} = U^{j,i} \subseteq N_{G^{j+1}}(y_{j+1})$. But this is immediate by construction. So P4(G^{j+1}) holds with $U^{j+1,i}$ playing the role of U_k^i. We now turn to P5(G^{j+1}). In what follows, $d_{G^r}^m$ is the missing degree with respect to the partition $(A_1^r, \ldots, A_k^r)$. Let $y \in V(G^{j+1})$. We have by construction that

$$d^m_{G^{j+1}}(y) = \begin{cases} |A_k^{j+1}| - d_{G^j}(y, A_k^j) - d_{G^j}(y, A_s^j) & \\ \quad -|Q(y_{j+1})| & \text{if } y = y_{j+1}, \\ d^m_{G^j}(y) + d_{Q(y_{j+1})}(y) - 1 & \text{if } y \in N_{\overline{G^j}}(y_{j+1}, A_s^j), \\ d^m_{G^j}(y) + d_{Q(y_{j+1})}(y) + 1 & \text{if } y \in N_{\overline{G_j}}(y_{j+1}, U^{j+1,s}) \setminus R(y_{j+1}), \\ d^m_{G^j}(y) + d_{Q(y_{j+1})}(y) & \text{otherwise.} \end{cases} \tag{7.14}$$

If $y \in Z \setminus \{y_{j+1}\}$, then y is isolated in $\bigcup_{i\in[k-1]} Q_i$ and hence in $Q(y_{j+1})$. So $d^m_{G^{j+1}}(y) \geqslant d^m_{G^j}(y)$. Thus we are done by P5(G^j) in this case. If $y \notin Z$, then, using $\Delta(\bigcup_{i\in[k-1]} Q_i) \leqslant 2\delta n$ from Proposition 7.2 and $Q(y_1), \ldots, Q(y_{j+1})$ are edge-disjoint, we have

$$d^m_{G^{j+1}}(y) \overset{(7.14)}{\leqslant} d^m_{G_1^\ell}(y) + \Delta\Big(\bigcup_{i\in[k-1]} Q_i\Big) + j + 1 \overset{P5(G_1^\ell)}{\leqslant} (\xi + 2\delta)n + j + 1,$$

as required. Moreover, by $K(1)$ and P3(G_1^ℓ), $d_{G^j}(y_{j+1}, A_k^j) \leqslant d_{G_1^\ell}(y_{j+1}, A_k) \leqslant \delta n$. Using (7.13) and (7.14), we have

$$\begin{aligned} d^m_{G^{j+1}}(y_{j+1}) &= |A_k^{j+1}| - d_{G^j}(y_{j+1}, A_k^j) - d_{G^j}(y_{j+1}, A_s^j) - |Q(y_{j+1})| \\ &\geqslant |A_k| - |Y| - 2\delta n - 2\gamma n \\ &\overset{P1(G_1^\ell)}{\geqslant} n - (k-1)cn - \beta n - 3\delta n - 2\gamma n \\ &\overset{(6.3)}{\geqslant} \alpha n > \xi n/2 - (j+1). \end{aligned} \tag{7.15}$$

Thus P5(G^{j+1}) holds. This completes the proof of $K(2, j+1)$.

Finally, we will show $K(3, j+1)$. For every $p \in [k-1]$ and $q \in [j+1]$, let

$$a_p^q := \sum_{t \in [k-1]\setminus\{p\}} |A_t^q|.$$

Then by (7.10), $a_s^j = a_s^{j+1}$. Observe that a triangle is in G^{j+1} but not G^j if and only if it contains an edge xy_{j+1}, where $x \in R(y_{j+1})$ or $x \in (Z_k \setminus Z_k^s) \cap A_k^j$ is a nonneighbour of y_{j+1} in G^j (this is precisely the set $T(y_{j+1})$); and a triangle is in G^j but not G^{j+1} if and only if it contains an edge uy_{j+1}, where $u \in N_{G^j}(y_{j+1}, A_s^j)$, or an edge $e \in Q(y_{j+1})$. Observe that there is no triangle in G^j that contains at least two edges from $E(G^j) \setminus E(G^{j+1})$. Indeed, this follows from (7.11) and the fact that $E(G^j[A_s^j]), E(G^j[R_k]) = \emptyset$ (due to $s \leqslant \ell$, Lemma 7.4(ii) and $K(1)$). Thus

$$\begin{aligned} K_3(G^{j+1}) - K_3(G^j) &\leqslant \sum_{e \in E(G^{j+1})\setminus E(G^j)} P_3(e, G^{j+1}) - \sum_{e \in E(G^j)\setminus E(G^{j+1})} P_3(e, G^j) \\ &\leqslant \sum_{x \in R(y_{j+1})} P_3(xy_{j+1}, G^{j+1}) - \sum_{y \in N_{G^j}(y_{j+1}, A_s^j)} P_3(yy_{j+1}, G^j) \\ &\quad + \sum_{z \in T(y_{j+1})} P_3(zy_{j+1}, G^{j+1}) - \sum_{e \in Q(y_{j+1})} P_3(e, G^j). \end{aligned}$$

We will estimate each summand separately. Let $y \in N_{G^j}(y_{j+1}, A_s^j)$. By $K(1, j+1)$ and the definition of $T(y_{j+1})$, we have that

$$\begin{aligned} P_3(yy_{j+1}, G^j) &\geqslant a_s^j + d_{G^j}(y_{j+1}, U^j \setminus U^{j,s}) + P_3(yy_{j+1}, G^j; R_k) \\ &= a_s^j + |U^j \setminus U^{j,s}| - |T(y_{j+1})| + P_3(yy_{j+1}, G^j; R_k). \end{aligned}$$

Now let $x \in R(y_{j+1})$. Then $d_{G^{j+1}}(y_{j+1}, A_s^j) = 0$ and $x \in R_k'$, so

$$P_3(xy_{j+1}, G^{j+1}) \leqslant a_s^j - d_{\overline{G^j}}(x, \bigcup_{i \in [k-1]\setminus\{s\}} A_i^j) + d_{G^{j+1}}(x, A_k^{j+1})$$

$$\leqslant a_s^j + d_G(x, A_k) \leqslant a_s^j + \Delta, \tag{7.16}$$

where we used Lemma 7.4(iv) to replace $d_{G_1^\ell}(x, A_k)$ by $d_G(x, A_k)$. Let $z \in T(y_{j+1})$. Let $t \in [k-1]\setminus\{s\}$ be such that $z \in Z_k^t$. Then, since $d_{G^{j+1}}(y_{j+1}, A_s^j) = 0$ and each of y_{j+1}, z has at most δn neighbours in A_k and $|A_t^{j+1}| = |A_t^j| \geqslant |A_t|$,

$$\begin{aligned} P_3(zy_{j+1}, G^{j+1}) &\leqslant \sum_{p\in[k-1]\setminus\{s,t\}} |A_p^j| + d_{G^{j+1}}(z, A_t^j) + d_{G^{j+1}}(z, A_k^{j+1}) \\ &\overset{P3(G^{j+1}),P5(G^{j+1})}{\leqslant} a_s^j - \xi n/2 + j + 1 + \delta n \leqslant a_s^j - \xi n/2 + 2\delta n. \end{aligned}$$

Let now $xy \in Q(y_{j+1})$, where $x \in R_s$ and $y \in R_k$. As $Q_s^0 \supseteq Q(y_{j+1})$, Proposition 7.2 implies that $P_3(xy, G_1^\ell) \geqslant a_s - 2\sqrt{\eta}n$. Then by $K(1)$,

$$P_3(xy, G^j) \geqslant P_3(xy, G_1^\ell) \geqslant a_s^j - |Y| - 2\sqrt{\eta}n \geqslant a_s^j - 2\delta n.$$

Before we upper bound $K_3(G^{j+1}) - K_3(G^j)$, we need some preliminary estimates. Let a, b, p be nonnegative integers such that $b \leqslant a$ and $p \leqslant 2\gamma n$. We claim that

$$\left(\frac{\xi a}{6\gamma} - b\right) p \leqslant \frac{\xi n}{3}(a-b). \tag{7.17}$$

Indeed, if $\frac{\xi a}{6\gamma} - b < 0$, then it trivially holds as $a \geqslant b$. Otherwise, $(\frac{\xi a}{6\gamma} - b)p \leqslant (\frac{\xi a}{6\gamma} - b)2\gamma n \leqslant \frac{\xi n}{3}(a-b)$ as desired.

Observe that $|U^j \setminus U^{j,s}|, d_{G^j}(y_{j+1}, U^j \setminus U^{j,s}), d_{G^j}(y_{j+1}, A_s^j)$ satisfy the conditions on a, b, p, respectively. Indeed, by Lemma 7.4(iv), $K(2)$ and the definition of Y, we have that

$$d_{G^j}(y_{j+1}, A_s^j) \leqslant d_{G_1^\ell}(y_{j+1}, A_s) + |Y| \overset{P3(G)}{\leqslant} 2\gamma n.$$

Now,

$$\begin{aligned} &K_3(G^{j+1}) - K_3(G^j) \\ &\quad\leqslant \sum_{y\in N_{G^j}(y_{j+1}, A_s^j)} \Big(\Delta - (|U^j \setminus U^{j,s}| - |T(y_{j+1})|) - P_3(yy_{j+1}, G^j; R_k)\Big) \\ &\qquad - |T(y_{j+1})| \cdot \xi n/3 \\ &\quad= d_{G^j}(y_{j+1}, A_s^j)\left(\Delta - d_{G^j}(y_{j+1}, U^j \setminus U^{j,s})\right) \\ &\qquad - \sum_{y\in N_{G^j}(y_{j+1}, A_s^j)} P_3(yy_{j+1}, G^j; R_k) \\ &\qquad - \left(|U^j \setminus U^{j,s}| - d_{G^j}(y_{j+1}, U^j \setminus U^{j,s})\right)\frac{\xi n}{3} \end{aligned}$$

$$\overset{(7.17)}{\leqslant} d_{G^j}(y_{j+1}, A_s^j)\left(\Delta - \frac{\xi}{6\gamma}|U^j \setminus U^{j,s}|\right)$$
$$- \sum_{y\in N_{G^j}(y_{j+1},A_s^j)} P_3(yy_{j+1}, G^j; R_k)$$
$$= \sum_{y\in N_{G^j}(y_{j+1},A_s^j)} \left(\Delta - \frac{\xi}{6\gamma}|U^j \setminus U^{j,s}| - P_3(yy_{j+1}, G^j; R_k)\right).$$

Observe that equality above holds only when equality in (7.16) holds. This happens only if $d_{\overline{G^j}}(x, \bigcup_{i\in[k-1]\setminus\{s\}} A_i^j) = 0$ for every $x \in R(y_{j+1})$; in other words, $G^j[R(y_{j+1}), \bigcup_{i\in[k-1]\setminus\{s\}} A_i^j]$ is complete. Recall that $R(y_{j+1}) = N_{G^{j+1}\setminus G^j}(y_{j+1}, R_k)$. So $K(3, j+1)$ holds. □

We can now derive some properties of the graph $G_2 := G_2^{k-1}$ obtained in Lemma 7.5, namely that its only bad edges have both endpoints in X, and G_2 does not have many more triangles than G_1. See the right-hand side of Figure 4 for an illustration of G_2 in the case $k = 3$. For all $i \in [k-1]$, we will let $A_i' := A_i \cup Y_i$ and

$$a_i' := \sum_{j\in[k-1]\setminus\{i\}} |A_j'| = n - |A_i'| - |A_k'|. \tag{7.18}$$

LEMMA 7.6. *There exists an (n, e)-graph G_2 on the same vertex set as $G_1 := G_1^{k-1}$ such that we have the following:*

(i) *G_2 has an $(A_1', \dots, A_k'; Z, 2\beta, \xi/3, 2\xi, \delta)$-partition with missing vector $\underline{m}^{(2)} = (m_1^{(2)}, \dots, m_{k-1}^{(2)})$, where $A_i' := A_i \cup Y_i$ for $i \in [k-1]$ and $A_k' := A_k \setminus Y = R_k \cup X$; also, $\alpha m_i^{(1)} \leqslant m_i^{(2)} \leqslant 2m_i^{(1)}$ for all $i \in [k-1]$.*

(ii) *If there are $i \in [k]$ and $xy \in E(G_2[A_i'])$, then $i = k$; furthermore, $x, y \in X$ and $xy \in E(G[A_k'])$.*

(iii) *For every $i \in [k-1]$ and every $z \in X_i$, we have that $d_{G_2}(z, A_i') \geqslant \gamma n$.*

(iv) *$K_3(G_2) \leqslant K_3(G_1) + \delta^{1/4}m^2/(3n)$.*

Proof. Let $q := |Y|$ and apply Lemma 7.5 to obtain $G_2 := G^q = G_2^{k-1}$ satisfying $K(1, q)$–$K(3, q)$. Write $\underline{m}^{(1)} = (m_1^{(1)}, \dots, m_{k-1}^{(1)})$. For $g \in [3]$, let $K(g)$ be the conjunction of the properties $K(g, 1)$–$K(g, q)$. Observe that $A_i' = A_i^q$ for all $i \in [k]$. Now $|Y| \leqslant |Z| \leqslant \delta n$, and so $q/n \leqslant \delta$. Thus, by $K(1, q)$, G_2 has an $(A_1', \dots, A_k'; Z, \beta+\delta, \xi/2-\delta, \xi+3\delta, \delta)$-partition and hence an $(A_1', \dots, A_k'; Z, 2\beta, \xi/3, 2\xi, \delta)$-partition.

Now, by $K(1)$,

$$\begin{aligned} m_i^{(2)} &= e(\overline{G_2}[A_i', A_k']) = e(\overline{G^q}[A_i \cup Y_i, A_k \setminus Y]) \\ &= e(\overline{G^q}[A_i, A_k \setminus Y] + \sum_{y \in Y_i} d_{\overline{G^q}}(y, A_k \setminus Y) \\ &= e(\overline{G_1}[A_i, A_k \setminus Y]) + \sum_{\substack{j \in [q] \\ s(j)=i}} |Q(y_j)| + \sum_{y \in Y_i} d_{\overline{G^q}}(y, A_k \setminus Y) \\ &= m_i^{(1)} - \sum_{y \in Y_i} \big(d_{\overline{G_1}}(y, A_i) - d_{\overline{G^q}}(y, A_k \setminus Y) \big) + \sum_{\substack{j \in [q] \\ s(j)=i}} |Q(y_j)|. \end{aligned}$$

Note further, using $|Y| \leqslant |Z| \leqslant \delta n$ by P3(G), that

$$\begin{aligned} &\sum_{y \in Y_i} \big(d_{\overline{G_1}}(y, A_i) - d_{\overline{G^q}}(y, A_k \setminus Y) \big) = \sum_{y \in Y_i} \big(d^m_{G_1}(y) - d^m_{G^q}(y) \big) \\ &\quad \leqslant \sum_{\substack{j \in [q] \\ s(j)=i}} \big(|A_i| - (d^m_{G^j}(y_j) - |Y|) \big) \overset{(7.15)}{\leqslant} \sum_{\substack{j \in [q] \\ s(j)=i}} (|A_i| - (1 - (k-1)c)n + 3\gamma n) \\ &\quad \overset{P1(G)}{\leqslant} |Y_i|(kc - 1 + 4\gamma)n \overset{(6.3)}{\leqslant} (c - \alpha)|Y_i|n. \end{aligned}$$

A similar calculation shows that the left-hand side is positive. Thus using $K(1)$ for the bound $|Q(y_j)| \leqslant \delta n$, we have $m_i^{(1)} - (c-\alpha)|Y_i|n \leqslant m_i^{(2)} \leqslant m_i^{(1)} + \delta n |Y_i|$. But the definition of Y_i and Lemma 7.4(iv) imply that

$$\begin{aligned} m_i^{(1)} &\geqslant |Y_i| \cdot \min_{y \in Y_i} d_{\overline{G_1}}(y, A_i) = |Y_i| \cdot \min_{y \in Y_i} d_{\overline{G}}(y, A_i) \\ &\overset{P1(G)}{\geqslant} |Y_i| \cdot (c - \beta - \gamma)n \geqslant |Y_i| \cdot (c - 2\gamma)n. \end{aligned}$$

Thus, using the fact that $c \leqslant \frac{1}{k-1} \leqslant \frac{1}{2}$ from (5.6),

$$\alpha \leqslant 1 - \frac{c - \alpha}{c - 2\gamma} \leqslant \frac{m_i^{(2)}}{m_i^{(1)}} \leqslant 1 + \frac{\delta}{c - 2\gamma} \leqslant 2.$$

This completes the proof of (i).

For (ii), the first part follows from $E(G_1[A_i]) = \emptyset$ due to Lemma 7.4(ii) and $K(1)$. For the second part, suppose $xy \in E(G_2[A_k'])$. Now, $Y \cap A_k' = \emptyset$ and $E(G_2[A_k']) \subseteq E(G_1[A_k])$, so every edge in $E(G_2[A_k'])$ is incident to a vertex of X. So $x \in X$, say. Suppose that $y \notin X$. Then $y \in A_k' \setminus X \subseteq R_k$. So xy is an edge of G_1 and hence of G by Lemma 7.4(iv). This is a contradiction to Proposition 6.12(i). This completes the proof of (ii). For (iii), note that for any $i \in [k-1]$ and any $z \in X_i$, G_1 and G_2 are identical in $[z, A_i]$. Thus, by Lemma 7.4(iv) and the definition of X, we have that $d_{G_2}(z, A'_i) \geqslant d_{G_2}(z, A_i) = d_G(z, A_i) \geqslant \gamma n$, as required.

Finally, for (iv),

$$\begin{aligned} K_3(G_2) - K_3(G_1) &= \sum_{j \in [q]} \left(K_3(G^j) - K_3(G^{j-1})\right) \\ &\overset{K(3,j)}{\leqslant} \sum_{j\in[q]} d_{G^{j-1}}(y_j, A^{j-1}_{s(j)}) \cdot \Delta \\ &\overset{K(1)}{\leqslant} \Delta \cdot \sum_{j\in[q]} (d_{G_1}(y_j, A_{s(j)}) + |Y|) \leqslant \Delta|Z|(\gamma n + |Z|) \\ &\overset{(6.31),(7.4)}{\leqslant} \frac{\delta^{1/3} m}{n} \cdot \frac{2m}{\xi n} \cdot 2\gamma n \leqslant \frac{\delta^{1/4} m^2}{3n}, \end{aligned}$$

as required. □

7.4. Transformation 3: removing bad X_i-X_i edges.

We have obtained a graph G_2 from G, which has the property that every bad edge has both endpoints in X. In the third transformation, we remove those bad edges whose endpoints both lie in X_i for some $i \in [k-1]$. The proof is very similar to the proofs of Lemmas 7.3 and 7.4.

For all $i \in [k-1]$ and $x, y \in X_i$, let

$$D(x) := d_{G_2}(x, X \setminus X_i) \text{ and } D(x, y) := |N_{G_2}(x, X \setminus X_i) \cap N_{G_2}(y, X \setminus X_i)|.$$

So $D(x) - D(x, y) \geqslant 0$ with equality if and only if the G_2-neighbourhood of x in $X \setminus X_i$ is a subset of y's.

LEMMA 7.7. *Let G_2 be any graph satisfying the conditions of Lemma 7.6. Let $f := |X|$ and let $x_1, \ldots, x_f$ be any ordering of X. For each $r \in [f]$, let $s(r)$ be such that $x_r \in X_{s(r)}$. Then there exists a sequence $G_2 =: G^0, G^1, \ldots, G^f =: G_3$ of graphs such that for all $j \in [f]$, we have the following:*

L(1,j): G^j is an (n, e)-graph and has an $(A'_1, \ldots, A'_k; Z, 2\beta, \xi/4, 2\xi, \delta)$-partition.

L(2,j): $E(G^j) \setminus E(G^{j-1}) = \{x_j x : x \in R(x_j)\}$, where $R(x_j) \subseteq R_{s(j)}$, and $E(G^{j-1}) \setminus E(G^j)$ is the set of $x_{j'} x_j \in E(G_2)$ with $s(j') = s(j)$ and $j' > j$. Thus $d_{G^j}(x_t, X_{s(t)}) = 0$ for all $t \in [j]$; $e(\overline{G^j}[A'_i, A'_k]) = e(\overline{G^{j-1}}[A'_i, A'_k])$ for all $i \neq s(j)$, and $e(\overline{G^j}[A'_{s(j)}, A'_k]) = e(\overline{G^{j-1}}[A'_{s(j)}, A'_k]) - d_{G^{j-1}}(x_j, X_{s(j)})$.

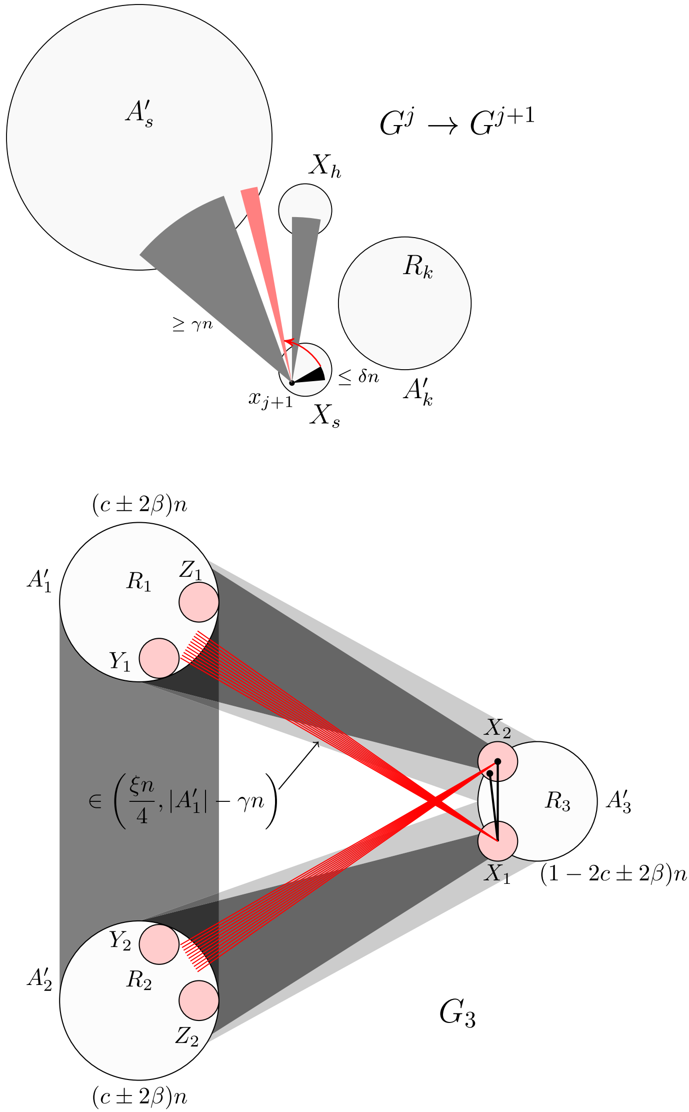

Figure 5. Transformation 3: $G_2 \to G_3$ (here $k = 3$). Top: A single step $G^j \to G^{j+1}$ as in Lemma 7.7, in which the black edges are replaced by the pink edges. Bottom: The final graph G_3 obtained in Lemma 7.8, in which X_1 and X_2 are now independent sets.

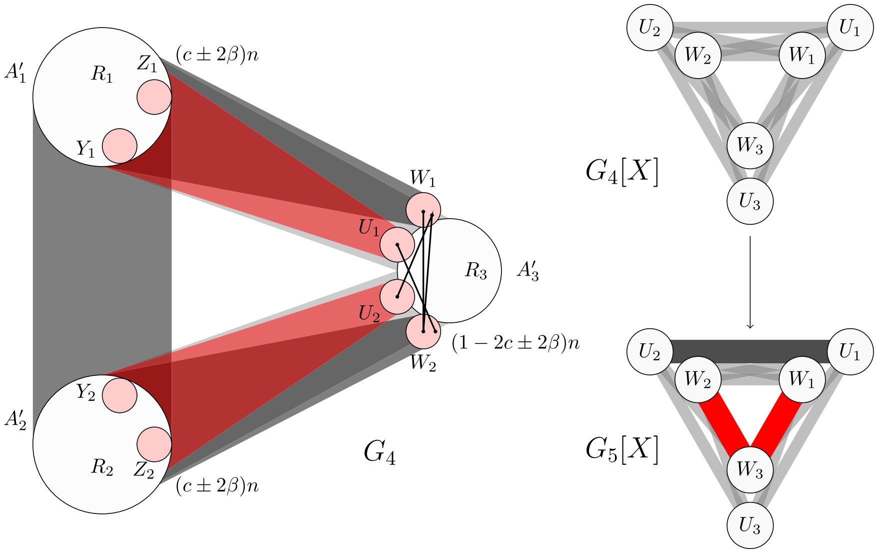

Figure 6. Transformations 4 and 5. Dark grey and red represent (almost) complete/empty bipartite pairs, respectively. Left: G_4 (here $k = 3$). The only bad edges lie in $[U_i, U_j \cup W_j]$ for some $ij \in \binom{[k-1]}{2}$. Right: $G_4 \to G_5$ in the case $k = 4$ and $I_1 = \{12\}$ and $I_2 = \{13, 23\}$.

L(3, j): $K_3(G^j) - K_3(G^{j-1}) \leqslant \sum_{y \in N_{G_2}(x_j, X_{s(j)} \setminus \{x_1,\ldots,x_{j-1}\})}(D(x_j) - D(y, x_j))$ *with equality only if* $K_3(x_j, G^{j-1}; X_{s(j)}) = 0$ *and* $N_{G^{j-1}}(y, A'_{s(j)}) \cap N_{G^{j-1}}(x_j, A'_{s(j)}) = \emptyset$ *for all* $y \in N_{G^{j-1}}(x_j, X_{s(j)})$.

Proof. Let $G^0 := G_2$. Suppose we have obtained $G^0, \ldots, G^j$ for some $j < f$ such that, for all $r \in [j]$, $L(1, r)$–$L(3, r)$ hold. Note that G^0 has an $(A'_1, \ldots, A'_k; Z, 2\beta, \xi/3, 2\xi, \delta)$-partition and hence an $(A'_1, \ldots, A'_k; Z, 2\beta, \xi/4, 2\xi, \delta)$-partition. For $g \in [3]$, let $L(g)$ denote the conjunction $L(g, 1), \ldots, L(g, j)$ of properties. We obtain G^{j+1} as follows. Let $s := s(j+1)$. Choose $R(x_{j+1}) \subseteq R_s \setminus N_{G^j}(x_{j+1}) \subseteq A'_s$ such that $|R(x_{j+1})| = d_{G^j}(x_{j+1}, X_s)$. Let us first see why this is possible. One consequence of $L(2)$ is that the neighbourhood of x_{j+1} in G^j can be obtained from its neighbourhood in $G^0 = G_2$ by removing its G_2-neighbours among $\{x_r : r \leqslant j \text{ and } s(r) = s\}$. Thus

$$d_{\overline{G^j}}(x_{j+1}, R_s) \overset{L(2)}{=} d_{\overline{G_2}}(x_{j+1}, R_s) \geqslant d_{\overline{G_2}}(x_{j+1}, A'_s) - |Z \cap A'_s| \overset{P5(G_2)}{\geqslant} \xi n/3 - \delta n \geqslant \delta n \overset{P5(G_2)}{\geqslant} |Z| \geqslant d_{G^j}(x_{j+1}, X_s).$$

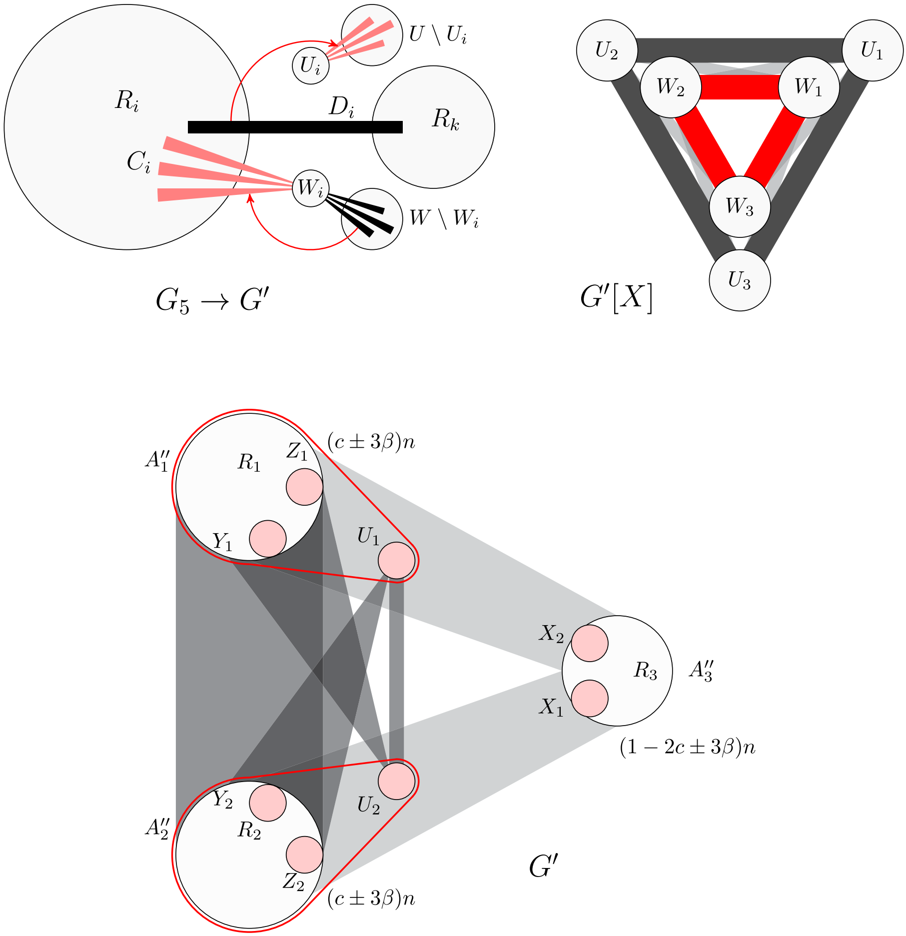

Figure 7. Transformation 6. Top: Transformation 6 at $X_i = U_i \cup W_i$. Bottom: G', in which the redistributed subsets of X are coloured pink (cf. G in Figure 2).

So $R(x_{j+1})$ exists. Now define G^{j+1} by setting $V(G^{j+1}) := V(G^j)$ and

$$E(G^{j+1}) := \big(E(G^j) \cup \{x_{j+1}x : x \in R(x_{j+1})\}\big) \setminus E(G^j[x_{j+1}, X_s]).$$

Thus G^{j+1} is obtained by replacing all bad edges of G^j between x_{j+1} and another vertex in X_s by the same number of missing edges of G^j that are between x_{j+1} and R_s. See the top half of Figure 5 for an illustration of the transformation $G^j \to G^{j+1}$.

We will now show that G^{j+1} satisfies $L(1, j+1), \dots, L(3, j+1)$, beginning with $L(1, j+1)$. By construction, G^{j+1} is an (n, e)-graph. To show that G^{j+1} has an $(A'_1, \dots, A'_k; Z, 2\beta, \xi/4, 2\xi, \delta)$-partition, we need to show that P1(G^{j+1})–P5(G^{j+1}) hold with the appropriate parameters. All properties except P5(G^{j+1}) are immediate. For P5, let $i \in [k]$ and let $y \in A'_i$ be arbitrary. Let $d^m_{G^j}, d^m_{G^{j+1}}$ denote the missing degree in G^j, G^{j+1} with respect to the partition $(A'_1, \dots, A'_k)$. We have that

$$d^m_{G^{j+1}}(y) = \begin{cases} d^m_{G^j}(y) - 1 & \text{if } y \in R(x_{j+1}), \\ d^m_{G^j}(y) - d_{G^j}(x_{j+1}, X_s) & \text{if } y = x_{j+1}, \\ d^m_{G^j}(y) & \text{otherwise.} \end{cases} \tag{7.19}$$

Thus if $y \in A'_i \setminus Z$, we have $d^m_{G^{j+1}}(y) \leqslant d^m_{G^j}(y) \leqslant 2\xi n$ since G^j has an $(A'_1, \dots, A'_k; Z, 2\beta, \xi/4, 2\xi, \delta)$-partition. It remains to consider the case $y = x_{j+1}$ (since missing degree is unchanged for all other vertices in Z). By the consequence of $L(2)$ stated above,

$$d^m_{G^j}(x_{j+1}) = d^m_{G_2}(x_{j+1}) \quad \text{and} \quad d_{G^j}(x_{j+1}, X_s) \leqslant |Z| \leqslant \delta n. \tag{7.20}$$

Thus

$$d^m_{G^{j+1}}(x_{j+1}) \overset{P5(G_2)}{\geqslant} \xi n/3 - \delta n \geqslant \xi n/4.$$

Thus P5(G^{j+1}) holds. We have shown that $L(1, j+1)$ holds. That $L(2, j+1)$ holds is clear from $L(2)$, the construction of G^{j+1} and (7.19).

For $L(3, j+1)$, observe that a triangle is in G^{j+1} but not G^j if and only if it contains an edge xx_{j+1}, where $x \in R(x_{j+1})$; and a triangle is in G^j but not G^{j+1} if and only if it contains an edge yx_{j+1}, where $y \in N_{G^j}(x_{j+1}, X_s)$. Observe also that there is no triangle in G^{j+1} that contains more than one vertex in $R(x_{j+1})$. Thus

$$\begin{aligned} K_3(G^{j+1}) = K_3(G^j) + \sum_{x \in R(x_{j+1})} P_3(xx_{j+1}, G^{j+1}) - \sum_{y \in N_{G^j}(x_{j+1}, X_s)} P_3(yx_{j+1}, G^j; \overline{X_s}) \\ - K_3(x_{j+1}, G^j; X_s). \end{aligned}$$

We will estimate each summand in turn. Fix $y \in N_{G^j}(x_{j+1}, X_s)$. By $L(1, j)$, P2(G^j) holds and, since $y, x_{j+1} \in X_s$, both of these vertices are incident to all of A'_t for $t \in [k-1] \setminus \{s\}$. So

$$\begin{aligned} P_3(yx_{j+1}, G^j; \overline{X_s}) &= a'_s + |N_{G^j}(y, X \setminus X_s) \cap N_{G^j}(x_{j+1}, X \setminus X_s)| \\ &\quad + |N_{G^j}(y, A'_s) \cap N_{G^j}(x_{j+1}, A'_s)| \\ &= a'_s + D(y, x_{j+1}) + |N_{G^j}(y, A'_s) \cap N_{G^j}(x_{j+1}, A'_s)|, \end{aligned} \tag{7.21}$$

where the last equality uses the fact that G^j and G_2 are identical at $[X_s, X \setminus X_s]$ for any $s \in [k-1]$ due to $L(2)$. Now fix $x \in R(x_{j+1})$. Then $d_{G^{j+1}}(x, A'_s) = d_{G_2}(x, A'_s) = 0$ and also $d_{G^{j+1}}(x_{j+1}, X_s) = 0$. By P4($G^{j+1}$), x is incident to every vertex in X_t for $t \neq s$. Recall that $d_{G^{j+1}}(x_{j+1}, R_k) = 0$. Indeed, $E(G[X, R_k]) = \emptyset$ due to Proposition 6.12(i), and it remains empty during the transformations $G \to G_1 \to G_2 \to G^q$ for any $q \in [f]$. Thus

$$\begin{aligned} P_3(xx_{j+1}, G^{j+1}) &= a'_s + P_3(xx_{j+1}, G^{j+1}; A'_k) = a'_s + \sum_{t \in [k-1] \setminus \{s\}} d_{G^{j+1}}(x_{j+1}, X_t) \\ &\overset{L(2)}{=} a'_s + D(x_{j+1}). \end{aligned}$$

Therefore

$$\begin{aligned} &K_3(G^{j+1}) - K_3(G^j) \\ &= \sum_{x \in R(x_{j+1})} P_3(xx_{j+1}, G^{j+1}) - \sum_{y \in N_{G^j}(x_{j+1}, X_s)} P_3(yx_{j+1}, G^j; \overline{X_s}) - K_3(x_{j+1}, G^j; X_s) \\ &\leqslant \sum_{y \in N_{G^j}(x_{j+1}, X_s)} (D(x_{j+1}) - D(y, x_{j+1}) - |N_{G^j}(y, A'_s) \cap N_{G^j}(x_{j+1}, A'_s)|) \\ &\quad - K_3(x_{j+1}, G^j; X_s) \\ &= \sum_{y \in N_{G_2}(x_{j+1}, X_s \setminus \{x_1, \ldots, x_j\})} (D(x_{j+1}) - D(y, x_{j+1}) - |N_{G^j}(y, A'_s) \cap N_{G^j}(x_{j+1}, A'_s)|) \\ &\quad - K_3(x_{j+1}, G^j; X_s), \end{aligned}$$

proving $L(3, j+1)$. □

Again we are now able to derive some properties of $G_3 := G^f$ obtained in Lemma 7.7, namely that every bad edge lies between X_i and X_j for some distinct i, j; and G_3 does not have many more triangles than G_2. The bottom half of Figure 5 shows G_3 in the case when $k = 3$.

LEMMA 7.8. *There exists an (n, e)-graph G_3 on the same vertex set as G_2 such that we have the following:*

(i) *G_3 has an $(A'_1, \ldots, A'_k; Z, 2\beta, \xi/4, 2\xi, \delta)$-partition with missing vector $\underline{m}^3 := (m_1^{(3)}, \ldots, m_{k-1}^{(3)})$ and $m_i^{(2)}/2 \leqslant m_i^{(3)} \leqslant m_i^{(2)}$, where $m_i^{(3)} = m_i^{(2)}$ if and only if $E(G_2[X_i]) = \emptyset$.*

(ii) *If there is $i \in [k]$ and $xy \in E(G_3[A'_i])$, then $i = k$ and there exists $\ell\ell' \in \binom{[k-1]}{2}$ such that $x \in X_\ell$ and $y \in X_{\ell'}$. Moreover, for all $st \in \binom{[k-1]}{2}$, we have $E(G_3[X_s, X_t]) = E(G_2[X_s, X_t])$ and $d_{G_3}(x, A'_i) \geqslant \gamma n$ for all $i \in [k-1]$ and $x \in X_i$.*

(iii) $K_3(G_3) - K_3(G_2) \leqslant |Z|^2 \cdot \max_{\substack{i\in[k-1]\\ x,y\in X_i}} (D(x) - D(x,y))$ *with equality only if for all* $i \in [k-1]$, *we have that* $G_2[X_i]$ *is triangle-free and* $N_{G_2}(x, A'_i) \cap N_{G_2}(y, A'_i) = \emptyset$ *for all* $xy \in E(G_2[X_i])$. *In particular,* $K_3(G_3) - K_3(G_2) \leqslant \sqrt{\delta} m^2/n$.

Proof. Let $f := |X|$ and apply Lemma 7.7 to G_2 to obtain $G_3 := G^f$ satisfying $L(1, f)$–$L(3, f)$. For $g \in [3]$, let $L(g)$ denote the conjunction of properties $L(g, 1)$–$L(g, f)$. By $L(1, f)$, G_3 has an $(A'_1, \ldots, A'_k; Z, 2\beta, \xi/4, 2\xi, \delta)$-partition. Also, for all $i \in [k-1]$,

$$\sum_{\substack{j\in[f]\\ s(j)=i}} d_{G^{j-1}}(x_j, X_i) = \sum_{\substack{j\in[f]\\ s(j)=i}} d_{G_2}(X_i \setminus \{x_1, \ldots, x_{j-1}\}) = e(G_2[X_i]).$$

Thus

$$\begin{aligned} m_i^{(3)} &= e(\overline{G^f}[A'_i, A'_k]) \overset{L(2,f)}{=} e(\overline{G_2}[A'_i, A'_k]) - \sum_{\substack{j\in[f]\\ s(j)=i}} d_{G^{j-1}}(x_j, X_i) \\ &= m_i^{(2)} - e(G_2[X_i]) \overset{P3(G_2)}{\geqslant} m_i^{(2)} - |X_i| \cdot \delta n \geqslant m_i^{(2)} - |X_i| \cdot \frac{\xi n}{6} \overset{P5(G_2)}{\geqslant} \frac{m_i^{(2)}}{2}, \end{aligned}$$

and also $m_i^{(3)} \leqslant m_i^{(2)}$ with equality holds if and only if $E(G_2[X_i]) = \emptyset$. This proves (i).

We now turn to (ii). By $L(2)$ and Lemma 7.6(ii), $E(G_3[A'_t]) = E(G_2[A'_t]) = \emptyset$ if $t \neq k$. Furthermore, $E(G_3[A'_k]) \subseteq E(G_2[A'_k])$. So if G_3 has a bad edge xy, both of its endpoints lie in X. But, for all $r \in [f]$, we have $d_{G^j}(x_r, X_{s(r)}) = 0$ for all $j \geqslant r$. So $E(G_3[X_i]) = \emptyset$ for all $i \in [k-1]$. Note that for any $x \in X_i$ with $i \in [k-1]$, after the transformations $G \to G_1 \to G_2 \to G_3$, we have $N_{G_3}(x, A_i) \supseteq N_G(x, A_i)$. Hence, by the definition of X,

$$d_{G_3}(x, A'_i) \geqslant d_{G_3}(x, A_i) \geqslant d_G(x, A_i) \geqslant \gamma n. \tag{7.22}$$

This proves (ii).

It remains to establish (iii). We have that

$$\begin{aligned} K_3(G_3) - K_3(G_2) &= \sum_{j\in[f]} \big(K_3(G^j) - K_3(G^{j-1})\big) \\ &\leqslant \sum_{j\in[f]} \sum_{y\in N_{G_2}(x_j, X_{s(j)}\setminus\{x_1,\ldots,x_{j-1}\})} (D(x_j) - D(y, x_j)) \\ &\leqslant |Z|^2 \cdot \max_{\substack{i\in[k-1]\\ x,y\in X_i}} (D(x) - D(x,y)) \end{aligned}$$

$$\overset{P3(G_2)}{\leqslant} |Z|^2 \cdot \delta n \overset{(6.31)}{\leqslant} \frac{4\delta m^2}{\xi^2 n} \leqslant \frac{\sqrt{\delta} m^2}{n}.$$

This together with $L(3)$ implies the inequality in (iii). Further, we have equality only if $K_3(x_j, G^{j-1}; X_{s(j)}) = 0$ for all $j \in [f]$, and $|N_{G^{j-1}}(y, A'_s) \cap N_{G^{j-1}}(x_j, A'_s)|$ for all $y \in N_{G^{j-1}}(x_j)$, where $s(j)$ is such that $x_j \in A'_{s(j)}$. This occurs if and only if $G_2[X_i]$ is triangle-free for all $i \in [k-1]$, and $N_{G_2}(x, A'_i) \cap N_{G_2}(y, A'_i) = \emptyset$, as required. □

7.5. Transformation 4: symmetrizing X_i-A'_i edges. Lemma 7.8(ii) implies that $D(x) = \sum_{t \in [k-1] \setminus \{i\}} d_{G_3}(x, X_t)$ for every $x \in X_i$, $i \in [k-1]$. Next we obtain an (n, e)-graph G_4 with the property that, for all $i \in [k-1]$ and all but at most one vertex $x \in X_i$, either $G_4[x, A'_i]$ is empty or it is almost complete (see the left-hand side of Figure 6).

LEMMA 7.9. *There exists an (n, e)-graph G_4 on the same vertex set as G_3 such that we have the following:*

(i) *G_4 has an $(A'_1, \ldots, A'_k; Z, 2\beta, \xi/5, 3\xi, \delta)$-partition; also, G_3 and G_4 can differ only at the union of $[X_i, A'_i]$ for $i \in [k-1]$.*

(ii) *For every $i \in [k-1]$, there exists a partition $X_i = U_i \cup W_i$ (into parts that may be empty) such that $d_{G_4}(w, A'_i) = |A'_i| - \xi n/5$ for all but at most one $w \in W_i$, which has at least $\xi n/5$ nonneighbours in A'_i, and $e(G_4[U_i, A'_i]) = 0$. Further, for all $i \in [k-1]$, if $U_i \neq \emptyset$, then $W_i \neq \emptyset$.*

(iii) *If there is $i \in [k]$ and $xy \in E(G_4[A'_i])$, then $i = k$ and there exists $st \in \binom{[k-1]}{2}$ such that $x \in X_s$ and $y \in X_t$, and further, $xy \in E(G_3[A'_k])$.*

(iv) *$K_3(G_4) \leqslant K_3(G_3)$; and if there exists $i \in [k-1]$ and $x, y \in X_i$ such that $D(x) \neq D(y)$, then $K_3(G_4) \leqslant K_3(G_3) - \xi n/20$.*

(v) *Let $\underline{m}^4 = (m_1^{(4)}, \ldots, m_{k-1}^{(4)})$ be the missing vector of G_4 with respect to $(A'_1, \ldots, A'_k)$. Then $m_i^{(4)} = m_i^{(3)}$ and $|U_i||A'_i| \leqslant m_i^{(4)}$ for all $i \in [k-1]$.*

Proof. Roughly speaking, we will obtain G_4 from G_3 by, for each $i \in [k-1]$, moving all X_i-A_i edges to be incident to vertices $x \in X_i$ such that $D(x)$ is minimal. Let $G^{1,0} := G_3$. For each $i \in [k-1]$, let $f_i := |X_i|$.

Set $i = 1$ and perform the following procedure.

(1) If $X_i = \emptyset$, then let $t_i := 0$ and go to Step (6). Otherwise, let $x_1^i, \ldots, x_{f_i}^i$ be an ordering of X_i such that $D(x_1^i) \leqslant \cdots \leqslant D(x_{f_i}^i)$. Suppose we have constructed $G^{i,0}, \ldots, G^{i,j}$ for some $j \geqslant 0$.

(2) Let $i^+ = i^+(j)$ be the largest $t \in [f_i]$ such that $d_{G^{i,j}}(x^i_t, A'_i) > 0$. Let $i^- = i^-(j)$ be the smallest $s \in [f_i]$ such that $d_{\overline{G^{i,j}}}(x^i_s, A'_i) > \xi n/5$.

(3) If $i^+ \leqslant i^-$, then set $t_i := j$ and go to Step (6).

(4) Choose $x \in N_{G^{i,j}}(x^i_{i^+}, A'_i)$ and $y \in N_{\overline{G^{i,j}}}(x^i_{i^-}, A'_i)$. Let $G^{i,j+1}$ be the graph on vertex set $V(G^{i,j})$ with

$$E(G^{i,j+1}) := E(G^{i,j}) \cup \{x^i_{i^-}, y\} \setminus \{x^i_{i^+}, x\}.$$

(5) Set $j := j + 1$ and go to Step (2).

(6) If $i = k - 1$, set $G_4 := G^{k-1,t_i}$ and STOP. Otherwise, set $G^{i+1,0} := G^{i,t_i}$, then set $i := i + 1$ and go to Step (1).

Observe that, by (7.22) and P5(G_3), for each $i \in [k-1]$ such that $X_i \neq \emptyset$ and for each $x \in X_i$, we have

$$\gamma n \leqslant d_{G_3}(x, A'_i) \leqslant |A'_i| - \xi n/4. \tag{7.23}$$

Thus in $G^{i,0}$, we have $i^+(0) = f_i \geqslant 1 = i^-(0)$. We need to show that the iteration terminates. Indeed, for each fixed $i \in [k-1]$, we have that $i^+ - i^-$ is a nonincreasing function of j, which is bounded above by f_i. Note further that $i^+ - i^-$ remains constant for at most n instances of Steps (2)–(4) since $d_{G^{i,j}}(x^i_{i^+}, A'_i)$ strictly decreases. Thus we reach Step (6) in a finite number t_i of steps for each $i \in [k-1]$. Thus we obtain the final graph G_4 in some finite number $t_1 + \cdots + t_{k-1}$ of steps, as required.

Recall that $E(G_3[X, R_k]) = E(G[X, R_k]) = \emptyset$. Then for all $i \in [k-1]$, $0 \leqslant j \leqslant t_i$, $x \in X_i$ and $u \in A'_i$, we have that

$$P_3(xu, G^{i,j}) = P_3(xu, G_3) = a'_i + d_{G_3}(x, X \setminus X_i) = a'_i + D(x).$$

This follows from the fact that the only edges that change lie between A'_ℓ and X_ℓ for some $\ell \in [k-1]$, and no such edge forms a triangle with xu. Together with the fact that Step (4) happens only when $i^+ > i^-$, we have

$$\begin{aligned} K_3(G^{i,j}) - K_3(G^{i,j-1}) &= P_3(x^i_{i^-}y, G^{i,j}) - P_3(x^i_{i^+}x, G^{i,j-1}) \\ &= D(x^i_{i^-}) - D(x^i_{i^+}) \leqslant 0. \end{aligned} \tag{7.24}$$

We will now prove (i)–(v). Clearly P1(G_4)–P4(G_4) hold with the same parameters. For P5(G_4), note that the missing degree of any $v \in V(G_4) \setminus Z$ changes by at most $|X| \leqslant \delta n$, so P5(G_3) implies that it is at most $3\xi n$, as required. For $i \in [k-1]$, every $v \in A'_i \cap Z$ has gained at most $|X| \leqslant \delta n$ neighbours in

A'_k, so, by P5(G_3), the missing degree of v in G_4 is at least $(\xi/4-\delta)n \geqslant \xi n/5$. For $v \in X_i \subseteq X = A'_k \cap Z$ for some $i \in [k-1]$, it follows from the construction that $d_{\overline{G_4}}(v, A'_i) \geqslant \xi n/5$. The last assertion follows from the construction. This completes the proof of (i).

We now prove (ii). If $X_i \neq \emptyset$, let

$$W_i = \{x^i_1, \ldots, x^i_{i^+(t_i)}\} \quad \text{and} \quad U_i := X_i \setminus W_i. \tag{7.25}$$

Then (ii) holds by construction. Property (iii) also holds by construction.

For (iv), let $\ell := \xi n/20$. Recall that for every $i \in [k-1]$ with $|X_i| \geqslant 2$, we have $i^+(0) = f_i \geqslant 2 > 1 = i^-(0)$. Then (7.23) and $\xi \ll \gamma$ imply that $t_i \geqslant \xi n/4 - \xi n/5 = \ell$ and for any $0 \leqslant j \leqslant \ell - 1$, we have $i^+(j) = f_i$ and $i^-(j) = 1$. Then (7.24) implies that

$$K_3(G_4) - K_3(G_3) = \sum_{\substack{i\in[k-1]\\ X_i\neq\emptyset}} \sum_{j\in[t_i]} \left(K_3(G^{i,j}) - K_3(G^{i,j-1})\right) \leqslant 0.$$

Furthermore, if there are $i \in [k-1]$ and $x, y \in X_i$ such that $D(x) \neq D(y)$, then $D(x^i_1) \leqslant D(x^i_{f_i}) - 1$. Then the observation above shows that in fact

$$\begin{aligned} K_3(G_4) - K_3(G_3) &\leqslant \sum_{0\leqslant j\leqslant \ell-1} \left(K_3(G^{i,j+1}) - K_3(G^{i,j})\right) \\ &\leqslant \ell \cdot (D(x^i_1) - D(x^i_{f_i})) \leqslant -\xi n/20. \end{aligned}$$

Finally, (v) is immediate by construction and the definition of U_i. □

7.6. Transformation 5: replacing $[W_i, W_j]$-edges with $[U_i, U_j]$-edges. The required partition of G' is obtained by moving U_i to A'_i for each $i \in [k-1]$, and for P2(G') to hold, we need that $G'[U_i, U_j]$ is complete. Using the next transformation, we obtain G_5 from G_4 by replacing $[W_i, W_j]$-edges with $[U_i, U_j]$-edges. Thus either we have the required property or $G_5[W_i, W_j]$ is empty. See the right-hand side of Figure 6 for an illustration.

LEMMA 7.10. *There exists an (n, e)-graph G_5 on the same vertex set as G_4 such that we have the following:*

(i) *G_5 has an $(A'_1, \ldots, A'_k; Z, 2\beta, \delta)$-partition.*

(ii) *Every pair $e \in E(G_4) \triangle E(G_5)$ has endpoints $x_s \in X_s$, $x_t \in X_t$ for some $st \in \binom{[k-1]}{2}$.*

(iii) *There is a partition* $I_1 \cup I_2$ *of* $\binom{[k-1]}{2}$ *such that for each* $ij \in I_1$, *we have* $e(\overline{G_5}[U_i, U_j]) = 0$; *and for each* $ij \in I_2$, *we have* $e(G_5[W_i, W_j]) = 0$.

(iv) $K_3(G_5) < K_3(G_4) + k^2\delta n + 2|Z|^3$.

Proof. Obtain a graph G_5 from G_4 as follows. For all $ij \in \binom{[k-1]}{2}$, let

$$f_{ij} := \min\{e(G_4[W_i, W_j]), e(\overline{G_4}[U_i, U_j])\}.$$

Let $F_{ij}^W \subseteq E(G_4[W_i, W_j])$ and $F_{ij}^U \subseteq E(\overline{G_4}[U_i, U_j])$ be such that $|F_{ij}^W| = |F_{ij}^U| = f_{ij}$. Let $V(G_5) := V(G_4)$ and

$$E(G_5) := E(G_4) \cup \bigcup_{ij \in \binom{[k-1]}{2}} F_{ij}^U \setminus \bigcup_{ij \in \binom{[k-1]}{2}} F_{ij}^W.$$

Clearly G_5 is an (n, e)-graph. Parts (i)–(iii) are also clear by construction (to define the partition in (iii), break ties arbitrarily).

It remains to prove part (iv). For this, we need to calculate the P_3-counts for those adjacencies that were changed by passing from G_4 to G_5. Recall from Lemma 7.8(ii) that for any $i \in [k-1]$, if $U_i \neq \emptyset$, then $W_i \neq \emptyset$. Note also that if $U_i = \emptyset$, then the adjacencies involving X_i are the same in G_4 and G_5. Thus, for fixed $ij \in \binom{[k-1]}{2}$, we may assume that $U_i, U_j \neq \emptyset$. Let $w_i \in W_i$ and $w_j \in W_j$ be arbitrary. Suppose that there exists a vertex $w_i' \in W_i$ with $d_{G_4}(w_i', A_i') \geqslant |A_i'| - \xi n/5$. Then, by P4($G_4$), w_i, w_i' are incident to every vertex in A_ℓ' with $\ell \in [k-1]\setminus\{i\}$, and w_j is incident to every vertex in A_ℓ' with $\ell \in [k-1]\setminus\{j\}$. So

$$P_3(w_i'w_j, G_4) \geqslant a_j' - \xi n/5.$$

Also,

$$P_3(w_iw_j, G_4) \geqslant P_3(w_iw_j, G_4; \overline{A_k'}) \geqslant a_i' - |A_j'| \overset{(7.18)}{=} a_j' - |A_i'|. \tag{7.26}$$

Let $u_i \in U_i$ and $u_j \in U_j$. Then $d_{G_5}(u_i, A_i'), d_{G_5}(u_j, A_j') = 0$ (since this holds in G_4), so

$$P_3(u_iu_j, G_5) \leqslant a_i' - |A_j'| + d_{G_4}(u_j, A_k') \overset{P1, P3(G_4)}{\leqslant} a_i' - (c - 2\beta - \delta)n \leqslant a_i' - cn/2. \tag{7.27}$$

Similarly, $P_3(u_iu_j, G_5) \leqslant a_j' - cn/2$. We have shown, for any $w_i \in W_i$, $w_j \in W_j$, $u_i \in U_i$ and $u_j \in U_j$ such that $d_{G_4}(w_\ell, A_\ell') \geqslant |A_\ell'| - \xi n/5$ for at least one $\ell \in \{i, j\}$, that

$$P_3(u_iu_j, G_5) - P_3(w_iw_j, G_4) \leqslant -cn/2 + \xi n/5 < -cn/3.$$

If we arbitrarily order F_{ij}^U as $\overline{e}_1, \ldots, \overline{e}_{f_{ij}}$ and F_{ij}^W as $e_1, \ldots, e_{f_{ij}}$, then we can write

$$K_3(G_5) - K_3(G_4) \leqslant \sum_{ij \in \binom{[k-1]}{2}} \sum_{\ell \in [f_{ij}]} (P_3(\overline{e}_\ell, G_5) - P_3(e_\ell, G_4)) + 2|Z|^3,$$

where $2|Z|^3$ bounds from above the error coming from the triangles in G_4 using at least two edges from $\bigcup_{ij \in \binom{[k-1]}{2}} F_{ij}^W$. Then the only ℓ for which the corresponding summand is potentially greater than $-cn/3$ is such that $e_\ell = w_i w_j$, where $w_t \in W_t$ for $t \in \{i, j\}$ and $d_{G_4}(w_t, A_t') < |A_t'| - \xi n/5$. Given any $u_i \in U_i$ and $u_j \in U_j$, we have in this case

$$P_3(u_i u_j, G_5) - P_3(w_i w_j, G_4) \overset{(7.26),(7.27)}{\leqslant} a_i' - |A_j'| + d_{G_4}(u_j, A_k') - (a_i' - |A_j'|) \leqslant \delta n.$$

But each W_t contains at most one such vertex by Lemma 7.9(ii), so the number of such summands is at most $\binom{k-1}{2}$. Thus we have

$$K_3(G_5) - K_3(G_4) \leqslant k^2 \delta n + 2|Z|^3,$$

proving (iv). □

7.7. Transformation 6 and the proof of Lemma 7.1. A final transformation of G_5 gives us the required graph G'. The transformation does the following. Let I_1, I_2 be defined as in Lemma 7.10. If ij is a pair in I_1, it replaces all $[W_i, W_j]$-edges with some missing edges in $[W_i, R_i]$. If ij is a pair in I_2, then it replaces some edges in $[R_i, R_k]$ with all missing edges in $[U_i, U_j]$. The resulting graph G' (see Figure 7) has the following properties: (i) an edge remains inside A_k' if and only if it is in $[U_i, W_j \cup U_j]$ for some $ij \in \binom{[k-1]}{2}$; (ii) for any $ij \in \binom{[k-1]}{2}$, $G'[U_i, U_j]$ is complete while $G'[W_i, W_j]$ is empty. Thus the new partition obtained by moving U_i to A_i' for all $i \in [k-1]$ satisfies P2.

Proof of Lemma 7.1. Apply Lemmas 7.3–7.10 to obtain (n, e)-graphs $G \to G_1 \to G_2 \to G_3 \to G_4 \to G_5$. We will obtain G' from G_5 as follows. For each $i \in [k-1]$, choose $C_i \subseteq E(\overline{G_5}[R_i, W_i])$ such that $|C_i| = e(G_5[W_i, \bigcup_{i\ell \in I_1 : \ell > i} W_\ell])$, and $D_i \subseteq E(G_5[R_k, R_i])$ such that $|D_i| = e(\overline{G_5}[U_i, \bigcup_{i\ell \in I_2 : \ell > i} U_\ell])$, each D_i is bipartite, and the collection of sets $V(D_i) \cap R_k$ is pairwise-disjoint over $i \in [k-1]$. Let

$$E(G') := \left(E(G_5) \cup \bigcup_{i \in [k-1]} C_i \cup \bigcup_{ij \in \binom{[k-1]}{2}} E(\overline{G_5}[U_i, U_j]) \right) \setminus$$

$$\left(\bigcup_{ij\in\binom{[k-1]}{2}} E(G_5[W_i, W_j]) \cup \bigcup_{i\in[k-1]} D_i \right).$$

So for each $i \in [k-1]$, we remove all $[W_i, W_j]$-edges with $j > i$ and replace them with missing $[R_i, W_i]$-edges (the set C_i); and we add all missing $[U_i, U_j]$-edges with $j > i$ and remove the same number of $[R_k, R_i]$-edges (the set D_i) to compensate (see Figure 7). Write $W = \bigcup_{i\in[k-1]} W_i$ and $U = \bigcup_{i\in[k-1]} U_i$. Observe that

$$\begin{aligned} e\left(G_5\left[W_i, \bigcup_{i\ell\in I_1:\ell>i} W_\ell\right]\right) &\leqslant e(G_5[W_i, W\backslash W_i]) \overset{P3(G_5)}{\leqslant} |W_i|\delta n < |W_i|(\xi/5-\delta)n \\ &\overset{P5(G_4)}{\leqslant} e(\overline{G_4}[W_i, A_i']) - |W_i||Z| \leqslant e(\overline{G_4}[W_i, R_i]) \\ &= e(\overline{G_5}[W_i, R_i]), \end{aligned}$$

where we used Lemma 7.10(ii) for the last equality. So C_i exists. On the other hand,

$$e\left(\overline{G_5}\left[U_i, \bigcup_{i\ell\in I_2:\ell>i} U_\ell\right]\right) \leqslant e(\overline{G_5}[U_i, U \setminus U_i]) \leqslant |Z|^2 \overset{(6.31)}{\leqslant} \eta n^2.$$

Note that, for every $v \in R_k$ and $i \in [k-1]$, we have

$$|R_i| \geqslant d_{G_5}(v, R_i) = d_{G_4}(v, R_i) \overset{P5(G_4)}{\geqslant} |A_i'| - \frac{\xi n}{5} - |Z| \overset{P1(G_4),(6.3)}{\geqslant} |R_k| \geqslant k\sqrt{\eta}n \overset{(6.31)}{\geqslant} k|Z|.$$

Thus we can choose D_i to be the union of stars with distinct centres at R_k and leaves in R_i such that $V(D_i) \cap R_k$ are pairwise-disjoint for all $i \in [k-1]$ as desired. There is no edge that is both added and removed as $W \cap U = \emptyset$, and

$$\begin{aligned} \sum_{ij\in\binom{[k-1]}{2}} e(G_5[W_i, W_j]) &= \sum_{i\in[k-1]} e\left(G_5\left[W_i, \bigcup_{i\ell\in I_1:\ell>i} W_\ell\right]\right) = \sum_{i\in[k-1]} |C_i|, \\ \sum_{ij\in\binom{[k-1]}{2}} e(\overline{G_5}[U_i, U_j]) &= \sum_{i\in[k-1]} e\left(\overline{G_5}\left[U_i, \bigcup_{i\ell\in I_2:\ell>i} U_\ell\right]\right) = \sum_{i\in[k-1]} |D_i|. \end{aligned} \tag{7.28}$$

Thus G' is an (n, e)-graph. By construction, we have the following:

(1) Every edge in $G'[A_k']$ is in $[U_i, W_j \cup U_j]$ for some $ij \in \binom{[k-1]}{2}$; furthermore, $G'[U_1, \ldots, U_{k-1}]$ is complete $(k-1)$-partite.

(2) The edge set of $G'[A'_i]$ is empty for all $i \in [k-1]$ (this follows from Lemmas 7.9(iii) and 7.10(ii) and that G_5 and G' are identical in A'_i for all $i \in [k-1]$).

(3) The edge set of $G'[A'_i, U_i]$ is empty for all $i \in [k-1]$ and the edge set of $G'[A'_j, U_i]$ is complete for all $j \in [k-1] \setminus \{i\}$ (this follows from Lemmas 7.9(ii) and 7.10(ii) and that G_5 and G' are identical in $[A'_i, U_i]$ for all $i \in [k-1]$).

With these observations, we can define the required partition of G' and prove (i). Indeed, let $A''_i := A'_i \cup U_i$ for all $i \in [k-1]$ and $A''_k := A'_k \setminus U$. Properties (1)–(3) imply that A''_i is independent for all $i \in [k]$.

We claim that G' has an $(A''_1, \ldots, A''_k; 3\beta)$-partition, that is, P1$(G')$ and P2(G') hold with the appropriate parameters. For P1(G'), clearly $A''_1, \ldots, A''_k$ is a partition of $V(G')$. Moreover, $\sum_{i \in [k-1]} |U_i| \leqslant |Z| \leqslant \delta n \leqslant \beta n$, so P1$(G_5)$ implies that P1(G') holds with parameter 3β.

For P2(G'), since $G'[A'_i, A'_j] = G_4[A'_i, A'_j]$ for $ij \in \binom{[k-1]}{2}$, it suffices to check that $G'[U_i, A''_j]$ is complete. By P4(G_4), we have that $G'[U_i, A'_j] = G_4[U_i, A'_j]$ is complete. But $G'[U_i, U_j]$ is also complete by Property (1). This proves P2(G'). We have shown that G' has an $(A''_1, \ldots, A''_k; 3\beta)$-partition.

Our next task is to bound the entries in the missing vector $\underline{m}' := (m'_1, \ldots, m'_{k-1})$ of G' with respect to $(A''_1, \ldots, A''_k)$. For each $i \in [k-1]$, we have

$$\begin{aligned} m'_i &= e(\overline{G'}[A''_i, A''_k]) = e(\overline{G'}[A'_i, A'_k \setminus U]) + e(\overline{G'}[U_i, A'_k \setminus U]) \\ &= e(\overline{G'}[A'_i, A'_k]) + e(\overline{G'}[U_i, A'_k \setminus U]) - e(\overline{G'}[U_i, A'_i]), \end{aligned} \tag{7.29}$$

where the last equality follows from $e(\overline{G'}[U, A'_i]) = e(\overline{G'}[U_i, A'_i])$, a consequence of Property (3). By Property (3), $e(\overline{G'}[U_i, A'_i]) = |U_i||A'_i|$. Note also that every transformation from G to G' preserves all adjacencies in $[X, R_k]$ (hence also $[U_i, R_k]$), which is empty in G. Together with $A'_k \setminus U = R_k \cup W$, this implies that

$$|U_i||R_k| \leqslant e(\overline{G'}[U_i, A'_k \setminus U]) \leqslant |U_i||A'_k|.$$

We then derive from (7.29) that

$$e(\overline{G'}[A'_i, A'_k]) - |U_i|(|A'_i| - |R_k|) \leqslant m'_i \leqslant e(\overline{G'}[A'_i, A'_k]) - |U_i|(|A'_i| - |A'_k|). \tag{7.30}$$

Lemma 7.10(ii) says that G_5 has the same number of edges between parts A'_i, A'_j as G_4 for all $1 \leqslant i < j \leqslant k$, and so implies that $e(\overline{G_5}[A'_i, A'_k]) = m_i^{(4)}$ for all $i \in [k-1]$. Then

$$e(\overline{G'}[A'_i, A'_k]) = e(\overline{G_5}[A'_i, A'_k]) - |C_i| + |D_i| = m_i^{(4)} - |C_i| + |D_i|. \tag{7.31}$$

Now, using P3(G_5),

$$\begin{aligned} |C_i| + |D_i| &\leqslant e(G_5[W]) + |U_i||Z| \leqslant e(G_5[A'_k]) + |Z|^2 \\ &\overset{(6.31)}{\leqslant} \frac{2m}{\xi n}(\delta n + \sqrt{\eta} n) \leqslant 2\sqrt{\delta} m. \end{aligned} \tag{7.32}$$

Lemma 7.9(v) implies that $m_i^{(4)} = m_i^{(3)}$ and $m_i^{(4)} \geqslant |U_i||A'_i|$ for all $i \in [k-1]$. Now,

$$|A'_i| - |A'_k| = |A'_i| - |R_k| \pm \delta n = |A'_i| - |A'_k| + |Z| \pm \delta n \overset{P3(G_5),P1(G_5)}{=} (kc-1)n \pm 5\beta n. \tag{7.33}$$

Thus

$$\begin{aligned} m'_i &\overset{(7.30),(7.31)}{\leqslant} m_i^{(4)} - |C_i| + |D_i| - |U_i|(|A'_i| - |A'_k|) \\ &\overset{(7.32),(7.33)}{\leqslant} m_i^{(4)} + 2\sqrt{\delta} m - |U_i|(kc - 1 \pm 5\beta)n \overset{(6.3)}{\leqslant} m_i^{(4)} + 2\sqrt{\delta} m. \end{aligned}$$

In the other direction,

$$\begin{aligned} m'_i &\overset{(7.30),(7.31)}{\geqslant} m_i^{(4)} - |C_i| + |D_i| - |U_i|(|A'_i| - |R_k|) \\ &\geqslant m_i^{(4)} - 2\sqrt{\delta} m - \frac{m_i^{(4)}}{|A'_i|} \cdot (kc - 1 + 5\beta)n \\ &\overset{P1(G_4)}{\geqslant} m_i^{(4)} - 2\sqrt{\delta} m - \frac{m_i^{(4)}}{(c - 2\beta)} \cdot (kc - 1 + 5\beta) \\ &= m_i^{(3)} \cdot \frac{1 - (k-1)c - 7\beta}{c - 2\beta} - 2\sqrt{\delta} m \\ &\overset{(6.3)}{\geqslant} m_i^{(3)} \cdot \frac{(k-1)\alpha - 7\beta}{c - 2\beta} - 2\sqrt{\delta} m. \end{aligned}$$

Then Lemmas 7.4, 7.6 and 7.8(i) imply that $\alpha m_i/4 \leqslant m_i^{(3)} \leqslant 2m_i$; thus,

$$\alpha^2 m_i - 2\sqrt{\delta} m \leqslant \frac{\alpha}{4} \cdot \frac{(k-1)\alpha - 7\beta}{c - 2\beta} \cdot m_i - 2\sqrt{\delta} m \leqslant m'_i \leqslant m_i^{(3)} + 2\sqrt{\delta} m \leqslant 2m_i + 2\sqrt{\delta} m,$$

as required.

It remains to bound $K_3(G') - K_3(G)$. To do so, we will first bound $K_3(G') - K_3(G_5)$. Let $i \in [k-1]$. Let $x_i \in R_i$ and $w_i \in W_i$ be arbitrary. Then $d_{G'}(x_i, A'_i) = d_{G_5}(x_i, A'_i) = 0$ and $d_{G'}(w_i, A'_k) \leqslant |U|$ by Properties (1) and (2). So $P_3(x_i w_i, G') \leqslant a'_i + |U| \leqslant a'_i + \delta n$ and hence

$$\max_{e \in C_i} P_3(e, G') \leqslant a'_i + \delta n. \tag{7.34}$$

Let $w_j \in W_j$ be arbitrary with $j \in [k-1] \setminus \{i\}$. Recall from Lemma 7.9(ii) that all vertices in W_i except at most one special vertex have G_4-degree in A'_i exactly $|A'_i| - \xi n/5$. Let $W' \subseteq W$ be the set of these special vertices from each W_i. Then $|W'| \leqslant k-1$. Further, define $E_{W\setminus W'} := E(G_5[W \setminus W'])$ to be the set of G_5-edges in $W \setminus W'$ and $E_{W'} := E(G_5[W]) - E_{W\setminus W'}$ to be the set of G_5-edges in W with at least one endpoint in W'. Note that

$$|E_{W'}| \leqslant |W'| \cdot |W| < k|Z| \overset{(6.31)}{\leqslant} \frac{2km}{\xi n}. \tag{7.35}$$

By P4(G_4) and the definition of W', we see that

$$P_3(w_i w_j, G_4; \overline{A'_k}) = \sum_{i=1}^{k-1} |A'_i| - 2\xi n/5 \quad \text{for all } w_i w_j \in E_{W\setminus W'}, \tag{7.36}$$

while for any $w_i w_j \in E_{W'}$, (7.26) holds. By Lemma 7.10(ii), for every $w \in W'$, we have $N_{G_5}(w, \overline{A'_k}) = N_{G_4}(w, \overline{A'_k})$, which in turn implies that the bounds in (7.26) and (7.36) hold also for $P_3(w_i w_j, G_5)$, that is,

$$P_3(w_i w_j, G_5) \geqslant a'_i - |A'_j| \overset{(7.18)}{=} a'_j - |A'_i| \quad \text{and} \tag{7.37}$$
$$P_3(w_i w_j, G_5; \overline{A'_k}) = \sum_{i=1}^{k-1} |A'_i| - 2\xi n/5 \quad \text{for all } w_i w_j \in E_{W\setminus W'}.$$

Let $x_k \in R_k$ and $y_i \in R_i$. By P2(G_5) (that is, Lemma 7.10(i)), $G_5[y_i, A'_\ell]$ is complete for all $\ell \in [k-1] \setminus \{i\}$. Moreover, Lemma 7.10(ii) implies that $d_{\overline{G_5}}(x_k, \overline{A'_k}) = d_{\overline{G_4}}(x_k, \overline{A'_k})$, which is at most $3\xi n$ by P5(G_4). Thus $P_3(x_k y_i, G_5) \geqslant a'_i - 3\xi n$, and so

$$\min_{e \in D_i} P_3(e, G_5) \geqslant a'_i - 3\xi n. \tag{7.38}$$

Let $u_i \in U_i$ and $u_j \in U_j$ for $j \in [k-1] \setminus \{i\}$. Then $d_{G'}(u_i, A'_i), d_{G'}(u_j, A'_j) = 0$ by (3) and $d_{G'}(u_i, A'_k), d_{G'}(u_j, A'_k) \leqslant |Z| \leqslant \delta n$ by (1). So

$$P_3(u_i u_j, G') \leqslant a'_i - |A'_j| + \delta n \overset{P1(G_5)}{\leqslant} a'_i - cn + 3\beta n. \tag{7.39}$$

Since for all $i \in [k-1]$ the graph $D_i \subseteq G_5[R_k, R_i]$ is bipartite and the D_i are pairwise vertex-disjoint, any triangle in G_5 that contains at least two edges in $\bigcup_{i\in[k-1]} D_i$ also contains an edge in $G_5[R_i]$ or $G_5[R_k]$ for some i. So there are no such triangles. Since $\bigcup_{i\in[k-1]} D_i \cap W = \emptyset$, the only possible triangles containing at least two edges from $E(G_5) \setminus E(G')$ lie in W, and there are at most $|Z|^3$ such triangles. Thus we can bound $K_3(G') - K_3(G_5)$ as follows:

$$K_3(G') - K_3(G_5) \leqslant \sum_{i\in[k-1]} \left(\sum_{e\in C_i} P_3(e, G') - \sum_{f\in E(G_5[W_i, \bigcup_{\ell>i} W_\ell])} P_3(f, G_5) \right)$$
$$+ \sum_{i\in[k-1]} \left(\sum_{f\in E(\overline{G_5}[U_i, \bigcup_{\ell>i} U_\ell])} P_3(f, G') - \sum_{e\in D_i} P_3(e, G_5) \right) + 2|Z|^3. \qquad (7.40)$$

Denote by Δ_W and Δ_U the first and second terms on the right-hand side of (7.40), respectively. If there is at most one nonempty U_i, then $\Delta_U = 0$. Otherwise, using (7.38) and (7.39), we have

$$\Delta_U \leqslant \sum_{i\in[k-1]} |D_i| \cdot (-cn + 3\beta n + 3\xi n) < 0.$$

We claim that $\Delta_W \leqslant \delta^{1/3} m^2/n$. To see this, note that if there is at most one nonempty W_i, then $\Delta_W = 0$, so assume not. Suppose first that $e(G_5[W]) = \sum_{i\in[k-1]} |C_i| \leqslant \delta^{1/3} m^2/n^2$, where the equality follows from (7.28) and the fact that $G_5[W_i, W_j] = \emptyset$. Then by (7.26) and (7.34),

$$\Delta_W \overset{(7.34),(7.37)}{\leqslant} \sum_{i\in[k-1]} |C_i| \cdot (a_i' + \delta n - a_i' + \max_{j\neq i,k} |A_j'|) \overset{P1(G_5)}{\leqslant} \sum_{i\in[k-1]} |C_i| \cdot (cn + 3\beta n)$$
$$\leqslant \frac{\delta^{1/3} m^2}{n^2} \cdot 2cn \leqslant \frac{\delta^{1/3} m^2}{n}.$$

We may then assume

$$e(G_5[W]) \geqslant \frac{\delta^{1/3} m^2}{n^2} \geqslant \delta^{1/3} \cdot C \cdot \frac{m}{n} \overset{(7.1)}{=} \frac{m}{\delta^{1/6} n}.$$

In this case, we need to estimate Δ_W more carefully making use of (7.37):

$$\begin{aligned}
\Delta_W &\leqslant |E_{W\setminus W'}| \cdot \left(\max_{j\neq k} a_j' + \delta n - \sum_{i=1}^{k-1} |A_i'| + \frac{2\xi n}{5} \right) + |E_{W'}| \cdot \left(\delta n + \max_{j\neq k} |A_j'| \right) \\
&\overset{P1(G_5)}{\leqslant} |E_{W\setminus W'}| \cdot \left(-\frac{cn}{2} \right) + |E_{W'}| \cdot 2cn = \frac{cn}{2} \cdot (4|E_{W'}| - |E_{W\setminus W'}|) \\
&= \frac{cn}{2} \cdot (5|E_{W'}| - e(G_5[W])) \overset{(7.35)}{\leqslant} \frac{cn}{2} \cdot \left(5 \cdot \frac{2km}{\xi n} - \frac{m}{\delta^{1/6} n} \right) < 0.
\end{aligned}$$

Therefore, we have

$$K_3(G') - K_3(G_5) \leqslant \Delta_W + \Delta_U + 2|Z|^3 \leqslant \frac{\delta^{1/3} m^2}{n} + 2|Z|^3.$$

Now, letting $G_0 := G$ and $G_6 := G'$ and using Lemmas 7.4(iii), 7.6(iv), 7.8(iii), 7.9(iv) and 7.10(iv) and the previous inequalities,

$$K_3(G') - K_3(G) = \sum_{i\in[6]}(K_3(G_i) - K_3(G_{i-1}))$$
$$\leqslant \left(\delta^{7/8} + \frac{\delta^{1/4}}{3} + \sqrt{\delta} + 0 + \delta^{1/3}\right)\frac{m^2}{n} + k^2\delta n + 4|Z|^3 \overset{(6.31)}{\leqslant} \frac{\delta^{1/4}m^2}{2n},$$

where we use the fact that $m > Cn$ to bound $k^2\delta n \leqslant k^2\delta m^2/(C^2 n) = k^2\delta^2 m^2/n$. This completes the proof of Lemma 7.1.

8. The intermediate case: finishing the proof

8.1. The intermediate case when m is large. In this section, we finish the proof of the intermediate case when

$$m \geqslant Cn. \tag{8.1}$$

8.1.1. Properties of G via G′ .

We will now use Lemma 7.1 to obtain some additional structural information about G, which will in turn enable us to redo the transformations in Section 7 more carefully. This will eventually imply that most exceptional sets X_i, Y_i are in fact empty. After this, one final 'global' transformation yields the result.

Apply Lemma 7.1 to G to obtain a k-partite graph G' with vertex partition $A''_1, \ldots, A''_k$ and missing vector $\underline{m'} = (m'_1, \ldots, m'_{k-1})$ satisfying Lemma 7.1(i)–(iii). Let $m' := \sum_{i\in[k-1]} m'_i$.

The first step is to use Lemma 4.19 to show that, in G', the parts $A''_1, \ldots, A''_{k-2}$ all have size within $o(m/n)$ of cn, the 'expected' size; and that the number of missing edges between these parts and A''_k is $o(m)$. Roughly speaking, this means that G' has edit distance $o(m)$ from a graph in $\mathcal{H}_1(n, e)$. Since $m'_i = \Theta(m_i) + o(m)$ for all $i \in [k-1]$, this information about missing edges in G' translates to G. Lemma 7.1(ii) clearly implies that

$$\frac{\alpha^2}{2} \leqslant \alpha^2 - 2k\sqrt{\delta} \leqslant \frac{m'}{m} \leqslant 2 + 2k\sqrt{\delta} \leqslant 3. \tag{8.2}$$

The next proposition shows that the smallest part A''_k of G' has to be noticeably larger than $(1-(k-1)c)n$ since the number of missing edges m' is large.

PROPOSITION 8.1. $|A''_k| \geqslant (1-(k-1)c)n + \frac{m'}{(kc-1)n}$.

Proof. Suppose, for a contradiction, that $|A''_k| < n - (k-1)cn + q$, where $q := \frac{m'}{(kc-1)n}$. Let $x := (k-1)cn - q$. Given $|A''_k|$, we certainly have

$$\sum_{ij \in \binom{[k-1]}{2}} |A''_i||A''_j| + (n - |A''_k|)|A''_k| \leqslant t_{k-1}(n - |A''_k|) + (n - |A''_k|)|A''_k|.$$

Recall that we assume

$$|A''_k| < n - x \overset{(8.2)}{\leqslant} (1-(k-1)c)n + \frac{3m}{(kc-1)n} \overset{(6.26)}{\leqslant} (1-(k-1)c+\sqrt{\eta})n \overset{(6.3)}{\leqslant} (c-\sqrt{\alpha})n.$$

As $(1-(k-1)c+\sqrt{\eta})+(k-1)(c-\sqrt{\alpha}) < 1$, we get from the above inequalities that $|A''_k| < n - x < n/k$. We know by Lemma 4.5 that $t_{k-1}(n - |A''_k|) + (n - |A''_k|)|A''_k|$ is an increasing function of $|A''_k|$ whenever $|A''_k| \leqslant n/k$. Thus we have $t_{k-1}(n - |A''_k|) + (n - |A''_k|)|A''_k| \leqslant t_{k-1}(x) + x(n-x)$. Therefore, since G' has no bad edges,

$$\begin{aligned}
e + m' &= \sum_{ij \in \binom{[k-1]}{2}} |A''_i||A''_j| + (n - |A''_k|)|A''_k| < t_{k-1}(x) + x(n-x) \\
&\leqslant \binom{k-1}{2}\left(\frac{x}{k-1}\right)^2 + x(n-x) \\
&= x\left(n - \frac{k}{2(k-1)}x\right) = (k-1)cn^2 - \binom{k}{2}c^2n^2 + (kc-1)qn - \frac{kq^2}{2(k-1)} \\
&\leqslant (k-1)cn^2 - \binom{k}{2}c^2n^2 + (kc-1)qn \overset{(4.10)}{=} e + (kc-1)qn = e + m',
\end{aligned}$$

a contradiction. □

LEMMA 8.2. *For all $j \in [k-2]$, the following hold.*

(i) $m_j \leqslant \delta^{1/6}m$.

(ii) $|Z_j \cup Z^j_k| \leqslant \delta^{1/7}m/(2n)$.

(iii) $||A''_j| - cn| \leqslant 6\delta^{1/9}m/n$ *and* $|A''_{k-1}| \leqslant cn - \alpha^2 m/(4cn)$.

Proof. Let $H := K^k_{\lfloor cn\rfloor,\ldots,\lfloor cn\rfloor,n-(k-1)\lfloor cn\rfloor}$ and let $B_1, \ldots, B_k$ be the parts of H, where $|B_i| = \lfloor cn\rfloor$ for all $i \in [k-1]$. We claim that there is an (n, e)-graph F, which one can obtain from H by removing at most $(k-1)^2cn$ edges from $H[B_{k-1}, B_k]$. Inequality (6.3) implies rather roughly that $|B_{k-1}||B_k| > (k-1)^2cn$, so it suffices to show that $e \leqslant E(H) \leqslant e + (k-1)^2cn$. Indeed, by (6.3), we have that $\lfloor cn\rfloor > n - (k-1)\lfloor cn\rfloor + k$, so

$$e = e(K^k_{cn,\ldots,cn,n-(k-1)cn}) \leqslant e(K^k_{\lfloor cn\rfloor,\ldots,\lfloor cn\rfloor,n-(k-1)\lfloor cn\rfloor}) = e(H)$$

$$= \binom{k-1}{2}\lfloor cn\rfloor^2 + (k-1)\lfloor cn\rfloor(n-(k-1)\lfloor cn\rfloor) \leqslant e + (k-1)^2cn,$$

as required.

We will apply Lemma 4.19 with G', $\{A''_i\}_{i\in[k]}$, F, $\lfloor cn\rfloor$, $(k-1)^2cn$, playing respectively the roles of G, $\{A_i\}_{i\in[k]}$, F, ℓ, d. Let $d_i := |A''_i| - \lfloor cn\rfloor$ for all $i \in [k-1]$ and $d_k := |A''_k| - n + (k-1)\lfloor cn\rfloor$. By Proposition 8.1, we have

$$d_k \geqslant \frac{m'}{(kc-1)n} - k \overset{(6.3)}{\geqslant} \frac{m'}{(c-(k-1)\alpha)n} - k \geqslant \frac{m'}{cn}. \tag{8.3}$$

Moreover, for all $i \in [k]$, Lemma 7.1(i) implies that

$$|d_i| \leqslant 4\beta n < \frac{\sqrt{2\alpha}n}{20k^3} \overset{(6.3)}{\leqslant} \frac{(kc-1)n}{20k^3} \leqslant \frac{\lfloor cn\rfloor - (n-(k-1)\lfloor cn\rfloor)}{12k^3}.$$

Then Lemma 4.19 can be applied with the parameters above to imply that

$$\begin{aligned} K_3(G) + \frac{\delta^{1/4}m^2}{2n} &\geqslant K_3(G') \\ &\geqslant K_3(F) + \sum_{t\in[k-1]} \frac{m'_t}{m'} \cdot \frac{k\lfloor cn\rfloor - n}{4}\left((d_t+d_k)^2 + \sum_{i\in[k-1]\setminus\{t\}} d_i^2\right) \\ &\quad - \frac{12(k-1)^4c^2n^2}{k\lfloor cn\rfloor - n}. \end{aligned}$$

Observe that each summand over $t \in [k-1]$ is nonnegative by (6.3). Bounding the last term, we have

$$\begin{aligned} 0 &\overset{(6.3)}{\leqslant} \frac{12(k-1)^4c^2n^2}{k\lfloor cn\rfloor - n} \leqslant \frac{14(k-1)^4c^2n}{kc-1} \overset{(6.3),(8.1)}{\leqslant} \frac{14k^4c^2m^2}{\sqrt{2\alpha}C^2n} \overset{(7.1)}{=} \frac{14k^4c^2\delta m^2}{\sqrt{2\alpha}n} \\ &\leqslant \frac{\delta^{7/8}m^2}{2n}. \end{aligned}$$

Furthermore,

$$\frac{k\lfloor cn\rfloor - n}{4} \overset{(6.3)}{\geqslant} \frac{\sqrt{2\alpha}n - k}{4} > \frac{\sqrt{\alpha}n}{4}.$$

Thus, for each $j \in [k-1]$, using the fact that $\delta^{7/8}/2 + \delta^{1/4}/2 \leqslant \delta^{1/4}$,

$$\frac{m'_j}{m'}\left((d_j+d_k)^2 + \sum_{i\in[k-1]\setminus\{j\}} d_i^2\right) \leqslant \frac{K_3(G) - K_3(F) + \frac{\delta^{1/4}m^2}{n}}{\frac{\sqrt{\alpha}n}{4}} \leqslant \frac{4\delta^{1/4}m^2}{\sqrt{\alpha}n^2}$$

$$\overset{(8.2)}{\leqslant} \frac{16\delta^{1/4}m'^2}{\alpha^{9/2}n^2} \leqslant \frac{\delta^{2/9}m'^2}{(k-1)n^2}.$$

So for all $ij \in \binom{[k-1]}{2}$, we have that

$$|d_j + d_k|, |d_i| \leqslant \frac{\delta^{1/9}m'}{n} \cdot \sqrt{\frac{m'}{(k-1)m'_j}}. \tag{8.4}$$

Suppose that $r \in [k-1]$ is such that $m'_r = \max_{j\in[k-1]} m'_j$. Then $m'_r \geqslant m'/(k-1)$. We have

$$|d_r + d_k| \overset{(8.4)}{\leqslant} \frac{\delta^{1/9}m'}{n} \quad \text{and} \quad |d_i| \leqslant \frac{\delta^{1/9}m'}{n}.$$

But by (8.3), $d_k \geqslant m'/(cn) > \delta^{1/9}m'/n$. So $d_r < 0$ and in fact $d_r = |A''_r| - \lfloor cn \rfloor \leqslant \delta^{1/9}m'/n - d_k$. Thus

$$\begin{aligned} |A''_r| &\overset{(8.3)}{<} \lfloor cn \rfloor - \left(\frac{1}{c} - \delta^{1/9}\right)\frac{m'}{n} \overset{(5.1)}{\leqslant} cn - \frac{m'}{2cn} \\ &\overset{(8.2)}{\leqslant} cn - \frac{\alpha^2 m}{4cn}; \quad \text{and} \end{aligned} \tag{8.5}$$

$$\big||A''_i| - cn\big| \overset{(8.4)}{\leqslant} \frac{2\delta^{1/9}m'}{n} \overset{(8.2)}{\leqslant} \frac{6\delta^{1/9}m}{n} \tag{8.6}$$

for all $i \in [k-1]\setminus\{r\}$. Suppose now that $m'_s \geqslant \delta^{1/5}m'$ for some $s \in [k-1]\setminus\{r\}$. Then applying (8.4) with $ij = rs$, we have

$$|A''_r| \geqslant \lfloor cn \rfloor - |d_r| \overset{(8.4)}{\geqslant} \lfloor cn \rfloor - \frac{\delta^{1/9}m'}{\sqrt{(k-1)\delta^{1/10}}n} > cn - \frac{4\delta^{1/90}}{\sqrt{(k-1)}n} > cn - \frac{\alpha^2 m}{4cn},$$

a contradiction to (8.5). Therefore, for all $s \in [k-1]\setminus\{r\}$, we have by Lemma 7.1(ii) that

$$\begin{aligned} m_s &\leqslant \frac{1}{\alpha^2}\left(m'_s + 2\sqrt{\delta}m\right) < \frac{1}{\alpha^2}\left(\delta^{1/5}m' + 2\sqrt{\delta}m\right) \overset{(8.2)}{\leqslant} \frac{1}{\alpha^2}\left(3\delta^{1/5}m + 2\sqrt{\delta}m\right) \\ &\leqslant \delta^{1/6}m. \end{aligned}$$

But $\max_{i\in[k-1]} m_i = m_{k-1} \geqslant m/(k-1)$, and so $r = k-1$. That is, $m_1, \ldots, m_{k-2} \leqslant \delta^{1/6}m$, as required for (i). By (6.31), we have for all $s \in [k-2]$ that

$$|Z_s \cup Z^s_k| \leqslant \frac{2\delta^{1/6}m}{\xi n} \leqslant \frac{\delta^{1/7}m}{2n},$$

proving (ii). Part (iii) follows from (8.5) and (8.6). □

Since the exceptional sets $Z_1, \ldots, Z_{k-2}$ and $Z_k^1, \ldots, Z_k^{k-2}$ are all small by the previous lemma, it is now easy to show that $G[R_1, R_k], \ldots, G[R_{k-2}, R_k]$ are all complete. That is, for all $i \in [k-2]$, every missing edge in $G[A_i, A_k]$ is incident to a vertex of Z.

LEMMA 8.3. *For every $i \in [k-2]$, $G[R_i, R_k]$ is complete.*

Proof. Let $x \in R_i$ and $y \in R_k$. By Proposition 6.12(i), $N_G(y, A_k) \subseteq Y$. By P3(G), $N_G(x, A_i) \subseteq Z_i$. Since $A''_j \supseteq A_j \cup Y_j$ for all $j \in [k-1]$, using Lemma 8.2(ii) and (iii) and $m \geqslant Cn$, we have that

$$\begin{aligned} P_3(xy, G) &\leqslant \sum_{j\in[k-1]\setminus\{i\}} |A_j| + |Z_i| + |Y| \leqslant \sum_{j\in[k-2]\setminus\{i\}} |A''_j| + |A''_{k-1}| + |Z_i \cup Z_k^i| \\ &\leqslant (k-3)\left(cn + \frac{6\delta^{1/9}m}{n}\right) + cn - \frac{\alpha^2 m}{4cn} + \frac{\delta^{1/7}m}{2n} \leqslant (k-2)cn - \frac{\alpha^2 m}{5cn} \\ &\overset{(8.1)}{\leqslant} (k-2)cn - \frac{\alpha^2 C}{5c} \overset{(7.1)}{\leqslant} (k-1)cn - 2k. \end{aligned}$$

Therefore $xy \in E(G)$ by (5.5). □

The previous two lemmas now imply very precise information about the sizes of the parts $A_1, \ldots, A_k$ in G. Indeed, we can calculate their sizes up to an $o(m/n)$ error term. Recall from (6.28) that $t = \frac{m}{(kc-1)n}$.

LEMMA 8.4. *The following hold for parts of G.*

$$|A_1|, \ldots, |A_{k-2}| = cn \pm \frac{\delta^{1/10}m}{n};$$
$$|A_{k-1}| = cn - t \pm \frac{\delta^{1/11}m}{n} \quad \textit{and}$$
$$|A_k| = n - (k-1)cn + t \pm \frac{\delta^{1/11}m}{n}.$$

Proof. For the first equation, recall that for all $i \in [k-1]$, Lemma 7.1(i) implies that $A_i \subseteq A''_i \subseteq A_i \cup Z_k^i$. If $j \in [k-2]$, then Lemma 8.2(iii) implies that $|A_j| \leqslant |A''_j| \leqslant cn + 6\delta^{1/9}m/n$. Using Lemma 8.2(ii) in addition, we see that also

$$|A_j| \geqslant |A''_j| - |Z_k^j| \geqslant cn - \frac{\delta^{1/10}m}{n},$$

as required. Therefore there is some $\tau \in \mathbb{R}$ such that

$$|A_{k-1}| = cn - \frac{\tau m}{n} \pm \frac{k\delta^{1/10}m}{n} \quad \text{and} \tag{8.7}$$

$$|A_k| = (1-(k-1)c)n + \frac{\tau m}{n} \pm \frac{k\delta^{1/10}m}{n}. \tag{8.8}$$

By Proposition 8.1, we have $|A_k| \geqslant |A''_k| \geqslant (1-(k-1)c)n + \frac{m'}{(kc-1)n}$. So (8.2) implies that $\tau \geqslant \frac{\alpha^2}{2(kc-1)}$. Let $\tilde{\delta} := k\delta^{1/10}m/n$. Then

$$\begin{aligned}
&e - e(G[A_{k-1}, A_k]) \\
&\quad = \sum_{ij \in \binom{[k-2]}{2}} |A_i||A_j| + (|A_{k-1}| + |A_k|) \sum_{i\in[k-2]} |A_i| + \sum_{i\in[k]} e(G[A_i]) - \sum_{i\in[k-2]} m_i \\
&\quad \overset{(6.27)}{=} \binom{k-2}{2}(cn \pm \tilde{\delta})^2 + (n-(k-2)cn \pm 2\tilde{\delta})((k-2)cn \pm \tilde{\delta}) \\
&\qquad \pm (\delta m + k\delta^{1/6}m) \\
&\quad = \binom{k-2}{2} c^2n^2 + (n-(k-2)cn)(k-2)cn \pm 3k^2\tilde{\delta}n \\
&\quad \overset{(4.10)}{=} e - cn(n-(k-1)cn) \pm 3k^3\delta^{1/10}m.
\end{aligned}$$

Here we used Lemma 8.2(i) to bound m_i for $i \in [k-2]$. We then have

$$e(G[A_{k-1}, A_k]) = cn(n-(k-1)cn) \pm 3k^3\delta^{1/10}m. \tag{8.9}$$

We claim that $\tau \leqslant 1/\delta$. So suppose for a contradiction that $\tau > 1/\delta$. Now, $|A_{k-1}| + |A_k| = n - \sum_{i\in[k-2]} |A_i| = (1-(k-2)c)n \pm \tilde{\delta}$. Further,

$$\begin{aligned}
|A_{k-1}| &\leqslant cn - \frac{\tau m}{n} - \tilde{\delta} \leqslant cn - \frac{m}{\delta n} - \tilde{\delta} \quad \text{and} \\
|A_k| &\geqslant (1-(k-1)c)n + \frac{\tau m}{n} - \tilde{\delta} \geqslant (1-(k-1)c)n + \frac{m}{\delta n} - \tilde{\delta}.
\end{aligned}$$

By (6.3), we have $|A_{k-1}| > |A_k|$. So the product $|A_{k-1}||A_k|$ is minimized when $|A_k|$ attains the upper bound above. So

$$\begin{aligned}
|A_{k-1}||A_k| &\geqslant \left(cn - \frac{m}{\delta n} - \tilde{\delta}\right)\left((1-(k-1)c)n + \frac{m}{\delta n} - \tilde{\delta}\right) \\
&\geqslant cn(n-(k-1)cn) + (kc-1)n \cdot \frac{m}{\delta n} - \frac{m^2}{\delta^2 n^2} - \tilde{\delta}n \\
&\overset{(6.3),(6.26)}{\geqslant} cn(n-(k-1)cn) + \sqrt{2\alpha} \cdot \frac{m}{\delta} - \frac{\eta m}{\delta^2} - k\delta^{1/10}m \\
&\geqslant cn(n-(k-1)cn) + \frac{m}{\sqrt{\delta}}.
\end{aligned}$$

But then, this implies that

$$\begin{aligned} e(G[A_{k-1}, A_k]) = |A_{k-1}||A_k| - m_{k-1} &\geqslant cn(n-(k-1)cn) + \frac{m}{\sqrt{\delta}} - m \\ &\geqslant cn(n-(k-1)cn) + m, \end{aligned}$$

contradicting (8.9). So $\tau \leqslant 1/\delta$, as claimed.

We now estimate $|A_{k-1}||A_k|$ again more carefully using that $\frac{\alpha^2}{2(kc-1)} \leqslant \tau \leqslant 1/\delta$. We have

$$\begin{aligned} |A_{k-1}||A_k| &= cn(n-(k-1)cn) + (kc-1)\tau m \\ &\quad + \left(-\frac{\tau^2 m^2}{n^2} \pm 2k\delta^{1/10} m + \frac{2\tau k \delta^{1/10} m^2}{n^2} + \frac{k^2 \delta^{1/5} m^2}{n^2} \right). \end{aligned}$$

But $m^2/n^2 \leqslant \eta m$ by (6.26) and $\tau \leqslant 1/\delta$, so the expression in the final parentheses is at most $3k\delta^{1/10}m$. So

$$|A_{k-1}||A_k| = cn(n-(k-1)cn) + (kc-1)\tau m \pm 3k\delta^{1/10} m. \tag{8.10}$$

As $m_{k-1} = (1 \pm k\delta^{1/6})m$ due to Lemma 8.2(i), we have

$$(1 \pm k\delta^{1/6})m = m_{k-1} = |A_{k-1}||A_k| - e(G[A_{k-1}, A_k]) \overset{(8.9),(8.10)}{=} (kc-1)\tau m \pm 4k^3\delta^{1/10} m.$$

Solving this for τ, we get

$$\frac{\tau m}{n} = \frac{m}{(kc-1)n} \pm \frac{\delta^{1/11} m}{2n} \overset{(6.28)}{=} t \pm \frac{\delta^{1/11} m}{2n}.$$

Combined with (8.7) and (8.8), this completes the proof of the lemma. □

The usefulness of G' is now exhausted, and we work only with G for the rest of the proof. The previous lemma implies that

$$a_i = \sum_{j \in [k-1] \setminus \{i\}} |A_j| = (k-2)cn - t \pm \frac{(k-2)\delta^{1/11} m}{n} \quad \text{for all } i \in [k-2]. \tag{8.11}$$

Armed with Lemmas 8.3 and 8.4, we can now 'redo' Transformations 1 and 2 of Section 7, in a slightly more careful fashion, to imply that $Z_i = Y_i = \emptyset$ for all $i \in [k-2]$.

PROPOSITION 8.5. *Let $i \in [k-2]$ and $z \in Z_i \cup Z_k^i$. Then $d_G(z, R_i) \geqslant t - \delta^{1/12} m/n > 0$.*

Proof. By P2(G), P3(G) and P5(G), every such z has at least ξn nonneighbours in A_k. Recall the definitions of R'_k and Δ in Section 7.1. We have

$$|R'_k \setminus N_G(z)| \geqslant d_{\overline{G}}(z, A_k) - |R_k \setminus R'_k| - |Z_k| \overset{(6.31)}{\geqslant} \xi n/2 - \sqrt{\eta}n \geqslant \xi n/3.$$

Thus we can choose $w \in R'_k \setminus N_G(z)$. Then $wz \in E(\overline{G})$ and so, by (5.5) and P2(G),

$$\begin{aligned}(k-2)cn - k \leqslant P_3(zw, G) &\overset{(7.3)}{\leqslant} a_i + d_G(z, A_i) + \Delta \\ &\overset{(7.4)}{\leqslant} a_i + |Z_i| + d_G(z, R_i) + \frac{\delta^{1/3}m}{n} \\ &\leqslant (k-2)cn - t + d_G(z, R_i) + \frac{k\delta^{1/11}m}{n},\end{aligned}$$

where the last inequality follows from Lemma 8.2(ii) and (8.11). Hence $d_G(z, R_i) \geqslant t - \delta^{1/12}m/n$, which is positive by (6.28). $\square$

LEMMA 8.6. *$Z_i = Y_i = \emptyset$ for all $i \in [k-2]$.*

Proof. Suppose that there exists $z \in Z_i$ for some $i \in [k-2]$. Let $z_1, \ldots, z_p$ be an arbitrary ordering of $Z \setminus Z_k$ such that $z := z_1$. Note that $N_G(z, A_i) \neq \emptyset$ due to Proposition 8.5. Now apply Lemma 7.3 to G and let F be the obtained (n, e)-graph G^1, which satisfies $J(1, 1)$–$J(3, 1)$. By $J(3, 1)$, we have that

$$0 \leqslant K_3(F) - K_3(G) \leqslant \sum_{y \in N_G(z, A_i)} \left(\Delta - |Z_k \setminus Z^i_k| - P_3(yz, G; R_k)\right) \tag{8.12}$$

$$\overset{(7.4)}{\leqslant} \sum_{y \in N_G(z, Z_i)} \frac{\delta^{1/3}m}{n} + \sum_{y \in N_G(z, R_i)} \left(\frac{\delta^{1/3}m}{n} - |Z_k \setminus Z^i_k| - d_G(z, R_k)\right). \tag{8.13}$$

Here, for all $y \in R_i$, since Lemma 8.3 implies that $R_k \subseteq N_G(y)$, we have $P_3(yz, G; R_k) = d_G(z, R_k)$. We must have $|Z_k \setminus Z^i_k| \leqslant \Delta \leqslant \delta^{1/3}m/n$, as otherwise the right-hand side of (8.12) is negative. So Lemma 8.2(ii) implies that

$$|Z_i \cup Z_k| = |Z_k \setminus Z^i_k| + |Z_i \cup Z^i_k| \leqslant \frac{\delta^{1/3}m}{n} + \frac{\delta^{1/7}m}{2n} \leqslant \frac{\delta^{1/7}m}{n}. \tag{8.14}$$

We will now bound $d_G(z, R_k)$. By P3(G), z has a nonneighbour u in R_i. Since $u \in R_i$, we have that $N_G(u, A_i) \subseteq Z_i$. Thus (5.5) then implies that

$$(k-2)cn - k \leqslant P_3(uz, G) \leqslant a_i + d_G(z, R_k) + |Z_i \cup Z_k|.$$

Thus

$$d_G(z, R_k) \geqslant (k-2)cn - a_i - \frac{2\delta^{1/7}m}{n} \overset{(8.11)}{\geqslant} t - \frac{\delta^{1/12}m}{n}. \tag{8.15}$$

Using Proposition 8.5, (8.14) and (8.15), the final upper bound in (8.13) is at most

$$\frac{\delta^{1/7}m}{n} \cdot \frac{\delta^{1/3}m}{n} + \left(\frac{\delta^{1/3}m}{n} - 0 - \left(t - \frac{\delta^{1/12}m}{n}\right)\right)\left(t - \frac{\delta^{1/12}m}{n}\right) \overset{(6.28)}{\leqslant} -\frac{t^2}{2}, \tag{8.16}$$

a contradiction. We have proved that $Z_i = \emptyset$, so $A_i = R_i$, for all $i \in [k-2]$.

Suppose now that there exists $y \in Y_i$ for some $i \in [k-2]$. Let $y_1, \ldots, y_q$ be an arbitrary ordering of $Y = \bigcup_{i \in [k-1]} Y_i$ (as in (6.32)) such that $y := y_1$. Observe that, since $Z_1 = \cdots = Z_{k-2} = \emptyset$, the graph G satisfies the conclusions of Lemma 7.4 when $\ell = k-2$. Therefore we can apply Lemma 7.5 with $k-2$, G playing the roles of ℓ, G_1^ℓ. Let F' be the obtained (n, e)-graph G^1, which satisfies $K(1, 1)$–$K(3, 1)$. Then $K(3, 1)$, (7.4) and Lemma 8.3 imply that

$$\begin{aligned} 0 \leqslant K_3(F') - K_3(G) &\leqslant \sum_{x \in N_G(y, R_i)} \left(\Delta - \frac{\xi}{6\gamma}|Z_k \setminus Z_k^i| - P_3(xy, G; R_k)\right) \\ &\leqslant \sum_{x \in N_G(y, R_i)} \left(\frac{\delta^{1/3}m}{n} - \frac{\xi}{6\gamma} \cdot |Z_k \setminus Z_k^i| - d_G(y, R_k)\right). \end{aligned}$$

Again by Proposition 8.5, $N_G(y, R_i) \neq \emptyset$. Therefore, as in (8.14), by Lemma 8.2, we have

$$\begin{aligned} |Z_i \cup Z_k| = |Z_k| &\leqslant |Z_k^{k-1}| + \frac{(k-2)\delta^{1/7}m}{2n} \\ &\leqslant \frac{6\gamma\delta^{1/3}m}{\xi n} + \frac{(k-2)\delta^{1/7}m}{2n} \leqslant \frac{k\delta^{1/7}m}{n}. \end{aligned} \tag{8.17}$$

We will now bound $d_G(y, R_k)$. By the definition of Y, y has a nonneighbour u in R_i. Then (5.5) implies that

$$(k-2)cn - k \leqslant P_3(uy, G) \leqslant a_i + d_G(y, R_k) + |Z_k|.$$

Thus

$$d_G(y, R_k) \overset{(8.17)}{\geqslant} (k-2)cn - k - a_i - \frac{k\delta^{1/7}m}{n} \overset{(8.11)}{\geqslant} t - \frac{\delta^{1/12}m}{n}.$$

But then, using Proposition 8.5 to bound $d_G(y, R_i)$, by a similar calculation to (8.16), we have

$$K_3(F') - K_3(G) \leqslant -t^2/2,$$

a contradiction. Thus $Y_i = \emptyset$ for all $i \in [k-2]$. □

We can now use the lemmas in this section to prove the following penultimate ingredient that we require. Let

$$A := \bigcup_{i\in[k-2]} R_i; \quad B := A_{k-1} \cup R_k \cup Z_k^{k-1} \quad \text{and} \quad X' := \bigcup_{i\in[k-2]} X_i. \tag{8.18}$$

Lemma 8.2(ii) implies that

$$|X'| \leqslant \frac{\delta^{1/8} m}{n}. \tag{8.19}$$

LEMMA 8.7. *The following properties hold for G.*

(i) *G has vertex partition $A \cup B \cup X'$; $G[A]$ is a complete $(k-2)$-partite graph with parts $R_1, \ldots, R_{k-2}$; and $G[A, B]$ is complete.*

(ii) *There exist $b_1 \leqslant b_2 \in \mathbb{N}$ such that $b_1 + b_2 = |B|$ and $(b_1 - 1)(b_2 + 1) < e(G[B]) \leqslant b_1 b_2$. Moreover, for all $x \in X_i$ with $i \in [k-2]$, we have*

$$K_3(x, G; \overline{X'}) \geqslant e(G[L_i]) + |L_i| b_1 + d_G(x, R_i)(|L_i| + b_1) + \alpha m,$$

where $L_i := A \setminus R_i$.

(iii) *For all $x \in X'$ we have*

$$d_G(x, Z_k^{k-1}) = t \pm \frac{2\delta^{1/12} m}{n}, \quad \text{and further } b_1 = cn \pm \frac{\delta^{1/13} m}{n}.$$

Proof. The previous lemma implies that $A_i = R_i$ and $Y_i = \emptyset$ for all $i \in [k-2]$. So $A \cup B \cup X'$ is a partition of $V(G)$. Property P3(G) implies that R_i is an independent set in G for all $i \in [k-2]$, which, together with P2(G), implies that $G[A]$ is a complete $(k-2)$-partite graph with parts $R_1, \ldots, R_{k-2}$. Properties P2(G), P4(G) and Lemma 8.3 imply that $G[A, B]$ is complete. This completes the proof of (i).

For (ii) and (iii), let $x \in X_i \subseteq X'$ for some $i \in [k-2]$. Proposition 6.12(i) implies that $E(G[X', R_k]) = \emptyset$. We need to determine $d_G(x, Z_k^{k-1})$ quite precisely. For this, let $u \in R_i$ be arbitrary. Then

$$\begin{aligned} P_3(ux, G) \quad &= \quad a_i + d_G(x, Z_k^{k-1}) \pm |X'| \\ &\overset{(8.11),(8.19)}{=} (k-2)cn - t + d_G(x, Z_k^{k-1}) \pm \frac{\delta^{1/12} m}{n}. \end{aligned} \tag{8.20}$$

Since $x \in X_i$, we have $d_{\overline{G}}(x, R_i) > 0$ by definition. Also, since $R_i = A_i$, we have $d_G(x, R_i) \geqslant \gamma n > 0$. That is, $N_G(x, R_i), N_{\overline{G}}(x, R_i) \neq \emptyset$. So (5.5) implies that the right-hand side of (8.20) lies in $[(k-2)cn - k, (k-2)cn + k]$. Thus

$$d_G(x, Z_k^{k-1}) = t \pm \frac{2\delta^{1/12} m}{n}. \tag{8.21}$$

Recall that, by P4(G), $G[A_{k-1}, X']$ is complete. Thus, all of the $m_{k-1} = (1 \pm k\delta^{1/6})m$ missing edges between A_{k-1} and A_k lie in B. Then Lemmas 8.2(i) and 8.4 imply that

$$\begin{aligned} e(G[B]) &= |A_{k-1}|(|A_k| - |X'|) - m_{k-1} + (e(G[A_{k-1}]) + e(G[A_k])) \\ &\overset{(6.27),(8.19)}{=} \left(cn - t \pm \frac{\delta^{1/11}m}{n}\right)\left(n - (k-1)cn + t \pm \frac{2\delta^{1/11}m}{n}\right) \\ &\qquad - m \pm k\delta^{1/6}m \pm \sqrt{\delta}m \\ &\overset{(6.28)}{=} (c - (k-1)c^2)n^2 \pm \delta^{1/12}m. \end{aligned} \tag{8.22}$$

Also,

$$|B| = |A_{k-1}| + |A_k| - |X'| = n - (k-2)cn \pm \frac{\delta^{1/12}m}{n}. \tag{8.23}$$

A simple calculation using (6.3), (6.26) and (8.22) shows that

$$e(G[B]) \leqslant \frac{1}{4}\left((1-(k-2)c)n - \frac{\delta^{1/12}m}{n}\right)^2 \leqslant \frac{|B|^2}{4}.$$

Thus there exist $b_1, b_2 \in \mathbb{N}$ such that $b_1 \leqslant b_2$ and

$$b_1 + b_2 = |B|$$

and

$$(b_1 - 1)(b_2 + 1) < e(G[B]) \leqslant b_1 b_2.$$

Suppose, for a contradiction, that $b_1 > cn + q$, where $q := \delta^{1/13}m/n$. Since the product $b_1 b_2$ is maximized when b_1, b_2 are as balanced as possible, while (6.3) and (8.23) imply that $2(cn + q) > |B|$, we have that

$$\begin{aligned} b_1 b_2 &< (cn+q)(|B| - cn - q) \overset{(8.23)}{\leqslant} (cn+q)(n - (k-1)cn - q) + \delta^{1/12}m \\ &\leqslant cn(n - (k-1)cn) - qn(kc - 1) + \delta^{1/12}m \\ &\overset{(6.3),(8.22)}{\leqslant} e(G[B]) - (\sqrt{2\alpha}\delta^{1/13} - 3\delta^{1/12})m \\ &\overset{(8.1)}{\leqslant} e(G[B]) - \sqrt{\alpha}\delta^{1/13}Cn \overset{(5.1),(7.1)}{<} e(G[B]) - 2n, \end{aligned}$$

a contradiction. Similarly, if $b_1 < cn - q$, then $b_1 b_2 > e(G[B]) + 2n$; consequently, $(b_1 - 1)(b_2 + 1) > e(G[B])$, a contradiction. Therefore

$$b_1 = cn \pm \frac{\delta^{1/13}m}{n} \quad \text{and so} \quad b_2 = n - (k-1)cn \pm \frac{2\delta^{1/13}m}{n}. \tag{8.24}$$

So

$$b_1 - |A_{k-1}| = t \pm \frac{2\delta^{1/13}m}{n}. \tag{8.25}$$

Recall from the statement of the lemma that $L_i = A \setminus R_i$. Now, $G[x, L_i \cup A_{k-1}]$ is complete by P4(G). Also, $G[L_i, A_{k-1}, R_i]$ is a complete tripartite graph by P4(G). Finally, $e(\overline{G}[A_{k-1}, Z_k^{k-1}]) \leqslant e(\overline{G}[A_{k-1}, A_k]) = m_{k-1}$ by definition. Write $\tilde{\delta} := 2\delta^{1/13}m/n$. Thus

$$
\begin{aligned}
&K_3(x, G; \overline{X'}) \\
&\quad \geqslant K_3\left(x, G; L_i \cup A_{k-1}\right) + K_3\left(x, G; R_i \cup Z_k^{k-1}, L_i \cup A_{k-1}\right) \\
&\qquad + K_3(x, G; R_i, Z_k^{k-1}) \\
&\quad \geqslant e(G[L_i \cup A_{k-1}]) + (d_G(x, R_i) + d_G(x, Z_k^{k-1}))(|L_i| + |A_{k-1}|) \\
&\qquad + d_G(x, R_i)d_G(x, Z_k^{k-1}) - m_{k-1} \\
&\quad \overset{(8.21),(8.25)}{\geqslant} e(G[L_i]) + |L_i|(b_1 - t - \tilde{\delta}) + (d_G(x, R_i) + t - \tilde{\delta})(|L_i| + b_1 - t - \tilde{\delta}) \\
&\qquad + d_G(x, R_i)(t - \tilde{\delta}) - m_{k-1} \\
&\quad \overset{(6.28),(8.24)}{\geqslant} e(G[L_i]) + |L_i|b_1 + d_G(x, R_i)(|L_i| + b_1) - m + \frac{cm}{kc - 1} \\
&\qquad - 5\tilde{\delta}n - 2\sqrt{\eta}m \\
&\quad \overset{(6.3)}{\geqslant} e(G[L_i]) + |L_i|b_1 + d_G(x, R_i)(|L_i| + b_1) \\
&\qquad + \left(\frac{(k-1)\alpha}{c - (k-1)\alpha} - 12\delta^{1/13}\right)m \\
&\quad \geqslant e(G[L_i]) + |L_i|b_1 + d_G(x, R_i)(|L_i| + b_1) + \alpha m,
\end{aligned}
$$

as required for (ii). Part (iii) follows immediately from (8.21) and (8.24). □

To complete the proof, we first observe that if $X' = \emptyset$, then we are done. Indeed, in this case, Lemma 8.7(i) and (ii) implies that G has a partition A, B, where $G[A]$ is complete $(k-2)$-partite, $G[A, B]$ is complete, and $e(G[B]) \leqslant t_2(|B|)$. Thus $K_3(G[B]) = g_3(|B|, e(B)) = 0$ and so $G \in \mathcal{H}_1(n, e)$, a contradiction. So we may assume that $X' \neq \emptyset$. Now we will perform a final global transformation on G to obtain an (n, e)-graph H, which has fewer triangles.

Proof of Theorem 1.7 in the intermediate case and when $m \geqslant Cn$. We may assume, as observed above, that $X' \neq \emptyset$. Choose b_1, b_2 as in Lemma 8.7(ii). Let B_1, B_2 be an arbitrary partition of B such that $|B_i| = b_i$ for $i \in [2]$. Let $v_1, \ldots, v_{b_1}$ be an ordering of B_1. Let $U \subseteq B_2$ have size $e(G[B]) - (b_1 - 1)b_2$. So $0 < |U| \leqslant b_2$. Let $x_1, \ldots, x_\ell$ be an arbitrary ordering of X'. For each $g \in [\ell]$, let $s(g) \in [k-2]$ be such that $x_g \in X_{s(g)}$. Choose an arbitrary set $T(x_g) \subseteq R_{s(g)}$

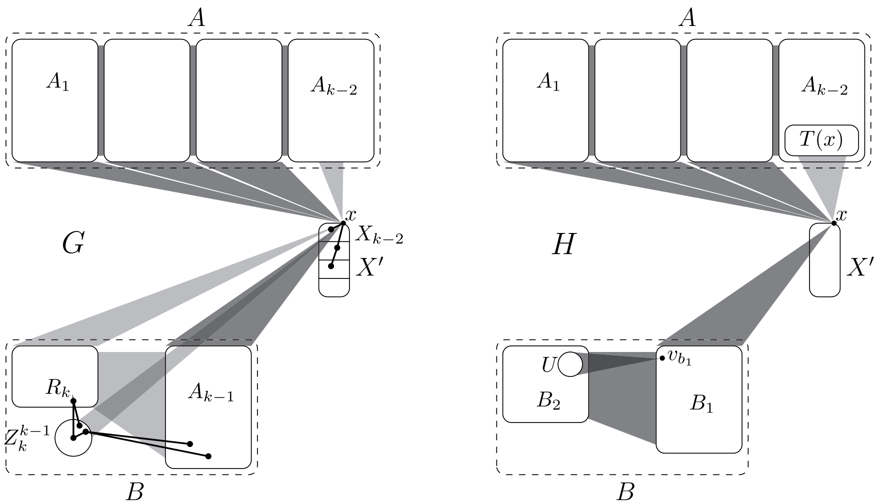

Figure 8. $G \to H$, from the perspective of a single $x \in X_{k-2} \subseteq X'$.

of size

$$\begin{aligned} |T(x_g)| &= d_G(x_g, B) + d_G(x_g, \{x_{g+1}, \ldots, x_\ell\}) + d_G(x_g, R_{s(g)}) - |B_1| \\ &= |A_{k-1}| + d_G(x_g, Z_k^{k-1}) + d_G(x_g, R_{s(g)}) - b_1 \pm |X'| \\ &\overset{(8.19),(8.25)}{=} d_G(x_g, R_{s(g)}) \pm \frac{3\delta^{1/13}m}{n}. \end{aligned} \tag{8.26}$$

Here we used the facts that $A_{k-1} \subseteq N_G(x_g)$ by P4(G); $R_k \cap N_G(x_g) = \emptyset$ by Proposition 6.12(i); and also Lemma 8.7(iii). But by the definition of $X_{s(g)}$, since $R_{s(g)} = A_{s(g)}$ by Lemma 8.6 and using $m \leqslant \eta n^2$ from (6.26), the right-hand side of (8.26) is at least $\gamma n - 3\delta^{1/13}\eta n \geqslant \gamma n/2$ and at most $|R_{s(g)}| - \xi n + 3\delta^{1/13}\eta n \leqslant |R_{s(g)}| - \xi n/2$. So $T(x_g)$ exists. Now define a new graph H by setting

$$E(H) := \Bigg(E(G) \cup \{v_i y : i \in [b_1 - 1], y \in B_2\} \cup \{v_{b_1} y : y \in U\} \\ \cup \bigcup_{x \in X'} \{xy : y \in B_1 \cup T(x)\} \Bigg) \Bigg\backslash \Bigg(E(G[B \cup X']) \cup \bigcup_{i \in [k-2]} E(G[X_i, R_i]) \Bigg).$$

Thus, informally, H is obtained from G by rearranging the edges in $G[B]$ to form a maximally unbalanced bipartition, and then for each $i \in [k-2]$ replacing the neighbours of $x \in X_i$ that lie in $X' \cup B \cup R_i$ with vertices in B_1, and then R_i. See Figure 8 for an illustration of G and H.

The following claim states some properties of H.

Claim 8.8. (i) *H is an (n, e)-graph such that $H[A, B]$ is complete; $H[A] = G[A]$ and $H[B]$ is bipartite with bipartition B_1, B_2, and $X' \neq \emptyset$. Also, $E(H[X', B_2]) = E(H[X']) = \emptyset$.*

(ii) *Let $T(G)$ be the set of triangles in G containing at least one vertex from X' and define $T(H)$ analogously. Then $|T(H)| \geqslant |T(G)|$.*

Proof of Claim. The first part of (i) follows from Lemma 8.7 and by the construction of H. Since $G[A, B] = H[A, B]$ are both complete, we have that

$$K_3(G) = K_3(G[A]) + K_3(G[B]) + |A|e(G[B]) + |B|e(G[A]) + |T(G)|; \text{ and}$$
$$K_3(H) = K_3(H[A]) + K_3(H[B]) + |A|e(H[B]) + |B|e(H[A]) + |T(H)|.$$

But $G[A] = H[A]$ and we also have $e(G[B]) = e(H[B])$ from the construction of H. Moreover, $H[B]$ is bipartite, so $K_3(H[B]) = 0$. Thus $0 \leqslant K_3(H) - K_3(G) = |T(H)| - |T(G)| - K_3(G[B])$. Then $|T(H)| \geqslant |T(G)|$, proving (ii). This completes the proof. □

In light of the claim, we will obtain a contradiction by showing that in fact $|T(H)| < |T(G)|$. Recall from Claim 8.8(i) that X' is an independent set in H, so there is no triangle in H involving more than one vertex in X', that is, $|T(H)| = \sum_{x \in X'} K_3(x, H; \overline{X'})$. By the inclusion–exclusion principle, we have

$$|T(H)| - |T(G)| \leqslant \sum_{x \in X'} (K_3(x, H; \overline{X'}) - K_3(x, G; \overline{X'})) + |X'|^2 \cdot n. \quad (8.27)$$

Let $x \in X'$ and let $i \in [k-2]$ be such that $x \in X_i$. Let us count the change in triangles involving x and two vertices in $\overline{X'} = A \cup B$.

Define $L_i := A \setminus R_i$ as in the proof of Lemma 8.7. By construction, we have the following:

(H1) $R_i \cup L_i \cup B_1 \cup B_2 \cup X'$ is a partition of $V(H)$, and B_1, B_2, R_i, X' are independent sets of H.

(H2) $L_i \cup B_1 \subseteq N_H(x)$ and $(B_2 \cup X') \cap N_H(x) = \emptyset$ and $H[R_i, L_i]$, $H[B, L_i]$ are complete bipartite graphs.

Thus

$$\begin{aligned} K_3(x, H; \overline{X'}) &\overset{(H1),(H2)}{=} K_3(x, H; L_i) + K_3(x, H; L_i, R_i) + K_3(x, H; L_i, B_1) \\ &\quad + K_3(x, H; R_i, B_1) \end{aligned}$$

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$$
\begin{aligned}
&\overset{(H2)}{=} \quad e(H[L_i]) + |L_i|b_1 + |T(x)|(|L_i| + b_1)\\
&\overset{(8.26)}{\leqslant} \quad e(G[L_i]) + |L_i|b_1 + d_G(x, R_i)(|L_i| + b_1) + \delta^{1/14}m\\
&\leqslant \quad K_3(x, G; \overline{X'}) - (\alpha - \delta^{1/14})m < K_3(x, G; \overline{X'}) - \alpha m/2,
\end{aligned}
$$

where we used Lemma 8.7(ii) for the penultimate inequality. This together with (8.27) implies that

$$
|T(H)| - |T(G)| \leqslant -|X'|(\alpha m/2 - |X'|n) \overset{(8.19)}{\leqslant} -|X'| \cdot (\alpha/4) \cdot m \leqslant -\alpha m/4,
$$

a contradiction to Claim 8.8(ii). Thus G is not a counterexample to Theorem 1.7, and we have proved Theorem 1.7 in this case.

8.2. The intermediate case when m is small. In this section, we will similarly obtain a contradiction to our assumption that G is a worst counterexample to Theorem 1.7 in the case when

$$
m < Cn. \tag{8.28}
$$

This case has a slightly different flavour from the rest of the proof. Indeed, in all other cases, we are eventually able to obtain from G an (n, e)-graph H with strictly fewer triangles than G, a contradiction. However, in the case when $m < Cn$, we can only guarantee an (n, e)-graph H with *at most* as many triangles as G but which lies in $\mathcal{H}(n, e)$. This is enough to prove that $g_3(n, e) = h(n, e)$, but not enough to prove that every extremal graph lies in $\mathcal{H}(n, e)$. This is not surprising, as when $m < Cn$, our graph G is very close indeed to a graph in $\mathcal{H}(n, e)$. Recall from the very beginning of the proof in Section 5 that our choice of extremal graph G was not arbitrary: we chose G according to three criteria (C1)–(C3), which ensure that G minimizes/maximizes certain graph parameters. Note that (C_3) has not affected the proof until now. In this part of the proof, we are required to analyse the transformations $G \to G^1 \to \cdots \to G^r \to H$ that take us from G to H. Using $K_3(G) = K_3(G^1) = \cdots = K_3(G^r) = K_3(H)$, we will show that for each i the graph G^i contradicts the choice of G according to (C1)–(C3) or $G^i \in \mathcal{H}(n, e)$. Then some additional work is required to show that this latter consequence implies that actually G itself lies in $\mathcal{H}(n, e)$, also a contradiction.

We follow all arguments until the end of Section 7. In particular, all definitions from Section 6.3 apply. Now (6.31) and (8.28) imply that Z has constant size, namely,

$$
|Z| \leqslant \frac{2C}{\xi}. \tag{8.29}
$$

Recall the definition of R'_k in Section 7.1. The number of $x \in A_k$ that have at least one neighbour in Z_k is at most

$$\sum_{z \in Z_k} d_G(z, A_k) \overset{P3(G)}{\leqslant} |Z_k|\delta n \leqslant \frac{2C\delta n}{\xi} = \frac{2\sqrt{\delta} n}{\xi} < \frac{\xi n}{2} = |R_k| - |R'_k|,$$

and so for all $x \in R'_k$, we have $d_G(x, Z_k) = 0$. Recalling the definition of $\varDelta$ in (7.3), we have

$$\varDelta = 0. \tag{8.30}$$

This will imply that Transformations 1–3 now do not increase the number of triangles. Thus by applying Lemmas 7.3, 7.5 and 7.8, we can easily obtain a graph G' with $K_3(G') = K_3(G)$ in which, for all $i \in [k-1]$, we have $E(G'[A_i]) = \emptyset$ (Lemma 8.14); $Y_i = \emptyset$ (Lemma 8.15); and $E(G'[X_i]) = \emptyset$ (Lemma 8.17). The final step is to further transform G' to another graph $G'' \in \mathcal{H}(n, e)$ with the same number of triangles. This proves that $g_3(n, e) = h(n, e)$. However, as mentioned above, we must prove that $G' \in \mathcal{H}(n, e)$. The next subsection contains some auxiliary results that we will need to achieve this.

8.2.1. Lemmas for characterizing extremal graphs. To compare G to some $H \in \mathcal{H}(n, e)$ that differs slightly from G, we need to compare our usual max-cut partition $A_1, \ldots, A_k$ of G with a canonical partition of H, which is $A^*_1, \ldots, A^*_{k-2}, B$ when $H \in \mathcal{H}_1(n, e)$ and $A^*_1, \ldots, A^*_k$ when $H \in \mathcal{H}_2(n, e)$. Recall that, given $U \subseteq V(G) = V(H)$, we say that G and H *only differ at* U if $E(G) \triangle E(H) \subseteq \binom{U}{2}$. The first lemma will be used in the case when $G' \in \mathcal{H}_1(n, e)$ (this is the easier case).

LEMMA 8.9. *Let $H \in \mathcal{H}_1(n, e)$ with $K_3(H) = K_3(G)$, $\varDelta(H[A_i]) \leqslant 2\gamma n$ for every $i \in [k]$ and $e(H[A_i, A_j]) > 0$ for every $ij \in \binom{[k]}{2}$. Then the following properties hold.*

(i) *If $A^*_1, \ldots, A^*_{k-2}, B$ is a canonical partition of H, then $B = A_p \cup A_q$ for some $pq \in \binom{[k]}{2}$ and there is a permutation σ of $[k]$ such that $A_i = A^*_{\sigma(i)}$ for all $i \in [k] \setminus \{p, q\}$. Furthermore, $e(\overline{H}[A_s, A_t]) > 0$ for some $st \in \binom{[k]}{2}$ only if $\{p, q\} = \{s, t\}$.*

(ii) *If H and G only differ at $A_{s'} \cup A_{t'}$, then $H[A_{s'}, A_{t'}]$ is complete.*

Proof. For (i), let $S \in \{A^*_1, \ldots, A^*_{k-2}, B\}$. Suppose for some $i \in [k]$, we have $A_i \cap S, A_i \cap \overline{S} \neq \emptyset$. Then, as $H[S, \overline{S}]$ is complete, there exists $v \in A_i$ with

$$d_H(v, A_i) \geqslant \frac{|A_i|}{2} \overset{P1(G)}{\geqslant} \frac{(c-\beta)n}{2} > 2\gamma n \geqslant \varDelta(H[A_i]),$$

a contradiction. So either $A_i \subseteq S$ or $A_i \subseteq \overline{S}$. Since $e(H[A_i, A_j]) > 0$ for every $ij \in \binom{[k]}{2}$, and every A_p^* with $p \in [k-2]$ is an independent set in H, A_p^* must contain exactly one A_i. This proves the first part of (i). Suppose now $e(\overline{H}[A_s, A_t]) > 0$ for some $st \in \binom{[k]}{2}$. Then the fact that $H[A_1^*, \ldots, A_{k-2}^*, B]$ is complete multipartite implies that $B = A_s \cup A_t$.

For (ii), suppose that H and G only differ at $A_{s'} \cup A_{t'}$ and $e(\overline{H}[A_{s'}, A_{t'}]) > 0$. Then by (i), we have $B = A_{s'} \cup A_{t'}$. So $H[\overline{B}] = G[\overline{B}]$ is complete $(k-2)$-partite and $H[B, \overline{B}] = G[B, \overline{B}]$ is complete. Since $K_3(H) = K_3(G)$, we have $K_3(G[B]) = K_3(H[B]) = 0$, so $G[B]$ is triangle-free. Thus $G \in \mathcal{H}_1(n, e)$ with canonical partition $A_1^*, \ldots, A_{k-2}^*, B$, contradicting the choice of G. $\square$

The next lemma analyses a graph $H \in \mathcal{H}_2(n, e)$ obtained by making some small changes to G.

LEMMA 8.10. *Let $H \in \mathcal{H}_2^{\min}(n, e) \backslash \mathcal{H}_1(n, e)$ be such that $|E(G) \triangle E(H)| \leqslant \delta n^2$ and $H[A_i, A_j]$ is complete for every $ij \in \binom{[k-1]}{2}$. Suppose that*

$$d := \max_{i \in [k]; v \in V(G)} |d_G(v, A_i) - d_H(v, A_i)| \leqslant \gamma n. \tag{8.31}$$

Let $A_1^*, \ldots, A_k^*$ be a canonical partition of H. Then $R_k \subseteq A_k^*$ and there exists a permutation σ of $[k]$ such that $|A_i \triangle A_{\sigma(i)}^*| \leqslant k\beta n$ for all $i \in [k]$, and the following properties hold:

(i) *If there exists $p \in [k-1]$ for which $Z_k = Z_k^p$, then $Z_k \subseteq A_{\sigma(p)}^* \cup A_k^*$. Moreover, there is $j \in [k-1]$ such that $A_{\sigma(i)}^* = A_i$ for all $i \in [k-1] \setminus \{j, p\}$, and if $j \neq p$, then $A_{\sigma(j)}^* \subseteq A_j \subseteq A_{\sigma(j)}^* \cup A_k^*$.*

(ii) *If $d \leqslant \delta n$ and $Y = \emptyset$, then $A_k \subseteq A_k^*$, and there is $j \in [k-1]$ such that $A_{\sigma(i)}^* = A_i$ for all $i \in [k-1] \setminus \{j\}$, and $A_{\sigma(j)}^* \subseteq A_j \subseteq A_{\sigma(j)}^* \cup A_k^*$.*

Proof. We require a claim.

CLAIM 8.11. *There exists a permutation σ of $[k]$ with $\sigma(k) = k$ such that the following hold:*

(1) *for all $i \in [k]$, we have $|A_i \triangle A_{\sigma(i)}^*| \leqslant k\beta n$;*

(2) *$R_k \subseteq A_k^*$;*

(3) *for all $i \in [k-1]$, we have $A_{\sigma(i)}^* \setminus A_i \subseteq A_k$ and $A_i \setminus A_{\sigma(i)}^* \subseteq A_k^*$;*

(4) *$A_j \subseteq A_{\sigma(j)}^*$ for all but at most one $j \in [k-1]$.*

Proof of Claim. We start with (1). Corollary 4.4(iii) implies that

$$\sum_{ij\in\binom{[k]}{2}} e(\overline{H}[A_i^*, A_j^*]) \leqslant n. \tag{8.32}$$

Further,

$$e(G[A_i]) \overset{(6.27)}{\leqslant} \delta m \overset{(8.28)}{\leqslant} \delta C n \overset{(7.1)}{=} \sqrt{\delta} n.$$

Suppose that there exist $i, j \in [k]$ such that $\beta n \leqslant |A_i \cap A_j^*| \leqslant |A_i| - \beta n$. Then

$$\begin{aligned}|E(G) \triangle E(H)| &\geqslant e(H[A_i \cap A_j^*, A_i \setminus A_j^*]) - e(G[A_i]) \\ &\overset{(8.32)}{\geqslant} |A_i \cap A_j^*||A_i \setminus A_j^*| - n - \sqrt{\delta} n \geqslant \frac{\beta^2 n^2}{2},\end{aligned}$$

a contradiction. Thus, for all $i, j \in [k]$, either $|A_i \cap A_j^*| \leqslant \beta n$ or $|A_i \cap A_j^*| \geqslant |A_i| - \beta n$. Since for all $i \in [k]$, we have

$$|A_i| \overset{P1(G)}{\geqslant} n - (k-1)cn - \beta n \overset{(6.3)}{\geqslant} (k-1)\alpha n - \beta n > k\beta n,$$

the first alternative cannot hold for every $j \in [k]$. Thus there is exactly one $j \in [k]$ for which $|A_i \cap A_j^*| \geqslant |A_i| - \beta n$. Suppose that there is $j \in [k]$ and $1 \leqslant i_1 < i_2 \leqslant k$ such that $|A_{i_p} \cap A_j^*| \geqslant |A_{i_p}| - \beta n$ for $p \in [2]$. Then

$$e(G[A_j^*]) \geqslant |A_{i_1} \cap A_j^*||A_{i_2} \cap A_j^*| - m \geqslant (|A_{i_1}| - \beta n)(|A_{i_2}| - \beta n) - \eta n^2 \overset{P1(G)}{>} 2\delta n^2,$$

and so $e(H[A_j^*]) > 0$, a contradiction. That is, there is a permutation σ of $[k]$ for which

$$|A_i \triangle A_{\sigma(i)}^*| = |A_i \setminus A_{\sigma(i)}^*| + |A_{\sigma(i)}^* \setminus A_i| \leqslant \beta n + \sum_{j\in[k]\setminus\{\sigma(i)\}} |A_j \cap A_{\sigma(i)}^*| \leqslant k\beta n. \tag{8.33}$$

Since

$$|A_k| \overset{P1(G)}{\leqslant} n - (k-1)cn + \beta n \overset{(6.3)}{\leqslant} cn - \sqrt{2\alpha} n + \beta n \overset{P1(G)}{<} |A_i| - \sqrt{\alpha} n$$

for all $i \in [k-1]$, we have $|A_{\sigma(k)}^*| \leqslant |A_{\sigma(i)}^*| - \sqrt{\alpha} n/2$ for all $i \in [k-1]$. Since $|A_1^*| \geqslant \cdots \geqslant |A_k^*|$, this implies that $\sigma(k) = k$. This proves (1).

Now, for all $i \in [k]$ and $v \in V(G)$, we have

$$|d_G(v, A_i) - d_H(v, A_{\sigma(i)}^*)| \overset{(8.31)}{\leqslant} d + |A_i \triangle A_{\sigma(i)}^*| \overset{(8.33)}{\leqslant} \gamma n + k\beta n \leqslant 2\gamma n. \tag{8.34}$$

For (2), let $x \in R_k$ and $i \in [k-1]$. We have

$$d_H(x, A^*_{\sigma(i)}) \overset{(8.34)}{\geqslant} d_G(x, A_i) - 2\gamma n \overset{P5(G)}{\geqslant} |A_i| - \xi n - 2\gamma n \overset{P1(G)}{\geqslant} (c - \beta - \xi - 2\gamma)n > 0.$$

Since $A^*_{\sigma(i)}$ is an independent set in H, we have that $v \notin A^*_{\sigma(i)}$. But $i \in [k-1]$ was arbitrary, so $v \in A^*_{\sigma(k)} = A^*_k$, proving (2).

For (3), suppose that $i \in [k-1]$ and there is some $v \in A^*_{\sigma(i)} \setminus A_i$. Then, since $A^*_{\sigma(i)}$ is independent in H, we have that

$$d_H(v, A_i) \overset{(8.31)}{\leqslant} d_G(v, A_i) + d \overset{(8.34)}{\leqslant} d_H(v, A^*_{\sigma(i)}) + 2\gamma n + d \leqslant 3\gamma n < (c - \beta)n \overset{P1(G)}{<} |A_i|.$$

But $H[A_i, A_j]$ is complete for all $ij \in \binom{[k-1]}{2}$, so $v \notin \bigcup_{j \in [k-1] \setminus \{i\}} A_j$. Thus $v \in A_k$, proving the first part of (3). For the second part, suppose that $v \in A_i \setminus A^*_{\sigma(i)}$ and let $j \in [k-1] \setminus \{i\}$. Then

$$d_H(v, A^*_{\sigma(j)}) \overset{(8.34)}{\geqslant} d_G(v, A_j) - 2\gamma n \overset{P2(G)}{=} |A_j| - 2\gamma n \overset{P1(G)}{>} (c - \beta - 2\gamma)n > 0.$$

So $u \notin A^*_{\sigma(j)}$ and so $u \in A^*_k$, completing the proof of (3).

Finally, for (4), suppose that there is $ij \in \binom{[k-1]}{2}$ for which there exist $v_i \in A_i \setminus A^*_{\sigma(i)}$ and $v_j \in A_j \setminus A^*_{\sigma(j)}$. Since $H[A_i, A_j]$ is complete, we have $v_i v_j \in E(H)$. But (3) implies that $v_i, v_j \in A^*_k$, a contradiction. This proves (4) and completes the proof of the claim. □

We will now prove Item (i) of the lemma. So suppose there is $p \in [k-1]$ for which $Z_k = Z^p_k$. Let $i \in [k-1] \setminus \{p\}$ and $y \in Z^p_k$. Then

$$d_H(y, A^*_{\sigma(i)}) \overset{(8.34)}{\geqslant} d_G(y, A_i) - 2\gamma n \overset{P4(G)}{=} |A_i| - 2\gamma n \overset{P1(G)}{\geqslant} (c - \beta - 2\gamma)n > 0.$$

Thus $y \notin A^*_{\sigma(i)}$ and so $y \in A^*_{\sigma(p)} \cup A^*_k$. Therefore, using Claim 8.11(2), $A_k = R_k \cup Z^p_k \subseteq A^*_{\sigma(p)} \cup A^*_k$. But, by Claim 8.11(3), for all $i \in [k-1]$, we have $A^*_{\sigma(i)} \setminus A_i \subseteq A_k \subseteq A^*_{\sigma(p)} \cup A^*_k$. Thus $A^*_{\sigma(i)} \subseteq A_i$ for all $i \in [k-1] \setminus \{p\}$. By Claim 8.11(4), this implies that there is $j \in [k-1]$ such that $A_i = A^*_{\sigma(i)}$ for all $i \in [k-1] \setminus \{j, p\}$. If $j \neq p$, then by Claim 8.11(3), $A^*_{\sigma(j)} \subseteq A_j \subseteq A^*_{\sigma(j)} \cup A^*_k$, completing the proof of (i).

For (ii), we may now assume that $d \leqslant \delta n$ and $Y = \emptyset$. Inequality (8.34) is replaced by the stronger statement

$$\begin{aligned} |d_G(v, A_i) - d_H(v, A^*_{\sigma(i)})| &\leqslant d + |A_i \triangle A^*_{\sigma(i)}| \\ &\leqslant \delta n + k\beta n \leqslant \sqrt{\beta} n \quad \text{for all } v \in V(G). \end{aligned} \tag{8.35}$$

Let $i \in [k-1]$ and $z \in Z_k = X$. Then, using the definition of X,

$$d_H(z, A^*_{\sigma(i)}) \overset{(8.35)}{\geqslant} d_G(z, A_i) - \sqrt{\beta} n \geqslant \gamma n - \sqrt{\beta} n > 0.$$

Thus $z \notin A^*_{\sigma(i)}$ and so $z \in A^*_k$. Combining this with Claim 8.11(2), we see that again $A_k = R_k \cup X \subseteq A^*_k$. Then Claim 8.11(3) implies that for all $i \in [k-1]$, we have $A^*_{\sigma(i)} \setminus A_i \subseteq A_k \subseteq A^*_k$ and so $A^*_{\sigma(i)} \subseteq A_i$. By Claim 8.11(4), there is $j \in [k-1]$ such that $A^*_{\sigma(i)} = A_i$ for all $i \in [k-1] \setminus \{j\}$; and $A^*_{\sigma(j)} \subseteq A_j \subseteq A^*_{\sigma(j)} \cup A^*_k$. This completes the proof of (ii). □

The final lemma in this subsection will be used to prove that, for all $i \in [k-1]$, we have $E(G[A_i]) = \emptyset$ (Lemma 8.14) and $Y_i = \emptyset$ (Lemma 8.15). Its proof uses part (i) of the previous lemma.

LEMMA 8.12. *Let $p \in [k-1]$ and $z \in A_p \cup A_k$ be such that $T := N_G(z, A_p)$ satisfies $1 \leqslant |T| \leqslant \gamma n$; and let $S \subseteq N_{\overline{G}}(z, R_k)$ satisfy $|S| = |T|$. Suppose further that $Z_k = Z^p_k$ and $P_3(yz, G; R_k) = 0$ for all $y \in T$, and $G[S, \bigcup_{i \in [k-1] \setminus \{p\}} A_i]$ is complete. Obtain H from G by replacing zy for all $y \in T$ with zx for all $x \in S$ and suppose that $K_3(H) = K_3(G)$. Then H is an (n, e)-graph, which does not lie in $\mathcal{H}_2(n, e) \setminus \mathcal{H}_1(n, e)$.*

Proof. Suppose that the lemma does not hold. Then by definition, H is an (n, e)-graph and so $H \in \mathcal{H}^{\min}_2(n, e) \setminus \mathcal{H}_1(n, e)$. Let $A^*_1, \ldots, A^*_k$ be a canonical partition of H. Clearly, $|E(G) \triangle E(H)| = |S| + |T| \leqslant 2\gamma n$ and for all $ij \in \binom{[k-1]}{2}$ we have $H[A_i, A_j] = G[A_i, A_j]$ is complete by P4(G); note also that (8.31) holds. So H satisfies the conditions of Lemma 8.10(i). Suppose without loss of generality that the permutation σ guaranteed by Lemma 8.10 is the identity permutation. By definition, H and G only differ at $A_p \cup A_k$. We will obtain a contradiction via the next claim.

CLAIM 8.13. *We have the following properties:*

(i) *$A^*_i = A_i$ for all $i \in [k-1] \setminus \{p\}$ and $R_k \subseteq A^*_k$;*

(ii) *$z \in A^*_p$ and $N_G(z, A_p) \cap A^*_p \neq \emptyset$;*

(iii) *there exists $j \in [k-1] \setminus \{p\}$ for which $G[A_j, R_k]$ is not complete.*

Proof of Claim. We first prove (i). By Lemma 8.10, $R_k \subseteq A^*_k$ and by Lemma 8.10(i), there exists $j \in [k-1]$ such that $A^*_i = A_i$ for all $i \in [k-1] \setminus \{j, p\}$ and $A^*_j \subseteq A_j$. We may assume that $j \neq p$ and there is some $v \in A_j \setminus A^*_j \subseteq A^*_k$, for otherwise we are done. Further, we have that $S \subseteq R_k \subseteq A_k^*$. Thus, recalling that A_k^* is an independent set in H and that G and H only differ at $A_p \cup A_k$,

$$0 = d_H(v, A_k^*) \geqslant d_H(v, S) = d_G(v, S),$$

a contradiction to the fact that $G[S, \bigcup_{i \in [k-1]\setminus\{p\}} A_i]$ is complete. This proves (i).

Thus $A_p^* \cup A_k^* = A_p \cup A_k =: B$. In particular, $H[B]$ is bipartite with bipartition A_p^*, A_k^*. Now, $d_H(z, A_k^*) \geqslant d_H(z, R_k) \geqslant |S| > 0$. Since A_k^* is an independent set in H, we have that $z \in A_p^*$. Suppose that $T \cap A_p^* = \emptyset$. Let G' be obtained from G by removing the edges xz for all $x \in T$. Then $G' \subseteq H$, and so $G'[B]$ is bipartite with bipartition A_p^*, A_k^*. Using that $T \cap A_p^* = \emptyset$ and $T \subseteq B$, we see that $T \subseteq A_k^*$. This together with $z \in A_p^*$ implies that $G[B]$ is bipartite (with bipartition A_p^*, A_k^*). But the fact that G and H only differ at $A_p \cup A_k$ and the definition of $\mathcal{H}_2(n, e)$ imply that $G[\overline{B}]$ is $(k-2)$-partite. Thus G is k-partite with partition $A_1^*, \ldots, A_k^*$. Then Corollary 4.4(i) implies that $G \in \mathcal{H}_2(n, e)$, a contradiction. Thus $T \cap A_p^* \neq \emptyset$. This proves (ii).

For (iii), suppose that $G[A_i, R_k]$ is complete for every $i \in [k-1] \setminus \{p\}$. Then, by P4($G$), we have that $A_k = R_k \cup Z_k^p$ is complete in G to $\bigcup_{j \in [k-1]\setminus\{p\}} A_j$. Further, P2($G$) implies that $G[A_i, A_j]$ is complete for all $ij \in \binom{[k-1]}{2}$. Thus $G[B, \overline{B}]$ is complete. Now, the facts that G and H only differ at $A_p \cup A_k$ and $K_3(H) = K_3(G)$ imply that $K_3(G[B]) = K_3(H[B]) = 0$ since $H[B]$ is bipartite. That is, $G[B]$ is triangle-free, so $G \in \mathcal{H}_1(n, e)$, a contradiction to (C1). This completes the proof of the claim. □

By part (iii) of the claim, we can choose $j \in [k-1] \setminus \{p\}$; $u \in A_j = A_j^*$ and $v \in R_k \subseteq A_k^*$ such that $uv \notin E(G)$. As $G[S, \bigcup_{i \in [k-1]\setminus\{p\}} A_i]$ is complete, we have $v \in R_k \setminus S$ and so $N_G(v) = N_H(v)$. By part (ii) of the claim, pick some $y \in T \cap A_p^*$. Since $H[\{v\}, A_j^*]$ is not complete, we have that $H[\{v\}, A_p^*]$ is complete by the definition of $\mathcal{H}_2(n, e)$. Thus $\{y, z\} \subseteq N_H(v) = N_G(v)$. But then $P_3(yz, G; R_k) \geqslant 1$, a contradiction. □

8.2.2. Refining the structure of G via Transformations 1–3. We now return to our extremal graph G and analyse the effects of Transformations 1–3 on the number of triangles to obtain additional structural information. To do this, we will apply each 'local' transformation once, changing edges at a single vertex to obtain a new graph G^1. This is the part of the proof at which we require the full strength of Lemmas 7.3, 7.5 and 7.8 to carefully analyse $K_3(G^1) - K_3(G)$. As we mentioned earlier, this turns out to now equal zero, and we show that $G^1 \in \mathcal{H}(n, e)$.

The first step is to apply Transformation 1 (Lemma 7.3) to show that the only bad edges in G lie in A_k.

LEMMA 8.14. *$E(G[A_i]) = \emptyset$ for all $i \in [k-1]$.*

Proof. Suppose to the contrary that $\bigcup_{i\in[k-1]} E(G[A_i]) \neq \emptyset$. Without loss of generality, assume $e(G[A_{k-1}]) > 0$. Then P3(G) implies that there is some $z \in Z_{k-1}$ with $d_G(z, A_{k-1}) \geqslant 1$. Let $z =: z_1, \ldots, z_p$ be an ordering of $Z \setminus Z_k$. Apply Lemma 7.3 to G to obtain an (n, e)-graph G^1 that satisfies $J(1, 1)$–$J(3, 1)$. Then $J(3, 1)$ implies that

$$K_3(G^1) - K_3(G) \leqslant \sum_{y\in N_G(z,A_{k-1})} \left(\Delta - |Z_k \setminus Z_k^{k-1}| - P_3(yz, G; R_k)\right) \overset{(8.30)}{\leqslant} 0.$$

As $K_3(G^1) \geqslant K_3(G)$, we have equality in the above. Then $J(3, 1)$ implies that $G[S, \bigcup_{i\in[k-2]} A_i]$ is complete, where $S := N_{G^1\setminus G}(z, R_k) \subseteq N_{\overline{G}}(z, R_k)$. Furthermore, $Z_k = Z_k^{k-1}$ and $P_3(yz, G; R_k) = 0$ for all $y \in N_G(z, A_{k-1})$.

By $J(2, 1)$, for all $i \in [k]$ and $v \in V(G)$, we have

$$|d_G(v, A_i) - d_{G^1}(v, A_i)| \leqslant d_G(z, A_{k-1}) \overset{P3(G)}{\leqslant} \delta n.$$

We also have that $\Delta(G^1[A_i]) \leqslant \Delta(G[A_i]) \leqslant \delta n$. Note that

$$\sum_{ij\in\binom{[k]}{2}} e(G^1[A_i, A_j]) = \sum_{ij\in\binom{[k]}{2}} e(G[A_i, A_j]) + d_G(z, A_{k-1}).$$

Since $K_3(G^1) = K_3(G)$, the choice of G, in particular (C2), implies that we must have $G^1 \in \mathcal{H}(n, e)$. But G^1 satisfies the properties of H in Lemma 8.12 with $p := k-1$, so $G^1 \in \mathcal{H}_1^{\min}(n, e)$. Then G^1 clearly satisfies the hypothesis of Lemma 8.9 and G^1 and G only differ at $A_{k-1} \cup A_k$. Lemma 8.9(ii) implies that $G^1[A_{k-1}, A_k]$ is complete. But

$$\begin{aligned} e(\overline{G^1}[A_{k-1}, A_k]) &\geqslant d_{\overline{G^1}}(z, R_k) = d_{\overline{G}}(z, R_k) - d_G(z, A_{k-1}) \\ &\geqslant d_{\overline{G}}(z, A_k) - |Z| - \Delta(G[A_{k-1}]) \overset{P5(G)}{\geqslant} \xi n - \delta n - \delta n \geqslant \xi n/2, \end{aligned}$$

a contradiction. This completes the proof of the lemma. □

The second step is to apply Transformation 2 (Lemma 7.5) to show that Y is empty. Then the only bad edges lie in A_k and by Lemma 6.12, they all have both endpoints in X. (By (8.29), this means that there are only constantly many bad edges.)

LEMMA 8.15. *$Y_i = \emptyset$ for all $i \in [k-1]$.*

Proof. Suppose, without loss of generality, that $Y_{k-1} \neq \emptyset$ and fix an arbitrary $y \in Y_{k-1}$. Let $\widehat{A}_i := A_i$ if $i \in [k-2]$, $\widehat{A}_{k-1} := A_{k-1} \cup \{y\}$ and $\widehat{A}_k := A_k \setminus \{y\}$. We may assume that $d_G(y, A_{k-1}) \geqslant 1$; otherwise, $\widehat{A}_1, \dots, \widehat{A}_k$ is a max-cut partition of G, which contradicts the choice of $A_1, \dots, A_k$, in particular (C3). Let $y =: y_1, y_2, \dots, y_q$ be an ordering of Y. Observe that G is a graph that satisfies the conclusions of Lemma 7.4 applied with $\ell := k-1$. Thus we can apply Lemma 7.5 to G with $\ell := k-1$ to obtain a graph G^1 satisfying $K(1, 1)$–$K(3, 1)$. By $K(3, 1)$,

$$K_3(G^1) - K_3(G) \leqslant \sum_{x \in N_G(y, A_{k-1})} \left(\Delta - \frac{\xi}{6\gamma} |Z_k \setminus Z_k^{k-1}| - P_3(xy, G; R_k) \right) \leqslant 0.$$

As $K_3(G^1) \geqslant K_3(G)$, we have equality in the above. Then $K(3, 1)$ implies that $G[S, \bigcup_{i \in [k-2]} A_i]$ is complete, where $S := N_{G^1 \setminus G}(y, R_k) \subseteq N_{\overline{G}}(y, R_k)$. Furthermore, $Z_k = Z_k^{k-1} = X_{k-1} \cup Y_{k-1}$ and $P_3(xy, G; R_k) = 0$ for all $x \in N_G(y, A_{k-1})$. Since $Z_k = X_{k-1} \cup Y_{k-1}$, by $K(1, 1)$, G^1 is obtained from G by replacing all edges from y to A_{k-1} with some nonedges from y to R_k, that is, $T(y)$ and $R(y)$ are empty. Also by $K(1, 1)$, we have that $\sum_{ij \in \binom{[k]}{2}} e(G^1[\widehat{A}_i, \widehat{A}_j]) \geqslant \sum_{ij \in \binom{[k]}{2}} e(G[A_i, A_j])$. Since $K_3(G^1) = K_3(G)$, we must have equality by (C2). But for all $i \in [k-1]$, we have $|\widehat{A}_i| \geqslant |\widehat{A}_k| = |A_k| - 1$, so (C3) implies that $G^1 \in \mathcal{H}^{\min}(n, e)$. Again, G^1 satisfies the properties of H in Lemma 8.12 with $k-1, y, G^1$ playing the roles of p, z, H, respectively. So we have that $G^1 \in \mathcal{H}_1^{\min}(n, e)$.

Let $A_1^*, \dots, A_{k-2}^*, B$ be a canonical partition of G^1. Note that G^1 satisfies the hypothesis of Lemma 8.9. Indeed,

$$\Delta(G^1[A_k]) \leqslant \Delta(G[A_k]) + d_G(y, A_{k-1}) \overset{P3(G)}{\leqslant} \delta n + \gamma n \leqslant 2\gamma n.$$

Further, G^1 and G only differ on $A_{k-1} \cup A_k$. Thus Lemma 8.9(ii) implies that $G^1[A_{k-1}, A_k]$ is complete. But by construction,

$$e(\overline{G^1}[A_{k-1}, A_k]) \geqslant d_{\overline{G^1}}(y, A_{k-1}) = |A_{k-1}|,$$

a contradiction. This completes the proof of the lemma. □

8.3. Obtaining a graph G_3. We will apply Lemma 7.8 to G to obtain a graph G_3 in which X_i is an independent set for all $i \in [k-1]$, but such that G_3 may contain constantly many more triangles than G. Then, applying further transformations to G_3, we deduce additional information about G.

Observe that by Propositions 8.14 and 8.15, G satisfies all the properties of G_2 in Lemma 7.6, so we can set $G_2 := G$ and, for all $i \in [k-1]$, set $A'_i := A_i$. Recall from the beginning of Section 7.4 that, for all $i \in [k-1]$ and $x, y \in X_i$, we define

$$D(x) := d_G(x, X \setminus X_i) \quad \text{and} \quad D(x, y) := |N_G(x, X \setminus X_i) \cap N_G(y, X \setminus X_i)|.$$

LEMMA 8.16. *Let G_3 be the (n, e)-graph obtained by applying Lemma 7.8 to G playing the role of G_2. Then, we have the following:*

(i) *G_3 has an $(A_1, \ldots, A_k; Z, 2\beta, \xi/4, 2\xi, \delta)$-partition and, for each $i \in [k-1]$, we have $e(\overline{G_3}[A_i, A_k]) \leqslant m_i$ with equality if and only if $E(G[X_i]) = \emptyset$.*

(ii) *For all $i \in [k-1]$, $E(G_3[A_i]) = \emptyset$ and $E(G_3[A_k]) = E(G[X_1, \ldots, X_k])$ and $d_{G_3}(x, A_i) \geqslant \gamma n$ for $x \in X_i$. Further, every pair in $E(G) \setminus E(G_3)$ lies in X_i for some $i \in [k-1]$, and every pair in $E(G_3) \setminus E(G)$ lies in $[X_i, A_i]$ for some $i \in [k-1]$.*

(iii) *For all $i \in [k-1]$ such that $X_i \neq \emptyset$, there exists $D_i \in \mathbb{N}$ such that $D(x) = D_i$ for all $x \in X_i$. Moreover, $P_3(xu, G_3) = a_i + D_i$ for all $x \in X_i$ and $u \in A_i$.*

(iv) *$K_3(G_3) \leqslant K_3(G) + |Z|^2 \cdot \max_{\substack{i \in [k-1] \\ x, y \in X_i}} (D_i - D(x, y))$ with equality only if $G[X_i]$ is triangle-free and $N_G(x, A_i) \cap N_G(y, A_i) = \emptyset$ for all $i \in [k-1]$ and $xy \in E(G[X_i])$.*

(v) *Let G' be such that $V(G') = V(G_3)$ and $E(G') \triangle E(G_3) \subseteq \bigcup_{i \in [k-1]} \{ax : a \in A_i, x \in X_i\}$ and $e(G'[X_i, A_i]) = e(G_3[X_i, A_i])$ for all $i \in [k-1]$. Then $K_3(G') = K_3(G_3)$.*

Proof. Parts (i) and (ii) and the fact that

$$K_3(G_3) \leqslant K_3(G) + |Z|^2 \cdot \max_{\substack{i \in [k-1] \\ x, y \in X_i}} (D(x) - D(x, y)) \tag{8.36}$$

with equality only if $G[X_i]$ is triangle-free and $N_G(x, A_i) \cap N_G(y, A_i) = \emptyset$ for all $i \in [k-1]$ and $xy \in E(G[X_i])$ follow immediately from Lemmas 7.8 and 7.7 L(2). Apply Lemma 7.9 to G_3 to obtain an (n, e)-graph G_4 on the same vertex set satisfying Lemma 7.9(i)–(v). Then, by Lemma 7.9(i), for every $xy \in E(G_3) \triangle E(G_4)$, there exists $i \in [k-1]$ such that $x \in X_i$ and $y \in A_i$. Let $i \in [k-1]$, $u \in A_i$ and $x \in X_i$. Then by Lemma 7.8(ii) and 7.9(i),(iii) we have, for $j \in \{3, 4\}$, that $d_{G_j}(u, A_i) = d_{G_j}(x, R_k) = d_{G_j}(x, X_i) = 0$ and $X \setminus X_i \subseteq N_{G_j}(u)$. So $P_3(xu, G_j) = a_i + D(x)$. Clearly, if G' is any graph as in (v), then these equalities also hold for G', in particular

$$P_3(xu, G') = a_i + D(x) = P_3(xu, G_j). \tag{8.37}$$

Suppose that there exist $i \in [k-1]$ and $x, y \in X_i$ such that $D(x) \neq D(y)$. Then Lemma 7.9(iv) implies that

$$K_3(G_4) \leqslant K_3(G_3) - \frac{\xi n}{20} \overset{(8.36)}{\leqslant} K_3(G) + |Z|^2 \cdot \max_{\substack{i \in [k-1] \\ x, y \in X_i}} (D(x) - D(x, y)) - \frac{\xi n}{20} \tag{8.38}$$

$$\leqslant K_3(G) + |Z|^3 - \frac{\xi n}{20} \overset{(8.29)}{\leqslant} K_3(G) + \frac{8C^3}{\xi^3} - \frac{\xi n}{20} < K_3(G) - \frac{\xi n}{30},$$

a contradiction. This proves (iii), and together with (8.36), we also obtain (iv). For (v), observe that there is no triangle in G_3 or G' that contains more than one A_i-X_i edge since A_i and X_i are independent sets in both graphs. Thus

$$K_3(G') - K_3(G_3) = \sum_{e \in E(G') \setminus E(G_3)} P_3(e, G') - \sum_{e \in E(G_3) \setminus E(G')} P_3(e, G_3) \overset{(8.37)}{=} 0,$$

where the last equality follows from the hypotheses on G' in (v) and (8.37). □

This allows us to conclude that G and G_3 are in fact the same graph.

LEMMA 8.17. *The following hold in G:*

(i) *For all $ij \in \binom{[k-1]}{2}$, the graph $G[X_i, X_j]$ is either complete or empty.*

(ii) *$G = G_3$, so $E(G[X_i]) = \emptyset$ for all $i \in [k-1]$.*

Proof. First, we will show the following claim.

CLAIM 8.18. *If $ij \in \binom{[k-1]}{2}$ is such that $E(G[X_i, X_j]) \neq \emptyset$, then*

$$e(G_3[X_i, A_i]) + e(G_3[X_j, A_j]) \leqslant cn + \sqrt{\beta} n. \tag{8.39}$$

Proof of Claim. To prove the claim, let $x \in X_i$ and $y \in X_j$ such that $xy \in E(G)$. Then $xy \in E(G_3)$ by Lemma 8.16(ii). By Lemma 8.16(v), we can obtain a graph G' from G_3 with the stated properties and such that

$$d_{G'}(x, A_i) = \min\{|A_i|, e(G_3[X_i, A_i])\} \quad \text{and} \tag{8.40}$$
$$d_{G'}(y, A_j) = \min\{|A_j|, e(G_3[X_j, A_j])\}.$$

That is, we obtain G' by moving as many X_i-A_i edges as possible to x, and similarly for y and X_j-A_j edges. By P4(G_3), x is complete to $\bigcup_{\ell \in [k-1] \setminus \{i\}} A_\ell$

in G_3 and y is complete to $\bigcup_{\ell \in [k-1]\setminus\{j\}} A_\ell$ in G_3. Thus the same is true in G'. Therefore, using Lemma 8.16(iv) and (v),

$$K_3(G') = K_3(G_3) \leqslant K_3(G) + |Z|^3 \overset{(8.29)}{\leqslant} K_3(G) + \frac{8C^3}{\xi^3} \leqslant K_3(G) + \frac{\beta n}{2}. \quad (8.41)$$

Corollary 4.18 applied with $p := \beta n/2$ implies that

$$(k-2)cn + \beta n \geqslant P_3(xy, G') \geqslant \sum_{\ell \in [k-1]\setminus\{i,j\}} |A_\ell| + d_{G'}(x, A_i) + d_{G'}(y, A_j)$$

and so

$$d_{G'}(x, A_i) + d_{G'}(y, A_j) \overset{P1(G)}{\leqslant} (k-2)cn + \beta n - (k-3)(cn - \beta n) \leqslant cn + \sqrt{\beta} n. \quad (8.42)$$

Now, P1(G) implies that $|A_i| + |A_j| \geqslant 2cn - 2\beta n > cn + \sqrt{\beta} n$, so without loss of generality from (8.40) we may suppose that $d_{G'}(x, A_i) = e(G_3[X_i, A_i])$. If $d_{G'}(y, A_j) = |A_j|$, then

$$e(G_3[X_i, A_i]) \leqslant cn + \sqrt{\beta} n - |A_j| \overset{P1(G)}{\leqslant} cn + \sqrt{\beta} n - (cn - \beta n) \leqslant 2\sqrt{\beta} n.$$

But this is a contradiction because $e(G_3[X_i, A_i]) \geqslant d_{G_3}(x, A_i) \geqslant \gamma n$ by Lemma 8.16(ii). Thus $d_{G'}(y, A_j) = e(G_3[X_j, A_j])$, and the claim follows from (8.42). $\square$

Suppose that (i) does not hold. Then there exist $ij \in \binom{[k-1]}{2}$; $xy \in E(G[X_i, X_j])$ and $x'y' \in E(\overline{G}[X_i, X_j])$ such that $x, x' \in X_i$ and $y, y' \in X_j$. These adjacencies are the same in G_3. Without loss of generality, we may assume that $x \neq x'$ (but it could be the case that $y = y'$). In particular, $|X_i| \geqslant 2$.

CLAIM 8.19. *There exists a graph G'' that satisfies Lemma 8.16(v) and such that*

$$d_{G''}(x', A_i) + d_{G''}(y', A_j) \leqslant cn - \xi n/5.$$

Proof of Claim. Let

$$p_i := e(G_3[X_i, A_i]) - 2\sqrt{\beta} n.$$

We claim that there is some G'' such that $E(G'') \triangle E(G_3) \subseteq \{av : a \in A_i, v \in X_i\}$ and $e(G''[X_i, A_i]) = e(G_3[X_i, A_i])$ in which $d_{G''}(x, A_i) = p_i$. To show that G'' exists, since $p_i < e(G_3[X_i, A_i])$ and $|X_i| \geqslant 2$, it suffices to show that $p_i \leqslant |A_i|$, then we can obtain G'' by moving all but $2\sqrt{\beta}n$ X_i-A_i edges to x. But this does indeed hold: Claim 8.18 implies that

$$p_i \leqslant cn - \sqrt{\beta}n \overset{P1(G)}{<} |A_i|,$$

as required. We have

$$e(G''[X_i, A_i]) = e(G_3[X_i, A_i]) = p_i + 2\sqrt{\beta}n = d_{G''}(x, A_i) + 2\sqrt{\beta}n.$$

Thus $d_{G''}(x', A_i) \leqslant 2\sqrt{\beta}n$. Furthermore,

$$d_{G''}(y', A_j) = d_{G_3}(y', A_j) \overset{P5(G_3)}{\leqslant} |A_j| - \xi n/4 \overset{P1(G)}{\leqslant} cn + \beta n - \xi n/4.$$

Then $d_{G''}(x', A_i) + d_{G''}(y', A_j) \leqslant cn + \beta n + 2\sqrt{\beta}n - \xi n/4 \leqslant cn - \xi n/5$, as required. □

Apply Claim 8.19 to obtain G''. Proposition 6.12(i) implies that $N_G(x')$ and $N_G(y')$ are disjoint from R'_k. This remains true with G replaced by G'', that is, we have that $N_{G''}(x') \cap R'_k = \emptyset$ and $N_{G''}(y') \cap R'_k = \emptyset$. Indeed, this follows from Lemma 8.16(ii) and that G'' and G_3 only differ on $[X_i, A_i]$. Thus

$$\begin{aligned} P_3(x'y', G'') &\leqslant \sum_{\ell \in [k-1]\setminus\{i,j\}} |A_\ell| + d_{G''}(x', A_i) + d_{G''}(y', A_j) + |Z| \\ &\overset{P1(G),(8.29)}{\leqslant} (k-3)(c+\beta)n + cn - \frac{\xi n}{5} + \frac{2C}{\xi} \\ &\leqslant (k-2)cn - \frac{\xi n}{6}. \end{aligned} \tag{8.43}$$

On the other hand, by Lemma 8.16(v) and the analogue of (8.41), $K_3(G'') = K_3(G_3) \leqslant K_3(G) + 8C^3/\xi^3$. As $x'y' \notin E(G'')$, Corollary 4.18 implies that $P_3(x'y', G'') \geqslant (k-2)cn - k - 8C^3/\xi^3$, contradicting (8.43). This completes the proof of (i).

We now turn to (ii). We claim first that $K_3(G_3) = K_3(G)$. Indeed, for all $i \in [k-1]$ and $x, y \in X_i$, we have

$$D(x, y) = \sum_{\substack{\ell \in [k-1]: \\ G_3[X_i, X_\ell] \text{ complete}}} |X_\ell| = D(x) = D(y) = D_i.$$

Then Lemma 8.16(iv) implies that $K_3(G_3) = K_3(G)$.

Recall $m^{(3)} = \sum_{ij \in \binom{[k]}{2}} e(\overline{G_3}[A_i, A_j])$ and Lemma 8.16(i) implies that $m^{(3)} \leqslant m$ with equality if and only if $E(G[X_i]) = \emptyset$ for all $i \in [k-1]$. Thus if

$m^{(3)} = m$, then Lemma 8.16(ii) implies that $G_3 = G$ as desired. We may then assume that $m^{(3)} < m$ and, without loss of generality, that $e(G[X_{k-1}]) > 0$. By Lemma 8.16(ii), this means that G_3 has more cross-edges with respect to $A_1, \ldots, A_k$ than G. As $K_3(G_3) = K_3(G)$, by the choice of G, in particular (C2) , we must have $G_3 \in \mathcal{H}(n, e)$.

For all $i \in [k-1]$ such that $X_i \neq \emptyset$, we have

$$e(\overline{G_3}[A_i, A_k]) = e(\overline{G}[A_i, A_k]) - e(G[X_i]) \overset{P3(G),P5(G)}{\geqslant} |X_i|(\xi n - \delta n) > 0. \quad (8.44)$$

Suppose first that $G_3 \in \mathcal{H}_1(n, e)$ and $A_1^*, \ldots, A_{k-2}^*, B$ is a canonical partition of G_3. By construction, G_3 satisfies the hypotheses of Lemma 8.9. Recall that $e(G[X_{k-1}]) > 0$, in particular, $X_{k-1} \neq \emptyset$. Then (8.44) and Lemma 8.9(i) imply that $B = A_{k-1} \cup A_k$, $G_3[A_i, B]$ is complete and $X_i = \emptyset$ for every $i \in [k-2]$. (There can only be one $i \in [k-1]$ such that $e(\overline{G_3}[A_i, A_k]) > 0$, so (8.44) and the fact that $X_{k-1} \neq \emptyset$ imply that $X_i = \emptyset$ for all $i \in [k-2]$.) But then G_3 and G only differ at $A_{k-1} \cup A_k$ and Lemma 8.9(ii) implies that $G_3[A_{k-1}, A_k]$ is complete, contradicting (8.44).

We may now assume that $G_3 \in \mathcal{H}_2^{\min}(n, e) \setminus \mathcal{H}_1(n, e)$ and let $A_1^*, \ldots, A_k^*$ be a canonical partition of G_3. We claim that G_3 satisfies the hypotheses of Lemma 8.10(ii). Indeed, by Lemma 8.16(ii), P5(G) and (8.29), $|E(G) \triangle E(G_3)| \leqslant |Z|^2 \leqslant \delta n^2$. Also, $G_3[A_i, A_j]$ is complete for all $ij \in \binom{[k-1]}{2}$ by P2(G_3). Finally, $d \leqslant |Z|^2 < \delta n$ and $Y = \emptyset$ by Proposition 8.15.

Recall that $X_{k-1} \neq \emptyset$. By Lemma 8.10(ii),

$$X_{k-1} \subseteq A_k \subseteq A_k^* \quad (8.45)$$

and there is a bijection $\sigma : [k-1] \to [k-1]$ and at most one $j \in [k-1]$ such that $A^*_{\sigma(\ell)} = A_\ell$ for all $\ell \in [k-1] \setminus \{j\}$, and $A^*_{\sigma(j)} \subseteq A_j \subseteq A^*_{\sigma(j)} \cup A_k^*$. Without loss of generality, assume that σ is the identity permutation. By P4(G_3), we have that $G_3[X_{k-1}, A_\ell]$ is complete for every $\ell \leqslant k-2$. But $X_{k-1} \subseteq A_k^*$, so $A_\ell \cap A_k^* = \emptyset$. Thus $A_\ell = A_\ell^*$. Therefore j can only be $k-1$ if it exists, that is, $A_{k-1}^* \subseteq A_{k-1} \subseteq A_{k-1}^* \cup A_k^*$. But $A_{k-1}^* \cup A_k^* = A_{k-1} \cup A_k$, so $A_k \subseteq A_k^*$. So

$$\begin{aligned} |A_{k-1}^*| - |A_k^*| \leqslant |A_{k-1}| - |A_k| &\overset{P1(G)}{\leqslant} (kc - 1 + 2\beta)n \quad (8.46) \\ &\overset{(6.3)}{<} (c - (k-1)\alpha)n + 2\beta n < (c - \alpha)n. \end{aligned}$$

Fix an arbitrary edge $xy \in E(G[X_{k-1}])$. Note that as $X \subseteq A_k \subseteq A_k^*$ is independent in G_3, for every $ij \in \binom{[k-1]}{2}$ we have that $[X_i, X_j]$ is empty in G_3, and hence also in G as they are identical at $\bigcup_{ij \in \binom{[k-1]}{2}} [X_i, X_j]$. So $D_i = 0$ for all $i \in [k-1]$. Since $K_3(G_3) = K_3(G)$, by Lemma 8.16(iv), we have that $G[X_i]$

is triangle-free for every $i \in [k-1]$, and $N_G(x, A_i) \cap N_G(y, A_i) = \emptyset$. That is, x and y have no common A_i-neighbour in G. So

$$e(\overline{G}[A_{k-1}, \{x, y\}]) \geqslant |A_{k-1}| \overset{P1(G)}{\geqslant} (c-\beta)n.$$

By (8.45), $\{x, y\} \subseteq X_{k-1} \subseteq A_k^*$, and recall that from G to G_3, at most $|Z|^2$ adjacencies are changed in $[A_{k-1}, X]$. Lemma 8.10 implies that $|A_{k-1} \setminus A_{k-1}^*| \leqslant |A_{k-1} \triangle A_{k-1}^*| \leqslant k\beta n$. So

$$\begin{aligned}
e(\overline{G_3}[A_{k-1}^*, A_k^*]) &\geqslant e(\overline{G_3}[A_{k-1}^*, \{x, y\}]) \geqslant e(\overline{G_3}[A_{k-1}, \{x, y\}]) - 2|A_{k-1} \setminus A_{k-1}^*| \\
&\geqslant e(\overline{G}[A_{k-1}, \{x, y\}]) - |Z|^2 - 2|A_{k-1} \setminus A_{k-1}^*| \\
&\overset{(8.29)}{\geqslant} (c-\beta)n - \frac{4C^2}{\xi^2} - 2k\beta n \\
&> (c-\alpha/2)n \overset{(8.46)}{>} |A_{k-1}^*| - |A_k^*| + 1,
\end{aligned}$$

contradicting Corollary 4.4(iii). This completes the proof of the lemma. □

For $ij \in \binom{[k-1]}{2}$, we write $X_i \sim X_j$ if $G[X_i, X_j]$ is complete and $X_i \not\sim X_j$ if $G[X_i, X_j]$ is empty (recall that exactly one of these holds for every pair ij by Lemma 8.17(i)). Thus for all $i \in [k-1]$,

$$D_i = \sum_{\ell \in [k-1] : X_\ell \sim X_i} |X_\ell|.$$

Proposition 8.20. *The following hold.*

(i) *Let $i, j \in [k-1]$ be such that $X_i, X_j \neq \emptyset$. Then $a_i + D_i = a_j + D_j$.*

(ii) *If G' is an (n, e)-graph with $E(G') \triangle E(G) \subseteq \bigcup_{i \in [k-1]} K[X_i, A_i]$, then $K_3(G') = K_3(G)$.*

Proof. Choose arbitrary $i, j \in [k-1]$ and $x \in X_i$ and $x' \in X_j$. We obtain (i) by performing a transformation on G. First observe that, by the definition of X and P5(G), we have $\gamma n \leqslant d(x, A_i) \leqslant |A_i| - \xi n$. So there exist sets $K(x) \subseteq N_G(x, A_i)$ and $\overline{K}(x) \subseteq N_{\overline{G}}(x, A_i)$ of size ξn, and equally sized subsets $K(x') \subseteq N_G(x', A_j)$ and $\overline{K}(x') \subseteq N_{\overline{G}}(x', A_j)$. Let J be obtained from G by adding $\{xv : v \in \overline{K}(x)\}$ and removing $\{x'u' : u' \in K(x')\}$. Let J' be obtained from G by adding $\{x'v' : v' \in \overline{K}(x')\}$ and removing $\{xu : u \in K(x)\}$. For all $a \in A_i$ and $a' \in A_j$, we have by Lemma 8.16(iii), Lemma 8.17(ii) and the constructions of J and J' that

$$P_3(xa, J) = P_3(xa, J') = P_3(xa, G) = a_i + D_i \quad \text{and}$$

$$P_3(x'a', J) = P_3(x'a', J') = P_3(x'a', G) = a_j + D_j.$$

Since A_i, A_j are independent sets in G by Proposition 8.14, there are no triangles in J containing both edges xv_1, xv_2 for distinct $v_1, v_2 \in \overline{K}(x)$; and no triangles in J containing both edges $x'v_1', x'v_2'$ for distinct $v_1', v_2' \in K(x')$. Thus

$$K_3(J) - K_3(G) = \sum_{v \in \overline{K}(x)} P_3(xv, J) - \sum_{u \in K(x')} P_3(x'u', G) = \xi n(a_i + D_i - a_j - D_j)$$

and similarly, $K_3(J') - K_3(G) = \xi n(a_j + D_j - a_i - D_i) = -(K_3(J) - K_3(G))$. If $a_i + D_i \neq a_j + D_j$, then either J or J' has at least ξn fewer triangles than G, a contradiction. Thus $a_i + D_i = a_j + D_j$ for all $i, j \in [k-1]$ for which $X_i, X_j \neq \emptyset$. This proves (i).

For (ii), it suffices to show that, for any $i, j \in [k-1]$, if G' is obtained from G by replacing one X_i-A_i edge e_i with one X_j-A_j edge e_j, then $K_3(G) = K_3(G')$. Then this can be iterated to obtain any required G'. But this follows from (i) since

$$\begin{aligned} K_3(G') - K_3(G) &= P_3(e_j, G') - P_3(e_i, G) = P_3(e_j, G) - P_3(e_i, G) \\ &= a_j + D_j - a_i - D_i = 0. \end{aligned} \qquad \square$$

It is now easy to complete the proof of Theorem 1.7 in the case under consideration.

Proof of Theorem 1.7 in the intermediate case and when $m < Cn$. Propositions 8.14 and 8.15 imply that $A_1, \ldots, A_{k-1}$ are independent sets in G and $Y = \emptyset$. By Proposition 6.12(i), every edge in $G[A_k]$ has both endpoints in X. Now Lemma 8.17 implies that $xy \in E(G[A_k])$ only if there are $ij \in \binom{[k-1]}{2}$ such that $x \in X_i$ and $y \in X_j$.

If $E(G[X]) = \emptyset$, then G is k-partite. But then we obtain a contradiction via Corollary 4.4(i). Thus we may choose $xy \in E(G[X])$ with $x \in X_i$ and $y \in X_j$ for some $ij \in \binom{[k-1]}{2}$. Note that $d_G(x, A_i) > 0$ by the definition (6.32) of X_i. Let G' be an (n, e)-graph obtained from G by successively replacing arbitrary x-A_i edges with arbitrary y-A_j nonedges until

(S1) $d_{G'}(x, A_i) = 1$; or

(S2) $d_{G'}(y, A_j) = |A_j|$ and $d_{G'}(x, A_i) \geqslant 1$.

We claim that in both cases, $d_{G'}(x, A_i) \leqslant \sqrt{\beta} n$. This is clearly true if (S1) holds. If (S2) holds, note that

$$\begin{aligned} (k-2)cn + k \overset{(5.5)}{\geqslant} P_3(xy, G) &\geqslant \sum_{\ell \in [k-1] \setminus \{i,j\}} |A_\ell| + d_G(x, A_i) + d_G(y, A_j) \\ &\overset{P1(G)}{\geqslant} (k-3)(c-\beta)n + d_G(x, A_i) + d_G(y, A_j). \end{aligned}$$

Thus

$$d_{G'}(x, A_i) = d_G(x, A_i) + d_G(y, A_j) - d_{G'}(y, A_j) \overset{(S2)}{\leqslant} cn + k\beta n - |A_j| \overset{P1(G)}{\leqslant} \sqrt{\beta} n,$$

as required. Note that $E(G') \triangle E(G) \subseteq K[X_i, A_i] \cup K[X_j, A_j]$. So by Proposition 8.20(ii), we have $K_3(G') = K_3(G)$. Recall that, by Proposition 6.12(i), in G and also in G', there is no edge between X and R_k. Then we can replace all x-A_i edges in G' with x-R_k nonedges to obtain a new graph G''. This is possible as

$$|R_k| \overset{P1(G), P3(G)}{\geqslant} (1-(k-1)c-\beta)n - |Z| \overset{(6.3),(6.31)}{\geqslant} \sqrt{\alpha} n \geqslant \sqrt{\beta} n \geqslant d_{G'}(x, A_i).$$

Fix arbitrary $u \in A_i$ and $u' \in R_k$. Note that $\bigcup_{\ell \in [k-1] \setminus \{i\}} A_\ell \subseteq N_G(x) \cap N_G(u)$ by P2(G) and P4(G). Further, $y \in N_G(u) \cap N_G(x)$ by the definition of $X_j \ni y$. Both of these statements also hold for G'. Thus $P_3(xu, G') \geqslant a_i + 1$. But $P_3(xu', G'') = a_i$ since $d_{G''}(x, A_i) = 0$ and every X-R_k edge is incident to x in G''. Thus

$$K_3(G'') - K_3(G) = K_3(G'') - K_3(G') \leqslant -1 \cdot d_{G'}(x, A_i) \overset{(S1),(S2)}{\leqslant} -1,$$

a contradiction.

This completes the proof of Theorem 1.7 in the intermediate case when $m < Cn$.

9. The boundary case

We have shown that no worst counterexample to Theorem 1.7 can satisfy (5.4) and (6.1). That is, we can assume that

$$t_k(n) - \alpha n^2 \leqslant e \leqslant t_k(n) - 1, \tag{9.1}$$

which we refer to as the *boundary case*. Let

$$r := t_k(n) - e \leqslant \alpha n^2. \tag{9.2}$$

So $r \geqslant 1$. Now, Lemmas 4.11 and 4.13 and (4.9) imply that $k(n,e) = k(2e/n^2)$ and

$$\frac{n}{k} + \sqrt{\frac{r}{\binom{k}{2}}} \leqslant cn \leqslant \frac{n}{k} + \sqrt{\frac{r+k/8}{\binom{k}{2}}} \quad \text{and so} \quad \frac{\sqrt{r}}{k} \leqslant cn - \frac{n}{k} \leqslant \sqrt{r}. \tag{9.3}$$

Therefore

$$\frac{n}{k} < cn \leqslant \frac{n}{k} + \sqrt{\alpha}n. \tag{9.4}$$

A useful consequence of this is that

$$1-(k-1)c \geqslant \frac{1}{k} - (k-1)\sqrt{\alpha} > \frac{1}{2k}. \tag{9.5}$$

9.1. The boundary case: approximate structure.

The first step is to obtain an analogue of Lemma 6.1. Let

$$D := 169k^{k+9}. \tag{9.6}$$

LEMMA 9.1 (Approximate structure). *Suppose that (9.1) holds. Let G be a worst counterexample as defined in Section 5.2 and let $A_1,\ldots,A_k$ be a max-cut partition of $V(G)$. Let $m := \sum_{ij\in\binom{[k]}{2}} e(\overline{G}[A_i,A_j])$ and $h := \sum_{i\in[k]} e(G[A_i])$. Then there exists $Z \subseteq V(G)$ such that G has a weak $(A_1,\ldots,A_k; Z, \sqrt{Dr}/n, \xi',\xi',\delta')$-partition in which, for all $i \in [k]$,*

$$\left||A_i| - \frac{n}{k}\right|, \left||A_i| - cn\right| \leqslant \sqrt{Dr}, \quad m \leqslant Dr \quad \textit{and} \quad h \leqslant \delta' m. \tag{9.7}$$

Recall from Section 4.5 that a weak partition requires that P1, P3 and P5 all hold with the appropriate parameters. Note that the partition in Lemma 9.1 is in terms of primed constants ξ',δ', which are both large compared to α, unlike ξ,δ in the intermediate case, which are small compared to α.

We will need the following analogue of Lemma 6.4, which is essentially the same as Theorem 2 in [26]. Since this theorem is not phrased in a way applicable to our situation, we reprove it here. In fact, this lemma applies for all, say, $r \leqslant \frac{n^2}{2k^2}$, but is only meaningful when $r = o(n^2)$.

LEMMA 9.2. *There exist integers $n_1,\ldots,n_k$ summing to n with $|n_i - n/k|$, $|n_i - cn| \leqslant 6k^{\frac{k+3}{2}}\sqrt{r}$ for all $i \in [k]$ such that $|E(G) \triangle E(K_{n_1,\ldots,n_k})| < 40k^{k+4}r$.*

Proof. Define $s \in \mathbb{R}$ by setting

$$e = \left(1 - \frac{1}{s}\right)\frac{n^2}{2} \quad \text{and so} \quad \frac{2r}{n^2} \leqslant \frac{1}{s} - \frac{1}{k} \leqslant \frac{2(r + k/8)}{n^2} \overset{(9.2)}{\leqslant} 3\alpha. \tag{9.8}$$

(Here we used Lemma 4.11.) For $0 \leqslant i \leqslant 3$, write N_i for the (unique) 3-vertex graph with exactly i edges, and write $N_i(G)$ for the number of induced copies of N_i in G. So, for example, $N_3(G) = K_3(G)$. We claim that

$$K_3(G) = \binom{s}{3}\left(\frac{n}{s}\right)^3 + \frac{1}{3}\left(\sum_{x \in V(G)} q_G(x)^2 + N_1(G)\right), \tag{9.9}$$

where $q_G(x) := 2e/n - d_G(x)$. This is a special case of inequality (14) in [26], but we repeat the simple proof of this case here for the reader's convenience.

For each edge f of G and $1 \leqslant i \leqslant 3$, let $n_{i,f}$ denote the number of vertices adjacent to exactly $i - 1$ vertices of f. Then for all $f \in E(G)$, we have $n_{1,f} + n_{2,f} + n_{3,f} = n - 2$, and $\sum_{f \in E(G)} n_{i,f} = i N_i(G)$. So

$$e(n-2) = 3N_3(G) + 2N_2(G) + N_1(G). \tag{9.10}$$

Additionally,

$$\begin{aligned}2(N_2(G) + 3N_3(G)) &= 2\sum_{v \in V(G)} \binom{d_G(v)}{2} = \sum_{v \in V(G)} 2\binom{2e/n - q_G(v)}{2} \\ &= \frac{4e^2}{n} + \sum_{v \in V(G)} q_G(v)^2 - 2e,\end{aligned}$$

where we used the fact that $\sum_{v \in V(G)} q_G(v) = 0$. Thus

$$en \overset{(9.10)}{=} -3N_3(G) + \frac{4e^2}{n^2} + \sum_{v \in V(G)} q_G(v)^2 + N_1(G).$$

So

$$\begin{aligned}K_3(G) = N_3(G) &= \frac{1}{3}\left(e \cdot \left(\frac{4e}{n} - n\right) + \sum_{v \in V(G)} q_G(v)^2 + N_1(G)\right) \\ &= \frac{1}{3}\left(\binom{s}{2}\left(\frac{n}{s}\right)^2\left(\frac{s-1}{2} \cdot \frac{4n}{s} - n\right) + \sum_{v \in V(G)} q_G(v)^2 + N_1(G)\right) \\ &= \binom{s}{3}\left(\frac{n}{s}\right)^3 + \frac{1}{3}\left(\sum_{v \in V(G)} q_G(v)^2 + N_1(G)\right),\end{aligned}$$

as required.

We now consider G. Certainly G has at most as many triangles as the (n, e)-graph obtained by deleting r edges between the two smallest classes of $T_k(n)$. By convexity, $K_3(T_k(n)) \leqslant \binom{k}{3}(n/k)^3$, so

$$K_3(G) \leqslant K_3(T_k(n)) - r\left(n - 2\left\lfloor \frac{n}{k} \right\rfloor\right) \overset{(9.8)}{\leqslant} \binom{s}{3}\left(\frac{n}{s}\right)^3 + rn + \frac{kn}{8}$$
$$\leqslant \binom{s}{3}\left(\frac{n}{s}\right)^3 + rkn.$$

Thus (9.9) implies that

$$\sum_{x \in V(G)} q_G(x)^2 \leqslant 3rkn \quad \text{and} \quad N_1(G) \leqslant 3rkn. \tag{9.11}$$

Let W be an arbitrary copy of K_k in G. Let A_W denote the set of vertices adjacent in G to at most $k-2$ vertices in W. Each vertex in A_W lies in at least one copy of N_1 (together with any pair of its missing neighbours in W). On the other hand, for every copy of N_1, its single edge lies in at most n^{k-2} copies of K_k. Thus

$$\sum_{W \subseteq G : W \cong K_k} |A_W| \leqslant N_1(G) \cdot n^{k-2} \leqslant 3rkn^{k-1}.$$

Denote by B_W the set of $xy \in E(\overline{G})$ such that $d_G(x, V(W)) = k-1$ and either (i) $d_G(y, V(W)) = k-1$, but $N_G(x, V(W)) \neq N_G(y, V(W))$, or (ii) $d_G(y, V(W)) = k$. Then for every $xy \in B_W$, there is $z \in V(W)$ such that x, y, z span a copy of N_1 in G, where x plays the role of the isolated vertex and there are two choices for z. On the other hand, there are at most $\binom{n-1}{k-1} \leqslant n^{k-1}/2$ copies of K_k that contain z. Thus

$$\sum_{W \subseteq G : W \cong K_k} |B_W| \leqslant N_1(G) \cdot 2 \cdot n^{k-1}/2 \leqslant 3rkn^k.$$

Let $q_W := \sum_{x \in V(W)} q_G(x)^2$. Any $x \in V(G)$ lies in at most n^{k-1} copies of K_k, so

$$\sum_{W \subseteq G : W \cong K_k} q_W \leqslant 3rkn^k.$$

Thus

$$\sum_{W \subseteq G : W \cong K_k} (n|A_W| + |B_W| + q_W) \leqslant 9rkn^k.$$

Now, G certainly contains many copies of K_k. For example, Theorem 1 in [26] implies that

$$K_k(G) \geqslant g_k(n, e) \geqslant \binom{s}{k}\left(\frac{n}{s}\right)^k \overset{(9.8)}{\geqslant} \frac{1}{2}\left(\frac{n}{k}\right)^k.$$

Thus, by averaging, there exists a copy W of K_k in G for which

$$|A_W| \leqslant \frac{18rk^{k+1}}{n}; \quad |B_W| \leqslant 18rk^{k+1}; \quad \text{and} \tag{9.12}$$
$$|q_G(x)| \leqslant 3\sqrt{2rk^{k+1}} \quad \text{for all } x \in V(W).$$

We will use this W to construct a partition of $V(G)$. Let $w_1, \ldots, w_k$ be the vertices of W. For all $i \in [k]$, let $C_i := \{x \in V(G) : N_{\overline{G}}(x, V(W)) = \{w_i\}\}$. Let also

$$C_0 := \{x \in V(G) : d_G(x, V(W)) = k\} \quad \text{and} \quad C_{k+1} := A_W.$$

So $C_0, \ldots, C_{k+1}$ is a partition of $V(G)$.

We will now estimate the sizes of each of these sets. We have that

$$|C_{k+1}| = |A_W| \leqslant \frac{18rk^{k+1}}{n} \overset{(9.2)}{\leqslant} 18k^{k+1}\sqrt{\alpha}\sqrt{r}. \tag{9.13}$$

Now, (9.12) implies that, for all $i \in [k]$,

$$\left| d_G(w_i) - \left(1 - \frac{1}{s}\right) n \right| = |q_G(w_i)| \leqslant 3\sqrt{2rk^{k+1}}.$$

But

$$d_G(w_i) = |C_0| + \sum_{j \in [k] \setminus \{i\}} |C_j| + d_G(w_i, C_{k+1}) = n - |C_i| \pm |C_{k+1}|,$$

so

$$\begin{aligned} |C_i| &= \frac{n}{s} \pm \left(3\sqrt{2rk^{k+1}} + |C_{k+1}|\right) \\ &\overset{(9.8),(9.13)}{=} \frac{n}{k} \pm \left(\frac{2(r + k/8)}{n} + 3\sqrt{2rk^{k+1}} + 18k^{k+1}\sqrt{\alpha}\sqrt{r}\right) \\ &\overset{(9.2)}{=} \frac{n}{k} \pm \left(3\sqrt{\alpha} + 3\sqrt{2k^{k+1}} + 18k^{k+1}\sqrt{\alpha}\right)\sqrt{r} = \frac{n}{k} \pm 5k^{\frac{k+1}{2}}\sqrt{r}. \end{aligned}$$

Thus $|C_0| \leqslant 5k^{\frac{k+3}{2}}\sqrt{r}$.

For each $i \in \{2, \ldots, k\}$, let $A_i := C_i$ and let $A_1 := C_0 \cup C_1 \cup C_{k+1}$. So, for all $i \in [k]$,

$$\begin{aligned} \Big||A_i| - cn\Big|, \left||A_i| - \frac{n}{k}\right| &\leqslant \left||C_i| - \frac{n}{k}\right| + |C_0| + |C_{k+1}| + \left|\frac{n}{k} - cn\right| \\ &\overset{(9.3)}{\leqslant} 5k^{\frac{k+1}{2}}\sqrt{r} + 5k^{\frac{k+3}{2}}\sqrt{r} + 18k^{k+1}\sqrt{\alpha}\sqrt{r} + \sqrt{r} \\ &\leqslant 6k^{\frac{k+3}{2}}\sqrt{r}. \end{aligned} \tag{9.14}$$

Let $ij \in \binom{[k]}{2}$ and $\overline{e} \in E(\overline{G}[A_i, A_j])$. Then, by definition, either $\overline{e} \in B_W$ or $\{x, y\} \cap A_W \neq \emptyset$ (note that any such $\overline{e}$ intersecting C_0 lies in B_W). Thus by (9.12) and (9.13), we have

$$\sum_{ij \in \binom{[k]}{2}} e(\overline{G}[A_i, A_j]) \leqslant |B_W| + |C_{k+1}|n \leqslant 36k^{k+1}r. \tag{9.15}$$

Let $d_i := n/k - |A_i|$ for all $i \in [k]$. Now, $\sum_{i \in [k]} d_i = 0$ and

$$\begin{aligned}\sum_{ij \in \binom{[k]}{2}} |A_i||A_j| &= \frac{1}{2}\left(n^2 - \sum_{i \in [k]}\left(\left(\frac{n}{k}\right)^2 - \frac{2d_i n}{k} + d_i^2\right)\right) \geqslant t_k(n) - k \cdot \max_{i \in [k]}\{d_i^2\} \\ &\overset{(9.14)}{\geqslant} e - 36k^{k+4}r.\end{aligned}$$

Thus

$$\begin{aligned}\sum_{i \in [k]} e(G[A_i]) = e - \sum_{ij \in \binom{[k]}{2}} (|A_i||A_j| - e(\overline{G}[A_i, A_j])) &\overset{(9.15)}{\leqslant} 36k^{k+4}r + 36k^{k+1}r \\ &\leqslant 38k^{k+4}r\end{aligned}$$

and so, letting $n_i := |A_i|$ for all $i \in [k]$, we have

$$|E(G) \triangle E(K_{n_1,\ldots,n_k})| \leqslant 36k^{k+1}r + 38k^{k+4}r < 40k^{k+4}r,$$

as required. □

The previous lemma together with Lemma 5.1 combine to prove Lemma 9.1.

Proof of Lemma 9.1. Choose a max-cut k-partition $V(G) = A_1 \cup \cdots \cup A_k$. Let

$$Z := \bigcup_{i \in [k]} \{z \in A_i : d_G(z, \overline{A_i}) \geqslant \xi' n\}. \tag{9.16}$$

(In fact, there can be no other choice for Z.) We need to show that P1(G) holds with parameter $\sqrt{Dr}/n$, P3(G) holds with parameter δ' and P5(G) holds with parameter ξ'.

Let $p := 6k^{\frac{k+3}{2}}\sqrt{r}$, $d := 40k^{k+4}r$ and $\rho := 40k^{k+4}\alpha$. Then

$$\begin{aligned}p^2 &= 36k^{k+3}r < d \leqslant \rho n^2 \quad \text{and} \quad 2\rho^{1/6} \leqslant 4k^{k/6+4/6}\alpha^{1/6} < \alpha^{1/7} < \frac{1}{2k} \\ &\overset{(9.5)}{<} 1 - (k-1)c.\end{aligned}$$

Thus, by Lemma 9.2, we can apply Lemma 5.1 with parameters p, d, ρ to imply that $A_1, \ldots, A_k$ satisfy conclusions (i)–(v) of Lemma 5.1.

Thus, by (i), $P1(G)$ holds with parameter $2k^2\sqrt{d}/n = 2\sqrt{40}k^{k/2+4}\sqrt{r}/n$. This together with (9.3) and (9.6) implies the required bound on $\left||A_i| - \frac{n}{k}\right|$ and thus $P1(G)$ holds with parameter $\sqrt{Dr}/n$. Lemma 5.1(ii) implies that

$$m := \sum_{ij \in \binom{[k]}{2}} e(\overline{G}[A_i, A_j]) \leqslant 2k^2\sqrt{d}(kc-1)n + d \overset{(9.3)}{\leqslant} (6\sqrt{40}k^{k/2+5} + 40k^{k+4})r \overset{(9.6)}{<} Dr. \tag{9.17}$$

For $P3(G)$, as in the intermediate case, every missing edge is incident to at most two vertices in Z, so

$$|Z| \leqslant \frac{2m}{\xi' n} \leqslant \frac{2Dr}{\xi' n} < \frac{2D\alpha n}{\xi'} \overset{(9.6)}{<} \delta' n. \tag{9.18}$$

Furthermore, Lemma 5.1(iii) implies that for every $i \in [k]$ and $e \in E(G[A_i])$, there is at least one endpoint x of e with

$$\begin{aligned} d_{\overline{G}}(x, \overline{A_i}) &\geqslant \frac{1}{2}\left((1-(k-1)c)n - 3k^2\sqrt{\rho n}\right) \overset{(9.5)}{\geqslant} \frac{1}{2}\left(\frac{1}{2k} - 3\sqrt{40}k^{k/2+4}\sqrt{\alpha}\right)n \\ &> \frac{n}{5k} > \xi' n. \end{aligned}$$

Thus $x \in Z$. The final part of $P3(G)$ follows from Lemma 5.1(iv) and the fact that $\alpha \ll \delta'$. Finally, $P5(G)$ holds immediately from the definition of Z. The assertion about m was proved in (9.17) and the assertion about h is an immediate consequence of Lemma 5.1(v) and the fact that $\alpha \ll \delta'$.

9.2. The boundary case: the remainder of the proof. Apply Lemma 9.1 to the worst counterexample G as defined in Section 5.2 (so G satisfies (C1)–(C3)). Now fix a weak $(A_1, \ldots, A_k; Z, \sqrt{Dr}/n, \xi', \xi', \delta')$-partition of G with Z (uniquely) defined as in (9.16) and define m as in the statement. For all $i \in [k]$, let

$$R_i := A_i \setminus Z.$$

As before, $P3(G)$ implies that R_i is an independent set for all $i \in [k]$. Suppose first that $Z = \emptyset$. Then G is a k-partite graph. So Corollary 4.4(i) implies that $G \in \mathcal{H}_2(n, e)$, a contradiction. Thus, exactly as in (9.18),

$$1 \leqslant |Z| \leqslant \frac{2m}{\xi' n} \quad \text{and} \quad \xi' \leqslant \frac{2m}{n}. \tag{9.19}$$

Given disjoint subsets $A, B \subseteq V(G)$, write $A \sim B$ if $G[A, B]$ is complete. For any $I \subseteq [k]$, write

$$R_I := \bigcup_{i \in I} R_i.$$

We would like to measure quite accurately the difference between $|R_I|/|I|$ and its 'expected' size cn for $I \neq \emptyset$ (recalling that cn, $n-(k-1)cn$ and n/k are all very close in the boundary case). Thus we define

$$\mathrm{diff}(I) := \left(\frac{|R_I|}{|I|} - cn\right)\frac{n}{m}, \quad \text{i.e. } |R_I| = \left(cn + \mathrm{diff}(I)\cdot\frac{m}{n}\right)|I|.$$

We will write $\mathrm{diff}(i)$ as shorthand for $\mathrm{diff}(\{i\})$. A trivial but useful observation is that, for pairwise-disjoint $I_1, \ldots, I_p \subseteq [k]$, we have

$$\min_{i\in[p]}\{\mathrm{diff}(I_i)\} \leqslant \mathrm{diff}(I_1 \cup \cdots \cup I_p) \leqslant \max_{i\in[p]}\{\mathrm{diff}(I_i)\}. \tag{9.20}$$

Note also that

$$\left(cn - \frac{m}{\alpha^{1/3}n}\right)k \overset{(9.3),(9.7)}{\geqslant} n + \sqrt{r} - \frac{kDr}{\alpha^{1/3}n} \overset{(9.2)}{\geqslant} n + \sqrt{r}\left(1 - kD\alpha^{1/6}\right) \overset{(9.6)}{>} n, \tag{9.21}$$

so we have the following:

($*$) If $I \subseteq [k]$ satisfies $\mathrm{diff}(I) \geqslant -1/\alpha^{1/3}$, then $|R_I| > |I|n/k$.

We cannot guarantee that P2(G) and P4(G) hold in this setting since there is no part that is significantly smaller than the other parts. However, the next lemma shows that an analogue of these properties holds.

LEMMA 9.3. *There exists a partition $Z = \bigcup_{I \in \binom{[k]}{k-2}} Z_I$ of Z such that, for all $ij \in \binom{[k]}{2}$, the following properties hold. We have $Z_{[k]\setminus\{i,j\}} \sim R_{[k]\setminus\{i,j\}}$, $Z_{[k]\setminus\{i,j\}} \subseteq A_i \cup A_j$ and for every $z \in Z_{[k]\setminus\{i,j\}} \cap A_i$, we have that $d_G(z, R_i) \leqslant \delta' n$ and $d_{\overline{G}}(z, R_j) \geqslant \xi' n/2$.*

Proof. Let $z \in Z$ be arbitrary, and let $i \in [k]$ be such that $z \in A_i$. By the definition of Z, there is some $j \in [k]\setminus\{i\}$ such that $d_{\overline{G}}(z, A_j) \geqslant \xi' n/k$. Let $I := [k]\setminus\{i, j\}$ and $x \in R_I$ be arbitrary, and let $h \in I$ be such that $x \in R_h$. Then

$$\begin{aligned} P_3(zx, G) &\leqslant d_G(z, A_i) + d_G(z, A_j) + d_G(x, A_h) + (n - |A_i| - |A_j| - |A_h|) \\ &\overset{P3(G),(9.7)}{\leqslant} 2\delta' n + n - 2\left(\frac{n}{k} - \sqrt{Dr}\right) - \frac{\xi' n}{k} \\ &\overset{(9.2)}{<} (k-2)\cdot\frac{n}{k} - \frac{\xi' n}{2k} \overset{(9.4)}{<} (k-2)cn - \frac{\xi' n}{3k}. \end{aligned}$$

Thus (5.5) implies that $xz \in E(G)$. Since x was arbitrary, we have shown that we can assign z to $Z_{[k]\setminus\{i,j\}}$. The second statement follows from P3(G), which says, since $z \in A_i$, that $d_G(z, R_i) \leqslant d_G(z, A_i) \leqslant \delta' n$ and P5(G), which together with the first statement says that $d_{\overline{G}}(z, R_j) \geqslant \xi' n - |Z| \geqslant (\xi' - \delta')n \geqslant \xi' n/2$. □

The next lemma shows that $\mathrm{diff}(I)$ can only be large when $|I| \leqslant k-2$.

LEMMA 9.4. *If* $I \subseteq [k]$ *has* $\mathrm{diff}(I) \geqslant -1/\alpha^{1/3}$, *then* $|I| \leqslant k-2$.

Proof. Note first that, by $(*)$, we have $\mathrm{diff}([k]) < -1/\alpha^{1/3}$. Suppose that there exists a set $I \in \binom{[k]}{k-1}$ such that $\mathrm{diff}(I) \geqslant -1/\alpha^{1/3}$. Without loss of generality, suppose that $I = [k-1]$. Let $q := \frac{m}{\alpha^{1/3} n}$. Then $(*)$ implies that $|R_I| \geqslant (k-1)n/k$. Since $\sum_{ij \in \binom{k}{2}} |A_i||A_j|$ is maximized when the parts A_i are as balanced as possible and $cn - q \geqslant n/k$ due to (9.3) and (9.7), we have

$$\begin{aligned} e + m - \sum_{i\in[k]} e(G[A_i]) &= \sum_{ij\in\binom{[k]}{2}} |A_i||A_j| \leqslant e(K^k_{cn-q,\ldots,cn-q,n-(k-1)(cn-q)}) \\ &= e - \binom{k}{2} q^2 + (k-1)q(kc-1)n \\ &\overset{(9.3)}{\leqslant} e + \frac{(k-1)m}{\alpha^{1/3} n} \cdot k\sqrt{\frac{r + k/8}{\binom{k}{2}}} \\ &\leqslant e + \frac{2km\sqrt{r+k}}{\alpha^{1/3} n} \overset{(9.2)}{\leqslant} e + 3k\alpha^{1/6} m. \end{aligned}$$

But then

$$\sum_{i\in[k]} e(G[A_i]) \geqslant (1 - 3k\alpha^{1/6})m > \sqrt{\delta'} m,$$

a contradiction to Lemma 9.1. □

We now show that if there is a missing edge between some R_i and R_j, where $i \neq j$, then the union of the other sets R_ℓ must be large.

LEMMA 9.5. *For all* $ij \in \binom{[k]}{2}$, *if* $R_i \not\sim R_j$, *then* $\mathrm{diff}([k] \setminus \{i,j\}) \geqslant -1/(2\alpha^{1/3})$.

Proof. Set $I := [k] \setminus \{i,j\}$. Since $R_i \not\sim R_j$, there exist $x \in R_i$ and $y \in R_j$ such that $xy \notin E(G)$. Then, since R_i and R_j are both independent sets in G,

$$(k-2)cn - k \overset{(5.5)}{\leqslant} P_3(xy, G) \leqslant |Z| + |R_I| \overset{(9.19)}{\leqslant} \frac{2m}{\xi' n} + |R_I|$$

and so

$$|R_I| \geqslant (k-2)cn - k - \frac{2m}{\xi' n} \overset{(9.19)}{\geqslant} (k-2)cn - \frac{2km}{\xi' n} \geqslant \left(cn - \frac{m}{2\alpha^{1/3}n}\right)|I|,$$

as required. □

Our next goal is to show that R_i is in fact small for *every* $i \in [k]$, which will in turn imply that $G[R_1, \ldots, R_k]$ is complete k-partite. To do this, we need the following lemma.

LEMMA 9.6. *For all* $i \in [k]$, *if* $\mathrm{diff}(i) \geqslant -1/(2\alpha^{1/3})$, *then there exists* $j \in [k]\backslash\{i\}$ *such that* $R_i \not\sim R_j$.

Proof. Let $i \in [k]$ such that $\mathrm{diff}(i) \geqslant -1/(2\alpha^{-1/3})$ be arbitrary. We begin by proving the following claim.

CLAIM 9.7. *It suffices to show that* $Z_I = \emptyset$ *for all* $I \in \binom{[k]\backslash\{i\}}{k-2}$.

Proof of Claim. Suppose that $Z_I = \emptyset$ for all $I \in \binom{[k]\backslash\{i\}}{k-2}$. Lemma 9.3 implies that $Z \sim R_i$. Suppose now that $R_i \sim R_j$ for all $j \in [k] \setminus \{i\}$. Thus $R_i \sim \overline{R_i}$, and R_i is an independent set. Let $n' := n - |R_i|$ and $e' := e(G[\overline{R_i}]) = e - n'(n-n')$. Note that $J := G[\overline{R_i}]$ satisfies $K_3(J) = g_3(n', e')$ (since otherwise we could replace it in G with an (n', e')-graph with fewer triangles to obtain an (n, e)-graph with fewer triangles than G, contradicting (C1)). Using (9.2), (9.7) and (9.19), we have

$$|R_i| = |A_i| \pm |Z| = \frac{n}{k} \pm \sqrt{Dr} \pm \frac{2m}{\xi' n} = \frac{n}{k} \pm \alpha^{1/3}n. \tag{9.22}$$

By (9.22), we have

$$n'(n-n') \geqslant \left(\frac{n}{k} - \alpha^{1/3}n\right)\left(\frac{k-1}{k}n + \alpha^{1/3}n\right) \geqslant \frac{k-1}{k^2}n^2 - \alpha^{1/3}\frac{k-1}{k}n^2.$$

Recall from the very beginning of Section 5.2 that $\alpha_{1.3}$ is the constant obtained by applying Theorem 1.3 with parameters k and $r := 3$. Together with $e < t_k(n) \leqslant (k-1)n^2/(2k)$, we have that

$$\begin{aligned} e' &= e - n'(n-n') \leqslant \frac{k-1}{k} \cdot \frac{n^2}{2} - \left(\frac{k-1}{k^2}n^2 - \alpha^{1/3}\frac{k-1}{k}n^2\right) \\ &= \frac{k-1}{k} \cdot \frac{n^2}{2}\left(1 - \frac{2}{k} + 2\alpha^{1/3}\right) \end{aligned}$$

$$\overset{(9.22)}{\leqslant} \frac{k-1}{2k}\left(\frac{k-2}{k}+2\alpha^{1/3}\right)\left(\frac{k}{k-1}n'+\alpha^{1/4}n'\right)^2 \leqslant t_{k-1}(n')+\alpha^{1/5}(n')^2$$
$$\overset{(5.1)}{\leqslant} t_{k-1}(n')+\alpha_{1.3}(n')^2 \tag{9.23}$$

and similarly $e' \geqslant t_{k-2}(n')+\alpha_{1.3}(n')^2$. So $k(n',e') \in \{k-1,k\}$. Further,

$$n' = n-|R_i| \overset{(9.22)}{\geqslant} \left(1-\frac{1}{k}\right)n-\alpha^{1/3}n \overset{(6.3)}{\geqslant} n/2 \geqslant n_0/2 \overset{(5.2)}{\geqslant} \max\{n_0(k-1,\alpha/3), n_{1.3}(k)\}.$$

Suppose first that $k(n',e') = k-1$. Then the minimality of k and the fact that $t_{k-2}(n')+\alpha(n')^2/3 \leqslant t_{k-2}(n')+\alpha_{1.3}(n')^2 \leqslant e' < t_{k-1}(n')$ implies that Theorem 1.7 holds for (n',e'), that is, $g_3(n',e') = h(n',e')$, and every extremal graph lies in $\mathcal{H}(n',e')$. So $J \in \mathcal{H}(n',e')$. If $J \in \mathcal{H}_1(n',e')$, then since G is obtained by adding an independent set R_i of vertices to J and adding every edge between R_i and $V(J)$, we have that $G \in \mathcal{H}_1(n,e)$, a contradiction to (C1). Otherwise, $J \in \mathcal{H}_2(n',e')$, and in particular, J is $(k-1)$-partite. So G is k-partite, and Corollary 4.4(i) implies that $G \in \mathcal{H}_2(n,e)$, again contradicting (C1).

Thus we may assume that $k(n',e') = k$. Theorem 1.3 implies that we can obtain a graph $F' \in \mathcal{H}_1(n',e')$ with canonical partition $A_1^{F'},\ldots,A_{k-2}^{F'},B^{F'}$ and $K_3(F') = K_3(G[\overline{R_i}])$. Let F be the graph obtained from G by replacing $G[\overline{R_i}]$ with F', so $K_3(F) = K_3(G)$. By Corollary 4.18, for every $xy \in E(F)$,

$$P_3(xy,F) \leqslant (k-2)cn+k \overset{(9.4)}{\leqslant} (k-2)\frac{n}{k}+\alpha^{1/3}n. \tag{9.24}$$

For each $j \in [k-2]$ for which $A_j^{F'}$ is nonempty, fix an arbitrary edge $x_jy_j \in F[A_j^{F'},R_i]$; then

$$P_3(x_jy_j,F) \geqslant n-|A_j^{F'}|-|R_i|,$$

which together with (9.22) and (9.24) implies that $|A_j^{F'}| \geqslant n/k-2\alpha^{1/3}n$. Similarly, for an edge x_By_B in $F[B^{F'}]$ (there must exist one such edge as otherwise $k(n,e) < k$), we have $P_3(x_By_B,F) \geqslant n-|B^{F'}|$. Hence, $|B^{F'}| \geqslant 2n/k-\alpha^{1/3}n$. But then

$$n = |R_i| + \sum_{j\in[k-2]}|A_j^{F'}|+|B^{F'}| \geqslant \frac{k+1}{k}n-\alpha^{1/4}n > n,$$

a contradiction. This completes the proof of the claim. □

Suppose now that there is some $I \in \binom{[k]\setminus\{i\}}{k-2}$ such that $Z_I \neq \emptyset$. Let $j \in [k]\setminus\{i\}$ be such that $[k]\setminus\{i,j\} = I$. Let $z \in Z_I$ and let $n_\ell := d_G(z,R_\ell)$ for all $\ell \in [k]$.

Lemma 9.3 implies that, for some $i', j' \in [k]$ with $\{i', j'\} = \{i, j\}$, we have $d_G(z, R_{i'}) \leqslant \delta' n$, $d_{\overline{G}}(z, R_{j'}) \geqslant \xi' n/2$ and, for all $\ell \in I$, we have $n_\ell = |R_\ell|$. Thus

$$\begin{aligned}
|R_\ell| - n_i - n_j &= |R_\ell| - n_{i'} - n_{j'} \geqslant |R_\ell| - \delta' n - \left(|R_j| - \frac{\xi' n}{2}\right) \\
&\geqslant \left(\frac{\xi'}{2} - \delta'\right) n - \left||A_\ell| - \frac{n}{k}\right| - \left||A_j| - \frac{n}{k}\right| - |Z| \\
&\overset{P3(G),(9.7)}{\geqslant} \left(\frac{\xi'}{2} - 2\delta'\right) n - 2\sqrt{Dr} \overset{(9.2)}{\geqslant} \left(\frac{\xi'}{2} - 2\delta' - 2\sqrt{D\alpha}\right) n \\
&\geqslant \frac{\xi' n}{3}.
\end{aligned} \tag{9.25}$$

Lemma 9.4 implies that $\mathrm{diff}([k] \setminus \{j\}) < -1/\alpha^{1/3}$. So, using (9.20) and the fact that $\mathrm{diff}(i) \geqslant -1/(2\alpha^{1/3})$, there exists $\ell \in [k] \setminus \{i, j\}$ such that $\mathrm{diff}(\ell) < -1/\alpha^{1/3}$, so

$$|R_\ell| < cn - \frac{m}{\alpha^{1/3} n} \leqslant |R_i| - \frac{m}{2\alpha^{1/3} n}. \tag{9.26}$$

Let $I' := [k] \setminus \{i, j, \ell\}$ and $W := R_i \cup R_j \cup R_\ell \cup Z$. Then

$$d_G(z, W) = n_i + n_j + |R_\ell| + d_G(z, Z), \tag{9.27}$$

$R_{I'} = \overline{W}$ and $\{z\} \sim R_{I'}$. Recalling that $n_\ell = |R_\ell|$ for all $\ell \in I$, we have that

$$K_3(z, G) \geqslant e(G[R_{I'}]) + |R_{I'}|(n_i + n_j + |R_\ell|) + |R_\ell|(n_i + n_j) - m. \tag{9.28}$$

We have

$$\begin{aligned}
d_G(z, W) &\overset{(9.27)}{=} n_i + n_j + n_\ell + d_G(z, Z) \\
&\overset{(9.25)}{\leqslant} 2|R_\ell| - \frac{\xi' n}{3} + |Z| \overset{P3(G)}{\leqslant} 2|R_\ell| - \frac{\xi' n}{4} \\
&\overset{(9.7)}{\leqslant} |R_i| + |R_j| + 2D\sqrt{r} + 2|Z| - \frac{\xi' n}{4} \\
&\overset{P3(G),(9.2)}{\leqslant} |R_i| + |R_j| + 2D\sqrt{\alpha} n + 2\delta' n - \frac{\xi' n}{4} \\
&\leqslant |R_i| + |R_j| - \frac{\xi' n}{5}.
\end{aligned}$$

Let $k_i := \min\{d_G(z, W), |R_i|\}$ and $k_j := \max\{d_G(z, W) - k_i, 0\}$. The previous equation implies that

$$k_i + k_j = d_G(z, W) \quad \text{and}$$
$$k_i k_j = \begin{cases} 0 & \text{if } d_G(z, W) \leqslant |R_i|, \\ |R_i|(d_G(z, W) - |R_i|) & \text{otherwise.} \end{cases} \tag{9.29}$$

Obtain a new graph G' from G as follows. Let $K_i \subseteq R_i$ with $|K_i| = k_i$ and $K_j \subseteq R_j$ with $|K_j| = k_j$ be arbitrary. Note that this is possible as $k_i \leqslant |R_i|$ and if $k_j > 0$, then $k_j \leqslant d_G(z, W) - |R_i| \leqslant |R_j| - \xi' n/5$. Let $V(G') := V(G)$ and

$$E(G') := \big(E(G) \cup \{zx : x \in K_i \cup K_j\}\big) \setminus \{zy : y \in N_G(z, W)\}.$$

That is, we obtain G' by changing the W-neighbourhood of z to a new neighbourhood of the same size by adding as many edges as possible to R_i and (if necessary) additional edges to R_j. Note that $N_{G'}(z, R_\ell \cup Z) = \emptyset$ and G' is an (n, e)-graph. We have

$$K_3(z, G') \leqslant e(G[R_{I'}]) + |R_{I'}| d_{G'}(z, W) + k_i k_j. \tag{9.30}$$

Suppose first that $d_G(z, W) > |R_i|$. Then by (9.29), we have

$$K_3(z, G') \leqslant e(G[R_{I'}]) + |R_{I'}| d_G(z, W) + |R_i|(d_G(z, W) - |R_i|)$$

and so

$$\begin{aligned}
&K_3(G') - K_3(G) = K_3(z, G') - K_3(z, G) \\
&\overset{(9.28)}{\leqslant} |R_{I'}|(d_G(z, W) - (n_i + n_j + |R_\ell|)) + |R_i|(d_G(z, W) - |R_i|) \\
&-|R_\ell|(n_i + n_j) + m \\
&\overset{(9.27)}{\leqslant} |R_{I'}||Z| + |R_i|(n_i + n_j + |R_\ell| + |Z| - |R_i|) - |R_\ell|(n_i + n_j) + m \\
&= |R_{I'}||Z| + (|R_i| - |R_\ell|)(n_i + n_j - |R_i|) + |Z||R_i| + m \\
&\overset{(9.25),(9.26)}{\leqslant} -(|R_i| - |R_\ell|)\left(|R_i| - |R_\ell| + \frac{\xi' n}{3}\right) + |Z|n + m \\
&\overset{(9.19),(9.26)}{\leqslant} -\frac{m\xi'}{7\alpha^{1/3}} + \frac{2m}{\xi'} + m < -\frac{2m}{\xi'} \overset{(9.19)}{\leqslant} -n,
\end{aligned}$$

a contradiction.

Therefore we may assume that $d_G(z, W) \leqslant |R_i|$. We need the following claim that n_j is large.

Claim 9.8. $n_j \geqslant \frac{km}{4\alpha^{1/3} n}$.

Proof of Claim. If $\mathrm{diff}(I) \geqslant -1/\alpha^{1/3}$, then since $\mathrm{diff}(i) \geqslant -1/(2\alpha^{1/3})$, we also have that $\mathrm{diff}(I \cup \{i\}) \geqslant -1/\alpha^{1/3}$, a contradiction to Lemma 9.4. So $\mathrm{diff}(I) < -1/\alpha^{1/3}$. The second part of Lemma 9.3 implies that there is some $u \in N_{\overline{G}}(z, R_i)$. Since R_i is an independent set in G, we have that

$$(k-2)cn - k \overset{(5.5)}{\leqslant} P_3(zu, G) \leqslant |Z| + n_j + |R_I|$$

and so, using the fact that $\mathrm{diff}(I) < -1/\alpha^{1/3}$,

$$\begin{aligned} n_j &\geqslant (k-2)cn - k - |Z| - |R_I| \\ &\overset{(9.19)}{\geqslant} (k-2)cn - k - \frac{2m}{\xi' n} - (k-2)\left(cn - \frac{m}{\alpha^{1/3}n}\right) \\ &\geqslant \left(\frac{k-2}{\alpha^{1/3}} - \frac{3k}{\xi'}\right)\frac{m}{n} \geqslant \frac{km}{4\alpha^{1/3}n}, \end{aligned}$$

completing the proof of the claim. □

Now (9.28)–(9.30) and Claim 9.8 imply that

$$\begin{aligned} K_3(z, G') - K_3(z, G) &\overset{(9.27)}{\leqslant} |R_{I'}||Z| + m - |R_\ell|(n_i + n_j) \\ &\overset{(9.7)}{\leqslant} n|Z| + m - \left(\frac{n}{k} - \sqrt{Dr} - |Z|\right)\cdot\frac{km}{4\alpha^{1/3}n} \\ &\leqslant \frac{2m}{\xi'} + m - \frac{n}{2k}\cdot\frac{km}{4\alpha^{1/3}n} \leqslant -\frac{m}{9\alpha^{1/3}} \overset{(9.19)}{<} 0, \end{aligned}$$

another contradiction. Thus there is no $z \in Z_I$, as required. □

The final ingredient is the following lemma, which states that every R_i is small; G induced on the union of the R_i is complete partite; and every $z \in Z$ has large degree into every R_i.

LEMMA 9.9. *The following hold in G:*

(i) *For all $i \in [k]$, we have* $\mathrm{diff}(i) < -1/(2\alpha^{1/3})$.

(ii) *$G[R_1 \cup \cdots \cup R_k]$ is a complete k-partite graph (with partition $R_1, \ldots, R_k$).*

(iii) *For all $i \in [k]$ and $z \in Z$, we have $d_G(z, R_i) \geqslant km/(9\alpha^{1/3}n)$.*

Proof. For (i), suppose that there is some $i \in [k]$ for which $\mathrm{diff}(i) \geqslant -1/(2\alpha^{-1/3})$. Apply Lemma 9.6 to obtain $j \in [k] \setminus \{i\}$ such that $R_i \not\sim R_j$. But Lemma 9.5 implies that $\mathrm{diff}([k] \setminus \{i, j\}) \geqslant -1/(2\alpha^{1/3})$. Thus $\mathrm{diff}([k] \setminus \{j\}) \geqslant -1/(2\alpha^{1/3})$, a contradiction to Lemma 9.4.

We now turn to (ii). Since R_i is an independent set in G for all $i \in [k]$, it suffices to show that $R_i \sim R_j$ for all $ij \in \binom{[k]}{2}$. If there is some $ij \in \binom{[k]}{2}$ for which this does not hold, then Lemma 9.5 implies that $\mathrm{diff}([k] \setminus \{i, j\}) \geqslant -1/(2\alpha^{1/3})$. Then, by averaging (that is, (9.20)), there is some $\ell \in [k] \setminus \{i, j\}$ for which $\mathrm{diff}(\ell) \geqslant -1/(2\alpha^{-1/3})$, contradicting (i).

For (iii), let $z \in Z$ be arbitrary. Lemma 9.3 implies that there is $I \in \binom{[k]}{k-2}$ such that $z \in Z_I$ (and so $z \sim R_I$). Let $ij \in \binom{[k]}{2}$ be such that $I = [k] \setminus \{i, j\}$ and for all $\ell \in [k]$, write $n_\ell := d_G(z, R_\ell)$. We only need to show that $n_i, n_j \geqslant (km)/(9\alpha^{1/3}n)$ since for all $\ell \in I$, we have

$$n_\ell = |R_\ell| \overset{(9.7)}{\geqslant} \frac{n}{k} - \sqrt{Dr} - |Z| > \frac{n}{2k} \overset{(9.6)}{>} \frac{kD\alpha^{2/3}n}{4} \overset{(9.2)}{\geqslant} \frac{kDr}{4\alpha^{1/3}n} \geqslant \frac{km}{9\alpha^{1/3}n}.$$

The second part of Lemma 9.3 implies that there exist $u_i \in N_{\overline{G}}(z, R_i)$ and $u_j \in N_{\overline{G}}(z, R_j)$. Then

$$(k-2)cn - k \overset{(5.5)}{\leqslant} P_3(zu_i, G) \leqslant |Z| + n_j + |R_I|$$

and so

$$\begin{aligned} n_j &\overset{(9.19)}{\geqslant} (k-2)cn - k - \frac{2m}{\xi' n} - \sum_{\ell \in I} |R_\ell| \\ &\overset{(i)}{\geqslant} (k-2)cn - \frac{2km}{\xi' n} - (k-2)\left(cn - \frac{m}{2\alpha^{1/3}n}\right) \geqslant \frac{km}{9\alpha^{1/3}n}, \end{aligned}$$

where we used the fact that $k \geqslant 3$. An identical proof works for n_i. □

Proof of Theorem 1.7 in the boundary case. We will show that $Z = \emptyset$, contradicting (9.19). Suppose not, and let $z \in Z$. Then Lemma 9.3 implies that there is $I \in \binom{[k]}{k-2}$ for which $z \in Z_I$. So $z \sim R_I$. Write $I = [k] \setminus \{i, j\}$ and suppose without loss of generality that $z \in A_i$. Let $n_\ell := d_G(z, R_\ell)$ for all $\ell \in [k]$. Let $F_{Z,j} := G[N_G(z, Z), N_G(z, R_j)]$ and $F_{Z,I} := G[N_G(z, Z), R_I]$. Then Lemma 9.9(ii) implies that

$$K_3(z, G) \geqslant e(G[R_I]) + |R_I|(n_i + n_j) + n_i n_j + e(F_{Z,j}) + e(F_{Z,I}).$$

We have

$$N_{\overline{G}}(z, R_j) \overset{P5(G)}{\geqslant} \xi' n - |Z| \overset{P3(G)}{>} \delta' n \geqslant d_G(z, R_i)$$

and hence we can choose a set $K_j \subseteq N_{\overline{G}}(z, R_j)$ with $|K_j| = d_G(z, R_i)$. Obtain a graph G' from G as follows. Let $V(G') := V(G)$ and $E(G') := (E(G) \cup \{zx : x \in K_j\}) \setminus \{zy : y \in N_G(z, R_i)\}$. Clearly, G' is an (n, e)-graph in which z has no neighbours in R_i, so

$$\begin{aligned} K_3(z, G') &\leqslant e(G'[R_I]) + |R_I| d_{G'}(z, R_j) + e(G'[N_{G'}(z, Z), R_I]) \\ &\quad + e(G'[N_{G'}(z, Z), N_{G'}(z, R_j)]) + |Z|^2 \\ &\leqslant e(G[R_I]) + |R_I|(n_i + n_j) + e(F_{Z,I}) + e(F_{Z,j}) + n_i|Z| + |Z|^2. \end{aligned}$$

Therefore, using Lemma 9.9(iii), we have

$$K_3(G') - K_3(G) \leqslant n_i(|Z| - n_j) + |Z|^2 \overset{(9.19)}{\leqslant} n_i\left(\frac{2m}{\xi' n} - \frac{km}{9\alpha^{1/3}n}\right) + \frac{4m^2}{(\xi')^2 n^2}$$
$$\leqslant -\frac{n_i m}{10\alpha^{1/3}n} + \frac{4m^2}{(\xi')^2 n^2} \leqslant \left(\frac{4}{(\xi')^2} - \frac{k}{90\alpha^{2/3}}\right)\frac{m^2}{n^2} \overset{(5.1)}{<} 0,$$

a contradiction. Thus $Z = \emptyset$, contradicting (9.19) as required.

This completes the proof of Theorem 1.7.

10. Concluding remarks

10.1. Related work. The more general *supersaturation problem* of determining $g_F(n,e)$, the minimum number of copies of F in an (n,e)-edge graph, is also an active area of research. The range of e for which $g_F(n,e) = 0$ is well understood. Indeed, given a fixed graph F, let $\mathrm{ex}(n, F)$ denote the maximum number of edges in an F-free n-vertex graph, that is, the maximum e for which $g_F(n,e) = 0$. Erdős and Stone [9] proved that $\mathrm{ex}(n, F) = t_{\chi(F)-1}(n) + o(n^2)$, where $\chi(F)$ is the chromatic number of F. The supersaturation phenomenon observed by Erdős and Simonovits [5] asserts that every (n,e)-graph G with $e \geqslant \mathrm{ex}(n, F) + \Omega(n^2)$ contains not just one copy of F, but in fact a positive proportion of all $|V(F)|$-sized vertex subsets in $V(G)$ span a copy of F. (This also extends to hypergraphs.)

We say that F is *critical* when there is an edge in F whose removal reduces the chromatic number. Observe that cliques are critical. Simonovits [40] showed that, for such F and large n, we have $\mathrm{ex}(n, F) = t_{\chi(F)-1}(n)$ and $T_{\chi(F)-1}(n)$ is the unique extremal graph. That is, $g_F(n,e) = 0$ if and only if $e \leqslant t_{\chi(F)-1}(n)$. Mubayi [29] showed that there is $c > 0$ such that, for large n, and $1 \leqslant \ell \leqslant cn$, we have

$$g_F(n, t_{\chi(F)-1}(n) + \ell) = (1 + o(1))\,\ell \cdot \mathrm{copy}(n, F),$$

where $\mathrm{copy}(n, F)$ is the minimum number of copies of F obtained by adding a single edge to $T_{\chi(F)-1}(n)$. (This can generally be computed easily for any fixed F.) Note that this result generalizes Erdős's result [7] from triangles (which are critical) to arbitrary critical F. Further, the error term can be removed in some cases, for example, when F is an odd cycle. Pikhurko and Yilma [35] generalized Mubayi's result by raising the upper bound cn on ℓ to $o(n^2)$.

The supersaturation problem for noncritical F with $\chi(F) \geqslant 3$ seems hard; for example, even the 'simplest' case when F consists of two triangles sharing a vertex poses considerable difficulties (see [19]).

The case of bipartite F is very different. A famous conjecture of Sidorenko [39] and Erdős–Simonovits [5] asserts, roughly speaking, that the minimal number of F-subgraphs is asymptotically attained by a random graph (we do not give a precise statement of the conjecture here). The conjecture is known to be true for trees, cycles, complete bipartite graphs, 'strongly tree-decomposable graphs' and others; see, for example, [2, 3, 15, 22, 24, 41].

A yet more general problem is the following. Let $\mathcal{F} := (F_1, \ldots, F_\ell)$ be a tuple of graphs with $v_1, \ldots, v_\ell$ vertices respectively. Let $F_i(G)$ denote the number of *induced* copies of F_i in a graph G, for all $i \in [\ell]$. To an n-vertex graph G, associate a vector $f_{\mathcal{F}}(G) := (F_1(G)/\binom{n}{v_1}, \ldots, F_\ell(G)/\binom{n}{v_\ell})$ of densities. What is the set $T(\mathcal{F}) \subseteq \mathbb{R}^\ell$ consisting of the accumulation points of $f_{\mathcal{F}}(G)$? When $\mathcal{F} = (K_2, K_r)$, it turns out that $T(\mathcal{F})$ has an upper and a lower bounding curve. The lower bounding curve of $T(\mathcal{F})$ is by definition $y = g_r(x)$, which by Reiher's clique density theorem [38] is a countable union of algebraic curves. The upper bounding curve is $y = x^{r/2}$, which is a consequence of the Kruskal–Katona theorem [20, 23]. This corresponds to the *maximum* r-clique density in a graph with given edge density. The shaded region in Figure 1 is $T(\mathcal{F})$ for $\mathcal{F} = (K_2, K_3)$.

The case $(F_1, F_2) = (K_3, \overline{K_3})$ was solved by Huang, Linial, Naves, Peled and Sudakov [18] (here the lower bounding curve is $x + y = 1/4$, due to Goodman [13]). Glebov, Grzesik, Hu, Hubai, Král' and Volec [11] studied the problem for every remaining pair (F_1, F_2) of three-vertex graphs. For larger graphs, the problem becomes extremely challenging. Some general results on the hardness of determining $T(\mathcal{F})$ were obtained by Hatami and Norine in [16, 17].

10.2. The range $\binom{n}{2} - \varepsilon n^2 < e \leqslant \binom{n}{2}$. Our main result, Theorem 1.6, determines $g_3(n, e)$ whenever $2e/n^2$ is bounded away from 1. There are a few obstacles to extending it to the remaining range $e = \binom{n}{2} - o(n^2)$. One is that Theorem 1.2 does not tell us anything meaningful in this range, as the error in its approximation is too large.

While it is trivial to determine $g_3(n, e)$ when $e \geqslant \binom{n}{2} - \lfloor n/2 \rfloor$ (with each extremal graph G being the complement of a matching) and this can be extended a bit further with some work, the problem seems to become very difficult in this regime quite quickly. In fact, the following observation shows that, under the assumption that $g_3 \equiv h^*$, pushing $\binom{n}{2} - e$ beyond $O(n)$ is as difficult as determining $g_3(n, e)$ for all pairs (n, e).

LEMMA 10.1. *Suppose that for every* $C > 0$, *there is* $n_0 > 0$ *such that* $g_3(n, e) = h^*(n, e)$ *for all* $n \geqslant n_0$ *and* $e \geqslant \binom{n}{2} - Cn$. *Then* $g_3(n, e) = h^*(n, e)$ *for* ***all*** $n, e \in \mathbb{N}$ *with* $e \leqslant \binom{n}{2}$.

Proof. Suppose on the contrary that some (n, e)-graph G satisfies $K_3(G) < h^*(n, e)$. Let $\boldsymbol{a}^* = \boldsymbol{a}^*(n, e)$. Our assumption for $C := n/2$ returns some n_0. Take ℓ such that $n' := \ell a_1^* + n$ is at least n_0. Let H be the complete partite graph with n' vertices, ℓ parts of size a_1^* and the last part, call it A, of size n. Let G' (respectively, H') be obtained from H by adding a copy of G (respectively, $H^*(n, e)$) into A. Each of these graphs has $e' := \binom{n'}{2} - \ell\binom{a_1^*}{2} - \binom{n}{2} + e$ edges, which are at least $\binom{n'}{2} - \frac{n}{2} n'$ because the maximum degree of the graph complement is at most n. Also, H' is isomorphic to $H^*(n', e')$: this follows by induction on $\ell \in \mathbb{N}$ using the easy claim that if we duplicate a largest part of any H^*-graph, then we get another H^*-graph. However, since A is complete to the rest of H, we have

$$K_3(G') - h^*(n', e') = K_3(G') - K_3(H') = K_3(G) - K_3(H^*(n, e)) < 0,$$

a contradiction to the choice of n_0. □

An interesting corollary of Proposition 1.5 and Lemma 10.1 is that the validity of Conjecture 1.4 for $r = 3$ will not be affected if we drop the assumption $n \geqslant n_0$.

10.3. Extensions. It would be very interesting to extend Theorem 1.7 to the $g_r(n, e)$-problem, as many parts of our proof extend when we minimize the number of r-cliques. A structure result for r-cliques with $r \geqslant 4$ (an analogue of Theorem 1.2) was recently proved by Kim, Liu, Pikhurko and Sharifzadeh [**21**].

A problem that may be more directly amenable to our method is as follows. Recall that $N_i(G)$ is the number of 3-subsets of $V(G)$ that induce exactly i edges, $0 \leqslant i \leqslant 3$. The question is to maximize $N_2(G)$ (the number of so-called *cherries*) in an (n, e)-graph for $n \geqslant n_0$. This problem was considered by Harangi [**14**], who obtained some partial results that were enough for his intended application. Note that for every (n, e)-graph G, we have (see (9.10))

$$e(n-2) = 3N_3(G) + 2N_2(G) + N_1(G).$$

Also, $N_1(H^*(n, e)) \leqslant m^* n = o(n^3)$. Since $H^*(n, e)$ asymptotically minimizes N_3 over (n, e)-graphs, it also asymptotically maximizes N_2. Furthermore, a stronger version of stability (that every almost N_2-extremal (n, e)-graph is $o(n^2)$-close to $H^*(n, e)$) can be easily derived from Theorem 1.2.

We hope that the method used here will be useful for further instances where one has to convert an asymptotic result into an exact one.

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List of Abbreviations and Symbols

Boundary case (BC)	$t_k(n) - \alpha n^2 < e \leqslant t_k(n) - 1$	123
(C1)–(C3)	Worst counterexample properties	37
Intermediate case (IC)	$t_{k-1}(n) + \alpha n^2 \leqslant e \leqslant t_k(n) - \alpha n^2$	42,123
$(V_1, \ldots, V_k; U, \beta)$-partition	(IC) P1(G), P2(G)	34
$(V_1, \ldots, V_k; U, \beta, \delta)$-partition	(IC) P1(G)–P4(G)	34
$(V_1, \ldots, V_k; U, \beta, \gamma_1, \gamma_2, \delta)$-partition	(IC) P1(G)–P5(G)	34
Weak $(V_1, \ldots, V_k; U, \beta, \gamma_1, \gamma_2, \delta)$-partition	(IC) P1(G), P3(G), P5(G)	34
α	(IC) $e \leqslant t_k(n) - \alpha n^2$; (BC) $e > t_k(n) - \alpha n^2$	37,42,123
$\alpha_{1.3}$	The minimum constant $\alpha(k)$ returned from Theorem 1.3 applied with $r = 3$ and $3 \leqslant k \leqslant 1/\varepsilon$	37,8
β	Deviation of $A_1, \ldots, A_{k-1}$ from cn is βn.	37
Δ	(IC) Maximum degree of $x \in R_k'$ into Z_k	62
δ	(IC) $\|Z\| \leqslant \delta n$; $h \leqslant \delta m$, $\Delta(G[A_i]) \leqslant \delta n$ for all $i \in [k]$	37
δ'	(BC) $\|Z\| \leqslant \delta' n$; $h \leqslant \delta' m$, $\Delta(G[A_i]) \leqslant \delta' n$ for all $i \in [k]$	37,124
ε	$e \leqslant \binom{n}{2} - \varepsilon n^2$	37
η	The number of missing edges m in G satisfies $m \leqslant \eta n^2$	37,58
γ	$x \in Z_k$ is in Y if and only if it has degree less than γn into its corresponding part	37,59
$\rho_4, \ldots, \rho_0$	Small constants used exclusively in the proof of Lemma 6.1	37,42
ξ	(IC) $z \in Z$ if and only if it has missing degree at least ξn	37,59
ξ'	(BC) $z \in Z$ if and only if it has missing degree at least $\xi' n$	37,128
(n, e)-graph	A graph with n vertices and e edges	2
$\boldsymbol{a}^*$	Length-k vector whose ith entry is the size of the ith part in $H^*(n, e)$	2
$\mathcal{G}^{\min}(n, e)$	Subfamily of a family $\mathcal{G}(n, e)$ of (n, e)-graphs which contain the fewest triangles	10
$\mathcal{H}^*(n, e)$	Family of (n, e)-graphs generated from $H^*(n, e)$	5
$\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}$	Auxiliary families of (n, e)-graphs. $\mathcal{H}(n, e) = \mathcal{H}_1(n, e) \cup \mathcal{H}_2(n, e)$	7
$\mathrm{diff}(I)$	$\|R_I\| = (cn + \mathrm{diff}(I) \cdot \frac{m}{n})\|I\|$	130
$\pm$	$x = a \pm b$ if $a - b \leqslant x \leqslant a + b$, where $b \geqslant 0$	10
$\underline{m} = (m_1, \ldots, m_{k-1})$	Missing vector, $m_i = e(\overline{G}[A_i, A_k])$	34,42,124
$\underline{m}'$	(IC) Missing vector of G'	60,93
$\underline{m}^{(i)}$ (respectively $\underline{m}^{(i,\ell)}$)	(IC) Missing vector of G_i (respectively of G_i^ℓ)	66,74,81
$A_1'', \ldots, A_k''$	(IC) Partition of G'	60,88
$A_1', \ldots, A_k'$	(IC) Partition of G_2	74
$A_1, \ldots, A_k$	Parts of G	42,58
$a_i, i \in [k-1]$	$a_i = \sum_{j \in [k-1] \setminus \{i\}} \|A_j\|$	61
c	Part ratio	24
D	(BC) $169k^{k+9}$	124
$D(x)$, $D(x, y)$	(IC) External X-degree of $x \in X$, common external X-degree of $x, y \in X$	76,116
$d_H^m(y)$	(IC) Missing degree of y into corresponding part	34
D_i	(IC) $D(x) = D_i$ for all $x \in X_i$, proved in Lemma 8.16	116
e	Number of edges in G	37
$e(K^\ell_{\alpha_1, \ldots, \alpha_\ell})$	Continuous edge count	24

f	$f(x) = (d_G(x)-(k-2)cn)(k-2)cn+\binom{k-2}{2}c^2n^2-K_3(x, G)$ for $x \in V(G)$	42
G	'Worst counterexample' graph with n vertices and e edges satisfying (C1)–(C3)	37
G''	(IC) Graph obtained in Lemma 7.1	60
G_i, G_i^ℓ	(IC) Graph obtained from G_{i-1} after Transformation i (applied with ℓ)	66,74,81
$g_r(n, e)$	Minimum number of r-cliques in an (n, e)-graph	2
h	Number of bad edges, $\sum_{i\in[k]} e(G[A_i])$	39,124
$h(n, e)$	Minimum number of triangles in graphs in $\mathcal{H}(n, e)$	8
$H^*(n, e)$	A conjectured extremal (n, e)-graph	2
$h^*(n, e)$	$K_3(H^*(n, e))$	2
k	Minimum $\ell \in \mathbb{N}$ such that $e \leqslant t_\ell(n)$	2
$K_3(K^\ell_{\alpha_1,\ldots,\alpha_\ell})$	Continuous triangle count	24
$K_3(x, G)$	Number of triangles in G containing vertex x	10
$K_3(x, G; A)$	Number of triangles in G containing vertex x and at least one other vertex in A	10
$K_3(x, G; A, A)$	Number of triangles in G containing vertex x and both other vertices in A	10
$K_r(H)$	Number of r-cliques in a graph H	2
m	Number of missing edges $m = \sum_{i\in[k-1]} m_i$	34,42,124
m^*	Number of missing edges in $H^*(n, e)$	2
n	Number of vertices in G	37
n_0	Sufficiently large, we require $n \geqslant n_0$	37
N_i	(BC) 3-vertex graph with i edges	125
P1(G)–P5(G)	(IC) Partition properties	34
$P_3(xy, G)$	Number of common neighbours of x, y in G	10
$P_3(xy, G; A)$	Number of common neighbours of x, y in G which lie in A	10
$Q_1, \ldots, Q_{k-1}$	(IC) $Q_i \subseteq E(G[R_i, R_k])$ carefully chosen	61
$q_G(x)$	(BC) $\frac{2e}{n} - d_G(x)$	125
r	(BC) $e = t_k(n) - r$	123
$R_1, \ldots, R_k$	$R_i := A_i \setminus Z$.	59,129
$R_I, I \subseteq [k]$	$\bigcup_{i\in I} R_i$	129
R'_k	(IC) A large subset of R_k	62
s	(BC) $e = (1 - \frac{1}{s})\binom{n}{2}$	125
t	(IC) $\frac{m}{(kc-1)n}$	59
$T_s(n), t_s(n)$	n-vertex s-partite Turán graph and the number of edges it contains	2
U_i, W_i	(IC) See Lemma 7.1	85
$X_1, \ldots, X_{k-1}$	(IC) $X_i := Z^i_k \setminus Y_i$	59
$Y_1, \ldots, Y_{k-1}$	(IC) $Y_i \subseteq Z^i_k$ contains elements with at most γn neighbours in A_i	59
Z	(IC) set of vertices z with $d^m_G(z) \geqslant \xi n$. Boundary case: $d^m_G(z) \geqslant \xi' n$	59,128
$Z_1, \ldots, Z_k$	(IC) $Z_i := A_i \cap Z$	59
$Z_I, I \in \binom{[k]}{k-2}$	(BC) $G[Z_I, R_I]$ complete	130
$Z^1_k, \ldots, Z^{k-1}_k$	(IC) union is Z_k, $G[Z^i_k, A_j]$ complete when $j \in [k-1]\setminus\{i\}$	59

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