HECKE ALGEBRAS AND THE LUSZTIG ISOMORPHISM

A.P. Fakiolas

Thesis submitted for the degree of Doctor of Philosophy at Warwick University

January, 1987

Mathematics Institute
University of Warwick
COVENTRY CV4 7AL

TO ANNE

TABLE OF CONTENTS

CHAPTER 6

ACKNOWLEDGEMENTS

I am especially grateful to my supervisor Professor R.W. Carter to whom I owe more than I can possibly express. He transmitted to me confidence in the mathematical field, while his critical analysis of the manuscript led to substantial improvements in the thesis.

My appreciation is extended to Professor J.A. Green for many useful conversations and especially for his constant encouragement after the M.Sc. examination.

I would like also to thank Terri Moss for her patience in typing the manuscript.

Finally I gratefully acknowledge the financial support given me by Alexander S. Onassis Public Benefit Foundation during the three years of my study at Warwick.

INTRODUCTION

Let G be a Chevalley group over a finite field with q elements and let B be a Borel subgroup of G. Let H(G,B) be the Hecke algebra of the pair (G,B). J. Tits showed that the Hecke algebra over $\mathbb{C}$ is isomorphic to the group algebra over $\mathbb{C}$ of the Weyl group. N. Iwahori conjectured that the Hecke algebra over $\mathbb{Q}$ of the pair (G,B) is isomorphic to the group algebra over $\mathbb{Q}$ of the Weyl group. Benson and Curtis proved that this conjecture is true whenever G is simple of type different from E_7, E_8. With the help of Springer they proved that the conjecture is no longer true when G is of type E_7, E_8. G. Lusztig constructed an explicit isomorphism from the Hecke algebra over $\mathbb{Q}(q^{\frac{1}{2}})$ to the group algebra over $\mathbb{Q}(q^{\frac{1}{2}})$ of the Weyl group.

The main purpose of this thesis is to investigate the general properties of this isomorphism. As a consequence of our investigation we introduce a way of obtaining orthogonal primitive idempotents inside the Hecke algebra.

This thesis has been divided into six chapters.

In Chapter 1 we recall some auxiliarly results about the structure of Coxeter groups and their associated Hecke algebras. We also recall the Kazhdan-Lusztig decomposition of a Coxeter group into left, right and two sided cells and we explain how the cells give rise to representations of the Coxeter groups and of the corresponding Hecke algebras.

Let W be a finite indecomposable Coxeter group satisfying a certain property (property (A)) for the structure of its two sided cells. We recall an explicit isomorphism from $H_{\mathbb{Q}(u^{\frac{1}{2}})}(W)$ to $\mathbb{Q}(u^{\frac{1}{2}})(W)$ constructed by G. Lusztig, where $\mathbb{Q}(u^{\frac{1}{2}})$ is the field of fractions of the polynomial ring $\mathbb{Q}[u^{\frac{1}{2}}]$.

The subsequent chapters are our own work.

In Chapter 2 we find an explicit formula for Lusztig's isomorphism in the case where $W = D_{2n}$ the dihedral groups. It turns out that these groups satisfy the required property (A). Here we achieve our results using classical properties of the Chebyshev polynomials of the second kind.

In Chapter 3 we investigate the centre of the Hecke algebra over the polynomial ring $\mathbb{Q}[u]$, following some ideas of R.W. Carter. These ideas give a natural basis for the centre of the Hecke algebras of dihedral groups and they lead to an interesting conjecture for the form of a basis of the centre of the Hecke algebra in the general case.

In Chapter 4 we find the images of the central basis elements of the Hecke algebra of dihedral type determined in the previous chapter, under Lusztig's isomorphism. Here we show that the images of these elements no longer involve $u^{\frac{1}{2}}$.

In Chapter 5 we prove results valid for arbitrary Hecke algebras. Here we show that the images of the generators T_s of the Hecke algebra under Lusztig's isomorphism Φ are given by $\Phi(T_s) = \frac{u-1}{2}\cdot 1 + \frac{u+1}{2}\cdot s + (u^{\frac{1}{2}}-1)^2 F_s$ for some $F_s \in \mathbb{Q}W$.

We give two independent proofs of this result. The second one is based on some conjectures made by R.W. Carter and uses the results of A. Gyoja for the irreducible representations of Coxeter groups and Hecke algebras.

We also show that if $c = \sum_{w \in W} a_w T_w$ is an element in the centre of the Hecke algebra with $a_w \in \mathbb{Q}[u]$, then in most of the cases the image of c under Lusztig's isomorphism Φ, belongs to $\mathbb{Q}[u](W)$.

In Chapter 6 we deal with the construction of orthogonal primitive idempotents inside the Hecke algebra. These ide mpotents are obtained naturally from the decomposition of a maximal commutative subalgebra inside the Hecke algebra. We shall achieve this decomposition in some special cases.

Finally we discuss some open questions which arise naturally from our work, and we make some conjectures which would allow these questions to be settled.

CHAPTER 1

1.1 Coxeter groups

We shall first state some well known results about Coxeter groups. A group W given by generators and relations as follows,

$$W = (W,S) = \langle s_i,\ i \in I,\ s_i^2 = 1,\ (s_i s_j)^{m_{ij}} = 1,\ m_{ij} \in \mathbb{Z},\ m_{ij} \geq 2 \quad i \neq j \text{ or } m_{ij} = \infty\rangle$$

is called a Coxeter group. When $m_{ij} = \infty$ we omit the relation $(s_i s_j)^{m_{ij}} = 1$. We put $S = \{s_i,\ i \in I\}$.

The Coxeter graph Γ associated to a given Coxeter group (W,S) is by definition a set of nodes labelled by the elements of S, together with a set Y of edges. An edge is a subset of S consisting of two elements, such that for $s_i, s_j \in S$ $\{s_i, s_j\} \in Y$, if and only if $m_{ij} \geq 3$. In this case the node i is joined to the node j by $m_{ij}-2$ bonds. If $m_{ij} = 2$, then $s_i s_j = s_j s_i$ and s_i, s_j are not joined by a bond.

A Coxeter group W is entirely determined up to isomorphism by its associated Coxeter graph.

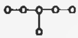
A Coxeter group (W,S) is defined to be indecomposable if its Coxeter graph is connected, i.e. for any s, t $\in$ S, there exists a sequence $s_0 = s, s_1, s_2, \ldots s_r = t$ in S for some $r \geq 0$ such that $\{s_{i-1}, s_i\}$ is an edge for every $1 \leq i \leq r$.

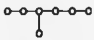
<u>Theorem 1.1</u> If (W,S) is an indecomposable Coxeter group of finite order, then its Coxeter graph has one of the following forms.

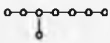
A_ℓ  ($\ell \geq 1$ nodes)

B_ℓ  ($\ell \geq 2$ nodes)

D_ℓ  ($\ell \geq 4$ nodes)

E_6 

E_7 

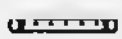
E_8 

F_4 

G_2 

H_3 

H_4 

$I_2^{(p)}$  ($p = 5$ or $p \geq 7$)

(p-2) bonds

For a proof see [5], page 193.

Assume that $W = \langle s_1, \ldots, s_\ell \mid s_i^2 = 1,\ (s_i s_j)^{m_{ij}} = 1,\ 1 \leq i, j \leq \ell \quad i \neq j \rangle$ $m_{ij} \geq 2$ if $i \neq j$, $m_{ii} = 1$ is a finite Coxeter group. Then W can be described as a group generated by reflections in a finite dimensional

Euclidean space. Let V be a vector space over the real field R of dimension ℓ with basis $\{a_1,\ldots,a_\ell\}$. We define a bilinear form on V by $\langle\ ,\ \rangle : V \times V \to R$ such that $(a_i,a_j) \to \langle a_i,a_j\rangle = -\operatorname{Cos}\frac{\pi}{m_{ij}}$ and extend by linearity. This form is symmetric since $m_{ij} = m_{ji}$, and $\langle a_i,a_i\rangle = 1$. Let H_i be the subspace of V defined by $H_i = \{v \in V: \langle a_i,v\rangle = 0\}$. Then, $\dim H_i = \ell-1$ and $V = Ra_i \oplus H_i$. We define a linear map $\tau_i:V \to V$ by $\tau_i(v) = v-2\langle a_i,v\rangle a_i$. Then $\tau_i(a_i) = -a_i$ and $\tau_i(v) = v$ for all $v \in H_i$. Thus τ_i is the reflection in the hyperplane H_i and so $\tau_i^2 = 1$. We a'so have that $(\tau_i\tau_j)^{m_{ij}} = 1$ if $i \neq j$, and that $\langle\tau_i(v),\tau_i(v')\rangle = \langle v,v'\rangle$ for all v, $v' \in V$. So there exists a homomorphism θ: $W \to \langle\tau_1,\ldots,\tau_\ell\rangle$ from W into the group of isometries of V given by $\theta(s_i) = \tau_i$.

The form $\langle v,v'\rangle$ on V can be shown to be non-singular and positive definite and so V may be regarded as Euclidean space. It can also be shown that W acts faitnfully on V and that V is an irreducible W-module. We call V the reflection representation of W.

For any element w of a Coxeter group (W,S) we define the length of w, denoted by $\ell(w)$, to be the smallest integer $q \geq 0$ such that w is a product of a sequence $(s_1,\ldots,s_q)$ of q elements of S. We define a reduced expression of w to be an expression $w = s_1,\ldots,s_q$ where $q = \ell(w)$.

Let $\Pi = \{a_1,\ldots,a_\ell\}$. Π is called the set of simple roots Let $\Phi = W(\Pi) = \{w(a_i),\ w \in W,\ a_i \in \Pi\}$. Φ is called the set of roots. It can be shown that each $a \in \Phi$ has the form $a = \sum_{i=1}^{\ell} \lambda_i a_i$ with each $\lambda_i \geq 0$ or each $\lambda_i \leq 0$.

Let $\Phi^+ = \{a \in \Phi: a = \sum_{i=1}^{\ell} \lambda_i a_i,\ \lambda_i \geq 0\ \forall i = 1,\ldots,\ell\}$, and let

$$\Phi^- = \{a \in \Phi : a = \sum_{i=1}^{\ell} \lambda_i a_i \ \lambda_i \le 0 \ \forall i = 1,\ldots,\ell\}.$$

Φ^+ is called the set of positive roots, Φ^- is called the set of negative roots, and clearly $\Phi = \Phi^+ \cup \Phi^-$.

For any $w \in W$ we denote by $n(w)$ the number of positive roots made into negative by w.

The following proposition provides some well known results about the $\ell(w)$.

<u>Proposition 1.2</u>

(i) For any $w \in W$, $n(w) = \ell(w)$

(ii) Let $s_i \in S$ $w \in W$ and $w = s_1,\ldots,s_q$ a reduced expression of w. Then there are two possibilities.

(a) $\ell(s_i w) = \ell(w) + 1$ and $(s_i, s_1, \ldots, s_q)$ is a reduced expression of $s_i w$. In this case $w^{-1}(a_i) \in \Phi^+$.

(b) $\ell(s_i w) = \ell(w) - 1$ and there exists a j, $1 \le j \le q$ such that $(s_1,\ldots,s_{j-1},s_{j+1},\ldots,s_q)$ is a reduced expression of $s_i w$ and $(s_i,s_1,\ldots,s_{j-1},s_{j+1},\ldots,s_q)$ is a reduced expression of w. In this case $w^{-1}(a_i) \in \Phi^-$.

(iii) If W is finite there exists a unique element of maximal length, denoted by w_0

(iv) Let $s,t \in S$, $w \in W$ such that $\ell(sw) = \ell(wt)$ and $\ell(swt) = \ell(w)$. Then $sw = wt$. (For a proof see [6], §2.2, and [6], pages 15, 18).

The Bruhat order relation on W is defined by: For any $y, w \in W$ we say $y \le w$ if there exist reduced expressions $y = s_{j_1} s_{j_2} \ldots s_{j_s}$ and

$\omega = s_1 s_2 \ldots s_q$ with all s_i lying in S, such that $(j_1, j_2, \ldots, j_s)$ is a subsequence of $1, 2, \ldots, q$.

A Coxeter group is called crystallographic if, for all $i \neq j$ $m_{ij} \in \{2,3,4,6\}$. Thus the indecomposable Coxeter groups which are crystallographic are of type A_ℓ, B_ℓ, D_ℓ, E_6, E_7, E_8, F_4, G_2.

1.2 The Hecke algebra of a finite Chevalley Group

Let L be a simple Lie algebra of finite dimension over $\mathbb{C}$. Then, there is a finite crystallographic Coxeter group W associated to L, called the Weyl group of L. For each such finite indecomposable crystallographic Coxeter group W, there is just one simple Lie algebra which has W as its Weyl group except when W has type B_ℓ $\ell \geq 3$, when there are two such Lie algebras, called B_ℓ, C_ℓ.

Any such Lie algebra has a Cartan decomposition $L = H \oplus \sum_{a \in \Phi} L_a$ where H is a maximal commutative subalgebra called a Cartan subalgebra of L and L_a is a 1-dimensional H-module of L. The set Φ of 1-dimensional representations of H arising in this way is called the set of roots of L. Φ has a subset Π, called a set of simple roots, such that each root in Φ is uniquely expressible as a linear combination of elements of Π with coefficients in $\mathbb{Z}$ which are either all non-negative or all non-positive. The set Φ decomposes in this way as $\Phi = \Phi^+ \cup \Phi^-$ where Φ^+, Φ^- are the positive and negative roots respectively. In this root system we no longer assume that roots are unit vectors, so in general this root system is different from the one defined in 1.1. (Proof of these results can be found in [16]).

We can choose a basis for L relative to a Cartan decomposition, called a Chevalley basis, whose elements are $\{h_a, a \in \Pi, e_a, a \in \Phi\}$.

The Lie product of any two elements in the basis, is a linear combination of basis elements with coefficients in Z.

For each $x \in L$ we define $adx : L \to L$ by $adx \cdot y = [x,y]$. This is a derivation. For each $\lambda \in \mathbb{C}$ and for each $a \in \Phi$, the map $ad(\lambda e_a) : L \to L$ is nilpotent. Thus we can form exp $ad(\lambda e_a)$ which is an automorphism of L. We write $\chi_a(\lambda) = \exp ad\,(\lambda e_a)$. The Chevalley basis has the property that the matrix $M_a(\lambda)$ of each $\chi_a(\lambda)$ with respect to this basis has entries which are of the form $b\lambda^i$ $i \geq 0$ $b \in Z$.

Now for any field K we can define a Lie algebra L_K over K by taking all K-combinations of elements in a Chevalley basis and taking Lie multiplication as before, interpreting the integers as elements of the prime subfield of K. L_K has a basis $\{\bar{h}_a, a \in \Pi, \bar{e}_a, a \in \Phi\}$, where $\bar{h}_a = h_a \otimes 1_K, \bar{e}_a = e_a \otimes 1_K$. For each $a \in \Phi$, $t \in K$, let $\bar{M}_a(t)$ be the matrix obtained by replacing the entry $b\lambda^i$ by $\bar{b}t^i$ where $\bar{b}$ is the element of the prime subfield of K corresponding to $b \in Z$.

Define $\bar{\chi}_a(t)$ to be the linear map of L_K into itself represented by the matrix $\bar{M}_a(t)$ with respect to the basis $\{\bar{h}_\alpha, a \in \Pi, \bar{e}_a, a \in \Phi\}$. Then it can be shown that $\bar{\chi}_a(t)$ is an automorphism of L_K, for each $a \in \Phi$, $t \in K$. The group of automorphisms of L_K generated by $\bar{\chi}_a(t)$ for all $a \in \Phi$, $t \in K$ is called the adjoint Chevalley group of type L over K. In particular if K is the finite field with q elements F_q, we obtain a finite Chevalley group which will be denoted by G(q).

For example if L is the simple Lie algebra $sl_n(\mathbb{C})$ of all $n \times n$ matrices of trace 0 over $\mathbb{C}$, G(q) will be the group $PSL_n(q)$ of all $n \times n$ matrices of determinant 1, factored by its centre. Let U(q) be the subgroup of G(q) generated by $X_a(\lambda)$ for all $a \in \Phi^+$ and all $\lambda \in F_q$. Let

$B(q) = N_{G(q)}(U(q))$, the normalizer of $U(q)$ in $G(q)$. In the previous example $U(q)$ is the subgroup of all upper unitriangular matrices and $B(q)$ is the subgroup of all upper triangular matrices. Let $1_{B(q)}$ be the trivial representation of $B(q)$ over $\mathbb{C}$, and let p be the representation of $G(q)$ induced by $1_{B(q)}$, i.e. $P = 1_{B(q)}^{G(q)}$. Let $e = \frac{1}{|B(q)|} \sum_{x \in B(q)} x$. Then $e \in \mathbb{C}B(q)$, $e^2 = e$, and $\mathbb{C}B(q)\cdot e$ is a left $B(q)$-module affording $1_{B(q)}$ while $\mathbb{C}G(q)\cdot e$ is a left $G(q)$-module affording $1_{B(q)}^{G(q)}$ (see [7], Proposition 11.21). We define the Hecke algebra of the pair $(G(q), B(q))$ to be the endomorphism algebra of the module $\mathbb{C}G(q)\cdot e$. We write $H_{\mathbb{C}}(q) = \mathrm{End}_{\mathbb{C}}(1_{B(q)}^{G(q)})$. Recall that the group algebra $\mathbb{C}G(q)$ is isomorphic as $\mathbb{C}$-algebra with the algebra of $\mathbb{C}$-valued functions $f:G(q) \to \mathbb{C}$ under convolution product, with the element $\sum_{x \in G(q)} a_x . x$ corresponding to the function f, defined by $f(x) = a_x$, $x \in G(q)$, $a_x \in \mathbb{C}$. If f,g are $\mathbb{C}$-valued functions on $G(q)$, their convolution product is defined as the function $f\cdot g : G(q) \to \mathbb{C}$ given by

$(f\cdot g)(x) = \sum_{y \in G(q)} f(xy^{-1})g(y)$.

Let $A = B(q)g\,B(q)$, $g \in G(q)$ and consider the $\mathbb{C}$-algebra of all formal linear combinations of $(B(q), B(q))$ double cosets, $\sum_A C_A . A$, $C_A \in \mathbb{C}$ under the following multiplication:

$$\text{Let } A = B(q)gB(q) = \dot{\bigcup_i} B(q)g_i,\ A' = B(q)g'B(q) = \dot{\bigcup_j} B(q)g_j'.$$

Define $A\cdot A' = \sum_C \mu_{A,A'}^C \cdot C$ where the summation is taken over all double cosets $C = B(q)\ cB(q)$ and $\mu_{A,A'}^C$ is the number of pairs (i,j) such that $B(q)g_ig_j' = B(q)\cdot c$. It can be shown that this is a well defined multiplication, in other words the number $\mu_{A,A'}^C$ is independent of the choice of the coset representatives g_i, g_j' , c. (See [14]). Moreover,

<u>Theorem 1.2.1</u>: The following four descriptions of the Hecke algebra of a finite Chevalley group G(q) with respect to the subgroup B(q) are equivalent:

(i) $\mathrm{End}_{\mathbb{C}}\,(1_{B(q)}^{G(q)})$

(ii) $e\cdot\mathbb{C}G(q)\cdot e$ where $e = \frac{1}{|B(q)|} \sum_{x \in B(q)} x$

(iii) Functions constant on (B(q),B(q)) double cosets, under convolution product.

(iv) The set of all formal linear combinations of the form $\sum_A C_A.A$, the summation being taken over the (B(q),B(q)) double cosets, with multiplication defined as above (Proof, see [10]).

<u>Theorem 1.2.2</u>: The dimension of $H_{\mathbb{C}}(q)$ is dim $H_{\mathbb{C}}(q) = |W|$, where W is the Weyl group of L. $H_{\mathbb{C}}(q)$ has a basis$\{v_w, w \in W\}$ such that if $w = s_1 \dots s_q$ is a reduced expression of w, then $v_w = v_{s_1} \dots v_{s_q}$. We write v_i for v_{s_i}, $i = 1,\dots,\ell$. Each v_i satisfies the quadratic relation

$$v_i^2 = q\cdot 1 + (q-1)v_i \quad (v_1 = 1 \text{ the identity of } H_{\mathbb{C}}(q)).$$

Finally $H_{\mathbb{C}}(q)$ has a presentation as $\mathbb{C}$-algebra given by

$$H_{\mathbb{C}}(q) = \langle v_1,\dots,v_\ell \mid v_i^2 = q\cdot 1 + (q-1)v_i,\ \underbrace{v_iv_j \dots}_{\leftarrow m_{ij} \rightarrow} = \underbrace{v_jv_i \dots}_{\leftarrow m_{ij} \rightarrow}$$

m_{ij} being the order of $s_is_j\rangle$.

(For the nature of the elements v_w $w \in W$ and for a proof see [10]).

Theorem 1.2.3. (J. Tits' deformation Theorem)

The Hecke algebra $H_{\mathbb{C}}(q)$ over $\mathbb{C}$ is isomorphic to the group algebra over $\mathbb{C}$ of the Weyl group: $H_{\mathbb{C}}(q) \simeq \mathbb{C}W$. (For a proof see [15] page 249).

We note that the product of any two basis elements v_w of $H_{\mathbb{C}}(q)$ is a $\mathbb{Z}$-combination of basis elements. So we have a subalgebra $H_{\mathbb{Z}}(q)$ of all Z-combinations of the basis elements v_w. However in this subalgebra the element v_w is not invertible since $v_i^{-1} = q^{-1}v_i + (q^{-1}-1)\cdot 1$. We therefore extend the ground ring to include q^{-1}.

Moreover, although Tits showed that $H_{\mathbb{C}}(q) \simeq \mathbb{C}W$, he did not give any explicit isomorphism between $H_{\mathbb{C}}(q)$ and $\mathbb{C}W$. Iwahori conjectured that $H_{\mathbb{Q}}(q) \simeq \mathbb{Q}W$ (see [10]). Benson and Curtis proved that this is true whenever the Chevalley group $G(q)$ is simple of type $\neq E_7, E_8$. (See [3]). It is not true when $G(q)$ has type E_7, E_8. (See [4]).

G. Lusztig constructed an explicit isomorphism between $H_{\mathbb{Q}(q^{\frac{1}{2}})}$ and $\mathbb{Q}(q^{\frac{1}{2}})(W)$. (See [12]).

The construction makes essential use of the Kazhdan-Lusztig theory.

1.3 The Kazhdan-Lusztig Theory

In this section we deal with the theory developed by Kazhdan and Lusztig in order to study representations of Hecke algebras. (See [11]). Let $(W,S) = \langle s_1,\dots,s_\ell \mid s_i^2 = 1,\ \underbrace{s_is_j\cdots}_{\leftarrow m_{ij}\rightarrow} = \underbrace{s_js_i\cdots}_{\leftarrow m_{ij}\rightarrow}\rangle$ to be a finite Coxeter group. With such a Coxeter system we associate an algebra $\mathcal{H}$ over the polynomial ring $\mathbb{Z}[u]$, u being an indeterminate over $\mathbb{Z}$, as follows. $\mathcal{H}$ has basis elements $\{T_w,\ w \in W\}$ and multiplication defined by the rules:

$$T_w . T_{w'} = T_{ww'} \text{ if } \ell(ww') = \ell(w) + \ell(w')$$

$$T_{s_i}^2 = u.T_1 + (u-1)\,T_{s_i} \qquad s_i \in S.$$

It can be shown that $\bar{H}$ has a presentation as an associative $Z[u]$-algebra given by generators and defining relations:

$$\langle T_{s_i} \mid T_{s_i}^2 = u.T_1 + (u-1)\,T_{s_i},\ \underbrace{T_{s_i} T_{s_j} \cdots}_{\leftarrow m_{ij} \rightarrow} = \underbrace{T_{s_j} T_{s_i} \cdots}_{\leftarrow m_{ij} \rightarrow} \rangle$$

m_{ij} being the order of $s_i s_j$, $i \neq j$. The idea of such an algebra is due to J. Tits, see [5] p. 55.

We extend the ground ring and we define the generic Hecke algebra $H(u)$ as follows: $H(u) = \bar{H} \otimes_{Z[u]} Z[u^{\frac{1}{2}}, u^{-\frac{1}{2}}]$. We put $A = Z[u^{\frac{1}{2}}, u^{-\frac{1}{2}}]$.

Our first step is to define representations of $H(u)$ with respect to a special basis. These representations are defined in terms of certain graphs.

Definition 1.3.0. Let $\mathbf{Z}$ be the ring of integers. A W-graph over $\mathbf{Z}$ is a set of vertices X together with a set of edges Y, an edge being a subset of X consisting of two elements and with two additional requirements:

(1) For each vertex $x \in X$ we are given a subset I_x of S and for each ordered pair of vertices y,x such that $\{y,x\} \in Y$ we are given an integer $\mu(y,x) \neq 0$.

(2) Let E be the free A-module with basis X. Then for any $s \in S$

$$\tau_s(x) = \begin{cases} -x & \text{if } s \in I_x \\ ux + u^{\frac{1}{2}} \displaystyle\sum_{\substack{y \in X \\ s \in I_y \\ (y,x) \in Y}} \mu(y,x)y & \text{if } s \notin I_x \end{cases}$$

defines an endomorphism of E, and there is a representation $\phi: H(u) \to End(E)$ such that $\phi(T_s) = \tau_s$, for each $s \in S$.

For any Coxeter group W we shall construct such a graph. Let $a \to \bar{a}$ be the involution of the ring $A = \mathbb{Z}[u^{\frac{1}{2}}, u^{-\frac{1}{2}}]$ defined by $\overline{u^{\frac{1}{2}}} = u^{-\frac{1}{2}}$. Now for every $s \in S$ the element T_s is invertible and $T_s^{-1} = u^{-1}T_s + (u^{-1}-1)T_1$. Therefore it makes sense to extend the involution above to a map $H(u) \to H(u)$ defined by $\overline{\sum_w a_w T_w} = \sum_w \bar{a}_w T_{w^{-1}}^{-1}$. This map preserves addition and multiplication but not scalar multiplication.

For any $w \in W$ we define $u_w = u^{\ell(w)}$, $\varepsilon_w = (-1)^{\ell(w)}$. Let $\leq$ be the Bruhat ordering on W, defined in §1.1. The following results are valid for an arbitrary Coxeter group W, however we restrict ourselves to a finite Coxeter group. For a proof of these results (see [11]).

<u>Theorem 1.3.1</u>. For any $w \in W$, there is a unique element $C_w \in H(u)$ such that:

$$\bar{C}_w = C_w \qquad \text{and}$$

$$C_w = \sum_{y \leq w} \varepsilon_y \varepsilon_w u_w^{\frac{1}{2}} u_y^{-1} \bar{P}_{y,w} T_y,$$

where $P_{y,w} \in A$ is a polynomial in u of degree $\leq \frac{1}{2}(\ell(w) - \ell(y) - 1)$ for $y < w$ and $P_{w,w} = 1$.

<u>Definition 1.3.2</u>. Given $y,w \in W$ we say $y \prec w$ if the following conditions are satisfied: $y < w$, $\varepsilon_w = -\varepsilon_y$ and $P_{y,w}$ is a polynomial in u of degree exactly $\frac{1}{2}(\ell(w) - \ell(y) - 1)$. In this case the leading coefficient of $P_{y,w}$ is denoted by $\mu(y,w)$. It is a non-zero integer.

The polynomials $P_{y,w}$ defined in Theorem 1.3.1 are given by the following inductive formula:

$$P_{y,w} = u^{1-c} P_{sy,v} + u^c P_{y,v} - \sum_{\substack{y \le z \prec v \\ sz < z}} \mu(z,v) u_z^{-\frac{1}{2}} u_v^{\frac{1}{2}} u^{\frac{1}{2}} P_{y,z} \quad (y \le w)$$

where $w = sv$, $s \in S$ with $\ell(w) = \ell(v) + 1$, $c = \begin{cases} 1 & \text{if } sy < y \\ 0 & \text{if } sy > y \end{cases}$ and we make the convention that $P_{x,v} = 0$ if $x \not\le v$. (See Theorem 1.1 in [11]).

<u>Remark</u>. The elements $\{C_w | w \in W\}$ defined in Theorem 1.3.1 provide a new basis for $H(u)$.

<u>Lemma 1.3.3</u>.

(i) For each $y < w$ with $\ell(w) = \ell(y) + 1$, we have $P_{y,w} = 1$. In particular we have $y \prec w$ and $\mu(y,w) = 1$

(ii) For each $y < w$ with $\ell(w) = \ell(y) + 2$ we have $P_{y,w} = 1$.

For further properties of the $P_{y,w}$ see Lemma 2.6 in [11].

<u>Lemma 1.3.4</u>. Let $\{C_w | w \in W\}$ as defined in Theorem 1.3.1, and let $s \in S$. Then: (i)

$$T_s \cdot C_v = \begin{cases} -C_v & \text{if } sv < v \\ uC_v + u^{\frac{1}{2}} C_{sv} + u^{\frac{1}{2}} \sum\limits_{\substack{z \prec v \\ sz < z}} \mu(z,v) C_z & \text{if } sv > v \end{cases}$$

(ii) Similarly interchanging left and right we have

$$C_v T_s = \begin{cases} -C_v & \text{if } vs < v \\ uC_v + u^{\frac{1}{2}} C_{vs} + u^{\frac{1}{2}} \sum\limits_{\substack{z < v \\ zs < z}} \mu(z,v) C_z & \text{if } vs > v \end{cases}$$

(See Proof of Theorem 1.3 in [11]).

Lemma 1.3.5. (i) Let $x,y \in W$, $s \in S$ be such that $x < y$, $sy < y$, $sx > x$. Then $P_{x,y} = P_{sx,y}$. Moreover $x \prec y$ if and only if $y = sx$, and this implies that $\mu(x,y) = 1$.

(ii) Let $x,y \in W$, $s \in S$ be such that $x < y$, $ys < y$, $xs > x$. Then $P_{x,y} = P_{xs,y}$. Moreover $x \prec y$ if and only if $y = xs$, and this implies that $\mu(x,y) = 1$.

(See Proof of Theorem 1.3 in [11]).

Now, Lemma 1.3.4(i) and 1.3.5(i) enable us to prove the following:

Theorem 1.3.6. Let Γ be the graph whose vertices are the elements of W and whose edges are the subsets of W of the form $\{y,w\}$ with $y \prec w$. For each $w \in W$, let $I_w = \mathcal{L}(w) =: \{s \in S : sw < w\}$. Then Γ, together with the assignment $w \to I_w$ and with the function μ given in Definition 1.3.2 is a W-graph. (See Theorem 1.3 in [11]).

Since Lemma 1.3.4 provides the action of the generators T_s on the basis $\{C_w | w \in W\}$, it is natural to ask what is the effect of any T_w on this basis.

The following result is stated for future purposes:

Lemma 1.3.7. Let $x, w \in W$. Then

$$T_w C_x = \sum_{\substack{z \in W \\ \ell(z) \equiv \ell(x) \bmod 2}} k_z(u) C_z + \sum_{\substack{z \in W \\ \ell(z) \not\equiv \ell(x) \bmod 2}} u^{\frac{1}{2}} \lambda_z(u) C_z$$

where $k_z(u), \lambda_z(u) \in \mathbb{Z}[u]$.

Proof. We shall use induction on $\ell(w)$.

If $\ell(w) = 0$ then $w = 1$ and our assertion holds. Assume by induction that our assertion holds for all elements w' with $\ell(w') < \ell(w)$. Let $w \in W$. We write $w = sv$ for some v with $\ell(v) - \ell(w) - 1$. We can apply our induction hypothesis on $T_v C_x$. So

$$T_w C_x = T_s(T_v C_x) = T_s[\sum_{\substack{z \in W \\ \ell(z) \equiv \ell(x) \bmod 2}} k_z(u) C_z + \sum_{\substack{z \in W \\ \ell(z) \not\equiv \ell(x) \bmod 2}} u^{\frac{1}{2}} \lambda_z(u) C_z]$$

where $k_z(u), \lambda_z(u) \in \mathbb{Z}[u]$.

Consider a C_z for which $\ell(z) = \ell(x) \bmod 2$. Then

$$k_z(u) T_s C_z = \begin{cases} -k_z(u) C_z & \text{if } sz < z \\ k_z(u)[uC_z + u^{\frac{1}{2}} C_{sz} + u^{\frac{1}{2}} \sum_{\substack{\gamma < z \\ s\gamma < \gamma}} \mu(\gamma, z) C_\gamma] & \text{if } sz > z. \end{cases}$$

Now when $\gamma < z$, then $\ell(z) \not\equiv \ell(\gamma) \bmod 2$, therefore $\ell(\gamma) \not\equiv \ell(x) \bmod 2$ and the coefficient of C_γ is $u^{\frac{1}{2}} g(u)$ for some $g(u) \in \mathbb{Z}[u]$. The coefficient of C_{sz} is also $u^{\frac{1}{2}} k_z(u)$ and $\ell(sz) \not\equiv \ell(x) \bmod 2$, while the coefficient of C_z is $uk_z(u)$.

Next consider a C_z for which $\ell(z) \not\equiv \ell(x) \bmod 2$. Then from the action of T_s on the summand $\sum_{\substack{z \in W \\ \ell(z)\not\equiv\ell(x) \bmod 2}} u^{\frac{1}{2}}\lambda_z(u)C_z$ we obtain a certain linear combination of C_β's as above and it is easy to check that those C_β for which $\ell(\beta) \not\equiv \ell(x) \bmod 2$ appear with coefficient $u^{\frac{1}{2}}f(u)$ for some $f(u) \in \mathbb{Z}[u]$, while those C_β for which $\ell(\beta) \equiv \ell(x) \bmod 2$ appear with coefficient $h(u)$, for some $h(u) \in \mathbb{Z}[u]$.

Therefore our lemma is now proved.

Corollary 1.3.7. The diagonal entries of the matrix which represents T_w with respect to the basis $\{C_x | x \in W\}$ consist only of polynomials in $\mathbb{Z}[u]$.

The following ideas are due to Kazhdan and Lusztig. (See [11]).

Definitions 1.3.8. Let $x, x' \in W$. We say that x, x' are joined, $(x - x')$, if either $x \prec x'$ or $x' \prec x$. We define a left preorder $\underset{L}{\leq}$ on W by saying that $x \underset{L}{\leq} x'$ if there exists a sequence of elements of $W: x = x_0, x_1, \ldots, x_n = x'$ such that for each i, $1 \leq i \leq n$, we have $x_{i-1} - x_i$ and there exists an $s \in S$ such that $sx_{i-1} < x_{i-1}$ but $sx_i > x_i$. We may then define an equivalence relation on W by saying $x \underset{L}{\sim} x'$ if $x \underset{L}{\leq} x' \underset{L}{\leq} x$. The equivalence classes with respect to the relation $\underset{L}{\sim}$ are called the left-cells of W. Similarly we define right cells by replacing the condition on $s \in S$, by $x_{i-1}s < x_{i-1}$ and $x_is > x_i$. The notation $x \underset{R}{\sim} x'$ means that x, x' are in the same right cell. Finally we can define 2-sided cells by replacing the condition on $s \in S$, by: either $sx_{i-1} < x_{i-1}$ and $sx_i > x_i$

or $x_{i-1}x < x_{i-1}$ and $x_is > x_i$.

The notation $\underset{LR}{\leq}$ means the 2-sided preorder and $x \underset{LR}{\sim} x'$ means that x, x' are inside the same 2-sided cell. Evidently, every left cell lies in a unique 2-sided cell and the same holds for every right cell. Each left cell, regarded as a full subgraph of the graph Γ (Theorem 1.3.6) with the same sets I_x and the same function μ is itself a W-graph. Therefore it gives rise to a representation of H(u). Nevertheless, this representation is not always irreducible. However,

Theorem 1.3.9. Let X be a left cell of $W = S_n$, and let Γ be the W-graph associated to X and let p be the representation of H(u) over the quotient field of A. Then p is irreducible, and the isomorphism class of the W-graph Γ depends only on the isomorphism class of p and not on X. (See Theorem 1.4 in [11]).

1.4 The Lusztig Isomorphism

Let W be a Coxeter group and S its set of reflections. Let E be the free $\mathbb{Q}[u^{\frac{1}{2}}]$ module with basis $\{e_w, w \in W\}$. Let H be the generic Hecke algebra over $\mathbb{Q}[u^{\frac{1}{2}}]$. We know that the formulae

$$T_s e_w = \begin{cases} -e_w & \text{if } sw < w \\ u e_w + u^{\frac{1}{2}} \sum\limits_{\substack{y \text{---} w \\ sy < y}} \tilde{\mu}(y,w) e_y & \text{if } sw > w \end{cases}$$

$$e_w T_t = \begin{cases} -e_w & \text{if } wt < w \\ u e_w + u^{\frac{1}{2}} \sum\limits_{\substack{y \text{---} w \\ yt < y}} \tilde{\mu}(y,w) e_y & \text{if } wt > w \end{cases}$$

define an (H,H) bimodule structure on E. $\tilde{\mu}(y,w) = \mu(y,w)$ if $y \prec w$ and $\tilde{\mu}(y,w) = \mu(w,y)$ if $w \prec y$.

We also define a left W-module structure on E by

$$s * e_w = \begin{cases} -e_w & \text{if } sw < w \\ e_w + \sum_{\substack{y - w \\ sy<y}} \tilde{\mu}(y,w)e_y & \text{if } sw > w \end{cases}$$

and a right W-module structure on E by

$$e_w * t = \begin{cases} -e_w & \text{if } wt < w \\ e_w + \sum_{\substack{y - w \\ yt<y}} \tilde{\mu}(y,w)e_y & \text{if } wt > w \end{cases}$$

We shall refer to the basis $\{e_w\ w \in W\}$ of E as the canonical basis of E.

Now, the left and right W-module structures on E commute with each other. However, the left H-module structure does not necessarily commute with the right W-module structure. For each 2-sided cell X of W we shall construct an H-module M_X of dimension $|X|$ over $\mathbb{Q}[u^{\frac{1}{2}}]$. In fact, for each 2-sided cell X of W, we consider the $\mathbb{Q}[u^{\frac{1}{2}}]$ submodule E_X of E, spanned by $\{e_w : w \underset{LR}{\leq} x$ for some $x \in X\}$. Inside E_X we consider E_X' the submodule spanned by $\{e_w : w \notin X, w \underset{LR}{\leq} x$ for some $x \in X\}$. Put $M_X = E_X / E_X'$. It is clear that M_X is an H-module of dimension $|X|$. Let $\mathrm{grad}(E) = \bigoplus_X M_X$, summed over all 2-sided cells X of W. It has a canonical basis, the images $\bar{e}_w$ of the elements e_w of E. It is clear that both E_X, E_X' are left H-submodules, left W-submodules, and right W-submodules. Therefore grad(E) is in a natural way a left H-module, a left W-module and a right W-module. It is clear that the left H action on the graded module is given by

$$T_s \tilde{e}_w = \begin{cases} -\tilde{e}_w & \text{if } sw < w \\ u\tilde{e}_w + u^{\frac{1}{2}} \sum\limits_{\substack{y \underset{LR}{\sim} w \\ sy < y}} \tilde{\mu}(y,w)\tilde{e}_y & \text{if } sw > w \end{cases}$$

and by specializing $u^{\frac{1}{2}} \to 1$ we obtain the action of the generators $s \in W$ on the graded module, and therefore the action of any $w \in W$.

<u>Definition 1.4.1</u>. We say that the Coxeter group W satisfies the property (A) if given $y, w \in W$ such that:

(i) $y — w$, (ii) $\{s \in S : sy < y\} \not\subset \{s \in S : sw < w\}$,

(iii) $\{s \in S : ys < y\} \not\subset \{s \in S : ws < w\}$, then y, w are not inside the same 2-sided cell of W.

<u>Lemma 1.4.2</u>. Assume that W has the property (A). Then the left H-module structure and the right W-module structure on grad(E) commute. (Proof: see Lemma 2.3 in [12].)

<u>Lemma 1.4.3</u>. Assume that W is a finite Weyl group. Then W satisfies the property (A). (Proof: See Lemma 4.1 in [12].)

The only known proof of this result uses the theory of primitive ideals in enveloping algebras. No elementary proof of this result is known.

Theorem 1.4.4. (Lusztig's isomorphism theorem). Assume that W is a finite indecomposable Weyl group.

(a) There is a unique homomorphism of $\mathbb{Q}[u^{\frac{1}{2}}]$-algebras $\phi: H \to \mathbb{Q}[u^{\frac{1}{2}}](W)$ such that for any $h \in H$ and any $w \in W$, the difference $he_w - \phi(h)e_w$ is a linear combination of e_y, y not in the same 2-sided cell with w.

(b) Let K be any field of characteristic zero, and χ any homomorphism of $\mathbb{Q}[u^{\frac{1}{2}}]$ into K, such that the specialized algebra $H_K = H \otimes K$ is semisimple. Then the specialized homomorphism of K-algebras $\phi_\chi: H_K \to KW$ is an isomorphism.

A more general version and a proof of this theorem is given in Theorem 3.1 in [2].

Remarks (1) The theorem above applies to any finite indecomposable Coxeter group W provided that W satisfies the property (A). In fact by taking $\chi:\mathbb{Q}[u^{\frac{1}{2}}] \to \mathbb{Q}(u^{\frac{1}{2}})$ the natural inclusion, then $H_{\mathbb{Q}(u^{\frac{1}{2}})} \simeq \mathbb{Q}(u^{\frac{1}{2}})(W)$.

(2) When W is a finite Weyl group and q is a prime power and $\mathbb{Q}(q^{\frac{1}{2}})$ is a field extension of $\mathbb{Q}$ of degree 1 or 2, then by taking $\chi:\mathbb{Q}[u^{\frac{1}{2}}] \to \mathbb{Q}(q^{\frac{1}{2}})$ such that $\chi(u^{\frac{1}{2}}) = q^{\frac{1}{2}}$ (the positive square root of q), we obtain $H_{\mathbb{Q}(q^{\frac{1}{2}})}(G(q),B(q)) \simeq \mathbb{Q}(q^{\frac{1}{2}})(W)$.

We next describe a procedure for an explicit construction of Lusztig's isomorphism.

Procedure: Let W be a finite indecomposable Coxeter group which satisfies the property (A). The graded module over $\mathbb{Q}(u^{\frac{1}{2}})$ when viewed as a left $H_{\mathbb{Q}(u^{\frac{1}{2}})}$ module is semisimple and affords the left regular representation of $H_{\mathbb{Q}(u^{\frac{1}{2}})}$. Let L be a field containing $\mathbb{Q}(u^{\frac{1}{2}})$ and assume that L is a splitting field for $H_{\mathbb{Q}(u^{\frac{1}{2}})}$. In fact such a field L can be chosen of the form $F(u^{\frac{1}{2}})$ where $F = \mathbb{Q}$ if W is crystallographic, (see [3], [4]), or $F = \mathbb{Q}(\sqrt{5})$ if W is of type H_3 (see [12].), or $F = \mathbb{Q}(\frac{1+\sqrt{5}}{2})$ if W is of type H_4 (See [2]), or $F = \mathbb{Q}\ (2 \cos \frac{2\pi}{n})$ if W is a dihedral group D_{2n} $n = 5$ or $n \geq 7$.

It is well known that H_L is also semisimple and therefore the graded module over L has a decomposition into a direct sum of left absolutely irreducible H_L submodules. Let

$$(D) : \mathrm{grad}(E) = V_{11} \oplus \dots \oplus V_{1d_1} \oplus \dots \oplus V_{t1} \oplus \dots \oplus V_{td_t}$$

be one such decomposition where $V_{ij} \simeq V_{rs}$ if and only if $i = r$, and for each $i \in \{1,\dots,t\}$, V_{ij} has dimension d_i, for every $j = 1,\dots,d_i$. We choose a basis of each constituent V_{ir} and in this way we obtain a basis adapted to this decomposition.

We also choose a full set of irreducible constituents, namely $X = \{V_{11}, V_{21},\dots,V_{t1}\}$.

In order to determine Lusztig's isomorphism ϕ, it is enough to determine the images under ϕ of the generators T_s, $s \in S$. By part (a) of the Theorem 1.4.4 we have that T_s and $\phi(T_s)$ act on the same way on the graded module. Fix an $s \in S$ and let $\phi(T_s) = \sum_{w \in W} c_w.w$, $c_w \in \mathbb{Q}(u^{\frac{1}{2}})$. We wish to determine the c_w, $w \in W$.

If $V_{i1} \in X$, $1 \leq i \leq t$ and has chosen basis $\{v_1^{(i)},\dots,v_{d_i}^{(i)}\}$ then $T_s\ v_\lambda^{(i)} = \phi(T_s)v_\lambda^{(i)}$, for every $\lambda = 1,\dots,d_i$. Moreover, for every $s \in S$, $T_s\ v_\lambda^{(i)} = \sum_{\mu=1}^{d_i} \gamma_{\mu,\lambda}^{(s)}\ (u^{\frac{1}{2}})v_\mu^{(i)}$, with $\gamma_{\mu,\lambda}^{(s)}\ (u^{\frac{1}{2}}) \in F(u^{\frac{1}{2}})$, $1 \leq \lambda \leq d_i$. When we consider the graded module as a left W-module with W-action obtained by specializing $u^{\frac{1}{2}} \to 1$ in the action of T_s, $s \in S$, then it affords the left regular representation of W. Now each V_{i1} $1 \le i \le t$ becomes an absolutely irreducible FW module.

By specializing $u^{\frac{1}{2}} \to 1$ in the matrices which represent T_s, $s \in S$ with respect to the basis adapted to the constituent $V_{i1} \in X$, $1 \le i \le t$, we obtain the matrices which represent the generators s and therefore the matrices which represent every $w \in W$.

Let $w \in W$ and let $(f_{k\lambda}^{(i)}(w))$, $k, \lambda \in \{1,\dots,d_i\}$ be the matrix which represent w on the FW module V_{i1}, $1 \le i \le t$.

Let $wv_\lambda^{(i)} = \sum_{\mu=1}^{d_i} f_{\mu\lambda}^{(i)}(w)v_\mu^{(i)}$, $1 \le \lambda \le d_i$. Then, $\Phi(T_s)v_\lambda^{(i)} = \sum_{w\in W} c_w(\sum_{\mu=1}^{d_i} f_{\mu\lambda}^{(i)}(w)v_\mu^{(i}$

Therefore by comparing coefficients of the basis elements $v_\mu^{(i)}$ on both sides of the equation $T_s\, v_\lambda^{(i)} = \Phi(T_s)v_\lambda^{(i)}$ we obtain d_i equations in the unknowns c_w, $w \in W$ and therefore from every $V_{i1} \in X$ we obtain d_i^2 such equations, $1 \le i \le t$.

Hence from the full set of irreducible constituents $\{V_{11},\dots,V_{t1}\}$ we obtain $\sum_{i=1}^{t} d_i^2 = |W|$ equations in the $|W|$ unknowns c_w, $w \in W$.

These equations are linearly independent (see 3.41 in [7]). and the solution of the system of these equations determines the c_w, $w \in W$ and therefore the image of T_s under Φ.

Example. Let W be the Weyl group of type A_2, $W \cong S_3$. W has a presentation: $\langle s_1,s_2 | s_1^2 = s_2^2 = 1,\ (s_1s_2)^3 = 1\rangle$. The 2-sided cells are $X_1 = \{1\}$, $X_2 = \{s_1,s_2,s_1s_2,\ s_2s_1\}$, $X_3 = \{s_1s_2s_1\}$.

The graded module has canonical basis $\{\bar{e}_w,\ w \in W\}$. The module M_{X_2} obtained by the 2-sided cell X_2 has an easy decomposition into a direct sum of two left irreducible $H_{Q(u^{\frac{1}{2}})}$ submodules, namely

$$M_{\chi_2} = \langle \bar{e}_{s_1}, \bar{e}_{s_2 s_1} \rangle \oplus \langle \bar{e}_{s_2}, \bar{e}_{s_1 s_2} \rangle, \text{ hence}$$

$$\mathrm{grad}(E) = \langle \bar{e}_1 \rangle \oplus \langle \bar{e}_{s_1}, \bar{e}_{s_2 s_1} \rangle \oplus \langle \bar{e}_{s_2}, \bar{e}_{s_1 s_2} \rangle \oplus \langle \bar{e}_{s_1 s_2 s_1} \rangle.$$

It is easy to check that the two summands appearing in the decomposition of M_{χ_2} afford equivalent representations of $H_{\mathbb{Q}(u^{\frac{1}{2}})}$.

Let $\phi(T_{s_1}) = \sum_{w \in W} c_w . w$. Then from the equations $T_{s_1} \bar{e}_w = \phi(T_{s_1}) \bar{e}_w$, $w \in \{1, s_1, s_2 s_1, s_1 s_2 s_1\}$ we obtain 6 equations in the 6 unknowns c_w, $w \in W$. The solution of these equations gives:

$$\phi(T_{s_1}) = \frac{u-1}{2} . 1 + \frac{u+1}{2} s_1 + \frac{(u^{\frac{1}{2}}-1)^2}{6} (-s_2 + s_1 s_2 - s_2 s_1 + s_1 s_2 s_1)$$

Similar work for T_{s_2} gives

$$\phi(T_{s_2}) = \frac{u-1}{2} . 1 + \frac{u+1}{2} s_2 + \frac{(u^{\frac{1}{2}}-1)^2}{6} (-s_1 + s_2 s_1 - s_1 s_2 + s_1 s_2 s_1)$$

(This example appears in [12]).

CHAPTER 2

The Lusztig isomorphism for Hecke algebras of dihedral type.

§2.1 General properties of the dihedral groups

The dihedral group of order 2n, n > 1 is defined by $D_{2n} = \langle s_1, s_2 | s_1^2 = s_2^2 = 1, (s_1s_2)^n = 1\rangle$. We put $S = \{s_1, s_2\}$.

The example at the previous chapter, demonstrates the Lusztig isomorphism for the Weyl group of type A_2 which is isomorphic to D_6. Therefore it is natural to ask the question, whether we can find explicitly this isomorphism for the Hecke algebras of dihedral type. In order to do so, we must check that the dihedral groups satisfy the property (A) of the definition 1.4.1.

For any $y, w \in D_{2n}$ we have: $y \leq w$ if and only if $y = w$ or $\ell(w) > \ell(y)$. We also have that if $\ell(w) - \ell(y) > 0$ with $\ell(w) - \ell(y) =$ odd, then either $\{s \in S : sw < w\} \not\subset \{s \in S : sy < y\}$ or $\{s \in S : ws < w\} \not\subset \{s \in S : ys < y\}$, but not both conditions hold. By lemma 1.3.5(i) and (ii) we have: $y \prec w$ if and only if either $w = s_iy$ or $w = ys_i$, for some $i = 1,2$.

<u>Lemma 2.1.1</u>. For any $y \leq w$ in D_{2n}, we have $P_{y,w} = 1$.

<u>Proof</u>.

We use induction on $\ell(w)$. The result is obvious if $\ell(w) = 0$. Assume that our lemma holds for all elements w' with $\ell(w') < \ell(w)$. Let $w \in D_{2n}$ and we may assume that $s_1w < w$, so $w = s_1v$, $\ell(v) = \ell(w) - 1$. We may also assume that $y \in D_{2n}$ is such that $\ell(w) - \ell(y) > 2$ (Lemma 1.3.3). The inductive formula which defines the polynomials $P_{y,w}$ gives

$$P_{y,w} = \begin{cases} u\,P_{s_1y,v} + P_{y,v} - u & \text{if } s_1y > y \\ P_{s_1y,v} + uP_{y,v} - u & \text{if } s_1y > y \end{cases}$$

By induction we have $P_{y,v} = P_{s_1y,v} = 1$ and so in both cases we obtain $P_{y,w} = 1$.

We can now easily determine the 2-sided cells of W. These are: $X_1 = \{1\}$, $X_2 = D_{2n} - \{1, w_0\}$, $X_3 = \{w_0\}$, where w_0 is the longest element of W. The cell X_2 contains two left cells which are $L_1 = \{w \in X_2 : \ell(ws_1) < \ell(w)\}$, $L_2 = \{w \in X_2 : \ell(ws_2) < \ell(w)\}$ and also contains two right cells $R_1 = \{w \in X_2 : \ell(s_1w) < \ell(w)\}$, $R_2 = \{w \in X_2 : \ell(s_2w) < \ell(w)\}$.

<u>Lemma 2.1.2</u>

D_{2n} satisfies the property (A).

<u>Proof</u>

Let y, w be two elements inside the cell X_2 such that y is joined to w, $y — w$. Then $\ell(w)-\ell(y)$ is odd or $\ell(y)-\ell(w)$ is odd. In this case we have that either $\{s \in S: sw < w\} \not\subset \{s \in S: sy < y\}$ or $\{s \in S: ws < w\} \not\subset \{s \in S: ys < y\}$, but not both conditions hold. So our lemma is proved.

Therefore the construction of Lusztig's isomorphism makes sense for the finite Coxeter groups of dihedral type.

The graded module in this case has canonical basis $\{\bar{e}_w, w \in D_{2n}\}$. The left action of the generators T_{s_i}, $i = 1,2$ on the canonical basis is as follows:

$$T_{s_1}\,\bar{e}_w = \begin{cases} -\bar{e}_w & \text{if } s_1w < w \\ u\bar{e}_w + u^{\frac{1}{2}}\,\bar{e}_{s_1w} + u^{\frac{1}{2}}\,\bar{e}_{s_2w} & \text{if } s_1w > w \end{cases}$$

and $w \notin \{1, s_2, s_1w_0\}$.

If $w = 1$, $T_{s_1}\bar{e}_1 = u\bar{e}_1$

If $w = s_2$, $T_{s_1}\bar{e}_{s_2} = u\bar{e}_{s_2} + u^{\frac{1}{2}}\bar{e}_{s_1s_2}$

If $w = s_1w_0$, $T_{s_1}\bar{e}_w = u\bar{e}_w + u^{\frac{1}{2}}\bar{e}_{s_2w}$.

Similarly, by interchanging the role of s_1, s_2 above, we obtain the action of T_{s_2} on the canonical basis.

By specializing $u^{\frac{1}{2}} \to 1$ we obtain the action of s_1, s_2 respectively.

§2.2 A set of polynomials $S_n(x) \in \mathbb{Z}[x]$.

We shall now introduce a set of polynomials $S_n(x) \in \mathbb{Z}[x]$ called the Chebyshev polynomials of the second kind, which play an essential role in the decomposition of Lusztig's graded module of dihedral type into a direct sum of left H irreducible submodules, and also in the determination of Lusztig's isomorphism in this case. Many properties of these polynomials and their relation with other families of Chebyshev polynomials can be found in [1], pages 774-8. These polynomials are defined as follows:

$$S_{-1}(x) = 0,\ S_0(x) = 1,\ S_1(x) = x,\ S_{k+1}(x) = xS_k(x) - S_{k-1}(x) \quad \forall k \geq 1.$$

The first few of these polynomials are:

$$S_2(x) = x^2-1,\quad S_3(x) = x^3-2x,\quad S_4(x) = x^4-3x^2+1,$$

$$S_5(x) = x^5-4x^3+3x,\ S_6(x) = x^6-5x^4+6x^2-1,\ S_7(x) = x^7-6x^5+10x^3-4x.$$

<u>Lemma 2.2.1</u> $S_n(x) = \sum_{k=0}^{\frac{n}{2}} (-1)^k \binom{n-k}{k} x^{n-2k}$ if n is even and

$$S_n(x) = \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n-k}{k} x^{n-2k} \text{ if n is odd.}$$

<u>Proof.</u>

We assume that n is odd and that $S_k(x)$ is given by the formula above for all $k \leq n$.

Then $S_{n+1}(x) = xS_n(x) - S_{n-1}(x) =$

$$x\left[\sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n-k}{k} x^{n-2k}\right] - \sum_{k=0}^{(n-1)/2} (-1)^k \binom{n-1-k}{k} x^{n-1-2k} =$$

$$= \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n-k}{k} x^{n+1-2k} - \sum_{k=0}^{(n-1)/2} (-1)^k \binom{n-1-k}{k} x^{n-1-2k} =$$

$$= x^{n+1} + \sum_{k=1}^{(n-1)/2} (-1)^k \binom{n-k}{k} x^{n+1-2k} - \sum_{k=1}^{(n+1)/2} (-1)^{k-1} \binom{n-k}{k-1} x^{n+1-2k} =$$

$$= x^{n+1} + \sum_{k=1}^{(n-1)/2} (-1)^k \left[\binom{n-k}{k} + \binom{n-k}{k-1}\right] x^{n+1-2k} + (-1)^{\frac{n+1}{2}} =$$

$$= \sum_{k=0}^{(n+1)/2} (-1)^k \binom{n+1-k}{k} x^{n+1-2k} .$$

Therefore $S_{n+1}(x)$ has also the desired form. The argument is similar if in our inductive hypothesis we assume that n is even.

<u>Lemma 2.2.2</u>. (i) The numbers $\rho_j = 2 \cos \frac{j\pi}{m}$, $1 \leq j \leq m-1$, are the zeros of the polynomial $S_{m-1}(x)$.

(ii) The numbers $\rho_j = 2 \cos \frac{2j\pi}{2m+1}$, $1 \leq j \leq m$, are the zeros of the polynomial $S_m(x) + S_{m-1}(x)$.

<u>Proof</u>. (i) Let $n = 2m$, and consider the polynomial $y^n - 1 = (y^2-1)Q(y)$ where $Q(y) = y^{2m-2} + y^{2m-4} + \ldots + y^2 + 1$. We write

$$Q(y) = y^{m-1}\left[(y^{m-1} + \frac{1}{y^{m-1}}) + (y^{m-3} + \frac{1}{y^{m-3}}) + \ldots\right]$$

$$\longrightarrow R_{m-1}(y + \frac{1}{y}) \longleftarrow$$

where $R_{m-1}(y + \frac{1}{y})$ finishes either with $y + \frac{1}{y}$ if m is even or with 1 if m is odd. We put $y + \frac{1}{y} = z$.

The zeros of $Q(y)$ are the numbers $y_j = e^{ij\pi/m}$, $j = 1,\ldots,2m-1$, $j \neq m$, and for each such y_j we have $Q(y_j) = y_j^{m-1}\, R_{m-1}\,(y_j + \frac{1}{y_j}) = 0$. Therefore the zeros of the polynomial $R_{m-1}(z)$ are the numbers $y_j + \frac{1}{y_j} = 2\cos\frac{j\pi}{m}$, $1 \leq j \leq m-1$.

Now, there is a recurrence relation which generates the polynomial $R_{m-1}(z)$. For, $y^\lambda + \frac{1}{y^\lambda} = (y^{\lambda-2} + \frac{1}{y^{\lambda-2}})\,(y^2 + \frac{1}{y^2}) - (y^{\lambda-4} + \frac{1}{y^{\lambda-4}})$

for every $\ \ = 3,4, \ldots, m-1$.

Hence $R_{m-1}(z) = (z^2-2)R_{m-3}(z) - R_{m-5}(z)$, with $R_0(z) = 1$. Now we claim that for all integers n, $R_n(z) = S_n(z)$. In fact for $n = 1$ it is true since $R_1(z) = z = S_1(z)$. Assume that $R_\lambda(z) = S_\lambda(z)$ for all integers $\lambda \leq k$. Then,

$$R_{k+1}(z) = (z^2-2)R_{k-1}(z) - R_{k-3}(z) = (z^2-2)S_{k-1}(z) - S_{k-3}(z) =$$

$$= z\,S_k(z) + (zS_{k-2}(z) - S_{k-3}(z)) - 2S_{k-1}(z) = zS_k(z) - S_{k-1}(z) = S_{k+1}(z).$$

In particular $R_{m-1}(z) = S_{m-1}(z)$, and so the zeros of the polynomial $S_{m-1}(z)$ are the numbers $\rho_j = 2\cos\frac{j\pi}{m}$ $1 \le j \le m-1$.

(ii) Let $n = 2m+1$ and consider the polynomial $y^n-1 = (y-1)Q(y)$ where $Q(y) = y^{2m} + \ldots + y+1$. The zeros of $Q(y)$ are the numbers $y_j = e^{2ij\pi/n}$, $1 \le j \le 2m$. We write

$$Q(y) = y^m\,[(y^m + \frac{1}{y^m}) + \ldots + (y + \frac{1}{y}) + 1]$$

$$\longrightarrow R_m(y + \frac{1}{y}) \longleftarrow$$

We put $y + \frac{1}{y} = z$ and we deduce as before that the zeros of $R_m(z)$ are the numbers $y_j + \frac{1}{y_j} = 2\cos\frac{2j\pi}{n}$ $1 \le j \le m$. Now the recurrence relation which generates $R_m(z)$ is

$$y^\lambda + \frac{1}{y^\lambda} = (y^{\lambda-1} + \frac{1}{y^{\lambda-1}})\,(y + \frac{1}{y}) - (y^{\lambda-2} + \frac{1}{y^{\lambda-2}}),\ \forall\lambda = 2,\ldots,m.$$

Hence $R_m(z) = zR_{m-1}(z) - R_{m-2}(z)$, with $R_0(z) = 1$, $R_1(z) = z + 1$. By induction we can prove that for all integers n, we have $R_n(z) = S_{n-1}(z) + S_n(z)$. For $n = 1$ it is true since $R_1(z) = z+1 = S_0(z) + S_1(z)$.

Assume it is true for all integers $\lambda \le k$. Then

$$R_{k+1}(z) = zR_k(z) - R_{k-1}(z) = z(S_k(z) + S_{k-1}(z)) - (S_{k-1}(z) + S_{k-2}(z))$$

$$= S_{k+1}(z) + S_k(z).$$

In particular $R_m(z) = S_m(z) + S_{m-1}(z)$, and hence the zeros of the polynomial

$S_m(z) + S_{m-1}(z)$ are the numbers $\rho_j = 2\cos\frac{2j\pi}{2m+1}$, $1 \le j \le m$.

Our lemma is now proved.

Lemma 2.2.3. (i) Let $n = 2m+1$, then $(x-2)\sum_{k=0}^{m-1}(n-2(k+1))S_k(x) = S_m(x) + S_{m-1}(x) - n$.

(ii) Let $n = 2m$, then $(x-2)\sum_{k=0}^{m-2}(n-2(k+1))S_k(x) = 2\,S_{m-1}(x) - n$.

Proof. (i) We have $(x-2)\sum_{k=0}^{m-1}(n-2(k+1))S_k(x) =$

$$= \sum_{k=0}^{m-1}(n-2(k+1))xS_k(x) - \sum_{k=0}^{m-1}2(n-2(k+1))S_k(x) =$$

$$= \sum_{k=0}^{m-1}(n-2(k+1))(S_{k+1}(x) + S_{k-1}(x)) - \sum_{k=0}^{m-1}2(n-2(k+1))S_k(x) =$$

$$= \sum_{k=0}^{m-1}(n-2(k+1))S_{k+1}(x) + \sum_{k=0}^{m-1}(n-2(k+1))S_{k-1}(x) - \sum_{k=0}^{m-1}2(n-2(k+1))S_k(x)$$

$$= \sum_{k=1}^{m}(n-2k)S_k(x) + \sum_{k=0}^{m-2}(n-2(k+2))S_k(x) - \sum_{k=0}^{m-1}2(n-2(k+1))S_k(x)$$

$$= \sum_{k=0}^{m-2}(n-2k)S_k(x) - n + 3S_{m-1}(x) + S_m(x) + \sum_{k=0}^{m-2}(n-2(k+2))S_k(x) -$$

$$- \sum_{k=0}^{m-2}2(n-2(k+1))S_k(x) - 2S_{m-1}(x) = S_m(x) + S_{m-1}(x) - n$$

$$(11)\quad (x-2)\sum_{k=0}^{m-2}(n-2(k+1))S_k(x) = \sum_{k=0}^{m-2}(n-2(k+1))xS_k(x) -$$

$$-\sum_{k=0}^{m-2} 2(n-2(k+1))S_k(x) =$$

$$= \sum_{k=0}^{m-2}(n-2(k+1))S_{k+1}(x) + \sum_{k=0}^{m-2}(n-2(k+1))S_{k-1}(x) - \sum_{k=0}^{m-2} 2(n-2(k+1))S_k(x) =$$

$$= \sum_{k=1}^{m-1}(n-2k)S_k(x) + \sum_{k=0}^{m-3}(n-2(k+2))S_k(x) - \sum_{k=0}^{m-2} 2(n-2(k+1))S_k(x)$$

$$= \sum_{k=0}^{m-3}(n-2k)S_k(x) - n + 4S_{m-2}(x) + 2S_{m-1}(x) + \sum_{k=0}^{m-3}(n-2(k+2))S_k(x) -$$

$$- \sum_{k=0}^{m-3} 2(n-2(k+1))S_k(x) - 4S_{m-2}(x) = 2S_{m-1}(x) - n.$$

Our lemma is now proved.

§2.3. The determination of Lusztig's isomorphism for Hecke algebras of dihedral type.

Our first step is to find a decomposition of Lusztig's graded module into a direct sum of left H-irreducible submodules.

First case: $n = 2m+1$. In this case we have two one-dimensional representations of H, namely $\sigma_0 : T_{s_i} \to u$, $i = 1,2$ and $\sigma_s : T_{s_i} \to -1$. These representations are afforded by the left H-submodules $M_0 = \langle \bar{e}_1 \rangle$ and $M_s = \langle \bar{e}_{w_0} \rangle$

respectively. Let N be the subspace spanned by $\langle \bar{e}_w,\ w \in W,\ w \neq 1, w_0 \rangle$. It is easy to verify that N is a left H-submodule complementary to $\langle \bar{e}_1 \rangle \oplus \langle \bar{e}_{w_0} \rangle$. We wish to decompose N into a direct sum of 2m 2-dimensional left H-submodules.

First we note that N has an obvious decomposition into the direct sum of two (n-1)-dimensional left H-submodules, namely

$$N_1 = \langle \bar{e}_{s_1}, \bar{e}_{s_2s_1}, \bar{e}_{s_1s_2s_1}, \ldots, \bar{e}_{(s_1s_2)^{m-1}s_1}, \bar{e}_{(s_2s_1)^m} \rangle \quad \text{and}$$

$$N_2 = \langle \bar{e}_{s_2}, \bar{e}_{s_1s_2}, \bar{e}_{s_2s_1s_2}, \ldots, \bar{e}_{(s_2s_1)^{m-1}s_2}, \bar{e}_{(s_1s_2)^m} \rangle.$$

We shall split N_1 (similarly N_2) into the direct sum of m 2-dimensional left H-modules. Consider the numbers $\rho_j = 2\cos\frac{2j\pi}{n}$ $1 \leq j \leq m$ and define the following sequence of real numbers:

$$a_0^{(j)} = -1,\ a_1^{(j)} = S_0(\rho_j) = 1, \ldots, a_{\lambda+1}^{(j)} = S_\lambda(\rho_j) + S_{\lambda-1}(\rho_j) \quad \lambda \in \{1,2,\ldots,m-1\}$$

(for the definition of the polynomials $S_n(x)$ see §2.2).

Next consider the following elements of the graded module:

$$u_j = a_1^{(j)}\bar{e}_{s_1} + a_2^{(j)}\bar{e}_{s_1s_2s_1} + \ldots + a_m^{(j)}\,\bar{e}_{(s_1s_2)^{m-1}s_1}$$

$$v_j = (a_1^{(j)} + a_2^{(j)})\bar{e}_{s_2s_1} + (a_2^{(j)} + a_3^{(j)})\bar{e}_{s_2s_1s_2s_1} + \ldots + (a_{m-1}^{(j)} + a_m^{(j)})\bar{e}_{(s_2s_1)^{m-1}} +$$

$$+\, a_m^{(j)}\,\bar{e}_{(s_2s_1)^m} \qquad 1 \leq j \leq m.$$

The number of these elements is 2m and they all lie inside the submodule $\bar{N}_1$.

We shall denote by $\mathbb{Q}_n$ the field $\mathbb{Q}\,(2\cos\frac{2\pi}{n})$.

<u>Proposition 2.3.1</u>. The elements u_j, v_j defined above are all linearly independent over $\mathbb{Q}_n(u^{\frac{1}{2}})$ and for each $1 \le j \le m$, the pair (u_j, v_j) spans a 2-dimensional left $H_{\mathbb{Q}_n(u^{\frac{1}{2}})}$-submodule, namely M_j. Moreover each M_j is irreducible and distinct j give rise to non-isomorphic such submodules.

<u>Proof</u>. From the definition of the numbers $a_\lambda^{(j)}$ above, we see that for every $1 \le k \le m-1$, $a_k^{(j)} + a_{k+1}^{(j)} = S_{k-1}(\rho_j) + S_{k-2}(\rho_j) + S_k(\rho_j) +$ $+ S_{k-1}(\rho_j) = 2S_{k-1}(\rho_j) + S_k(\rho_j) + S_{k-2}(\rho_j) = (2+\rho_j)S_{k-1}(\rho_j)$.
We also note that if $\lambda = m-1$ then $a_m^{(j)} = S_{m-1}(\rho_j) + S_{m-2}(\rho_j) =$ $(2+\rho_j)S_{m-1}(\rho_j)$, for $S_m(\rho_j) + S_{m-1}(\rho_j) = 0$, i.e. $\rho_j S_{m-1}(\rho_j) - S_{m-2}(\rho_j) + S_{m-1}(\rho_j)$ $= 0$, i.e. $S_{m-2}(\rho_j) = (1+\rho_j)S_{m-1}(\rho_j)$.

Assume that $\lambda_1, \lambda_2, \dots, \lambda_m$, $\mu_1, \mu_2, \dots, \mu_m$ are elements in $\mathbb{Q}_n(u^{\frac{1}{2}})$ such that $\sum_{j=1}^m \lambda_j u_j + \sum_{j=1}^m \mu_j v_j = 0$. **(R)**. We shall show that $\lambda_j = \mu_j = 0 \ \forall j = 1, \dots, m$. Since each u_j is by definition a linear combination of basis elements $\bar{e}_w$ which do not appear in the expression of v_j, we can concentrate separately on the coefficients λ_j and μ_j. For simplicity we put $w_1 = \bar{e}_{s_1}$, $w_2 = \bar{e}_{s_1 s_2 s_1}, \dots, w_m = \bar{e}_{(s_1 s_2)^{m-1} s_1}$. Then, in the expression $\sum_{j=1}^m \lambda_j u_j$ the coefficient of a typical element w_k $1 \le k \le m$ is of the form $\sum_{j=1}^m \lambda_j a_k^{(j)}$. Since the elements w_k are linearly independent, the relation **(R)** above implies that $\sum_{j=1}^m \lambda_j a_k^{(j)} = 0$ for every $1 \le k \le m$. Thus, we have a system of m-homogeneous linear equations of the form E_k: $\sum_{j=1}^m \lambda_j a_k^{(j)} = 0$, in the m-unknowns λ_j. I claim that any solution of these homogeneous equations say (Σ) : $\{E_k : \sum_{j=1}^{m} \lambda_j a_k^{(j)} = 0 \; 1 \le k \le m\}$ is also a solution of the system of homogeneous equations (Σ') : $\{E_k' : \sum_{j=1}^{m} \lambda_j \rho_j^{k-1} = 0,\; 1 \le k \le m\}$. In fact if $k = 1$, the equation E_1 of the system (Σ) is $E_1 : \lambda_1 + \dots + \lambda_m = 0$, because we have defined $a_1^{(j)} = 1,\ \forall\ 1 \le j \le m$. Assume that

the λ_i satisfy the equations $E_1', E_2', \dots, E_k'$. Then we look at the equation $E_{k+1} : \sum_{j=1}^{m} \lambda_j a_{k+1}^{(j)} = 0$. We recall that $a_{k+1}^{(j)} = S_k(\rho_j) + S_{k-1}(\rho_j) = \rho_j^k +$ linear combination of lower powers of ρ_j, for every $1 \le j \le m$. Therefore by induction the λ_j satisfy the equation $E_{k+1}' : \sum_{j=1}^{m} \lambda_j \rho_j^k = 0$.

Now, the determinant of the coefficients of the λ_j in the homogeneous system (Σ') is the Vandermonde determinant

$$\Delta = \begin{vmatrix} 1 & 1 & \dots & 1 \\ \rho_1 & \rho_2 & & \rho_m \\ \rho_1^{m-1} & \rho_2^{m-1} & & \rho_m^{m-1} \end{vmatrix} = \prod_{m \ge i > j \ge 1} (\rho_i - \rho_j) \ne 0, \text{ since } \rho_i \ne \rho_j \text{ if } i \ne j$$

Therefore $\lambda_j = 0 \quad \forall\ j = 1, \dots, m$.

The argument is similar, for the coefficients μ_j, $j = 1, \dots, m$.

We put $z_1 = \bar{e}_{s_2}$, $z_2 = \bar{e}_{s_2 s_1 s_2 s_1}, \dots, z_m = \bar{e}_{(s_2 s_1)^{m-1} s_2}$

In the expression $\sum_{j=1}^{m} \mu_j v_j$, the coefficient of a typical element z_k, $1 \le k \le m$ is of the form $\sum_{j=1}^{m} \mu_j \,(a_k^{(j)} + a_{k+1}^{(j)})$, $1 \le k \le m$ and we showed that for each $1 \le k \le m$, $a_k^{(j)} + a_{k+1}^{(j)} = (2+\rho_j) S_{k-1}(\rho_j)$. (Note that $a_{m+1}^{(j)} = S_m(\rho_j) + S_{m-1}(\rho_j) = 0$).

Since the elements z_k are linearly independent, the relation (R) implies that

$\sum_{j=1}^{m} \mu_j(a_k^{(j)} + a_{k+1}^{(j)}) = 0 \;\; \forall\, 1 \le k \le m$. Now, we have a system of m-homogeneous equations of the form (Σ) : $\{E_k = \sum_{j=1}^{m} \mu_j(a_k^{(j)} + a_{k+1}^{(j)}) = 0,\ 1 \le k \le m\}$, in the unknowns μ_j.

I claim that any solution of (Σ) is also a solution of (Σ') : $\{E'_k = \sum_{j=1}^{m} \mu_j(2 + \rho_j)\rho_j^{k-1} = 0 \;\; 1 \le k \le m\}$. In fact if $k = 1$ this is true since $a_1^{(j)} + a_2^{(j)} = 2 + \rho_j$. Assume that for all integers $\lambda \le k-1$, the μ_i satisfy the equations E'_λ. Then we look at the equation E_k : $\sum_{j=1}^{m} \mu_j(a_k^{(j)} + a_{k+1}^{(j)}) = 0$. We have $a_k^{(j)} + a_{k+1}^{(j)} = (2 + \rho_j)S_{k-1}(\rho_j) = (2+\rho_j)[\rho_j^{k-1} +$ + linear combination of lower powers of $\rho_j]$. Therefore, by induction we obtain that μ_i satisfy the equation $E'_k = \sum_{j=1}^{m} \mu_j(2 + \rho_j)\rho_j^{k-1} = 0$. Now the determinant of the unknowns μ_j in the system (Σ') is

$$\begin{vmatrix} 2+\rho_1 & 2+\rho_2 & \cdots & 2+\rho_m \\ (2+\rho_1)\rho_1 & (2+\rho_2)\rho_2 & \cdots & (2+\rho_m)\rho_m \\ \cdots & \cdots & \cdots & \cdots \\ (2+\rho_1)\rho_1^{m-1} & (2+\rho_2)\rho_2^{m-1} & \cdots & (2+\rho_m)\rho_m^{m-1} \end{vmatrix} = \prod_{j=1}^{m} (2+\rho_j) \prod_{m \ge i > j \ge 1} (\rho_i - \rho_j) \ne 0$$

since $\rho_j \ne -2 \;\forall\, j = 1,\ldots,m$ and $\rho_i \ne \rho_j$ if $i \ne j$. Therefore we obtain $\mu_j = 0 \;\; \forall\, j = 1,\ldots,m$.

So, the elements $\{u_j, v_j \;\; 1 \le j \le m\}$ are linearly independent over $\mathbb{Q}_n(u^{\lambda})$ and hence they form a basis for the submodule N_1.

Next we show that for each j, the pair (u_j, v_j) spans a 2-dimensional left $H_{Q_n}(u^{\frac{1}{2}})$-module. We shall show that the 2-dimensional subspace spanned by (u_j, v_j) is invariant under the action of T_{s_1} and T_{s_2}.

By recalling the action of the generators T_{s_i} on the graded module (see §2.1) we have that,

$$T_{s_1}.u_j = -u_j, \; T_{s_2}.u_j = u\,u_j + u^{\frac{1}{2}}v_j$$

$$T_{s_2}.v_j = -v_j$$

$$T_{s_1}.v_j = uv_j + u^{\frac{1}{2}}(a_1^{(j)}+a_2^{(j)})\bar{e}_{s_1} + u^{\frac{1}{2}}(a_1^{(j)} + 2a_2^{(j)} + a_3^{(j)})\bar{e}_{s_1s_2s_1} + \cdots$$

$$+ \cdots + u^{\frac{1}{2}}(a_{m-2}^{(j)} + 2a_{m-1}^{(j)} + a_m^{(j)})\bar{e}_{(s_1s_2)^{m-2}s_1} + u^{\frac{1}{2}}(a_{m-1}^{(j)} + 2a_m^{(j)})\bar{e}_{(s_1s_2)^{m-1}s_1}$$

The following relations hold:

$$a_1^{(j)} + a_2^{(j)} = 2 + \rho_j \quad \forall\ 1 \le j \le m$$

$$a_k^{(j)} + 2a_{k+1}^{(j)} + a_{k+2}^{(j)} = a_k^{(j)} + a_{k+1}^{(j)} + a_{k+1}^{(j)} + a_{k+2}^{(j)} = (2+\rho_j)S_{k-1}(\rho_j) + (2+\rho_j)S_k(\rho_j)$$

$$= (2+\rho_j)(S_k(\rho_j) + S_{k-1}(\rho_j)) = (2+\rho_j)a_{k+1}^{(j)} \quad \text{for every } 1 \le k \le m-2\,.$$

$$a_{m-1}^{(j)} + 2a_m^{(j)} = a_{m-1}^{(j)} + a_m^{(j)} + a_m^{(j)} = (2+\rho_j)S_{m-2}(\rho_j) + a_m^{(j)} =$$

$$= (2+\rho_j)S_{m-2}(\rho_j) + (2+\rho_j)S_{m-1}(\rho_j) = (2+\rho_j)(S_{m-2}(\rho_j)+S_{m-1}(\rho_j))$$

$$= (2+\rho_j)a_m^{(j)}.$$

Therefore $T_{s_1}.v_j = uv_j + (2+\rho_j)u^{\frac{1}{2}}\,u_j$.

So, the subspace spanned by (u_j, v_j) is a left $H_{Q_n}(u^{\frac{1}{2}})$-submodule of the graded module.

Let $M_j = \langle u_j, v_j \rangle$. Then the submoudle N_1 decomposes as:

$$N_1 = \bigoplus_{j=1}^{m} \langle u_j, v_j \rangle = \bigoplus_{j=1}^{m} M_j \; .$$

Moreover each M_j is an irreducible left H-module, because otherwise it would have a non-trivial one-dimensional left H-submodule. However, the graded module affords the regular representation of H. We have two one-dimensional representations of H, afforded by the submodules $\langle \bar{e}_1 \rangle$, and $\langle \bar{e}_{w_0} \rangle$ respectively, and they can appear only once.

Finally the matrix which rperesents T_{s_1} on M_j with respect to the basis $\{u_j, v_j\}$ is

$$\begin{pmatrix} -1 & (2 + \rho_j)u^{\frac{1}{2}} \\ 0 & u \end{pmatrix}$$

and the matrix which represents T_{s_2} on M_j with respect to the same basis is

$$\begin{pmatrix} u & 0 \\ u^{\frac{1}{2}} & -1 \end{pmatrix}$$

Therefore, the matrix which represents $T_{s_1 s_2}$ on M_j is

$$\begin{pmatrix} u(1 + \rho_j) & -u^{\frac{1}{2}}(2 + \rho_j) \\ u^{3/2} & -u \end{pmatrix}.$$

Hence the character of this element is $u\rho_j$, $1 \le j \le m$. Since $\rho_j \ne \rho_{j'}$ if $j \ne j'$ we conclude that $M_j \not\cong M_{j'}$, if $j \ne j'$. Our proposition is now proved.

The set $\{M_0, M_s, M_j, 1 \le j \le m\}$ is a full set of left irreducible H modules since the sum of the squares of the degrees is $4m + 2 = 2n$.

With a similar argument we can decompose the submodule N_2 into a direct sum of left irreducible H-submodules, namely $N_2 = \bigoplus_{j=1}^{m} \langle \hat{u}_j, \hat{v}_j \rangle$ where

$$\hat{u}_j = a_1^{(j)}\bar{e}_{s_2} + a_2^{(j)}\bar{e}_{s_2s_1s_2} + \dots + a_m^{(j)} e_{(s_2s_1)^{m-1}s_2},$$

$$\hat{v}_j = (a_1^{(j)} + a_2^{(j)})\bar{e}_{s_1s_2} + (a_2^{(j)} + a_3^{(j)})\bar{e}_{s_1s_2s_1s_2} + \dots + (a_{m-1}^{(j)} + a_m^{(j)})e_{(s_1s_2)^{m-1}} +$$

$$+ a_m^{(j)} e_{(s_1s_2)^m}$$

and the $a_n^{(j)}$ are defined as before.

The action of T_{s_1}, T_{s_2} on $\hat{u}_j, \hat{v}_j$ is given by:

$$\left.\begin{array}{ll} T_{s_1}.\hat{u}_j = u\hat{u}_j + u^{\frac{1}{2}}\hat{v}_j, & T_{s_1}.\hat{v}_j = -\hat{v}_j \\ T_{s_2}.\hat{u}_j = -\hat{u}_j & T_{s_2}.\hat{v}_j = u\hat{v}_j + u^{\frac{1}{2}}(2 + \rho_j)\hat{u}_j \end{array}\right\} \; 1 \le j \le m$$

So, eventually we have a decomposition of grad(E) over $\mathbb{Q}_n(u^{\frac{1}{2}})$ into: $\text{grad}(E) = M_0 \bigoplus_{j=1}^{m} M_j \bigoplus_{j=1}^{m} \hat{M}_j \oplus M_s$ of left irreducible H-submodules. In fact, for every $1 \le j \le m$, $M_j \simeq \hat{M}_j$.

This will become clearer in the lines below .

We note that $2 + \rho_j = 2(1 + \cos \frac{2j\pi}{n}) = 4 \cos^2 \frac{j\pi}{n}, j = 1,\dots,m.$

Thus: $T_{s_1}.u_j = -u_j \qquad T_{s_1}.v_j = uv_j + 4u^{\frac{1}{2}} \cos^2 \frac{j\pi}{n}\, u_j$

$T_{s_2}.u_j = uu_j + u^{\frac{1}{2}}v_j \qquad T_{s_2}.v_j = -v_j.$

We replace u_j by $u_j' = 2\cos\frac{j\pi}{n}u_j$ to obtain

$$T_{s_1}.u_j' = -u_j', \quad T_{s_1}.v_j = uv_j + 2u^{\frac{1}{2}}\cos\frac{j\pi}{n}u_j'$$

$$T_{s_2}.u_j' = uu_j' + 2u^{\frac{1}{2}}\cos\frac{j\pi}{n}v_j, \quad T_{s_2}.v_j = -v_j$$

Therefore the matrices which represent T_{s_i} $i = 1,2$ on each M_j with respect to the basis $\{u_j', v_j\}$ are:

$$T_{s_1} \to \begin{pmatrix} -1 & 2u^{\frac{1}{2}}\cos\frac{j\pi}{n} \\ 0 & u \end{pmatrix}, \quad T_{s_2} \to \begin{pmatrix} u & 0 \\ 2u^{\frac{1}{2}}\cos\frac{j\pi}{n} & -1 \end{pmatrix}$$

Similarly we replace $\hat{u}_j$ by $\hat{u}_j' = 2u^{\frac{1}{2}}\cos\frac{j\pi}{n}\hat{u}_j$, to obtain that the matrices which represent T_{s_i}, $i = 1,2$ on $\hat{M}_j$, with respect to the basis $\{\hat{u}_j', \hat{v}_j\}$ are:

$$T_{s_1} \to \begin{pmatrix} u & 0 \\ 2u^{\frac{1}{2}}\cos\frac{j\pi}{n} & -1 \end{pmatrix}, \quad T_{s_2} \to \begin{pmatrix} -1 & 2u^{\frac{1}{2}}\cos\frac{j\pi}{n} \\ 0 & u \end{pmatrix}$$

$$1 \le j \le m.$$

It is easy to verify that the matrices which represent T_{s_i}, $i = 1,2$ on $\hat{M}_j$ are obtained by conjugation by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ from the matrices which represent T_{s_i}, $i = 1,2$ on M_j.

<u>Second case: $n = 2m$.</u> In this case we have four 1-dimensional representations of $H_{\mathbb{Q}(u^{\frac{1}{2}})}$ which are:

$\sigma_0 : T_{s_1} \to u,\ T_{s_2} \to u, \qquad \sigma_1 : T_{s_1} \to -1,\ T_{s_2} \to u,$

$\sigma_2 : T_{s_1} \to u,\ T_{s_2} \to -1 \qquad \sigma_s : T_{s_1} \to -1,\ T_{s_2} \to -1.$

The representations σ_0, σ_s are afforded by the H-submodules $M_0 = \langle \bar{e}_1 \rangle$ and $M_s = \langle \bar{e}_{w_0} \rangle$ respectively.

Let M_1 be the subspace spanned by the element $\theta_1 = \sum_{k=1}^{m} (-1)^{k-1} \bar{e}_{(s_1s_2)^{k-1}s_1}$.

By recalling the action of T_{s_i}, $i = 1,2$ on the graded module, we have that

$T_{s_1}\theta_1 = -\theta_1$.

We also have that for every $k \in \{2,\dots,m-1\}$,

$T_{s_2}.\bar{e}_{(s_1s_2)^{k-1}s_1} = u\,\bar{e}_{(s_1s_2)^{k-1}s_1} + u^{\frac{1}{2}}\,\bar{e}_{(s_2s_1)^k} + u^{\frac{1}{2}}\,\bar{e}_{(s_2s_1)^{k-1}}$, while for

$k = 1$ we have $T_{s_2}.\bar{e}_{s_1} = u\bar{e}_{s_1} + u^{\frac{1}{2}}\,\bar{e}_{s_2s_1}$, and for $k = m$ we have

$T_{s_2}\,\bar{e}_{(s_1s_2)^{m-1}s_1} = u\,\bar{e}_{(s_1s_2)^{m-1}s_1} + u^{\frac{1}{2}}\,\bar{e}_{(s_2s_1)^{m-1}}.$

Thus, $T_{s_2}.\theta_1 =$

$$= u \sum_{k=1}^{m} (-1)^{k-1}\,\bar{e}_{(s_1s_2)^{k-1}s_1} + u^{\frac{1}{2}}\,\bar{e}_{s_2s_1} + u^{\frac{1}{2}} \sum_{k=2}^{m-1} (-1)^{k-1}(\bar{e}_{(s_2s_1)^{k-1}} + \bar{e}_{(s_2s_1)^k})$$

$$+ (-1)^{m-1}\,u^{\frac{1}{2}}\,\bar{e}_{(s_2s_1)^{m-1}} = u\theta_1 + u^{\frac{1}{2}}\,\bar{e}_{s_2s_1} +$$

$$+ u^{\frac{1}{2}} \sum_{k=2}^{m-1} (-1)^{k-1}\,\bar{e}_{(s_2s_1)^{k-1}} + u^{\frac{1}{2}} \sum_{k=3}^{m} (-1)^{k-2}\,\bar{e}_{(s_2s_1)^{k-1}} + (-1)^{m-1}u^{\frac{1}{2}}\bar{e}_{(s_2s_1)^{m-1}}$$

$$= u\theta_1.$$

Therefore $M_i = \langle\theta_i\rangle$ affords the representation σ_i. We put

$\theta_2 = \sum_{k=1}^{m} (-1)^{k-1} \bar{e}_{(s_2s_1)^{k-1}s_2}$, and a similar argument (interchanging the

role of s_1,s_2) shows that $M_2 = \langle\theta_2\rangle$ affords the representation σ_2.

Let $a_k = \bar{e}_{(s_1s_2)^{k-1}s_1} + \bar{e}_{(s_1s_2)^k s_1}$, $b_k = \bar{e}_{(s_2s_1)^{k-1}s_2} + \bar{e}_{(s_2s_1)^k s_2}$

$\gamma_k = \bar{e}_{(s_1s_2)^k}$, $\delta_k = \bar{e}_{(s_2s_1)^k}$, $k = 1,\ldots,m-1$.

It is clear that these elements are linearly independent over $\mathbb{Q}(u^{\frac{1}{2}})$.
Moreover,

$T_{s_1} a_k = -a_k$,

$T_{s_2} a_k = ua_k + 2u^{\frac{1}{2}} \bar{e}_{(s_2s_1)^k} + u^{\frac{1}{2}} \bar{e}_{(s_2s_1)^{k-1}} + u^{\frac{1}{2}} \bar{e}_{(s_2s_1)^{k+1}}$, for

every $k \in \{2,\ldots,m-2\}$, while for $k = 1$ we have

$T_{s_2} a_1 = ua_1 + 2u^{\frac{1}{2}} \bar{e}_{s_2s_1} + u^{\frac{1}{2}} \bar{e}_{(s_2s_1)^2}$, and for $k = m-1$ we have,

$T_{s_2} a_{m-1} = ua_{m-1} + 2u^{\frac{1}{2}} \bar{e}_{(s_2s_1)^{m-1}} + u^{\frac{1}{2}} \bar{e}_{(s_2s_1)^{m-2}}$.

Similarly, by interchanging the role of s_1,s_2 we obtain the action of T_{s_i} on b_k, $i = 1,2$, $k = 1,\ldots,m-1$. Finally $T_{s_1}\cdot\gamma_k = -\gamma_k$, $k = 1,\ldots,m-1$, and $T_{s_2} \gamma_k = u\gamma_k + u^{\frac{1}{2}} \bar{e}_{(s_2s_1)^{k-1}s_2} + u^{\frac{1}{2}} \bar{e}_{(s_2s_1)^k s_2}$ $k = 1,\ldots,m-1$, i.e.
$T_{s_2} \gamma_k = u\gamma_k + u^{\frac{1}{2}} b_k$, $k = 1,\ldots,m-1$.

By interchanging the role of s_1, s_2 we obtain that

$$T_{s_2}\,\delta_k = -\delta_k,\quad T_{s_1}\,\delta_k = u\,\delta_k + u^{\frac{1}{2}}\,a_k,\ k = 1,\dots,m-1.$$

Therefore $V = \langle a_k, \beta_k, \gamma_k, \delta_k\rangle$, $k = 1,\dots,m-1$, is an H-submodule of the graded module, with dimension 4m-4, and it is readily seen that V is a complementary submodule inside the graded module of the direct sum $M_0 \oplus M_1 \oplus M_2 \oplus M_s$, where the 1-dimensional H-submodules M_0, M_1, M_2, M_s are defined above.

We shall next decompose V into a direct sum of 2m-2 2-dimensional left H-submodules.

Firstly V has an obvious decomposition into the direct sum of two (n-2)-dimensional submodules, namely $V = \tilde{V}_1 \oplus \tilde{V}_2$, where

$$\tilde{V}_1 = \langle \bar{e}_{s_1} + \bar{e}_{s_1s_2s_1},\dots,\bar{e}_{(s_1s_2)^{m-2}s_1} + \bar{e}_{(s_1s_2)^{m-1}s_1},\ \bar{e}_{s_2s_1},\ \bar{e}_{(s_2s_1)^2},\dots,\bar{e}_{(s_2s_1)^{m-1}}\rangle$$

and

$$\tilde{V}_2 = \langle \bar{e}_{s_2} + \bar{e}_{s_2s_1s_2},\dots,\bar{e}_{(s_2s_1)^{m-2}s_2} + \bar{e}_{(s_2s_1)^{m-1}s_2},\bar{e}_{s_1s_2},\bar{e}_{(s_1s_2)^2},\dots,\bar{e}_{(s_1s_2)^{m-1}}\rangle$$

We shall decompose $\tilde{V}_1$ (similarly $\tilde{V}_2$) into a direct sum of m-1 2-dimensional left H-submodules.

Consider the real numbers $\rho_j = 2\cos\frac{j\pi}{m}$ $1 \le j \le m-1$, and for every j define the following sequence of real numbers:

$$a_0^{(j)} = 0,\ a_1^{(j)} = 1 = S_0(\rho_j),\ a_2^{(j)} = S_1(\rho_j)\ \dots,\ a_\lambda^{(j)} = S_{\lambda-1}(\rho_j),$$
$$\lambda \in \{1,2,\dots,m-1\}.$$

We also consider the following elements inside the graded module .

$$u_j = a_1^{(j)}(\bar{e}_{s_1} + \bar{e}_{s_1s_2s_1}) + a_2^{(j)}(\bar{e}_{s_1s_2s_1} + \bar{e}_{s_1s_2s_1s_2s_1}) + \ldots + a_{m-1}^{(j)}(\bar{e}_{(s_1s_2)^{m-2}s_1}$$

$$+ \bar{e}_{(s_1s_2)^{m-1}s_1})$$

$$v_j = (2a_1^{(j)} + a_2^{(j)})\bar{e}_{s_2s_1} + (a_1^{(j)} + 2a_2^{(j)} + a_3^{(j)})\bar{e}_{(s_2s_1)^2} + \ldots +$$

$$+ (a_{m-3}^{(j)} + 2a_{m-2}^{(j)} + a_{m-1}^{(j)})\bar{e}_{(s_2s_1)^{m-2}} + (a_{m-2}^{(j)} + 2a_{m-1}^{(j)})\bar{e}_{(s_2s_1)^{m-1}}$$

for every $1 \le j \le m-1$.

The number of these elements is 2m-2 and they lie inside the submodule $\tilde{V}_1$.

We shall show the following.

Proposition 2.3.2: The elements u_j, v_j defined above, are all linearly independent over $\mathbb{Q}_n(u^{\frac{1}{2}})$, and for each $1 \le j \le m-1$, the pair (u_j, v_j) spans a 2-dimensional left $H_{\mathbb{Q}_n(u^{\frac{1}{2}})}$-submodule, namely V_j. Moreover each V_j is irreducible and distinct j give rise to non-isomorphic such submodules.

Proof: From the definition of $a_i^{(j)}$ we have the following relations:

$$2a_1^{(j)} + a_2^{(j)} = 2 + \rho_j$$

$a_{m-2}^{(j)} + 2a_{m-1}^{(j)} = S_{m-3}(\rho_j) + 2S_{m-2}(\rho_j)$. Recall that $S_{m-1}(\rho_j) = 0 =$

$\rho_j S_{m-2}(\rho_j) - S_{m-3}(\rho_j)$. Hence $a_{m-2}^{(j)} + 2a_{m-1}^{(j)} = (2+\rho_j)S_{m-2}(\rho_j)$

and for every $1 \le k \le m-3$, we also have that $a_k^{(j)} + 2a_{k+1}^{(j)} + a_{k+2}^{(j)} =$

$= S_{k-1}(\rho_j) + 2S_k(\rho_j) + S_{k+1}(\rho_j) = 2S_k(\rho_j) + \rho_j S_k(\rho_j) = (2+\rho_j)S_k(\rho_j)$.

From now on, the argument is entirely similar with the case when n is odd.

We assume that $\sum_{j=1}^{m-1} \lambda_j u_j + \sum_{j=1}^{m-1} \mu_j v_j = 0$ (R), $\lambda_j, \mu_j \in Q_n(u^{\frac{1}{2}})$ and we shall show that $\lambda_j = \mu_j = 0$ for every $1 \le j \le m-1$. We put again $w_1 = \bar{e}_{s_1} + \bar{e}_{s_1 s_2 s_1}, \ldots, w_{m-1} = \bar{e}_{(s_1 s_2)^{m-2} s_1} + \bar{e}_{(s_1 s_2)^{m-1} s_1}$. Then, in the expression $\sum_{j=1}^{m-1} \lambda_j u_j$ the coefficient of a typical element w_k $1 \le k \le m-1$ has the form $\sum_{j=1}^{m-1} \lambda_j a_k^{(j)}$, $1 \le k \le m-1$. Since the elements w_k are linearly independent, the relation (R) implies that $\sum_{j=1}^{m-1} \lambda_j a_k^{(j)} = 0$ for every $1 \le k \le m-1$. So, we obtain a system of m-1 homogeneous equations in m-1 unknowns the λ_j whose equations are of the form $E_k : \sum_{j=1}^{m-1} \lambda_j a_k^{(j)} = 0$, $1 \le k \le m-1$. With an argument similar to the one when n is odd, we can show that any solution of these homogeneous equations is also a solution of the system of homogeneous equations

$$(\Sigma') = \{E'_k : \sum_{j=1}^{m-1} \lambda_j \rho_j^{k-1} = 0,\ 1 \le k \le m-1\}.$$

The determinant of the coefficients in the later system is the Vandermonde determinant $\Delta = \prod_{m-1 \ge i > j \ge 1} (\rho_i - \rho_j) \ne 0$, since $\rho_i \ne \rho_j$ if $i \ne j$. Hence $\lambda_j = 0 \ \forall\ 1 \le j \le m-1$.

We also put $z_1 = \bar{e}_{s_2 s_1}, \ldots, z_{m-1} = \bar{e}_{(s_2 s_1)^{m-1}}$. Then in the expression $\sum_{j=1}^{m-1} \mu_j v_j$, the coefficient of a typical element z_k has the form

$$\sum_{j=1}^{m-1} \mu_j (a_{k-1}^{(j)} + 2a_k^{(j)} + a_{k+1}^{(j)}),\ 1 \le k \le m-1, \text{ with } a_0^{(j)} = 0,\ a_m^{(j)} = S_{m-1}(\rho_j) = 0.$$

Hence we obtain again a system $(\Sigma) = \begin{cases} E_k = \sum_{j=1}^{m-1} \mu_j(a_{k-1}^{(j)} + 2a_k^{(j)} + a_{k+1}^{(j)}) = 0 \\ \forall\ 1 \le k \le m-1 \end{cases}$

and by induction we can prove that any solution of the system (Σ) is also a solution of the system

$$(\Sigma') = \{E_k' = \sum_{j=1}^{m-1} \mu_j(2+\rho_j)\rho_j^{k-1} = 0, 1 \le k \le m-1\}.$$

The determinant of the coefficients in the latter system is
$\prod_{j=1}^{m-1} (2+\rho_j) \prod_{m-1 \ge i > j \ge 1} (\rho_i - \rho_j) \ne 0$ since $\rho_j \ne -2\ \forall\ j = 1,\dots,m-1$, and $\rho_i \ne \rho_j$ if $i \ne j$.

Hence $\mu_j = 0\ \ \forall 1 \le j \le m-1$.

Therefore the elements (u_j, v_j) $j = 1,\dots,m-1$ are linearly independent and so they form a basis for the submodule $\tilde{V}_1$. Let $V_j = \langle u_j, v_j \rangle$, $1 \le j \le m-1$. Then it can be shown that V_j is a left $H_{\mathcal{C}_n}(u^{\frac{1}{2}})$ module. In fact by recalling the left action of T_{s_i}, $i = 1,2$ on the graded module, we have that

$$T_{s_1} u_j = -u_j, \qquad T_{s_2} u_j = u\, u_j + u^{\frac{1}{2}} v_j$$

$$T_{s_2} v_j = -v_j, \qquad T_{s_1} v_j = uv_j + u^{\frac{1}{2}}(2+\rho_j)u_j$$

By replacing u_j by $u_j' = 2\cos\frac{j\pi}{n} u_j$ we obtain

$$\left.\begin{array}{ll} T_{s_1}u_j' = -u_j', & T_{s_2} u_j' = u\, u_j' + 2u^{\frac{1}{2}} \cos\frac{j\pi}{n} v_j \\ T_{s_2}v_j = -v_j, & T_{s_1} v_j = uv_j + 2u^{\frac{1}{2}} \cos\frac{j\pi}{n}\ u_j' \end{array}\right\} \qquad 1 \le j \le m-1$$

For the same reason as in the case n odd we have that each V_j is an irreducible left $H_{\theta_n}(u^{\frac{1}{2}})$ submodule, that $V_j \not\cong V_{j'}$, and that the set $\{M_0, M_1, M_2, M_s, V_j, 1 \le j \le m-1\}$ is a full set of left irreducible $H_{\theta_n}(u^{\frac{1}{2}})$ modules.

Our proposition is now proved.

The matrices which represent T_{s_i}, $i = 1,2$ on V_j with respect to the basis (u_j', v_j) are

$$T_{s_1} \to \begin{pmatrix} -1 & 2u^{\frac{1}{2}} \cos \frac{j\pi}{n} \\ 0 & u \end{pmatrix}, \quad T_{s_2} \to \begin{pmatrix} u & 0 \\ 2u^{\frac{1}{2}} \cos \frac{j\pi}{n} & -1 \end{pmatrix} \quad 1 \le j \le m-1.$$

With a similar argument we can find a decomposition of the submodule $\tilde{V}_2$, namely $\tilde{V}_2 = \bigoplus_{j=1}^{m-1} \langle \hat{u}_j, \hat{v}_j \rangle$, where $\hat{u}_j$, $\hat{v}_j$ are obtained by interchanging the role of s_1, s_2 in u_j, v_j respectively. Thus

$$\hat{u}_j = a_1^{(j)}(\bar{e}_{s_2} + \bar{e}_{s_2 s_1 s_2}) + \ldots + a_{m-1}^{(j)} (\bar{e}_{(s_2 s_1)^{m-2} \cdot s_2} + \bar{e}_{(s_2 s_1)^{m-1} \cdot s_2})$$

$$\hat{v}_j = (2a_1^{(j)} + a_2^{(j)})\bar{e}_{s_1 s_2} + \ldots + (a_{m-3}^{(j)} + 2a_{m-2}^{(j)} + a_{m-1}^{(j)})\bar{e}_{(s_1 s_2)^{m-2}} + (a_{m-2}^{(j)} + 2a_{m-1}^{(j)})\bar{e}_{(s_1 s_2)^{m-1}}$$

Each $\hat{V}_j = \langle \hat{u}_j, \hat{v}_j \rangle$ is irreducible left $H_{\theta_n}(u^{\frac{1}{2}})$ module, and $\hat{V}_j \cong V_j$, $1 \le j \le m-1$. The matrices with represent T_{s_i}, $i = 1,2$ on $\hat{V}_j$ with respect to the basis $(\hat{u}_j', \hat{v}_j)$, where $\hat{u}_j' = 2 \cos \frac{j\pi}{n} \hat{u}_j$, are

$$T_{s_1} \to \begin{pmatrix} u & 0 \\ 2u^{\frac{1}{2}} \cos \frac{j\pi}{n} & -1 \end{pmatrix}, \quad T_{s_2} \to \begin{pmatrix} -1 & 2u^{\frac{1}{2}} \cos \frac{j\pi}{n} \\ 0 & u \end{pmatrix}$$

Eventually we have $\text{grad}(E) = M_0 \oplus M_1 \oplus M_2 \oplus M_s \bigoplus_{j=1}^{m-1} V_j \bigoplus_{j=1}^{m-1} \hat{V}_j$.

Theorem 2.3.1: Let W be a finite Coxeter group of dihedral type, $W = D_{2n}$ and let Φ be the Lusztig isomorphism between the generic Hecke algebra $H_{\mathbb{Q}(u^{\frac{1}{2}})}$ of dihedral type and the group algebra over $\mathbb{Q}(u^{\frac{1}{2}})$ of D_{2n}. Define

$$\Xi_k = -(s_2s_1)^{k-1} s_2 + (s_1s_2)^k - (s_2s_1)^k + (s_1s_2)^k s_1 \quad \text{and}$$

$$\hat{\Xi}_k = -(s_1s_2)^{k-1}s_1 + (s_2s_1)^k - (s_1s_2)^k + (s_2s_1)^k s_2.$$

(i) If $n = 2m+1$, then the images of the generators T_{s_i}, $i = 1,2$ of the generic Hecke algebra under Φ are given by:

$$\Phi(T_{s_1}) = \frac{u-1}{2}.1 + \frac{u+1}{2} s_1 + \frac{(u^{\frac{1}{2}}-1)^2}{2n} \sum_{k=1}^{m} (n-2k)\Xi_k$$

$$\Phi(T_{s_2}) = \frac{u-1}{2}.1 + \frac{u+1}{2} s_2 + \frac{(u^{\frac{1}{2}}-1)^2}{2n} \sum_{k=1}^{m} (n-2k)\hat{\Xi}_k$$

(ii) If $n = 2m$, then

$$\Phi(T_{s_1}) = \frac{u-1}{2}.1 + \frac{u+1}{2} s_1 + \frac{(u^{\frac{1}{2}}-1)^2}{2n} \sum_{k=1}^{m-1} (n-2k)\Xi_k$$

$$\Phi(T_{s_2}) = \frac{u-1}{2}.1 + \frac{u+1}{2} s_2 + \frac{(u^{\frac{1}{2}}-1)^2}{2n} \sum_{k=1}^{m-1} (n-2k)\hat{\Xi}_k$$

Proof: In the previous pages we have established a decomposition of the graded module over $\mathbb{Q}_n(u^{\frac{1}{2}})$ into a direct sum of left irreducible H-submodules, for both the cases when n is odd and n is even.

(i) Let $n = 2m+1$. Then $\mathrm{grad}(E) = M_0 \bigoplus_{j=1}^{m} M_j \bigoplus_{j=1}^{m} \hat{M}_j \oplus M_s$ where $M_0 = \langle \bar{e}_1 \rangle$ affords the representation $T_{s_i} \to u$ $i = 1,2$, $M_s = \langle \bar{e}_{w_0} \rangle$ affords the representation $T_{s_i} \to -1$, $i = 1,2$, $\dim M_j = 2$ and $M_j \not\simeq M_{j'}$, if $j \neq j'$, $M_j \simeq \hat{M}_j$ $1 \le j \le m$. Moreover we established that the matrices which represent T_{s_i} on M_j with respect to the basis $\{u_j', v_j\}$ are

$$T_{s_1} \to \begin{pmatrix} -1 & 2u^{\frac{1}{2}} \cos \frac{j\pi}{n} \\ 0 & u \end{pmatrix}, \quad T_{s_2} \to \begin{pmatrix} u & 0 \\ 2u^{\frac{1}{2}} \cos \frac{j\pi}{n} & -1 \end{pmatrix}$$

and that the matrices which represent T_{s_i} on $\hat{M}_j$ with respect to the basis $\{\hat{u}_j', \hat{v}_j'\}$ are

$$T_{s_1} \to \begin{pmatrix} u & 0 \\ 2u^{\frac{1}{2}} \cos \frac{j\pi}{n} & -1 \end{pmatrix} \qquad T_{s_2} \to \begin{pmatrix} -1 & 2u^{\frac{1}{2}} \cos \frac{j\pi}{n} \\ 0 & u \end{pmatrix} \quad 1 \le j \le m.$$

It is enough to show that T_{s_i} and $\phi(T_{s_i})$ act on the same way on each M_j, $\hat{M}_j$, and on each 1-dimensional submodule. By specializing $u \to 1$ in the matrices which represent T_{s_i} above, we obtain the matrices which represent s_i, $i = 1,2$.

It is trivial to verify that T_{s_i} and $\phi(T_{s_i})$ $i = 1,2$ act on the same way on the submodules M_0 and M_s.

In the sequel the terminology blocks of the first kind, means the matrices which represent a specific element of the group $W = D_{2n}$ on M_j, $j = 1,\dots,m$, and blocks of the second kind, the matrices of the same element on $\bar{M}_j$, $j = 1,\dots,m$. I claim that the blocks of the first kind which represent the element $(s_1 s_2)^k$, $k = 1,\dots,m$ are given by

$$X_j = \begin{bmatrix} S_k(\rho_j) + S_{k-1}(\rho_j) & -2\cos\frac{j\pi}{n} S_{k-1}(\rho_j) \\ 2\cos\frac{j\pi}{n} S_{k-1}(\rho_j) & -(S_{k-1}(\rho_j) + S_{k-2}(\rho_j)) \end{bmatrix}$$

and the blocks of the second kind by

$$Y_j = \begin{bmatrix} -(S_{k-1}(\rho_j) + S_{k-2}(\rho_j)) & 2\cos\frac{j\pi}{n} S_{k-1}(\rho_j) \\ -2\cos\frac{j\pi}{n} S_{k-1}(\rho_j) & S_k(\rho_j) + S_{k-1}(\rho_j) \end{bmatrix},$$

where $\rho_j = 2\cos\frac{2j\pi}{n}$, $1 \le j \le m$.

We concentrate on the blocks of the first kind.

For $k = 1$ we have $s_1 \rightarrow \begin{pmatrix} -1 & 2\cos\frac{j\pi}{n} \\ 0 & 1 \end{pmatrix}$, $s_2 \rightarrow \begin{pmatrix} 1 & 0 \\ 2\cos\frac{j\pi}{n} & -1 \end{pmatrix}$

therefore $s_1 s_2 \rightarrow \begin{pmatrix} 4\cos^2\frac{j\pi}{n} - 1 & -2\cos\frac{j\pi}{n} \\ 2\cos\frac{j\pi}{n} & -1 \end{pmatrix}$

Since $2 + \rho_j = 4\cos^2\frac{j\pi}{n}$, $S_0(\rho_j) = 1$, $S_1(\rho_j) = \rho_j$, $S_{-1}(\rho_j) = 0$, the assertion holds.

Assume, by induction, that the blocks of the first kind which represent $(s_1 s_2)^\lambda$ are of the form

$$\begin{bmatrix} S_\lambda(\rho_j) + S_{\lambda-1}(\rho_j) & -2\cos\frac{j\pi}{n} S_{\lambda-1}(\rho_j) \\ 2\cos\frac{j\pi}{n} S_{\lambda-1}(\rho_j) & -(S_{\lambda-1}(\rho_j) + S_{\lambda-2}(\rho_j)) \end{bmatrix}$$

Then, $(s_1 s_2)^{\lambda+1} = (s_1 s_2)^\lambda (s_1 s_2)$ is represented by

$$\begin{bmatrix} S_\lambda(\rho_j) + S_{\lambda-1}(\rho_j) & -2\cos\frac{j\pi}{n} S_{\lambda-1}(\rho_j) \\ 2\cos\frac{j\pi}{n} S_{\lambda-1}(\rho_j) & -(S_{\lambda-1}(\rho_j) + S_{\lambda-2}(\rho_j)) \end{bmatrix} \begin{bmatrix} 1 + \rho_j & -2\cos\frac{j\pi}{n} \\ 2\cos\frac{j\pi}{n} & -1 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \text{where}$$

$$a_{11} = (1 + \rho_j)[S_\lambda(\rho_j) + S_{\lambda-1}(\rho_j)] - 4\cos^2\frac{j\pi}{n} S_{\lambda-1}(\rho_j) =$$

$$S_\lambda(\rho_j) + S_{\lambda-1}(\rho_j) + \rho_j S_\lambda(\rho_j) + \rho_j S_{\lambda-1}(\rho_j) - (2 + \rho_j)S_{\lambda-1}(\rho_j) =$$

$$= \rho_j S_\lambda(\rho_j) - S_{\lambda-1}(\rho_j) + S_\lambda(\rho_j) = S_\lambda(\rho_j) + S_{\lambda+1}(\rho_j) .$$

$$a_{12} = -2\cos\frac{j\pi}{n} S_\lambda(\rho_j) .$$

$$a_{21} = 2\cos\frac{j\pi}{n}(1 + \rho_j)S_{\lambda-1}(\rho_j) - 2\cos\frac{j\pi}{n}(S_{\lambda-1}(\rho_j) + S_{\lambda-2}(\rho_j)) =$$

$$= 2\cos\frac{j\pi}{n}[\rho_j S_{\lambda-1}(\rho_j) - S_{\lambda-2}(\rho_j)] = 2\cos\frac{j\pi}{n} S_\lambda(\rho_j).$$

$$a_{22} = -4\cos^2\frac{j\pi}{n} S_{\lambda-1}(\rho_j) + S_{\lambda-1}(\rho_j) + S_{\lambda-2}(\rho_j) = -(2+\rho_j)S_{\lambda-1}(\rho_j) +$$

$$+ S_{\lambda-1}(\rho_j) + S_{\lambda-2}(\rho_j)$$

$$= - S_{\lambda-1}(\rho_j) - [\rho_j S_{\lambda-1}(\rho_j) - S_{\lambda-2}(\rho_j)] = - (S_{\lambda-1}(\rho_j) + S_\lambda(\rho_j))$$

Hence our assertion holds for every k.

To obtain the matrices Y_j we conjugate the matrices X_j by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

By symmetry, the blocks of the first kind which represent $(s_2 s_1)^k$, $k = 1,\dots,m$ are

$$Y_j = \begin{bmatrix} -(S_{k-1}(\rho_j) + S_{k-2}(\rho_j)) & 2\cos\frac{j\pi}{n} S_{k-1}(\rho_j) \\ -2\cos\frac{j\pi}{n} S_{k-1}(\rho_j) & S_k(\rho_j) + S_{k-1}(\rho_j) \end{bmatrix}$$

and the blocks of the second kind are

$$X_j = \begin{bmatrix} S_k(\rho_j) + S_{k-1}(\rho_j) & -2\cos\frac{j\pi}{n} S_{k-1}(\rho_j) \\ 2\cos\frac{j\pi}{n} S_{k-1}(\rho_j) & -(S_{k-1}(\rho_j) + S_{k-2}(\rho_j)) \end{bmatrix}$$

Simple matrix multiplication gives the blocks of the first kind of the element $(s_1 s_2)^k . s_1$, $k = 0,\dots,m$, which are

$$Z_j = \begin{bmatrix} -(S_k(\rho_j) + S_{k-1}(\rho_j)) & 2\cos\frac{j\pi}{n} S_k(\rho_j) \\ -2\cos\frac{j\pi}{n} S_{k-1}(\rho_j) & S_k(\rho_j) + S_{k-1}(\rho_j) \end{bmatrix}$$

and the blocks of the second kind which are

$$W_j = \begin{bmatrix} S_k(\rho_j) + S_{k-1}(\rho_j) & -2\cos\frac{j\pi}{n} S_{k-1}(\rho_j) \\ 2\cos\frac{j\pi}{n} S_k(\rho_j) & -(S_k(\rho_j) + S_{k-1}(\rho_j)) \end{bmatrix}$$

By symmetry, we obtain the blocks of the first kind which represent $(S_2 s_1)^k . s_2, k = 0,\ldots,m-1$ and which are

$$W_j = \begin{bmatrix} S_k(\rho_j) + S_{k-1}(\rho_j) & -2\cos\frac{j\pi}{n} S_{k-1}(\rho_j) \\ 2\cos\frac{j\pi}{n} S_k(\rho_j) & -(S_k(\rho_j) + S_{k-1}(\rho_j)) \end{bmatrix}$$

and the blocks of the second kind which are

$$Z_j = \begin{bmatrix} -(S_k(\rho_j) + S_{k-1}(\rho_j)) & 2\cos\frac{j\pi}{n} S_k(\rho_j) \\ -2\cos\frac{j\pi}{n} S_{k-1}(\rho_j) & S_k(\rho_j) + S_{k-1}(\rho_j) \end{bmatrix}$$

Now, for every $1 \le k \le m$, let $R_k^{(j)}$ be the matrix which represents the element of the group algebra Ξ_k on M_j. Then,

$$R_k^{(j)} = \begin{bmatrix} -S_{k-1}(\rho_j) - S_{k-2}(\rho_j) & 2\cos\frac{j\pi}{n} S_{k-2}(\rho_j) \\ -2\cos\frac{j\pi}{n} S_{k-1}(\rho_j) & S_{k-1}(\rho_j) + S_{k-2}(\rho_j) \end{bmatrix} +$$

$$+ \begin{bmatrix} S_k(\rho_j) + S_{k-1}(\rho_j) & -2\cos\frac{j\pi}{n} S_{k-1}(\rho_j) \\ 2\cos\frac{j\pi}{n} S_{k-1}(\rho_j) & -(S_{k-1}(\rho_j) + S_{k-2}(\rho_j)) \end{bmatrix}$$

$$+ \begin{bmatrix} S_{k-1}(\rho_j) + S_{k-2}(\rho_j) & -2\cos\frac{j\pi}{n} S_{k-1}(\rho_j) \\ 2\cos\frac{j\pi}{n} S_{k-1}(\rho_j) & -(S_k(\rho_j) + S_{k-1}(\rho_j)) \end{bmatrix}$$

$$+ \begin{bmatrix} -(S_k(\rho_j) + S_{k-1}(\rho_j)) & 2\cos\frac{j\pi}{n} S_k(\rho_j) \\ -2\cos\frac{j\pi}{n} S_{k-1}(\rho_j) & S_k(\rho_j) + S_{k-1}(\rho_j) \end{bmatrix}$$

Hence $R_k^{(j)} = \begin{bmatrix} 0 & A_k^{(j)} \\ 0 & 0 \end{bmatrix} \quad 1 \le j \le m$

where $A_k^{(j)} = 2\cos\frac{j\pi}{n}[S_k(\rho_j) + S_{k-2}(\rho_j)] - 4\cos\frac{j\pi}{n} S_{k-1}(\rho_j)$.

Furthermore, $\rho_j A_{k-1}^{(j)} - A_{k-2}^{(j)} = 2\rho_j \cos\frac{j\pi}{n}[S_{k-1}(\rho_j) + S_{k-3}(\rho_j)]$

$$- 4\rho_j \cos\frac{j\pi}{n} S_{k-2}(\rho_j) -$$

$$- 2\cos\frac{j\pi}{n}[S_{k-2}(\rho_j) + S_{k-4}(\rho_j)] + 4\cos\frac{j\pi}{n} S_{k-3}(\rho_j) =$$

$$= 2\cos\frac{j\pi}{n}[\rho_j S_{k-1}(\rho_j) - S_{k-2}(\rho_j) + \rho_j S_{k-3}(\rho_j) - S_{k-4}(\rho_j)]$$

$$- 4\cos\frac{j\pi}{n}[\rho_j S_{k-2}(\rho_j) - S_{k-3}(\rho_j)]$$

$$= A_k^{(j)}.$$

Therefore $R_k^{(j)} = \rho_j R_{k-1}^{(j)} - R_{k-2}^{(j)} \quad 1 \le k \le m$. When $k = 1$, then $A_1^{(j)} = 2\cos\frac{j\pi}{n}(\rho_j - 2)$.

Assume that for all integers $\lambda \le k$, we have $A_\lambda^{(j)} = S_{\lambda-1}(\rho_j)A_1^{(j)}$. Then

$$A_{k+1}^{(j)} = \rho_j A_k^{(j)} - A_{k-1}^{(j)} = \rho_j S_{k-1}(\rho_j)A_1^{(j)} - S_{k-2}(\rho_j)A_1^{(j)}$$

$$= (\rho_j S_{k-1}(\rho_j) - S_{k-2}(\rho_j))\, A_1^{(j)} = S_k(\rho_j)A_1^{(j)} .$$

Therefore, the matrix which represents the element of the group algebra $\sum_{k=1}^{m} (n-2k)\Xi_k$ on M_j is $\sum_{k=1}^{m} (n-2k)R_k^{(j)} = \begin{bmatrix} 0 & \sum_{k=1}^{m} (n-2k)A_k^{(j)} \\ 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & \sum_{k=1}^{m} (n-2k)S_{k-1}(\rho_j)A_1^{(j)} \\ 0 & 0 \end{bmatrix}$$

Now $\sum_{k=1}^{m} (n-2k)S_{k-1}(\rho_j)A_1^{(j)} = 2\cos\frac{j\pi}{n}(\rho_j-2)\sum_{k=1}^{m}(n-2k)S_{k-1}(\rho_j) =$

$= -2n\cos\frac{j\pi}{n}$ (by Lemma 2.2.2(ii) and 2.2.3(i)).

Hence the matrix which represents the element of the group algebra:

$\frac{u-1}{2}.1 + \frac{u+1}{2}s_1 + \frac{(u^{\frac{1}{2}}-1)^2}{2n}\sum_{k=1}^{m}(n-2k)\Xi_k$, on M_j is:

$$\frac{u-1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{u+1}{2}\begin{pmatrix}-1 & 2\cos\frac{j\pi}{n}\\ 0 & 1\end{pmatrix} + \frac{(u^{\frac{1}{2}}-1)^2}{2n}\begin{pmatrix}0 & -2n\cos\frac{j\pi}{n}\\ 0 & 0\end{pmatrix}$$

$= \begin{pmatrix}-1 & 2u^{\frac{1}{2}}\cos\frac{j\pi}{n}\\ 0 & u\end{pmatrix}$ for every $j = 1,\dots,m$.

Therefore T_{s_1} acts on every M_j in the same way as $\phi(T_{s_1})$ does.

We next put $\hat{R}_k^{(j)}$ to be the matrix which represents the element Ξ_k on $\hat{M}_j$. It turns out that $\hat{R}_k^{(j)} = \begin{bmatrix} 0 & 0 \\ A_k^{(j)} & 0 \end{bmatrix}$ and therefore the matrix which represents the element of the group algebra

$\frac{u-1}{2}\,1 + \frac{u+1}{2}\,s_1 + \frac{(u^{\frac{1}{2}}-1)^2}{2n} \sum_{k=1}^{m} (n-2k)\Xi_k$ on $\hat{M}_j$ is

$$\frac{u-1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{u+1}{2}\begin{bmatrix} 1 & 0 \\ 2\cos\frac{j\pi}{n} & -1 \end{bmatrix} + \frac{(u^{\frac{1}{2}}-1)^2}{2n}\begin{bmatrix} 0 & 0 \\ -2n\cos\frac{j\pi}{n} & 0 \end{bmatrix} =$$

$= \begin{pmatrix} u & 0 \\ 2u^{\frac{1}{2}}\cos\frac{j\pi}{n} & -1 \end{pmatrix}$. So T_{s_1} also acts on $\hat{M}_j$ in the same way as $\phi(T_{s_1})$ does. By symmetry (interchanging the role of s_1, s_2) we obtain the image $\phi(T_{s_2})$ and so part (i) of our theorem is proved.

The proof of part (ii) of our theorem, is entirely similar, since the matrices which represent T_{s_i} $i = 1,2$ on the irreducible constituents of the graded module when $n = 2m$, have the same form as in the case $n = 2m+1$.

In this case, the matrix which represents the element

$\sum_{k=1}^{m-1} (n-2k)\Xi_k$ on V_j, $1 \le j \le m-1$, is $\begin{bmatrix} 0 & \sum_{k=1}^{m-1}(n-2k)A_k^{(j)} \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & \sum_{k=1}^{m-1}(n-2k)S_{k-1}(\rho_j)A_1^{(j)} \\ 0 & 0 \end{bmatrix}$ where $\rho_j = 2\cos\frac{j\pi}{m}$, and

$A_1^{(j)} = 2\cos\frac{j\pi}{n}(\rho_j - 2)$.

Now Lemma 2.2.2(i) and 2.2.3(ii) give

$$\sum_{k=1}^{m-1} (n-2k)S_{k-1}(\rho_j)A_1^{(j)} = -2n \cos \frac{j\pi}{n} \quad 1 \le j \le m-1.$$

Therefore we can verify that T_{s_i}, $i = 1,2$ acts on every irreducible constituent V_j and $\hat{V}_j$, $1 \le j \le m-1$, on the same way as $\phi(T_{s_i})$. This is also true when we consider each one of the four 1-dimensional representations, afforded by the submodules M_0, M_1, M_2, M_s. It is obvious when we consider the submodules M_0 and M_s.

For the representation $\sigma_1 : s_1 \to -1,\ s_2 \to 1$, we have that

$$\Xi_k \quad \left\{ \begin{array}{l} -1 - 1 + 1 + 1 = 0 \text{ if } k = \text{odd} \\ \\ +1 + 1 - 1 - 1 = 0 \text{ if } k = \text{even} \end{array} \right\} \quad 1 \le k \le m-1$$

and therefore:

$$\phi(T_{s_1}) \to -1.$$

For the same representation, we have that

$$\hat{\Xi}_k \to \left\{ \begin{array}{l} 1 - 1 + 1 - 1 = 0 \text{ if } k = \text{odd} \\ -1 + 1 - 1 + 1 = 0 \text{ if } k = \text{even} \end{array} \right\}$$

and therefore $\phi(T_{s_2}) \to u$.

The argument is similar for the representation $\sigma_2 : s_1 \to 1,\ s_2 \to -1$. Thus our theorem is proved.

We illustrate the situation by giving some examples.

(1) $W(A_1) \simeq S_2 : \phi(T_{s_1}) = \frac{u-1}{2} 1 + \frac{u+1}{2} s_1$

(2) $W = D_4$: $\phi(T_{s_1}) = \frac{u-1}{2} 1 + \frac{u+1}{2} s_1$.

$$\phi(T_{s_2}) = \frac{u-1}{2} 1 + \frac{u+1}{2} s_2$$

(3) $W(A_2) \simeq S_3$. This example appears at the end of Chapter I.

(4) $W(B_2) \simeq D_8$ $(n = 4,\ m = 2)$

$$\phi(T_{s_1}) = \frac{u-1}{2} 1 + \frac{u+1}{2} s_1 + \frac{(u^{1/2}-1)^2}{4} (-s_2 + s_1s_2-s_2s_1+s_1s_2s_1)$$

$$\phi(T_{s_2}) = \frac{u-1}{2} 1 + \frac{u+1}{2} s_2 + \frac{(u^{1/2}-1)^2}{4} (-s_1+s_2s_1-s_1s_2+s_2s_1s_2)$$

(5) $W = D_{10}$ $(n = 5,\ m = 2)$.

$$\phi(T_{s_1}) = \frac{u-1}{2} 1 + \frac{u+1}{2} s_1 + 3\,\frac{(u^{1/2}-1)^2}{10}\ (-s_2+s_1s_2-s_2s_1+s_1s_2s_1) + $$
$$+ \frac{(u^{1/2}-1)^2}{10} (-s_2s_1s_2 + s_1s_2s_1s_2 - s_2s_1s_2s_1 + s_1s_2s_1s_2s_1)$$

$$\phi(T_{s_2}) = \frac{u-1}{2} 1 + \frac{u+1}{2} s_2 + 3\,\frac{(u^{1/2}-1)^2}{10}\ (-s_1+s_2s_1-s_1s_2+s_2s_1s_2) + $$
$$+ \frac{(u^{1/2}-1)^2}{10} (-s_1s_2s_1 + s_2s_1s_2s_1 - s_1s_2s_1s_2 + s_2s_1s_2s_1s_2)$$

(6) $W(G_2) \simeq D_{12}$: $(n = 6,\ m = 3)$

$$\Phi(T_{s_1}) = \frac{u-1}{2}.1 + \frac{u+1}{2} s_1 + \frac{(u^{1/2}-1)^2}{3} (-s_2+s_1s_2-s_2s_1 + s_1s_2s_1) +$$

$$+ \frac{(u^{1/2}-1)^2}{6} (-s_2s_1s_2 + s_1s_2s_1s_2 - s_2s_1s_2s_1 + s_1s_2s_1s_2s_1)$$

$$\Phi(T_{s_2}) = \frac{u-1}{2} 1 + \frac{u+1}{2} s_2 + \frac{(u^{1/2}-1)^2}{3} (-s_1 + s_2s_1 - s_1s_2 + s_2s_1s_2) +$$

$$+ \frac{(u^{1/2}-1)^2}{6} (-s_1s_2s_1 + s_2s_1s_2s_1 - s_1s_2s_1s_2 + s_2s_1s_2s_1s_2).$$

<u>Remark</u>: We note that when n is even, the images $\Phi(T_{s_i})$, $i = 1,2$ do not involve all the elements of the group D_{2n}. In fact, in the images of $\Phi(T_{s_i})$ in the examples above, when n is even the elements w_0 and s_iw_0 do not appear at all.

CHAPTER 3

The centre of the generic Hecke algebra

Let (W,S) be a finite Coxeter system and let H be the generic Hecke algebra over the polynomial ring $\mathbb{Q}[u]$, associated to (W,S). Let Z(H) be the centre of H.

The following result is rather well known. We include a proof for the sake of completeness.

Lemma 3.1: Let $c = \sum_{w \in W} a_w T_w$ be an element of H. Then c lies in the centre of H if and only if the following two conditions hold:

$C_1 : a_{sws} = a_w$ if $\ell(sws) = \ell(w)$, $s \in S$, and

$C_2 : a_w = ua_{sws} - (u-1)a_{sw}$ if $\ell(sws) = \ell(w) + 2$, $s \in S$.

Proof: It is clear that c lies in the centre of H if and only if $cT_s = T_s c$ $\forall s \in S$. We fix a $s \in S$ and we denote by $C_W(s)$ the centralizer of s in W. We can write

$$c = \sum_{\substack{\ell(sw)>\ell(w)\\ \ell(ws)>\ell(w)\\ w \notin C_W(s)}} a_w T_w + \sum_{\substack{\ell(sw)>\ell(w)\\ \ell(ws)<\ell(w)\\ w \notin C_W(s)}} a_w T_w + \sum_{\substack{\ell(sw)<\ell(w)\\ \ell(ws)>\ell(w)\\ w \notin C_W(s)}} a_w T_w + \sum_{\substack{\ell(sw)<\ell(w)\\ \ell(ws)<\ell(w)\\ w \notin C_W(s)}} a_w T_w + \sum_{w \in C_W(s)} a_w T_w$$

Now $cT_s = T_s c$ if and only if

$$\sum_{\substack{\ell(sw)>\ell(w)\\ \ell(ws)>\ell(w)\\ w\notin C_W(s)}} a_w T_{ws} + \sum_{\substack{\ell(sw)>\ell(w)\\ \ell(ws)<\ell(w)\\ w\notin C_W(s)}} a_w(u\, T_{ws} + (u-1)T_w) + \sum_{\substack{\ell(sw)<\ell(w)\\ \ell(ws)>\ell(w)\\ w\notin C_W(s)}} a_w T_{ws} +$$

$$+ \sum_{\substack{\ell(sw)<\ell(w)\\ \ell(ws)<\ell(w),\ w\notin C_W(s)}} a_w (u\, T_{ws} + (u-1)\, T_w) =$$

$$= \sum_{\substack{\ell(sw)>\ell(w)\\ \ell(ws)>\ell(w)\\ w\notin C_W(s)}} a_w T_{sw} + \sum_{\substack{\ell(sw)>\ell(w)\\ \ell(ws)<\ell(w)\\ w\notin C_W(s)}} a_w T_{sw} + \sum_{\substack{\ell(sw)<\ell(w)\\ \ell(ws)>\ell(w)\\ w\notin C_W(s)}} a_w(u\, T_{sw} + (u-1)T_w) +$$

$$+ \sum_{\substack{\ell(sw)<\ell(w)\\ \ell(ws)<\ell(w),\ w\notin C_W(s)}} a_w (u\, T_{sw} + (u-1)T_w).$$

This condition is equivalent to

$$\sum_{\substack{\ell(sws)=\ell(w)+2\\ w\notin C_W(s)}} a_w T_{ws} + \sum_{\substack{\ell(sws)=\ell(w)+2\\ w\notin C_W(s)}} a_{ws} (u\, T_w + (u-1)T_{ws}) + \sum_{\substack{\ell(sws)<\ell(ws)\\ \ell(ws)<\ell(w)\\ w\notin C_W(s)}} a_{ws} T_w$$

$$+ \sum_{\substack{\ell(sws)=\ell(w)+2 \\ w \notin C_W(s)}} a_{sws} \; (u\,T_{sw} + (u-1)T_{sws}) \; =$$

$$\sum_{\substack{\ell(sws)=\ell(w)+2 \\ w \notin C_W(s)}} a_w\, T_{sw} \; + \sum_{\substack{\ell(sw)<\ell(w) \\ \ell(sws)<\ell(sw) \\ w \notin C_W(s)}} a_{sw}\; T_w + \sum_{\substack{\ell(sws)=\ell(w)+2 \\ w \notin C_W(s)}} a_{sw}(uT_w+(u-1)T_{sw}) + \sum_{\substack{\ell(sws)=\ell(w)+2 \\ w \notin C_W(s)}} a_{sws}(u\,T_{ws}+(u-1)T_{sws})$$

We now compare coefficients of the basis elements T_w, $w \in W$ on both sides of the relation $cT_s = T_s c$, to obtain

$$a_w = u\, a_{sws} - (u-1)a_{ws} \quad \text{if } \ell(sws) = \ell(w) + 2$$

$$a_w = u\, a_{sws} - (u-1)a_{sw} \quad \text{if } \ell(sws) = \ell(w) + 2$$

$$ua_{ws} = ua_{sw} \text{ if } \ell(sws) = \ell(w) + 2,$$

and for every $w \in \{w : w \notin C_W(s) \text{ with } \ell(sws) < \ell(ws) \text{ and } \ell(ws) < \ell(w)\}$ $= \{w : w \notin C_W(s) \text{ with } \ell(sw)<\ell(w) \text{ and } \ell(sws)<\ell(sw)\}$, we have

$$a_{ws} = a_{sw}.$$

In the latter case, by replacing w by sw we obtain that $a_w = a_{sws}$ for w such that $\ell(sw)>\ell(w)$ and $\ell(ws)<\ell(w)$, and by replacing w by ws, we also obtain that $a_w = a_{sws}$ for w such that $\ell(sw) < \ell(w)$ and $\ell(ws) > \ell(w)$. Thus $a_w = a_{sws}$ for w such that $\ell(sws) = \ell(w)$.

Our lemma is now proved.

The following ideas are due to R.W. Carter.

<u>Definition 3.1</u>: Let $w, w' \in W$. We say that w, w' are strongly conjugate if there exists a sequence $x_1,\dots,x_n$, $x_i \in W$, $i = 1,\dots,n$ with $w = x_1,\dots,x_n = w'$ such that for all i, $x_{i+1} = sx_is$ for some $s \in S$ with $\ell(x_{i+1}) = \ell(x_i)$.

Thus, if $c = \sum_{w \in W} a_w T_w \in Z(H)$ and if w, w' are strongly conjugate then condition C_1 implies $a_w = a_{w'}$.

The relation of being strongly conjugate is an equivalence relation inside each conjugacy class and so each conjugacy class is a disjoint union of strong conjugacy classes.

If C is a strong conjugacy class we write a_C for the coefficient a_w for some $w \in C$. Thus, if $c = \sum_w a_w T_w \in Z(H)$, we have $c = \sum_C a_C T_C$, the summation being taken over all strong conjugacy classes C, with $T_C = \sum_{w \in C} T_w$.

Let C, C' be two strong conjugacy classes inside a given conjugacy class. We say that C' covers C if, $\exists\, w \in C$, $s \in S$ such that $sws \in C'$ with $\ell(sws) = \ell(w) + 2$.

Suppose that C' covers C. Let D be a strong conjugacy class. We say that D is an intermediate class for the pair (C,C') if $\exists\, w \in C$, $s \in S$ such that $sw \in D$, and $sws \in C'$. It is clear that D belongs to a different conjugacy class from the given one which contains C and C'. If C' covers C, and D is an intermediate class for (C',C), then condition C_2 gives $a_C = ua_{C'} - (u-1)a_D$.

We define a partial ordering on strong conjugacy classes by saying that $C < C'$ if there exists a sequence of strong conjugacy classes $\{C_1,\dots,C_k\}$

with $C = C_1,\ldots,C_k = C'$ such that C_{i+1} covers C_i $\forall\ 1 \le i \le k-1$. We define the length of a strong conjugacy class $\ell(C) = \ell(w)$ for all $w \in C$. By C_M we denote a strong conjugacy class of maximal length inside a given conjugacy class. Thus every strong conjugacy class C can be joined to a C_M by means of a sequence of strong conjugacy classes $C_1,\ldots,C_\lambda$, with $C_1 = C$, $C_\lambda = C_M$ and with the property that either C_i covers C_{i+1} or C_{i+1} covers C_i $\forall 1 \le i \le \lambda - 1$.

Therefore we can always express the coefficient a_C in terms of a_{C_M} for some maximal strong conjugacy class C_M and in fact $a_C = u^e a_{C_M}$ + linear combination of other $a_{C'}$, $e \in \mathbb{Z}$. Nevertheless this can be done in many different ways.

Remarks: (i) If w, w' are strongly conjugate, then $a_w = a_{w'}$. The converse is not true. For instance if s, s' $\in$ S and s, s are inside the same conjugacy class, then clearly s, s' are not strongly conjugate. However $a_s = a_{s'}$. For s, s' are conjugate if and only if, there exists a sequence $(s_1,\ldots,s_q)$ with $s_1 = s$, $s_q = s'$ such that $s_i s_{i+1}$ has finite odd order $\forall\ 1 \le i \le q-1$. Now, if $s_i s_{i+1}$ has odd order $n_i = 2\theta + 1$, then by repeated application of the condition C_2 and using the fact that the elements $(s_i s_{i+1})^\lambda$, $(s_{i+1} s_i)^\lambda$ are strongly conjugate for every $\lambda = 1,\ldots,\theta$, we obtain $a_{s_i} = a_{s_i+1}$ $\forall\ i = 1,\ldots,q-1$. Thus $a_s = a_{s'}$. (ii) It is not true that inside a given conjugacy class, the elements of maximal length are strongly conjugate.

For instance, when $W = S_6$ the elements (34)(1526), (34) (1625), (16) (2435), (16) (2534), are conjugate and they have the same length 14, which is maximal length for their conjugacy class. However, they fall into two strong conjugacy classes namely

$C_{M_1} = \{(34)(1526), (34)(1625)\}$, and $C_{M_2} = \{(16)(2435),$

$(16)(2534)\}$.

When $W = D_{2n}$, the theory above enables us to find a natural basis for the centre of the generic Hecke algebra.

Proposition 3.1: (i) Let $n = 2m+1$, and let $H(D_{2n})$ be the generic Hecke algebra over the polynomial ring $\mathbb{Q}[u]$. Then, a basis for the centre $Z(H)$ is given by the following set

$$T_1,\ T_{(s_1s_2)^k} + T_{(s_2s_1)^k} - (u-1)\sum_{\lambda=1}^{k} u^{\lambda-1}[T_{(s_1s_2)^{k-\lambda}s_1} + T_{(s_2s_1)^{k-\lambda}s_2}],$$

$$T_{w_0} + \sum_{\lambda=1}^{m} u^{\lambda}\,[T_{(s_1s_2)^{m-\lambda}s_1} + T_{(s_2s_1)^{m-\lambda}s_2}],\quad 1 \le k \le m$$

(ii) Let $n = 2m$. Then, if m is even, a basis for the centre of $H(D_{2n})$ is given by the set

$$T_1,\ T_{(s_1s_2)^k} + T_{(s_2s_1)^k} - (u-1)\sum_{\lambda=1}^{k} u^{\lambda-1}[T_{(s_1s_2)^{k-\lambda}s_2} + T_{(s_2s_1)^{k-\lambda}s_2}],$$

$$T_{(s_1s_2)^{m-1}s_1} + \dots + u^{m-1}\,T_{s_2},\ T_{(s_2s_1)^{m-1}s_2} + \dots + u^{m-1}\,T_{s_1},\ T_{w_0}$$

$1 \le k \le m-1$

and when m is odd by the set

$$T_1 \cdot T_{(s_1s_2)^k} + T_{(s_2s_1)^k} - (u-1) \sum_{\lambda=1}^{k} u^{\lambda-1} [T_{(s_1s_2)^{k-\lambda}s_1} + T_{(s_2s_1)^{k-\lambda}s_2}],$$

$$T_{(s_2s_1)^{m-1}s_2} + \dots + u^{m-1} T_{s_2} \cdot T_{(s_1s_2)^{m-1}s_1} + \dots + u^{m-1} T_{s_1} \cdot T_{w_0}$$

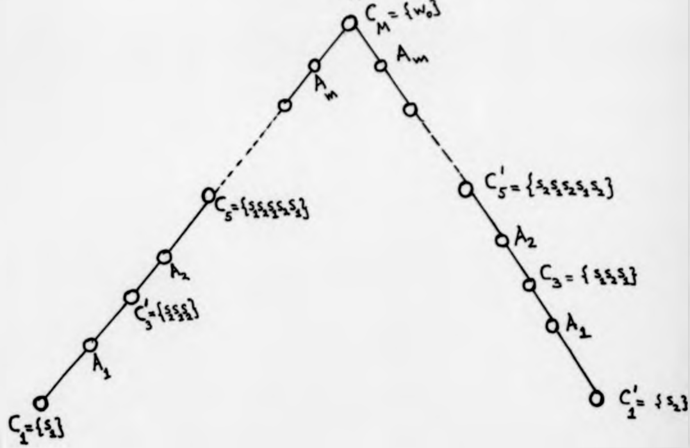
$1 \le k \le m-1.$

<u>Proof</u>: (i) When $n = 2m + 1$, the dimension of the centre of H is $m + 2$. The conjugacy classes of W are given by $A_0 = \{1\}$,

$$A_j = \{(s_1s_2)^j, (s_2s_1)^j\} \quad j = 1,\dots,m, \quad A_{m+1} = \{s_1,s_2,s_1s_2s_1,\dots,w_0\}.$$

Let $c = \sum_w a_w T_w \in Z(H)$. We note that $(s_1s_2)^j$, $(s_2s_1)^j$ are strongly conjugate $\forall\, j = 1,\dots,m$. So each A_j is itself a strong conjugacy class and therefore $a_{(s_1s_2)^j} = a_{(s_2s_1)^j}$, $j = 1,\dots,m$.

The other strong conjugacy classes are: $C_i = \{(s_1s_2)^{i-1}s_1\}$ $i = 1,\dots,m$ $C_i' = \{(s_2s_1)^{i-1}.s_2\}$ $i = 1,\dots,m$, and $C_M = \{w_0\}$. The partial ordering inside the conjugacy class A_{m+1} is given by the graph:



By repeated application of the relation $a_C = ua_{C'} - (u-1)a_D$ whenever C' covers C we obtain: (also using that $a_{(s_1s_2)^j} = a_{(s_2s_1)^j} \ \forall\ 1 \le j \le m$)

$$a_{s_1} = u^m a_{w_0} - \sum_{j=1}^{m} u^{m-j}(u-1)a_{(s_2s_1)^{m-j+1}}$$

$$a_{s_1s_2s_1} = u^{m-1}a_{w_0} - f^{(121)}_{(m)} \quad \text{where } f^{(121)}_{(m)} = \begin{cases} 0 & \text{if } m = 1 \\ \sum_{j=2}^{m} u^{m-j}(u-1)a_{(s_2s_1)^{m-j+2}} & \text{if } m \ge 2 \end{cases}$$

$$a_{s_1s_2s_1s_2s_1} = u^{m-2}a_{w_0} - f^{(12121)}_{(m)} \text{ where } f^{(12121)}_{(m)} = \begin{cases} 0 & \text{if } m \le 2 \\ \sum_{j=1}^{m} u^{m-j}(u-1)a_{(s_2s_1)^{m-j+3}} & \text{if } m \ge 3 \end{cases}$$

. . . .

$$a_{(s_1s_2)^{m-1}\cdot s_1} = u\, a_{w_0} - (u-1)a_{(s_2s_1)^m}$$

By interchanging the role of s_1, s_2 we obtain the coefficients $a_{(s_2s_1)^{i-1}\cdot s_2}$ $\forall\ i = 1,2,\dots,m$. Hence $a_{(s_1s_2)^{i-1}\cdot s_1} = a_{(s_2s_1)^{i-1}\cdot s_2}$ $\forall\ 1 \le i \le m$.

Let $c = \sum_{w \in D_{2n}} a_w T_w \in Z(H)$. Then

$$c = a_1T_1 + a_{s_1}[T_{s_1} + T_{s_2}] + a_{s_1s_2}[T_{s_1s_2} + T_{s_2s_1}] + a_{s_1s_2s_1}[T_{s_1s_2s_1} + T_{s_2s_1s_2}]$$

$$+ \dots + a_{(s_1s_2)^m}[T_{(s_1s_2)^m} + T_{(s_2s_1)^m}] + a_{w_0}T_{w_0} =$$

$$= a_1T_1 + (u^m a_{w_0} - \sum_{j=1}^{m} u^{m-j}(u-1)a_{(s_1s_2)^{m-j+1}})[T_{s_1}+T_{s_2}] + a_{s_1s_2}[T_{s_1s_2} + T_{s_2s_1}] +$$

$$+ (u^{m-1}a_{w_0} - f^{(121)}_{(m)})[T_{s_1s_2s_1} + T_{s_2s_1s_2}] + a_{(s_1s_2)^2}[T_{(s_1s_2)^2} + T_{(s_2s_1)^2}] +$$

$$+ \dots + a_{(s_1s_2)^m}[T_{(s_1s_2)^m} + T_{(s_2s_1)^m}] + a_{w_0}T_{w_0} =$$

$$a_1T_1 + a_{s_1s_2}[T_{s_1s_2} + T_{s_2s_1} - (u-1)(T_{s_1} + T_{s_2})] +$$

$$+ a_{(s_1s_2)^2}[T_{(s_1s_2)^2} + T_{(s_2s_1)^2} - u(u-1)(T_{s_1} + T_{s_2}) - (u-1)(T_{s_1s_2s_1} + T_{s_2s_1s_2})] +$$

$$+ \dots + a_{(s_1s_2)^m}\begin{bmatrix} T_{(s_1s_2)^m} + T_{(s_2s_1)^m} - (u-1)(T_{(s_1s_2)^{m-1}s_1} + T_{(s_2s_1)^{m-1}s_2}) - \\ -u(u-1)(T_{(s_1s_2)^{m-2}s_1} + T_{(s_2s_1)^{m-2}s_2}) - \dots - u^{m-1}(u-1)(T_{s_1} + T_{s_2}) \end{bmatrix}$$

$$+ a_{w_0}[T_{w_0} + u(T_{(s_1s_2)^{m-1}s_1} + T_{(s_2s_1)^{m-1}s_2}) + \dots + u^m(T_{s_1} + T_{s_2})].$$

Therefore part (i) of the proposition is proved.

(ii) Let $W = D_{2n}$, $n = 2m$, and say m is even. In this case, the conjugacy classes are: $A_0 = \{1\}$,

$A_j = \{(s_1s_2)^j, (s_2s_1)^j\}$, $1 \le j \le m-1$,

$A_m = \{s_1, s_2s_1s_2, (s_1s_2)^2s_1, \dots, (s_2s_1)^{m-1}.s_2\}$.

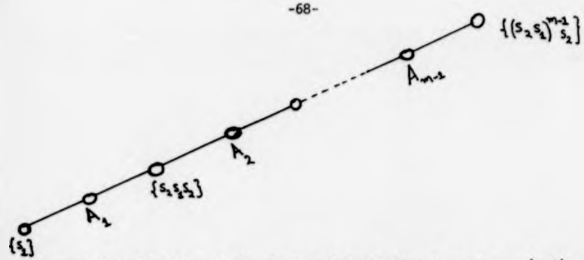
$A_{m+1} = \{s_2, s_1s_2s_1, (s_2s_1)^2s_2, \dots, (s_1s_2)^{m-1}.s_1\}$.

$A_{m+2} = \{w_0\}$.

The dimension of $Z(H)$ is $m + 3$.

Each element inside the conjugacy class A_m forms by itself a strong conjugacy class, and each A_j $1 \le j \le m-1$ is a strong conjugacy class.

The partial ordering inside the conjugacy class A_m is given by the graph:



By repeated application of the fundamental relation $a_C = ua_{C'} - (u-1)a_D$, whenever C' covers C, and using the fact that each A_j is itself a strong conjugacy class, we obtain:

$$a_{s_1} = u^{m-1}\, a_{(s_2s_1)^{m-1}.s_2} - \sum_{j=1}^{m-1} u^{m-j-1}(u-1)a_{(s_1s_2)^{m-j}}$$

$$a_{s_2s_1s_2} = u^{m-2}\, a_{(s_2s_1)^{m-1}.s_2} - f^{(212)}_{(m)} \text{ where } f^{(212)}_{(m)} = \begin{cases} 0 \text{ if } m-1 < 2 \\ \sum_{j=1}^{m-1} u^{m-j-1} a_{(s_1s_2)^{m-j+1}} \\ \quad \text{if } m-1 \geq 3 \end{cases}$$

. . . .

$$a_{(s_1s_2)^{m-2}.s_1} = u\, a_{(s_2s_1)^{m-1}.s_2} - (u-1)a_{(s_1s_2)^{m-1}}$$

By interchanging the role of s_1, s_2 we obtain similar relations from the partial ordering on strong conjugacy classes inside the conjugacy class A_{m+1}, for the coefficients a_w, $w \in A_{m+1}$. Now if $c = \sum_{w \in D_{2n}} a_w T_w \in Z(H)$, then we replace the coefficients a_w for $w \in A_m \cup A_{m+1}$ by the corresponding relations which express any a_w in terms of $a_{(s_2s_1)^{m-1}s_2}$ or $a_{(s_1s_2)^{m-1}s_1}$. By gathering coefficients together in the expression $\sum_{w \in D_{2n}} a_w T_w$, we obtain a basis for Z(H) of the desired form.

When m is odd, then the conjugacy classes are:

$$A_0 = \{1\},\ A_j,\ 1 \le j \le m-1,\ \ A_m = \{s_1, s_2s_1s_2, \ldots, (s_1s_2)^{m-1}s_1\}\ ,$$

$$A_{m+1} = \{s_2, s_1s_2s_1, \ldots, (s_2s_1)^{m-1}s_2\},\ A_{m+2} = \{w_0\}.$$

With a similar argument we obtain a basis for Z(H), and our proposition is now proved.

We note that under the specialization $u \to 1$, the basis above specializes to the class sums.

Let $W = S_n$ the symmetric group and let H be the generic Hecke algebra of symmetric type over the polynomial ring $\mathbb{Q}[u,u^{-1}]$. James and Dipper defined elements called generalized Murphy operators by

$$L_0 = T_1,$$

$$L_1 = \frac{1}{u} T_{s_1}$$

$$L_2 = \frac{1}{u} T_{s_2} + \frac{1}{u^2} T_{s_1s_2s_1}$$

$$\vdots$$

$$L_{n-1} = \frac{1}{u} T_{s_{n-1}} + \frac{1}{u^2} T_{s_{n-2}s_{n-1}s_{n-2}} + \cdots + \frac{1}{u^{n-1}} T_{s_1\ldots s_{n-1}\ldots s_1}\ .$$

By specializing $u \to 1$ we obtain the standard Murphy operators (See [13]).

James and Dipper proved (see [8]) the following:

(i) $L_0, L_1, \ldots, L_{n-1}$ commute with each other

(ii) The algebra they generate contains the centre of H. Using Murphy's construction for the centre of the group algebra of the symmetric group (see [13]) they showed that the centre of H consists of the symmetric polynomials in the $L_0, L_1, \ldots, L_{n-1}$.

Moreover they showed that for any partition λ of $n, \lambda \vdash n$ say $\lambda = (\lambda_1, \lambda_2, \ldots)$ there exists a unique basis up to a scalar multiple of the centre of H say $\{y^{(\lambda)}, \lambda \vdash n\}$ with the following properties:

(i) The coefficients of the T_w,s involved in $y^{(\lambda)}$ belong to $\mathbb{Z}[u]$

(ii) $y^{(\lambda)}$ involves T_{u_λ}, where $u_\lambda = (1\ 2\ \ldots\ \lambda_1)\ (\lambda_1 + 1\ \ldots\ \lambda_1 + \lambda_2)\ \ldots$ and does not involve T_{u_μ} for $\mu \neq \lambda$.

(iii) The coefficient of T_{u_λ} is a power of u.

In fact their construction implies that $y^{(\lambda)}$ does not involve any other element T_w with $\ell(w)$ minimal and w belongs to a different conjugacy class from the one determined by the partition λ, and that the basis $\{y^{(\lambda)}, \lambda \vdash n\}$ specializes to the class sums under the specialization $u \to 1$.

<u>Examples</u>: $W = S_3$, $s_1 = (12)$, $s_2 = (23)$.

The conjugacy classes of W are:

$\{1\}$, $\{s_1s_2, s_2s_1\}$, $\{s_1, s_2, s_1s_2s_1\}$.

$\lambda = 3 \quad u_\lambda = (123) = s_1s_2$

$\lambda = 21 \quad u_\lambda = (12)\,(3) = s_1$

$\lambda = 111 \; u_\lambda = (1)\,(2)\,(3) = 1$

A Dipper-James basis is given by the set of elements

$$\{T_1,\ u(T_{s_1} + T_{s_2}) + T_{s_1s_2s_1},\ u(T_{s_1s_2} + T_{s_2s_1}) + (u-1)T_{s_1s_2s_1}\}$$

$W = S_4,\quad s_1 = (12),\quad s_2 = (23),\ s_3 = (34)$

The conjugacy classes of W are:

(4): $\{s_1s_3s_2,\ s_2s_1s_3,\ s_3s_2s_1,\ s_1s_2s_3,\ s_1s_2s_1s_3s_2,\ s_2s_3s_1s_2s_1\}$

(31): $\{s_1s_2,\ s_2s_1,\ s_2s_3,\ s_3s_2,\ s_1s_3s_2s_1,\ s_1s_2s_1s_3,\ s_1s_2s_3s_2,\ s_2s_3s_2s_1\}$

(22): $\{s_1s_3,\ s_2s_1s_3s_2,\ s_1s_2s_1s_3s_2s_1$

(211): $\{s_1,\ s_2,\ s_3,\ s_1s_2s_1,\ s_2s_3s_2,\ s_1s_2s_3s_2s_1\}$

(1111): $\{1\}$.

$\lambda = 4 \qquad u_\lambda = (1234) = s_1s_2s_3$

$\lambda = 31 \qquad u_\lambda = (123)(4) = s_1s_2$

$\lambda = 22 \qquad u_\lambda = (12)(34) = s_1s_3$

$\lambda = 211 \qquad u_\lambda = (12)(3)(4) = s_1$

$\lambda = 1111 \qquad u_\lambda = (1)(2)(3)(4) = 1.$

A Dipper-James basis is given by the set of elements $\{y^{(\lambda)}, \lambda \models 4\}$

$$y^{(4)} = u^3(T_{s_1s_2s_3} + T_{s_3s_2s_1} + T_{s_1s_3s_2} + T_{s_2s_1s_3}) +$$

$$+ (u^3-u^2+u)(T_{s_1s_2s_1s_3s_2} + T_{s_2s_3s_1s_2s_1}) + (2u^2-u-1)T_{s_1s_2s_1s_3s_2s_1} +$$

$$+ u^2(u-1)(T_{s_1s_2s_1s_3} + T_{s_2s_3s_2s_1} + T_{s_1s_2s_3s_2} + T_{s_1s_3s_2s_1} + T_{s_2s_1s_3s_2})$$

$$+ u(u-1)^2\, T_{s_1s_2s_3s_2s_1}$$

$$y^{(31)} = u^3(T_{s_1s_2} + T_{s_2s_1} + T_{s_2s_3} + T_{s_3s_2}) +$$

$$+ u^2(T_{s_1s_3s_2s_1} + T_{s_1s_2s_3s_2} + T_{s_2s_3s_2s_1} + T_{s_1s_2s_1s_3}) + u^2(u-1)(T_{s_1s_2s_1} + T_{s_2s_3s_2})$$

$$+ 2u(u-1)T_{s_1s_2s_3s_2s_1} + u(u-1)(T_{s_1s_2s_1s_3s_2} + T_{s_2s_3s_1s_2s_1}) +$$

$$+ (u-1)^2\, T_{s_1s_2s_1s_3s_2s_1}$$

$$y^{(22)} = u^2 T_{s_1s_3} + u\,T_{s_2s_1s_3s_2} + T_{s_1s_2s_1s_3s_2s_1}$$

$$y^{(211)} = u^2(T_{s_1} + T_{s_2} + T_{s_3}) + u(T_{s_1s_2s_1} + T_{s_2s_3s_2}) + T_{s_1s_2s_3s_2s_1}$$

$$y^{(1111)} = T_1.$$

We next provide an alternative basis for the centre of $H(S_4)$ based on the partial ordering on strong conjugacy classes.

The strong conjugacy classes inside S_4 are: (indexed by the conjugacy classes inside which they occur)

$\{1\} = C_{1^4}$

$\{s_1\} = C_{21^2,A}$, $\{s_2\} = C_{21^2,B}$, $\{s_3\} = C_{21^2,C}$

$\{s_1s_2, s_2s_1\} = C_{31,A}$, $\{s_2s_3, s_3s_2\} = C_{31,B}$,

$\{s_1s_3\} = C_{2^2,A}$, $\{s_1s_2s_1\} = C_{21^2,D}$, $\{s_2s_3s_2\} = C_{21^2,E}$,

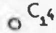
$\{s_1s_3s_2, s_2s_1s_3, s_3s_2s_1, s_1s_2s_3\} = C_{4,A}$

$\{s_2s_3s_2s_1, s_1s_2s_3s_2, s_1s_3s_2s_1, s_1s_2s_1s_3\} = C_{31,C}$

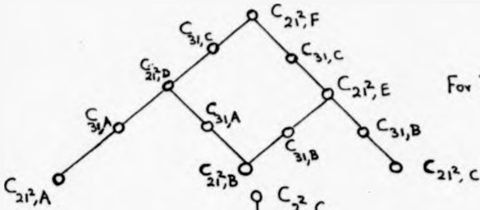
$\{s_2s_1s_3s_2\} = C_{2^2,B}$, $\{s_1s_2s_3s_2s_1\} = C_{21^2,F}$

$\{s_1s_2s_1s_3s_2, s_2s_1s_3s_2s_1\} = C_{4,B}$, $\{s_1s_2s_1s_3s_2s_1\} = C_{2^2,C}$

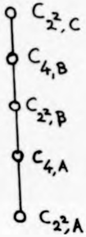
The partial ordering on the strong conjugacy classes is given by:



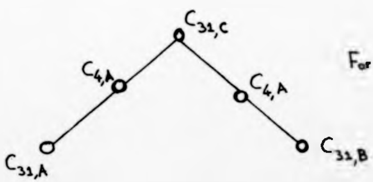
For the conjugacy class(1



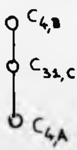
For the conjugacy class (21^2)



For the conjugacy class (2^2)



For the conjugacy class (31)



For the conjugacy class (4)

A basis for the centre of $H(S_4)$ is given by the set of elements

$$v^{(4)} = T_{s_1s_2s_1s_3s_2} + T_{s_2s_1s_3s_2s_1} + u(T_{s_1s_3s_2} + T_{s_2s_1s_3} + T_{s_1s_2s_3} + T_{s_3s_2s_1}) -$$

$$-(u-1)T_{s_2s_1s_3s_2} - 2u(u-1)T_{s_1s_3} - u(u-1)(T_{s_1s_2} + T_{s_2s_1} + T_{s_2s_3} + T_{s_3s_2}) +$$

$$+ u(u-1)^2 (T_{s_1} + T_{s_2} + T_{s_3})$$

$$v^{(31)} = T_{s_1s_3s_2s_1} + T_{s_1s_2s_1s_3} + T_{s_1s_2s_3s_2} + T_{s_2s_3s_2s_1} +$$

$$+ (u^2-u+1)(T_{s_1s_2} + T_{s_2s_1} + T_{s_2s_3} + T_{s_3s_2}) -$$

$$-(u-1)(T_{s_1s_3s_2} + T_{s_2s_1s_3} + T_{s_1s_2s_3} + T_{s_3s_2s_1}) + (u-1)^2 T_{s_1s_3} -$$

$$-(u-1)(T_{s_1s_2s_1} + T_{s_2s_3s_2}) - (u^3-u^2+u-1)(T_{s_1} + T_{s_2} + T_{s_3})$$

$$v^{(22)} = T_{s_1s_2s_1s_3s_2s_1} + u\, T_{s_2s_1s_3s_2} + u^2\, T_{s_1s_3}$$

$$v^{(211)} = T_{s_1s_2s_3s_2s_1} + u(T_{s_1s_2s_1} + T_{s_2s_3s_2}) + u^2(T_{s_1} + T_{s_2} + T_{s_3})$$

$$v^{(1111)} = T_1.$$

<u>Remark</u>: In contrast with the Dipper-James basis given above, this basis has the property that for each partition $\lambda \vdash 4$, the element $v^{(\lambda)}$ has the form:

$$\sum_{\ell(w) = \text{maximal inside the conjugacy class determined by } \lambda} T_w$$

+ linear combination of T_w's where T_w is not of maximal length in any other conjugacy class different from the one determined by λ.

CHAPTER 4

The determination of Lusztig's isomorphism on the centre Z(H) of the generic Hecke algebra of dihedral type

In Chapter 3, we found a basis for the centre Z(H) of the generic Hecke algebra of dihedral type (see Proposition 3.1). In this chapter, we shall determine the images of the basis elements under the Lusztig isomorphism

The canonical basis of $H(D_{2n})$ is given by:

$$\{T_1,\ T_{(s_1s_2)^k},\ T_{(s_2s_1)^k},\ T_{(s_1s_2)^\lambda s_1},\ T_{(s_2s_1)^\lambda s_2},\ T_{w_0}\}\ ,$$

where $k = 1,\dots,m$ if $n = 2m+1$ and $k = 1,\dots,m-1$ if $n = 2m$, and $\lambda = 0,1,\dots,m-1$. In §2.3 we achieved a decomposition of the graded module grad(E) of dihedral type over $Q_n(u^{\frac{1}{2}})$ into a direct sum of left H-irreducible submodules, for both cases $n = 2m+1$ and $n = 2m$, by means of the Chebyshev polynomials of the second kind.

To these decompositions we adapt the basis $\{\bar{e}_1,\ u_j',\ v_j,\ \hat{u}_j',\ \hat{v}_j,\ 1 \le j \le m,\ \bar{e}_{w_0}\}$, for the case $n = 2m+1$ and $\{\bar{e}_1, \theta_1, \theta_2,\ u_j',\ v_j,\ \hat{u}_j',\ \hat{v}_j,\ 1 \le j \le m-1,\ \bar{e}_{w_0}\}$, for the case $n = 2m$. (See also §2.3, for the definition of the basis elements in both cases n odd and n even).

We first determine the matrices which represent each element of the canonical basis of $H(D_{2n})$ on each direct summand of the decompositions mentioned above with respect to the corresponding basis. We concentrate on the 2-dimensional summands whose number is 2m if $n = 2m + 1$, and $2m-2$ if $n = 2m$.

The terminology blocks of the first kind and blocks of the second kind is as in the proof of Theorem 2.3.1, and will be adopted here, for the elements of the canonical basis of $H(D_{2n})$.

Lemma 4.1 (i). The blocks of the first kind which represent $T_{(s_1 s_2)^k}$, $k = 1,\ldots,m$, are given by:

$$X_j^{(u)} = \begin{bmatrix} u^k[S_k(\rho_j) + S_{k-1}(\rho_j)] & -2u^{2k-1/2} \cos \frac{j\pi}{n} S_{k-1}(\rho_j) \\ 2u^{2k+1/2} \cos \frac{j\pi}{n} S_{k-1}(\rho_j) & -u^k[S_{k-1}(\rho_j) + S_{k-2}(\rho_j)] \end{bmatrix}$$

where $\rho_j = 2 \cos \frac{2j\pi}{n}$, $j = 1,\ldots,m$ if $n = 2m+1$, or $j=1,\ldots,m-1$ if $n = 2m$, and the polynomials $S_k(x)$ are defined in §2.2.

The blocks of the second kind which represent $T_{(\bar{s}_1 \bar{s}_2)^k}$, $k = 1,\ldots,m$, are given by:

$$Y_j^{(u)} \begin{bmatrix} -u^k[S_{k-1}(\rho_j) + S_{k-2}(\rho_j)] & 2u^{2k+1/2} \cos \frac{j\pi}{n} S_{k-1}(\rho_j) \\ -2u^{2k-1/2} \cos \frac{j\pi}{n} S_{k-1}(\rho_j) & u^k[S_k(\rho_j) + S_{k-1}(\rho_j)] \end{bmatrix}$$

$j = 1,\ldots,m$ if $n = 2m+1$, or $j = 1,\ldots,m-1$ if $n = 2m$.

(ii) The blocks of the first kind which represent $T_{(s_2 s_1)^k}$, $k = 1,\ldots,m$ if $n = 2m+1$, or $k = 1,\ldots,m-1$ if $n = 2m$, are given by the $Y_j^{(u)}$, and the blocks of the second kind by the $X_j^{(u)}$.

(iii) The blocks of the first kind which represent the element $T_{(s_1s_2)^k s_1}$ $k = 0,1,\ldots,m$ if $n = 2m+1$ or $k = 0,1,\ldots,m-1$ if $n = 2m$ are given by:

$$Z_j^{(u)} = \begin{bmatrix} -u^k[S_k(\rho_j) + S_{k-1}(\rho_j)] & 2u^{2k+1/2} \cos \frac{j\pi}{n} S_k(\rho_j) \\ -2u^{2k+1/2} \cos \frac{j\pi}{n} S_{k-1}(\rho_j) & u^{k+1}[S_k(\rho_j) + S_{k-1}(\rho_j)] \end{bmatrix}$$

$j = 1,\ldots,m$ if $n = 2m+1$ or $j = 1,\ldots,m-1$ if $n = 2m$.

The blocks of the second kind which represent $T_{(s_1s_2)^k s_1}$ are given by

$$W_j^{(u)} = \begin{bmatrix} u^{k+1}[S_k(\rho_j) + S_{k-1}(\rho_j)] & -2u^{2k+1/2} \cos \frac{j\pi}{n} S_{k-1}(\rho_j) \\ 2u^{2k+1/2} \cos \frac{j\pi}{n} S_k(\rho_j) & -u^k[S_k(\rho_j) + S_{k-1}(\rho_j)] \end{bmatrix}$$

$j = 1,\ldots,m$ if $n = 2m+1$, or $j = 1,\ldots,m-1$ if $n = 2m$.

(iv) The blocks of the first kind which represent the element $T_{(s_2s_1)^k s_2}$ $k = 0,1,\ldots,m-1$ are the $W_j^{(u)}$ and the blocks of the second kind are the $Z_j^{(u)}$.

Proof: We concentrate on the blocks of the first kind. For $k = 1$, the result is true since $T_{s_1} \to \begin{bmatrix} -1 & 2u^{\frac{1}{2}} \cos \frac{j\pi}{n} \\ 0 & u \end{bmatrix}$, and $T_{s_2} \to \begin{bmatrix} u & 0 \\ 2u^{\frac{1}{2}} \cos \frac{j\pi}{n} & -1 \end{bmatrix}$

Therefore $T_{s_1 s_2} \rightarrow \begin{bmatrix} u[4\cos^2\frac{j\pi}{n} - 1] & -2u^{\frac{1}{2}}\cos\frac{j\pi}{n} \\ 2u^{3/2}\cos\frac{j\pi}{n} & -u \end{bmatrix}$

where $4\cos^2\frac{j\pi}{n} - 1 = (2+\rho_j) - 1 = 1 + \rho_j = S_0(\rho_j) + S_1(\rho_j)$.

Assume by induction that the blocks of the first kind, which represent $T_{(s_1 s_2)^\lambda}$ are of the form:

$$\begin{bmatrix} u^\lambda[S_\lambda(\rho_j) + S_{\lambda-1}(\rho_j)] & -2u^{2\lambda-1/2}\cos\frac{j\pi}{n} S_{\lambda-1}(\rho_j) \\ 2u^{2\lambda+1/2}\cos\frac{j\pi}{n} S_{\lambda-1}(\rho_j) & -u^\lambda[S_{\lambda-1}(\rho_j) + S_{\lambda-2}(\rho_j)] \end{bmatrix}$$

Then $T_{(s_1 s_2)^{\lambda+1}} = T_{(s_1 s_2)^\lambda} T_{s_1 s_2}$ is represented by

$$\begin{bmatrix} u^\lambda[S_\lambda(\rho_j) + S_{\lambda-1}(\rho_j)] & -2u^{2\lambda-1/2}\cos\frac{j\pi}{n} S_{\lambda-1}(\rho_j) \\ 2u^{2\lambda+1/2}\cos\frac{j\pi}{n} S_{\lambda-1}(\rho_j) & -u^\lambda[S_{\lambda-1}(\rho_j)+S_{\lambda-2}(\rho_j)] \end{bmatrix} \begin{bmatrix} u(1+\rho_j) & -2u^{\frac{1}{2}}\cos\frac{j\pi}{n} \\ 2u^{3/2}\cos\frac{j\pi}{n} & -u \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}^{(u)} & a_{12}^{(u)} \\ a_{21}^{(u)} & a_{22}^{(u)} \end{bmatrix}$$, where

$$a_{11}^{(u)} = u^{\lambda+1}(1 + \rho_j)[S_\lambda(\rho_j) + S_{\lambda-1}(\rho_j)] - 4u^{\lambda+1} \cos^2 \frac{j\pi}{n} S_{\lambda-1}(\rho_j) =$$

$$= u^{\lambda+1}(1 + \rho_j)[S_\lambda(\rho_j) + S_{\lambda-1}(\rho_j)] - u^{\lambda+1}(2 + \rho_j)S_{\lambda-1}(\rho_j) =$$

$$= u^{\lambda+1}[S_\lambda(\rho_j) + S_{\lambda-1}(\rho_j) + \rho_j S_\lambda(\rho_j)+\rho_j S_{\lambda-1}(\rho_j) - 2S_{\lambda-1}(\rho_j)-\rho_j S_{\lambda-1}(\rho_j)] =$$

$$= u^{\lambda+1}[\rho_j S_\lambda(\rho_j) - S_{\lambda-1}(\rho_j) + S_\lambda(\rho_j)] = u^{\lambda+1}[S_{\lambda+1}(\rho_j) + S_\lambda(\rho_j)].$$

$$a_{12}^{(u)} = - 2u^{2\lambda+1/2} \cos \frac{j\pi}{n} [S_\lambda(\rho_j) + S_{\lambda-1}(\rho_j)] + 2u^{2\lambda+1/2} \cos \frac{j\pi}{n} S_{\lambda-1}(\rho_j) =$$

$$= - 2u^{2\lambda+1/2} \cos \frac{j\pi}{n} S_\lambda(\rho_j) \quad .$$

$$a_{21}^{(u)} = 2u^{2\lambda+3/2} \cos \frac{j\pi}{n} (1+\rho_j)S_{\lambda-1}(\rho_j) - 2u^{2\lambda+3/2} \cos \frac{j\pi}{n} [S_{\lambda-1}(\rho_j) + S_{\lambda-2}(\rho_j)]$$

$$= 2u^{2\lambda+3/2} \cos \frac{j\pi}{n} [\rho_j S_{\lambda-1}(\rho_j) - S_{\lambda-2}(\rho_j)] = 2u^{2\lambda+3/2} \cos \frac{j\pi}{n} S_\lambda(\rho_j) \quad .$$

$$a_{22}^{(u)} = - 4 \cos^2 \frac{j\pi}{n} u^{\lambda+1} S_{\lambda-1}(\rho_j) + u^{\lambda+1}[S_{\lambda-1}(\rho_j) + S_{\lambda-2}(\rho_j)] =$$

$$= - (2+\rho_j)u^{\lambda+1}S_{\lambda-1}(\rho_j) + u^{\lambda+1}[S_{\lambda-1}(\rho_j) + S_{\lambda-2}(\rho_j)] =$$

$$= -u^{\lambda+1}[\rho_j S_{\lambda-1}(\rho_j) - S_{\lambda-2}(\rho_j)+S_{\lambda-1}(\rho_j)] = -u^{\lambda+1}[S_\lambda(\rho_j) + S_{\lambda-1}(\rho_j)].$$

Thus the blocks of first kind which represent the element $T_{(s_1 s_2)^{\lambda+1}}$ have also the required form. To obtain the matrices $Y_j^{(u)}$ we conjugate the matrices $X_j^{(u)}$ by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Part (ii) can be proved in a similar way. Finally matrix multiplication by T_{s_1} and T_{s_2} gives (iii) and (iv) respectively, so our lemma is now proved.

Remark (4.1). We note that the blocks of the first kind, which represent $T_{(s_1s_2)^k}$ and $T_{(s_2s_1)^k}$ are mutually obtained from one another by conjugation by the matrix $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. The situation is similar for the elements $T_{(s_1s_2)^k s_1}$ and $T_{(s_2s_1)^k s_2}$.

We shall now distinguish between two cases.

First case: $n = 2m + 1$.

In this case, a basis of the centre $Z(H)$ of the generic Hecke algebra is given by the following set of elements:

$$v_0 = T_1,\ v_k = T_{(s_1s_2)^k} + T_{(s_2s_1)^k} - (u-1)\sum_{\lambda=1}^{k} u^{\lambda-1}[T_{(s_1s_2)^{k-\lambda}s_1} + T_{(s_2s_1)^{k-\lambda}s_2}],$$

$k = 1,\dots,m,$

$$v_{m+1} = T_{w_0} + \sum_{\lambda=1}^{m} u^{\lambda}\,[T_{(s_1s_2)^{m-\lambda}s_1} + T_{(s_2s_1)^{m-\lambda}s_2}].$$

We recall the decomposition of the graded moudle into a direct sum of irreducible left H-submodules. This decomposition is,

$\mathrm{grad}(E) = M_0 \bigoplus_{j=1}^{m} M_j \bigoplus_{j=1}^{m} \hat{M}_j \oplus M_s$, where M_0, M_s afford the 1-dimensional representations $\sigma_0 : T_{s_i} \to u$, $i = 1,2$, and $\sigma_s : T_{s_i} \to -1$, $i = 1,2$ respectively, and M_j is a 2-dimensional H-submodule with basis $\{u_j', v_j\}$, which is isomorphic to the 2-dimensional H-submodule $\hat{M}_j$ which has basis $\{\hat{u}_j', \hat{v}_j\}$, for every $j = 1,\dots,m$. Our first step is to establish the action of the basis elements of the centre on the irreducible submodules appearing in the decomposition of grad(E), with respect to the above basis.

It is well known that each element of the centre is represented on an irreducible constituent of dimension say d, by a scalar multiple of the identity matrix I_d.

The elements v_k is represented on M_0 by:

$$2u^{2k} - (u-1)\sum_{\lambda=1}^{k} u^{\lambda-1}\, 2u^{2(k-\lambda)+1} = 2u^k,\ k = 1,\dots,m,$$

and it is also represented on M_s by:

$$2 - (u-1)\sum_{\lambda=1}^{k} u^{\lambda-1}(-2) = 2u^k,\ k = 1,\dots,m.$$

The element v_{m+1} is represented on M_0 by:

$$u^{2m+1} + \sum_{\lambda=1}^{m} u^{\lambda}\, 2\, u^{2(m-\lambda)+1} = u^{2m+1} + 2\sum_{\lambda=1}^{m} u^{2m-\lambda+1} =$$

$= u^{2m+1} + 2u^{2m} + 2u^{2m-1} + \dots + 2u^{m+1}$, and on M_s by:

$$-1 - 2\sum_{\lambda=1}^{m} u^{\lambda} = -2u^m - 2u^{m-1} - \dots - 2u - 1.$$

Let $\mu_j^{(k)} \times I_2$ be the matrix which represents v_k on the 2-dimensional constituent M_j, $j = 1,\dots,m$ ($\hat{\mu}_j^{(k)} \times I_2$ the corresponding matrix on $\hat{M}_j$). We shall determine $\mu_j^{(k)}$. We write v_k as:

$$v_k = T_{(s_1s_2)^k} + T_{(s_2s_1)^k} - (u-1)\sum_{\ell=0}^{k-1} u^{k-(\ell+1)}[T_{(s_1s_2)^{\ell}s_1} + T_{(s_2s_1)^{\ell}s_2}] .$$

We recall the matrices which represent the elements T_w, $w \in D_{2n}$ on the blocks of the first kind (see Lemma 4.1) and we concentrate on the diagonal entries of these matrices.

It turns out that the diagonal entries of the matrix which represents $T_{(s_1s_2)^k} + T_{(s_2s_1)^k}$ have the form $u^k[S_k(\rho_j) - S_{k-2}(\rho_j)]$ and the diagonal entries of the matrix which represents $-(u-1)\sum_{\ell=0}^{k-1} u^{k-(\ell+1)}[T_{(s_1 s_2)^{\ell} s_1} + T_{(s_2 s_1)^{\ell} s_2}]$ have the form $-u^{k-1}(u-1)^2 \sum_{\ell=0}^{k-1} [S_\ell(\rho_j) + S_{\ell-1}(\rho_j)]$. Therefore

$$\mu_j^{(k)} = u^k[S_k(\rho_j)-S_{k-2}(\rho_j)]-u^{k-1}(u-1)^2[S_{k-1}(\rho_j)+2S_{k-2}(\rho_j)+\ldots+2S_1(\rho_j)+2S_0(\rho_j)].$$

The matrix $\underline{\mu}_j^{(k)} \times I_2$ is obtained by conjugating $\mu_j^{(k)} \times I_2$ by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and so $\underline{\mu}_j^{(k)} = \mu_j^{(k)}$.

Similarly let $\mu_j^{(m+1)} \times I_2$ be the matrix which represents v_{m+1} on M_j, $1 \le j \le m$ (or $\underline{\mu}_j^{(m+1)} \times I_2$ the corresponding matrix on $\hat{M}_j$). With a similar argument, and taking into account that $S_m(\rho_j) + S_{m-1}(\rho_j) = 0$, we find that

$$\mu_j^{(m+1)} = \underline{\mu}_j^{(m+1)} = u^m(u-1)[S_{m-1}(\rho_j) + 2S_{m-2}(\rho_j) + \ldots + 2S_1(\rho_j) + 2S_0(\rho_j)].$$

We shall now simplify the expressions $\mu_j^{(k)}, \mu_j^{(m+1)}$, $1 \le j \le m$.
In order to do so, we introduce certain families of polynomials. The Chebyshev polynomials of the first kind $T_k(x)$ are defined by:
$T_0(x) = 1$, $T_1(x) = x$, $T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x)$, $\forall k \ge 1$ and they have the property that for every θ, $T_k(\cos\theta) = \cos k\theta$. Clearly $T_k(1) = 1$ and so $T_k(x) - 1$ is divisible by $x-1$. The Chebyshev polynomials of the second kind have been already defined by $S_0(x) = 1$, $S_1(x) = x$, $S_{k+1}(x) = xS_k(x)-S_{k-1}(x)$ $\forall k \ge 1$, and they have the property that $S_k(2\cos\theta) = \frac{\sin(k+1)\theta}{\sin\theta}$ [see [1], pp.776-8
Define $V_k(x)$ by $V_0(x) = 2$, $V_1(x) = x$, $V_k(x) = S_k(x) - S_{k-2}(x)$ $\forall k \ge 2$.
We can easily show by induction that $V_k(x) = 2T_k(x/2)$. Define

$$R_k(x) = \frac{T_{k+1}(x) - 1}{x-1} \quad \text{for every } k \ge 0.$$

Let $\theta_j = \frac{2j\pi}{n}$, and let $\hat{\Delta}_k^{(j)} = S_{k-1}(2\cos\theta_j)+2S_{k-2}(2\cos\theta_j)+\ldots+2S_0(2\cos\theta_j)$, $1 \le j \le m$, $1 \le k \le m$. Then

Lemma 4.2: (i) $V_k(2\cos\theta_j) = 2\cos k\theta_j \quad 1 \le k \le m, \ 1 \le j \le m$

(ii) $$\hat{\Delta}_k^{(j)} = \sum_{\mu=1}^{k-1} 2\mu\cos(k-\mu)\theta_j + k, \ 1 \le j \le m, \ 1 \le k \le m.$$

Proof: (i) This follows from the fact that $V_k(x) = 2T_k(x/2)$ and $T_k(\cos\theta_j) = \cos k\theta_j$.

(ii) From the definition of $R_k(x)$ and using the fact that $T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x)$, we can easily show that $R_{k+1}(x) = 2xR_k(x) - R_{k-1}(x) + 2$, for every k.

By induction, we can also prove that (using also the fact that $S_{k+1}(x) = x\,S_k(x) - S_{k-1}(x)$)

$$R_{k-1}(x) = S_{k-1}(2x) + 2S_{k-2}(2x) + \ldots + 2S_1(2x) + 2S_0(2x), \ 1 \le k \le m.$$

Therefore,

$$\hat{\Delta}_k^j = R_{k-1}(\cos\theta_j) = \frac{1-\cos(k\theta_j)}{1-\cos\theta_j}$$

Moreover $(1-\cos\theta_j)(\sum_{\mu=1}^{k-1} 2\mu\cos(k-\mu)\theta_j + k) =$

$$\sum_{\mu=1}^{k-1} 2\mu\cos(k-\mu)\theta_j + k - k\cos\theta_j - \sum_{\mu=1}^{k-1} 2\mu\cos(k-\mu)\theta_j\cos\theta_j =$$

$$\sum_{\mu=1}^{k-1} 2\mu\cos(k-\mu)\theta_j + k - k\cos\theta_j - \sum_{\mu=1}^{k-1} \mu\cos(k-\mu+1)\theta_j - \sum_{\mu=1}^{k-1} \mu\cos(k-\mu-1)\theta_j =$$

$= 1 - \cos(k\theta_j)$. Our lemma is now proved.

We next show the central character table of the generic Hecke algebra of dihedral type for the case $n = 2m + 1$.

	V_0	V_1	V_2	...	V_m	V_{m+1}
M_0	1	$2u$	$2u^2$	...	$2u^m$	$u^{2m+1}+2u^{2m}+2u^{2m-1}+\ldots+2u^{m+1}$
				...		
M_s	1	$2u$	$2u^2$	...	$2u^m$	$-2u^m-2u^{m-1}-\ldots-2u-1$
				...		
M_1	1	$u\Delta_1^1-(u-1)^2\hat{\Delta}_1^1$	$u^2\Delta_2^1-u(u-1)^2\hat{\Delta}_2^1$	...	$u^m\Delta_m^1-u^{m-1}(u-1)^2\hat{\Delta}_m^1$	$u^m(u-1)\hat{\Delta}_m^1$
				...		
M_2	1	$u\Delta_1^2-(u-1)^2\hat{\Delta}_1^2$	$u^2\Delta_2^2-u(u-1)^2\hat{\Delta}_2^2$	...	$u^m\Delta_m^2-u^{m-1}(u-1)^2\hat{\Delta}_m^2$	$u^m(u-1)\hat{\Delta}_m^2$
				...		
M_3	1	$u\Delta_1^3-(u-1)^2\hat{\Delta}_1^3$	$u^2\Delta_2^3-u(u-1)^2\hat{\Delta}_2^3$	...	$u^m\Delta_m^3-u^{m-1}(u-1)^2\hat{\Delta}_m^3$	$u^m(u-1)\hat{\Delta}_m^3$
⋮	⋮	⋮	⋮	...	⋮	⋮
M_m	1	$u\Delta_1^m-(u-1)^2\hat{\Delta}_1^m$	$u^2\Delta_2^m-u(u-1)^2\hat{\Delta}_2^m$	...	$u^m\Delta_m^m-u^{m-1}(u-1)^2\hat{\Delta}_m^m$	$u^m(u-1)\hat{\Delta}_m^m$

In the table above, M_0, M_s, M_j, $1 \le j \le m$ is a full set of left irreducible H-modules, $\{v_0, v_k, 1 \le k \le m, v_{m+1}\}$ is a basis of the centre of the generic Hecke algebra,

$$\Delta_k^j = 2\cos\frac{2kj\pi}{n}, \quad 1 \le k \le m, \quad 1 \le j \le m,$$

$$\hat{\Delta}_k^j = R_{k-1}(\cos\theta_j), \quad 1 \le k \le m, \quad 1 \le j \le m.$$

The entries of this table, represent the scalars according to which the basis elements of the centre of H act on the irreducible constituents.

By specializing $u \to 1$, we obtain the central character table of the dihedral group D_{2n}, $n = 2m + 1$.

	$[v_0]$	$[v_1]$	$[v_2]$	. . . .	$[v_m]$	$[v_{m+1}]$
M_0	1	2	2	. . . .	2	n
M_s	1	2	2		2	-n
M_1	1	$2\cos\frac{2\pi}{n}$	$2\cos\frac{4\pi}{n}$		$2\cos\frac{2m\pi}{n}$	0
M_2	1	$2\cos\frac{4\pi}{n}$	$2\cos\frac{8\pi}{n}$		$2\cos\frac{4m\pi}{n}$	0
M_3	1	$2\cos\frac{6\pi}{n}$	$2\cos\frac{12\pi}{n}$		$2\cos\frac{6m\pi}{n}$	0
⋮		⋮	⋮		⋮	⋮
M_m	1	$2\cos\frac{2m\pi}{n}$	$2\cos\frac{4m\pi}{n}$		$2\cos\frac{2m^2\pi}{n}$	0

Notation: $[v_0]$, $[v_k]$, $k=1,\ldots,m$, $[v_{m+1}]$, denote the class sums, and the entries of the table represent the scalars, according to which, the class sums act on the irreducible constituents. We now recall the following auxiliary result (see [7], page 213).

Let G be any finite group with s conjugacy classes K_j, $1 \le j \le s$, and let $Z = (X_j^{(i)})$ be the character table of the group G. Let $C = (c_j^i)$ be the central character matrix, where $c_j^i = \frac{h_j X_j^{(i)}}{d_i}$, d_i being the degree of $X^{(i)}$, h_j the cardinality of the conjugacy class K_j, $i,j \in \{1,\ldots,s\}$. Then Z is an invertible matrix with inverse matrix $\bar{Z}$ whose (i,j) entry is $\frac{h_i\ \bar{X}_i^{(j)}}{|G|}$, $\bar{X}_i^{(j)}$ being the complex conjugate of $X_i^{(j)}$.

In our case all characters are real valued. We have two characters of degree 1 namely $\chi^{(1)}$, $\chi^{(2)}$ and m characters of degree 2. With the aid of the orthogonality relations we can verify that the matrix C determined by the central character table of the group W is invertible with inverse matrix $\bar{C}$ where

$$\bar{C} = \left[\begin{array}{c|c|c|c|cc}
\frac{1}{2n} & \frac{1}{2n} & \frac{2}{n} & \frac{2}{n} & & \frac{2}{n} \\
\frac{1}{2n} & \frac{1}{2n} & \frac{2}{n}\cos\frac{2\pi}{n} & \frac{2}{n}\cos\frac{4\pi}{n} & \cdots\cdots & \frac{2}{n}\cos\frac{2m\pi}{n} \\
\frac{1}{2n} & \frac{1}{2n} & \frac{2}{n}\cos\frac{4\pi}{n} & \frac{2}{n}\cos\frac{8\pi}{n} & \cdots\cdots & \frac{2}{n}\cos\frac{4m\pi}{n} \\
\frac{1}{2n} & \frac{1}{2n} & \frac{2}{n}\cos\frac{6\pi}{n} & \frac{2}{n}\cos\frac{12\pi}{n} & \cdots\cdots & \frac{2}{n}\cos\frac{6m\pi}{n} \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
\frac{1}{2n} & \frac{1}{2n} & \frac{2}{n}\cos\frac{2m\pi}{n} & \frac{2}{n}\cos\frac{4m\pi}{n} & \cdots\cdots & \frac{2}{n}\cos\frac{2m^2\pi}{n} \\
\frac{1}{2n} & -\frac{1}{2n} & 0 & 0 & & 0
\end{array}\right]$$

Every entry of the i^{th} column $1 \le i \le m+2$ of the matrix $\bar{C}$ is given by the corresponding entry in the i^{th} row of the character table of the group D_{2n} multiplied by the d_i ($d_i = 2$, $3 \le i \le m+2$) and divided by the order of the group.

Now if z is a typical element of the basis $(v_0, v_k, 1 \le k \le m, v_{m+1})$ of the centre of H, then z acts on every irreducible constituent of the graded module according to the information given by the central character table of H. If ϕ is Lusztig's isomorphism, then $\phi(z)$ is a certain linear combination of the class sums, and each class sum acts on the irreducible constituents according to the information provided by the central character table of the group D_{2n}, $n = 2m + 1$.

Furthermore z and $\Phi(z)$ act in the same way on the graded module and therefore the coefficients appearing in the expression of $\Phi(z)$ as linear combination of the class sums are polynomials in the indeterminate u with rational coefficients.

We shall determine these coefficients explicitly.

In fact if $A = (a_{ir})$, $i, r \in \{1,\dots,m+2\}$ is the matrix describing Lusztig's isomorphism on the centre of H with respect to the basis $\{v_0, v_k, 1 \le k \le m, v_{m+1}\}$, then $A = \bar{C}\Lambda$, where $\bar{C}$ is the inverse of the central character matrix C of the group D_{2n} and Λ is the matrix determined by the central character table of H.

We now compute the entries a_{ir}, $i, r \in \{1,2,\dots,m+2\}$. We shall make use of the trigonometric identities:

$2 \cos a \cos b = \cos(a + b) + \cos(a - b)$ and

$\frac{1}{2} + \cos x + \cos 2x + \dots + \cos px = \frac{1}{2} \; \frac{\sin(p+\frac{1}{2})x}{\sin \frac{x}{2}}$.

Let $i = 2$, $r = 1$. Then $a_{11} = \frac{1}{2n} + \frac{1}{2n} + m\,\frac{2}{n} = \frac{2m+1}{n} = 1$. If $i \in \{2,3,\dots,m+1\}$ and $r = 1$, then $a_{ir} = \frac{1}{2n} + \frac{1}{2n} + \frac{2}{n} \sum_{j=1}^{m} \cos\,(i-1)\,\frac{2j\pi}{n} = \frac{1}{n} + \frac{2}{n}\,(-\frac{1}{2}) = 0$.

If $i = m+2$ and $r < m+2$ then $a_{ir} = 0$.

If $i = m+2$ and $r = m+2$, then $a_{ir} = \frac{1}{2n}\,(u+1)(u^{n-1} + \dots + u+1)$.

If $i = 1$ $r \in \{2,3,\dots,m+1\}$, then

$$a_{ir} = \frac{2u^{r-1}}{2n} + \frac{2u^{r-1}}{2n} + \frac{2u^{r-1}}{n} \sum_{j=1}^{m} \Delta^j_{r-1} - \frac{2}{n}\,u^{r-2}(u-1)^2 \sum_{j=1}^{m} \hat{\Delta}^j_{r-1} \cdot$$

$$\Delta^j_{r-1} = 2 \cos \frac{2(r-1)j\pi}{n}, \text{ thus } \sum_{j=1}^{m} \Delta^j_{r-1} = 2(-\tfrac{1}{2}) = -1 .$$

$$\hat{\Delta}^j_{r-1} = 2 \cos(r-2)\frac{2j\pi}{n} + 4 \cos(r-3)\frac{2j\pi}{n} + \dots + (2r-4)\cos \frac{2j\pi}{n} + (r-1), \text{ hence}$$

$$\sum_{j=1}^{m} \hat{\Delta}^{j}_{r-1} = 2\sum_{j=1}^{m}\cos(r-2)\frac{2j\pi}{n} + 4\sum_{j=1}^{m}\cos(r-3)\frac{2j\pi}{n}+\ldots+(2r-4)\sum_{j=1}^{m}\cos\frac{2j\pi}{n} + m(r-1)$$

$$= (2+4+\ldots+2r-4)\left(-\frac{1}{2}\right) + m(r-1) = m(r-1) - \frac{(r-1)(r-2)}{2} = \frac{(r-1)(n-r+1)}{2}.$$

Thus $a_{ir} = -\dfrac{u^{r-2}(u-1)^2(r-1)(n-r+1)}{n}$, $i = 1$, $r \in \{2,3,\ldots,m+1\}$.

Let $i = 1$, $r = m+2$. Then

$$a_{1m+2} = \frac{1}{2n}[u^{2m+1} + 2u^{2m}+\ldots+ 2u^{m+1}-2u^{m}-2u^{m-1}-\ldots-2u-1] + \frac{2}{n}u^{m}(u-1)\sum_{j=1}^{m}\hat{\Delta}^{j}_{m}$$

Now $\sum_{j=1}^{m}\hat{\Delta}^{j}_{m} = (2+4+\ldots+2m-2)\left(-\frac{1}{2}\right) + m^2 = m^2 - \dfrac{m(m-1)}{2} = \dfrac{m(m+1)}{2}$.

So $a_{1m+2} = \dfrac{1}{2n}[u^{2m+1}+2u^{2m}+\ldots+2u^{m+1}-2u^{m}-2u^{m-1}-\ldots-2u-1 + 2m(m+1)u^{m}(u-1)] =$

$$= \frac{1}{2n}(u-1)\left[\sum_{\lambda=0}^{m-1}(2\lambda+1)u^{2m-\lambda} + (n+2m(m+1))u^{m} + \sum_{\lambda=1}^{m}(2m-2\lambda+1)u^{m-\lambda}\right].$$

If $i > 1$, $r \in \{2,3,\ldots,m+1\}$, $i = r$, then

$$a_{1r} = \frac{2u^{r-1}}{2n} + \frac{2u^{r-1}}{2n} + \frac{2u^{r-1}}{n}\sum_{j=1}^{m} 2\cos\frac{2(r-1)j\pi}{n}\cdot\cos\frac{(r-1)2j\pi}{n} -$$

$$-\frac{2}{n}u^{r-2}(u-1)^2\sum_{j=1}^{m}\cos\frac{2(r-1)j\pi}{n}\cdot\hat{\Delta}^{j}_{r-1}.$$

Now $\sum_{j=1}^{m} 2\cos\dfrac{2(r-1)j\pi}{n}\cdot\cos\dfrac{2(r-1)j\pi}{n} = \sum_{j=1}^{m}\cos\dfrac{4(r-1)j\pi}{n} + m = m - \dfrac{1}{2} = \dfrac{2m-1}{2}$.

Also $\sum_{j=1}^{m}\cos\dfrac{2(r-1)j\pi}{n}\hat{\Delta}^{j}_{r-1} =$

$$= \sum_{j=1}^{m}\cos\frac{2(r-1)j\pi}{n}\left[2\cos(r-2)\frac{2j\pi}{n} + 4\cos(r-3)\frac{2j\pi}{n}+\ldots+(2r-4)\cos\frac{2j\pi}{n} + (r-1)\right]$$

$$= \sum_{j=1}^{m}\left[\sum_{\lambda=1}^{r-2}\lambda\left[\cos(2r-\lambda-2)\frac{2j\pi}{n} + \cos\lambda\frac{2j\pi}{n}\right]\right] + (r-1)\sum_{j=1}^{m}\cos\frac{(r-1)2j\pi}{n}$$

$$= -(1+2+3+\ldots+r-2) - \frac{(r-1)}{2} = -\frac{(r-1)(r-2)}{2} - \frac{(r-1)}{2} =$$

$$= -\frac{(r-1)^2}{2}.$$

Hence $a_{ir} = \frac{2u^{r-1}}{n} + \frac{2u^{r-1}}{n}\left(\frac{2m-1}{2}\right) + u^{r-2}(u-1)^2\frac{(r-1)^2}{n} =$

$= u^{r-1} + \frac{(r-1)^2}{n}u^{r-2}(u-1)^2$, $i > 1$, $r \in \{2,3,\ldots,m+1\}$, $i = r$.

Let $i > 1$, $i \neq m+2$, $r = m+2$. Then $a_{im+2} =$

$= \frac{1}{2n}[u^{2m+1} + 2u^{2m} + \ldots + 2u^{m+1} - 2u^m - 2u^{m-1} - \ldots - 2u-1] +$

$+ \frac{2}{n}u^m(u-1)\sum_{j=1}^{m}\cos(i-1)\frac{2j\pi}{n}\hat{\Delta}_m^j.$

Now $\cos(i-1)\frac{2j\pi}{n}\hat{\Delta}_m^j = \cos(i-1)\frac{2j\pi}{n}\left[\sum_{k=1}^{m}(2k)\cos(m-k)\frac{2j\pi}{n} + m\right].$

We can have $i - 1 = m-k$ if $k = m - i + 1$. Thus,

$$\cos(i-1)\frac{2j\pi}{n}\hat{\Delta}_m^j = \cos(i-1)\frac{2j\pi}{n}\left[\sum_{k=1}^{m-i}(2k)\cos(m-k)\frac{2j\pi}{n} + 2(m-i+1)\cos(i-1)\frac{2j\pi}{n} + \sum_{k=m-i+2}^{m-1}(2k)\cos(m-k)\frac{2j\pi}{n} + m\right]$$

$$= \sum_{k=1}^{m-i} k\cdot\left[\cos(i+m-k-1)\frac{2j\pi}{n} + \cos(m-i-k+1)\frac{2j\pi}{n}\right] +$$

$$+ (m-i+1)\left[1 + \cos\frac{4(i-1)j\pi}{n}\right] + \sum_{k=m-i+2}^{m-1} k\left[\cos(i+m-k-1)\frac{2j\pi}{n} + \cos(m-i-k+1)\frac{2j\pi}{n}\right] +$$

$+ m\cos(i-1)\frac{2j\pi}{n}$. Thus, $\sum_{j=1}^{m}\cos(i-1)\frac{2j\pi}{n}\hat{\Delta}_m^j =$

$$= -\sum_{k=1}^{m-i} k + (m-i+1)\left(m - \frac{1}{2}\right) - \sum_{k=m-i+2}^{m-1} k + m\left(-\frac{1}{2}\right)$$

$$= -\sum_{k=1}^{m-1} k + (m-i+1) + \frac{(m-i+1)(2m-1)}{2} - \frac{m}{2} =$$

$$= -\frac{m(m-1)}{2} + \frac{(m-i+1)}{2}\cdot n - \frac{m}{2} = \frac{1}{2}[n(m-i+1)-m^2].$$

Hence a_{im+2} =

$$= \frac{1}{2n}(u-1)[u^{2m}+3u^{2m-1}+5u^{2m-2}+\ldots+(2m-1)u^{m+1}+(n+2n(m-i+1)-2m^2)u^m+(2m-1)u^{m-1}+\ldots+3u+1]$$

$$= \frac{1}{2n}(u-1)[\sum_{\lambda=0}^{m-1}(2\lambda+1)u^{2m-\lambda}+(n+2n(m-i+1)-2m^2)u^m+\sum_{\lambda=1}^{m}(2m-2\lambda+1)u^{m-\lambda}]$$

with $i > 1$, $i \neq m+2$.

Finally let $i > 1$, $r \in \{2,3,\ldots,m+1\}$ $i \neq r$. Then,

$$a_{ir} = \frac{2u^{r-1}}{2n} + \frac{2u^{r-1}}{2n} + \frac{2u^{r-1}}{n}\sum_{j=1}^{m} 2\cos\frac{(i-1)2j\pi}{n}\cdot\cos(r-1)\frac{2j\pi}{n} -$$

$$- \frac{2}{n}u^{r-2}(u-1)^2\sum_{j=1}^{m}\cos(i-1)\frac{2j\pi}{n}\cdot\hat{\Delta}^j_{r-1}.$$

Now $\sum_{j=1}^{m} 2\cos(i-1)\frac{2j\pi}{n}\cos(r-1)\frac{2j\pi}{n} = -\frac{1}{2}-\frac{1}{2} = -1$.

$$\sum_{j=1}^{m}\cos(i-1)\frac{2j\pi}{n}\hat{\Delta}^j_{r-1} = \sum_{j=1}^{m}\cos(i-1)\frac{2j\pi}{n}[\sum_{\lambda=1}^{r-2}(2\lambda)\cos(r-(\lambda+1))\frac{2j\pi}{n}+(r-1)].$$

So, if $i > r$ then $\cos(i-1)\frac{2j\pi}{n} \neq \cos(r-\lambda-1)\frac{2j\pi}{n}$ $\forall\ \lambda = 1,\ldots,r-2$, because $i-1 = r-\lambda-1$ if $r-i > 0$. Thus when $i > r$, then

$$\sum_{j=1}^{m}\cos(i-1)\frac{2j\pi}{n}\hat{\Delta}^j_{r-1} =$$

$$= \sum_{j=1}^{m}[\sum_{\lambda=1}^{r-2}\lambda.\cos(r+i-\lambda-2)\frac{2j\pi}{n}] + \sum_{j=1}^{m}[\sum_{\lambda=1}^{r-2}\lambda\cos\frac{(i-r+\lambda)2j\pi}{n}] + (r-1)\sum_{j=1}^{m}\cos\frac{(i-1)2j\pi}{n}$$

$$= -(1+2+\ldots+r-2)+(r-1)(-\frac{1}{2}) = -\frac{(r-1)(r-2)}{2}-\frac{(r-1)}{2} = -\frac{(r-1)^2}{2}.$$

Hence $a_{ir} = u^{r-2}(u-1)^2\frac{(r-1)^2}{n}$ $i > r$, $i \neq r$.

If $r > i$, then $\cos(r-\lambda-1)\frac{2j\pi}{n} = \cos(i-1)\frac{2j\pi}{n}$ for $\lambda = r-i$.

Now $\cos(i-1)\frac{2j\pi}{n}\,\tilde{\Delta}^{j}_{r-1} =$

$$= \cos(i-1)\frac{2j\pi}{n}\left[\sum_{\lambda=1}^{r-i-1}(2\lambda)\cos(r-\lambda-1)\frac{2j\pi}{n} + 2(r-i)\cos(i-1)\frac{2j\pi}{n} + \sum_{\lambda=r-i+1}^{r-2}(2\lambda)\cos(r-\lambda-1)\frac{2j\pi}{n} + (r-1)\right]$$

So $\sum_{j=1}^{m}\cos(i-1)\frac{2j\pi}{n}\,\tilde{\Delta}^{j}_{r-1} =$

$$= \sum_{j=1}^{m}\Big[\sum_{\lambda=1}^{r-i-1}\lambda\,[\cos(r+i-\lambda-2)\frac{2j\pi}{n} + \cos(r-i-\lambda)\frac{2j\pi}{n}]\Big] +$$

$$+ (r-i)\sum_{j=1}^{m}\Big(1 + \cos\frac{4(i-1)j\pi}{n}\Big) +$$

$$+ \sum_{j=1}^{m}\Big[\sum_{\lambda=r-i+1}^{r-2}\lambda\,[\cos(r+i-\lambda-2)\frac{2j\pi}{n} + \cos(r-i-\lambda)\frac{2j\pi}{n}]\Big] +$$

$$+ (r-1)\sum_{j=1}^{m}\cos\frac{(i-1)\,2j\pi}{n} =$$

$$= -\sum_{\lambda=1}^{r-i-1}\lambda + (r-i)\,(m - \tfrac{1}{2}) - \sum_{\lambda=r-i+1}^{r-2}\lambda \quad - \frac{(r-1)}{2} =$$

$$- \sum_{\lambda=1}^{r-2}\lambda + (r-i) + \frac{(r-i)}{2}(2m-1) - \frac{(r-1)}{2} =$$

$$= -\frac{(r-2)(r-1)}{2} + \frac{(r-i)}{2}\cdot n - \frac{(r-1)}{2} = \frac{(r-i)}{2}\cdot n - \frac{(r-1)^2}{2}\ .$$

Hence, when $i \neq r$, $r > i$, then

$$a_{ir} = -\frac{2}{n}\,u^{r-2}(u-1)^2\Big[\frac{(r-i)}{2}\cdot n - \frac{(r-1)^2}{2}\Big] =$$

$$= u^{r-2}(u-1)^2\Big[\frac{(r-1)^2}{n} - (r-i)\Big].$$

For the convenience of the reader we summarize our calculations in the following matrix.

We put $K(1,m+2) = \sum_{\lambda=0}^{m-1} (2\lambda+1)u^{2m-\lambda} + (n+2m(m+1))u^{m} + \sum_{\lambda=1}^{m} (2m-2\lambda+1)u^{m-\lambda}$,

and $K(i,m+2) = \sum_{\lambda=0}^{m-1} (2\lambda+1)u^{2m-\lambda} + (n+2n(m-i+1)-2m^{2})u^{m} + \sum_{\lambda=1}^{m} (2m-2\lambda+1)u^{m-\lambda}$

for every $i = 2,3,\ldots,m+1$.

	1	$-\frac{1}{n}(u-1)^2(n-1)$	$-\frac{2}{n}u(u-1)^2(n-2)$	$-\frac{3}{n}u^2(u-1)^2(n-3)$	$-\frac{m}{n}u^{m-1}(u-1)^2(n-m)$	$\frac{1}{2n}(u-1)K(1,m+2)$
	0	$u+\frac{1}{n}(u-1)^2$	$\longrightarrow$	$u^{r-2}(u-1)^2[\frac{(r-1)^2}{n}-(r-2)]$, $3 \le r \le m+1$	$\longleftarrow$	$\frac{1}{2n}(u-1)K(2,m+2)$
	0	$\frac{1}{n}(u-1)^2$	$u^2+\frac{4}{n}u(u-1)^2$ $\longrightarrow$	$u^{r-2}(u-1)^2[\frac{(r-1)^2}{n}-(r-3)]$, $4 \le r \le m+1$	$\longleftarrow$	$\frac{1}{2n}(u-1)K(3,m+2)$
	⋮		⋱		.	⋮
ith row $2 \le i \le m+1$	0	$\frac{1}{n}u^{r-2}(u-1)^2(r-1)^2$ $i > r=2,\dots,i-1$		$u^{r-1}+\frac{1}{n}u^{r-2}(u-1)^2(r-1)^2$ $r=2,3,\dots,m+1$	$u^{r-2}(u-1)^2[\frac{(r-1)^2}{n}-(r-i)]$ $i < r=i+1,\dots,m+1$	$\frac{1}{2n}(u-1)K(i,m+2)$
(m+2) row	0	$\longrightarrow$	0	$\longleftarrow$		$\frac{1}{2n}(u+1)[u^{n-1}+\dots+u+1]$

When we specialize $u \to 1$, the matrix above becomes the identity matrix as it should be.

We give some examples to illustrate the situation

(i) $W = D_6$, $n = 3 = 2.1 + 1$, $m = 1$.

A basis of the centre of the generic Hecke algebra H is given by:

$$v_o = T_1,\ v_1 = T_{s_1s_2} + T_{s_2s_1} - (u-1)(T_{s_1} + T_{s_2}),$$

$$v_2 = T_{s_1s_2s_1} + u(T_{s_1} + T_{s_2}).$$

The matrix which describes the Lusztig isomorphism on the centre of $H(D_6)$ with respect to this basis is given by:

$$\begin{bmatrix} 1 & -\frac{2}{3}(u-1)^2 & \frac{1}{6}(u-1)(u^2+7u+1) \\ 0 & u+\frac{1}{3}(u-1)^2 & \frac{1}{6}(u-1)(u^2+u+1) \\ 0 & 0 & \frac{1}{6}(u+1)(u^2+u+1) \end{bmatrix}$$

The information given by this matrix is that:

$$\phi(v_o) = 1,$$

$$\phi(v_1) = -\frac{2}{3}(u-1)^2\cdot 1 + (u + \frac{1}{3}(u-1)^2)(s_1s_2 + s_2s_1)$$

$$\phi(v_2) = \frac{1}{6}(u-1)(u^2+7u+1)\cdot 1 + \frac{1}{6}(u-1)(u^2+u+1)(s_1s_2 + s_2s_1) +$$

$$+ \frac{1}{6}(u+1)(u^2+u+1)(s_1 + s_2 + s_1s_2s_1)$$

(ii) $W = D_{10}$, $n = 5 = 2.2 + 1$, $m = 2$.

A basis of the centre of H is given by:

$v_0 = T_1,\ v_1 = T_{s_1s_2} + T_{s_2s_1} - (u-1)(T_{s_1} + T_{s_2}),$

$v_2 = T_{(s_1s_2)^2} + T_{(s_2s_1)^2} - (u-1)(T_{s_1s_2s_1} + T_{s_2s_1s_2}) - u(u-1)(T_{s_1} + T_{s_2})$

$v_3 = T_{s_1s_2s_1s_2s_1} + u(T_{s_1s_2s_1} + T_{s_2s_1s_2}) + u^2(T_{s_1} + T_{s_2}).$

The matrix which describes the Lusztig isomorphism Φ on the centre of $H(D_{10})$ with respect to this basis is given by:

$$\begin{bmatrix} 1 & -\frac{4}{5}(u-1)^2 & -\frac{6}{5}u(u-1)^2 & \frac{1}{10}(u-1)(u^4+3u^3+17u^2+3u+1) \\ 0 & u+\frac{1}{5}(u-1)^2 & -\frac{1}{5}u(u-1)^2 & \frac{1}{10}(u-1)(u^4+3u^3+7u^2+3u+1) \\ 0 & \frac{1}{5}(u-1)^2 & u^2+\frac{4}{5}u(u-1)^2 & \frac{1}{10}(u-1)(u^4+3u^3-3u^2+3u+1) \\ 0 & 0 & 0 & \frac{1}{10}(u+1)(u^4+u^3+u^2+u+1) \end{bmatrix}$$

Second case $n = 2m$:

When m is even, a basis of the centre of $H(D_{2n})$ is given by:

$v_0 = T_1,\ v_k = T_{(s_1s_2)^k} + T_{(s_2s_1)^k} - (u-1)\sum_{\lambda=1}^{k} u^{\lambda-1}[T_{(s_1s_2)^{k-\lambda}s_1} + T_{(s_2s_1)^{k-\lambda}s_2}],$

$k = 1, 2, \ldots, m-1,$

$v_m = T_{(s_1s_2)^{m-1}s_1} + u\,T_{(s_2s_1)^{m-2}s_2} + \ldots + u^{m-1}T_{s_2}$

$v_{m+1} = T_{(s_2s_1)^{m-1}s_2} + u\,T_{(s_1s_2)^{m-2}s_1} + \ldots + u^{m-1}T_{s_1}$

$v_{m+2} = T_{w_0}$

and when m is odd by: $v_0,\ v_k,\ k = 1,\ldots,m-1,$

$$v_m = T_{(s_1s_2)^{m-1}_{s_1}} + u\,T_{(s_2s_1)^{m-2}_{s_2}} + \dots + u^{m-1}T_{s_1}$$

$$v_{m+1} = T_{(s_2s_1)^{m-1}_{s_2}} + uT_{(s_1s_2)^{m-2}_{s_1}} + \dots + u^{m-1}T_{s_2}$$

$$v_{m+2} = T_{w_0} .$$

We recall the decomposition of the graded module into a direct sum of irreducible left H-submodules. This decomposition is,

$\mathrm{grad}(E) = M_0 \oplus M_s \oplus M_1 \oplus M_2 \bigoplus_{j=1}^{m-1} V_j \bigoplus_{j=1}^{m-1} \hat{V}_j$, where M_0, M_s, M_1, M_2 afford the 1-dimensional representations $\sigma_0 : T_{s_i} \to u$, $i = 1,2$, $\sigma_s : T_{s_i} \to -1$. $\sigma_1 : T_{s_1} \to -1$, $T_{s_2} \to u$, $\sigma_2 : T_{s_1} \to u$, $T_{s_2} \to -1$, respectively.

V_j is a 2-dimensional H-submodule with basis $\{u_j', v_j\}$, which is isomorphic to the 2-dimensional H-submodule $\hat{V}_j$ which has basis $\{\hat{u}_j', \hat{v}_j\}$, $j = 1,\dots,m-1$ (see §2.3 for the definition of u_j', v_j, $\hat{u}_j'$, $\hat{v}_j$, $j = 1,\dots,m-1$).

We first establish the action of the central basis elements on the several irreducible constituents with respect to the corresponding basis adapted to them. For the elements v_k, $k = 1,\dots,m-1$, we already know that on both M_0 and M_s, they are respected by $2u^k$.

If k is odd, then, on the M_1 v_k is respresented by $-(u^{k+1} + u^{k-1})$.

If k is even, then, on M_1 v_k is represented by

$$2u^k - (u-1) \sum_{\substack{\lambda=1 \\ \lambda=\text{odd}}}^{k-1} u^{\lambda-1}[u^{k-\lambda} - u^{k-\lambda+1}] - (u-1) \sum_{\substack{\lambda=2 \\ \lambda=\text{even}}}^{k} u^{\lambda-1}[u^{k-\lambda+1} - u^{k-\lambda}] =$$

$$= 2u^k - (u-1)\Big[\sum_{\substack{\lambda=1 \\ \lambda=\text{odd}}}^{k-1} (u^{k-1} - u^k) + \sum_{\substack{\lambda=2 \\ \lambda=\text{even}}}^{k} (u^k - u^{k-1})\Big] = 2u^k.$$

Similarly we can verify that on M_2, v_k is represented by $-(u^{k+1} + u^{k-1})$ if k is odd, and by $2u^k$ if k is even. The element v_m is represented on M_0 by $u^{2m-1} + u^{2m-2} + \dots + u^m$, and on M_s by $-(u^{m-1} + u^{m-2} + \dots + 1)$, for both cases m even and m odd.

When m is even, the element

$$v_m = \sum_{\substack{\lambda=1\\ \lambda=\text{odd}}}^{m-1} u^{\lambda-1}\, T_{(s_1s_2)^{m-\lambda}s_1} + \sum_{\substack{\lambda=2\\ \lambda=\text{even}}}^{m} u^{\lambda-1}\, T_{(s_2s_1)^{m-\lambda}s_2}.$$ Thus, v_m is represented

on M_1 by $\sum_{\substack{\lambda=1\\ \lambda=\text{odd}}}^{m-1} u^{\lambda-1} u^{m-\lambda} + \sum_{\substack{\lambda=2\\ \lambda=\text{even}}}^{m} u^{\lambda-1} u^{m-\lambda+1} = \frac{m}{2}(u^m + u^{m-1})$ and on

M_2, v_m is represented by $-\frac{m}{2}(u^m + u^{m-1})$. When m is odd then

$$v_m = \sum_{\substack{\lambda=1\\ \lambda=\text{odd}}}^{m} u^{\lambda-1}\, T_{(s_1s_2)^{m-\lambda}s_1} + \sum_{\substack{\lambda=2\\ \lambda=\text{even}}}^{m-1} u^{\lambda-1}\, T_{(s_2s_1)^{m-\lambda}s_2}$$ and now v_m is

represented on M_1 by

$$-\sum_{\substack{\lambda=1\\ \lambda=\text{odd}}}^{m} u^{\lambda-1}\, u^{m-\lambda} - \sum_{\substack{\lambda=2\\ \lambda=\text{even}}}^{m-1} u^{\lambda-1}\, u^{m-\lambda+1} = -\frac{m+1}{2} u^{m-1} - \frac{m-1}{2} u^m$$ and on M_2 by

$$\sum_{\substack{\lambda=1\\ \lambda=\text{odd}}}^{m} u^{\lambda-1} u^{m-\lambda+1} + \sum_{\substack{\lambda=2\\ \lambda=\text{even}}}^{m-1} u^{\lambda-1} u^{m-\lambda} = \frac{m+1}{2} u^m + \frac{m-1}{2} u^{m-1}.$$

The element v_{m+1} is represented on M_0 by $u^{2m-1} + u^{2m-2} + \ldots + u^m$ and on M_s by $-(u^{m-1} + u^{m-2} + \ldots + 1)$, for both cases m even and m odd.

With a similar argument as for the element v_m we can verify that when m is even v_{m+1} is represented on M_1 by $-\frac{m}{2}(u^m + u^{m-1})$ and on M_2 by $\frac{m}{2}(u^m + u^{m-1})$, while when m is odd v_{m+1} is represented by $\frac{m+1}{2}u^m + \frac{m-1}{2}u^{m-1}$ and by $-\frac{m+1}{2}u^{m-1} - \frac{m-1}{2}u^m$ on M_1 and M_2 respectively.

Finally the element $v_{m+2} = T_{w_0}$ is represented by:

u^{2m}, 1, u^m, u^m, on M_0, M_s, M_1, M_2 respectively (when m is even) and by: u^{2m}, 1, $-u^m$, $-u^m$ on the same modules, when m is odd. We next determine the action of the central basis elements on each 2-dimensional irreducible submodule. This has already been done for the elements v_k, $k = 1,\dots,m-1$, with the only difference that now $\rho_j = 2\cos\frac{2j\pi}{n} = 2\cos\frac{j\pi}{m}$, $1 \le j \le m-1$. We also recall that ρ_j are the zeros of the polynomial $S_{m-1}(x)$ (see Lemma 2.2.2(i)).

Assume m is even and write $v_m = \sum_{\substack{i=2\\ i=\text{even}}}^{m} u^{m-1}\, T_{(s_1s_2)^{i-1}s_1} + \sum_{\substack{i=1\\ i=\text{odd}}}^{m-1} u^{m-1} T_{(s_2s_1)^{i-1}s_2}$.

We recall the matrices which represent the elements T_w, $w \in D_{2n}$ on the blocks of the first kind and we concentrate on the diagonal entries. Let $\lambda_m^{(j)} \times I_2$ be the matrix which represents v_m on the 2-dimensional constituent V_j, $1 \le j \le m-1$, ($\lambda_{2m}^{(j)} \times I_2$ the corresponding matrix on $\hat{V}_j$).

It turns out that the diagonal entries of the matrix which represents v_m have the form

$$u^m \sum_{\substack{i=1\\ i=\text{odd}}}^{m-1} [S_{i-1}(\rho_j) + S_{i-2}(\rho_j)] - u^{m-1} \sum_{\substack{i=2\\ i=\text{even}}}^{m} [S_{i-1}(\rho_j) + S_{i-2}(\rho_j)], \text{ and}$$

$$u^m \sum_{\substack{i=2\\ i=\text{even}}}^{m} [S_{i-1}(\rho_j) + S_{i-2}(\rho_j)] - u^{m-1} \sum_{\substack{i=1\\ i=\text{odd}}}^{m-1} [S_{i-1}(\rho_j) + S_{i-2}(\rho_j)]$$

Moreover,

$$u^m \sum_{\substack{i=1\\ i=\text{odd}}}^{m-1} [S_{i-1}(\rho_j) + S_{i-2}(\rho_j)] - u^{m-1} \sum_{\substack{i=2\\ i=\text{even}}}^{m} [S_{i-1}(\rho_j) + S_{i-2}(\rho_j)] =$$

$$= u^m \sum_{\substack{i=2\\ i=\text{even}}}^{m} [S_{i-2}(\rho_j) + S_{i-3}(\rho_j)] - u^{m-1} \sum_{\substack{i=2\\ i=\text{even}}}^{m} [S_{i-1}(\rho_j) + S_{i-2}(\rho_j)] =$$

$$= u^{m-1}(u-1) \sum_{\substack{i=2\\ i=\text{even}}}^{m} S_{i-2}(\rho_j) + u^m \sum_{\substack{i=4\\ i=\text{even}}}^{m} S_{i-3}(\rho_j) - u^{m-1} \sum_{\substack{i=2\\ i=\text{even}}}^{m-2} S_{i-1}(\rho_j)$$

(since $S_{-1}(x) = 0$ and $S_{m-1}(\rho_j) = 0$)

$$= u^{m-1}(u-1)\sum_{\substack{i=2\\ i=\text{even}}}^{m} S_{i-2}(\rho_j) + u^{m-1}(u-1)\sum_{\substack{i=2\\ i=\text{even}}}^{m-2} S_{i-1}(\rho_j) =$$

$$= u^{m-1}(u-1)[S_0(\rho_j) + S_1(\rho_j) + \dots + S_{m-3}(\rho_j) + S_{m-2}(\rho_j)],\ 1 \le j \le m-1.$$

Similarly we can calculate that

$$u^{m}\sum_{\substack{i=2\\ i=\text{even}}}^{m} [S_{i-1}(\rho_j) + S_{i-2}(\rho_j)] - u^{m-1}\sum_{\substack{i=1\\ i=\text{odd}}}^{m-1} [S_{i-1}(\rho_j) + S_{i-2}(\rho_j)]$$

$$= u^{m-1}(u-1)[S_0(\rho_j) + S_1(\rho_j) + \dots + S_{m-3}(\rho_j) + S_{m-2}(\rho_j)]$$

Therefore

$\lambda_m^{(j)} = u^{m-1}(u-1)[S_0(\rho_j) + S_1(\rho_j) + \dots + S_{m-3}(\rho_j) + S_{m-2}(\rho_j)]$, and since $\underline{\lambda}_m^{(j)} \times I_2$ is obtained by conjugating $\lambda_m^{(j)} \times I_2$ by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we also have $\underline{\lambda}_m^{(j)} = \lambda_m^{(j)}$, $1 \le j \le m-1$.

An entirely similar calculation shows that when m is odd

$v_m = \sum_{\substack{i=1\\ i=\text{odd}}}^{m} u^{m-1}\, T_{(s_1s_2)^{i-1}s_1} + \sum_{\substack{i=2\\ i=\text{even}}}^{m-1} u^{m-1}\, T_{(s_2s_1)^{i-1}s_2}$, is represented on V_j and $\hat{V}_j$ by the same multiple of the identity matrix I_2. The same multiple of the identity matrix I_2 appears when we consider the element v_{m+1} in both cases m even and m odd.

Finally by Lemma 4.2 using the fact that $S_{m-1}(\rho_j) = 0$, $1 \le j \le m-1$, we obtain that

$$S_0(\rho_j) + S_1(\rho_j) + \dots + S_{m-2}(\rho_j) = \frac{1}{2}\,\frac{1 - \cos(m\theta_j)}{1 - \cos\theta_j}\ ,\ \theta_j = \frac{j\pi}{m}$$

$$= \begin{bmatrix} 0 \quad \text{if } j = \text{even} \\ \\ \dfrac{1}{1 - \cos\theta_j} \quad \text{if } j = \text{odd} \end{bmatrix} \qquad 1 \le j \le m-1$$

To summarize our calculations, we conclude that the elements v_m and v_{m+1} for both cases m even and m odd, are represented on each 2-dimensional constitutent V_j, $\hat{V}_j$, $1 \le j \le m-1$ by $\lambda_m^{(j)} \times I_2$ and $\lambda_{m+1}^{(j)} \times I_2$ respectively, where

$$\lambda_m^{(j)} = \lambda_{m+1}^{(j)} = \begin{bmatrix} 0 \quad \text{if } j = \text{even} \\ \dfrac{u^{m-1}(u-1)}{1 - \cos\theta_j} \quad \text{if } j = \text{odd.} \end{bmatrix}$$

Finally the matrix which represents T_{w_0} on V_j is (see Lemma 4.1)

$$\begin{bmatrix} u^m[S_m(\rho_j) + S_{m-1}(\rho_j)] & -2u^{2m-1/2}\cos\frac{j\pi}{n} S_{m-1}(\rho_j) \\ 2u^{2m+1/2}\cos\frac{j\pi}{n} S_{m-1}(\rho_j) & -u^m[S_{m-1}(\rho_j) + S_{m-2}(\rho_j)] \end{bmatrix}$$

Using the fact that $S_{m-1}(\rho_j) = 0$ and $S_m(\rho_j) = \rho_j S_{m-1}(\rho_j) - S_{m-2}(\rho_j) = -S_{m-2}(\rho_j)$, we obtain that this matrix above is $\lambda_{w_0}^{(j)} \times I_2$ where $\lambda_{w_0}^{(j)} = -u^m S_{m-2}(\rho_j)$.

Moreover $S_{m-2}(\rho_j) = \dfrac{\sin(m-1)\theta_j}{\sin\theta_j} = -\cos(m\theta_j) = \begin{bmatrix} 1 & \text{if } j = \text{odd} \\ -1 & \text{if } j = \text{even.} \end{bmatrix}$

Therefore $\lambda_{w_0}^{(j)} = \begin{bmatrix} -u^m & \text{if } j = \text{odd} \\ u^m & \text{if } j = \text{even.} \end{bmatrix}$

We next exhibit the central character table of $H(D_{2n})$, $n = 2m$, for both cases m even and m odd.

Central Character Table of $H(D_{2n})$ $n = 2m$, m even

	v_0	v_1	v_2	v_{m-1}	v_m	v_{m+1}	v_{m+2}
M_0	1	$2u$	$2u^2$	$\ldots\ 2u^{m-1}$	$u^{2m-1}+\ldots+u^m$	$u^{2m-1}+\ldots+u^m$	u^{2m}
M_s	1	$2u$	$2u^2$	$\ldots\ 2u^{m-1}$	$-(u^{m-1}+\ldots+u+1)$	$-(u^{m-1}+\ldots+u+1)$	1
M_1	1	$-(u^2+1)$	$2u^2$	$\ldots\ -(u^m+u^{m-2})$	$\frac{m}{2}(u^m+u^{m-1})$	$-\frac{m}{2}(u^m+u^{m-1})$	u^m
M_2	1	$-(u^2+1)$	$2u^2$	$\ldots\ -(u^m+u^{m-2})$	$-\frac{m}{2}(u^m+u^{m-1})$	$\frac{m}{2}(u^m+u^{m-1})$	u^m
V_1	1	$u\Delta_1^1-(u-1)^2\hat{\Delta}_1^1$	$u^2\Delta_2^1-u(u-1)^2\hat{\Delta}_2^1$	$\ldots\ u^{m-1}\Delta_{m-1}^1-u^{m-2}(u-1)^2\hat{\Delta}_{m-1}^1$	$u^{m-1}(u-1)\frac{1}{1-\cos\theta}$	$u^{m-1}(u-1)\frac{1}{1-\cos\theta}$	$-u^m$
V_2	1	$u\Delta_1^2-(u-1)^2\tilde{\Delta}_1^2$	$u^2\Delta_2^2-u(u-1)^2\tilde{\Delta}_2^2$	$\ldots\ u^{m-1}\Delta_{m-1}^2u^{m-2}(u-1)^2\tilde{\Delta}_{m-1}^2$	0	0	u^m
V_3	1	$u\Delta_1^3-(u-1)^2\tilde{\Delta}_1^3$	$u^2\Delta_2^3-u(u-1)^2\tilde{\Delta}_2^3$	$\ldots\ u^{m-1}\Delta_{m-1}^3-u^{m-2}(u-1)^2\tilde{\Delta}_{m-1}^3$	$u^{m-1}(u-1)\frac{1}{1-\cos 3\theta}$	$u^{m-1}(u-1)\frac{1}{1-\cos 3\theta}$	$-u^m$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
V_{m-1}	1	$u\Delta_1^{m-1}-(u-1)^2\tilde{\Delta}_1^{m-1}$	$u^2\Delta_2^{m-1}-u(u-1)^2\tilde{\Delta}_2^{m-1}$	$\ldots\ u^{m-1}\Delta_{m-1}^{m-1}-u^{m-2}(u-1)^2\tilde{\Delta}_{m-1}^{m-1}$	$u^{m-1}(u-1)\frac{1}{1-\cos(m-1)\theta}$	$u^{m-1}(u-1)\frac{1}{1-\cos(m-1)\theta}$	$-u^m$

In the table above $\theta = \frac{\pi}{m}$, $\Delta_k^j = 2\cos\frac{kj\pi}{m}$, $1 \le j \le m-1$, and $\hat{\Delta}_k^j = R_{k-1}(\cos\theta_j)$, $1 \le k \le m-1$, $\theta_j = \frac{j\pi}{m}$. $\{M_0, M_s, M_1, M_2, V_j, 1 \le j \le m-1\}$ is a full set of left irreducible H-modules, $\{v_i, 0 \le i \le m+2\}$ is a basis of the centre of H, and the entries of the table represent the scalars according to which the basis elements of the centre, act on the irreducible constituents.

By specializing $u \to 1$ we obtain the central character table of $W = D_{2n}$, $n = 2m$, m even which determines the central character matrix $C =$

$$\begin{bmatrix}
1 & 2 & 2 & \cdots & 2 & m & m & 1 \\
1 & 2 & 2 & \cdots & 2 & -m & -m & 1 \\
1 & -2 & 2 & \cdots & -2 & m & -m & 1 \\
1 & -2 & 2 & \cdots & -2 & -m & m & 1 \\
1 & 2\cos\frac{\pi}{m} & 2\cos\frac{2\pi}{m} & \cdots & 2\cos\frac{(m-1)\pi}{m} & 0 & 0 & -1 \\
1 & 2\cos\frac{2\pi}{m} & 2\cos\frac{4\pi}{m} & \cdots & 2\cos\frac{2(m-1)\pi}{m} & 0 & 0 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 2\cos\frac{(m-1)\pi}{m} & 2\cos\frac{2(m-1)\pi}{m} & \cdots & 2\cos\frac{(m-1)^2\pi}{m} & 0 & 0 & -1
\end{bmatrix}$$

The inverse of the central character matrix C is given by $\bar{C}$ =

$$
\begin{array}{l}
\\ \\ \\ \\
m^{th}\ \text{row} \rightarrow \\
\\ \\ \\
\end{array}
\left[
\begin{array}{c|c|c|c|c|c|c}
\frac{1}{2n} & \frac{1}{2n} & \frac{1}{2n} & \frac{1}{2n} & \frac{1}{m} & \cdots\cdots & \frac{1}{m} \\[2ex]
\frac{1}{2n} & \frac{1}{2n} & -\frac{1}{2n} & -\frac{1}{2n} & \frac{1}{m}\cos\frac{\pi}{m} & \cdots\cdots & \frac{1}{m}\cos\frac{(m-1)\pi}{m} \\[2ex]
\frac{1}{2n} & \frac{1}{2n} & \frac{1}{2n} & \frac{1}{2n} & \frac{1}{m}\cos\frac{2\pi}{m} & \cdots\cdots & \frac{1}{m}\cos\frac{2(m-1)\pi}{m} \\[2ex]
\frac{1}{2n} & \frac{1}{2n} & -\frac{1}{2n} & -\frac{1}{2n} & \frac{1}{m}\cos\frac{3\pi}{m} & \cdots\cdots & \frac{1}{m}\cos\frac{3(m-1)\pi}{m} \\[2ex]
\frac{1}{2n} & \frac{1}{2n} & -\frac{1}{2n} & -\frac{1}{2n} & \frac{1}{m}\cos\frac{(m-1)\pi}{m} & \cdots\cdots & \frac{1}{m}\cos\frac{(m-1)^2\pi}{m} \\[2ex]
\frac{1}{2n} & -\frac{1}{2n} & \frac{1}{2n} & -\frac{1}{2n} & 0 & \cdots\cdots & 0 \\[2ex]
\frac{1}{2n} & -\frac{1}{2n} & -\frac{1}{2n} & \frac{1}{2n} & 0 & \cdots\cdots & 0 \\[2ex]
\frac{1}{2n} & \frac{1}{2n} & \frac{1}{2n} & \frac{1}{2n} & -\frac{1}{m} & \frac{1}{m}\ \cdots\cdots & \cdots\ -\frac{1}{m}
\end{array}
\right]
$$

Every entry of the i^{th} column of the matrix $\bar{C}$ is given by the corresponding entry in the i^{th} row of the character table of the group D_{2n}, $n = 2m$, m even, multiplied by the degree d_i and divided by the order of the group.

Central character table of $H(D_{2n})$, $n = 2m$, m odd.

	v_0	v_1	...	v_{m-1}	v_m	v_{m+1}	v_{m+2}
M_0	1	$2u$	...	$2u^{m-1}$	$u^{2m-1}+\ldots+u^m$	$u^{2m-1}+\ldots+u^m$	u^{2m}
M_s	1	$2u$	...	$2u^{m-1}$	$-(u^{m-1}+\ldots+u+1)$	$-(u^{m-1}+\ldots+u+1)$	1
M_1	1	$-(u^2+1)$	...	$2u^{m-1}$	$-\frac{m+1}{2}u^{m-1}-\frac{m-1}{2}u^m$	$\frac{m+1}{2}u^m+\frac{m-1}{2}u^{m-1}$	$-u^m$
M_2	1	$-(u^2+1)$	...	$2u^{m-1}$	$\frac{m+1}{2}u^m+\frac{m-1}{2}u^{m-1}$	$-\frac{m+1}{2}u^{m-1}-\frac{m-1}{2}u^m$	$-u^m$
V_1	1	$u\Delta_1^1-(u-1)^2\hat{\Delta}_1^1$	...	$u^{m-1}\Delta_{m-1}^1-u^{m-2}(u-1)^2\hat{\Delta}_{m-1}^1$	$u^{m-1}(u-1)\frac{1}{1-\cos\theta}$	$u^{m-1}(u-1)\frac{1}{1-\cos\theta}$	$-u^m$
V_2	1	$u\Delta_1^2-(u-1)^2\hat{\Delta}_1^2$	...	$u^{m-1}\Delta_{m-1}^2-u^{m-2}(u-1)^2\hat{\Delta}_{m-1}^2$	0	0	u^m
V_3	1	$u\Delta_1^3-(u-1)^2\hat{\Delta}_1^3$	...	$u^{m-1}\Delta_{m-1}^3-u^{m-2}(u-1)^2\hat{\Delta}_{m-1}^3$	$u^{m-1}(u-1)\frac{1}{1-\cos 3\theta}$	$u^{m-1}(u-1)\frac{1}{1-\cos 3\theta}$	$-u^m$
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
V_{m-1}	1	$u\Delta_1^{m-1}-(u-1)^2\hat{\Delta}_1^{m-1}$	...	$u^{m-1}\Delta_{m-1}^{m-1}-u^{m-2}(u-1)^2\hat{\Delta}_{m-1}^{m-1}$	0	0	u^m

By specializing $u \to 1$, we obtain the Central Character table of the group $W = D_{2n}$, $n = 2m$, m odd, which determines the central character matrix $C' =$

$$
\left[\begin{array}{c|c|c|c|c|c}
1 & 2 & 2 & m & m & 1 \\
1 & 2 & 2 & -m & -m & 1 \\
1 & -2 & 2 & -m & m & -1 \\
1 & -2 & 2 & m & -m & -1 \\
1 & 2\cos\frac{\pi}{m} & 2\cos\frac{2\pi}{m} & \cdots\ 0 & 0 & -1 \\
1 & 2\cos\frac{2\pi}{m} & 2\cos\frac{4\pi}{m} & \cdots\ 0 & 0 & 1 \\
1 & 2\cos\frac{3\pi}{m} & 2\cos\frac{6\pi}{m} & \cdots\ 0 & 0 & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 2\cos\frac{(m-1)\pi}{m} & 2\cos\frac{2(m-1)\pi}{m} & \cdots\ 0 & 0 & 1
\end{array}\right]
$$

The inverse of the central character matrix C', is $\bar{C}'$ =

$$
\begin{array}{c}
\\ \\ \\ \\ m^{th} \longrightarrow \text{row} \\ \\ \\ \\
\end{array}
\left[
\begin{array}{c|c|c|c|c|c|c}
\frac{1}{2n} & \frac{1}{2n} & \frac{1}{2n} & \frac{1}{2n} & \frac{1}{m} & \cdots & \frac{1}{m} \\
\frac{1}{2n} & \frac{1}{2n} & -\frac{1}{2n} & -\frac{1}{2n} & \frac{1}{m}\cos\frac{\pi}{m} & \cdots & \frac{1}{m}\cos\frac{(m-1)\pi}{m} \\
\frac{1}{2n} & \frac{1}{2n} & \frac{1}{2n} & \frac{1}{2n} & \frac{1}{n}\cos\frac{2\pi}{m} & \cdots & \frac{1}{m}\cos\frac{2(m-1)\pi}{m} \\
\frac{1}{2n} & \frac{1}{2n} & -\frac{1}{2n} & -\frac{1}{2n} & \frac{1}{m}\cos\frac{3\pi}{m} & \cdots & \frac{1}{m}\cos\frac{3(m-1)\pi}{m} \\
\frac{1}{2n} & \frac{1}{2n} & \frac{1}{2n} & \frac{1}{2n} & \frac{1}{m}\cos\frac{(m-1)\pi}{m} & \cdots & \frac{1}{m}\cos\frac{(m-1)^2\pi}{m} \\
\frac{1}{2n} & -\frac{1}{2n} & -\frac{1}{2n} & \frac{1}{2n} & 0 & \cdots & 0 \\
\frac{1}{2n} & -\frac{1}{2n} & \frac{1}{2n} & -\frac{1}{2n} & 0 & \cdots & 0 \\
\frac{1}{2n} & \frac{1}{2n} & -\frac{1}{2n} & -\frac{1}{2n} & -\frac{1}{m} & \cdots & \frac{1}{m}
\end{array}
\right]
$$

Let $A = (a_{ir})$ $i, r \in \{1,2,\ldots,m+3\}$ be the matrix describing Lusztig's isomorphism on the centre of the generic Hecke algebra $H(D_{2n})$ with respect to the basis $\{v_0, v_k, 1 \le k \le m-1, v_m, v_{m+1}, v_{m+2}\}$. Then $A = \bar{C}\,\Lambda$ (or $A = \bar{C}'\Lambda'$) where $\bar{C}$ is the inverse of the central character matrix C (when m is even), $\bar{C}'$ the inverse of the central character matrix C' (when m is odd), and Λ, Λ' the matrix determined by the central character table of $H(D_{2n})$ for the cases m even and m odd respectively.

The calculations are entirely similar as in the case n odd and they are omitted. We provide the result of our calculations in a matrix form in which we use the following notation.

$$K(1,m+1) = K(1,m+2) = \frac{1}{2n}(u-1)\Big[\sum_{\lambda=2}^{m}(\lambda-1)u^{2m-\lambda}+(m^2+m)u^{m-1} + \sum_{\lambda=2}^{m}(m-\lambda+1)u^{m-\lambda}\Big]$$

$$K(i,m+1) = K(i,m+2) = \frac{1}{2n}(u-1)\Big[\sum_{\lambda=2}^{m}(\lambda-1)u^{2m-\lambda} +(m^2-n(i-1)+m)u^{m-1}+ \sum_{\lambda=2}^{m}(m-\lambda+1)u^{m-\lambda}\Big]$$

for every $i = 2,3,\ldots,m$.

$$K(m+1,m+1) = K(m+2,m+2) = \frac{1}{2n}[u^{n-1} + \ldots + 1] + \frac{1}{4}(u^m + u^{m-1})$$

$$K(m+1,m+2) = K(m+2,m+1) = \frac{1}{2n}[u^{n-1} + \ldots + 1] - \frac{1}{4}(u^m + u^{m-1})$$

$$K(m+3,m+1) = K(m+3,m+2) = \frac{1}{2n}(u-1)\Big[\sum_{\lambda=2}^{m}(\lambda-1)u^{2m-\lambda}-(m^2-m)u^{m-1} + \sum_{\lambda=2}^{m}(m-\lambda+1)u^{m-\lambda}\Big]$$

	1	$-\frac{1}{n}(u-1)^2(n-1)$	$-\frac{2}{n}u(u-1)^2(n-2)$	$-\frac{3}{n}u^2(u-1)^2(n-3)$ .... $-\frac{(m^2-1)}{n}u^{m-2}(u-1)^2$	$K(1,m+1)$	$K(1,m+2)$	$\frac{1}{2n}(u^m-1)^2$
	0	$u+\frac{1}{n}(u-1)^2$	→ $u^{r-2}(u-1)^2[\frac{(r-1)^2}{n}-(r-2)]$, $3 \le r \le m$ ←		$K(2,m+1)$	$K(2,m+2)$	$\frac{1}{2n}(u^m-1)^2$
	0	$\frac{1}{n}(u-1)^2$	$u^2+\frac{4}{n}u(u-1)^2$	→ $u^{r-2}(u-1)^2[\frac{(r-1)^2}{n}-(r-2)]$, $4 \le r \le m$ ←	$K(3,m+1)$	$K(3,m+2)$	$\frac{1}{2n}(u^m-1)^2$
i^{th} row $2 \le i \le m$	0	$\frac{1}{n}u^{r-2}(u-1)^2(r-1)^2$ $i > r = 2,\dots,i-1$	$u^{r-1}+\frac{1}{n}u^{r-2}(u-1)^2(r-1)^2$ $i = r = 2,3,\dots,m$	$u^{r-2}(u-1)^2[\frac{(r-1)^2}{n}-(r-i)]$ $i < r = i+1,\dots,m$	$K(i,m+1)$	$K(i,m+2)$	$\frac{1}{2n}(u^m-1)^2$
m+1 row	0	→ 0 ←			$K(m+1,m+1)$	$K(m+1,m+2)$	$\frac{u^n-1}{2n}$
m+2 row	0	→ 0 ←			$K(m+2,m+1)$	$K(m+2,m+2)$	$\frac{u^n-1}{2n}$
m+3 row	0	→ $\frac{1}{n}u^{r-2}(u-1)^2(r-1)^2$ $r = 2,3,\dots,m.$ ←			$K(m+3,m+1)$	$K(m+3,m+2)$	$u^m+\frac{1}{2n}(u^m-1)^2$

When we specialize $u \to 1$ the matrix above specializes to be identity matrix as it should be.

We illustrate the situation by giving some examples.

(i) $W = D_8$, $n = 4$, $m = 2$.

A basis for the centre of $H(D_8)$ is given by the set of elements

$$v_0 = T_1,\quad v_1 = T_{s_1s_2} + T_{s_2s_1} - (u-1)[T_{s_1} + T_{s_2}],$$

$$v_2 = T_{s_1s_2s_1} + uT_{s_2},\quad v_3 = T_{s_2s_1s_2} + uT_{s_1},\quad v_4 = T_{s_1s_2s_1s_2}$$

The matrix A is given by:

$$\begin{bmatrix} 1 - \frac{3}{4}(u-1)^2 & \frac{1}{8}(u-1)(u^2+6u+1) & \frac{1}{8}(u-1)(u^2+6u+1) & \frac{1}{8}(u^2-1)^2 \\ 0 \quad u + \frac{1}{4}(u-1)^2 & \frac{1}{8}(u-1)(u+1)^2 & \frac{1}{8}(u-1)(u+1)^2 & \frac{1}{8}(u^2-1)^2 \\ 0 \qquad 0 & \frac{1}{8}(u+1)^3 & \frac{1}{8}(u+1)(u-1)^2 & \frac{u^4-1}{8} \\ 0 \qquad 0 & \frac{1}{8}(u+1)(u-1)^2 & \frac{1}{8}(u+1)^3 & \frac{u^4-1}{8} \\ 0 \quad \frac{1}{4}(u-1)^2 & \frac{1}{8}(u-1)^3 & \frac{1}{8}(u-1)^2 & u^2 + \frac{1}{8}(u^2-1)^2 \end{bmatrix}$$

The information given by this matrix is that

$$\phi(v_0) = 1$$

$$\phi(v_1) = -\frac{3}{4}(u-1)^2\cdot 1 + [u + \tfrac{1}{4}(u-1)^2](s_1s_2 + s_2s_1) + \frac{1}{4}(u-1)^2(s_1s_2)^2$$

$$\phi(v_2) = \frac{1}{8}(u-1)(u^2+6u+1)\cdot 1 + \frac{1}{8}(u-1)(u+1)^2(s_1s_2 + s_2s_1) + \\ + \frac{1}{8}(u+1)^3(s_1s_2s_1 + s_2) + \frac{1}{8}(u+1)(u-1)^2(s_2s_1s_2 + s_1) + \\ + \frac{1}{8}(u-1)^3(s_1s_2)^2$$

$$\phi(v_3) = \frac{1}{8}(u-1)(u^2+6u+1)\cdot 1 + \frac{1}{8}(u-1)(u+1)^2(s_1s_2 + s_2s_1) + \\ + \frac{1}{8}(u+1)(u-1)^2(s_1s_2s_1 + s_2) + \frac{1}{8}(u+1)^3(s_2s_1s_2 + s_1) + \frac{1}{8}(u-1)^2(s_1s_2)^2$$

$$\phi(v_4) = \frac{1}{8}(u^2-1)^2\cdot 1 + \frac{1}{8}(u^2-1)^2\cdot(s_1s_2 + s_2s_1) + \frac{u^4-1}{8}(s_1s_2s_1 + s_2) \\ + \frac{u^4-1}{8}(s_2s_1s_2 + s_1) + [u^2 + \frac{1}{8}(u^2-1)^2](s_1s_2)^2$$

(ii) $W = D_{12}$, $n = 6$, $m = 3$.

A basis for the centre of $H(D_{12})$ is given by the set of elements:

$$v_0 = T_1,\ v_1 = T_{s_1s_2} - T_{s_2s_1} - (u-1)[T_{s_1} + T_{s_2}],$$

$$v_2 = T_{s_1s_2s_1s_2} + T_{s_2s_1s_2s_1} - (u-1)[T_{s_1s_2s_1} + T_{s_2s_1s_2}] - u(u-1)[T_{s_1} + T_{s_2}],$$

$$v_3 = T_{s_1s_2s_1s_2s_1} + u\, T_{s_2s_1s_2} + u^2\, T_{s_1},$$

$$v_4 = T_{s_2s_1s_2s_1s_2} + uT_{s_1s_2s_1} + u^2T_{s_2},\ \ v_5 = T_{s_1s_2s_1s_2s_1s_2}$$

$$
\begin{bmatrix}
1 & -\frac{5}{6}(u-1)^2 & -\frac{4}{3}u(u-1)^2 & \frac{1}{12}(u-1)(u^4+2u^3+12u^2+2u+1) & \frac{1}{12}(u-1)(u^4+2u^3+12u^2+2u+1) & \frac{1}{12}(u^3-1)^2 \\
0 & +\frac{1}{6}(u-1)^2 & -\frac{1}{3}u(u-1)^2 & \frac{1}{12}(u-1)(u^4+2u^3+6u^2+2u+1) & \frac{1}{12}(u-1)(u^4+2u^3+6u^2+2u+1) & \frac{1}{12}(u^3-1)^2 \\
0 & \frac{1}{6}(u-1)^2 & u^2+\frac{2}{3}u(u-1)^2 & \frac{1}{12}(u-1)(u^4+2u^3+2u+1) & \frac{1}{12}(u-1)(u^4+2u^3+2u+1) & \frac{1}{12}(u^3-1)^2 \\
0 & 0 & 0 & \frac{1}{12}(u^5+u^4+4u^3+4u^2+u+1) & \frac{1}{12}(u^5+u^4-2u^3-2u^2+u+1) & \frac{u^6-1}{12} \\
0 & 0 & 0 & \frac{1}{12}(u^5+u^4-2u^3-2u^2+u+1) & \frac{1}{12}(u^5+u^4+4u^3+4u^2+u+1) & \frac{u^6-1}{12} \\
0 & \frac{1}{6}(u-1)^2 & \frac{2}{3}u(u-1)^2 & \frac{1}{12}(u-1)(u^4+2u^3-6u^2+2u+1) & \frac{1}{12}(u-1)(u^4+2u^3-6u^2+2u+1) & u^3+\frac{1}{12}(u^3-1)^2
\end{bmatrix}
$$

CHAPTER 5

The general form of Lusztig's isomorphism and its restriction to the centre of the generic Hecke algebra

In this chapter we shall generalize the results of Chapters 2 and 4. In §1.4, we gave a procedure for the determination of Lusztig's isomorphism. It is clear that the larger the order of the group W becomes, the harder it is to find an explicit formula for this isomorphism. We wish to find some information which will simplify this procedure and enable us to establish a general formula for this isomorphism. The starting point of our investigation was the determination of Lusztig's isomorphism for the symmetric group S_4.

Let W be the symmetric group S_4, given by a presentation

$W = \langle s_1, s_2, s_3 : s_1^2 = s_2^2 = s_3^2 = 1,\ (s_1s_2)^3 = (s_2s_3)^3 = 1,\ s_1s_3 = s_3s_1\rangle.$

There are ten left cells in W, given by:

$X_0 = \{1\},\ L_1 = \{s_1, s_2s_1, s_3s_2s_1\},\ L_2 = \{s_1s_2, s_2, s_3s_2\},$

$L_3 = \{s_1s_2s_3, s_2s_3, s_3\},\ M_1 = \{s_1s_3, s_2s_1s_3\},$

$M_2 = \{s_1s_3s_2, s_2s_1s_3s_2\},\ N_1 = \{s_2s_3s_1s_2s_1, s_3s_1s_2s_1, s_1s_2s_1\},$

$N_2 = \{s_2s_3s_2, s_1s_2s_3s_2, s_1s_2s_1s_3s_2\},$

$N_3 = \{s_2s_3s_2s_1, s_1s_3s_2s_1s_3, s_1s_2s_1s_3\},\ X_4 = \{s_1s_2s_1s_3s_2s_1\}.$

There are five two-sided cells in W given by:

$X_0,\ X_1 = L_1 \cup L_2 \cup L_3,\ X_2 = M_1 \cup M_2,\ X_3 = N_1 \cup N_2 \cup N_3,\ X_4$.

We consider the free $\mathbb{Q}[u^{\frac{1}{2}}]$ module E with basis $\{e_w, w \in S_4\}$, and we make it into left H-module with action described in §1.4 where H is the generic Hecke algebra over the ring $\mathbb{Q}[u^{\frac{1}{2}}]$. Then we construct the graded module grad(E) (see also §1.4) with canonical basis $\{\bar{e}_w, w \in S_4\}$. We know that grad(E) affords the left regular representation of H.

Now, each left cell gives rise to a W-graph according to Theorem 1.3.6. The W-graphs arising from the left cells are

The circles represent the vertices, and inside each circle we describe the indices i for which $s_i w < w$, $i = 1,2,3$, and w is the vertex represented by the corresponding cycle. The function μ is identically 1 and it is omitted.

Each such W-graph gives rise to a representation of H over $\mathbb{Q}(u^{\frac{1}{2}})$ which by Theorem 1.3.9 is irreducible.

For each left-cell say C, we consider the subspace V_C of grad(E) spanned by $\{\bar{e}_w, w \in C\}$. Then V_C is an irreducible left $H_{\mathbb{Q}(u^{\frac{1}{2}})}$ module.

We next provide the matrices which represent T_{s_i}, $i = 1,2,3$ on the various modules V_C, with bases C

T_{s_1} is represented on V_{X_0}, V_{L_1}, V_{M_1}, V_{N_1}, V_{X_4} by the matrices:

$$(u), \quad \begin{bmatrix} -1 & u^{\frac{1}{2}} & 0 \\ 0 & u & 0 \\ 0 & 0 & u \end{bmatrix}, \quad \begin{bmatrix} -1 & u^{\frac{1}{2}} \\ 0 & u \end{bmatrix}, \quad \begin{bmatrix} u & 0 & 0 \\ u^{\frac{1}{2}} & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad (-1)$$

respectively.

The matrices representing T_{s_2} on these modules are

$$(u), \quad \begin{bmatrix} u & 0 & 0 \\ u^{\frac{1}{2}} & -1 & u^{\frac{1}{2}} \\ 0 & 0 & u \end{bmatrix} \quad \begin{bmatrix} u & 0 \\ u^{\frac{1}{2}} & -1 \end{bmatrix} \quad \begin{bmatrix} -1 & u^{\frac{1}{2}} & 0 \\ 0 & u & 0 \\ 0 & u^{\frac{1}{2}} & -1 \end{bmatrix}, \quad (-1)$$

and the matrices representing T_{s_3} are

$$(u), \quad \begin{bmatrix} u & 0 & 0 \\ 0 & u & 0 \\ 0 & u^{\frac{1}{2}} & -1 \end{bmatrix} \quad \begin{bmatrix} -1 & u^{\frac{1}{2}} \\ 0 & u \end{bmatrix} \quad \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & u^{\frac{1}{2}} \\ 0 & 0 & u \end{bmatrix}, \quad (-1)$$

Let $\phi(T_{s_i})$ be the images of the generators T_{s_i}, $i = 1,2,3$, under the Lusztig isomorphism ϕ.

Let $\phi(T_{s_i}) = \sum_{w \in S_4} c_w^{(i)} \cdot w$, $c_w^{(i)} \in \mathbb{Q}(u^{\frac{1}{2}})$. The generators s_i are represented on the various modules V_C with bases C according to the matrices obtained by the ones above by specializing $u^{\frac{1}{2}} \to 1$. Therefore we can work out the matrices which represent every $w \in S_4$ with respect to these bases. Using the fact that T_{s_i} and $\phi(T_{s_i})$ act on the same way on the graded module and comparing coefficients of the basis elements on both sides of the equation $T_{s_i}\bar{e}_w = \phi(T_{s_i})\bar{e}_w$ we obtain from each irreducible representation of degree $d \in \{1,3,2,3,1\}$, d^2 equations in the $c_w^{(i)}$, $w \in S_4$, $i = 1,2,3$. Thus we obtain a total of 24 equations in 24 unknowns $c_w^{(i)}$. These equations are linearly independent and the solution of the system of these equations gives

$$\phi(T_{s_1}) = \frac{u-1}{2}\cdot 1 + \frac{u+1}{2}\cdot s_1 + \frac{(u^{\frac{1}{2}}-1)^2}{6}[-s_2 + s_1s_2 - s_2s_1 + s_1s_2s_1]$$

$$+ \frac{(u^{\frac{1}{2}}-1)^2}{12}[-s_2s_3s_2 + s_1s_2s_3s_2 - s_2s_3s_2s_1 + s_1s_2s_3s_2s_1] +$$

$$\frac{(u^{\frac{1}{2}}-1)^2}{24}[-s_2s_3 + s_3s_2 - s_1s_3s_2 + s_3s_2s_1 - s_2s_3s_1 + s_1s_2s_3 - s_1s_3s_2s_1 + s_1s_2s_1s_3]$$

$$\phi(T_{s_2}) = \frac{u-1}{2}\cdot 1 + \frac{u+1}{2} s_2 + \frac{(u^{\frac{1}{2}}-1)^2}{6}[-s_1 - s_3 - s_1s_2 + s_2s_1 + s_2s_3 - s_3s_2 + s_1s_2s_1 + s_2s_3s_2]$$

$$+ \frac{(u^{\frac{1}{2}}-1)^2}{12}[-s_3s_2s_1 - s_1s_2s_3 - s_1s_3s_2s_1 + s_1s_2s_1s_3 - s_1s_2s_3s_2 + s_2s_3s_2s_1 + s_1s_2s_1s_3s_2 + s_2s_3s_2s_1s_2]$$

$$\phi(T_{s_3}) = \frac{u-1}{2}\cdot 1 + \frac{u+1}{2} s_3 + \frac{(u^{\frac{1}{2}}-1)^2}{6}[-s_2 + s_3s_2 - s_2s_3 + s_2s_3s_2] +$$

$$\frac{(u^{\frac{1}{2}}-1)^2}{12}[-s_1s_2s_1 + s_3s_1s_2s_1 - s_2s_1s_2s_3 + s_3s_2s_1s_2s_3] +$$

$$+ \frac{(u^{\frac{1}{2}}-1)^2}{24}[-s_2s_1 + s_1s_2 - s_3s_1s_2 + s_1s_2s_3 - s_2s_1s_3 + s_3s_2s_1 - s_3s_1s_2s_3 + s_2s_3s_2s_1]$$

We note that the coefficients of 1_{S_4} and s_1 in the images of $\phi(T_{s_1})$ are $\frac{u-1}{2}$ and $\frac{u+1}{2}$ respectively and that $c_w^{(i)} = c_{ws_i}^{(i)}$ while $c_w^{(i)} = -c_{s_iw}^{(i)}$. Thus Thus if $w \in C_W(s_i)$, $w \neq 1, s_i$, $i = 1,2,3$, then $c_w^{(i)} = 0$, where $C_W(s_i)$ denotes the centralizer of s_i in W. (Compare these remarks with the formulae giving the isomorphism ϕ for the dihedral case). Thus the coefficients of s_3, s_1s_3 in $\phi(T_{s_1})$, and the coefficients of $s_1s_2s_3s_2s_1$, $s_1s_2s_1s_3s_2s_1$ in $\phi(T_{s_2})$ and the coefficients of s_1, s_1s_3 in $\phi(T_{s_3})$ are all zero. There are certainly other elements w for which $c_w^{(i)} = 0$.

We now prove the following:

<u>Theorem 5.1</u>: Let W be a finite indecomposable Weyl group, and ϕ be the Lusztig isomorphism between the generic Hecke algebra over $\mathbb{Q}(u^{\frac{1}{2}})$ and the group algebra of W over $\mathbb{Q}(u^{\frac{1}{2}})$. Let $\phi(T_s) = \sum_{w \in W} c_w . w$, $c_w \in \mathbb{Q}(u^{\frac{1}{2}})$. Then

(i) $c_1 = \frac{u-1}{2}$, $c_s = \frac{u+1}{2}$, $c_w = c_{ws}$, $c_w = -c_{sw}$ $\forall\, w \neq 1, s$

(ii) $\phi(T_s) = \frac{u-1}{2} \cdot 1 + \frac{u+1}{2} s + (u^{\frac{1}{2}}-1)^2 F_s$, where $F_s \in \mathbb{Q}W$, F_s does not involve 1, s, and F_s satisfies the properties $F_s^2 = 0$, $sF_s = -F_s$, $F_s s = F_s$.

<u>Proof</u>: We consider the graded module over $\mathbb{Q}(u^{\frac{1}{2}})$ with the canonical basis $\{\bar{e}_w, w \in W\}$. We fix an $s \in S$ and let $W_1 = \{w \in W: sw < w\}$, $W_2 = \{w \in W : sw > w\}$. Then $W = W_1 \cup W_2$, and if $d_i = |W_i|$, $i = 1,2$, then $d_i = \frac{|W|}{2}$ since the map $W \to W$ such that $w \to sw$ is clearly a bijection.

We order the canonical basis so that $\{\bar{e}_w, w \in W\} = \{\bar{e}_w, w \in W_1\} \cup \{\bar{e}_w . w \in W_2\}$, and then, the matrix which represents T_s on grad(E) with respect to this ordering of the canonical basis is (considering T_s acting on grad(E) by the left)

$$A(T_s) = \begin{bmatrix} -I_{d_1} & u^{\frac{1}{2}} E_s \\ 0 & uI_{d_1} \end{bmatrix}$$

The action of T_s on $\bar{e}_w$ is given in §1.4. I_{d_1} is the identity matrix of size d_1. 0 is the zero matrix of the same size and E_s is a matrix also of the same size whose entries are integers.

We consider the image of T_s under ϕ and we write $\phi(T_s) = c_1 \cdot 1 + \sum_{w \neq 1} c_w . w$. When we view the grad(E) as a left H-module and as a left W-module with actions described in §1.4, then it affords the left regular representation of H and W respectively. We also recall that T_s and $\phi(T_s)$ act on grad(E) in the same way. Since the trace of any $w \in W$, $w \neq 1$ for the regular representation of W is zero, we obtain that the trace of the matrix $A(T_s)$ is $\frac{u-1}{2}|W| = c_1|W|$, therefore $c_1 = \frac{u-1}{2}$. We next write $\phi(T_s) = f_s(u) + u^{\frac{1}{2}} g_s(u)$, where $f_s(u)$, $g_s(u)$ belong to $\mathbb{Q}(u)(W)$.

We also write
$$A(T_s) = \begin{bmatrix} -I_{d_1} & 0 \\ 0 & uI_{d_1} \end{bmatrix} + u^{\frac{1}{2}} \begin{bmatrix} 0 & E_s \\ 0 & 0 \end{bmatrix}$$

Therefore the elements of the group algebra $f_s(u)$, $g_s(u)$ are represented on grad(E) with respect to the ordered canonical basis as above, by the matrices:

$$f_s(u) \to \begin{bmatrix} -I_{d_1} & 0 \\ 0 & uI_{d_1} \end{bmatrix}, \qquad g_s(u) \to \begin{bmatrix} 0 & E_s \\ 0 & 0 \end{bmatrix}$$

By specializing $u^{\frac{1}{2}} \to 1$ we obtain the matrix which represents the generator s, so

$$s \to \begin{bmatrix} -I_{d_1} & E_s \\ 0 & I_{d_1} \end{bmatrix} = \begin{bmatrix} -I_{d_1} & 0 \\ 0 & I_{d_1} \end{bmatrix} + \begin{bmatrix} 0 & E_s \\ 0 & 0 \end{bmatrix}$$

Thus $g_s(1) \to \begin{bmatrix} 0 & E_s \\ 0 & 0 \end{bmatrix}$

Thus $g_s(u)$, and $g_s(1)$ act on the same way on grad(E), and so $g_s(u) = g_s(1) \in \mathbb{Q}W$.

Let a be the following element inside $\mathbb{Q}(u^{\frac{1}{2}})(W)$.

$$a = \frac{u-1}{2}\cdot 1 + \frac{u+1}{2}\, s - \frac{u+1}{2}\, g_s(1) + u^{\frac{1}{2}} g_s(1).$$

Then, the matrix which represents a on grad(E) with respect to the ordered canonical basis as above, is

$$\begin{bmatrix} (\frac{u-1}{2} - \frac{u+1}{2})\, I_{d_1} & (\frac{u+1}{2} - \frac{u+1}{2} + u^{\frac{1}{2}})\, E_s \\ 0 & (\frac{u-1}{2} + \frac{u+1}{2}) I_{d_1} \end{bmatrix}$$

$$= \begin{bmatrix} -I_{d_1} & u^{\frac{1}{2}} E_s \\ 0 & uI_{d_1} \end{bmatrix}.$$

Therefore $\phi(T_s) = \frac{u-1}{2}\cdot 1 + \frac{u+1}{2}\, s - \frac{u+1}{2}\, g_s(1) + u^{\frac{1}{2}} g_s(1)$.

Put $F_s = -\frac{1}{2}\, g_s(1)$ to obtain

$\phi(T_s) = \frac{u-1}{2}\, 1 + \frac{u+1}{2}\, s + (u^{\frac{1}{2}} - 1)^2\, F_s$, where $F_s \in \mathbb{Q}W$, F_s does not involve 1.

Now the matrix which represents F_s on grad(E) with respect to the ordered canonical basis is $-\frac{1}{2} \begin{bmatrix} 0 & E_s \\ 0 & 0 \end{bmatrix}$. Thus, the matrix which represents F_s^2 is the zero matrix so $F_s^2 = 0$.

Suppose that $F_s = c_s'.s + \sum_{w \neq 1, s} c_w'.w$. Then, the matrix which represents sF_s on grad(E) with respect to the ordered canonical basis is

$$\begin{bmatrix} -I_{d_1} & E_s \\ 0 & I_{d_1} \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{2}E_s \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}E_s \\ 0 & 0 \end{bmatrix}$$

Thus $sF_s = -F_s$ i.e. $c_s'.1 + \sum_{w\neq 1,s} c_w' \, sw = -c_s'.s - \sum_{w\neq 1,s} c_w' \cdot w$ i.e.

$c_s'.1 + \sum_{w\neq 1,s} c_{sw}' \, w = -c_s'.s - \sum_{w\neq 1,s} c_w'.w$, and therefore we obtain that

$c_s' = 0$ and $c_{sw}' = -c_w'$, $w \neq 1,s$. Therefore F_s does not involve the element s as well.

The matrix which represents the element $F_s.s$ is

$$\begin{bmatrix} 0 & -\frac{1}{2}E_s \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -I_{d_1} & E_s \\ 0 & I_{d_1} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2}E_s \\ 0 & 0 \end{bmatrix}$$

Thus $F_s s = F_s$ and so $c_w' = c_{ws}'$ $w \neq 1,s$.

Our theorem is now proved.

Now for every generator s_i of the group W, we write

$$\phi(T_{s_i}) = \frac{u-1}{2}\,1 + \frac{u+1}{2}\,s_i + (u^{\frac{1}{2}}-1)^2\,F_{s_i} =$$

$= \frac{u-1}{2}\,1 + \frac{u+1}{2}\,s_i + (u+1-2u^{\frac{1}{2}})F_{s_i}$. We put $g_i =: \; g_{s_i} = -2F_{s_i}$ and

$f_i =: f_{s_i}(u) = \frac{u-1}{2}\,1 + \frac{u+1}{2}\,s_i - \frac{u+1}{2}\,g_i$.

We note that $f_{s_i}(u)$ depends upon u, while g_{s_i} does not.

<u>Proposition 5.2</u>: (1) $g_i^2 = 0$, $s_i g_i = -g_i$, $g_i s_i = g_i$

(2) $f_i^2 = u1 + (u-1)f_i$, $g_i f_i + f_i g_i = (u-1)g_i$, $g_i f_i g_i = 0$, $f_i g_i f_i = -ug_i$.

(3) $g_i g_j - g_j g_i = g_i s_j - s_j g_i + s_i g_j - g_j s_i + s_j s_i - s_i s_j$, for $i \neq j$.

(4) $\underbrace{f_if_jf_i\ \dots}_{\leftarrow m_{ij} \rightarrow} = \underbrace{f_jf_if_j\ \dots}_{\leftarrow m_{ij} \rightarrow}$, for $m_{ij} \in \{2,4,6\}$, $i \neq j$ and

$f_if_jf_i - f_jf_if_j = u(f_j - f_i)$ $i \neq j$.

<u>Proof</u>: (1) It is clear from the fact that $g_i = -2F_{s_i}$ and $s_iF_{s_i} = -F_{s_i}$, $F_{s_i}s_i = F_{s_i}$, $F_{s_i}^2 = 0$.

(2) The matrix which represents f_i on grad(E) with respect to the canonical basis (the basis elements being properly ordered), is $\begin{bmatrix} -I_{d_1} & 0 \\ 0 & uI_{d_1} \end{bmatrix}$ and the matrix which represents g_i is $\begin{bmatrix} 0 & E_{d_1} \\ 0 & 0 \end{bmatrix}$. Therefore, f_i^2 is represented by $\begin{bmatrix} I_{d_1} & 0 \\ 0 & u^2I_{d_1} \end{bmatrix}$ and the element $u \cdot 1 + (u-1)f_i$ is also represented by the same matrix, so $f_i^2 = u \cdot 1 + (u-1)f_i$. The element $g_if_i + f_ig_i$ is represented by the matrix

$$\begin{bmatrix} 0 & E_{d_1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -I_{d_1} & 0 \\ 0 & uI_{d_1} \end{bmatrix} + \begin{bmatrix} -I_{d_1} & 0 \\ 0 & uI_{d_1} \end{bmatrix} \begin{bmatrix} 0 & E_{d_1} \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & (u-1)E_{d_1} \\ 0 & 0 \end{bmatrix}, \text{ so } g_if_i + f_ig_i = (u-1)g_i,$$

and therefore $g_if_ig_i = 0$ since $g_i^2 = 0$, while $f_ig_if_i = (u-1)f_ig_i - f_i^2g_i =$ $= (u-1)f_ig_i - (u \cdot 1 + (u-1)f_i)g_i = -ug_i$.

(3) We write $W = W_1 \cup W_2 \cup W_3 \cup W_4$ where

$W_1 = \{w \in W : s_i w < w, s_j w < w\}$, $W_2 = \{w \in W : s_i w < w, s_j w > w\}$

$W_3 = \{w \in W : s_i w > w, s_j w < w\}$, $W_4 = \{w \in W : s_i w > w, s_j w > w\}$

Let d_i be the cardinality of W_i, $i = 1,2,3,4$, and we order the elements of the canonical basis of grad(E) according to the decomposition of W above.

The matrices which represent s_i, g_i with respect to this basis are respectively

$$\begin{bmatrix} -I_{d_1} & 0 & A^{(i)} & C^{(i)} \\ 0 & -I_{d_2} & B^{(i)} & D^{(i)} \\ 0 & 0 & I_{d_3} & 0 \\ 0 & 0 & 0 & I_{d_4} \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & A^{(i)} & C^{(i)} \\ 0 & 0 & B^{(i)} & D^{(i)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where $A^{(i)}$, $B^{(i)}$, $C^{(i)}$, $D^{(i)}$ are matricesof size $d_1 \times d_3$, $d_2 \times d_3$, $d_1 \times d_4$, $d_2 \times d_4$ respectively.

Therefore the element $s_i - g_i$ is represented by the matrix

$$\begin{bmatrix} -I_{d_1} & & & \\ & -I_{d_2} & & \\ & & I_{d_3} & \\ & & & I_{d_4} \end{bmatrix}.$$

The matrices which represent s_j, g_j are respectively

$$\begin{bmatrix} -I_{d_1} & A^{(j)} & 0 & C^{(j)} \\ 0 & I_{d_2} & 0 & 0 \\ 0 & B^{(j)} & -I_{d_3} & D^{(j)} \\ 0 & 0 & 0 & I_{d_4} \end{bmatrix} \text{ and } \begin{bmatrix} 0 & A^{(j)} & 0 & C^{(j)} \\ 0 & 0 & 0 & 0 \\ 0 & B^{(j)} & 0 & D^{(j)} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

where $A^{(j)}$, $B^{(j)}$, $C^{(j)}$, $D^{(j)}$ are matrices of size $d_1 \times d_2$, $d_3 \times d_2$, $d_1 \times d_4$, $d_3 \times d_4$ respectively.

Therefore the element $s_j - g_j$ is represented by the matrix

$$\begin{bmatrix} -I_{d_1} & & & \\ & I_{d_2} & & \\ & & -I_{d_3} & \\ & & & I_{d_4} \end{bmatrix}$$

Hence the elements $s_i - g_i$ and $s_j - g_j$, $i \neq j$ commute, i.e.

$(s_i-g_i)(s_j-g_j) = (s_j-g_j)(s_i-g_i)$ so

$s_is_j - s_ig_j - g_is_j + g_ig_j = s_js_i - s_jg_i - g_js_i + g_jg_i$, or

$g_ig_j - g_jg_i = g_is_j - s_jg_i + s_ig_j - g_js_i + s_js_i - s_is_j$ $i \neq j$.

(4) Let $i \neq j$. The elements f_i, f_j are represented by the matrices

$$\begin{bmatrix} -I_{d_1} & & & \\ & -I_{d_2} & & \\ & & uI_{d_3} & \\ & & & uI_{d_4} \end{bmatrix} \quad \text{and}$$

$$\begin{bmatrix} -I_{d_1} & & & \\ & uI_{d_2} & & \\ & & -I_{d_3} & \\ & & & uI_{d_4} \end{bmatrix}.$$

Hence $f_if_j = f_jf_i$ for $i \neq j$ and $\underbrace{f_if_jf_i \ldots}_{m_{ij}} = \underbrace{f_jf_if_j \ldots}_{m_{ij}}$ for $m_{ij} \in \{4,6\}$,

while $f_if_jf_i - f_jf_if_j = f_i^2f_j - f_j^2f_i = (u1 + (u-1)f_i)f_j - (u1 + (u-1)f_j)f_i =$

$= u(f_j - f_i)$, $i \neq j$.

We next provide an alternative proof of Theorem 5.1 based on some conjectures made by R.W. Carter.

This proof illuminates the procedure described in §1.4, for the determination of Lusztig's isomorphism. It has an interesting connection with the orthogonality relations for group characters, and it provides information which is going to be used in the next chapter.

We recall (see definition 1.3.0) the definition of a W-graph over $\mathbb{Z}$, for a Coxeter group W.

Definition 5.0: We say that a W-graph is even, if there is a map $\mathrm{sgn} : X \to \{-1,1\}$ such that $\mu(y,x)\mathrm{sgn}(x)\mathrm{sgn}(y) = -\mu(y,x)$ for any distinct $x,y \in X$.

Let W be a finite crystallographic Coxeter group, and let H be the generic Hecke algebra over the field $\mathbb{Q}(u^{\frac{1}{2}})$. We shall make use of the following:

Theorem 5.3: (1) Every irreducible H-module is afforded by a W-graph over $\mathbb{Z}$.
(2) An irreducible H-module is afforded by an even W-graph over $\mathbb{Z}$, if and only if it is not exceptional.

(For the definition of an exceptional representation and also for a proof of this theorem, see [9]).

We now consider the graded module grad(E) over $\mathbb{Q}(u^{\frac{1}{2}})$, associated to a finite crystallographic Coxeter group (i.e. a Weyl group), and we view it as a left H-module. We know that grad(E) affords the left regular representation of H and as it is also semisimple, it has a decomposition into a direct sum of left absolutely irreducible H-submodules (H splits over $\mathbb{Q}(u^{\frac{1}{2}})$).

Let $\mathrm{grad}(E) = V_{11} \oplus \dots \oplus V_{1d_1} \oplus \dots \oplus V_{t1} \oplus \dots \oplus V_{td_t}$ be one such decomposition, where each V_{ir} has dimension d_i, $1 \le i \le t$, $1 \le r \le d_i$ and $V_{ir} \simeq V_{js}$ if and only if $i = j$.

Let $X = \left\{V_{11},\ldots,V_{t1}\right\}$ be a full set of irreducible constituents. For the sake of simplicity we relabel the members of X by putting $V_{i1} =: V_i \quad \forall 1 \le i \le t$.

According to Theorem 5.3(1), for each $V_r \in X$, $1 \le r \le t$ we can choose a basis $B_r = \{v_1,\ldots,v_{d_r}\}$, such that if $v_k \in B_r$, $1 \le k \le d_r$, and T_s is a generator of H, then

$$T_s\, v_k = \begin{cases} -v_k & \text{if } s \in I(v_k) \\ uv_k + u^{\frac{1}{2}} \displaystyle\sum_{\substack{v_i \in B_r \\ s \in I(v_i),(v_i,v_k) \in Y}} \mu(v_i,v_k)v_i & \text{if } s \notin I(v_k) \end{cases}$$

with $\mu(v_i,v_k) \in Z$ and Y being the set of edges.

We fix an $s \in S$ and suppose that among the basis elements of B_r, d_{r_1} elements v_k are such that $s \in I(v_k)$, $1 \le k \le d_{r_1}$ and that d_{r_2} elements v_k are such that $s \notin I(v_k)$. Then $d_{r_1} + d_{r_2} = d_r$ and we can arrange the elements of B_r in such a way so that the matrix which represents T_s, s fixed, with respect to B_r has the form

$$A_r(T_s) = \begin{bmatrix} -I_{d_{r_1}} & u^{\frac{1}{2}} E_s \\ 0 & uI_{d_{r_2}} \end{bmatrix}$$

whose $I_{d_{r_1}}$, $I_{d_{r_2}}$ are the identity matrices of size d_{r_1}, d_{r_2} respectively, 0 is the zero matrix of size $d_{r_2} \times d_{r_1}$, and E_s is a $d_{r_1} \times d_{r_2}$ matrix whose entries are integers.

Let $B_r = \{v_1,\ldots,v_{d_{r_1}}, v_{d_{r_1}+1},\ldots,v_{d_r}\}$, be this arrangement of the basis elements of B_r, with $s \in I(v_k)$, $1 \le k \le d_{r_1}$ and $s \notin I(v_k)$, $d_{r_1} < k \le d_r$. With respect to this arrangement of the basis elements of B_r, the matrix which represents $T_{s'}$, $s' \neq s$ is not necessarily of the same form, but its entries still involve only polynomials in $Z[u^{\frac{1}{2}}]$. By specializing $u^{\frac{1}{2}} \to 1$ we obtain the matrices which represent the generators of the group W, and therefore the matrices which represent every $w \in W$ with respect to B_r.

Let s fixed as above, and let $(f_{ij}^{(r)}(w))$, $i,j \in \{1,\ldots,d_r\}$, $1 \le r \le t$ be the matrix which represents w on V_r with respect to B_r.

Let $v_k \in B_r$, $1 \le k \le d_r$ and let $wv_k = \sum_{i=1}^{d_r} f_{ik}^{(r)}(w)v_i$, $f_{ik}^{(r)}(w) \in \mathbb{Z}$.

Suppose that $\phi(T_s) = \sum_{w \in W} c_w w$ is the image of T_s under the Lusztig isomorphism ϕ, $c_w \in \mathbb{Q}(u^{\frac{1}{2}})$. Then $\phi(T_s)v_k = \sum_{w \in W} c_w\left(\sum_{i=1}^{d_r} f_{ik}^{(r)}(w)v_i\right) = T_s v_k$, since T_s and $\phi(T_s)$ act on the same way on the graded module.

Therefore, by comparing coefficients of the basis elements v_i on both sides of the equation above, we obtain d_r equations in the unknowns c_w, $w \in W$ of the form

$$\sum_{w \in W} f_{ik}^{(r)}(w)c_w = \lambda \ , \ \lambda \in \{-1, 0, u, \theta u^{\frac{1}{2}}, \theta \in \mathbb{Z}\}.$$

Hence from the constituent $V_r \in X$, $1 \le r \le t$ we obtain in this way d_r^2 equations in the unknowns c_w, $w \in W$.

We now recall the form of the matrix $A_r(T_s)$ which represents T_s on V_r. All diagonal positions in this matrix are u or -1. Also u, -1 do not occur except on the diagonal. We also emphasize the fact that every position of this matrix, gives rise to a certain equation in the unknowns c_w, $w \in W$ in a way we have described above.

The striking thing about these equations is that some of them behave better than the others.

The following definition and conjectures appearing in Proposition 5.4 are due to R.W. Carter.

Definition 5.1: An equation $\sum_{w \in W} f^{(r)}_{ik}(w)c_w = \lambda$. $i,k \in \{1,\dots,d_r\}$ $1 \le r \le t$, in the unknowns c_w, $w \in W$ is called amenable with respect to s, if the form $\sum_{w \in W} f^{(r)}_{ik}(w)c_w$ is a linear combination of c_1, c_s, $c_w + c_{sw}$, $c_w - c_{ws}$ for all $w \in W$ with $w \ne 1,s$.

Proposition 5.4: (1) Every position -1 on the diagonal gives rise to an amenable equation.

(2) Every position which occurs in the same column as a -1 on the diago nal gives rise to an amenable equation.

(3) Every position u on the diagonal gives rise to an amenable equation.

(4) Every position which occurs in the same column and row as a u on the diagonal gives rise to an amenable equation.

Proof: In order to prove (1) and (2) we consider an element $v_k \in B_r$ such that $s \in I(v_k)$. Then $T_s v_k = -v_k$, $1 \le k \le d_{r_1}$. We write

$$\Phi(T_s) = c_1.1 + c_s.s + \sum_{\substack{w \in W \\ ws > w \\ w \ne 1}} c_w w + \sum_{\substack{w \in W \\ ws < w \\ w \ne s}} c_w.w =$$

$$= c_1.1 + c_s.s + \sum_{\substack{w \in W \\ ws > w \\ w \ne 1}} c_w.w + \sum_{\substack{w \in W \\ ws > w \\ w \ne 1}} c_{ws}.ws.$$

Thus,

$$\Phi(T_s)v_k = c_1 v_k + c_s(s.v_k) + \sum_{\substack{w \in W \\ ws > w \\ w \ne 1}} c_w(w.v_k) + \sum_{\substack{w \in W \\ ws > w \\ w \ne 1}} c_{ws}(ws).v_k =$$

$$= c_1 v_k - c_s v_k + \sum_{\substack{w \in W \\ ws > w \\ w \ne 1}} c_w(w.v_k) - \sum_{\substack{w \in W \\ ws > w \\ w \ne 1}} c_{ws}(w.v_k) =$$

$$= c_1 v_k - c_s v_k + \sum_{\substack{w \in W \\ ws>w \\ w \neq 1}} c_w \left(\sum_{i=1}^{d_r} f_{ik}^{(r)}(w) v_i \right) - \sum_{\substack{w \in W \\ ws>w \\ w \neq 1}} c_{ws} \left(\sum_{i=1}^{d_r} f_{ik}^{(r)}(w) v_i \right)$$

Therefore, by comparing coefficients on both sides of the equation

$T_s \cdot v_k = \phi(T_s) \cdot v_k = -v_k$, $1 \leq k \leq d_{r_1}$, we obtain

$$c_1 - c_s + \sum_{\substack{w \in W \\ ws>w \\ w \neq 1}} f_{kk}^{(r)}(w)(c_w - c_{ws}) = -1 \text{ and}$$

$$\sum_{\substack{w \in W \\ ws>w \\ w \neq 1}} f_{ik}^{(r)}(w)(c_w - c_{ws}) = 0 \text{ if } i \neq k,\ 1 \leq i \leq d_r$$

Both these equations are amenable with respect to s.

In order to prove (3) and (4) we consider an element $v_k \in B_r$ with $d_{r_1} < k \leq d_r$, i.e. $s \notin I(v_k)$. We now write

$$\phi(T_s) = c_1.1 + c_s.s + \sum_{\substack{w \in W \\ sw>w \\ w \neq 1}} c_w.w + \sum_{\substack{w \in W \\ sw<w \\ w \neq s}} c_w\, w$$

$$= c_1.1 + c_s s + \sum_{\substack{w \in W \\ sw>w \\ w \neq 1}} c_w w + \sum_{\substack{w \in W \\ sw>w \\ w \neq 1}} c_{sw} sw$$

So $\phi(T_s).v_k = c_1 v_k + c_s(s.v_k) + \sum_{\substack{w \in W \\ sw>w \\ w \neq 1}} c_w(w.v_k) + \sum_{\substack{w \in W \\ sw>w \\ w \neq 1}} c_{sw}(sw).v_k$

We recall that when $s \notin I(v_k)$, then

$$T_s v_k = u v_k + u^{\frac{1}{2}} \sum_{\substack{v_i \in B_r \\ s \in I(v_i) \\ \{v_i, v_k\} \in Y}} \mu(v_i, v_k).v_i$$

So $s \cdot v_k = v_k + \sum_{i=1}^{d_{r_1}} \mu(v_i,v_k)v_i, \quad \mu(v_i,v_k) \in \mathbb{Z}.$

We put $\mu(v_i,v_k) = \mu_{ik}$, so $s \cdot v_k = v_k + \sum_{i=1}^{d_{r_1}} \mu_{ik} v_i$.

Let $w \in W$ with $sw > w, w \neq 1$ and let $w \cdot v_k = \sum_{i=1}^{d_{r_1}} f_{ik}^{(r)}(w)v_i + \sum_{i=d_{r_1}+1}^{d_r} f_{ik}^{(r)}(w)v_i$

Then $(sw)v_k = -\sum_{i=1}^{d_{r_1}} f_{ik}^{(r)}(w)v_i + \sum_{i=d_{r_1}+1}^{d_r} f_{ik}^{(r)}(w)(v_i + \sum_{j=1}^{d_{r_1}} \mu_{ji} v_j)$.

Then $\phi(T_s) \cdot v_k = c_1 v_k + c_s(v_k + \sum_{i=1}^{d_{r_1}} \mu_{ik} v_i) +$

$$+ \sum_{\substack{w\in W\\ sw>w, w\neq 1}} c_w\Big(\sum_{i=1}^{d_{r_1}} f_{ik}^{(r)}(w)v_i + \sum_{i=d_{r_1}+1}^{d_r} f_{ik}^{(r)}(w)v_i\Big) +$$

$$+ \sum_{\substack{w\in W\\ sw>w, w\neq 1}} c_{sw}\Big[-\sum_{i=1}^{d_{r_1}} f_{ik}^{(r)}(w)v_i + \sum_{i=d_{r_1}+1}^{d_r} f_{ik}^{(r)}(w)(v_i + \sum_{j=1}^{d_{r_1}} \mu_{ji} v_j)\Big].$$

By comparing coefficients on both sides of the equation

$T_s v_k = \phi(T_s)v_k$, $d_{r_1} < k \le d_r$ we obtain:

$$c_1 + c_s + \sum_{\substack{w\in W\\ sw>w,\ w\neq 1}} f_{kk}^{(r)}(w)(c_w + c_{sw}) = u$$

and

$$\sum_{\substack{w\in W\\ sw>w, w\neq 1}} f_{ik}^{(r)}(w)(c_w + c_{sw}) = 0 \text{ if } i \neq k,\ d_{r_1} < i \le d_r$$

Both these equations are amenable with respect to s. Our proposition is now proved.

Remark 5.4: From the proof of (3) and (4) of Proposition 5.4 we see that if v_i, v_k are such that $\{v_i, v_k\} \in Y$, Y being the set of edges, then by comparing the coefficient of the element v_i on both sides of $T_s \cdot v_k = \phi(T_s) \cdot v_k$, we obtain the equation: $c_s\ \mu_{ik}$ + integral linear combination of $c_{w \neq 1,s} = u^{\frac{1}{2}}\ \mu_{ik}$, $\mu_{ik} \neq 0$, while if $\{v_i, v_k\} \notin Y$, $1 \leq i \leq d_{r_1}$ we obtain the equation: Integral linear combination of $c_{w \neq 1,s} = 0$.

We shall show that both types of these equations in the unknowns c_w are not amenable with respect to s.

Proposition 5.5: The total number of equations of type (1), (2), (3), (4) of Proposition 5.4, is equal to $|C_W(s)| + \frac{3}{4}(|W| - |C_W(s)|)$, and this number is the same as the total number of linearly independent combinations of c_1, c_s, $c_w + c_{sw}$, $c_w - c_{ws}$, where s is fixed and $w \in W$, $w \neq 1,s$. $C_W(s)$ denotes the centralizer of s in W.

Proof: The number of entries of the matrix $A_r(T_s)$ which represents T_s on the constituent V_r with respect to the ordered basis B_r, which give rise to amenable equations of type (1), (2), (3), (4) is $d_{r_1}^2 + d_{r_2}^2 + d_{r_1} d_{r_2}$. Thus, the total number of equations of these types obtained from all the inequivalent irreducible constituents is $\sum_{r=1}^{t} d_{r_1}^2 + \sum_{r=1}^{t} d_{r_2}^2 + \sum_{r=1}^{t} d_{r_1} d_{r_2}$.
The trace of the matrix $A_r(T_s)$ is: Trace $A_r(T_s) = ud_{r_2} - d_{r_1}$. By specializing $u \to 1$ we obtain the trace of the matrix which represents s on V_r with respect to the ordered basis B_r, $1 \leq r \leq t$. Let $C_W(s)$ be the centralizer of s in W, K_i the conjugacy class of s, $\chi_i^{(r)}$ be the irreducible character which corresponds to the constituent V_r, defined by $\chi_i^{(r)} =: \chi_i^{(r)}(x)$, $x \in K_i$.
The second orthogonality relation gives: $|C_W(s)| = \sum_{r=1}^{t} \chi_i^{(r)} \overline{\chi_i^{(r)}}$, where $\overline{\chi_i^{(r)}}$ is the complex conjugate of $\chi_i^{(r)}$. Here we have $\overline{\chi_i^{(r)}} = \chi_i^{(r)}$ because $s^2 = 1$.

Hence $|C_W(s)| = \sum_{r=1}^{t} (\chi_i^{(r)})^2 = \sum_{r=1}^{t} (d_{r_2} - d_{r_1})^2 = \sum_{r=1}^{t} d_{r_2}^2 + \sum_{r=1}^{t} d_{r_1}^2 - 2 \sum_{r=1}^{t} d_{r_1} d_{r_2}$.

So, $-|C_W(s)| = 2 \sum_{r=1}^{t} d_{r_1} d_{r_2} - \sum_{r=1}^{t} d_{r_2}^2 - \sum_{r=1}^{t} d_{r_1}^2$.

So, $|W| - |C_W(s)| = \sum_{r=1}^{t} d_r^2 + 2 \sum_{r=1}^{t} d_{r_1} d_{r_2} - \sum_{r=1}^{t} d_{r_2}^2 - \sum_{r=1}^{t} d_{r_1}^2 =$

$= \sum_{r=1}^{t} (d_{r_1} + d_{r_2})^2 + 2 \sum_{r=1}^{t} d_{r_1} d_{r_2} - \sum_{r=1}^{t} d_{r_2}^2 - \sum_{r=1}^{t} d_{r_1}^2 = 4 \sum_{r=1}^{t} d_{r_1} d_{r_2}$.

Hence $|C_W(s)| + \frac{3}{4} (|W| - |C_W(s)|) = \sum_{r=1}^{t} d_{r_1}^2 + \sum_{r=1}^{t} d_{r_2}^2 + \sum_{r=1}^{t} d_{r_1} d_{r_2}$.

Next we calculate the number of linearly independent combinations of $c_1, c_s, c_w + c_{sw}, c_w - c_{ws}$, $w \in W$, $w \neq 1,s$.
If $w \in C_W(s)$, $w \neq 1,s$, then w gives rise to two such combinations, namely $c_w + c_{sw}, c_w - c_{ws}$. For $sw = ws$ and the element sw gives rise to the combinations $c_{sw} + c_w, c_{sw} - c_{sws} = c_{ws} - c_w$, and these are linearly dependent on those already obtained. Thus, for every $w \in C_W(s)$, $w \neq 1,s$, the pair (w,sw) contributes to two such linearly independent combinations, and therefore the whole of $C_W(s)$, contributes to $|C_W(s)|$ such linearly independent combinations (counting c_1, c_s as well).

Now, if $w \in W$, $w \notin C_W(s)$, the quadruple (w, sw, ws, sws) contributes to three such linearly independent combinations, namely $c_w + c_{sw}, c_w - c_{ws}, c_{ws} + c_{sws}$. For, the combination $c_{sw} - c_{sws} = (c_w + c_{sw}) - (c_w - c_{ws}) - (c_{ws} + c_{sws})$ and this is a linear combination on those already obtained. Therefore, the total number of linearly independent combinations of $c_1, c_s, c_w + c_{sw}, c_w - c_{ws}$, $w \in W$, $w \neq 1,s$, is exactly $|C_W(s)| + \frac{3}{4} (|W| - |C_W(s)|)$, and our proposition is now proved.

We recall that these equations of type (1), (2), (3), (4) are all linearly independent as they are obtained from inequivalent irreducible representations of the group W (see procedure in §1.4 for the determination of Lusztig's isomorphism ϕ). Hence

Corollary 5.5: Both types of equations in the c_w, $w \in W$, obtained by the inequivalent irreducible representations of W, mentioned in the Remark 5.4, are not amenable with respect to s. For, otherwise we would have a system of N say linearly independent equations in M unknowns the c_1, c_s, $c_w + c_{sw}$, $c_w - c_{ws}$, $w \neq 1,s$, with $N > M$. The matrix of this system has rank at most M. So the rank is exactly M and therefore the remaining $N-M$ amenable equations with respect to s, would be linearly dependent on the previous M. This is a contradiction.

For example in case $W = S_4$, $s = s_1$, the first 0 in the third column of the first 3 x 3 matrix which represents T_{s_1}, is the right-hand side of the following equation:

$$c_{s_1s_2} + c_{s_1s_2s_1} + c_{s_1s_3s_2} - c_{s_1s_2s_3} + c_{s_1s_3s_2s_1} - c_{s_1s_2s_1s_3} + c_{s_2s_1s_3s_2} - c_{s_1s_2s_3s_2} - c_{s_1s_2s_3s_2s_1} - c_{s_1s_2s_1s_3s_2} + c_{s_2s_3s_2s_1s_2} - c_{s_1s_2s_1s_3s_2s_1}.$$

This equation is not amenable with respect to s_1.

Also, the u^1 appearing in this matrix is the right-hand side of the equation:

$$c_{s_1} - c_{s_1s_2} + c_{s_2s_1} + c_{s_1s_3} - c_{s_1s_2s_1} - c_{s_1s_3s_2} + c_{s_3s_2s_1} + c_{s_2s_1s_3} - c_{s_1s_3s_2s_1} - c_{s_2s_1s_3s_2} + c_{s_2s_3s_2s_1} - c_{s_2s_3s_2s_1s_2}.$$

This is also not amenable with respect to s_1.

We can now give an alternative proof to Theorem 5.1.

Second proof: Part (i) is now an immediate consequence of Propositions 5.4 and 5.5. For, we put $A = |C_W(s)| + \frac{3}{4}(|W| - |C_W(s)|)$. Then, the system of A-amenable equations with respect to s in the A-unknowns c_1, c_s, $c_w + c_{sw}$, $c_w - c_{ws}$, $w \neq 1,s$ has a unique solution. Such a solution is:

$c_1 = \frac{u-1}{2}$, $c_s = \frac{u+1}{2}$, $c_w + c_{sw} = 0$, $c_w - c_{ws} = 0$, $w \neq 1,s$.

So this is the only one.

In order to prove part (ii) we consider the set of $|W|$ equations in the $|W|$ unknowns c_w, $w \in W$, obtained from the several inequivalent irreducible representations of W. From these equations we omit those coming from the irreducible constituents which afford the representations of H

$\sigma_0 : T_s \to u \;\; \forall s \in S$, and $\sigma_s : T_s \to -1 \; \forall s \in S$. These equations are

$\sum_{w \in W} c_w = u$ and $\sum_{w \in W} (-1)^{\ell(w)} c_w = -1$ respectively.

So we are left with $|W| - 2$ linearly independent equations whose type is one of the following (see Proposition 5.4 and Remark 5.4)

$$c_1 + c_s + \text{integral linear combination of } \underset{w \neq 1,s}{c_w} = u$$

$$c_1 - c_s + \text{integral linear combination of } \underset{w \neq 1,s}{c_w} = -1$$

$$\lambda c_s + \text{integral linear combination of } \underset{w \neq 1,s}{c_w} = \lambda u^{\frac{1}{2}},\ \lambda \in \mathbb{Z},\ \lambda \neq 0$$

$$\text{integral linear combination of } \underset{w \neq 1,s}{c_w} = 0.$$

In these equations we replace c_1, c_s by $\frac{u-1}{2}$ and $\frac{u+1}{2}$ respectively and we divide the third type of equations by the non zero integer λ. Thus we obtain a system of $|W| - 2$ linearly independent equations in the $|W| - 2$ unknowns c_w, $w \neq 1,s$.

The coefficients of the unknowns c_w, $w \neq 1,s$ are rationals and the right-hand side of these equations is now either 0 or $(u^{\frac{1}{2}} -1)^2$. This system has a unique solution in the c_w, $w \neq 1,s$ and therefore we obtain that each c_w, $w \neq 1,s$ is a rational multiple of $(u^{\frac{1}{2}}-1)^2$. Thus

$\phi(T_s) = \frac{u-1}{2}\cdot 1 + \frac{u+1}{2}\cdot s + (u^{\frac{1}{2}}-1)^2 F_s$ where $F_s \in \mathbb{Q}W$ and F_s does not involve $1,s$.

Moreover, let $F_s = \sum_{w \neq 1,s} c'_w w$, $c'_w \in \mathbb{Q}$. Then,

$$sF_s = s(\sum_{w\neq 1,s} c'_w . w) = \sum_{w\neq 1,s} c'_w sw = \sum_{w\neq 1,s} c'_{sw} . w = -\sum_{w\neq 1,s} c'_w . w = -F_s,$$

and

$$F_s . s = (\sum_{w\neq 1,s} c'_w . w)s = \sum_{w\neq 1,s} c'_w ws = \sum_{w\neq 1,s} c'_{ws} w = \sum_{w\neq 1,s} c'_w . w = F_s.$$

Thus $(sF_s)^2 = sF_s\, sF_s = sF_s^2 = -F_s^2$ and $(sF_s)^2 = F_s^2$, so $F_s^2 = 0$.

Corollary 5.6: (i) If $w \in C_W(s)$, $w \neq 1,s$ then $c_w = 0$

(ii) $$\sum_{\substack{\ell(w)=\text{odd}\\ w\neq s}} c_w = \sum_{\substack{\ell(w)=\text{even}\\ w\neq 1}} c_w = 0$$

Proof: (i) is obvious and for (ii) we note that since $c_w = c_{ws}$, $w \neq 1,s$

we have $$\sum_{\substack{\ell(w)=\text{odd}\\ w\neq s}} c_w = \sum_{\substack{\ell(w)=\text{even}\\ w\neq 1}} c_w .$$

The equation which is obtained from the one dimensional constituent which affords the representation $T_s \to u$ $\forall s \in S$, is

$$c_1 + c_s + \sum_{w\neq 1,s} c_w = u, \text{ so } \quad 2\sum_{\substack{\ell(w)=\text{even}\\ w\neq 1}} c_w = 0, \text{so} \quad \sum_{\substack{\ell(w)=\text{even}\\ w\neq 1}} c_w = 0$$

Proposition 5.7: Let W be a finite indecomposable Coxeter group not of type E_7, E_8, H_3, H_4 and let H be the generic Hecke algebra over the polynomial ring $\mathbb{Q}[u]$ associated with W. Let $c = \sum_{w\in W} a_w T_w$, $a_w \in \mathbb{Q}[u]$ be an element of the centre of H. Then, the image $\phi(c)$ of c under the Lusztig isomorphism ϕ belongs to $\mathbb{Q}[u](W)$.

Proof: The case where W is a Coxeter group of dihedral type has been treated in Chapter 4, so we may assume that W is a crystallographic Coxeter group. We consider the graded module grad(E) over $\mathbb{Q}(u^{\frac{1}{2}})$ as a left $H_{\mathbb{Q}(u^{\frac{1}{2}})}$-module with action described in §1.4. Then it has a decomposition into a direct sum of left $H_{\mathbb{Q}(u^{\frac{1}{2}})}$ irreducible submodules each one occurring with multiplicity equal to its dimension. Let V be such an irreducible submodule. Then V can be afforded by an even W-graph over Z (see Theorem 5.3(2)). In other words, there exists a basis X of V such that for any $x \in X$ and $s \in W$ we have

$$T_s x = \begin{cases} -x & \text{if } s \in I(x) \\ ux + u^{\frac{1}{2}} \sum\limits_{\substack{y \in X \\ s \in I(y)}} \mu(y,x)y & \text{if } s \notin I(x) \end{cases}$$

with $(y,x) \in Z$ and there is also a map sgn : $X \to \{-1,1\}$ such that $\mu(y,x)\,\mathrm{sgn}(y)\mathrm{sgn}(x) = -\mu(y,x)$ for any distinct x,y in X. We shall show that for any $w \in W$, $T_w x = \sum\limits_{\substack{z \in X \\ \mathrm{sgn}(z)\mathrm{sng}(x)=1}} \lambda_z(u)z + \sum\limits_{\substack{z \in X \\ \mathrm{sgn}(z)\mathrm{sgn}(x)=-1}} u^{\frac{1}{2}} k_z(u)z$, where $\lambda_z(u)$, $k_z(u)$ belong to Z[u]. As in Lemma 1.3.7 we argue by induction on $\ell(w)$. When $\ell(w) = 0$, $w = 1$ and we have nothing to prove. We assume that our assertion holds for all elements w' with $\ell(w') < \ell(w)$ and let $w = sv$ with $\ell(v) = \ell(w)-1$. Then by induction we have $T_v x = \sum\limits_{\substack{z \in X \\ \mathrm{sgn}(z)\mathrm{sgn}(x)=1}} \lambda_z(u)z + \sum\limits_{\substack{z \in X \\ \mathrm{sgn}(z)\mathrm{sgn}(x)=-1}} u^{\frac{1}{2}}k_z(u)z =$

$$= \sum_{\substack{z \in X \\ s \in I(z),\mathrm{sgn}(z)\mathrm{sgn}(x)=1}} \lambda_z(u)z + \sum_{\substack{z \in X \\ s \notin I(z),\mathrm{sgn}(z)\mathrm{sgn}(x)=1}} \lambda_z(u)z +$$

$$+ \sum_{\substack{z \in X \\ s \in I(z),\mathrm{sgn}(z)\mathrm{sgn}(x)=-1}} u^{\frac{1}{2}}k_z(u)z + \sum_{\substack{z \in X \\ s \notin I(z),\mathrm{sgn}(z)\mathrm{sgn}(x)=-1}} u^{\frac{1}{2}}k_z(u)z$$

Therefore $T_s\, T_v x = - \sum\limits_{\substack{z \in X \\ s \in I(z),\mathrm{sgn}(z)\mathrm{sgn}(x)=1}} \lambda_z(u)z +$

$$+ \sum_{\substack{z \in X \\ s \notin I(z) \\ \mathrm{sgn}(z)\mathrm{sgn}(x)=1}} \lambda_z(u)\,[uz + u^{\frac{1}{2}} \sum_{\substack{y \in X \\ s \in I(y)}} \mu(y,z)y] - \sum_{\substack{z \in X \\ s \in I(z)\mathrm{sgn}(z)\mathrm{sgn}(x)=-1}} u^{\frac{1}{2}}k_z(u)z +$$

$$+ \sum_{\substack{z \in X \\ s \notin I(z), sgn(z)sgn(x)=-1}} u^{\frac{1}{2}} k_z(u) \, [uz + u^{\frac{1}{2}} \sum_{\substack{y' \in X \\ s \in I(y')}} \mu(y',z)y']$$

Now we note that the coefficient of y in the expression above is a polynomial of the form $u^{\frac{1}{2}} f_y(u)$, $f_y(u) \in Z[u]$ and these y have sgn(y) = - sgn(z), so sgn(y) = - sgn(x) while the coefficient of y' is a polynomial $g_{y'}(u) \in Z[u]$ and these y' have sgn(y') = sgn(x). So our induction is now complete.

In particular the diagonal entries of the matrix which represents any T_w, $w \in W$ with respect to the basis X are polynomials in Z[u]. Hence if $c = \sum_{w \in W} a_w T_w$, $a_w \in Q[u]$ is an element in the centre of H, then c is represented on the irreducible module V of dimension say d with respect to the basis X, by a scalar multiple of the identity matrix I_d, and therefore this scalar is necessarily a polynomial in Q[u]. Since c and $\phi(c)$ act on the same way on the graded module we conclude that $\phi(c)$ belongs to Q[u](W).

Remark 5.8: The result above fails if W is one of the Coxeter groups E_7, E_8, H_3, H_4. For instance let W be the Coxeter group of type H_3. A decomposition of the graded module over $Q(u^{\frac{1}{2}}, \sqrt{5})$ into a direct sum of irreducible left H-submodules and a W-graph for each one of them is provided in [12] page 496-7. It can be shown that $H_3 \simeq A_5 \times C_2$ (see [5] Ch. 6, page 231, exercise 11), where A_5 is the alternating group on 5 symbols. Therefore the centre of H_3 is a cyclic group of order 2, say Z = {1,a}, $a^2 = 1$.

The element a is represented on every irreducible representation of W by a scalar multiple of the identity matrix, and since $a^2 = 1$ this scalar is either 1 or -1. When we consider the reflection module V (see Ch. I) which is a faithful module, then this scalar must be -1. So a transforms every positive root into a negative one, and therefore $a = w_0$, the element of maximal length in W, so $Z = \{1, w_0\}$.

There are three fundamental reflections s_1, s_2, s_3, and we know that for every $i = 1,2,3$, there exists an element $x_i \in W$ such that $w_o = s_i x_i$ with $\ell(w_o) = \ell(s_i x_i) = \ell(x_i) + 1$. Since w_o commutes with s_i we also have $w_o = x_i s_i$. Now $T_{s_i} T_{w_o} = T_{s_i} T_{s_i x_i} = uT_{x_i} + (u-1)T_{w_o}$ and

$$T_{w_o} T_{s_i} = T_{x_i s_i} T_{s_i} = uT_{x_i} + (u-1)T_{w_o}.$$

So T_{w_o} commutes with every T_{s_i} $i = 1,2,3$, therefore T_{w_o} belongs to the centre of H.

Nevertheless there exist irreducible representations of H on which T_{w_o} is represented by $u^{15/2}$ or $-u^{15/2}$ (see [12], page 497).

The following result relates an algebra defined by Gyoja (see [9]) to the Lusztig isomorphism. We denote Gyoja's algebra over the polynomial ring $\mathbb{Z}[u^{\frac{1}{2}}]$, associated to a Coxeter group W by $G(W)_{\mathbb{Z}[u^{\frac{1}{2}}]}$, $u^{\frac{1}{2}}$ being an indeterminate over $\mathbb{Z}$. This is an algebra given by the following presentation. For every generator s of W, $G(W)_{\mathbb{Z}[u^{\frac{1}{2}}]}$ has generators s(0) and s(1) subject to the relations

$$s(0)^2 = s(0) \qquad \text{(R)}$$

$$s(0)s'(0) = s'(0)s(0)$$

$$s(0)s(1) = s(1)$$

$$s(1)s(0) = 0$$

together with additional relations given as follows:

Let $\tilde{T}_s$ be an element of $G(W)_{\mathbb{Z}[u^{\frac{1}{2}}]}$ defined by $\tilde{T}_s = -s(0)+u(1-s(0))+u^{\frac{1}{2}}s(1)$.

Then, we require that the elements $\tilde{T}_s$ satisfy the homogeneous Coxeter relations, i.e. $\underbrace{\tilde{T}_{s_i} \tilde{T}_{s_j} \tilde{T}_{s_i} \ldots}_{m_{ij}} = \underbrace{\tilde{T}_{s_j} \tilde{T}_{s_i} \tilde{T}_{s_j} \ldots}_{m_{ij}}$, m_{ij} being the order of $s_i s_j$, $i \neq j$

The relations (R) above imply that $s(1)^2 = 0$ and this enables us to show that $\bar{T}_s$ also satisfies the quadratic relation i.e. $\bar{T}_s^2 = u1 + (u-1)\bar{T}_s$. Thus, we have an algebra homomorphism $\phi_u : H_{Z[u^{\frac12}]} \to G(W)_{Z[u^{\frac12}]}$ defined by

$\phi_u(T_s) = \bar{T}_s$, $s \in S$,

where H is the generic Hecke algebra over $Z[u^{\frac12}]$ associated to W.

Let E be Lusztig's graded module over Z, with canonical basis $\{e_w \; w \in W\}$. Gyoja showed that E can be made into a left (similarly right) G(W) module by defining

$$s(0)e_w = \begin{cases} e_w & \text{if } sw < w \\ 0 & \text{if } sw > w \end{cases} \qquad (R_1)$$

and

$$s(1)e_w = \begin{cases} \sum\limits_{y \underset{LR}{\sim} w,\ sy<y} \tilde{\mu}(y,w)e_y & \text{if } sw > w \\ 0 & \text{if } sw < w. \end{cases} \qquad (R_2)$$

The interpretation of $y \underset{LR}{\sim} w$ and $\tilde{\mu}(y,w)$ is given in §1.4. This action gives an action of $\tilde{T}_s$ on $E_{\mathbb{Q}(u^{\frac12})}$ because we can easily verify that

$$\tilde{T}_s\, e_w = \begin{cases} -e_w & \text{if } sw < w \\ ue_w + u^{\frac12} \sum\limits_{y \underset{LR}{\sim} w \;\; sy<y} \tilde{\mu}(y,w)e_y & \text{if } sw > w \end{cases}$$

Therefore the left and right G(W) action on E induces a left and right $H_{\mathbb{Q}(u^{\frac12})}$ action on $E_{\mathbb{Q}(u^{\frac12})}$ by defining $T_s\, e_w =: \phi_u(T_s)e_w = \bar{T}_s e_w$.

We know that $E_{\mathbb{Q}(u^{\frac12})}$ affords the two-sided regular representation of $H_{\mathbb{Q}(u^{\frac12})}$. Hence the map ϕ_u is injective and so we can regard $H_{\mathbb{Q}(u^{\frac12})}$ as a subalgebra of $G(W)_{\mathbb{Q}(u^{\frac12})}$. Let $\mathrm{End}_o\,(E_{\mathbb{Q}(u^{\frac12})})$ be the endomorphism of E which commute with the right $H_{\mathbb{Q}(u^{\frac12})}$ action. Since $E_{\mathbb{Q}(u^{\frac12})}$ affords the two sided regular representation of $H_{\mathbb{Q}(u^{\frac12})}$, the left $H_{\mathbb{Q}(u^{\frac12})}$ action on $E_{\mathbb{Q}(u^{\frac12})}$ gives rise to an algebra isomorphism $a : H_{\mathbb{Q}(u^{\frac{1}{2}})} \simeq End_0(E_{\mathbb{Q}(u^{\frac{1}{2}})})$.

Moreover Gyoja showed that the left and right G(W) action on E commute (see [9] Lemma 2.11).

Let $b : G(W) \to End_0(E_{\mathbb{Q}(u^{\frac{1}{2}})})$ be the algebra homomorphism defined by the left G(W) action on $E_{\mathbb{Q}(u^{\frac{1}{2}})}$. Then the map $a^{-1}b$ restricted to $H_{\mathbb{Q}(u^{\frac{1}{2}})}$ is the identity map, i.e. $a^{-1} b|_{H_{\mathbb{Q}(u^{\frac{1}{2}})}} = 1_{H_{\mathbb{Q}(u^{\frac{1}{2}})}}$, and hence the map b is surjective.

<u>Proposition 5.9</u>: Let W be a finite crystallographic Coxeter group and let $\phi(T_s) = f_s(u) + u^{\frac{1}{2}}g_s$, be the image of the generator T_s of the generic Hecke algebra over $\mathbb{Q}(u^{\frac{1}{2}})$ under the Lusztig isomorphism, where $f_s(u) = \frac{u-1}{2}.1 + \frac{u+1}{2}.s - \frac{u+1}{2} g_s$ and $g_s \in \mathbb{Q}W$. Then, there exists a surjective homomorphism of algebras $\theta: G_{\mathbb{Q}}(W) \to \mathbb{Q}W$ such that

$$\theta(s(0)) = \frac{1}{2}(1 - s + g_s) \text{ and } \theta(s(1)) = g_s .$$

<u>Proof</u>: By specializing $u^{\frac{1}{2}} \to 1$ we obtain an algebra homomorphism $\phi_1 : \mathbb{Q}W \to G_{\mathbb{Q}}(W)$ such that $\phi_1(s) = -s(0) + (1-s(0)) + s(1)$.

Therefore the left and right action of the element $-s(0) + (1-s(0)) + s(1)$ on $E_{\mathbb{Q}}$ gives rise to a left and right action of the group algebra $\mathbb{Q}W$ on $E_{\mathbb{Q}}$. In fact $E_{\mathbb{Q}}$ affords the two sided regular representation of W. Therefore, the left W action on $E_{\mathbb{Q}}$ induces an algebra isomorphism $\bar{a} : \mathbb{Q}W \simeq End_0(E_{\mathbb{Q}})$ where $End_0(E_{\mathbb{Q}})$ are the endomorphisms of $E_{\mathbb{Q}}$ which commute with the right W action. Let $\bar{b}$ be the algebra homomorphism induced by the left $G_{\mathbb{Q}}(W)$ action on $E_{\mathbb{Q}}$. Then $\bar{b} : G_{\mathbb{Q}}(W) \to End_0(E_{\mathbb{Q}})$ is surjective, since $\bar{a}^{-1} \bar{b}|_{\mathbb{Q}W} = 1_{\mathbb{Q}W}$. Hence, under the surjective map $\bar{a}^{-1} \bar{b} : G_{\mathbb{Q}}(W) \to \mathbb{Q}W$, every element g of $G_{\mathbb{Q}}(W)$ maps to an element of $\mathbb{Q}W$ which is determined by the property that induces the same element of $End_0(E_{\mathbb{Q}})$ as the element g. Such an element inside $\mathbb{Q}W$ is unique. We put $\theta = \bar{a}^{-1} \bar{b}$.

Now we consider the endomorphisms of E_q induced by the elements $s(0)$ and $s(1)$ (see relations (R_1) and (R_2) above).

Let $\phi(T_s) = f_s(u) + u^{\frac{1}{2}} g_s$, $g_s \in QW$,

$$f_s(u) = \frac{u-1}{2}\cdot 1 + \frac{u+1}{2}\cdot s - \frac{u+1}{2}\, g_s = u\cdot 1 - \frac{u+1}{2}(1 - s + g_s) .$$

We recall that with respect to a suitable arrangement of the elements of the canonical basis of E, the matrices which represent $f_s(u)$, g_s are respectively

$$\begin{bmatrix} -I & 0 \\ 0 & uI \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & E_s \\ 0 & 0 \end{bmatrix}$$

(see, first proof of Theorem 5.1).

By specializing $u^{\frac{1}{2}} \to 1$ we obtain the matrix which represents $s \in W$ and which is $\begin{bmatrix} -I & E_s \\ 0 & I \end{bmatrix}$

Now if $sw < w$ then $\frac{1}{2}(1-s+g_s).e_w = \frac{1}{2} e_w - \frac{1}{2}(-e_w) + 0.e_w = e_w$, while if $sw > w$, then $\frac{1}{2}(1-s+g_s).e_w = \frac{1}{2} e_w - \frac{1}{2} e_w + 0.e_w = 0$.

In other words, the element $\frac{1}{2}(1-s+g_s)$ induces the same endomorphism of E_q as the element $s(0)$ and hence $\theta(s(0)) = \frac{1}{2}(1-s+g_s)$.

We also have that if $sw < w$ then $g_s.e_w = 0 = s(1)e_w$, while if $sw > w$ then

$T_s\, e_w = ue_w + u^{\frac{1}{2}} \sum_{y \underset{LR}{\prec} w,\ sy<y} \bar{\mu}(y,w)e_y$ (see left action of H on the graded module, in §1.4).

Hence $g_s e_w = \sum_{y \underset{LR}{\prec} w,\ sy<y} \bar{\mu}(y,w)e_y = s(1)e_w$ if $sw > w$

In other words, the element g_s induces the same endomorphism of E_q, as the element $s(1)$, and hence $\theta(s(1)) = g_s$. Our proposition is now proved.

CHAPTER 6

6.1 A maximal commutative subalgebra of the generic Hecke algebra.

Let W be a finite crystallographic Coxeter group and let H be the generic Hecke algebra over $K = \mathbb{Q}(u^{\frac{1}{2}})$, which is a splitting field for H. Let $V_1, V_2, \ldots, V_s$ be a full set of left irreducible H-modules with K-dimensions d_i, $i = 1, \ldots, s$ respectively. According to Theorem 5.3(1), every V_i can be afforded by a W-graph over $\mathbb{Z}$. Such a W-graph determines for each V_i a K-basis with properties described in the previous chapter, and therefore we obtain in this way a full set of irreducible matrix representations for H, namely $\Lambda_1, \Lambda_2, \ldots, \Lambda_s$.

Since H is semisimple we obtain a K-algebra isomorphism

$$\pi : H \to \coprod_{i=1}^{s} M_{d_i}(K) \quad \text{such that}$$

$$\pi(h) = (\Lambda_1(h), \ldots, \Lambda_s(h)), \quad \forall h \in H.$$

Inside H we define $M(u) =: \{h \in H$ such that $\Lambda_i(h)$ is a diagonal matrix $\forall\, i = 1, \ldots, s\}$. Then clearly $M(u)$ is a maximal commutative subalgebra of H.

It is clear that the K-dimension of $M(u)$ is $\sum_{i=1}^{s} d_i$. It is also important to emphasize that the definition of $M(u)$ depends on the chosen W-graph.

On the other hand we define inside H a subset

$$L(u) = \left\{ \sum_{w \in W} c_w T_w \in H \text{ such that } c_w - c_{ws} + uc_{sw} - u\, c_{sws} = 0 \text{ for all pairs } (w,s)\ w \in W,\ s \in S \text{ such that } \ell(sws) = \ell(w) + 2. \right\}$$

It is clear by its definition that $L(u)$ does not depend on any W-graph. There seems to be an interesting connection between $M(u)$ and $L(u)$.

<u>Proposition 6.1</u>: Let $M(u)$, $L(u)$ defined as above. Then $M(u) \subseteq L(u)$.

<u>Proof</u>: Let $h = \sum_{w \in W} c_w T_w$ be an element of $M(u)$ and we fix a generator $s \in S$. We know that for any $w \in W$ we have either $\ell(sws) = \ell(w)$ or $\ell(sws) = \ell(w) \pm 2$. If $w \notin C_W(s)$ and if $\ell(sws) = \ell(w)$ then $\ell(sw) \neq \ell(ws)$ (see Proposition 1.1 (iv)). Therefore we can write

$$h = \sum_{\substack{w \in W \\ \ell(sws)=\ell(w)+2}} c_w T_w + \sum_{\substack{w \in W,\ w \notin C_W(s) \\ \ell(sws)=\ell(w)}} c_w T_w + \sum_{\substack{w \in W \\ \ell(sws)=\ell(w)-2}} c_w T_w + \sum_{\substack{w \in W \\ w \in C_W(s)}} c_w T_w =$$

$$= \sum_{\substack{w \in W \\ \ell(sws)=\ell(w)+2}} c_w T_w + \sum_{\substack{w \in W,\ w \notin C_W(s) \\ \ell(sw) > \ell(w) \\ \ell(ws) < \ell(w)}} c_w T_w + \sum_{\substack{w \in W,\ w \notin C_W(s) \\ \ell(sw) < \ell(w) \\ \ell(ws) > \ell(w)}} c_w T_w +$$

$$+ \sum_{\substack{w \in W \\ \ell(sws)=\ell(w)-2}} c_w T_w + \sum_{\substack{w \in W,\ w \in C_W(s) \\ \ell(ws) > \ell(w)}} c_w T_w + \sum_{\substack{w \in W,\ w \in C_W(s) \\ \ell(ws) < \ell(w)}} c_w T_w =$$

$$= \sum_{\substack{w \in W \\ \ell(sws)=\ell(w)+2}} c_w T_w + \sum_{\substack{w \in W \\ \ell(sws)=\ell(w)+2}} c_{ws} T_{ws} + \sum_{\substack{w \in W \\ \ell(sws)=\ell(w)+2}} c_{sw} T_{sw} + \sum_{\substack{w \in W \\ \ell(sws)=\ell(w)+2}} c_{sws} T_{sws} +$$

$$+ \sum_{\substack{w \in C_W(s) \\ \ell(ws) > \ell(w)}} c_w T_w + \sum_{\substack{w \in C_W(s) \\ \ell(ws) > \ell(w)}} c_{ws} T_{ws} .$$

Let B_r be the basis determined by a W-graph and adapted to V_r $1 \le r \le s$. Then we can arrange suitably the basis elements (as in Chapter 5), so that the matrix which represents T_s, s fixed, with respect to B_r has the form

$$\begin{bmatrix} -I_{d_{r_1}} & u^{\frac{1}{2}} E_s \\ 0 & uI_{d_{r_2}} \end{bmatrix}$$

where $I_{d_{r_1}}$ and $I_{d_{r_2}}$ are the identity matrices of size d_{r_1} and d_{r_2} respectively, E_s is a $d_{r_1} \times d_{r_2}$ matrix and 0 is the $d_{r_2} \times d_{r_1}$ zero matrix, and $d_{r_1} + d_{r_2} = d_r$. Let $\begin{bmatrix} A_w^r & B_w^r \\ K_w^r & L_w^r \end{bmatrix}$ be the matrix which represents an element T_w, $\ell(sws) = \ell(w) + 2$ on V_r with respect to B_r, where A_w^r is a $d_{r_1} \times d_{r_1}$ matrix, B_w^r is a $d_{r_1} \times d_{r_2}$ matrix, K_w^r is a $d_{r_2} \times d_{r_1}$ matrix, and L_w^r is a $d_{r_2} \times d_{r_2}$ matrix.

Then, the matrix which represents T_{ws} is

$$\begin{bmatrix} -A_w^r & u^{\frac{1}{2}}A_w^rE_s + uB_w^r \\ -K_w^r & u^{\frac{1}{2}}K_w^rE_s + uL_w^r \end{bmatrix} \qquad \ell(sws) = \ell(w)+2$$

the matrix which represents T_{sw} is

$$\begin{bmatrix} -A_w^r + u^{\frac{1}{2}}E_sK_w^r & -B_w^r + u^{\frac{1}{2}}E_sL_w^r \\ uK_w^r & uL_w^r \end{bmatrix} \qquad \ell(sws) = \ell(w)+2$$

and the matrix which represents T_{sws}, $\ell(sws) = \ell(w)+2$ is

$$\begin{bmatrix} A_w^r - u^{\frac{1}{2}}E_sK_w^r & -u^{\frac{1}{2}}A_w^rE_s - uB_w^r + uE_sK_w^rE_s + u^{3/2}E_sL_w^r \\ -uK_w^r & u^{3/2}K_w^rE_s - u^2L_w^r \end{bmatrix}$$

If $w \in C_w(s)$ with $\ell(ws) > \ell(w)$, then since $T_{ws} = T_{sw}$, we have that

$$\begin{bmatrix} -A_w^r & u^{\frac{1}{2}}A_w^rE_s + uB_w^r \\ -K_w^r & u^{\frac{1}{2}}K_w^rE_s + uL_w^r \end{bmatrix} = \begin{bmatrix} -A_w^r + u^{\frac{1}{2}}E_sK_w^r & -B_w^r + u^{\frac{1}{2}}E_sL_w^r \\ uK_w^r & uL_w^r \end{bmatrix}$$

Therefore, if $w \in C_w(s)$, $\ell(ws) > \ell(w)$, then $K_w^r = 0$. Hence, the matrix which represents h on V_r with respect to B_r has the form

$$\begin{bmatrix} * & * \\ \sum_{w \in W : \ell(sws)=\ell(w)+2} (c_w - c_{ws} + uc_{sw} - uc_{sws})K_w^r & * \end{bmatrix}$$

Having assumed that $h \in M(u)$, we must have that

$$\sum_{w \in W \,:\, \ell(sws)=\ell(w)+2} (c_w - c_{ws} + u\, c_{sw} - uc_{sws}) K_w^r = 0.$$

As we have mentioned above, for each w with $\ell(sws) = \ell(w)+2$, K_w^r is a $d_{r_2} \times d_{r_1}$ matrix. We consider the quantities $c_w - c_{ws} + uc_{sw} - uc_{sws}$ as unknowns, $w \notin C_w(s)$, $\ell(sws) = \ell(w)+2$. From each V_r, $1 \leq r \leq s$ we obtain $d_{r_1} d_{r_2}$ homogeneous equations in these unknowns, hence from the full set of irreducible H-modules $\{V_1,\ldots,V_s\}$ we obtain $\sum_{r=1}^{s} d_{r_1} d_{r_2}$ homogeneous equations in these unknowns.

On the other hand, the number of these unknowns is the same as the number of distinct quadruples (w,ws,sw,sws), $w \notin C_w(s)$, which is equal to $\frac{|W| - |C_w(s)|}{4}$. The latter number is equal to $\sum_{r=1}^{s} d_{r_1} d_{r_2}$ (see Proposition 5.8).

Let W^1 be the subset of W defined by $W^1 = \{w \in W : \ell(sws) = \ell(w)+2\}$. Let $K_w^r = (f_{ij}^{(r)}(T_w))$ $w \in W^1$, $1 \leq r \leq s$. We shall show that the functions $f_{ij}^{(r)} : T_w \to f_{ij}^{(r)}(T_w)$, $w \in W^1$ for all r, i, j are linearly independent over K.

In fact, let $\sum_{i,j,r} a_{ij}^{(r)} f_{ij}^{(r)}(T_w) = 0$ for all $w \in W^1$. We note that $-f_{ij}^{(r)}(T_w) = f_{ij}^{(r)}(T_{ws})$, $uf_{ij}^{(r)}(T_w) = f_{ij}^{(r)}(T_{sw})$, and $-uf_{ij}^{(r)}(T_w) = f_{ij}^{(r)}(T_{sws})$ for $w \in W^1$. We recall that if $w \in C_w(s)$ then $K_w^{(r)} = 0$, while if $w \notin C_w(s)$ then it gives rise to quadruple $(x,xs,\ sx,\ sxs)$ with $\ell(sxs) = \ell(x) + 2$ and $x \in \{w,ws,sw,sws\}$.

Hence if $\sum_{i,j,r} a_{ij}^{(r)} f_{ij}^{(r)}(T_w) = 0$ for all $w \in W^1$, then $\sum_{i,j,r} a_{ij}^{(r)} f_{ij}^{(r)}(T_w) = 0$ for all $w \in W$, and so $\sum_{i,j,r} a_{ij}^{(r)} f_{ij}^{(r)}(h) = 0$ for all $h \in H$.
So, (by 3.41 in [7]) we obtain $a_{ij}^{(r)} = 0$, for all i,j,r.

Thus our system of $\sum_{r=1}^{s} d_{r_1} d_{r_2}$ homogeneous equations in the same number of unknowns, the $c_w - c_{ws} + uc_{sw} - uc_{sws}$, $\ell(sws) = \ell(w)+2$ has only the trivial solution.

In other words $c_w - c_{ws} + uc_{sw} - uc_{sws} = 0$ for all w with $\ell(sws) = \ell(w)+2$.

Our proposition is now proved.

Remark: The argument above applies to any finite Coxeter group, not necessarily crystallographic, provided that every irreducible H_K module can be afforded by a W-graph, and K is a splitting field of H.

Now let ℓ_I be the maximum number of linearly independent expressions of the form $c_w - c_{ws} + uc_{sw} - uc_{sws}$ for all pairs (w,s) $w \in W$, $s \in S$ such that $\ell(sws) = \ell(w) + 2$.

In other words ℓ_I is the rank of the matrix determined by a certain number of homogeneous equations of the form $c_w - c_{ws} + uc_{sw} - uc_{sws} = 0$, for all pairs (w,s) such that $\ell(sws) = \ell(w)+2$, in the $|W|$ unknowns c_w, $w \in W$.

Then, the K dimension of $L(u)$ is $|W| - \ell_I \geq \sum_{i=1}^{s} d_i$. By specializing $u^{\frac{1}{2}} \to 1$, every V_i becomes a left irreducible $\mathbb{Q}W$ module, and we can similarly define $M(1) = \{\bar{h} \in \mathbb{Q}W$ such that $\Lambda_i(\bar{h})$ is a diagonal matrix $\forall$ $i = 1,\ldots,s\}$, and $L(1) = \{ \sum_{w\in W} c_w.w,\ c_w \in \mathbb{Q}$ such that $c_w - c_{ws} + c_{sw} - c_{sws} = 0$, for all pairs (W,s) $w \in W$, $s \in S$ such that $\ell(sws) = \ell(w)+2\}$.

An entirely similar argument as in Proposition 6.1 shows (by specializing $u^{\frac{1}{2}} \to 1$) that $M(1) \subseteq L(1)$.

Let $\tilde{\ell}_I$ be the rank of the matrix determined by a certain number of homogeneous equations of the form $c_w - c_{ws} + c_{sw} - c_{sws} = 0$ for all pairs (w,s) such that $\ell(sws) = \ell(w)+2$, in the $|W|$ unknowns c_w, $w \in W$. Then the dimension of $L(1)$ is $|W| - \bar{\ell}_I \geq \sum_{i=1}^{s} d_i$. We next recall that the group algebra $\mathbb{Q}W$ is isomorphic as a $\mathbb{Q}$-algebra with the algebra of $\mathbb{Q}$-valued functions $f:W \to \mathbb{Q}$, under convolution product, with the element $\sum_{w\in W} c_w w$ corresponding to the function f, defined by $f(w) = c_w$, $w \in W$, $c_w \in \mathbb{Q}$.

If f, g are $\mathbb{Q}$-valued functions on W, their convolution product is defined as the function $f \cdot g : W \to \mathbb{Q}$ given by

$$(f \cdot g)(w) = \sum_{z \in W} f(wz^{-1})g(z).$$

Proposition 6.2: (1) L(1) is a subalgebra of $\mathbb{Q}W$.

(2) $\dim L(u) \leq \dim L(1)$.

Proof: (1) It is enough to show that L(1) is closed under multiplication. We note that every element of L(1) determines a function $f : W \to \mathbb{Q}$ such that $f(w) - f(ws) + f(sw) - f(sws) = 0$ for all $w \in W$, $s \in S$.

Let f,g be two such functions. We shall show that the function $h : W \to \mathbb{Q}$, defined by $h(w) = \sum_{x \in W} f(wx^{-1})g(x)$ has also the property $h(w)-h(ws)+h(sw)-h(sws)=0$ for all $w \in W$, $s \in S$.

In fact $h(w)-h(ws)+h(sw)-h(sws) =$

$$= \sum_{x \in W} f(wx^{-1})g(x) - \sum_{x \in W} f(wsx^{-1})g(x) + \sum_{x \in W} f(swx^{-1})g(x) - \sum_{x \in W} f(swsx^{-1})g(x)$$

$$= \sum_{x \in W} f(wx^{-1})g(x) - \sum_{x \in W} f(wx^{-1})g(xs) + \sum_{x \in W} f(swx^{-1})g(x) - \sum_{x \in W} f(swx^{-1})g(xs)$$

$$= \sum_{x \in W} (f(wx^{-1}) + f(swx^{-1}))(g(x) - g(xs)) =$$

$$= \sum_{\substack{x \in W \\ \ell(sx)>\ell(x)}} (f(wx^{-1})+f(swx^{-1}))(g(x)-g(xs)) + \sum_{\substack{x \in W \\ \ell(sx)<\ell(x)}} f(wx^{-1})+f(swx^{-1}))(g(x)-g(xs)$$

$$= \sum_{\substack{x \in W \\ \ell(sx)>\ell(x)}} (f(wx^{-1}) + f(swx^{-1}))\ (g(x) - g(xs)) +$$

$$+ \sum_{\substack{x \in W \\ \ell(sx)>\ell(x)}} (f(wx^{-1}s) + f(swx^{-1}s))\ (g(sx) - g(sxs)) =$$

$$= \sum_{\substack{x \in W \\ \ell(sx)>\ell(x)}} (f(wx^{-1})-f(wx^{-1}s)+f(swx^{-1}) - f(swx^{-1}s))(g(x) -g(xs)) = 0.$$

(2) In order to prove (2) we observe that the matrix say $\tilde{A}$ determined by the homogeneous equations $c_w - c_{ws} + c_{sw} - c_{sws} = 0$ for all pairs (w,s) such that $\ell(sws) = \ell(w) + 2$ in the $|W|$ unknowns c_w, $w \in W$ has entries 1, -1 or 0. Let $\tilde{r}$ be the rank of $\tilde{A}$. Then there exists an $\tilde{r} \times \tilde{r}$ minor whose determinant $\tilde{D}$ is non zero. Let $\tilde{B}$ be the $\tilde{r} \times \tilde{r}$ matrix with determinant $\tilde{D}$.

On the other hand, the matrix say A determined by the homogeneous equations $c_w - c_{ws} + uc_{sw} - uc_{sws} = 0$ for all pairs (w,s) such that $\ell(sws) = \ell(w) + 2$, has entries 1, -1, u, -u. Let B be the $\tilde{r} \times \tilde{r}$ matrix inside A which under the specialization $u \to 1$ specializes to the matrix $\tilde{B}$. If D is the determinant of B, then D is a polynomial in u, namely $\phi(u)$. Moreover $\phi(u) \neq 0$ since $\phi(1) \neq 0$. Therefore if r is the rank of A, we have $r \geq \tilde{r}$. Hence $\dim L(u) = |W| - r \leq |W| - \tilde{r} = \dim L(1)$.

Our proposition is now proved.

In the case where $L(1)$ is a set of commutative elements then $M(1) = L(1)$ and so by Proposition 6.1 and 6.2(2) we also have $M(u) = L(u)$ and so $M(u)$ does not depend on a chosen W graph.

It is also clear that $M(u) \simeq K^{\lambda}$, (K-algebra isomorphism) where $\lambda = \sum_{i=1}^{s} d_i$.

Thus $M(u)$ is a semisimple K-algebra and the identity $1_{M(u)}$ has a unique decomposition into a sum of orthogonal primitive idempotents, namely $1 = \sum_{i=1}^{\lambda} e_i$, $e_i \in M$. We shall determine this orthogonal idempotent decomposition in some special cases.

In the cases to follow it turns out that $M(u) = L(u)$.

Let W be the symmetric group S_4. A W-graph for each irreducible H module isgiven in Chapter 5, and the sum of the degrees of the irreducible representations is 10. The order $|C_W(s_i)|$ of the centralizer of s_i in W, i = 1,2,3 is 4 and therefore the number of all expressions of the form $c_w - c_{ws_i} + uc_{s_iw} - uc_{s_iws_i}$, $\ell(s_iws_i) = \ell(w)+2$, i = 1,2,3 is equal to $\sum_{i=1}^{3} \frac{|W| - |C_W(s_i)|}{4} = 15$.

There is one non trivial relation between these expressions namely,

$$(c_{s_2} - c_{s_2s_1} + uc_{s_1s_2} - uc_{s_1s_2s_1}) - (c_{s_2} - c_{s_2s_3} + uc_{s_3s_2} - uc_{s_2s_3s_2}) +$$

$$+ (c_{s_2s_1} - c_{s_2s_1s_3} + uc_{s_3s_2s_1} - uc_{s_2s_3s_2s_1}) - u(c_{s_1s_2} - c_{s_1s_2s_3} + uc_{s_1s_3s_2} -$$

$$- uc_{s_1s_2s_3s_2}) + u(c_{s_1s_2s_1} - c_{s_1s_2s_1s_3} + uc_{s_1s_3s_2s_1} - uc_{s_1s_2s_3s_2s_1}) -$$

$$- (c_{s_2s_3} - c_{s_2s_1s_3} + uc_{s_1s_2s_3} - uc_{s_1s_2s_1s_3}) +$$

$$+ u(c_{s_3s_2} - c_{s_3s_2s_1} + uc_{s_1s_3s_2} - uc_{s_1s_3s_2s_1}) -$$

$$- u(c_{s_2s_3s_2} - c_{s_2s_3s_2s_1} + uc_{s_1s_2s_3s_2} - uc_{s_1s_2s_3s_2s_1}) = 0$$

Hence there are 14 linearly independent expressions of the form

$c_w - c_{ws_i} + uc_{s_iw} - uc_{s_iws_i}$, $\ell(s_iws_i) = \ell(w) + 2$, $i = 1,2,3$ and so

the dimension of $L(u)$, is 10. Hence $M(u) = L(u)$. A basis for $L^{(u)}$ is given

by the following set of elements:

$$v_0 = T_1$$

$$v_1 = T_{s_2} - uT_{s_1} - uT_{s_3} + T_{s_2s_1} + T_{s_2s_3} - uT_{s_1s_3} + T_{s_2s_1s_3}$$

$$v_2 = -uT_{s_1} + u^2T_{s_2} - uT_{s_3} - uT_{s_1s_2} - uT_{s_3s_2} + T_{s_1s_3} + T_{s_1s_3s_2}$$

$$v_3 = (1+u)T_{s_1} - (u+u^2)T_{s_2} + (1+u)T_{s_1s_2} - uT_{s_2s_3} + T_{s_1s_2s_3} - uT_{s_2s_3s_2} + T_{s_1s_2s_3s_2}$$

$$v_4 = -(u^2+u^3)T_{s_1} + (u+u^2)T_{s_2} + (u+u^2)T_{s_2s_1} - uT_{s_3s_2} + T_{s_2s_3s_2} - uT_{s_3s_2s_1} +$$

$$+ T_{s_2s_3s_2s_1}$$

$$v_5 = (u^2+u)T_{s_2} - (u^2+u^3)T_{s_3} - uT_{s_1s_2} + (u+u^2)T_{s_2s_3} + T_{s_1s_2s_1} -$$

$$- uT_{s_1s_2s_3} + T_{s_1s_2s_1s_3}$$

$$v_6 = -(u+u^2)T_{s_2} + (1+u)T_{s_3} - uT_{s_2s_1} + (1+u)T_{s_3s_2} - uT_{s_1s_2s_1} +$$

$$+ T_{s_3s_2s_1} + T_{s_1s_3s_2s_1}$$

$v_7 = u^2T_{s_1} + u^2T_{s_2} + u^2T_{s_3} + uT_{s_2s_1s_2} + uT_{s_2s_3s_2} + T_{s_1s_2s_3s_2s_1}$

$v_8 = (u+u^3)T_{s_1} - 2u^2T_{s_2} + (u+u^3)T_{s_3} + uT_{s_1s_2} - u^2T_{s_2s_1} - u^2T_{s_2s_3} +$

$+ uT_{s_3s_2} + (u-u^2)T_{s_1s_3} + uT_{s_3s_2s_1} + uT_{s_1s_2s_3} + (1-u)T_{s_2s_1s_3s_2} +$

$+ T_{s_1s_2s_1s_3s_2} + T_{s_2s_3s_2s_1s_2}$

$v_9 = T_{s_1s_2s_1s_3s_2s_1} + uT_{s_2s_1s_3s_2} + u^2T_{s_1s_3}.$

In order to find this basis, we consider a typical element $h = \sum_{w\in S_4} c_wT_w$ of $L(u)$ and with the aid of the relations $c_w - c_{ws_i} + uc_{s_iw} - uc_{s_iws_i} = 0$, $i = 1,2,3$, we express the coefficients c_w in terms of $c_{w'}$ where w' have bigger length than w. We also make use of the fact that $c_{s_2s_1s_3s_2s_1} = c_{s_1s_2s_1s_3s_2}$ which is a consequence of the relations:

$c_{s_2s_1s_3s_2} - c_{s_2s_1s_3s_2s_1} + uc_{s_1s_2s_1s_3s_2} - uc_{s_1s_2s_1s_3s_2s_1} = 0$ and

$c_{s_2s_1s_3s_2} - c_{s_1s_2s_1s_3s_2} + uc_{s_2s_1s_3s_2s_1} - uc_{s_1s_2s_1s_3s_2s_1} = 0.$

Then, we substitute the expressions of the c_w obtained in this way, in h to obtain $h = \sum_{i=0}^{9} c_w^{(1)}v_i$, for certain c_w.

Finally we determine a system of orthogonal primitive idempotents for the Hecke algebra of the group S_4. This system arises from the decomposition of $M(u)$, as described above, and consists of the following set of elements:

$$e_0 = \frac{1}{(u+1)(u^2+u+1)(u^3+u^2+u+1)} \sum_{w\in S_4} T_w$$

$$e_1^{(1)} = \frac{1}{(u+1)(u^3+u^2+u+1)} (uT_1-T_{s_1})(T_1 + T_{s_2} + T_{s_3} + T_{s_3s_2} + T_{s_2s_3} + T_{s_2s_3s_2})$$

$$e_2^{(1)} = \frac{1}{u(u+1)(u^3+u^2+u+1)}(uT_1-T_{s_2})(uT_1+T_{s_1s_3s_2})(T_1 + T_{s_1} + T_{s_3} + T_{s_1s_3})$$

$$e_3^{(1)} = \frac{1}{(u+1)(u^3+u^2+u+1)} (uT_1-T_{s_3})(T_1 + T_{s_1} + T_{s_2} + T_{s_1s_2} + T_{s_2s_1} + T_{s_1s_2s_1})$$

$$e_4 = \frac{1}{u(u+1)^2(u^2+u+1)} (u^2T_1-uT_{s_1}-uT_{s_3} + T_{s_1s_3})(uT_1 + T_{s_2s_1s_3})(T_1 + T_{s_2})$$

$$e_5 = \frac{1}{u(u+1)^2(u^2+u+1)} (uT_1-T_{s_2})(u^2T_1 - T_{s_1s_3s_2})(T_1 + T_{s_1} + T_{s_3} + T_{s_1s_3})$$

$$e_1^{(2)} = \frac{1}{(u+1)(u^3+u^2+u+1)}(u^3T_1-u^2T_{s_2}-u^2T_{s_3}+uT_{s_2s_3}+uT_{s_3s_2}-T_{s_2s_3s_2})(T_1 + T_{s_1})$$

$$e_2^{(2)} = \frac{1}{u(u+1)(u^3+u^2+u+1)} (u^2T_1-uT_{s_1}-uT_{s_3}+T_{s_1s_3})(u^2T_1-T_{s_2s_1s_3})(T_1 + T_{s_2})$$

$$e_3^{(2)} = \frac{1}{(u+1)(u^3+u^2+u+1)} (u^3T_1-u^2T_{s_1}-u^2T_{s_2}+uT_{s_1s_2} + uT_{s_2s_1} - T_{s_1s_2s_1})(T_1 + T_{s_3})$$

$$e_9 = \frac{1}{(u+1)(u^2+u+1)(u^3+u^2+u+1)} \sum_{w\in S_4} (-1)^{\ell(w)} u^{6-\ell(w)} T_w$$

The element e_0 is determined by the fact that it is represented by (1) on the one dimensional submodule which affords the representation $T_{s_i} \to u$, $i = 1,2,3$, and by the zero matrix on every other irreducible submodule.

Similarly the element e_9 is represented by (-1) on the irreducible submodule which affords the representation $T_{s_i} \to -1$, $i = 1,2,3$, and by the zero matrix on every other irreducible submodule.

The elements $e_j^{(1)}$ $j = 1,2,3$ are represented on the first three dimensional irreducible submodule V_L (see Chapter 5 for the definition of V_L) by the diagonal matrix which has 1 in the j entry and 0 elsewhere, and by the zero matrix on every other irreducible submodule.

The elements $e_j^{(2)}$ $j = 1,2,3$ are determined similarly by their action on the second three dimensional irreducible submodule V_N (see also Chapter 5).

Finally the elements e_4, e_5 are represented respectively on the two dimensional irreducible submodule V_M by the matrices $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ and $\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$ and by the zero matrix on every other irreducible submodule.

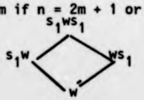
§6.2 A system of orthogonal primitive idempotents inside the generic Hecke algebra of dihedral type.

In this paragraph we shall make use of the results in Chapter 2. In that Chapter we had established a decomposition of Lusztig's graded module over $\mathbb{Q}_n(u^{\frac{1}{2}})$ associated to the dihedral group D_{2n}, into a direct sum of left irreducible H-submodules, where H is the generic Hecke algebra of dihedral type over $\mathbb{Q}_n(u^{\frac{1}{2}})$, each one of them being afforded by a W-graph.

When n = 2m+1 we had obtained a decomposition of grad(E) as $\mathrm{grad}(E) = M_0 \bigoplus_{j=1}^{m} M_j \bigoplus_{j=1}^{m} \hat{M}_j \oplus M_s$, where $\{M_0, M_j, j = 1,\dots,m, M_s\}$ is a full set of left irreducible H-modules, and to this decomposition we adapted the basis $B_0 = \{\bar{e}_1, u_j', v_j, \hat{u}_j', \hat{v}_j, \bar{e}_{w_0}\}$ where $M_j = \langle u_j', v_j\rangle$, $\hat{M}_j = \langle \hat{u}_j', \hat{v}_j\rangle$, $j = 1,\dots,m$.

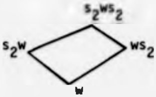
When n = 2m we had achieved a decomposition of grad(E) as $\mathrm{grad}(E) = M_0 \oplus M_s \oplus M_1 \oplus M_2 \bigoplus_{j=1}^{m-1} V_j \bigoplus_{j=1}^{m-1} \hat{V}_j$, where $\{M_0, M_s, M_1, M_2, V_j, j = 1,\dots,m-1\}$ is a full set of left irreducible H submodules, and to this decomposition we had adapted a basis $B_1 = \{\bar{e}_1, \bar{e}_{w_0}, \theta_1, \theta_2, u_j', v_j, \hat{u}_j', \hat{v}_j, j = 1,\dots,m-1\}$ where $V_j = \langle u_j', v_j\rangle, \hat{V}_j = \langle \hat{u}_j', \hat{v}_j\rangle$, $j = 1,\dots,m-1$.
For the definition of the elements of B_0 and B_1 see also Chapter 2.

By defining M(u) and L(u) for the case $W = D_{2n}$, and since $\mathbb{Q}_n(u^{\frac{1}{2}})$ is a splitting field of H and every irreducible H submodule appearing in these decompositions above is being afforded by a W-graph, we have that $M(u) \subsetneq L(u)$.
Let $h = \sum_{w \in D_{2n}} c_w T_w$ be an element of L(u). Now for each $w \in \{(s_2s_1)^{k-1}_{s_2}\}$ $k = 1,\dots,m$ if $n = 2m + 1$ or $k = 1,\dots,m-1$ if $n = 2m$ we consider the

quadruples 

and for each $w \in \{(s_1s_2)^{k-1}_{s_1}\}$ $k = 1,\dots,m$

if $n = 2m+1$, or $k = 1,\dots,m-1$ if $n = 2m$ we consider the quadruples



Each of these w appears at the bottom of such a quadruple only once and it is related to elements of length bigger than the length of w. Therefore it is clear that the expressions $c_w - c_{ws_1} + uc_{s_1w} - uc_{s_1ws_1}$ for $w \in \{(s_2s_1)^{k-1}s_2\}$ and $c_w - c_{ws_2} + uc_{s_2w} - uc_{s_2ws_2}$ for $w \in \{(s_1s_2)^{k-1}s_1\}$ are all linearly independent. The number of these expressions is $2m$ if $n = 2m+1$ or $2m-2$ if $n = 2m$. Thus the dimension of $L(u)$ is $(4m+2) - 2m = 2m + 2$ if $n = 2m + 1$, or $4m - (2m-2) = 2m+2$ if $n = 2m$. In both cases the dimension of $L(u)$ is equal to the sum of the degrees of the irreducible representations of H and hence $M(u) = L(u)$.

In order to establish a decomposition of $1_{M(u)}$ into a sum of orthogonal primitive idempotents, we need some properties of the polynomials $S_n(x)$.

We recall that the polynomials $S_n(x)$ are defined by

$$S_0(x) = 1,\ S_1(x) = x,\ S_{n+1}(x) = xS_n(x) - S_{n-1}(x),\ \forall n \geq 1.$$

In Chapter 2 we showed that the numbers $\rho_j = 2\cos\frac{j\pi}{m}$ $1 \leq j \leq m-1$ are the roots of the polynomial $S_{m-1}(x)$, while the numbers $\rho_j = 2\cos\frac{2j\pi}{2m+1}$ $1 \leq j \leq m$ are the roots of the polynomial $S_m(x) + S_{m-1}(x)$.

<u>Lemma 6.3</u>: (Sum formulae):

(1) Let $\rho_k = 2\cos\frac{2k\pi}{2m+1}$, $1 \leq k \leq m$. Then $\sum\limits_{\lambda=0}^{m-1} (\lambda+1)S_\lambda(\rho_k) = \prod\limits_{\substack{j=1 \\ j\neq k}}^{m} (\rho_k - \rho_j)$.

(2) Let $\rho_k = 2\cos\frac{k\pi}{m}$ $1 \leq k \leq m-1$, and let m be odd. Then

$$\sum_{\substack{\lambda=1 \\ \lambda=\text{odd}}}^{m-2} (\lambda+1)S_\lambda(\rho_k) = \prod_{\substack{j=1 \\ j\neq k}}^{m-1} (\rho_k - \rho_j).$$

If m is even, then

$$\sum_{\substack{\lambda=1 \\ \lambda=\text{odd}}}^{m-1} \lambda S_{\lambda-1}(\rho_k) = \prod_{\substack{j=1 \\ j\neq k}}^{m-1} (\rho_k-\rho_j)$$

Proof: (1) We know that the numbers ρ_k are the roots of the polynomial $S_m(x) + S_{m-1}(x)$.

If m is even, then $S_m(x) + S_{m-1}(x) = \sum_{k=0}^{m/2} (-1)^k \binom{m-k}{k} x^{m-2k} + \sum_{k=0}^{(m-2)/2} (-1)^k \binom{m-1-k}{k} x^{m-1-2k}$, while if m is odd, then

$$S_m(x) + S_{m-1}(x) = \sum_{k=0}^{(m-1)/2} (-1)^k \binom{m-k}{k} x^{m-2k} + \sum_{k=0}^{(m-1)/2} (-1)^k \binom{m-1-k}{k} x^{m-1-2k}.$$

Let σ_λ be the λ elementary symmetric function on the ρ_k, $1 \leq k \leq m$, given by

$$\sigma_\lambda = \sum_{1 \leq i_1 < i_2 < \dots < i_\lambda \leq m} \rho_{i_1} \rho_{i_2} \cdots \rho_{i_\lambda}$$

Then $\sigma_\lambda = \begin{cases} (-1)^k \binom{m-k}{k} & \text{if } \lambda = 2k \quad k \geq 1 \\ (-1)^{k+1} \binom{m-1-k}{k} & \text{if } \lambda = 2k+1, \quad k \geq 0. \end{cases}$

We know that there exists at most one polynomial of degree $\leq m-1$, say $f_m(x)$ which at the m different points $\rho_1, \dots, \rho_m$, assumes given values $f_m(\rho_k) = \prod_{\substack{j=1 \\ j\neq k}}^{m} (\rho_k-\rho_j)$. By Lagrange's interpolation formula there is always one polynomial of degree $\leq m-1$ which assumes the given values at these points. It is the polynomial

$$f_m(x) = \sum_{k=1}^{m} f_m(\rho_k) \frac{(x-\rho_1)\dots(x-\rho_{k-1})(x-\rho_{k+1})\dots(x-\rho_m)}{(\rho_k-\rho_1)\dots(\rho_k-\rho_{k-1})(\rho_k-\rho_{k+1})\dots(\rho_k-\rho_m)}.$$

By putting $f_m(\rho_k) = \prod_{\substack{j=1 \\ j\neq k}}^{m} (\rho_k-\rho_j)$, we obtain $f_m(x) = \sum_{k=1}^{m} \prod_{\substack{j=1 \\ j\neq k}}^{m} (x-\rho_j)$. We shall show that $f_m(x) = \sum_{\lambda=0}^{m-1} (\lambda+1)S_\lambda(x)$ and then (1) follows. Let $1 \le i_1 < i_2 < \dots < i_r \le m$, $r \in \{1,\dots,m-1\}$. Then every summand of $f_m(x)$ contributes to the expression $(-1)^r \rho_{i_1}\rho_{i_2}\cdots\rho_{i_r}$ except from the summands which correspond to $k = i_1, i_2, \dots, i_r$.

Thus $f_m(x) = mx^{m-1} + \sum_{\lambda=1}^{m-1} (-1)^\lambda (m-\lambda)\sigma_\lambda x^{m-\lambda-1}$. We assume that m is even (the argument is entirely similar if m is odd). We substitute the expressions for σ_λ in $f_m(x)$ to obtain

$$f_m(x) = mx^{m-1} + \sum_{\substack{\lambda=2\\ \lambda=2k}}^{m-2} (-1)^\lambda (m-\lambda)(-1)^k \binom{m-k}{k} x^{m-\lambda-1} +$$

$$+ \sum_{\substack{\lambda=1\\ \lambda=2k+1}}^{m-1} (-1)^\lambda (m-\lambda)(-1)^{k+1} \binom{m-1-k}{k} x^{m-\lambda-1} =$$

$$= \sum_{k=1}^{m/2} (-1)^{k-1}(m-2k+2)\binom{m-k+1}{k-1} x^{m-2k+1} +$$

$$+ \sum_{k=1}^{m/2} (-1)^{k-1}(m-2k+1)\binom{m-k}{k-1} x^{m-2k}.$$

Therefore $f_{m-1}(x) = \sum_{k=1}^{m/2} (-1)^{k-1}(m-2k+1)\binom{m-k}{k-1} x^{m-2k} +$

$$+ \sum_{k=1}^{(m-2)/2} (-1)^{k-1}(m-2k)\binom{m-1-k}{k-1} x^{m-2k-1}.$$

Thus $f_m(x) - f_{m-1}(x) = \sum_{k=1}^{m/2} (-1)^{k-1}(m-2k+2)\binom{m-k+1}{k-1} x^{m-2k+1} -$

$$- \sum_{k=1}^{(m-2)/2} (-1)^{k-1}(m-2k)\binom{m-1-k}{k-1} x^{m-2k-1} = mx^{m-1} + m \sum_{k=1}^{(m-2)/2} (-1)^k \binom{m-1-k}{k} x^{m-2k-1} =$$

$$= mS_{m-1}(x).$$

Therefore $f_m(x) = \sum_{\lambda=0}^{m-1} (\lambda+1)S_\lambda(x)$.

Thus part (1) is now proved.

(2) The argument is similar. We assume that m is odd, the proof being similar if m is even.

We recall that the numbers $\rho_j = 2\cos\frac{j\pi}{m}$, $1 \le j \le m-1$ are the zeros of the polynomial $S_{m-1}(x) = \sum_{k=0}^{m-1/2} (-1)^k \binom{m-1-k}{k} x^{m-1-2k}$. We know that there exists at most one polynomial of degree $\le m-2$, say $H_{m-2}(x)$, which at m-1 different points $\rho_1,\dots,\rho_{m-1}$ assumes given values $H_{m-2}(\rho_k) = \prod_{\substack{j=1\\ j\ne k}}^{m-1} (\rho_k-\rho_j)$.

By Lagrange's interpolation formula there is always one polynomial of degree $\le m-2$ which assumes the given values at these points. It is the polynomial

$$H_{m-2}(x) = \sum_{k=1}^{m-1} H_{m-2}(\rho_k) \frac{(x-\rho_1)\dots(x-\rho_{k-1})(x-\rho_{k+1})\dots(x-\rho_{m-1})}{(\rho_k-\rho_1)\dots(\rho_k-\rho_{k-1})(\rho_k-\rho_{k+1})\dots(\rho_k-\rho_{m-1})} =$$

$$= \sum_{k=1}^{m-1} \prod_{\substack{j=1\\ j\ne k}}^{m-1} (x-\rho_j).$$

Let $\sigma_\lambda = \sum_{1\le i_1<i_2<\dots<i_\lambda\le m-1} \rho_{i_1}\rho_{i_2}\dots\rho_{i_\lambda}$ be the λ elementary symmetric function on the ρ_j. Then

$$\sigma_\lambda = \begin{cases} 0 & \text{if } \lambda = 2k+1 \quad k \ge 0 \\ (-1)^k \binom{m-1-k}{k} & \text{if } \lambda = 2k. \end{cases}$$

If $1 \le i_1 < i_2 < \dots < i_r \le m-1$, $r \in \{1,2,\dots,m-2\}$, then every summand of $H_{m-2}(x)$ contributes to $(-1)^r \rho_{i_1}\rho_{i_2}\dots\rho_{i_r}$ except from the summands which correspond to $k = i_1, i_2,\dots,i_r$. Thus, $H_{m-2}(x) = (m-1)x^{m-2} +$

$$+ \sum_{\lambda=1}^{m-2} (-1)^\lambda(m-\lambda-1)\,\sigma_\lambda\, x^{m-\lambda-2} = (m-1)x^{m-2} + \sum_{\substack{\lambda=2\\ \lambda=2k}}^{m-3} (-1)^\lambda(m-\lambda-1)\,\sigma_\lambda\, x^{m-\lambda-2} \text{ (by}$$

taking into account that m is odd and $\sigma_\lambda = 0$ for the odd values of λ). By replacing the expressions for σ_λ above we obtain

$$H_{m-2}(x) = (m-1)x^{m-2} + \sum_{k=1}^{(m-3)/2} (-1)^k (m-2k-1)\binom{m-1-k}{k}x^{m-2k-2} =$$

$$= \sum_{k=1}^{(m-1)/2} (-1)^{k-1}(m-2k+1)\binom{m-k}{k-1}x^{m-2k}.$$

Hence $H_{m-4}(x) = \sum_{k=1}^{(m-3)/2} (-1)^{k-1}(m-2k-1)\binom{m-k-2}{k-1}x^{m-2k-2} =$

$$= \sum_{k=2}^{m-1/2} (-1)^{k-2}(m-2k+1)\binom{m-k-1}{k-2}x^{m-2k}.$$

We also have that $S_{m-2}(x) = \sum_{k=1}^{(m-1)/2} (-1)^{k-1}\binom{m-k-1}{k-1}x^{m-2k}$.

Now we show that $H_{m-2}(x) - H_{m-4}(x) = (m-1)S_{m-2}(x)$, and then (2) follows.

In fact

$$H_{m-2}(x) - H_{m-4}(x) = \sum_{k=1}^{(m-1)/2} (-1)^{k-1}(m-2k+1)\binom{m-k}{k-1}x^{m-2k} -$$

$$- \sum_{k=2}^{(m-1)/2} (-1)^{k-2}(m-2k+1)\binom{m-k-1}{k-2}x^{m-2k} =$$

$$= (m-1)x^{m-2} + \sum_{k=2}^{(m-1)/2} (-1)^{k-1}(m-2k+1)\left[\binom{m-k}{k-1} + \binom{m-k-1}{k-2}\right]x^{m-2k} =$$

$$= (m-1)x^{m-2} + \sum_{k=2}^{(m-1)/2} (-1)^{k-1}(m-2k+1)\left[\frac{(m-k)!}{(k-1)!(m-2k+1)!} + \frac{(m-k-1)!}{(k-2)!(m-2k+1)!}\right]x^{m-2k}$$

$$= (m-1)x^{m-2} + \sum_{k=2}^{(m-1)/2} (-1)^{k-1}\left[\frac{(m-k)!}{(k-1)!(m-2k)!} + \frac{(m-k-1)!}{(k-2)!(m-2k)!}\right]x^{m-2k} =$$

$$= (m-1)x^{m-2} + \sum_{k=2}^{(m-1)/2} (-1)^{k-1} \frac{(m-k-1)!}{(k-1)!(m-2k)!} (m-k+k-1)x^{m-2k} =$$

$$= (m-1)S_{m-2}(x).$$

Our lemma is now proved.

Lemma 6.4: (Orthogonality formulae)

(1) Let $\rho_k = 2\cos\frac{2k\pi}{2m+1}$, $1 \le k \le m$. Then $\sum_{\lambda=0}^{m-1} S_{m-1-\lambda}(\rho_k)(S_\lambda(\rho_j) + S_{\lambda-1}(\rho_j)) = 0$,

for every $1 \le j \le m$, $j \ne k$.

(2) Let $\rho_k = 2\cos\frac{k\pi}{m}$, $1 \le k \le m-1$. Then $\sum_{\lambda=0}^{m-2} S_{m-2-\lambda}(\rho_k)\, S_\lambda(\rho_j) = 0$, for every

$1 \le j \le m-1$, $j \ne k$.

Proof: (1) We recall that the roots of the polynomial $S_m(x) + S_{m-1}(x)$ are the numbers $\rho_j = 2\cos\frac{2j\pi}{2m+1}$, $1 \le j \le m$. We fix a k, $1 \le k \le m$ and we shall show that

$$\frac{S_m(x) + S_{m-1}(x)}{x-\rho_k} = \sum_{\lambda=0}^{m-1} S_{m-1-\lambda}(\rho_k)(S_\lambda(x) + S_{\lambda-1}(x)).$$

In fact $(x-\rho_k)\sum_{\lambda=0}^{m-1} S_{m-1-\lambda}(\rho_k)(S_\lambda(x) + S_{\lambda-1}(x)) =$

$$(x-\rho_k)S_{m-1}(\rho_k) + (x-\rho_k)\sum_{\lambda=2}^{m} S_{m-\lambda}(\rho_k)(S_{\lambda-1}(x) + S_{\lambda-2}(x)).$$

Now for every $\lambda \ge 2$ we have $S_\lambda(x) = xS_{\lambda-1}(x) - S_{\lambda-2}(x)$, and so $x[S_{\lambda-1}(x) + S_{\lambda-2}(x)] = S_\lambda(x) + S_{\lambda-1}(x) + S_{\lambda-2}(x) + S_{\lambda-3}(x)$. We also have $-\rho_k S_{m-\lambda}(\rho_k) = -S_{m-\lambda+1}(\rho_k) - S_{m-\lambda-1}(\rho_k)$ $\forall\ \lambda = 2,3,\ldots,m$. Therefore,

$$(x-\rho_k)\sum_{\lambda=0}^{m-1} S_{m-1-\lambda}(\rho_k)(S_\lambda(x) + S_{\lambda-1}(x)) =$$

$$= (x-\rho_k)S_{m-1}(\rho_k) + \sum_{\lambda=2}^{m} S_{m-\lambda}(\rho_k)(S_\lambda(x) + S_{\lambda-1}(x)) + \sum_{\lambda=2}^{m} S_{m-\lambda}(\rho_k)(S_{\lambda-2}(x)+S_{\lambda-3}(x))$$

$$- \sum_{\lambda=2}^{m} S_{m-\lambda+1}(\rho_k)(S_{\lambda-1}(x) + S_{\lambda-2}(x)) - \sum_{\lambda=2}^{m} S_{m-\lambda-1}(\rho_k)(S_{\lambda-1}(x) + S_{\lambda-2}(x))$$

$$= (x-\rho_k)S_{m-1}(\rho_k) + \sum_{\lambda=2}^{m-1} S_{m-\lambda}(\rho_k)(S_\lambda(x) + S_{\lambda-1}(x)) + S_m(x) + S_{m-1}(x) +$$

$$+ S_{m-2}(\rho_k) + \sum_{\lambda=3}^{m} S_{m-\lambda}(\rho_k)(S_{\lambda-2}(x) + S_{\lambda-3}(x)) - S_{m-1}(\rho_k)(1 + x) -$$

$$- \sum_{\lambda=2}^{m-1} S_{m-\lambda}(\rho_k)(S_\lambda(x) + S_{\lambda-1}(x)) - \sum_{\lambda=3}^{m} S_{m-\lambda}(\rho_k)(S_{\lambda-2}(x) + S_{\lambda-3}(x)) =$$

$$= S_m(x) + S_{m-1}(x) + S_{m-2}(\rho_k) - \rho_k S_{m-1}(\rho_k) - S_{m-1}(\rho_k) =$$

$$= S_m(x) + S_{m-1}(x) + \rho_k \, S_{m-1}(\rho_k) - S_m(\rho_k) - \rho_k \, S_{m-1}(\rho_k) - S_{m-1}(\rho_k) =$$

$$= S_m(x) + S_{m-1}(x).$$

Now part (1) is proved.

(2) We recall that the numbers $\rho_j = 2\cos\frac{j\pi}{m}$ $1 \le j \le m-1$ are the roots of the polynomial $S_{m-1}(x)$. We fix a k, $1 \le k \le m-1$ and we shall show that

$$\frac{S_{m-1}(x)}{x-\rho_k} = \sum_{\lambda=0}^{m-2} S_\lambda(x) S_{m-2-\lambda}(\rho_k). \qquad \text{In fact}$$

$$(x-\rho_k) \sum_{\lambda=0}^{m-2} S_\lambda(x) S_{m-2-\lambda}(\rho_k) = (x-\rho_k) \sum_{\lambda=2}^{m} S_{\lambda-2}(x) S_{m-\lambda}(\rho_k) =$$

$$= x \sum_{\lambda=2}^{m} S_{\lambda-2}(x) S_{m-\lambda}(\rho_k) - \rho_k \sum_{\lambda=2}^{m} S_{\lambda-2}(x) S_{m-\lambda}(\rho_k).$$

Moreover, $xS_{\lambda-2}(x) = S_{\lambda-1}(x) + S_{\lambda-3}(x)$ $\forall \lambda = 2,\ldots,m$ and

$-\rho_k \, S_{m-\lambda}(\rho_k) = - S_{m-\lambda+1}(\rho_k) - S_{m-\lambda-1}(\rho_k)$. Therefore,

$$(x-\rho_k) \sum_{\lambda=0}^{m-2} S_\lambda(x) S_{m-2-\lambda}(\rho_k) =$$

$$= \sum_{\lambda=2}^{m} S_{m-\lambda}(\rho_k)(S_{\lambda-1}(x) + S_{\lambda-3}(x)) - \sum_{\lambda=2}^{m} S_{\lambda-2}(x)(S_{m-\lambda+1}(\rho_k) + S_{m-\lambda-1}(\rho_k)) =$$

$$= S_{m-1}(x) + S_{m-3}(x) + \sum_{\lambda=2}^{m-1} S_{m-\lambda}(\rho_k)(S_{\lambda-1}(x) + S_{\lambda-3}(x)) - S_{m-2}(x) S_1(\rho_k) - S_{m-3}(x)(S_2(\rho_k)+1)$$

$$- \sum_{\lambda=2}^{m-2} S_{\lambda-2}(x)(S_{m-\lambda+1}(\rho_k) + S_{m-\lambda-1}(\rho_k)) =$$

$$= S_{m-1}(x) + S_{m-3}(x) + \sum_{\lambda=2}^{m-1} S_{m-\lambda}(\rho_k) S_{\lambda-1}(x) + \sum_{\lambda=2}^{m-1} S_{m-\lambda}(\rho_k) S_{\lambda-3}(x) - S_{m-2}(x) S_1(\rho_k) -$$

$$- S_{m-3}(x)S_2(\rho_k) - S_{m-3}(x) - \sum_{\lambda=1}^{m-3} S_{\lambda-1}(x)S_{m-\lambda}(\rho_k) - \sum_{\lambda=3}^{m-1} S_{\lambda-3}(x)S_{m-\lambda}(\rho_k) =$$

$$= S_{m-1}(x) + \sum_{\lambda=2}^{m-3} S_{m-\lambda}(\rho_k)S_{\lambda-1}(x) + S_2(\rho_k)S_{m-3}(x) + S_1(\rho_k)S_{m-2}(x) +$$

$$+ \sum_{\lambda=3}^{m-1} S_{m-\lambda}(\rho_k)S_{\lambda-3}(x) - S_{m-2}(x)S_1(\rho_k) - S_{m-3}(x)S_2(\rho_k) - \sum_{\lambda=2}^{m-3} S_{\lambda-1}(x)S_{m-\lambda}(\rho_k) -$$

$$- S_{m-1}(\rho_k) - \sum_{\lambda=3}^{m-1} S_{\lambda-3}(x)S_{m-\lambda}(\rho_k) = S_{m-1}(x).$$

Now our lemma is proved.

Lemma 6.5: (Multiplication formula)

$$S_i(x)S_j(x) = \sum_{k=0}^{i} S_{j-i+2k}(x) \quad \forall\ i \le j.$$

Proof: We use induction on i. If $i = 0$, it is obvious. Assume it is true for all integers $\le i-1$. Then

$$S_i(x)S_j(x) = [xS_{i-1}(x)-S_{i-2}(x)]S_j(x) = xS_{i-1}(x)S_j(x) - S_{i-2}(x)S_j(x) =$$

$$= x \sum_{k=0}^{i-1} S_{j-i+1+2k}(x) - \sum_{k=0}^{i-2} S_{j-i+2+2k}(x).$$

Moreover $xS_{j-i+1+2k}(x) = S_{j-i+2+2k}(x) + S_{j-i+2k}(x)$. Hence

$$S_i(x)S_j(x) = \sum_{k=0}^{i-1} S_{j-i+2+2k}(x) + \sum_{k=0}^{i-1} S_{j-i+2k}(x) - \sum_{k=0}^{i-2} S_{j-i+2+2k}(x) =$$

$$= \sum_{k=0}^{i} S_{j-i+2k}(x).$$

Lemma 6.6: $\displaystyle\sum_{\lambda=0}^{m-1} S_{m-1-\lambda}(x)(S_\lambda(x) + S_{\lambda-1}(x)) = \sum_{\lambda=0}^{m-1} (\lambda+1)S_\lambda(x).$

Proof: Assume m is odd, the argument being similar if m is even. We have

$$\sum_{\lambda=0}^{m-1} S_{m-1-\lambda}(x)\,(S_\lambda(x) + S_{\lambda-1}(x)) =$$

$$= \sum_{\lambda=0}^{m-1} S_\lambda(x)S_{m-1-\lambda}(x) + \sum_{\lambda=1}^{m-1} S_{\lambda-1}(x)S_{m-1-\lambda}(x) =$$

$$= 2 \sum_{\substack{i<j \\ i+j=m-1 \\ i\in\{0,1,\dots,\frac{m-3}{2}\}}} S_i(x)S_j(x) + \left[S_{\frac{m-1}{2}}(x)\right]^2 + 2\sum_{\substack{i<j \\ i+j=m-2 \\ i\in\{0,1,\dots,\frac{m-3}{2}\}}} S_i(x)S_j(x)$$

Now, the number of pairs (i,j), $i < j$, $i + j = m-1$, $i \in \{0,1,\dots,\frac{m-3}{2}\}$ is clearly $\frac{m-1}{2}$. Each such pair gives the number $j-i \in \{2,4,\dots,m-3,m-1\}$, and therefore, (by means of 6.5), for a given $\lambda \in \{1,3,\dots,m-2\}$, the polynomial $S_{m-\lambda}(x)$ appears in the product $S_i(x)S_j(x)$, unless if $j-i > m-\lambda$. The number of pairs (i,j) such that $j-i > m-\lambda$ is clearly $\frac{\lambda-1}{2}$. Hence the number of pairs (i,j), $i < j$, $i + j = m-1$, $i \in \{0,1,\dots,\frac{m-3}{2}\}$, for which the corresponding product $S_i(x)S_j(x)$ contributes to $S_{m-\lambda}(x)$, $\lambda \in \{1,3,\dots,m-2\}$ is $\frac{m-\lambda}{2}$.

Similarly, the number of pairs (i,j), $i < j$, $i+j = m-2$, $i \in \{0,1,\dots,\frac{m-3}{2}\}$ for which the corresponding product $S_i(x)S_j(x)$ contributes to $S_{m-\lambda}(x)$, $\lambda \in \{2,4,\dots,m-1\}$ is $\frac{m-\lambda+1}{2}$. Hence,

$$\sum_{\lambda=0}^{m-1} S_\lambda(x)S_{m-1-\lambda}(x) + \sum_{\lambda=1}^{m-1} S_{\lambda-1}(x)S_{m-1-\lambda}(x) =$$

$$= \sum_{\substack{\lambda=1 \\ \lambda=\text{odd}}}^{m-2} (m-\lambda)S_{m-\lambda}(x) + \sum_{k=0}^{(m-1)/2} S_{2k} + \sum_{\substack{\lambda=2 \\ \lambda=\text{even}}}^{m-1} (m-\lambda+1)S_{m-\lambda}(x) = \sum_{\lambda=0}^{m-1} (\lambda+1)S_\lambda(x).$$

Lemma 6.7:

$$\sum_{k=1}^{m-2} \Big(\sum_{\lambda=k+2}^{m} (-1)^{\lambda-k} S_{m-\lambda}(x)\Big)(S_{k-1}(x) + S_k(x)) + \sum_{\lambda=2}^{m} (-1)^\lambda S_{m-\lambda}(x) =$$

$$= \sum_{i\geq 0} (m-1-2i)S_{m-2-2i}(x) \quad \begin{pmatrix}\text{finishes with } S_0(x) \text{ if } m \text{ is even, or with } 2S_1(x) \\ \text{if } m \text{ is odd.}\end{pmatrix}$$

Proof: For the sake of simplicity we call $C_{m-2}(x)$ the left hand side of the equality we wish to show.

We also put $D_r(x) = \sum_{\lambda=r}^{m} (-1)^{\lambda-r} S_{m-\lambda}(x) \quad \forall\, r \in \{3,4,\dots,m\}$.

Hence $C_{m-2}(x)$ becomes

$$C_{m-2}(x) = \sum_{r=3}^{m} D_r(x)(S_{r-2}(x) + S_{r-3}(x)) + \sum_{\lambda=2}^{m} (-1)^{\lambda} S_{m-\lambda}(x) =$$

$$= D_3(x)+D_3(x)S_1(x) + \sum_{r=4}^{m-1} D_r(x)S_{r-2}(x)+S_{m-2}(x) + \sum_{r=4}^{m-1} D_r(x)S_{r-3}(x)+S_{m-3}(x)+ \sum_{\lambda=2}^{m}(-1)^{\lambda}S_{m-\lambda}($$

$$= D_3(x)+D_3(x)S_1(x) + \sum_{r=4}^{m-2} D_r(x)S_{r-2}(x) + (S_1(x)-1)S_{m-3}(x) + S_{m-2}(x)+D_4(x)S_1(x) +$$

$$+ \sum_{r=4}^{m-2} D_{r+1}(x)S_{r-2}(x) + S_{m-3}(x) + \sum_{\lambda=2}^{m}(-1)^{\lambda}S_{m-\lambda}(x) =$$

$$= D_3(x) + S_1(x)(D_3(x)+D_4(x))+ \sum_{r=4}^{m-2} S_{r-2}(x)(D_r(x)+D_{r+1}(x))+(S_1(x)-1)S_{m-3}(x) +$$

$$+ S_{m-2}(x) + S_{m-3}(x) + \sum_{\lambda=2}^{m}(-1)^{\lambda} S_{m-\lambda}(x) =$$

$$= D_3(x) + 2S_1(x)S_{m-3}(x) + S_{m-2}(x) + \sum_{r=4}^{m-2} S_{r-2}(x)S_{m-r}(x) + \sum_{\lambda=2}^{m}(-1)^{\lambda}S_{m-\lambda}(x).$$

Assume m is odd, then

$$C_{m-2}(x) = D_3(x) + 2S_1(x)S_{m-3}(x) + S_{m-2}(x) + \sum_{r=1}^{m-5} S_{r+1}(x)S_{m-3-r}(x) + \sum_{\lambda=2}^{m}(-1)^{\lambda}S_{m-\lambda}(x)$$

$$= D_3(x) + 2S_1(x)S_{m-3}(x) + S_{m-2}(x) + 2\sum_{r=1}^{(m-5)/2} S_{r+1}(x)S_{m-3-r}(x) + \sum_{\lambda=2}^{m}(-1)^{\lambda}S_{m-\lambda}(x) =$$

$$= D_3(x) + S_{m-2}(x) + \sum_{\lambda=2}^{m}(-1)^{\lambda}S_{m-\lambda}(x) + 2\sum_{\substack{i<j \\ i+j=m-2 \\ i\in\{1,2,\dots,\frac{m-3}{2}\}}} S_i(x)S_j(x)$$

if m is even, then

$$C_{m-2}(x) = D_3(x) + S_{m-2}(x) + \sum_{\lambda=2}^{m}(-1)^{\lambda}S_{m-\lambda}(x)+ 2\sum_{\substack{i<j \\ i+j=m-2 \\ i\in\{1,2,\dots,\frac{m-4}{2}\}}} S_i(x)S_j(x)+ \left[S_{\frac{m-2}{2}}(x)\right]^2$$

Moreover $D_3(x) + \sum_{\lambda=2}^{m}(-1)^\lambda S_{m-\lambda}(x) = \sum_{\lambda=3}^{m}(-1)^{\lambda-3}S_{m-\lambda}(x) + \sum_{\lambda=2}^{m}(-1)^\lambda S_{m-\lambda}(x) = S_{m-2}(x)$.

Therefore

$$C_{m-2}(x) = 2 \sum_{\substack{i<j \\ i+j=m-2 \\ i\in\{0,1,\dots,\frac{m-3}{2}\}}} S_i(x)S_j(x) \quad \text{if } m \text{ is odd}$$

or

$$C_{m-2}(x) = 2 \sum_{\substack{i<j \\ i+j=m-2 \\ i\in\{0,1,\dots,\frac{m-4}{2}\}}} S_i(x)S_j(x) + \left[S_{\frac{m-2}{2}}(x)\right]^2, \quad \text{if } m \text{ is even.}$$

When m is odd, then the number of pairs (i,j) such that $i < j$, $i + j = m-2$, $i \in \{0,1,\dots,\frac{m-3}{2}\}$, is $\frac{m-1}{2}$.

Each such pair gives the number $j-i \in \{1,3,\dots,m-4,m-2\}$ and therefore for a given $\lambda \in \{2,4,\dots,m-1\}$, the product $S_i(x)S_j(x)$ (see Lemma 6.5) contributes to $S_{m-\lambda}(x)$, unless if $j-i > m-\lambda$. Clearly the number of pairs (i,j) such that $j-i > m-\lambda$ is $\frac{\lambda-2}{2}$, and therefore the number of pairs (i,j), $i < j$, $i + j = m-2$, $i \in \{0,1,\dots,\frac{m-3}{2}\}$ for which the corresponding product $S_i(x)S_j(x)$ contributes to $S_{m-\lambda}(x)$ is $\frac{m-1}{2} - \frac{\lambda-2}{2} = \frac{m-\lambda+1}{2}$, $\lambda \in \{2,4,\dots,m-1\}$.
Therefore, when m is odd

$$C_{m-2}(x) = 2 \sum_{\substack{\lambda=2 \\ \lambda=\text{even}}}^{m-1} \frac{(m-\lambda+1)}{2} S_{m-\lambda}(x) = \sum_{\substack{\lambda=2 \\ \lambda=\text{even}}}^{m-1} (m-\lambda+1)S_{m-\lambda}(x).$$

When m is even, a similar argument shows that the number of pairs (i,j), $i < j$, $i + j = m-2$, $i \in \{0,1,\dots,\frac{m-4}{2}\}$, such that the corresponding product $S_i(x)S_j(x)$ contributes to $S_{m-\lambda}(x)$, $\lambda \in \{2,4,\dots,m-2\}$ is $\frac{m-2}{2} - \frac{\lambda-2}{2} = \frac{m-\lambda}{2}$. In this case

$$C_{m-2}(x) = \sum_{\substack{\lambda=2 \\ \lambda=\text{even}}}^{m-2} (m-\lambda)S_{m-\lambda}(x) + \left[S_{\frac{m-2}{2}}(x)\right]^2 =$$

$$\sum_{\substack{\lambda=2 \\ \lambda=\text{even}}}^{m-2} (m-\lambda)S_{m-\lambda}(x) + \sum_{k=0}^{(m-2)/2} S_{2k}(x) = \sum_{\substack{\lambda=2 \\ \lambda=\text{even}}}^{m} (m-\lambda+1)S_{m-\lambda}(x).$$

We are going to apply the results stated in the previous lemmas in order to obtain a system of orthogonal primitive idempotents of the generic Hecke algebra of dihedral type. This system of idempotents is obtained from the decomposition of M(u), which in this case is the same as L(u), as we have shown before. It was easier to conjecture this system of idempotents in the case n is odd, n = 2m+1, where we get irrational values for the $\rho_j = 2\cos\frac{2j\pi}{n}$, $1 \le j \le m$ rather quickly. The conjecture for the case n = 2m was made after we had worked out the case n = 10!. We adopt the following notation:

$\mathbb{Q}_n =: \mathbb{Q}(2\cos\frac{2\pi}{n})$, $\rho_j = 2\cos\frac{2j\pi}{n}$, $j \in \{1,\dots,m\}$ if n = 2m+1, or $j \in \{1,\dots,m-1\}$ if n = 2m

$$Q_k = uT_{(s_2s_1)^{k-1}} - T_{(s_1s_2)^{k-1}_{s_1}} + uT_{(s_2s_1)^{k-1}_{s_2}} - T_{(s_1s_2)^k} \quad 1 \le k \le m$$

$$\hat{Q}_k = uT_{(s_1s_2)^{k-1}} - T_{(s_2s_1)^{k-1}_{s_2}} + uT_{(s_1s_2)^{k-1}_{s_1}} - T_{(s_2s_1)^k} \quad 1 \le k \le m.$$

We shall denote the generic Hecke algebra of dihedral type over the field $\mathbb{Q}_n(u^{\frac{1}{2}})$, by $H(D_{2n})$. We now state the following.

<u>Theorem 6.8</u>: (1) Let n = 2m+1. Then the following set of elements inside $H(D_{2n})$ forms a system of orthogonal primitive idempotents whose sum is $1_{H(D_{2n})}$.

$$e_o = \frac{1}{(u+1)(u^{n-1}+\dots+u+1)} \sum_{w\in D_{2n}} T_w$$

$$e_\sigma = \frac{1}{(u+1)(u^{n-1}+\dots+u+1)} \sum_{w\in D_{2n}} (-1)^{\ell(w)} u^{n-\ell(w)} T_w$$

and for every $1 \le j \le m$, the pair $\{e_1^{(j)}, e_2^{(j)}\}$ where

$$e_1^{(j)} = \frac{1}{u^{m-1}\prod_{\substack{i=1\\ i\neq j}}^{m}(\rho_j-\rho_i)(u^2-u\rho_j+1)} \sum_{k=1}^{m} u^{m-k} S_{m-k}(\rho_j)Q_k$$

$$e_2^{(j)} = \frac{1}{u^{m-1} \prod_{\substack{i=1 \\ i\neq j}}^{m} (\rho_j-\rho_i)(u^2-u\rho_j+1)} \sum_{k=1}^{m} u^{m-k} S_{m-k}(\rho_j)\hat{Q}_k.$$

(2) Let $n = 2m$. Then the following set of elements inside $H(D_{2n})$ forms a system of orthogonal primitive idempotents whose sum is $1_{H(D_{2n})}$.

$$e_0 = \frac{1}{(u+1)(u^{n-1}+\ldots+u+1)} \sum_{w\in D_{2n}} T_w$$

$$e_\sigma = \frac{1}{(u+1)(u^{n-1}+\ldots+u+1)} \sum_{w\in D_{2n}} (-1)^{\ell(w)} u^{n-\ell(w)} T_w$$

$e_{\sigma_1} = \frac{1}{mu^{m-1}(u+1)^2} \sum_{w\in D_{2n}} (-1)^{\ell_1(w)} u^{m-\ell_1(w)} T_w$, where $\ell_1(w)$ is the number of s_1's in a reduced expression of $w \in D_{2n}$.

$e_{\sigma_2} = \frac{1}{mu^{m-1}(u+1)^2} \sum_{w\in D_{2n}} (-1)^{\ell_2(w)} u^{m-\ell_2(w)} T_w$, where $\ell_2(w)$ is the number of s_2's in a reduced expression of $w \in D_{2n}$, and for every $1 \leq j \leq m-1$, the pair $\{e_1^{(j)}, e_2^{(j)}\}$ where

$$e_1^{(j)} = \frac{1}{mu^{m-1} \prod_{\substack{i=1 \\ i\neq j}}^{m-1} (\rho_j-\rho_i)(u^2-u\rho_j+1)} \times$$

$$\times \sum_{k=0}^{m-1} u^{m-k-1}[m \sum_{\lambda=k+2}^{m} (-1)^{\lambda-k} S_{m-\lambda}(\rho_j) + (-1)^{k-1} \sum_{\lambda=2}^{m} (-1)^{\lambda}(\lambda-1)S_{m-\lambda}(\rho_j)]Q_{k+1}$$

and

$$e_2^{(j)} = \frac{1}{mu^{m-1} \prod_{\substack{i=1 \\ i\neq j}}^{m-1} (\rho_j-\rho_i)(u^2-u\rho_j+1)} \times$$

$$\times \sum_{k=0}^{m-1} u^{m-k-1}[m \sum_{\lambda=k+2}^{m} (-1)^{\lambda-k} S_{m-\lambda}(\rho_j) + (-1)^{k-1} \sum_{\lambda=2}^{m} (-1)^{\lambda}(\lambda-1)S_{m-\lambda}(\rho_j)]\hat{Q}_{k+1}$$

<u>Proof</u>: We shall make use of the decomposition of the graded module grad(E) into a direct sum of left irreducible H-submodules (see at the beginning of §6.2), and we consider the basis B_0 adapted to this decomposition. We also recall that the submodules $M_0 = \langle\bar{e}_1\rangle$, $M_s = \langle\bar{e}_{w_0}\rangle$, $M_j = \langle u_j', v_j\rangle$, $1 \le j \le m$ form a full set of left irreducible H-submodules. Let $\Lambda_0, \Lambda_s, \Lambda_j$, $1 \le j \le m$ be the irreducible matrix representations obtained in this way, with degrees $d_0 = d_s = 1$, $d_j = 2$, $1 \le j \le m$. Then, under the isomorphism Π(see beginning of §6.1), we have

$$\Pi(h) = (\Lambda_0(h),\ \Lambda_1(h),\ldots,\Lambda_m(h),\Lambda_s(h)),\ \forall\ h \in H.$$

We first consider the elements $e_1^{(j)}$, $1 \le j \le m$ and we shall show that $e_1^{(j)}$ is represented on M_j by the matrix $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ and by the zero matrix on every other irreducible constituent. It is clear by their definition that Q_k belongs to $L(u) = M(u)$, and so $e_1^{(j)}$ belongs to $L(u)$. Therefore $e_1^{(j)}$ is represented by a diagonal matrix on every irreducible constituent. Now

$$Q_k = uT_{(s_2s_1)^{k-1}} - T_{(s_1s_2)^{k-1}s_1} + uT_{(s_2s_1)^{k-1}s_2} - T_{(s_1s_2)^k} \qquad k \ge 1.$$

Thus, on the module M_0, which affords the representation $T_{s_i} \to u$, $i = 1,2$, each Q_k is represented by: $uu^{2k-2} - u^{2k-1} + uu^{2k-1} - u^{2k} = 0$. So $e_1^{(j)}$ is represented by (0) on M_0. Similarly each Q_k is represented on M_s which affords the representation $T_{s_i} \to -1$, $i = 1,2$, by:

$u(-1)^{2k-2} - (-1)^{2k-1} + u(-1)^{2k-1} - (-1)^{2k} = 0$, and therefore $e_1^{(j)}$ is also represented by (0) on M_s.

By recalling the matrices which represent the elements T_w on M_λ, $1 \le \lambda \le m$ (see Lemma 4.1) we obtain that each Q_k is represented on M_λ by the diagaon matrix $\begin{bmatrix}A_k^{(\lambda)} & 0\\ 0 & B_k^{(\lambda)}\end{bmatrix}$ where

$$B_k^{(\lambda)} = u(u^{k-1}(S_{k-1}(\rho_\lambda) + S_{k-2}(\rho_\lambda))) - u^k(S_{k-1}(\rho_\lambda) + S_{k-2}(\rho_\lambda)) -$$

$$- u(u^{k-1}(S_{k-1}(\rho_\lambda) + S_{k-2}(\rho_\lambda))) + u^k(S_{k-1}(\rho_\lambda) + S_{k-2}(\rho_\lambda)) = 0,$$

while the $A_k^{(\lambda)}$ are given by the following relations

$$A_1^{(\lambda)} = u^2-u\rho_\lambda + 1, \quad A_2^{(\lambda)} = u(1 +\rho_\lambda)(u^2-u\rho_\lambda + 1),$$

$$A_k^{(\lambda)} = u(-u^{k-1}(S_{k-2}(\rho_\lambda) + S_{k-3}(\rho_\lambda))) + u^{k-1}(S_{k-1}(\rho_\lambda) + S_{k-2}(\rho_\lambda)) +$$

$$+ u(u^k(S_{k-1}(\rho_\lambda) + S_{k-2}(\rho_\lambda))) - u^k(S_k(\rho_\lambda) + S_{k-1}(\rho_\lambda)) =$$

$$= -u^k(S_{k-2}(\rho_\lambda) + S_{k-3}(\rho_\lambda)) + (u^{k-1} + u^{k+1})(S_{k-1}(\rho_\lambda) + S_{k-2}(\rho_\lambda)) -$$

$$- u^k(S_k(\rho_\lambda) + S_{k-1}(\rho_\lambda)), \text{ for } k = 3,4,\ldots,m.$$

Moreover $S_k(\rho_\lambda) = \rho_\lambda S_{k-1}(\rho_\lambda) - S_{k-2}(\rho_\lambda)$ for all $k = 3,\ldots,m$ and therefore

$$S_{k-2}(\rho_\lambda) + S_{k-3}(\rho_\lambda) = \rho_\lambda(S_{k-1}(\rho_\lambda) + S_{k-2}(\rho_\lambda)) - S_k(\rho_\lambda) - S_{k-1}(\rho_\lambda).$$

Hence $A_k^{(\lambda)} = -u^k\rho_\lambda (S_{k-1}(\rho_\lambda) + S_{k-2}(\rho_\lambda)) + (u^{k-1} + u^{k+1})(S_{k-1}(\rho_\lambda) + S_{k-2}(\rho_\lambda)) =$

$= u^{k-1}(u^2-u\rho_\lambda+1)(S_{k-1}(\rho_\lambda) + S_{k-2}(\rho_\lambda))$, $k = 3,\ldots,m$, and so we eventually have $A_k^{(\lambda)} = u^{k-1}(u^2-u\rho_\lambda+1)(S_{k-1}(\rho_\lambda) + S_{k-2}(\rho_\lambda))$ for $1 \le k \le m$. Therefore the element $e_1^{(j)}$ is represented on M_λ, $1 \le \lambda \le m$ by the matrix $\begin{bmatrix} F^{(\lambda)} & 0 \\ 0 & 0 \end{bmatrix}$, where

$$F^{(\lambda)} = \frac{1}{u^{m-1} \prod\limits_{\substack{i=1 \\ i\neq j}}^{m} (\rho_j-\rho_i)(u^2-u\rho_j+1)} u^{m-1}(u^2-u\rho_\lambda+1) \sum_{k=1}^{m} S_{m-k}(\rho_j)(S_{k-1}(\rho_\lambda)+S_{k-2}(\rho_\lambda)).$$

Now if $\lambda \neq j$, then Lemma 6.4(1) implies $F^{(\lambda)} = 0$. If $\lambda = j$ then Lemmas 6.6 and 6.3(1) imply $F^{(\lambda)} = 1$. We next consider the elements $e_2^{(j)}$, $1 \le j \le m$ and we shall show that $e_2^{(j)}$ is represented on M_j by the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and by the zero matrix on every other irreducible constituent.

It is easy to verify that it is represented by the zero matrix on M_0, M_s, simply by looking at the action of $\hat{Q}_k$ on these modules, $1 \le k \le m$, which turns out to be the zero action. We recall (see Remark 4.1), that the matrices which represent the elements $T_{(s_1 s_2)^\mu}$ and $T_{(s_2 s_1)^\mu}$, $\mu = 1,\dots,m$ on the constituent M_λ, $1 \le \lambda \le m$, are obtained from one another by conjugation by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

The same is true for the elements $T_{(s_1 s_2)^\mu s_1}$ and $T_{(s_2 s_1)^\mu s_2}$ $\mu = 0,1,\dots,m-1$.

Therefore the matrix which represents $\hat{Q}_k$ on M_λ is given by conjugating the matrix which represents Q_k on M_λ by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

In other words the matrix which represents $\hat{Q}_k$ on M_λ is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F^{(\lambda)} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & F^{(\lambda)} \end{bmatrix} .$$

Thus $e_2^{(j)}$ is represented by the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ on M_j and by the zero matrix on every other irreducible constituent.

Finally we consider the elements e_0 and e_σ. We put $P(u) = (u+1)(u^{n-1}+\dots+u+1)$

We write $e_0 = \dfrac{1}{P(u)} \displaystyle\sum_{\substack{w \\ s_i w > w}} (T_w + T_{s_i w})$ for $i = 1,2$. Then

$T_{s_i} e_0 = \dfrac{1}{P(u)} \displaystyle\sum_{\substack{w \in W \\ s_i w > w}} (T_{s_i w} + uT_w + (u-1)T_{s_i w}) = ue_0$ and therefore He_0

affords the representation $T_{s_i} \to u$, $i = 1,2$.

Moreover e_0 belongs to the centre of H and since $T_w . e_0 = u^{\ell(w)} e_0$ for every $w \in W$, we have $e_0^2 = e_0$. We similarly write the element e_σ as

$$e_\sigma = \frac{1}{P(u)}\Big[\sum_{\substack{w \\ s_1 w > w}} (-1)^{\ell(w)} u^{n-\ell(w)}\, T_w + \sum_{\substack{w \\ s_1 w > w}} (-1)^{\ell(w)+1} u^{n-\ell(w)-1}\, T_{s_1 w}\Big].$$

Thus $T_{s_i} e_\sigma =$

$$\frac{1}{P(u)}\Big[\sum_{\substack{w \\ s_1 w > w}} (-1)^{\ell(w)} u^{n-\ell(w)} T_{s_1 w} + \sum_{\substack{w \\ s_1 w > w}} (-1)^{\ell(w)+1} u^{n-\ell(w)-1} (uT_w + (u-1)T_{s_1 w})\Big]$$

$= -e_\sigma$, for $i = 1,2$.

Therefore He_σ affords the representation $T_{s_i} \to -1$, $i = 1,2$. We also have that e_σ belongs to the centre of H and since $T_w e_\sigma = (-1)^{\ell(w)} e_\sigma$ for every $w \in W$, we have $e_\sigma^2 = e_\sigma$. It is now clear that the simple components of H are given by He_0, $He_1^{(j)} \oplus He_2^{(j)}$, $1 \le j \le m$ and He_σ.

Moreover He_0 annihilates every other component different from itself. Therefore e_0 is represented by the zero matrix on every irreducible module different from M_0, which affords the representation $T_{s_i} \to u$, $i = 1,2$.

The argument is similar for the element e_σ.

So part (1) of our theorem is now proved.

(2). Let $n = 2m$. We now make use of the decomposition of grad(E) as

$\text{grad}(E) = M_0 \oplus M_s \oplus M_1 \oplus M_2 \bigoplus_{j=1}^{m-1} V_j \bigoplus_{j=1}^{m-1} \hat{V}_j$, where $\{M_0, M_S, M_1, M_2, V_j, 1 \le j \le m-1\}$ is a full set of left irreducible H-modules. To this decomposition we adopt a basis $B_1 = \{\bar{e}_1, \bar{e}_{w_0}, \theta_1, \theta_2, u_j', v_j, \hat{u}_j', \hat{v}_j, 1 \le j \le m-1\}$ with $V_j = \langle u_j', v_j \rangle$, $\hat{V}_j = \langle \hat{u}_j', \hat{v}_j \rangle$, $j = 1,\dots,m-1$. Let $\Lambda_0, \Lambda_s, R_1, R_2, \Lambda_j$, $1 \le j \le m-1$, be the irreducible matrix representations obtained in this way with underlying modules M_0, M_s, M_1, M_2, V_j and degrees $d_0 = d_s = d_1 = d_2 = 1$, $f_j = 2$, $1 \le j \le m-1$. Then under the isomorphism Π we have $\Pi(h) = (\Lambda_0(h), \Lambda_s(h), R_1(h), R_2(h), \Lambda_1(h), \dots, \Lambda_{m-1}(h))$, for every $h \in H$. We first consider the element $e_1^{(j)}$, $1 \le j \le m-1$ and we shall show that $e_1^{(j)}$ is represented on V_j with respect to the adopted basis $\{u_j', v_j\}$ by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and by the zero matrix on every other irreducible module V_j, $j \neq j'$, M_0, M_s, M_1, M_2. In order to do so, we need the action of the quantities Q_k, $1 \leq k \leq m$ on the several irreducible modules. This has already been established for the 2-dimensional modules V_λ, $1 \leq \lambda \leq m-1$ (see case n odd) and for the modules M_0, M_s which afford the representations $T_{s_i} \to u$ and $T_{s_i} \to -1$, $i = 1,2$ respectively. We have to consider the action of Q_k on M_1, M_2 which afford the representations $T_{s_1} \to -1$, $T_{s_2} \to u$ and $T_{s_1} \to u$ $T_{s_2} \to -1$ respectively.

On the module M_1, Q_k is represented by: $(-1)^{k-1} u^k - (-1)^k u^{k-1} + (-1)^{k-1} u^{k+1} - (-1)^k u^k = (-1)^{k-1} u^{k-1}(u+1)^2$, and on the module M_2 is represented by: $(-1)^{k-1} u^k - (-1)^{k-1} u^k + (-1)^k u^k - (-1)^k u^k = 0$.

To summarize the action of Q_k, $1 \leq k \leq m$ on the several irreducible modules we have:

$M_0 : Q_k \to 0$, $M_s : Q_k \to 0$, $M_1 : Q_k \to (-1)^{k-1} u^{k-1}(u+1)^2$,

$M_2 : Q_k \to 0$, and on V_ℓ, $1 \leq \ell \leq m-1$, Q_k is represented by the matrix

$$\begin{bmatrix} A_k^{(\ell)} & 0 \\ 0 & 0 \end{bmatrix}, \text{ where}$$

$A_k^{(\ell)} = u^{k-1}[S_{k-1}(\rho_\ell) + S_{k-2}(\rho_\ell)](u^2 - u\rho_\ell+1)$, $1 \leq k \leq m$, with $A_m^{(\ell)} = u^{m-1}S_{m-2}(\rho_\ell)(u^2-u\rho_\ell+1)$ (since $S_{m-1}(\rho_\ell) = 0$).

We adopt the notation $G_j =: mu^{m-1} \prod_{i=1,\, i\neq j}^{m-1} (\rho_i-\rho_j)(u^2-u\rho_j+1)$.

We must investigate the action of $e_1^{(j)}$ only on the modules M_1 and V_ℓ, $1 \leq \ell \leq m-1$. By taking into account the way in which Q_k acts on M_1, $1 \leq k \leq m$, we obtain that $e_1^{(j)}$ is represented on M_1 by:

$$\frac{u^{m-1}(u+1)^2}{G_j} \sum_{k=0}^{m-1} (-1)^k [m \sum_{\lambda=k+2}^{m} (-1)^{\lambda-k} S_{m-\lambda}(\rho_j) + (-1)^{k-1} \sum_{\lambda=2}^{m} (-1)^\lambda (\lambda-1) S_{m-\lambda}(\rho_j)] =$$

$$= \frac{u^{m-1}(u+1)^2}{G_j} [m \sum_{k=0}^{m-1} \sum_{\lambda=k+2}^{m} (-1)^\lambda S_{m-\lambda}(\rho_j) + \sum_{k=0}^{m-1} (-1)^{2k-1} (\sum_{\lambda=2}^{m} (-1)^\lambda (\lambda-1) S_{m-\lambda}(\rho_j))]$$

$$= \frac{u^{m-1}(u+1)^2}{G_j} \; m[\sum_{k=0}^{m-1} \sum_{\lambda=k+2}^{m} (-1)^{\lambda} S_{m-\lambda}(\rho_j) - \sum_{\lambda=2}^{m} (-1)^{\lambda}(\lambda-1) S_{m-\lambda}(\rho_j)].$$

In the last expression the coefficient of $S_{m-\lambda}(\rho_j)$, $\lambda \in \{2,\dots,m\}$ is

$$\frac{mu^{m-1}(u+1)^2}{G_j} [\sum_{k=0}^{\lambda-2} (-1)^{\lambda} - (-1)^{\lambda}(\lambda-1)] = 0.$$

Hence $e_1^{(j)}$ is represented by the zero matrix on the one dimensional module M_1.

Using a similar argument as in the case n odd, we find that $e_1^{(j)}$ is represented on the two dimensional module V_ℓ, $1 \le \ell \le m-1$ by the diagonal matrix $\begin{bmatrix} F^{(\ell)} & 0 \\ 0 & 0 \end{bmatrix}$ where $F^{(\ell)} = \frac{1}{G_j}$ ×

$$\sum_{k=0}^{m-1} u^{m-k-1} [m \sum_{\lambda=k+2}^{m} (-1)^{\lambda-k} S_{m-\lambda}(\rho_j) + (-1)^{k-1} \sum_{\lambda=2}^{m} (-1)^{\lambda}(\lambda-1) S_{m-\lambda}(\rho_j)] u^k (S_{k-1}(\rho_\ell) + S_k(\rho_\ell))(u^2 - u\rho_\ell + 1$$

So, $F^{(\ell)} = \frac{u^{m-1}(u^2-u\rho_\ell+1)}{G_j}$ ×

$$\sum_{k=0}^{m-1} [m \sum_{\lambda=k+2}^{m} (-1)^{\lambda-k} S_{m-\lambda}(\rho_j) + (-1)^{k-1} \sum_{\lambda=2}^{m} (-1)^{\lambda}(\lambda-1) S_{m-\lambda}(\rho_j)](S_{k-1}(\rho_\ell) + S_k(\rho_\ell)).$$

Or, $F^{(\ell)} = \frac{u^{m-1}(u^2-u\rho_\ell+1)}{G_j}$ ×

$$\Big[\sum_{\lambda=2}^{m} (-1)^{\lambda}(m+1-\lambda) S_{m-\lambda}(\rho_j) + m \sum_{k=1}^{m-2} \Big(\sum_{\lambda=k+2}^{m} (-1)^{\lambda-k} S_{m-\lambda}(\rho_j)\Big)(S_{k-1}(\rho_\ell) + S_k(\rho_\ell)) +$$

$$+ \sum_{\lambda=2}^{m} (-1)^{\lambda}(\lambda-1) S_{m-\lambda}(\rho_j))[\sum_{k=1}^{m-2} (-1)^{k-1}(S_{k-1}(\rho_\ell) + S_k(\rho_\ell))] +$$

$$+ (-1)^{m-2} \Big(\sum_{\lambda=2}^{m} (-1)^{\lambda}(\lambda-1) S_{m-\lambda}(\rho_j)\Big) S_{m-2}(\rho_\ell). \Big]$$

Moreover $\sum_{k=1}^{m-2} (-1)^{k-1}(S_{k-1}(\rho_\ell) + S_k(\rho_\ell)) = 1 + S_1(\rho_\ell) - S_1(\rho_\ell) + \sum_{k=2}^{m-3} (-1)^{k-1} S_k(\rho_\ell) +$

$$+ \sum_{k=3}^{m-2} (-1)^{k-1} S_{k-1}(\rho_\ell) + (-1)^{m-3} S_{m-2}(\rho_\ell) = 1 + (-1)^{m-3} S_{m-2}(\rho_\ell).$$

Hence $F^{(\ell)} = \frac{mu^{m-1}(u^2-u\rho_\ell+1)}{G_j} \times$

$$[\sum_{k=1}^{m-2}(\sum_{\lambda=k+2}^{m}(-1)^{\lambda-k}S_{m-\lambda}(\rho_j))(S_{k-1}(\rho_\ell)+S_k(\rho_\ell)) + \sum_{\lambda=2}^{m}(-1)^{\lambda}S_{m-\lambda}(\rho_j)].$$

Now if $\ell = j$, then Lemmas 6.7 and 6.3(2) imply $F^{(\ell)} = 1$. Next we assume that $\ell \neq j$, and let $D_r(\rho_j) = \sum_{\lambda=r}^{m}(-1)^{\lambda-r}S_{m-\lambda}(\rho_j)$ for every $r \in \{3,4,\dots,m\}$. (See Lemma 6.7). Then

$$\sum_{k=1}^{m-2}(\sum_{\lambda=k+2}^{m}(-1)^{\lambda-k}S_{m-\lambda}(\rho_j))(S_{k-1}(\rho_\ell)+S_k(\rho_\ell)) + \sum_{\lambda=2}^{m}(-1)^{\lambda}S_{m-\lambda}(\rho_j) =$$

$$\sum_{r=3}^{m}D_r(\rho_j)(S_{r-2}(\rho_\ell)+S_{r-3}(\rho_\ell)) + \sum_{\lambda=2}^{m}(-1)^{\lambda}S_{m-\lambda}(\rho_j) =$$

$$\rho_\ell D_3(\rho_j) + D_3(\rho_j) + \sum_{r=4}^{m-1}D_r(\rho_j)S_{r-2}(\rho_\ell) + S_{m-2}(\rho_\ell) + \sum_{r=3}^{m-2}D_{r+1}(\rho_j)S_{r-2}(\rho_\ell) +$$

$$+ S_{m-3}(\rho_\ell) + \sum_{\lambda=2}^{m}(-1)^{\lambda}S_{m-\lambda}(\rho_j) =$$

$$= \rho_\ell D_3(\rho_j) + D_3(\rho_j) + \sum_{r=4}^{m-1}D_r(\rho_j)S_{r-2}(\rho_\ell) + S_{m-2}(\rho_\ell) + \rho_\ell D_4(\rho_j) +$$

$$+ \sum_{r=4}^{m-2}D_{r+1}(\rho_j)S_{r-2}(\rho_\ell) + S_{m-3}(\rho_\ell) + \sum_{\lambda=2}^{m}(-1)^{\lambda}S_{m-\lambda}(\rho_j) =$$

$$= D_3(\rho_j) + \rho_\ell(D_3(\rho_j) + D_4(\rho_j)) + \sum_{r=4}^{m-2}S_{r-2}(\rho_\ell)(D_r(\rho_j) + D_{r+1}(\rho_j)) +$$

$$+ D_{m-1}(\rho_j)S_{m-3}(\rho_\ell) + S_{m-2}(\rho_\ell) + S_{m-3}(\rho_\ell) + \sum_{\lambda=2}^{m}(-1)^{\lambda}S_{m-\lambda}(\rho_j) =$$

$$= \sum_{\lambda=3}^{m}(-1)^{\lambda-3}S_{m-\lambda}(\rho_j) + \rho_\ell S_{m-3}(\rho_j) + \sum_{r=4}^{m-2}S_{r-2}(\rho_\ell)S_{m-r}(\rho_j) + (\rho_j-1)S_{m-3}(\rho_\ell) +$$

$$+ S_{m-2}(\rho_\ell) + S_{m-3}(\rho_\ell) + \sum_{\lambda=2}^{m}(-1)^{\lambda}S_{m-\lambda}(\rho_j) =$$

$$= \sum_{r=2}^{m}S_{r-2}(\rho_\ell)S_{m-r}(\rho_j) = 0 \text{ (by Lemma 6.4(2))}.$$

Hence if $\ell \neq j$, $1 \le \ell \le m-1$, then $F^{(\ell)} = 0$.

Therefore we have established that $e_1^{(j)}$ is represented on the constituent V_j, $1 \le j \le m-1$ by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and by the zero matrix on every other irreducible constituent V_ℓ, $\ell \ne j$, M_0, M_s, M_1, M_2.

The argument is entirely similar when we consider the elements $e_2^{(j)}$, $1 \le j \le m-1$. The action of the quadruples $\hat{Q}_k$ on the four one dimensional submodules M_0, M_s, M_1, M_2 is given by:

$$M_0 : \hat{Q}_k \to 0,\ M_s : \hat{Q}_k \to 0,\ M_1 : \hat{Q}_k \to 0,\ M_2 : \hat{Q}_k \to (-1)^{k-1}\, u^{k-1}(u+1)^2,$$

$1 \le k \le m$, and therefore $e_2^{(j)}$ is represented by the zero matrix on every one of them.

On the other hand, $\hat{Q}_k$ is represented on each V_ℓ, $1 \le \ell \le m-1$ by a matrix which is obtained by conjugating the matrix which represents Q_k on V_ℓ, by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Hence $\hat{Q}_k$ is represented on V_ℓ by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F^{(\ell)} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & F^{(\ell)} \end{bmatrix}$$

Thus $e_2^{(j)}$ is represented by the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ on V_j and by the zero matrix on every V_ℓ, $\ell \ne j$, $1 \le j \le m-1$.

We finally consider the elements e_0, e_σ, e_{σ_1}, e_{σ_2}. We have already shown (see the case n odd), that He_0, He_σ afford the one dimensional representations $T_{s_i} \to u$ and $T_{s_i} \to -1$ $i = 1,2$ respectively. Thus we have to consider only the elements e_{σ_1}, e_{σ_2}.

For every $w \in D_{2n}$ let $\ell_1(w)$, $\ell_2(w)$ be the number of s_1's (respectively s_2's) in a reduced expression of w.

We write

$$e_{\sigma_1} = \frac{1}{mu^{m-1}(u+1)^2}\left[\sum_{w:s_1w>w}(-1)^{\ell_1(w)}u^{m-\ell_1(w)}T_w + \sum_{w:s_1w>w}(-1)^{\ell_1(w)+1}u^{m-\ell_1(w)-1}T_{s_1w}\right]$$

$$= \frac{1}{mu^{m-1}(u+1)^2}\left[\sum_{w:s_2w>w}(-1)^{\ell_1(w)}u^{m-\ell_1(w)}T_w + \sum_{w:s_2w>w}(-1)^{\ell_1(w)}u^{m-\ell_1(w)}T_{s_2w}\right]$$

Therefore,

$$T_{s_1}e_{\sigma_1} = \frac{1}{mu^{m-1}(u+1)^2}\Big[\sum_{w:s_1w>w}(-1)^{\ell_1(w)}u^{m-\ell_1(w)}T_{s_1w} + \sum_{w:s_1w>w}(-1)^{\ell_1(w)+1}u^{m-\ell_1(w)-1}$$

$$(uT_w + (u-1)T_{s_1w})\Big]$$

$= -e_{\sigma_1}$ and,

$$T_{s_2}e_{\sigma_1} = \frac{1}{mu^{m-1}(u+1)^2}\Big[\sum_{w:s_2w>w}(-1)^{\ell_1(w)}u^{m-\ell_1(w)}T_{s_2w} + \sum_{w:s_2w>w}(-1)^{\ell_1(w)}u^{m-\ell_1(w)}$$

$$(uT_w + (u-1)T_{s_2w})\Big]$$

$= ue_{\sigma_1}$.

Hence He_{σ_1} affords the representation $T_{s_1} \to -1$, $T_{s_2} \to u$. Similarly by considering pairs (w,ws_1) or (w,ws_2) we can verify that $e_{\sigma_1}T_{s_1} = -e_{\sigma_1}$, $e_{\sigma_1}T_{s_2} = ue_{\sigma_1}$, and so e_{σ_1} is a central element inside H. Similarly we write

$$e_{\sigma_2} = \frac{1}{mu^{m-1}(u+1)^2}\left[\sum_{w:s_2w>w}(-1)^{\ell_2(w)}u^{m-\ell_2(w)}T_w + \sum_{w:s_2w>w}(-1)^{\ell_2(w)+1}u^{m-\ell_2(w)-1}T_{s_2w}\right]$$

$$= \frac{1}{mu^{m-1}(u+1)^2}\left[\sum_{w:s_1w>w}(-1)^{\ell_2(w)}u^{m-\ell_2(w)}T_w + \sum_{w:s_1w>w}(-1)^{\ell_2(w)}u^{m-\ell_2(w)}T_{s_1w}\right]$$

and we can easily show that $T_{s_1}e_{\sigma_2} = e_{\sigma_2}T_{s_1} = ue_{\sigma_2}$, and $T_{s_2}e_{\sigma_2} = e_{\sigma_2}T_{s_2} = -e_{\sigma_2}$. Hence e_{σ_2} belongs to the centre of H and He_{σ_2} affords the representation $T_{s_1} \to u$, $T_{s_2} \to -1$. Moreover

$T_w e_{\sigma_1} = (-1)^{\ell_1(w)}u^{\ell_2(w)}$ and $T_w e_{\sigma_2} = (-1)^{\ell_2(w)}u^{\ell_1(w)}$ for every $w \in D_{2n}$.

We also write $W = \{1\} \cup A_1 \cup A_2 \cup A_3 \cup \{(s_1s_2)^m\}$, where

$A_1 = \{(s_1s_2)^j, (s_2s_1)^j,\ 1 \le j \le m-1\}$, $A_2 = \{(s_1s_2)^{j-1}s_1,\ 1 \le j \le m\}$,

$A_3 = \{(s_2s_1)^{j-1}s_2,\ 1 \le j \le m\}$ Thus

$$e_{\sigma_1}^2 = \frac{1}{mu^{m-1}(u+1)^2}\left(\sum_{w\in W} u^{m+\ell_2(w)-\ell_1(w)}\right)e_{\sigma_1}, \text{ with}$$

$$\sum_{w\in W} u^{m+\ell_2(w)-\ell_1(w)} = u^m+(2m-2)u^m+mu^{m-1}+mu^{m+1}+u^m = mu^{m-1}(u+1)^2.$$

So $e_{\sigma_1}^2 = e_{\sigma_1}$. Similarly we can show that $e_{\sigma_2}^2 = e_{\sigma_2}$.

Now the simple components of H are clearly

He_0, He_σ, He_{σ_1}, He_{σ_2}, $He_1^{(j)} \oplus He_2^{(j)}$, $1 \le j \le m-1$.

Each such component annihilates every other component different from itself, so the element e_0 is represented by the zero matrix on every irreducible module different from M_0 which affords the representation $T_{s_i} \to u$, $i = 1,2$. The argument is similar for the elements e_σ, e_{σ_1}, e_{σ_2}. Our theorem is now proved.

We next give some examples to illustrate the situation.

(1) $W(A_2) \simeq D_6$, $n = 3 = 2.1 + 1$, $m = 1$, $\rho_1 = 2\cos\frac{2\pi}{3} = -1$

$$e_0 = \frac{1}{(u+1)(u^2+u+1)}\sum_{w\in D_6} T_w,\quad e_\sigma = \frac{1}{(u+1)(u^2+u+1)}\sum_{w\in D_6}(-1)^{\ell(w)}u^{3-\ell(w)}T_w$$

$$e_1^{(1)} = \frac{1}{u^2+u+1}[uT_1-T_{s_1}+uT_{s_2}-T_{s_1s_2}] = \frac{1}{u^2+u+1}(uT_1-T_{s_1})(T_1+T_{s_2})$$

$$e_2^{(1)} = \frac{1}{u^2+u+1}[uT_1-T_{s_2}+uT_{s_1}-T_{s_2s_1}] = \frac{1}{u^2+u+1}(uT_1-T_{s_2})(T_1+T_{s_1})$$

(2) $W(B_2) \simeq D_8$, $n = 4 = 2.2$, $m = 2$, $\rho_1 = 2\cos\frac{\pi}{2} = 0$

$$e_0 = \frac{1}{(u+1)(u^3+u^2+u+1)}\sum_{w\in D_8} T_w,\quad e_\sigma = \frac{1}{(u+1)(u^3+u^2+u+1)}\sum_{w\in D_8}(-1)^{\ell(w)}u^{4-\ell(w)}T_w$$

$$e_{\sigma_1} = \frac{1}{2u(u+1)^2}[u^2T_1 - uT_{s_1} + u^2T_{s_2} - uT_{s_1s_2} - uT_{s_2s_1} + T_{s_1s_2s_1} - uT_{s_2s_1s_2} + T_{s_1s_2s_1s_2}]$$

$$e_{\sigma_2} = \frac{1}{2u(u+1)^2}[u^2T_1 - uT_{s_2} + u^2T_{s_1} - uT_{s_2s_1} - uT_{s_1s_2} + T_{s_2s_1s_2} - uT_{s_1s_2s_1} + T_{s_1s_2s_1s_2}]$$

$$e_1^{(1)} = \frac{1}{2u(u+1)^2}[u^2T_1 - uT_{s_1} + u^2T_{s_2} - uT_{s_1s_2} + uT_{s_2s_1} \qquad - T_{s_1s_2s_1} + uT_{s_2s_1s_2} - T_{s_1s_2s_1s_2}]$$

$$= \frac{1}{2u(u+1)^2}(u^2T_1 - uT_{s_1} + uT_{s_2s_1} - T_{s_1s_2s_1})(T_1 + T_{s_2}) =$$

$$= \frac{1}{2u(u+1)^2}(uT_1 - T_{s_1})(uT_1 \qquad + T_{s_2s_1})(T_1 + T_{s_2})$$

$$e_2^{(1)} = \frac{1}{2u(u+1)^2}[u^2T_1 - uT_{s_2} + u^2T_{s_1} - uT_{s_2s_1} + uT_{s_1s_2} - T_{s_2s_1s_2} + uT_{s_1s_2s_1} - T_{s_1s_2s_1s_2}]$$

$$= \frac{1}{2u(u+1)^2}[u^2T_1 - uT_{s_2} + uT_{s_1s_2} - T_{s_2s_1s_2})(T_1 + T_{s_1}) =$$

$$= \frac{1}{2u(u+1)^2}(uT_1 - T_{s_2})(uT_1 + T_{s_1s_2})(T_1 + T_{s_1}).$$

(3) $W = D_{10}$, $n = 5 = 2.2 + 1$, $m = 2$, $\rho_1 = 2\cos\frac{2\pi}{5}$, $\rho_2 = 2\cos\frac{4\pi}{5}$.

$$e_0 = \frac{1}{(u+1)(u^4+u^3+u^2+u+1)}\sum_{w\in D_{10}} T_w,\quad e_\sigma = \frac{1}{(u+1)(u^4+u^3+u^2+u+1)}\sum_{w\in D_{10}} (-1)^{\ell(w)}u^{5-\ell(w)}T_w$$

$$e_1^{(1)} = \frac{1}{u(\rho_1-\rho_2)(u^2-u\rho_1+1)}[u\rho_1(uT_1 - T_{s_1} + uT_{s_2} - T_{s_1s_2}) + uT_{s_2s_1} - T_{s_1s_2s_1} + uT_{s_2s_1s_2} - T_{(s_1s_2)^2}]$$

$$= \frac{1}{u(\rho_1-\rho_2)(u^2-u\rho_1+1)}(uT_1 - T_{s_1})(u\rho_1T_1 + T_{s_2s_1})(T_1 + T_{s_2})$$

$$e_2^{(1)} = \frac{1}{u(\rho_1-\rho_2)(u^2-u\rho_1+1)}[u\rho_1(uT_1 - T_{s_2} + uT_{s_1} - T_{s_2s_1}) + uT_{s_1s_2} - T_{s_2s_1s_2} + uT_{s_1s_2s_1} - T_{(s_2s_1)^2}]$$

$$= \frac{1}{u(\rho_1-\rho_2)(u^2-u\rho_1+1)}(uT_1 - T_{s_2})(u\rho_1T_1 + T_{s_1s_2})(T_1 + T_{s_1})$$

$$e_1^{(2)} = \frac{1}{u(\rho_2-\rho_1)(u^2-u\rho_2+1)}[u\rho_2(uT_1-T_{s_1}+uT_{s_2}-T_{s_1s_2})+uT_{s_2s_1}-T_{s_1s_2s_1}+uT_{s_2s_1s_2}-T_{(s_1s_2)^2}]$$

$$= \frac{1}{u(\rho_2-\rho_1)(u^2-u\rho_2+1)}(uT_1-T_{s_1})(u\rho_2T_1+T_{s_2s_1})(T_1+T_{s_2})$$

$$e_2^{(2)} = \frac{1}{u(\rho_2-\rho_1)(u^2-u\rho_2+1)}[u\rho_2(uT_1-T_{s_2}+uT_{s_1}-T_{s_2s_1})+uT_{s_1s_2}-T_{s_2s_1s_2}+uT_{s_1s_2s_1}-T_{(s_2s_1)^2}]$$

$$= \frac{1}{u(\rho_2-\rho_1)(u^2-u\rho_2+1)}(uT_1-T_{s_2})(u\rho_2T_1+T_{s_1s_2})(T_1+T_{s_1})$$

(4) $W(G_2) \simeq D_{12}$, $n = 6 = 2.3$, $m = 3$, $\rho_1 = 2\cos\frac{\pi}{3} = 1$, $\rho_2 = 2\cos\frac{2\pi}{3} = -1$.

$$e_0 = \frac{1}{(u+1)(u^5+u^4+u^3+u^2+u+1)}\sum_{w\in D_{12}} T_w ,$$

$$e_\sigma = \frac{1}{(u+1)(u^5+u^4+u^3+u^2+u+1)}\sum_{w\in D_{12}} (-1)^{\ell(w)} u^{6-\ell(w)} T_w$$

$$e_{\sigma_1} = \frac{1}{3u^2(u+1)^2}\Big[u^3T_1-u^2T_{s_1}+u^3T_{s_2}-u^2T_{s_1s_2}-u^2T_{s_2s_1}+uT_{s_1s_2s_1}-u^2T_{s_2s_1s_2} + uT_{(s_1s_2)^2}+uT_{(s_2s_1)^2}-T_{(s_1s_2)^2s_1}+uT_{(s_2s_1)^2s_2}-T_{(s_1s_2)^3}\Big]$$

$$e_{\sigma_2} = \frac{1}{3u^2(u+1)^2}\Big[u^3T_1-u^2T_{s_2}+u^3T_{s_1}-u^2T_{s_2s_1}-u^2T_{s_1s_2}+uT_{s_2s_1s_2}-u^2T_{s_1s_2s_1} + uT_{(s_2s_1)^2}+uT_{(s_1s_2)^2}-T_{(s_2s_1)^2s_2}+uT_{(s_1s_2)^2s_1}-T_{(s_1s_2)^3}\Big]$$

$$e_1^{(1)} = \frac{1}{6u^2(u^2-u+1)}\Big[u^2(uT_1-T_{s_1}+uT_{s_2}-T_{s_1s_2})+2u(uT_{s_2s_1}-T_{s_1s_2s_1}+uT_{s_2s_1s_2}-T_{(s_1s_2)^2}) + uT_{(s_2s_1)^2}-T_{(s_1s_2)^2s_1}+uT_{(s_2s_1)^2s_2}-T_{(s_1s_2)^3}\Big]$$

$$= \frac{1}{6u^2(u^2-u+1)}(uT_1-T_{s_1})(u^2T_1+2uT_{s_2s_1}+T_{(s_2s_1)^2})(T_1+T_{s_2})$$

$$e_2^{(1)} = \frac{1}{6u^2(u^2-u+1)}\Big[u^2(uT_1-T_{s_2}+uT_{s_1}-T_{s_2s_1})+2u(uT_{s_1s_2}-T_{s_2s_1s_2}+uT_{s_1s_2s_1}-T_{(s_2s_1)^2})+$$

$$+uT_{(s_1s_2)^2}-T_{(s_2s_1)^2s_2}+uT_{(s_1s_2)^2s_1}-T_{(s_1s_2)^3}\Big]$$

$$= \frac{1}{6u^2(u^2-u+1)}(uT_1-T_{s_2})(u^2T_1+2uT_{s_1s_2}+T_{(s_1s_2)^2})(T_1+T_{s_1})$$

$$e_1^{(2)} = \frac{1}{2u^2(u^2+u+1)}[u^2(uT_1-T_{s_1}+uT_{s_2}-T_{s_1s_2})-uT_{(s_2s_1)^2}+T_{(s_1s_2)^2_{s_1}}-uT_{(s_2s_1)^2_{s_2}}+T_{(s_1s_2)^3}]$$

$$= \frac{1}{2u^2(u^2+u+1)}(uT_1-T_{s_1})\big(u^2T_1-uT_{(s_2s_1)^2}+T_{(s_1s_2)^2_{s_1}}\big)(T_1+T_{s_2})$$

$$e_2^{(2)} = \frac{1}{2u^2(u^2+u+1)}[u^2(uT_1-T_{s_2}+uT_{s_1}-T_{s_2s_1})-uT_{(s_1s_2)^2}+T_{(s_2s_1)^2_{s_2}}-uT_{(s_1s_2)^2_{s_1}}+T_{(s_1s_2)^3}]$$

$$= \frac{1}{2u^2(u^2+u+1)}(uT_1-T_{s_2})(u^2T_1-uT_{(s_1s_2)^2}+T_{(s_2s_1)^2_{s_2}})(T_1+T_{s_1}).$$

OPEN QUESTIONS

We now mention two open questions which arise naturally from our work.

(A): In Chapter 3 we investigate the centre of H following some ideas of R.W. Carter. These ideas give a natural basis for the centre of Hecke algebras of dihedral type, and under the specialization $u \to 1$ this basis specializes to the class sums.

These elements are parametrized by the conjugacy classes of the group and for each conjugacy class C a typical element z_C of this basis has the form: $z_C = \sum_{\ell(w)=\text{maximal inside } C} T_w$ + linear combination of other T_w's not involving any T_w with w of maximal length in any other conjugacy class different from C.

In the same chapter we determine a basis of the same form for the case $W = S_4$.

It is therefore natural to make the following conjecture:

Let H be the Hecke algebra over the polynomial ring $\mathbb{Q}[u]$ associated to a finite indecomposable Coxeter group W. Then there exists a basis $\{ Z_C | C$ runs over the conjugacy classes of W$\}$ of the centre of H, where each Z_C has the form $z_C = \sum_{\ell(w)=\text{maximal inside } C} T_w$ + linear combination of other T_w's, not involving any T_w with w of maximal length in any other conjugacy class different from C.

Furthermore the coefficient of each T_w involved in z_C belongs to $\mathbb{Z}[u]$, and this basis specializes to the class sums under the specialization $u \to 1$.

(B): In Chapter 6 we have introduced a maximal commutative subalgebra M(u) inside the generic Hecke algebra. The definition of this subalgebra depends upon a chosen W-graph.

Nevertheless we have evidence that M(u) does not in fact depend on the choice of the W-graph. To prove this it would be enough to show that M(u) = L(u). In order to show this it would be sufficient to prove that the subalgebra L(1) of $\mathbb{Q}W$ is commutative. We conjecture that this is true for all finite Coxeter groups W.

The validity of this conjecture together with the results of Proposition 6.1 and 6.2(ii) would imply that M(u) = L(u). This result might be of significant help in the effort to decompose the identity of M(u) into a sum of orthogonal primitive ideompotents.

We have checked the truth of this conjecture when W is a dihedral group and when W is the symmetric group S_4.

REFERENCES

[1] M. Abramowitz and I. Stegun: Handbook of Mathematical functions, Dover.

[2] D. Alvis and G. Lusztig: The representations and generic degrees of the Hecke algebra of type H_4, J. Reine Angew. Math, 336 (1982), 201-212.

[3] C.T. Benson and C.W. Curtis: On the degrees and rationality of certain characters of finite Chevalley groups, Trans. Amer. Math. Soc. 165 (1972), 251-273.

[4] ————————: Corrections and additions to "on the degrees and rationality of certain characters of finite Chevalley groups, Trans. Amer. Math. Soc., 202 (1975), 405-406.

[5] N. Bourbaki: Groupes et algébres de Lie, IV, V, VI, Herman, Paris (1968).

[6] R.W. Carter: Simple groups of Lie type, John Wiley, London (1972).

[7] C.W. Curtis and I. Reiner: Methods of Representation theory with application to finite groups and orders, John Wiley, New York (1981).

[8] R. Dipper and G.D. James: Blocks and idempotents of Hecke algebras of general linear groups (Preprint 1985).

[9] A. Cyoja: On the existence of a W-graph for an irreducible Representation of a Coxeter group, J. Algebra 86 (1984), 422-438.

[10] N. Iwahori: On the structure of a Hecke ring of a Chevalley group over a finite field, J. Fac. Sci. Univ. Tokyo, 10 (1964), 215-236.

[11] D. Kazhdan and G. Lusztig: Representations of Coxeter groups and Hecke algebras, Invent. Math. 53 (1979), 165-184.

[12] G. Lusztig: On a theorem of Benson and Curtis, J. Algebra, 71 (1981), 490-498.

[13] G.E. Murphy: The idempotents of the Symmetric group and Nakayama's conjecture, J. Algebra 81 (1983), 258-265.

[14] G. Shimura: Sur les integrales attachées aux formes automorphes, J. Math. Soc. Japan, 11 (1959), 291-311.

[15] R. Steinberg: Lectures on Chevalley groups, Yale University (1967).

[16] I.N. Stewart: Lie Algebras, Lecture notes in Matheamtics, 127 (1970), Springer.