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A Dominant Strategy, Double Clock Auction with Estimation-Based Tâtonnement*

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Abstract

The price mechanism is fundamental to economics but difficult to reconcile with incentive compatibility and individual rationality. We introduce a double clock auction for a homogeneous good market with multi-dimensional private information and multi-unit traders that is deficit-free, ex post individually rational, constrained efficient, and makes sincere bidding a dominant strategy equilibrium. Under a weak dependence and an identifiability condition, our double clock auction is also asymptotically efficient. Asymptotic efficiency is achieved by estimating demand and supply using information from the bids of traders that have dropped out and following a tâtonnement process that adjusts the clock prices based on the estimates.

Keywords: Deficit free, dominant strategy mechanisms, double clock auctions, individual rationality, multi-dimensional types, privacy preservation, reserve prices, VCG mechanism.
JEL Classification: C72, D44, D47, D82.

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1 Introduction

The study of price formation and market making with variable demand and supply and a focus on the efficient allocation of resources has a long tradition in economics. Walras (1874) proposed a procedure, called *tâtonnement*, in which buyers and sellers quote their demands and supplies at a given price to an auctioneer that increases the price if there is excess demand and decreases it if there is excess supply, with transactions only taking place when equilibrium is reached. One important problem with the Walrasian tâtonnement is that agents do not have an incentive to indicate truthfully their demand and supply schedules, as their bidding affects the final price.[1] In his landmark paper, Vickrey (1961) showed that it is possible to elicit the true demands and supplies and implement the efficient allocation, using a generalization of the static auction that bears his name. Observing that it runs a deficit and hence must be financed by an outside source, Vickrey was skeptical about the practical relevance of the market mechanism he proposed, calling it "inordinately expensive" for the market maker. Vickrey did not see an easy way to modify it so as to avoid the deficit, preserve the truth telling property and achieve an approximately efficient allocation, noting (Vickrey, 1961, p.13-14):

> It is tempting to try to modify this scheme in various ways that would reduce or eliminate this cost of operation while still preserving the tendency to optimum resource allocation. However, it seems that all modifications that do diminish the cost of the scheme either imply the use of some external information as to the true equilibrium price or reintroduce a direct incentive for misrepresentation of the marginal-cost and marginal-value curves. To be sure, in some cases the impairment of optimum allocation would be small relative to the reduction in cost, but, unfortunately, the analysis of such variations is extremely difficult; ...

In this paper, we propose a novel double-clock auction that induces price taking behavior by all buyers and sellers at all times and hence elicits revelation of the true quantities demanded and supplied, without running a deficit. We do so for a general environment in which traders have multi-unit demands and supplies and multi-dimensional private information about their marginal values and costs. We view our double-clock auction as a possible solution to the challenges identified by Vickrey. Under mild regularity conditions, we show that our double clock auction generates an outcome converging to the efficient allocation as the number of traders grows at rate $1/n$, where n is the number of traders.

As emphasized by Ausubel (2004), two fundamental prescriptions for practical auction design derived from the auction literature are that the prices paid by an agent ought to be as

[1] The substantial impact on social welfare of strategic behavior in tâtonnement mechanisms was discussed by Babaioff et al. (2014). As they pointed out, tâtonnement mechanisms "are used, for example, in the daily opening of the New York Stock Exchange and the call market for copper and gold in London."

independent as possible from her own bids, and that the auction should be structured in an open, dynamic fashion, so as to convey clear price information to bidders and to preserve the privacy of the winners' valuations. Under the latter property, market participants are protected from hold up by the designer, because they do not reveal their willingness to pay on units they trade, and the designer is protected from the often substantial political and public risk of ex post regret – not knowing the agents' willingness to pay makes it difficult if not impossible to claim that there was "money left on the table."[2]

Our double-clock auction (DCA) satisfies both prescriptions. It consists of a descending clock price for sellers and an ascending clock price for buyers. At every point in the DCA, traders indicate the number of units they are active, or bid, on, with activity meaning that this is the number of units they supply (demand) if they are sellers (buyers). There is a monotone activity rule that stipulates that in the course of the auction a trader can only *decrease* her activity. Once an agent's activity has dropped to zero, the agent is said to have dropped out (or exited). Based on information obtained only from agents who have exited, the DCA *estimates* supply and demand and, at any point in the process, sets target prices that are such that estimated excess demand is zero. If a given target price is reached without any additional exits, this target price becomes the reserve price. If an additional exit occurs before the target is reached, supply and demand are estimated anew, the target price is adjusted, and the DCA proceeds as before until the earliest of two points in time – both clock prices reach the target price, or an additional trader drops out.

Once both clock prices reach the target price, this price becomes the *reserve*, and the quantities supplied and demanded by all remaining *active* traders are used to determine whether buyers or seller are on the long side of the market at the reserve. If aggregate quantity demanded equals aggregate quantity supplied at the reserve, then all trades are executed at this price. Otherwise, agents on the long side participate in an Ausubel (2004) auction, starting at the reserve. We show that sincere bidding by each agent is a dominant strategy equilibrium in the DCA. By construction, it never runs a deficit. It is ex post individually rational and constrained efficient in the sense that the units it trades are procured at minimum cost and allocated to buyers to maximize value. We also provide conditions under which the DCA is asymptotically efficient. Asymptotic efficiency obtains, for example, in the *order statistics model* (Burdett and Woodward, 2020), according to which each buyer (seller) draws a number of values (costs) independently from the same distribution equal to its maximum demand (capacity).

Our paper relates to the literature on dominant strategy mechanisms in the tradition of

[2]The practice of mechanism design and historical experience with auctions offer plenty of examples of such public outcry. The 1990 spectrum license auction in New Zealand is one famous example of political risk due to ex post regret (see, for example, McMillan, 1994, or Milgrom, 2004). That static, sealed bid, mechanisms are prone to the bidders' hold-up problem was known by stamp collectors before the middle of the 20th century (Lucking-Reiley, 2000).

Vickrey (1961), Clarke (1971) and Groves (1973). There are particularly close connections to papers that develop deficit-free dominant strategy mechanisms such as Hagerty and Rogerson (1987) and McAfee (1992). We provide a detailed discussion of these in Section 4, after we have formally introduced our double-clock auction and derived its key properties. Our paper, and the double-clock auction we design, obviously draws inspiration from the extended body of research that has emphasized advantages of clock auctions, such as Ausubel (2004, 2006), Ausubel, Cramton and Milgrom (2006), Perry and Reny (2005), Bergemann and Morris (2007), Levin and Skrzypacz (2016), Li (2017), Sun and Yang (2009, 2014), and Milgrom and Segal (2019).[3] Perry and Reny (2005, p.568), for example, argue that "simultaneous auction formats tend to treat information as if it were costless to collect and costless to provide" while dynamic auctions economize on the information collected.

Our paper also relates to the recent and growing literature on mechanism design with estimation initiated by Baliga and Vohra (2003) and Segal (2003).[4] In that literature, the designer's objective is profit-maximization, and hence the objects to be estimated are hazard rates and virtual types. In contrast, our market maker's objective is social surplus, without running a deficit, and so the object to be estimated is, like in Kojima and Yamashita (2017) the Walrasian price. A more detailed discussion of the connection and differences to the paper of Kojima and Yamashita, which was concurrently written with a previous draft of the present one, is deferred to Section 4.

Of course, the very idea of a tâtonnement process to discover market clearing prices dates back to Walras (1874), and so our paper is also tightly connected to the literature on the decentralized micro-foundations of competitive equilibrium, such as Satterthwaite and Williams (1989, 2002), Rustichini, Satterthwaite, and Williams (1994), and Cripps and Swinkels (2006) as well as to Reny and Perry (2006), who study the related question of the foundations of rational expectation equilibrium.[5] Our double-clock auction can be viewed as providing a centralized micro-foundation in which the "Walrasian" auctioneer does the heavy lifting while endowing agents with dominant strategies. Rather than getting rid of the Walrasian auctioneer, it fills her role with substance.

The remainder of the paper is organized as follows. Section 2 provides the setup. In Section 3, we introduce the DCA and derive its key properties. Section 4 provides a comparison of different mechanisms in the small and a discussion of the most closely related literature. Section

[3] Ausubel's (2004) proposed a clock implementation of the VCG mechanism for the case of homogeneous goods. For subsequent generalizations to the case of heterogenous objects, see Ausubel (2006), and Sun and Yang (2009, 2014).

[4] See also Loertscher and Marx (2019a), who develop a prior-free clock auction that is asymptotically profit-maximizing in an environment with single-unit traders and independently distributed types.

[5] Other papers on the convergence to competitive equilibrium in the single-unit case include Gresik and Satterthwaite (1989) who looked at optimal trading mechanisms, Yoon (2001) who studied a double auction with participation fees and Tatur (2005), who introduced a double auction with a fixed fee. For the multi-unit case, Yoon (2008) introduced the participatory Vickrey-Clarke-Groves mechanism.

5 introduces conditions under which the double-clock auction is asymptotically efficient, and Section 6 concludes the paper.

2 The Setup

There is a set $\mathcal{N} = \{1, ..., N\}$ of buyers, and a set $\mathcal{M} = \{1, ..., M\}$ of sellers of a homogeneous good. In Section 5, to study convergence to efficiency, we will proportionally expand the sets of buyers and sellers to $\mathcal{N} = \{1, ..., nN\}$ and $\mathcal{M} = \{1, ..., nM\}$ and we will let n go to infinity.

Denote by $\boldsymbol{v}^b = \left(v_1^b, ..., v_{k_B}^b\right)$ the valuation, or type, of buyer $b \in \mathcal{N}$, where $v_k^b \in [0, 1]$ is buyer b's marginal value for the k-th unit of the good and k_B is an upper bound on each buyer's demand. Denote by $\boldsymbol{c}^s = \left(c_1^s, ..., c_{k_S}^s\right)$ the cost, or type, of seller $s \in \mathcal{M}$, where $c_k^s \in [0, 1]$ is seller s's cost for producing, or giving up the use of, the k-th unit and k_S is an upper bound on each seller's capacity.[6] Let $\boldsymbol{v} = \left(\boldsymbol{v}^1, ..., \boldsymbol{v}^N\right) = \left(\boldsymbol{v}^b, \boldsymbol{v}^{-b}\right)$ be the profile of valuations, $\boldsymbol{c} = \left(\boldsymbol{c}^1, ..., \boldsymbol{c}^M\right) = \left(\boldsymbol{c}^s, \boldsymbol{c}^{-s}\right)$ be the profile of costs, and $\boldsymbol{\theta} = (\boldsymbol{v}, \boldsymbol{c}) = \left(\boldsymbol{v}^b, \boldsymbol{\theta}^{-b}\right) = \left(\boldsymbol{c}^s, \boldsymbol{\theta}^{-s}\right)$. We assume diminishing marginal values and increasing marginal costs; that is, for all $b \in \mathcal{N}$, all $k \in \{1, .., k_B - 1\}$, we have $v_k^b \geqslant v_{k+1}^b$ and, for all $s \in \mathcal{M}$, all $k \in \{1, .., k_S - 1\}$, we have $c_k^s \leqslant c_{k+1}^s$. A buyer b receiving q goods at unit prices $p_1^b, ..., p_q^b$ obtains payoff $\sum_{k=1}^q \left(v_k^b - p_k^b\right)$; a buyer receiving no units and making no payments has zero payoff. Similarly, a seller s selling q goods at prices $p_1^s, ..., p_q^s$ obtains payoff $\sum_{k=1}^q \left(p_k^s - c_k^s\right)$; a seller receiving no payments and selling no units has zero payoff. The payoff functions and the upper bounds on traders' capacities are common knowledge, but marginal values and marginal costs are private information of each trader.[7]

The mechanism we propose has an open bid, clock format. As ours is a setting with active buyers and sellers (as opposed to a one-sided auction), the mechanism is a *double clock auction*; that is, it will be run with an ascending clock on the buyers' side and a descending clock on the sellers' side. This implies that the mechanism is privacy preserving; that is, it does not reveal the marginal values or marginal costs of the units that are traded.[8] Our mechanism is robust in the sense of Bergemann and Morris (2005), because it satisfies *dominant strategy* incentive compatibility, so that agents do not need well specified beliefs about the other agents' types in order to bid optimally. It can be specified without making use of detailed a priori information about agents' types and beliefs, so to some extent our mechanism is detail free in the sense of Wilson (1987), except that beliefs about values and costs are needed by the auctioneer to estimate demand and supply and determine which of the two clocks should be running at each point in time.

[6] The assumption that values and costs are in $[0, 1]$ is just a normalization.

[7] Our results remain valid when traders have complete information about all marginal values and costs. This is because in the DCA we introduce traders have the dominant strategy of bidding sincerely.

[8] See Engelbrecht-Wiggans and Kahn (1991), Naor et al. (1999), Ausubel (2004) and Milgrom and Segal (2019) for discussions of the importance of this requirement.

Denote the individualized price vector of agent i by $\boldsymbol{p}^i(\boldsymbol{\theta}^{-i}) = \left(p_0^i(\boldsymbol{\theta}^{-i}), ..., p_{k_i}^i(\boldsymbol{\theta}^{-i})\right)$, where $p_k^i\left(\boldsymbol{\theta}^{-i}\right)$ is the price buyer i must pay (seller i is paid) for the k-th unit of the good.[9] Using the convention $v_0^b = c_0^s = 0$ for all b and s, let the quantities traded by each buyer $b \in \mathcal{N}$ and seller $s \in \mathcal{M}$ at their personalized prices be:

$$q^b(\boldsymbol{p}^b(\boldsymbol{\theta}^{-b}), \boldsymbol{v}^b) = \underset{0 \leqslant q \leqslant k_B}{\arg\max} \sum_{k=0}^{q} \left(v_k^b - p_k^b(\boldsymbol{\theta}^{-b})\right) \quad \text{and}$$

$$q^s(\boldsymbol{p}^s(\boldsymbol{\theta}^{-s}), \boldsymbol{c}^s) = \underset{0 \leqslant q \leqslant k_S}{\arg\max} \sum_{k=0}^{q} \left(p_k^s(\boldsymbol{\theta}^{-s}) - c_k^s\right)$$

Let $q_B(\boldsymbol{\theta}) = \sum_{b \in \mathcal{N}} q^b(\boldsymbol{p}^b(\boldsymbol{\theta}^{-b}), \boldsymbol{v}^b)$ be the total quantity acquired by buyers and $q_S(\boldsymbol{\theta}) = \sum_{s \in \mathcal{M}} q^s(\boldsymbol{p}^s(\boldsymbol{\theta}^{-s}), \boldsymbol{c}^s)$ be the total quantity sold by sellers.

A mechanism is *feasible* if for every $\boldsymbol{\theta}$, $q_B(\boldsymbol{\theta}) = q_S(\boldsymbol{\theta})$.[10]

Given that the outside option has zero value for every agent, a mechanism satisfies *ex post individual rationality* if for all b, $\boldsymbol{\theta} = \left(\boldsymbol{v}^b, \boldsymbol{\theta}^{-b}\right)$ and for all s, $\boldsymbol{\theta} = (\boldsymbol{c}^s, \boldsymbol{\theta}^{-s})$:

$$p_0^b(\boldsymbol{\theta}^{-b}) \leqslant 0\,; \qquad p_0^s(\boldsymbol{\theta}^{-s}) \geqslant 0\,.$$

The profit a mechanism generates at $\boldsymbol{\theta}$ is:

$$\Pi(\boldsymbol{\theta}) = \sum_{b \in \mathcal{N}} \sum_{q^b=0}^{q^b(\boldsymbol{p}^b(\boldsymbol{\theta}^{-b}))} p_{q^b}^b(\boldsymbol{\theta}^{-b}) - \sum_{s \in \mathcal{M}} \sum_{q^s=0}^{q^s(\boldsymbol{p}^s(\boldsymbol{\theta}^{-s}))} p_{q^s}^s(\boldsymbol{\theta}^{-s})\,;$$

a mechanism is *deficit free* if for all $\boldsymbol{\theta}$, $\Pi(\boldsymbol{\theta}) \geqslant 0$.

The performance of any allocation mechanism that targets welfare maximization must be evaluated in term of its efficiency level. In our setting, full ex post efficiency, which implies feasibility, requires that for all possible type profiles the buyers with the highest marginal valuations trade with the sellers with the lowest marginal costs and that the total quantity traded is $q_B(\boldsymbol{\theta}) = q_S(\boldsymbol{\theta}) = q_{CE}(\boldsymbol{\theta})$, where $q_{CE}(\boldsymbol{\theta})$ is a Walrasian (competitive equilibrium) quantity associated with $\boldsymbol{\theta}$:[11]

$$\max\left\{q \in \{0, ..., K\} : v_{(q)} > c_{[q]}\right\} \leqslant q_{CE}\left(\boldsymbol{\theta}\right) \leqslant \max\left\{q \in \{0, ..., K\} : v_{(q)} \geqslant c_{[q]}\right\}.$$

In our setting, dominant strategy incentive compatibility and ex post efficiency are satisfied if and only if the mechanism is a Groves mechanism (e.g., see Holmström, 1979) and ex post

[9] By the taxation and revelation principles (see Rochet, 1985, and Myerson, 1979), any dominant strategy mechanism is strategically equivalent to a "direct" price mechanism that sets an individualized marginal price vector for each agent as a function of the other agents' types and lets each agent decide how many units to trade at the specified prices.

[10] If there was free disposal, we could weaken the feasibility condition to $q_B(\boldsymbol{\theta}) \leqslant q_S(\boldsymbol{\theta})$, but this would not help in any substantial way in the design of our DCA.

[11] Given a vector $\boldsymbol{x}$, we denote by $x_{(i)}$ its i-th highest element and by $x_{[i]}$ its i-th lowest element. Thus, $x_{(q)} = x_{[m+1-q]}$ if the vector contains m elements. We also adopt the notational convention that $v_{(0)} = 1$ and $c_{[0]} = 0$, which implies that $q_{CE}(\boldsymbol{\theta})$ is well defined.

individual rationality and deficit minimization further restrict the mechanism to be a VCG mechanism. The VCG mechanism is not deficit free. Indeed, in Loertscher and Mezzetti (2019), we have shown that in the setting of a market for a homogeneous good the two-sided VCG auction runs a deficit on each trade and the total deficit does not vanish as the number of traders grows large. While it is not possible to construct a mechanism that is ex post efficient and deficit free, efficiency is an important feature of an allocation mechanism. Thus, we require our double clock auction to satisfy two efficiency properties, constrained efficiency and asymptotic efficiency.

A mechanism is *constrained efficient* if, given the total quantity traded $q(\boldsymbol{\theta}) = q_B(\boldsymbol{\theta}) = q_S(\boldsymbol{\theta})$, the trades completed are the most valuable ones – those associated with the $q(\boldsymbol{\theta})$-th highest marginal values and the $q(\boldsymbol{\theta})$-th lowest marginal costs. Constrained efficiency is an appealing property of the price mechanism in competitive and oligopolistic markets.

The total welfare at $\boldsymbol{\theta}$ generated by a mechanism is given by the gains of trade:

$$W(\boldsymbol{\theta}) = \sum_{b \in \mathcal{N}} \sum_{q^b=0}^{q^b(\boldsymbol{p}^b(\boldsymbol{\theta}^{-b}))} v^b_{q^b}(\boldsymbol{p}^b(\boldsymbol{\theta}^{-b})) - \sum_{s \in \mathcal{M}} \sum_{q^s=0}^{q^s(\boldsymbol{p}^s(\boldsymbol{\theta}^{-s}))} c^s_{q^s}(\boldsymbol{p}^s(\boldsymbol{\theta}^{-s})) \,.$$

Let $q^b_{CE}(\boldsymbol{\theta})$ and $q^s_{CE}(\boldsymbol{\theta})$ be the quantity traded by buyer b and seller s in a Walrasian equilibrium. Under a fully efficient allocation, total welfare at $\boldsymbol{\theta}$ is:

$$W_{CE}(\boldsymbol{\theta}) = \sum_{b \in \mathcal{N}} \sum_{q^b=0}^{q^b_{CE}(\boldsymbol{\theta})} v^b_{q^b}(\boldsymbol{\theta}) - \sum_{s \in \mathcal{M}} \sum_{q^s=0}^{q^s_{CE}(\boldsymbol{\theta})} c^s_{q^s}(\boldsymbol{\theta}) \,,$$

Thus, the percentage welfare loss at $\boldsymbol{\theta}$ is $\mathcal{L}(\boldsymbol{\theta}) = 1 - \frac{W(\boldsymbol{\theta})}{W_{CE}(\boldsymbol{\theta})}$. Let $\mathbb{P}_{\phi_*}$ be the probability measure determining the true marginal values and costs (i.e., $\boldsymbol{\theta}$) and $\mathbb{E}_{\phi_*}$ be the expectation operator with respect to $\mathbb{P}_{\phi_*}$.[12] For $\rho > 0$, we say that a mechanism is *asymptotically efficient at rate* $1/n^\rho$ if the expected percentage welfare loss converges to zero at rate $1/n^\rho$ as the size of the market n goes to infinity; that is, if there is a constant $L > 0$ such that for all n: $\mathbb{E}_{\phi_*}[\mathcal{L}(\boldsymbol{\theta})] \leqslant L/n^\rho$. Our double clock auction will be constrained efficient and asymptotically efficient at rate $1/n$.

3 The Dominant Strategy Double Clock Auction

Our DCA uses a price adjustment process inspired by Walras' tâtonnement. Its starting clock prices are $p^B = 0$ for buyers and $p^S = 1$ for sellers. Each buyer starts with a quantity demanded equal to k_B and each seller starts with a quantity supplied equal to k_S. When they are permitted to take an action, buyers and sellers may reduce their quantity demanded or supplied by any

[12]The true probability distribution is not known by the auctioneer or by the traders. There is a set Φ indexing the possible probability measures $\mathbb{P}_\phi$, with $\phi \in \Phi$. See Section 5 for details.

non-negative integer, but they can never demand or supply less than zero. Let $\mathcal{N}_{\mathcal{O}}(p^B)$ be the set of buyers whose quantity demanded is zero when the buyers clock price reaches p^B and let $\mathcal{M}_{\mathcal{O}}(p^S)$ be the set of sellers whose quantity supplied is zero when the sellers clock price reaches p^S. These two sets are the traders who have irrevocably dropped out of the DCA; these traders cannot re-enter and will trade zero units. The only active traders after the clock prices have reached p^B and p^S are the buyers in the set $\mathcal{N}_{\mathcal{A}}(p^B) = \mathcal{N}\backslash\mathcal{N}_{\mathcal{O}}(p^B)$ and the sellers in the set $\mathcal{M}_{\mathcal{A}}(p^S) = \mathcal{M}\backslash\mathcal{M}_{\mathcal{O}}(p^S)$.

An important novelty of our DCA is that, rather than by the "true" (or revealed) excess demand as in Walras' tâtonnement, the tâtonnement process is driven by rounds in which the auctioneer uses some procedure to derive *estimated* excess demand. To derive the main result of this section, Theorem 1, and design an estimation procedure which gives no trader an incentive to misrepresent her true demand and supply, it suffices to assume that when the current clock prices are p^B and p^S, the auctioneer only uses information from traders in the sets $\mathcal{N}_{\mathcal{O}}(p^B)$ and $\mathcal{M}_{\mathcal{O}}(p^S)$. That is, the auctioneer uses the history of demand and supply reductions of all traders that have dropped out of the DCA, but she does not use any information from the traders who are still active.[13] The estimation procedure used to prove the asymptotic efficiency of the DCA will be explained in detail in Section 5.

3.1 Definition

The DCA starts in round 0 of the estimation state.

STEP ES: ESTIMATION STATE.

- Let $t \in \{0, 1, 2, ...\}$ indicate the estimation round. Set $p_0^B = 0$ and $p_0^S = 1$. In each estimation round t the auctioneer uses the history of previous drop-outs to estimate demand and supply at the current prices $p_t^B < p_t^S$:[14]
  - If estimated demand exceeds estimated supply at the current prices, then set a target price $p_t^{TB} \leqslant p_t^S$ such that for $p_t^{TB} < p_t^S$ estimated demand at p_t^{TB} equals estimated supply at p_t^S; go to STEP BC.
  - If estimated supply exceeds estimated demand at the current prices, then set a target price $p_t^{TS} \geqslant p_t^B$ such that for $p_t^{TS} > p_t^B$ estimated supply at p_t^{TS} equals estimated demand at p_t^B; go to STEP SC.

[13] Although our mechanism is different, the idea of using only information from losing bidders is not novel, as it is the basis for price formation in a single-unit English auction and its strategic equivalence with the second-price auction. Brooks (2013) also exploits this idea.

[14] In round 0 the auctioneer may use any arbitrary prior estimate of the demand and supply functions.

- If estimated demand equals estimated supply at the current prices, then set the target price p_t^T at which estimated demand equals estimated supply; go to Step Dc.

Step Bc: Buyers' clock state.

- The buyer price p^B increases continuously starting from p_t^B.
- At any price level p^B, each active buyer $b \in \mathcal{N}_{\mathcal{A}}(p^B)$ decides whether to reduce her demand.
  - If the demand of one of the active buyers becomes zero at price $p^B < p_t^{TB}$ or at price $p^B = p_t^{TB} < p_t^S$, set $p_{t+1}^B = p^B$ and $p_{t+1}^S = p_t^S$, go to Step Es.
  - If at price $p_t^{TB} < p_t^S$ the demand of none of the active buyers has become zero, go to Step Dc.
  - If price p^B reaches $p_t^{TB} = p_t^S$, then set the reserve price $r = p_t^{TB}$, go to Step Ls.

Step Sc: Sellers clock state.

- The seller price p^S decreases continuously starting from p_t^S.
- At any price level p^S, each active seller $s \in \mathcal{M}_{\mathcal{A}}(p^S)$ decides whether to reduce her supply.
  - If the supply of one of the active sellers becomes zero at price $p^S > p_t^{TS}$ or at price $p^S = p_t^{TS} > p_t^B$, set $p_{t+1}^S = p^S$, and $p_{t+1}^B = p_t^B$, go to Step Es.
  - If at price $p_t^{TS} > p_t^B$ the supply of none of the active sellers has become zero, go to Step Dc.
  - If price p^S reaches $p_t^{TS} = p_t^B$, then set the reserve price $r = p_t^{TS}$, go to Step Ls.

Step Dc: Double clock state.

- The buyer price p^B increases continuously starting from p_t^B and the seller price p^S decreases continuously starting from p_t^S; prices change in such a way that equality of estimated demand and supply is maintained and that they would reach the target price p_t^T simultaneously.
- At any price level p^B, each active buyer decides whether to reduce her demand; at any price level p^S, each active seller decides whether to reduce her supply.
  - If the demand of one of the active buyers or the supply of one of the active sellers becomes zero at prices $p^B < p_t^T$ and $p^S > p_t^T$, set $p_{t+1}^B = p^B$ and $p_{t+1}^S = p^S$; go to Step Es.

- If prices p^B and p^S reach p_t^T, then set the reserve price $r = p_t^T$; go to Step Ls.

Step Ls: Long side determination state.

- Let $q^b(r)$ be the quantity demanded by buyer b and $q^s(r)$ the quantity supplied by seller s at the reserve price r. Total quantities demanded and supplied are: $q^B(r) = \sum_{b \in \mathcal{N}} q^b(r)$ and $q^S(r) = \sum_{s \in \mathcal{M}} q^s(r)$. Select $q(r) = \min\{q^B(r), q^S(r)\}$ as the aggregate quantity traded.
  - If $q^B(r) = q^S(r)$, then allocate to each buyer and seller the quantity she demands or supplies; charge buyer and pay seller r for each unit demanded or supplied; end the DCA.
  - If $q^B(r) > q^S(r)$ (i.e., buyers are on the long side of the market), then allocate to each seller the quantity she supplies at r and pay her r for each unit; go to Step Ab.
  - If $q^B(r) < q^S(r)$ (i.e., sellers are on the long side of the market), then allocate to each buyer the quantity she demands at r and charge her r for each unit; go to Step As.

Step Ab: Ausubel buyers auction state.

- The buyer price p^B increases continuously starting from r.
- At any price level p^B, each active buyer $b \in \mathcal{N}_{\mathcal{A}}(p^B)$ decides whether to reduce her demand.
  - When the total quantity demanded at price p^B reaches $q^B(p^B) = q(r)$, allocate units and charge prices to buyers as in an Ausubel auction (i.e., the VCG unit prices bounded by the reserve price); end the DCA.

Step As: Ausubel sellers auction state.

- The seller price p^S decreases continuously starting from r.
- At any price level p^S, each active seller $s \in \mathcal{M}_{\mathcal{A}}(p^S)$ decides whether to reduce her supply.
  - When the total quantity supplied at price p^S is $q^S(p^S) = q(r)$, stop: allocate units and pay prices to sellers as in a reverse Ausubel auction; end the DCA.

Two observations are worth making. First, at all points in the DCA the only information available to the active traders are the state and the current clock prices.[15] Second, in the long side determination state the auctioneer uses information from the active traders to determine the aggregate quantity traded. As we shall prove in Theorem 1, this does not introduce an incentive for traders to misrepresent their true demands and supplies.

3.2 Properties

We say that an agent engages in *sincere bidding* if she expresses her quantity demanded or supplied truthfully. That is, buyer b bids sincerely if for any buyers clock price p^B her demand is q^b such that $v^b_{q^b} \geqslant p^B \geqslant v^b_{q^b+1}$ and seller s bids sincerely if for any sellers clock price p^S her supply q^s is such that $c^s_{q^s} \leqslant p^S \leqslant c^s_{q^s+1}$.

We now show that the DCA is feasible, deficit free, ex post individually rational, constrained efficient and dominant strategy incentive compatible.

Theorem 1. *Sincere bidding by each agent is a dominant strategy equilibrium in the DCA. The DCA is also feasible, deficit free, ex post individually rational and constrained efficient.*

Proof. By construction, the DCA is feasible as the quantity traded is determined by the short side of the market at the reserve price, and it is deficit free since the minimum price paid by buyers (the reserve price r) is equal to the maximum price paid to sellers (also the reserve price r). Ex post individual rationality holds since each trader may guarantee herself the outside option payoff by dropping out of the bidding. Constrained efficiency holds because, under sincere bidding, for any given quantity to be traded q, the trades that are completed are those associated with the q highest marginal values and the q lowest marginal costs.

Because of the symmetry of buyers and sellers, to save space we will just argue that sincere bidding is a dominant strategy for buyers. First observe that if by bidding sincerely buyer b ends up dropping out and not buying any unit, then no alternative strategy could increase her payoff, as it could only make her acquire units at a price above their marginal value.

Second, suppose that by bidding sincerely buyer b acquires at least one unit. This means that she acquires all units having marginal value above the reserve price r. To see that no strategy can increase the buyer's payoff requires the following observations: (i) she cannot affect the reserve price if she adopts a strategy leading to positive trades; (ii) she suffers a payoff reduction if she uses a strategy leading to no trades; (iii) she cannot improve her payoff while remaining on the same side of the market; (iv) if she is on the short side of the market

[15] There are two reasons why no bid information about the other agents is revealed to a trader. First, it makes the bidding environment straightforward; much like in the Walrasian analysis of competitive markets, all information that an agent has is the price she faces. Second, as shown by Theorem 1, it makes sincere bidding by all agents, as defined below, a dominant strategy equilibrium. As in Ausubel (2004), if we allowed either full or aggregate bid information, then sincere bidding would be an ex post perfect equilibrium.

by bidding sincerely, she can only raise the prices she pays if she ends up on the long side by adopting a different strategy; and (v) if she is on the long side of the market by bidding sincerely, she would do at least as well by using the strategy of dropping out on all units at price r in the Ausubel auction, then by reducing demand before the reserve price is determined in order to end up on the short side, but the strategy of dropping out on all units at price r in the Ausubel auction is dominated by sincere bidding. □

4 Discussion and Comparisons

Vickrey (1961) first noted that developing mechanisms for two-sided allocation problems that minimize inefficiencies, do not run a deficit and require no prior information about the true equilibrium price is "extremely difficult". For a bilateral trade setting à la Myerson and Satterthwaite (1983) with the buyer and seller drawing their value and cost independently from distributions with overlapping support, Hagerty and Rogerson (1987) showed that the best the market maker can do subject to dominant strategy incentive compatibility, ex post individual rationality and budget balance is to post an exogenously given price and let the buyer and seller decide if they want to trade at that price. With only one agent on each side of the market, there is simply no way of endogenizing the price at which trade occurs without giving up on dominant strategies (see also Čopič and Ponsati, 2016, and Čopič, 2017).

McAfee (1992) proposed a mechanism that embeds this insight in a setup with multiple *single-unit* traders whose values and costs are elements of the $[0, 1]$-interval. In any round t with the same number of buyers and sellers, the mechanism posts a price p_t like Hagerty and Rogerson in the bilateral trade setting. It then runs a double-clock auction, with the sellers' clock price p^S decreasing from its starting point (which is 1 at the beginning of the mechanism) and the buyers' price p^B increasing (and equal to 0 at the outset). If no agent exits by the time both clocks reach the posted price p_t (i.e., by the time $p^B = p^S = p_t$), then all active agents trade at p_t. The posted price in each round does not depend on the values of the active agents, but it depends on the prices at which the last buyer and seller dropped out; McAfee's double auction thus endogenizes the posted price of Hagerty and Rogerson. More precisely, if at the beginning of round t the clock prices are p^B and p^S, then the posted price is set at $p_t = \frac{p^B+p^S}{2}$.[16]

If the numbers of buyers and sellers are not the same, either at the outset or after a trader drops out, the mechanism runs a single-clock auction; only the clock price on the long side is moved until the number of active agents is the same on both sides of the market. If this happens when the buyer's clock price is lower than the seller's, then the mechanism selects a new posted price in the interval (p^B, p^S) and runs the double clock auction again. If equality

[16] Any choice of a posted price equal to $\alpha_t p^B + (1 - \alpha_t)\, p^S$, with $\alpha_t \in [0, 1]$ would work equally well with regards to the incentive compatibility and individual rationality constraints.

in the number of buyers and sellers is reached when $p^B > p^S$, then the remaining active traders trade at those prices; buyers pay p^B sellers receive p^S.

McAfee's double clock auction endows traders with dominant strategies and it either implements trading of the efficient quantity, which happens if trade occurs at a posted price p_t, or it just excludes the single least efficient trade, which happens if trade occurs at prices $p^B > p^S$. Although McAfee does not refer to *estimation*, the posted price p_t is naturally interpreted as the price at which the estimated demand function and the estimated supply function are equal. For example, take estimated demand and supply to be linear starting from the current number N_t of buyers and sellers and the current clock prices p^B and p^S, with quantity demanded at price $p \geqslant p^B$ being $N_t - \lambda\left(p - p^B\right)$ and quantity supplied at price $p \leqslant p^S$ being $N_t + \lambda\left(p - p^S\right)$ for some $\lambda > 0$; then demand equals supply at $p_t = \frac{p^B+p^S}{2}$.[17]

Similarly, McAfee's mechanism can be viewed as entering an Ausubel auction phase when $p^B = p^S$ and the number of traders on the long side exceeds the number of traders on the short side (with probability one only by one trader): with single-unit traders, the single-clock, Ausubel auction on the long side is simply a clock implementation of the second-price Vickrey auction, determining the trading price at the drop-out price of the first trader that exits, the most competitive losing bid. Our DCA can thus be viewed as an extension of McAfee's (1992) double auction to traders with *multi-unit* demand and supply.[18] It is worth recalling that in standard auction formats multi-unit buyers and seller have an incentive to reduce their demands and supplies so as to manipulate the prices at which they trade (e.g., see Ausubel et al., 2014). Unlike the DCA, many apparently intuitive generalizations of McAfee's double auction, that rely on counting the number of drop-outs or on excluding the least efficient trades, either give traders incentives to misrepresent their marginal values or do not guarantee asymptotic efficiency.

The most prominent alternative to our and McAfee's approach to estimating equilibrium prices and basing allocations and transfers on these estimates is to use *random splitting* mechanisms, where agents are randomly split in submarkets and data from the other markets is used to set the price to post in a given market. In a paper that was developed independently and at the same time as the first version of this paper, Kojima and Yamashita (2017) use this method. Their focus is different from ours, as they also target efficiency, but in a setting with interdependent values, when the type of each trader is single dimensional and a single crossing

[17] Natural extensions would be to allow the slope coefficients to vary across the two sides of the market, which in the notation used in footnote 16 would yield $\alpha_t \neq 1/2$, and to estimate these coefficients based on information from the agents who have dropped out. In turn, this would make α_t a non-degenerate function of the values and costs of the agents who have dropped out. As we discuss in the Conclusions, even in McAfee's mechanism this use of estimation has the drawback of making shill bidding profitable.

[18] McAfee also proposed a simultaneous bid version of his mechanism, which has been extended in the operation research and computer science literature (see Chu, 2009, and Segal-Halev et al., 2017, for recent contributions and references). None of these extensions has considered a setting with multi-dimensional types, or has used a double clock format.

condition holds, so as to escape from the impossibility results that plague ex post implementation (e.g., see Jehiel et al., 2006) when two-stage mechanisms as in Mezzetti (2004) are not allowed.[19]

A consequence of dividing agents randomly into submarkets is that, unlike our mechanism, the random splitting mechanisms are neither constrained efficient nor clock implementable. Yet Kojima and Yamashita have established for their mechanism (like we do for the DCA in the next section) that asymptotic efficiency obtains as the number of traders grow large. Since in general little is known about the performance in the small of asymptotically efficient double auction mechanisms, we will now compare the efficiency properties of McAfee's and the DCA with the random splitting mechanism for two simple examples in which traders have independent private values. Rather than determining the winner of a horse race, our goal is to gain some understanding of the differences in the two approaches.

We start from the case with two buyers and two sellers with unit demand and supply, $N = M = 2$, $k_B = k_S = 1$. In this case the DCA coincides with McAfee's double clock auction. To simplify the analysis, we assume that values and costs are all drawn from the same distribution F. With $N = M = 2$ the random splitting mechanism creates two submarkets, each with one buyer and one seller, has all agents report their types, and uses the reports from one market to post a price in the other market. Because in each market i the price is exogenous, it follows that reporting truthfully is a dominant strategy.

The expected social welfare generated by McAfee's double auction can be divided into three components. The first, denoted by W^{T1} is the welfare created by the most valuable trade (i.e., the trade between the seller with the lowest cost and the buyer with the highest value) when efficiency requires that two trades be completed, with each of the two sellers selling her good and each of the buyers acquire one unit. The second component, denoted by W^{T2}, is the expected payoff from the least efficient trade (i.e., the trade between the seller with the highest cost and the buyer with the lowest value) when it is efficient that two trades be completed. The third welfare component, denoted by W^{E1}, is the expected welfare when it is efficient only to complete one trade and $p_1 = \frac{v_{(2)}+c_{[2]}}{2}$ is the price posted after the highest cost seller drops out at price $c_{[2]}$ and the lowest value seller drops out at $v_{(2)} < c_{[2]}$. The expected welfare in McAfee's double auction with two single-unit traders on each side of the market is thus $W^M = W^{T1} + W^{T2} + W^{E1}$.[20]

The random splitting mechanism separates the two unit-demand buyers and two unit-supply sellers into two markets, denoted by A and B. Letting (v^i, c^i) be the reported types in market $i \in \{A, B\}$, we set the price posted in market j, p^j, with $j \neq i$ equal to the midpoint of the

[19]Random splitting mechanism are related to the random sampling approach frequently used in the computer science literature and first introduced in economics by Baliga and Vohra (2003) and Segal (2003) to study a monopolist seller whose goal is to extract maximum profit from single-unit traders.

[20]The formal definition of these welfare components are given in the proof of Proposition 1 in Appendix A.

Walrasian price gap in market i, that is, $p^j = \frac{v^i + c^i}{2}$.[21]

Denote the expected welfare in this random splitting mechanism by W^{RS}. In the following proposition (proven in Appendix A), we compare welfare in McAfee's mechanism and the random splitting mechanism.

Proposition 1. *Assume $k_B = k_S = 1$ and $N = M = 2$. For any distribution F, we have*

$$W^M > W^{RS} = W^{E1}.$$

In other words, with four, symmetric, single-unit traders, the random splitting mechanism generates the same welfare as total welfare when it is efficient to complete a single trade. Relative to McAfee's mechanism and the DCA, it loses the welfare from the most efficient trade when it is efficient to trade both units, and the welfare from the least efficient trade whenever such trade is completed at the original reserve price $p_0 = \frac{1}{2}$. Intuitively, when efficiency dictates to complete both trades, if the high value buyer is matched with the high cost seller and the low value buyer is matched with the low cost seller, the resulting prices in the random splitting mechanism could be such that no trade is completed, while at least one trade is always completed in the DCA in this case.

Let us now consider the case of multi-unit demand and supply, for which McAfee's double auction is not defined. As explained above, the DCA can be viewed as an extension of McAfee's design to this environment. We do not need to make this assumption for any of our formal results, but one natural way to think about multi-unit traders is that they emerge from the conglomeration of multiple single-unit traders, for example, via mergers, acquisitions, or joint ventures (see e.g., Loertscher and Marx, 2019b). If, as in our example in this section, the single-unit trader setup has buyers and sellers draw their types independently from the same distribution F, then in the multi-unit setup in which a trader has a capacity of $k_B = k_S = K$, each is naturally characterized by K draws from F. Hence, the buyer's highest marginal value is distributed according to F^K and the seller's lowest marginal cost according to $1 - (1 - F)^K$, and so on. We refer to this model as the *order statistics model.*

Using this order statistics model, we next consider what might be the simplest specification with multi-unit traders that permits a comparison between our DCA and random splitting mechanisms. We assume that each trader has a capacity of $k_B = k_S = K = 2$ and, as in the single-unit case, $N = M = 2$. Moreover, we let F to be uniform on $[0, 1]$. Table 1 provides a summary of the performance of various mechanisms in the small for the case when there are either $N = 1$ or $N = 2$ pairs of buyers and sellers present, each agents drawing her type(s) according to the order statistics model from the uniform distributions with a capacity $K \in \{1, 2\}$. Capacities K and the number of pairs present N are common knowledge. Derivations of and details for the numbers in Table 1 are provided in Appendix A.

[21] Note that p^j is a Walrasian price for market i irrespective of whether $v^i \geqslant c^i$ or $v^i < c^i$.

Comparisons				
Mechanisms	SB	DCA	RS	McAfee
$K = 1,\ N = 1$	0.84	0.75	n.a.	0.75
$K = 1,\ N = 2$	0.94	0.82	0.52	0.82
$K = 2,\ N = 1$	n.a.	0.91	n.a.	n.a.
$K = 2,\ N = 2$	n.a.	0.71	0.65	n.a.

Table 1: Table entries are social surplus under a given mechanism divided by social surplus under ex post efficiency for F uniform. *SB* is the second-best mechanism (Myerson and Satterthwaite 1983; Gresik and Satterthwaite, 1989); *DCA* is our *double clock auction*; *RS* is the random splitting mechanism, and *McAfee* is McAfee's (1992) double auction. For $N = 1$ and $K = 1$, McAfee and the DCA reduce to the posted price mechanism of Hagerty and Rogerson (1987) with $p_0 = 1/2$; for $N = 1$ and $K = 2$ the DCA again reduces to a posted price mechanism with $p_0 = 1/2$ (note that because $N = 1$, there is no rationing assumption required).

There are clearly many limits to how much one can read into the computations displayed in Table 1. Nevertheless, for $K = 1$ and $N = 2$, the superior performance of McAfee's relative to the RS mechanism is noteworthy. It can be explained by the fact that by splitting the market into two, the RS reduces—and in the $N = 2$ case, eliminates—the benefits from sorting; that is, it reduces the ability to match high value buyers with low cost sellers.

While we compute welfare under McAfee's mechanism using the initial posted price $p_0 = 1/2$, it is also worth mentioning that if p_0 were set equal to 1 or 0, so that no gains from trade occur at the exogenous price, McAfee's mechanism (or, for that matter, the DCA) still achieves 0.77 of welfare under ex post efficiency for $K = 1$ and $N = 2$. Intuitively, McAfee's mechanism is better able to exploit the increasing returns to scale inherent in market making as larger markets can always replicate what smaller, stand-alone, markets do, and sometimes do better.[22]

The superior sorting properties of the DCA relative to RS are also apparent in the specification with $N = M = K = 2$, where the DCA achieves 0.71 of first-best welfare while the RS mechanism achieves only 0.65. For the DCA we compute expected welfare by taking as a reserve price the price at which the first trader drops out.[23] For the RS mechanism we take the midpoint of the Walrasian price gap in one market to set the price in the other, just as we did in the case with $K = 1$. This price is an unbiased estimate of the Walrasian price in the limit economy when $N \to \infty$. The sorting advantage of the DCA is mitigated by the fact that the reserve price is a biased estimator of the limit Walrasian price, as it is either the lowest cost of

[22] Denoting by $W(N_A + N_B, M_A + M_B)$ the expected welfare under first best with $N_A + N_B$ single-unit buyers and $M_A + M_B$ single-unit sellers, each drawing their types independently from F, we have, for any positive integers N_A, N_B, M_A, M_B, $W(N_A + N_B, M_A + M_B) > W(N_A, M_A) + W(N_B, M_B)$.

[23] Thus, the welfare reported in Table 1 for the DCA excludes any welfare generated when no trader has dropped when both clocks reach the initial posted price $p_0 = 1/2$. This welfare component is however small.

the seller or the highest value of the buyer who is first to drop out. Note however that, since F is uniform, the ex-ante expected price is an unbiased estimator. This hints at the asymptotic efficiency of the DCA, Theorem 2 in the next section. As the number of traders increases any estimation advantage of the RS mechanism disappears.

Intuitively, and as the preceding discussion suggests, the DCA should perform well relative to RS in environments in which sorting (or matching) is important. As an additional example, imagine that there are very many buyers, each demanding a few units, and only a few, different, sellers, each supplying many units. Then the reserve price in the DCA will be determined by the last buyer to drop out and will be a good approximation of the Walrasian price in the aggregate market, while the RS mechanism will set prices that are different from the aggregate Walrasian price as they will reflect the Walrasian prices in different sub-markets with different "supply functions".

To further corroborate the notion that the DCA does well when matching is important, we now extend the setup and provide a brief comparative statics exercise that varies the distributions F and G from which buyers and sellers draw their types. Among other things, as F and G differ in this experiment, this will also demonstrate that the main insight of Proposition 1 extends beyond setups where buyers and sellers draw their types from identical distributions. To focus on the simplest case that permits meaningful comparisons between DCA and RS, we assume $K = 1$ and $N = M = 2$, which, of course, implies that the DCA is equivalent to McAfee's mechanism. The distributions we use for this exercise are

$$F(v) = 1 - (1 - v)^a \quad \text{and} \quad G(c) = c^a, \tag{1}$$

where $a > 0$.[24] The Walrasian price p^W in the limit as $N \to \infty$ is $1/2$, which is easily seen to be true by solving $1 - F(p) = G(p)$ for p. Panel (a) in Figure 1 illustrates the effects of increasing a on F and G. Both F and G become worse distributions, with the buyer having a lower value and the seller having a higher cost with higher probability. Thus, as a increases, a bilateral matching is ever less likely to induce trade. In this sense, the parameter $a > 0$ measures the importance of matching. More formally, let

$$\begin{aligned} w_1(a) &= \int_0^1 \int_0^v (v - c) f(v) g(c) dc dv \quad \text{and} \\ w_\infty(a) &= \int_{1/2}^1 v f(v) dv - \int_0^{1/2} c g(c) dc = \int_0^{1/2} (G(x) + 1 - F(1/2 + x)) dx \end{aligned}$$

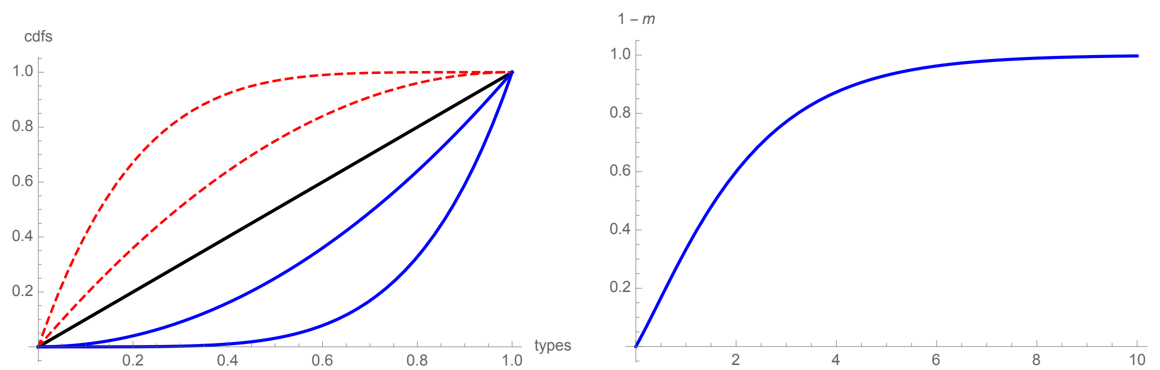
be, respectively, first-best welfare in bilateral trade setting (that is, with $N = 1 = M$) and the welfare per buyer-seller pair in the continuum limit with $N \to \infty$ (that is, $w_\infty(a)$ is social

[24] This family is convenient because it implies linear virtual type functions. Among other things, this permits closed-form expressions for the social surplus under the second-best mechanism for bilateral trade problems, as we show in Appendix A.

surplus per buyer-seller pair in the continuum limit economy). The ratio

$$m(a) = \frac{w_1(a)}{w_\infty(a)}$$

then measures the importance of matching: it gives the percentage of limit welfare that is achieved when there is only one buyer-seller pair present. The larger is this ratio, the less important is matching. For example, for $a = 1$, both F and G are uniform, and we have $w_1(a) = 1/6$ and $w_\infty(a) = 1/4$, so $m(a) = 2/3$. In other words, with $N = 1$ and F and G uniform, we already have 66% of all the per-trader pair welfare of a perfectly thick market (i.e. of $N \to \infty$). This means that the gains from better matching are (very) limited. As Panel (b) in Figure 1 illustrates, $1 - m(a)$ is an increasing function of a that goes to 1 as a increases.[25] Thus, as a increases, the distributions have a longer tail, and the importance of matching increases.[26]



(a) F and G (b) Importance of matching

Figure 1: Panel (a): $F(v)$ (red, dashed) and $G(c)$ (blue, solid) for $a = 2, 5$. The solid black line is the uniform, corresponding to $a = 1$. The farther way from the 45-degree line, the larger is a. Panel (b): $1 - m(a)$ as a function of a.

Table 2 summarizes the results for the family of distributions in (1) for different values of a.[27] As expected, as matching becomes more important (that is, as a increases), the performance of the DCA relative to the RS mechanisms improves further. Interestingly, however, relative to first-best, both mechanisms perform worse as a increases. As we show in Appendix A, where we

[25]In Appendix A, we provide the closed-form solution for $m(a)$ and show that $m(a)$ is decreasing and satisfies $m(0) = 1$ and $\lim_{a\to\infty} m(a) = 0$.

[26]Obviously, this measure neglects incentives compatibility, individual rationality and budget constraints. However, and less obviously, as we show in Appendix A, relative to gains from improved matching due to increases in market size when a is large (and $m(a)$ is small), these constraints are of second-order importance insofar as, for any $a > 0$, the ratio of second-best welfare in the bilateral trade problem, $w_1^{SB}(a)$ to $w_1(a)$ is bounded from below by $2/e \approx 0.73$ whereas the lower bound for $m(a)$ is 0. As the inefficiency of the second-best mechanism decreases in market size (see, e.g., Gresik and Satterthwaite, 1989), this inefficiency is largest in the bilateral trade setting.

[27]These results were obtained numerically using the same formulas as for Table 1.

Performance in relation to importance of matching				
a	W^{RS}/W^*	W^{DCA}/W^*	W^{RS}/W^{DCA}	$m(a)$
1	0.523	0.822	0.633	0.667
2	0.343	0.644	0.533	0.400
5	0.200	0.405	0.501	0.069
10	0.151	0.300	0.500	0.003

Table 2: The performance of the RS mechanism and DCA for $K = 1$ and $N = 2$ for the parameterized family of distributions in (1) for different values of a. (With $K = 1$, DCA is equivalent to McAfee's mechanism.) The first row repeats what we know from Table 1 with the exception of the last entry. The fourth column is obtained by dividing the second by the third. Other than the last column, table entries are social surplus ratios.

study the behaviour of the second-best mechanism for the bilateral trade problem as a function of a, this reflects, in accentuated form, the feature that the incentive problem worsens as a increases in the sense that $w_1^{SB}(a)/w_1(a)$ decreases in a, where $w_1^{SB}(a)$ is the social surplus under the second-best mechanism for $N = 1$.

5 Asymptotic Efficiency of the DCA

To prove the asymptotic efficiency of the DCA in the general model, we now endow the auctioneer with a model of the random process generating traders' valuations, allowing the number of traders to grow large. Thus, as foreshadowed in Section 2, the sets of buyers $\mathcal{N}$ and sellers $\mathcal{M}$ now contain, respectively, nN and nM elements, and we study the limit equilibrium outcome as $n \to \infty$.

Given an integer n, we assume that the marginal values and costs of the agents are drawn from one of the feasible probability measures $\mathbb{P}_\phi^n$. The set of indexes Φ determines the set of feasible measures and the index $\phi \in \Phi$ specifies an element of the set. We assume that Φ is a compact subset of a metric space and that $\mathbb{P}_\phi^n$ is continuous as a function of ϕ (an assumption that is trivially satisfied if Φ is a finite set). For example, consider the *ordinal statistics, conditionally independent model (OS-CI)*, an extension of the model used in the examples of Section 4, in which the marginal values of buyers and sellers correspond to the ordinal statistics of k_B and k_S independent draws from the same atomless distributions with positive densities in $[0, 1]$. However, to model conditional independence, we now also allow for there to be a set of distributions F_{ϕ^B} for buyers and G_{ϕ^S} for sellers (e.g., $F_{\phi^B}(v) = v^{\phi^B}$ and $G_{\phi^S}(v) = v^{\phi^S}$ with $0 < \underline{\phi} \leqslant \phi^B, \phi^S \leqslant \overline{\phi}$), in which case the probability measure $\mathbb{P}_\phi^n$, with $\phi = (\phi^B, \phi^S)$, is simply the product measure of the $nN \times k_B$ product measure of F_{ϕ^B} and the $nM \times k_S$ product measure of G_{ϕ^S}.

For the general model, we will make three assumptions about the probability measures

$\mathbb{P}^n_\phi$. Two of them are automatically satisfied in the OS-CI model; a sufficient condition for the third condition—called *identifiability*—to be satisfied is that: (*i*) $\phi_1^B \neq \phi_2^B$ implies that for all $p \in (0,1]$ it is $F_{\phi_1^B}(v) \neq F_{\phi_2^B}(v)$ for a positive Lebesgue measure set of values $v \in (0,p]$; and (*ii*) $\phi_1^S \neq \phi_2^S$ implies that for all $p \in [0,1)$ it is $G_{\phi_1^S}(c) \neq G_{\phi_2^S}(c)$ for a positive Lebesgue measure set of values $c \in [p,1)$.

Let $\mathbf{1}(\cdot)$ be the indicator function and define the true demand and supply for the k-th unit by buyer b and seller s at price p as: $D_k^b(p) = \mathbf{1}\left(v_k^b \geqslant p\right)$ and $S_k^s(p) = \mathbf{1}\left(c_k^s \leqslant p\right)$. We denote the demand at price p for the k-th unit of the buyers who are still active at price p^B by $D_k^{\mathcal{N}_{\mathcal{A}}(p^B)}(p)$, and of those who have dropped out by $D_k^{\mathcal{N}_{\mathcal{O}}(p^B)}(p)$. Adding the two we obtain the aggregate demand for the k-th unit $D_k^{\mathcal{N}}(p)$, which allows us to define aggregate demand at price p as $D^{\mathcal{N}}(p) = \sum_{k=1}^{k_B} D_k^{\mathcal{N}}(p)$. Similarly, we denote the supply at price p for the k-th unit of the active sellers at price p^S by $S_k^{\mathcal{M}_{\mathcal{A}}(p^S)}(p)$ and of those who have dropped out by $S_k^{\mathcal{M}_{\mathcal{O}}(p^S)}(p)$; aggregate supply for the k-th unit is denoted by $S_k^{\mathcal{M}}(p)$ and aggregate supply at price p is $S^{\mathcal{M}}(p) = \sum_{k=1}^{k_S} S_k^{\mathcal{M}}(p)$.

Given any probability measure $\mathbb{P}^n_\phi$, any possible event $\mathcal{Z}$ describing the information obtained from buyers and sellers that have dropped out when prices p^B, p^S are reached, and any random variable X which is measurable with respect to such drop-outs information, let $\mathbb{E}^n_\phi\left[X \mid \mathcal{Z}\right]$ be the conditional expectation of X and $\mathbb{E}^n_\phi\left[\mathbb{E}^n_\phi\left[X \mid \mathcal{Z}\right]\right] = \mathbb{E}^n_\phi\left[X\right]$ be the unconditional expectation. Thus, for example, $\mathbb{E}^n_\phi\left[D^{\mathcal{N}}(p) \mid \mathcal{Z}\right]$ is expected aggregate demand at price p conditional on $\mathcal{Z}$ and $\mathbb{E}^n_\phi\left[D^{\mathcal{N}}(p)\right]$ is unconditional expected aggregate demand at price p.

The first assumption we make guarantees that with a large number of traders the per capita demand and supply functions are strictly monotone.

Assumption 1. (*Monotonicity of Demand and Supply*) *There exist w and W with $0 < w < W$ such that:*

(i) *For all $p \in [0,1]$, all $\epsilon \in [0, 1-p]$, all n, and all $\phi \in \Phi$, we have:*

$$wn\epsilon \leqslant \mathbb{E}^n_\phi\left[D^{\mathcal{N}}(p)\right] - \mathbb{E}^n_\phi\left[D^{\mathcal{N}}(p+\epsilon)\right] \leqslant Wn\epsilon\,. \tag{2}$$

(ii) *For all $p \in [0,1]$, all $\epsilon \in [0, p]$, all n, and all $\phi \in \Phi$, we have:*

$$wn\epsilon \leqslant \mathbb{E}^n_\phi\left[S^{\mathcal{M}}(p)\right] - \mathbb{E}^n_\phi\left[S^{\mathcal{M}}(p-\epsilon)\right] \leqslant Wn\epsilon\,. \tag{3}$$

Assumption 1 holds in the OS-CI model since there $\mathbb{E}^n_{\phi^B}[D^{\mathcal{N}}(p)] = nNk_B[1 - F_{\phi^B}(p)]$ and $\mathbb{E}^n_{\phi^S}[S^{\mathcal{M}}(p)] = nMk_S G_{\phi^S}(p)$. More generally, a sufficient condition for Assumption 1 to hold is that the probability measures $\mathbb{P}^n_\phi$ are absolutely continuous with respect to Lebesgue measure and their Radon-Nikodym derivatives (densities) are bounded away from zero and finite.[28]

[28] The requirement that $wn\epsilon \leqslant \mathbb{E}^n_\phi\left[D^{\mathcal{N}}(p)\right] - \mathbb{E}_\phi\left[D^{\mathcal{N}}(p+\epsilon)\right]$ and $wn\epsilon \leqslant \mathbb{E}^n_\phi\left[S^{\mathcal{M}}(p)\right] - \mathbb{E}_\phi\left[S^{\mathcal{M}}(p-\epsilon)\right]$ is essentially the same as the assumption of No Asymptotic Gaps in Cripps and Swinkels (2006), while the requirement that $\mathbb{E}^n_\phi\left[D^{\mathcal{N}}(p)\right] - \mathbb{E}_\phi\left[D^{\mathcal{N}}(p+\epsilon)\right] \leqslant Wn\epsilon$ and $\mathbb{E}^n_\phi\left[S^{\mathcal{M}}(p)\right] - \mathbb{E}_\phi\left[S^{\mathcal{M}}(p-\epsilon)\right] \leqslant Wn\epsilon$ is the counterpart of their No Asymptotic Atoms assumption.

As we want to allow for traders' values to be correlated to same extent, even after conditioning on the true index ϕ_*, the second assumption we make is needed to guarantee that the law of large numbers holds for demands and supplies. We borrow the concept of weak independence from the statistical literature (e.g., see Bradley, 2005, and Dedecker et al., 2007); it requires that, for any given index $\phi \in \Phi$, the covariances among the marginal values of two traders vanish as the distance between them, as measured by their position in an ordered list, grows large.[29] Given a probability measure $\mathbb{P}^n_\phi$, consider the following covariances:

$$\alpha^{ij}_k(p;\phi) = \mathbb{P}^n_\phi\left(D^i_k(p) = D^j_k(p) = 1\right) - \mathbb{P}^n_\phi\left(D^i_k(p) = 1\right)\mathbb{P}_\phi\left(D^j_k(p) = 1\right) ;$$
$$\beta^{ij}_k(p;\phi) = \mathbb{P}^n_\phi\left(S^i_k(p) = S^j_k(p) = 1\right) - \mathbb{P}^n_\phi\left(S^i_k(p) = 1\right)\mathbb{P}_\phi\left(S^j_k(p) = 1\right) .$$

Note that $\alpha^{ij}_k(p;\phi)$ and $\beta^{ij}_k(p;\phi)$ are bounded above by $1/4$ and below by $-1/4$. If the individual demands at p of buyers i and j are independent conditional on ϕ as in the OS-CI model, or if individual demands are deterministic, then $\alpha^{ij}_k(p;\phi) = 0$; similarly, if the individual supplies of sellers i and j at p are independent conditional on ϕ, or if individual supplies are deterministic, then $\beta^{ij}_k(p;\phi) = 0$. In both cases Assumption 2 holds.

Assumption 2. (*Weak Dependence of Individual Demands and Supplies*)

(i) *There exists $\Delta_B < \infty$ and a permutation $b \to i$ of the buyers' names such that, for all $p \in (0,1)$, all $k \in \{1,...,k_B\}$, all n, all $i \in \mathcal{N}$ and all $\phi \in \Phi$:*

$$\sum_{j\in\mathcal{N},\ j>i} \alpha^{ij}_k(p;\phi) \leqslant \Delta_B \,. \tag{4}$$

(ii) *There exists $\Delta_S < \infty$ and a permutation $s \to i$ of the sellers' names such that, for all $p \in (0,1)$, all $k \in \{1,...,k_S\}$, all n, all $i \in \mathcal{M}$ and all $\phi \in \Phi$:*

$$\sum_{j\in\mathcal{M},\ j>i} \beta^{ij}_k(p;\phi) \leqslant \Delta_S \,. \tag{5}$$

The bite of Assumption 2 comes as the number of buyers and sellers grows large; it requires that there is a listing of buyers, and one of sellers, under which the covariance between the demands of any buyer b and buyer $b+\tau$, and seller s and $s+\tau$, vanishes as the distance τ between the position in the list of the two buyers, and the two sellers, grows large. Assumption 2 holds more generally than in the OS-CI model. For example suppose that, conditional on $\phi \in [a,b]$, the marginal values of buyer 1 are independently drawn from the probability distribution $F_\phi(v)$, while the marginal values of traders $i > 1$ are independently drawn from $F_\phi(v)$ with probability $0 < \lambda < 1$ and with the remaining probability they are either: (A) identical to the marginal

[29] Cripps and Swinkels (2006) and Peters and Severinov (2006) use different assumptions that are closely related to the related statistical literature on mixing conditions.

values drawn by trader $i-1$, or (B) identical to the marginal values drawn by trader 1. In case (A) Assumption 2 holds for buyers, while in case (B) it fails. More generally, one may think of buyers as linked in a network with values of connected types being correlated. Case (A) corresponds to a line network, case (B) to a star. Extending this example, if for all n the network is fully connected, then Assumption 2 also fails.

Before stating the third assumption, recall that when the buyers clock price in the DCA is p^B and the sellers clock price is p^S, to estimate the parameter ϕ the data available to the auctioneer are the true demands and supplies of the traders that have dropped out of the DCA, that is, of the buyers and sellers in the sets $\mathcal{N}_{\mathcal{O}}(p^B)$ and $\mathcal{M}_{\mathcal{O}}(p^S)$. We assume that the auctioneer computes the parameter ϕ that minimizes the integrated square distance between true and expected per capita demand and supply of the traders that have dropped out; that is, she solves the minimum distance problem:[30]

$$\min_{\phi\in\Phi}\left(\int_0^{p^B}\left(\frac{D^{\mathcal{N}_{\mathcal{O}}(p^B)}(p)-\mathbb{E}_\phi\big[D^{\mathcal{N}_{\mathcal{O}}(p^B)}(p)\big]}{n}\right)^2 dp+\int_{p^S}^1\left(\frac{S^{\mathcal{M}_{\mathcal{O}}(p^S)}(p)-\mathbb{E}_\phi\big[S^{\mathcal{M}_{\mathcal{O}}(p^S)}(p)\big]}{n}\right)^2 dp\right). \tag{6}$$

For any given event $\mathcal{Z}$ describing the information obtained from the traders that have dropped out when the DCA has reached prices p^B and p^S, let $\phi(\mathcal{Z})$ be the solution of the minimum distance problem.[31] Estimated demand and supply then are $\mathbb{E}^n_{\phi(\mathcal{Z})}\big[D^{\mathcal{N}}(p) \mid \mathcal{Z}\big]$ and $\mathbb{E}^n_{\phi(\mathcal{Z})}\big[S^{\mathcal{M}}(p) \mid \mathcal{Z}\big]$.

Convergence to efficiency requires that the estimation procedure be informative about the true stochastic process generating the data (i.e., marginal values and costs). Thus, like in any statistical or econometric model, we need an identifiability assumption on the admissible probability measures, Assumption 3 below, which guarantees, loosely speaking, that the data available are sufficient to determine the true value of ϕ. Let $\mathbb{P}^n_{\phi_*}$ be the true probability measure from which values and costs are drawn. In combination with the operation of the DCA, $\mathbb{P}^n_{\phi_*}$ determines the distribution of the reserve price. Indeed, the reserve price only depends on the event $\mathcal{Z}$ describing the information obtained from traders that have dropped out of the DCA; to emphasize this dependency and the fact that the reserve price is a random variable, we will now denote by $R_{\mathcal{Z}}$ the reserve price when the event is $\mathcal{Z}$.

Assumption 3. (*Identifiability*) *Suppose the vectors of valuations of the nN buyers and nM sellers are drawn according to the probability measure $\mathbb{P}^n_{\phi_*}$. Let $\mathcal{Z}$ be the event describing the*

[30] The mean square distance is the "right" distance because to prove Theorem 2 we will use the convergence in mean square to their expectations of aggregate demand and supply at the reserve price.

[31] Since Φ is a compact subset of a metric space and $\mathbb{P}^n_\phi$ is a continuous function of ϕ, the minimizer $\phi(\mathcal{Z})$ of (6) exists. The size of the set Φ does not matter for our results, but it would affect the computability of the estimator $\phi(\mathcal{Z})$. As long as the probability measures are well behaved functions of ϕ, for the purpose of computation Φ could be approximated by a finite grid.

information obtained from the traders that have dropped out when the DCA has reached the reserve price $R_{\mathcal{Z}}$. For $\phi \in \Phi$ and $\phi \neq \phi_*$, let $\mathbb{P}^n_\phi$ be any other feasible probability measure. There exists $\zeta > 0$ such that:

$$\left(\frac{\mathbb{E}^n_{\phi_*}\big[D^{\mathcal{N}}(R_{\mathcal{Z}}) \mid \mathcal{Z}\big] - \mathbb{E}^n_{\phi}\big[D^{\mathcal{N}}(R_{\mathcal{Z}}) \mid \mathcal{Z}\big]}{n}\right)^2 + \left(\frac{\mathbb{E}^n_{\phi_*}\big[S^{\mathcal{M}}(R_{\mathcal{Z}}) \mid \mathcal{Z}\big] - \mathbb{E}^n_{\phi}\big[S^{\mathcal{M}}(R_{\mathcal{Z}}) \mid \mathcal{Z}\big]}{n}\right)^2$$

$$\leqslant \zeta \cdot \mathbb{E}^n_{\phi_*}\Bigg[\int_0^{R_{\mathcal{Z}}} \left(\frac{\mathbb{E}^n_{\phi_*}\big[D^{\mathcal{N}_{\mathcal{O}}(R_{\mathcal{Z}})}(p)\big] - \mathbb{E}^n_{\phi}\big[D^{\mathcal{N}_{\mathcal{O}}(R_{\mathcal{Z}})}(p)\big]}{n}\right)^2 dp$$
$$+ \int_{R_{\mathcal{Z}}}^1 \left(\frac{\mathbb{E}^n_{\phi_*}\big[S^{\mathcal{M}_{\mathcal{O}}(R_{\mathcal{Z}})}(p)\big] - \mathbb{E}^n_{\phi}\big[S^{\mathcal{M}_{\mathcal{O}}(R_{\mathcal{Z}})}(p)\big]}{n}\right)^2 dp\Bigg] \tag{7}$$

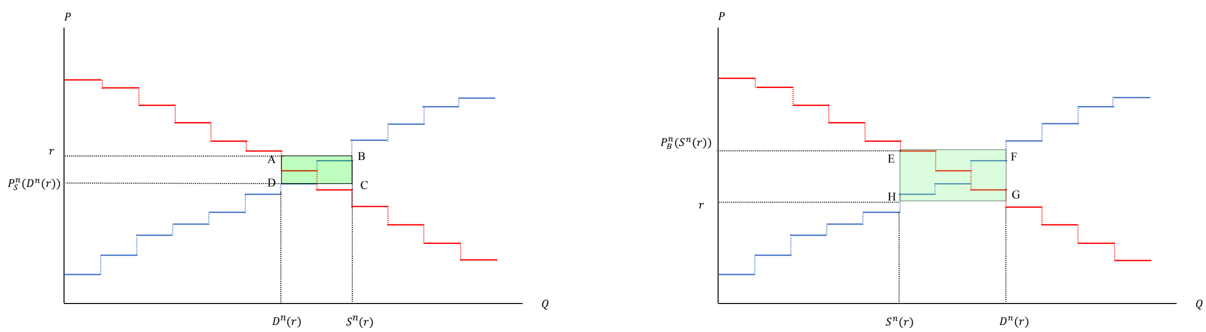
The identifiability condition requires that the difference between true expected demand and supply and expected demand and supply according to a different probability measure at the reserve price $R_{\mathcal{Z}}$, conditional on the event $\mathcal{Z}$, is bounded by some multiple of the expected demand and supply distance of the buyers and sellers that have dropped out at $R_{\mathcal{Z}}$.

To understand Assumption 3, consider the OS-CI model and, to simplify the exposition, assume there is a unique, known, distribution from which the sellers' costs are drawn, so that the second terms on both sides of (7) vanish. Let $n_{\mathcal{A};\mathcal{Z}}N$ be the number of buyers still active at the reserve price $R_{\mathcal{Z}}$. Then, conditional on the event $\mathcal{Z}$, all active buyers demand at least one unit plus an additional number of units equal to the number of the other $k_B - 1$ independent draws that are above $R_{\mathcal{Z}}$. In other words, expected demand conditional on $\mathcal{Z}$ by an active buyer when the index is ϕ_* is: $1 + (k_B - 1)[1 - F_{\phi^*}(R_{\mathcal{Z}})]$. It then follows that the left hand side of (7) is: $\left(\frac{n_{\mathcal{A};\mathcal{Z}}N}{n}(k_B - 1)[F_\phi(R_{\mathcal{Z}}) - F_{\phi_*}(R_{\mathcal{Z}})]\right)^2$, which is less than $N^2(k_B-1)^2[F_\phi(R_{\mathcal{Z}})-F_{\phi_*}(R_{\mathcal{Z}})]^2$. We may follow the same approach to compute the integrand on the right hand side of (7), after first noting that there is no conditioning on the event $\mathcal{Z}$ apart from the number $n_{\mathcal{O};\mathcal{Z}}N = (n - n_{\mathcal{A};\mathcal{Z}})N$ of buyers who have dropped out; note that the expected number of buyers that drop out by the time any price r is reached is $nNF_{\phi_*}(r)^{k_B}$. Thus, the integrand on the right hand side of (7) is: $\left(\frac{n_{\mathcal{O};\mathcal{Z}}N}{n}k_B[F_\phi(p) - F_{\phi_*}(p)]\right)^2$. Thus, if (i) $\phi \neq \phi_*$ implies that for all $p \in (0,1]$ it is $F_\phi(v) \neq F_{\phi_*}(v)$ for a positive Lebesgue measure set of values $v \in (0,p]$, then Assumption 3 holds in the OS-CI model.

The first, trivial, way in which Assumption 3 would fail is if for all feasible probability measures, all buyers had the highest possible value for the first unit, $v^b_1 = 1$ for all b, and all sellers had the lowest possible cost for the first unit $c^s_1 = 0$ for all s. In such a case there would be no drop-outs at any interior reserve price and the right hand side of (7) would always equal zero.[32] More generally, for Assumption 3 to fail the active traders at the reserve price must be unpredictably different from the inactive traders. Thus, in a similar vein to the example just

[32]Note however that if the auctioneer knows this information, she could allocate the first unit from sellers to buyers at an arbitrary price and then run the DCA.

discussed, suppose for simplicity that all sellers' values are independently drawn from the same distribution G while buyers are first drawn to be *weak* or *strong*; weak buyers draw all their values independently from the same distribution F, while strong buyers value the first unit at 1 and the marginal values for all other units are independently drawn from a distribution F_ϕ, with $\phi \in \Phi$. Now there will be traders that drop out (both sellers and weak buyers), but their values provide no information about the values of the strong buyers.



(a) Buyers on short side

(b) Sellers on short side

Figure 2: Illustration of bounds on welfare losses. Panel (a): Buyers are on short side at r. Panel (b): Sellers are on short side at r.

We are now ready to prove the asymptotic efficiency of the DCA. Denote by $P_B^n(q) = \{\min p : D^{\mathcal{N}}(p') \leqslant q \leqslant D^{\mathcal{N}}(p) \text{ for all } p' > p\}$ the inverse realized market demand and by $P_S^n(q) = \{\max p : S^{\mathcal{M}}(p') \leqslant q \leqslant S^{\mathcal{M}}(p) \text{ for all } p' < p\}$ the inverse realized market supply. Consider the demand and supply diagram in Figure 2, with r being the realized reserve price. When buyers are on the short side of the market – i.e., when $D^{\mathcal{N}}(r) < S^{\mathcal{M}}(r')$ for some $r' < r$, as in Panel (a) – the quantity traded in the DCA is $q(r) = D^{\mathcal{N}}(r)$; let $P_S^n\left(D^{\mathcal{N}}(r)\right) < r$ be the price at which supply is equal to $D^{\mathcal{N}}(r)$. The difference between efficient and realized welfare, $W_{CE}(\boldsymbol{\theta}) - W(\boldsymbol{\theta})$, is bounded above by the area of the shaded rectangle ABCD. Thus, the welfare difference is at most the area of this rectangle; that is, $[r - P_S^n\left(D^{\mathcal{N}}(r)\right)] \cdot [S^{\mathcal{M}}(r) - D^{\mathcal{N}}(r)]$. Similarly, when sellers are on the short side of the market – i.e., when $S^{\mathcal{M}}(r) < D^{\mathcal{N}}(r')$ for some $r' > r$ as in Panel (b) of Figure 2 – the quantity traded is $q(r) = S^{\mathcal{M}}(r)$; let $P_B^n\left((S^{\mathcal{M}}(r)\right)$ be the price at which demand would be equal to $S^{\mathcal{M}}(r)$. The welfare difference is now bounded above by $[P_B^n\left(S^{\mathcal{M}}(r)\right) - r] \cdot [D^{\mathcal{N}}(r) - S^{\mathcal{M}}(r)]$, the area of the rectangle EFGH.

In the proof of Theorem 2 we show that the ratio of the area of the rectangle ABCD (or EFGH) to total welfare, and hence the expected percentage welfare loss, converges to zero at rate $1/n$.

Theorem 2. *Under Assumptions 1, 2 and 3, the expected percentage welfare loss in the DCA converges to zero at rate* $1/n$ *as* $n \to \infty$.

To prove Theorem 2 we need to establish that the expected distance between demand and supply at the reserve price r reached by the DCA is "small". The proof strategy is to observe that an upper bound on the expected distance between demand and supply is given by a multiple of the highest of three expected distances, all of which are small. The first is the expected distance between demand and expected demand at r (given the true index ϕ_*). The second is the expected distance between supply and expected supply at r (again, given the true index ϕ_*). The proof that these two expected distances are small appeals to the law of large numbers, Corollary 1 in Appendix B, and only requires monotonicity and weak dependence of demand and supply, that is, Assumptions 1 and 2. The third expected distance is the expected distance between estimated demand and estimated supply; that is, the expected magnitude of estimated excess demand. The claim that this expected distance is small is in Lemma 2 in Appendix B. This is the only part of the proof of Theorem 2 that requires our identifiability condition, Assumption 3.

The rate of convergence to efficiency in Theorem 2 is $1/n$ because the auctioneer uses the empirical distribution of values and costs of the traders that have dropped out to estimate demand and supply, and the empirical distribution converges to the true distribution at rate $1/\sqrt{n}$. In McAfee (1992) the rate of convergence is $1/n^2$ as the gap between demand and supply is never more than one unit and there is no need to use the empirical distribution to estimate demand and supply with single-unit traders. Thus, in McAfee's mechanism not only the percentage welfare loss, but also the total welfare loss goes to zero as the number of traders increases. The literature on the k-double auction (see Rustichini, Satterthwaite, and Williams, 1994, and Cripps and Swinkels, 2006) has also obtained convergence to efficiency at rate $1/n^2$. In that literature, no estimation procedure is needed as the auctioneer is passive and the *traders know the true distribution* of values and costs when computing their equilibrium strategies. The k-double auction literature puts the burden of aggregating information on the traders' knowledge of the true distribution and their ability to compute and coordinate on an equilibrium.[33] In contrast, our DCA puts the burden of aggregating information on the auctioneer, making the traders' strategy straightforward. Finally, we should mention that the splitting mechanism in Kojima and Yamashita (2017) converges to efficiency at rate $1/n^{1/6}$. Their and our rate of convergence, however, are not not easily comparable, as the models are different; they assume interdependent values and single dimensional types determining the shape of a trader's valuation.

[33] In the case of unit demand and supply it is well known that there are a continuum of equilibria. In the case of multi-unit demands and supplies it is only known that a mixed strategy equilibrium exists, but no such equilibrium has yet been found.

6 Conclusions

Progress in research is made one step at a time; in this paper, we have proposed an estimation-based market design for a homogeneous good market which targets efficiency. In contrast to Walrasian tâtonnement, from which it draws inspiration, it maintains dominant strategy incentive compatibility throughout by making all agents price-takers at all times.

Importantly, our DCA achieves this while accommodating multi-unit traders, which is of relevance in practice. Of course, it can be criticized on the ground that it does not perform well (e.g., in the large) in environments different from those we have studied. This is however true of any mechanism; for example, McAfee's mechanism has nice incentive and asymptotic properties with single-unit traders, but these properties are not robust to the introduction of multi-unit traders with multi-dimensional types. Similarly, our DCA may be vulnerable to shill bidding if the designer cannot prevent agents from registering under multiple identities, because it estimates target prices and eventually the reserve price based on the values and costs of the traders that have dropped out. To see this simply, reconsider the variant of McAfee's mechanism with single-unit traders mentioned in footnote 17, in which the target price is a non-degenerate function of the exit prices of all agents who have become inactive. By registering multiple times and dropping out early on the shills an agent may be able to affect the price at which she trades in her favor. Thus, robustness to shill bidding is not a problem specific to our mechanism but applies to the entire literature on mechanism design with estimation.

Our DCA design is quite flexible, and can be modified in several ways, depending on the goals and constraints facing the designer, while preserving the property that sincere bidding is a dominant strategy equilibrium.

First, the DCA generates a budget surplus, because it runs an Ausubel auction on the long side of the market. While in many practical applications (e.g., double auctions run by governments or public agencies) running a surplus is acceptable, or even desirable, a budget surplus could be avoided by using a rationing procedure instead of an Ausubel auction. Suppose that after selecting the reserve price the auctioneer randomly selects a priority order of the traders on the long side of the market and fulfills their demands or supplies according to the drawn priority, up to the quantity determined on the short side of the market. All traders are charged or paid the reserve price for each unit they receive or provide. This modification does not change the incentive properties of the DCA, as no trader can affect the reserve price unless they drop out. They also cannot profitably affect the quantity traded. Thus, Theorem 1 continues to hold and the modified DCA balances the budget. However, this comes at the cost of giving up constrained efficiency and slowing the convergence to efficiency as the number of traders grows. Consider the case when buyers are on the long side of the market. The number of efficient trades that are not completed is still given by the difference between demand and supply at the reserve price $R_{\mathcal{Z}}$, $D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}})$. However, as the non-completed trades are randomly selected among buyers with marginal values above r, the upper bound on the welfare loss is now (some multiple of) $D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}})$. Thus, an upper bound on the expected percentage efficiency loss is $\mathbb{E}^n_{\phi_*}\left[\frac{\sqrt{(D^{\mathcal{N}}(R_{\mathcal{Z}})-S^{\mathcal{M}}(R_{\mathcal{Z}}))^2}}{n}\right]$. Since Lemma 2 in Appendix B proves that $\mathbb{E}^n_{\phi_*}\left[\mathbb{E}^n_{\phi_*}\left[\frac{D^{\mathcal{N}}(R_{\mathcal{Z}})-S^{\mathcal{M}}(R_{\mathcal{Z}})}{n} \mid \mathcal{Z}\right]^2\right]$ converges to zero at rate $1/n$, it follows from Jensen's inequality that the expected percentage efficiency loss of the DCA with rationing converges to zero at rate $1/\sqrt{n}$.

Second, our DCA can be modified to allow for the incorporation of constraints on the aggregate quantities subsets of bidders may be allocated or may procure, such as a cap on the number of units a subgroup of buyers may acquire in total. Quantity constraints like these may arise for a number of reasons, such as antitrust concerns or technological constraints.

Third, in the DCA the auctioneer selects target prices in each estimation round and the clock state is determined so as to achieve equality of estimated demand and supply; the goal is to reach a unique reserve price at which estimated excess demand is zero. A profit maximizing intermediary could instead set target prices and clock states so as to target equality of estimated marginal cost and marginal benefit (derived from estimated demand and supply), with the goal of reaching two different reserve prices, one for buyers and one for sellers, at which estimated marginal revenue equals estimated marginal cost and estimated demand equals estimated supply. Call *maximum profit* the profit that would be generated if the profit maximizing intermediary knew demand and supply, but was constrained to select two prices, a uniform price for buyers and a uniform price for sellers.[34] With an additional monotonicity assumption on marginal cost and benefit, we conjecture that such a modified DCA would be asymptotically profit maximizing; that is, the percentage profit loss relative to maximum profit would convergences to zero as the number of traders grows. We leave a proper investigation to future research.

In future research, it would also be important to expand the setup to allow for heterogenous commodities or incorporate versions of the assignment model.[35] The latter is simpler than what we have studied here, insofar as agents trade at most one unit, but the challenges arise because there is no natural ordering of agents according to their types. One could also depart from the two-sided setup we considered here by studying an asset market model in which every agent is endowed with some units while having demand for more units. This setup takes away the market maker's ability to separate traders a priori into buyers and sellers.

[34] Extending Myerson (1981) optimal single-side auction in a Bayesian setting to the case of buyers with multi-unit demand is still an open problem; a fortiori, we do not know the Bayesian mechanism (or, for that matter, the dominant strategy mechanism) that maximizes the intermediary profit in a setting with multi-unit demands and supplies and no restrictions on the prices the intermediary may charge.

[35] See Ausubel (2006) and Shapley and Shubik (1972), respectively.

Appendix A

Proof of Proposition 1. We denote by $v_{(i)}$ the i-th highest value and by $c_{[i]}$ the i-th lowest cost. We let: $f_{(2:2)}\left(v_{(2)}\right)$ be the unconditional density of $v_{(2)}$; $f_{(1:2)}\left(v_{(1)} \,|\, v_{(2)}\right)$ be the density of $v_{(1)}$ conditional on $v_{(2)}$; $f_{[2:2]}(c_{[2]})$ be the unconditional density of $c_{[2]}$ and $f_{[1:2]}\left(c_{[1]} \,|\, c_{[2]}\right)$ be the density of $c_{[1]}$ conditional on $c_{[2]}$.

We are now ready to define the three welfare components in McAfee's mechanism. Integrating for $v_{(2)} \geqslant c_{[2]}$, first we have

$$W^{T1} = \int_0^1 \int_{c_{[2]}}^1 \int_0^{c_{[2]}} \int_{v_{(2)}}^1 (v_{(1)} - c_{[1]}) f_{(1:2)}(v_{(1)}|v_{(2)}) f_{[1:2]}(c_{[1]}|c_{[2]}) f_{(2:2)}(v_{(2)}) f_{[2:2]}(c_{[2]}) dv_{(1)} dc_{[1]} dv_{(2)} dc_{[2]} ,$$

and second we have

$$W^{T2} = \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 (v_{(2)} - c_{[2]}) f_{(2:2)}(v_{(2)}) f_{[2:2]}(c_{[2]}) dv_{(2)} dc_{[2]}.$$

Third, integrating over $v_{(2)} < c_{[2]}$ with $p_1 = \frac{v_{(2)} + c_{[2]}}{2}$ we have

$$W^{E1} = \int_0^1 \int_0^{c_{[2]}} \int_0^{p_1} \int_{p_1}^1 (v_{(1)} - c_{[1]}) f_{(1:2)}(v_{(1)}|v_{(2)}) f_{[1:2]}(c_{[1]}|c_{[2]}) f_{(2:2)}(v_{(2)}) f_{[2:2]}(c_{[2]}) dv_{(1)} dc_{[1]} dv_{(2)} dc_{[2]}.$$

Letting f be the density of F, recall that the densities and conditional densities of the order statistics out of a population $N = M = 2$ are $f_{[2:2]}(c) = 2f(c)F(c)$, $f_{(2:2)}(v) = 2f(v)[1 - F(v)]$, $f_{[1:2]}(x \,|\, c) = \frac{f(x)}{F(c)}$, $f_{(1:2)}(y \,|\, v) = \frac{f(y)}{1-F(v)}$. Making use of these definitions yields

$$W^{E1} = 4 \int_0^1 \int_0^c \int_0^{\frac{v+c}{2}} \int_{\frac{v+c}{2}}^1 (y - x) f(y) f(x) f(v) f(c) dy dx dv dc.$$

We now prove that $W^{E1} = W^{RS}$. The proposition then follows from the fact that $W^{T1} > 0$, $W^{T2} > 0$.

Observe that in the random sampling mechanism in each market i welfare w_i^{RS} is

$$w_i^{RS} = \int_0^1 \int_0^1 \int_0^{\frac{v^j + c^j}{2}} \int_{\frac{v^j + c^j}{2}}^1 (v^i - c^i) f(v^i) f(c^i) f(v^j) f(c^j) dv^i dc^i dv^j dc^j,$$

where v^j, c^j are the realizations in the other market $j \neq i$. Thus, dropping the superscripts on v^j, c^j and setting $v^i = y, c^i = x$, we have:

$$\begin{aligned} W^{RS} &= 2 \int_0^1 \int_0^1 \int_0^{\frac{v+c}{2}} \int_{\frac{v+c}{2}}^1 (y - x) f(y) f(x) f(v) f(c) dy dx dv dc \\ &= 2 \int_0^1 \int_0^c \left[\int_0^{\frac{v+c}{2}} \int_{\frac{v+c}{2}}^1 (y - x) f(y) f(x) dy dx \right] f(v) f(c) dv dc \\ &\quad + 2 \int_0^1 \int_c^1 \left[\int_0^{\frac{v+c}{2}} \int_{\frac{v+c}{2}}^1 (y - x) f(y) f(x) dy dx \right] f(v) f(c) dv dc. \end{aligned}$$

By changing the order of integration of c and v, the second term on the right hand side is equal to

$$2\int_0^1\int_0^v\left[\int_0^{\frac{v+c}{2}}\int_{\frac{v+c}{2}}^1(y-x)f(y)f(x)dydx\right]f(c)f(v)dcdv$$

It is thus clear, by swapping the variables c and v, that the second term is equal to the first term on the right hand side, and hence $W^{RS}=W^{E1}$, which is what we wanted to show. $\square$

Background for Table 1

We used Wolfram Mathematica to perform the computations presented in the text and discussed in this appendix. The files are available from the authors.

Ex post efficiency. As all the numbers in the table refer to fractions of social surplus under ex post efficiency, we first need to compute social surplus under ex post efficiency for given $N=M$ and $k_B=k_S=K$.

For any $N\geqslant 1$ and $K=1$, expected social surplus under ex post efficiency, denoted $W(n)$, can be computed as

$$W(N)=\sum_{j=1}^N W_j,$$

where for $j\in\{1,\dots,N\}$

$$W_j=\int_0^1\int_0^{v_{(1)}}\dots\int_{c_{[j-1]}}^{v_{(j)}}(v_{(j)}-c_{[j]})f_{[j:N]}(c_{[j]}|c_{[j-1]})f_{(j:N)}(v_{(j)}|v_{(j-1)})\dots f_{(1:N)}(v_{(1)})f_{[1:N]}(c_{[1]})\\ dc_{[j]}dv_{(j)}\dots dc_{[1]}dv_{(1)}$$

is the expected social surplus created by the j-th most valuable trade and where $f_{[j:N]}(c_{[j]}|c_{[j-1]})$ is the conditional density of the j-th lowest of N independent draws from F, conditional on this draw being greater than the $j-1$-th lowest, and $f_{(j:N)}(v_{(j)}|v_{(j-1)})$ is the conditional density of the j-th highest of N independent draws from F, conditional on this draw being smaller than the $j-1$-th highest. For $K>1$, denote expected social surplus by $W(N,K)$. Replacing N by $\hat{N}=NKk$, one obtains $W(N,K)=W(\hat{N})$ as the expression of expected social surplus in the setting with N buyers and sellers each with capacity K.

With F uniform and $N=K=1$, expected social surplus under ex post efficiency is $\int_0^1\int_0^v(v-c)dcdv=1/6$. With $N=2$ and $K=1$, and equivalently for $N=1$ and $K=2$, it is $2/5$. With $N=K=2$, or equivalently for $N=4$ and $K=1$, it is $\frac{8}{9}$.

SB: As is well known, the Myersonian mechanism design machinery works only for one-dimensional types, which means that the second-best mechanism is only known for the cases with $K=1$. For $\alpha\in[0,1]$, letting $\Psi_\alpha^B(v)=v-\alpha(1-F(v))/f(v)$ and $\Psi_\alpha^S(c)=c+\alpha F(v)/f(v)$ be the α-weighted virtual types, which we assume to be increasing in v and c, the second-best mechanism has the allocation rule of inducing trade by all pairs $i\in\{0,\dots,N\}$ with values

$v_{(i)}$ and costs $c_{[i]}$ such that $\Psi^B_{\alpha*}(v_{(i)}) - \Psi^S_{\alpha*}(c_{[i]}) \geqslant 0$, where no trade occurs if $\Psi^B_{\alpha*}(v_{(1)}) - \Psi^S_{\alpha*}(c_{[1]}) < 0$ and where α^* is the smallest number α such that

$$\mathbb{E}_\alpha\left[\sum_{i=1}^{N}\left(\Psi_1^B(v_{(i)}) - \Psi_1^S(c_{[i]})\right)\right] = 0,$$

with the expectation accounting for the allocation rule parameterized by α. Accordingly, second-best welfare $W^{SB}(N)$ is

$$W^{SB}(N) = \mathbb{E}_{\alpha*}\left[\sum_{i=1}^{N}\left(v_{(i)} - c_{[i]}\right)\right].$$

See Myerson and Satterthwaite (1983) and Gresik and Satterthwaite (1989) for derivations of the second-best mechanism. For $N = 1$ and F uniform on $[0,1]$, $\mathbb{E}_\alpha\left[\sum_{i=0}^{N}\left(\Psi_1^B(v_{(i)} - \Psi_1^S(c_{[i]})\right)\right] = (3\alpha-1)/(6(1+\alpha)^3)$, so $\alpha^* = 1/3$, implying a second best welfare of $\int_{1/4}^{1}\int_0^{v-1/2}(v-c)dcdv = 9/64$, which is 27/32 of first-best welfare. For $N = 2$, one can show that $\alpha^* = 0.225629$, yielding a second-best welfare of 0.377463.[36] This is 0.944 times first-best welfare of 2/5.

RS: The formula for welfare in the RS mechanism with two markets and single-unit capacity ($N = 2$, $K = 1$) is given in the proof of Proposition 1. With two markets and traders with a capacity of two ($N = K = 2$) and the estimated price being equal to the average Walrasian price, the expected welfare in the RS mechanism is

$$\begin{aligned} W^{RS} = 2\int_0^1 \Bigg[\int_p^1\int_0^p &(v_{(1)} - c_{[1]})f_{(1:2)}(v_{(1)})f_{[1:2]}(c_{[1]})dv_{(1)}dc_{[1]} \\ + \int_p^1\int_0^p &(v_{(2)} - c_{[2]})f_{(2:2)}(v_{(2)})f_{[2:2]}(c_{[2]})dv_{(2)}dc_{[2]}\Bigg] h(p)(p)dp, \end{aligned}$$

where $h(p)$ is the density of the midpoint of the Walrasian price gap in a market with one buyer and one seller, each with a capacity of 2 drawing their types independently from F; that is, it is the density of the average of the second and third order statistic out of four draws; see e.g., Loertscher and Mezzetti (2019). Thus, for F uniform on $[0,1]$, the density of p is

$$h(p) = \begin{cases} 24p^2 - 32p^3, & p \in [0, 1/2] \\ -8 + 48p - 72p^2 + 32p^3, & p \in (1/2, 1]. \end{cases}$$

DCA: The formula for $N = 2$ and $K = 1$ is in the proof of Proposition 1. For $N = K = 2$, we stack the deck against the DCA and assume there is no exogenous initial reserve price p_0. Then the DCA has to proceed until one agent drops out and the reserve price in the ensuing Ausubel auction is determined by the lowest cost of the first seller that drops out, or the highest

[36] Analytical expressions for second-best welfare and for α^* are available but not particularly insightful. Welfare for given α is $(2(3 + 15\alpha + 25\alpha^2 + 15\alpha^3))/(15(1+\alpha)^5)$ while $\alpha^* = (-30 + (29700 - 1350\sqrt{434})^{1/3} + 35^{2/3}(2(22 + \sqrt{434})^{1/3}))/90$. Second-best welfare is then obtained by plugging in α^*.

value of the first buyer that drops out. When it is the first seller to drop out that determines the price, this is either the second lowest out of four (first event), or the third lowest out of four (second event) value draws. The first of these two events is twice as likely as the second, because the seller that drops out could have either the second and the third or the second and the fourth lowest values. While the only assignment of values in the event when the price is the third lowest seller value is that the seller that drops out has the third and fourth lowest values. Each of the possible value assignments to the seller are equally likely. Thus welfare is:

$$\begin{aligned} W_S^{DA} = &\frac{2}{3}\int_0^1 \left[\int_p^1 \int_0^p (v_{(1)} - c_{[1]}) f_{(1:4)}(v_{(1)}) f_{[1:1]}(c_{[1]} \,|\, c_{[1]} < p) dc_{[1]} dv_{(1)}\right] f_{[2:4]}(p) dp \\ &+ \frac{1}{3}\int_0^1 \left[\int_p^1 \int_0^p (v_{(1)} - c_{[1]}) f_{(1:4)}(v_{(1)}) f_{[1:2]}(c_{[1]} \,|\, c_{[2]} < p) dc_{[1]} dv_{(1)}\right] f_{[3:4]}(p) dp \\ &+ \frac{1}{3}\int_0^1 \left[\int_p^1 \int_0^p (v_{(2)} - c_{[2]}) f_{(2:4)}(v_{(2)}) f_{[2:2]}(c_{[2]} \,|\, c_{[2]} < p) dc_{[2]} dv_{(2)}\right] f_{[3:4]}(p) dp, \end{aligned}$$

where, for F uniform, $f_{[1:1]}(c_{[1]} \,|\, c_{[1]} < p) = 1/p$, $f_{[1:2]}(c_{[1]} \,|\, c_{[2]} < p) = 2(p - c_{[1]})/p^2$ and $f_{[2:2]}(c_{[2]} \,|\, c_{[2]} < p) = 2c_{[2]}/p^2$.

When it is the first buyer that drops out to determine the price, the argument is analogous and welfare is:

$$\begin{aligned} W_B^{DA} = &\frac{2}{3}\int_0^1 \left[\int_0^p \int_p^1 (v_{(1)} - c_{[1]}) f_{[1:4]}(c_{[1]}) f_{(1:1)}(v_{(1)} \,|\, v_{(1)} > p) dv_{(1)} dc_{[1]}\right] f_{(2:4)}(p) dp \\ &+ \frac{1}{3}\int_0^1 \left[\int_0^p \int_p^1 (v_{(1)} - c_{[1]}) f_{[1:4]}(c_{[1]}) f_{(1:2)}(v_{(1)} \,|\, v_{(2)} > p) dv_{(1)} dc_{[1]}\right] f_{(3:4)}(p) dp \\ &+ \frac{1}{3}\int_0^1 \left[\int_0^p \int_p^1 (v_{(2)} - c_{[2]}) f_{[2:4]}(c_{[2]}) f_{(2:2)}(v_{(2)} \,|\, v_{(2)} > p) dv_{(2)} dc_{[2]}\right] f_{(3:4)}(p) dp, \end{aligned}$$

with, for F uniform, $f_{(1:1)}(v_{(1)} \,|\, v_{(1)} > p) = 1/(1-p)$, $f_{(1:2)}(v_{(1)} \,|\, v_{(2)} > p) = 2(v_{(1)} - p)/(1-p)^2$, and $f_{(2:2)}(v_{(2)} \,|\, v_{(2)} > p) = 2(1 - v_{(2)})/(1-p)^2$.

When it is equally likely that the first drop out is a seller or a buyer (which is true for the uniform and any distribution that is symmetric around 1/2), welfare is $(W_S^{DA} + W_B^{DA})/2$. In the uniform case, we have also

$$W_S^{DA} = W^{DA} = \frac{17}{27}.$$

Background for Table 2 and the Importance of Matching

It is straightforward to show that limit welfare is $w_\infty(a) = 1/(2^a(1+a))$ while welfare in the bilateral trade setting is $w_1(a) = \frac{a\Gamma(a)\Gamma(2+a)}{(1+a)\Gamma(2(1+a))}$, where $\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}dx$. Hence,

$$m(a) = \frac{2^a a\Gamma(a)\Gamma(2+a)}{\Gamma(2(1+a))},$$

which has the properties mentioned in the text. The *second-best* mechanism for the bilateral trade problem with the distributions in (1) is characterized by the allocation that induces trade if $\Psi^B_\alpha(v) = v - \alpha(1-v)/a \geqslant c + \alpha c/a = \Psi^S_\alpha(c)$ for the smallest value of α such that $\mathbb{E}_\alpha[\Psi^B_1(v) - \Psi^S_1(c)]$, where the expectation accounts for this allocation rule (see, e.g., Myerson and Satterthwaite, 1983, and Background for Table 1). Letting $\alpha^*(a)$ be this value, one can show that

$$\alpha^*(a) = \frac{a}{1+2a}.$$

For example, for $a = 1$, the uniform distribution for both buyers and sellers, we have $\alpha^*(1) = 1/3$. Accordingly, social surplus under the second-best mechanism for the bilateral trade problem parameterized by a is

$$w_1^{SB}(a) = \mathbb{E}_{\alpha^*(a)}[v - c] = \frac{a\left(\frac{1+a}{1+2a}\right)^{-2a}\Gamma(a)\Gamma(2+a)}{4^a(1+a)^2\Gamma(1+2a)}.$$

As $w_1(a) = (2a\Gamma(a)\Gamma(2+a))/\Gamma(3+2a)$, it follows that

$$\frac{w_1^{SB}(a)}{w_1(a)} = \frac{2^{-(1+2a)}\left(\frac{1+a}{1+2a}\right)^{-2a}\Gamma(3+2a)}{(1+2a)^2\Gamma(1+2a)}.$$

As shown in Figure 3, this is a decreasing function of a. Moreover, one can show that $\lim_{a\to\infty}\frac{w_1^{SB}(a)}{w_1(a)} = \frac{2}{e} \approx 0.736$.

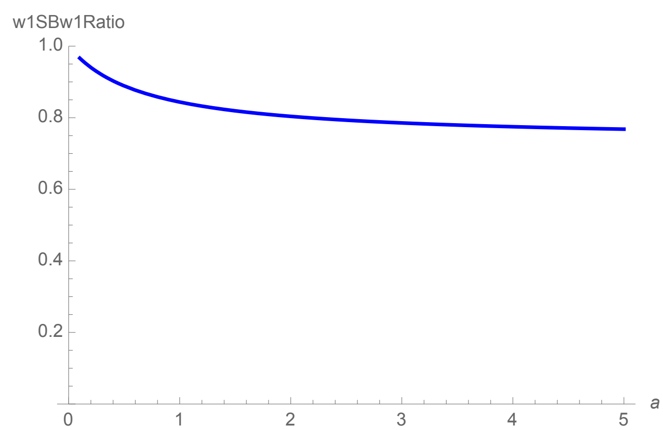


Figure 3: The ratio $w_1^{SB}(a)/w_1(a)$.

Appendix B

Lemma 1. *Suppose Assumptions 1 and 2 hold and the valuations and costs of the nN buyers and nM sellers are drawn according to the probability measure $\mathbb{P}^n_\phi$. Then there exists $\Delta < \infty$ such that, for all $p, p^B, p^S \in (0,1)$, $k \in \{1, ..., k_B\}$ or $k \in \{1, ..., k_S\}$, and all $\phi \in \Phi$:*[37]

$$\mathbb{E}^n_\phi\Big[\Big(\frac{D^{\mathcal{N}}_k(p) - \mathbb{E}_\phi\big[D^{\mathcal{N}}_k(p)\big]}{n}\Big)^2\Big] \leqslant \frac{\Delta}{n}\,, \tag{8}$$

$$\mathbb{E}^n_\phi\Big[\Big(\frac{D^{\mathcal{N}_\mathcal{O}(p^B)}_k(p) - \mathbb{E}_\phi\big[D^{\mathcal{N}_\mathcal{O}(p^B)}_k(p)\big]}{n_O(p^B)}\Big)^2\Big] \leqslant \frac{\Delta}{n_O(p^B)}\,, \tag{9}$$

$$\mathbb{E}^n_\phi\Big[\Big(\frac{S^{\mathcal{M}}_k(p) - \mathbb{E}_\phi\big[S^{\mathcal{M}}_k(p)\big]}{n}\Big)^2\Big] \leqslant \frac{\Delta}{n}\,, \tag{10}$$

$$\mathbb{E}^n_\phi\Big[\Big(\frac{S^{\mathcal{M}_\mathcal{O}(p^S)}_k(p) - \mathbb{E}_\phi\big[S^{\mathcal{M}_\mathcal{O}(p^S)}_k(p)\big]}{m_O(p^S)}\Big)^2\Big] \leqslant \frac{\Delta}{m_O(p^S)}\,. \tag{11}$$

Proof. We will only prove (8), as the the proofs of (9) − (11) are analogous. We have:

$$\begin{aligned}
&\mathbb{E}^n_\phi\Big[\Big(\frac{D^{\mathcal{N}}_k(p) - \mathbb{E}^n_\phi\big[D^{\mathcal{N}}_k(p)\big]}{n}\Big)^2\Big] \\
&= \frac{1}{n^2}\mathbb{E}^n_\phi\Big[\Big(\sum_{i\in\mathcal{N}}\Big(D^i_k(p) - \mathbb{E}^n_\phi\big[D^i_k(p)\big]\Big)\Big)^2\Big] \\
&= \frac{1}{n^2}\cdot\sum_{i\in\mathcal{N}}\Bigg(\mathbb{E}^n_\phi\Big[\Big(D^i_k(p) - \mathbb{E}^n_\phi\big[D^i_k(p)\big]\Big)^2\Big] \\
&\qquad + 2\sum_{j\in\mathcal{N},\, j>i}\mathbb{E}^n_\phi\Big[\Big(D^i_k(p) - \mathbb{E}^n_\phi\big[D^i_k(p)\big]\Big)\Big(D^j_k(p) - \mathbb{E}^n_\phi\big[D^j_k(p)\big]\Big)\Big]\Bigg) \\
&= \frac{1}{n^2}\cdot\sum_{i\in\mathcal{N}}\Bigg(\mathbb{E}^n_\phi\Big[\Big(D^i_k(p) - \mathbb{E}^n_\phi\big[D^i_k(p)\big]\Big)^2\Big] \\
&\qquad + 2\sum_{j\in\mathcal{N},\, j>i}\Big(\mathbb{E}^n_\phi\big[D^i_k(p)D^j_k(p)\big] - \mathbb{E}^n_\phi\big[D^i_k(p)\big]\mathbb{E}^n_\phi\big[D^j_k(p)\big]\Big)\Bigg) \\
&= \frac{1}{n^2}\cdot\sum_{i\in\mathcal{N}}\Bigg(\mathbb{E}^n_\phi\Big[\Big(D^i_k(p) - \mathbb{E}^n_\phi\big[D^i_k(p)\big]\Big)^2\Big] + 2\sum_{j\in\mathcal{N},\, j>i}\alpha^{ij}_k(p;\phi)\Bigg) \\
&\leqslant \frac{1}{n^2}\cdot\Big(n + 2n\Delta_B\Big),
\end{aligned}$$

where the inequality follows from Assumption 2. Setting $\Delta = 1 + 2\Delta_B$ concludes the proof. □

[37]Expectations in (9) and (11) are taken given the identities of the inactive traders in $\mathcal{N}_\mathcal{O}(p^B)$ and $\mathcal{M}_\mathcal{O}(p^S)$.

Corollary 1. *Suppose the valuations and costs of the nN buyers and nM sellers are drawn according to the probability measure $\mathbb{P}_\phi^n$ and Assumptions 1 and 2 hold, then there exists $\Delta < \infty$ such that, for all $p, p^B, p^S \in (0,1)$, and all $\phi \in \Phi$:*

$$\mathbb{E}_\phi^n\Big[\Big(\frac{D^{\mathcal{N}}(p) - \mathbb{E}_\phi^n\big[D^{\mathcal{N}}(p)\big]}{n}\Big)^2\Big] \leqslant \frac{\Delta}{n}\,, \tag{12}$$

$$\mathbb{E}_\phi^n\Big[\Big(\frac{D^{\mathcal{N}_{\mathcal{O}}(p^B)}(p) - \mathbb{E}_\phi^n\big[D^{\mathcal{N}_{\mathcal{O}}(p^B)}(p)\big]}{n_O(p^B)}\Big)^2\Big] \leqslant \frac{\Delta}{n_O(p^B)}\,, \tag{13}$$

$$\mathbb{E}_\phi^n\Big[\Big(\frac{S^{\mathcal{M}}(p) - \mathbb{E}_\phi^n\big[S^{\mathcal{M}}(p)\big]}{n}\Big)^2\Big] \leqslant \frac{\Delta}{n}\,, \tag{14}$$

$$\mathbb{E}_\phi^n\Big[\Big(\frac{S^{\mathcal{M}_{\mathcal{O}}(p^S)}(p) - \mathbb{E}_\phi^n\big[S^{\mathcal{M}_{\mathcal{O}}(p^S)}(p)\big]}{m_O(p^S)}\Big)^2\Big] \leqslant \frac{\Delta}{m_O(p^S)}\,. \tag{15}$$

Proof. Define $Y_k^{\mathcal{N}}(p) = D_k^{\mathcal{N}}(p) - \mathbb{E}_\phi\big[D_k^{\mathcal{N}}(p)\big]$. It is:

$$\begin{aligned}
&\mathbb{E}_\phi^n\Big[\Big(\frac{D^{\mathcal{N}}(p) - \mathbb{E}_\phi^n\big[D^{\mathcal{N}}(p)\big]}{n}\Big)^2\Big] \\
&\quad = \mathbb{E}_\phi^n\Big[\Big(\sum_{k=1}^{k_B} \frac{D_k^{\mathcal{N}}(p) - \mathbb{E}_\phi^n\big[D_k^{\mathcal{N}}(p)\big]}{n}\Big)^2\Big] \\
&\quad = \mathbb{E}_\phi^n\Big[\Big(\sum_{k=1}^{k_B} \frac{Y_k^{\mathcal{N}}(p)}{n}\Big)^2\Big] \\
&\quad = \sum_{k=1}^{k_B} \mathbb{E}_\phi^n\Big[\Big(\frac{Y_k^{\mathcal{N}}(p)}{n}\Big)^2\Big] + 2\sum_{k=1}^{k_B}\sum_{h=k+1}^{k_B} \mathbb{E}_\phi^n\Big[\Big(\frac{Y_k^{\mathcal{N}}(p)}{n}\Big)\Big(\frac{Y_h^{\mathcal{N}}(p)}{n}\Big)\Big] \\
&\quad \leqslant \sum_{k=1}^{k_B} \mathbb{E}_\phi^n\Big[\Big(\frac{Y_k^{\mathcal{N}}(p)}{n}\Big)^2\Big] + \underbrace{2\sum_{k=1}^{k_B}\sum_{h=k+1}^{k_B} \mathbb{E}_\phi^n\Big[\Big(\frac{Y_k^{\mathcal{N}}(p)}{n}\Big)^2\Big]^{1/2} \cdot \mathbb{E}_\phi^n\Big[\Big(\frac{Y_h^{\mathcal{N}}(p)}{n}\Big)^2\Big]^{1/2}}_{\text{by Hölder's inequality}} \\
&\quad \leqslant (k_B)^2 \cdot \max_{k \in \{1,\dots,k_B\}} \Big\{\mathbb{E}_\phi\Big[\Big(\frac{Y_k^{\mathcal{N}}(p)}{n}\Big)^2\Big]\Big\} \\
&\quad = (k_B)^2 \cdot \max_{k \in \{1,\dots,k_B\}} \Big\{\mathbb{E}_\phi\Big[\Big(\frac{D_k^{\mathcal{N}}(p) - \mathbb{E}_\phi^n\big[D_k^{\mathcal{N}}(p)\big]}{n}\Big)^2\Big]\Big\}.
\end{aligned}$$

Then (12) follows from Lemma 1. The proofs of (13) – (15) are analogous. □

Corollary 2. *Suppose the valuations and costs of the nN buyers and nM sellers are drawn according to the probability measure $\mathbb{P}_\phi^n$ and Assumptions 1 and 2 hold, then there exists $\Delta < \infty$ such that, for all $p, p^B, p^S \in (0,1)$, all events $\mathcal{Z}$ determining the set of buyers and sellers that* *have dropped out when prices are p^B and p^S and all $\phi \in \Phi$:*

$$\mathbb{E}_\phi^n\Big[\Big(\frac{D^{\mathcal{N}}(p) - \mathbb{E}_\phi^n\big[D^{\mathcal{N}}(p) \mid \mathcal{Z}\big]}{n}\Big)^2\Big] \leqslant \frac{\Delta}{n}\,, \tag{16}$$

$$\mathbb{E}_\phi^n\Big[\Big(\frac{S^{\mathcal{M}}(p) - \mathbb{E}_\phi^n\big[S^{\mathcal{M}}(p) \mid \mathcal{Z}\big]}{n}\Big)^2\Big] \leqslant \frac{\Delta}{n}\,. \tag{17}$$

Proof. We only prove (16), as the the proof of (17) is analogous. It is:

$$\begin{aligned}
&\mathbb{E}_\phi^n\Big[\Big(\frac{D^{\mathcal{N}}(p) - \mathbb{E}_\phi^n\big[D^{\mathcal{N}}(p) \mid \mathcal{Z}\big]}{n}\Big)^2\Big] \\
&\quad = \mathbb{E}_\phi^n\Big[\frac{1}{n^2}\cdot\Big(D^{\mathcal{N}}(p)^2 + \mathbb{E}_\phi^n\big[D^{\mathcal{N}}(p) \mid \mathcal{Z}\big]^2 - 2D^{\mathcal{N}}(p)\mathbb{E}_\phi^n\big[D^{\mathcal{N}}(p) \mid \mathcal{Z}\big]\Big)\Big] \\
&\quad = \frac{1}{n^2}\cdot\Big(\mathbb{E}_\phi^n\Big[D^{\mathcal{N}}(p)^2\Big] + \mathbb{E}_\phi^n\Big[\mathbb{E}_\phi^n\big[D^{\mathcal{N}}(p) \mid \mathcal{Z}\big]^2\Big] - \underbrace{2\mathbb{E}_\phi^n\Big[\mathbb{E}_\phi^n\Big[D^{\mathcal{N}}(p)\mathbb{E}_\phi^n\big[D^{\mathcal{N}}(p) \mid \mathcal{Z}\big] \,|\mathcal{Z}\Big]\Big]}_{\text{by iterated expectations}}\Big) \\
&\quad = \frac{1}{n^2}\cdot\Big(\mathbb{E}_\phi^n\Big[D^{\mathcal{N}}(p)^2\Big] - \mathbb{E}_\phi^n\Big[\mathbb{E}_\phi^n\big[D^{\mathcal{N}}(p) \mid \mathcal{Z}\big]^2\Big]\Big) \\
&\quad \leqslant \mathbb{E}_\phi^n\Big[\Big(\frac{D^{\mathcal{N}}(p)}{n}\Big)^2\Big] - \underbrace{\mathbb{E}_\phi^n\Big[\frac{D^{\mathcal{N}}(p)}{n}\Big]^2}_{\text{by Jensen's inequality}} \\
&\quad = \mathbb{E}_\phi^n\Big[\Big(\frac{D^{\mathcal{N}}(p) - \mathbb{E}_\phi^n\big[D^{\mathcal{N}}(p)\big]}{n}\Big)^2\Big] \leqslant \underbrace{\frac{\Delta}{n}}_{\text{by Corollary 1.}}
\end{aligned}$$

□

Proof of Theorem 2. Take ϕ_* to be the true index; that is, take $\mathbb{P}_{\phi_*}^n$ to be the true probability measure determining values and costs. Define δ_S as $\delta_S = [r - P_S^n\left(D^{\mathcal{N}}(r)\right)]$, where r is the realized reserve price. By Assumption 1, $nw\delta_S \leqslant \mathbb{E}_{\phi_*}^n[S^{\mathcal{M}}(r) - S^{\mathcal{M}}(r-\delta_S)] = \mathbb{E}_{\phi_*}^n[S^{\mathcal{N}}(r) - D^{\mathcal{N}}(r)]$; thus, when buyers are on the short side, we have that $\mathbb{E}_{\phi_*}^n\left[\left|D^{\mathcal{N}}(r) - S^{\mathcal{M}}(r)\right|\right]^2/nw$ is an upper bound of the area of the rectangle ABCD in Fig. 2(a), which in turn is an upper bound of the expected welfare loss when the reserve price is r. Similarly, define $\delta_B = [P_B^n\left(S^{mathcalM}(r)\right) - r]$. By Assumption 1, $nw\delta_B \leqslant \mathbb{E}_{\phi_*}^n[D^{\mathcal{N}}(r) - D^{\mathcal{N}}(r+\delta_B)] = \mathbb{E}_{\phi_*}^n[D^{\mathcal{N}} - S^{\mathcal{M}}(r)]$; thus, when sellers are on the short side of the market, $\mathbb{E}_{\phi_*}^n\frac{1}{nw}\left[\left|D^{\mathcal{N}}(r) - S^{\mathcal{M}}(r)\right|\right]^2$ is also an upper bound of the expected welfare loss, as it is an upper bound of the area of the rectangle EFGH in Fig. 2(b).

Recall that the reserve price $R_{\mathcal{Z}}$ only depends on the event $\mathcal{Z}$ describing the information obtained from traders that have dropped out of the DCA. The percentage welfare loss is

$\mathcal{L}(\boldsymbol{\theta}) = \frac{(W_{CE}(\boldsymbol{\theta})-W(\boldsymbol{\theta}))/n}{W_{CE}(\boldsymbol{\theta})/n}$ and the per capita efficient welfare is finite, has finite variance and its expectation converges to a finite level as $n \to \infty$. By Assumption 1, we may then conclude that to prove that the expected percentage efficiency loss $\mathbb{E}^n_{\phi_*}[\mathcal{L}(\boldsymbol{\theta})]$ converges to zero at rate $1/n$, it is sufficient to prove that the expectation of the numerator of $\mathcal{L}(\boldsymbol{\theta})$, which is bounded above by $\frac{1}{n}\mathbb{E}^n_{\phi_*}\frac{1}{n}\left[\mathbb{E}^n_{\phi_*}\left[|D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}})| \,\middle|\, \mathcal{Z}\right]\right]^2$, converges to zero at rate $1/n$, where the inside expectation is taken over demand and supply conditional on $\mathcal{Z}$ and the outside expectation is over events $\mathcal{Z}$ and hence the random reserve price $R_{\mathcal{Z}}$. That is, we must prove that $\mathbb{E}^n_{\phi_*}\left[\mathbb{E}^n_{\phi_*}\left[\frac{|D^{\mathcal{N}}(R_{\mathcal{Z}})-S^{\mathcal{M}}(R_{\mathcal{Z}})|}{n} \,\middle|\, \mathcal{Z}\right]\right]^2 \leqslant \frac{L}{n}$ for some constant $L > 0$ and all n. By Jensen's inequality: $\mathbb{E}^n_{\phi_*}\left[\mathbb{E}^n_{\phi_*}\left[\frac{|D^{\mathcal{N}}(R_{\mathcal{Z}})-S^{\mathcal{M}}(R_{\mathcal{Z}})|}{n} \,\middle|\, \mathcal{Z}\right]\right]^2 \leqslant \mathbb{E}^n_{\phi_*}\left[\mathbb{E}^n_{\phi_*}\left[\left(\frac{D^{\mathcal{N}}(R_{\mathcal{Z}})-S^{\mathcal{M}}(R_{\mathcal{Z}})}{n}\right)^2 \,\middle|\, \mathcal{Z}\right]\right]$ and hence it suffices to show that for some $L > 0$ and all n: $\mathbb{E}^n_{\phi_*}\left[\mathbb{E}^n_{\phi_*}\left[\left(\frac{D^{\mathcal{N}}(R_{\mathcal{Z}})-S^{\mathcal{M}}(R_{\mathcal{Z}})}{n}\right)^2 \,\middle|\, \mathcal{Z}\right]\right] \leqslant \frac{L}{n}$.

For all $r \in [0,1]$, define expected excess demand at r as $X^{\mathcal{N}}_{\phi^*}(r;\mathcal{Z}) = \mathbb{E}^n_{\phi^*}\left[D^{\mathcal{N}}(r) - S^{\mathcal{M}}(r) \mid \mathcal{Z}\right]$. Note that for all $\mathcal{Z}$ and $R_{\mathcal{Z}} \in [0,1]$:

$$\begin{aligned}
&\mathbb{E}^n_{\phi_*}\left[\mathbb{E}^n_{\phi_*}\left[\left(\frac{D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}})}{n}\right)^2 \,\middle|\, \mathcal{Z}\right]\right] \\
&= \mathbb{E}^n_{\phi_*}\left[\mathbb{E}^n_{\phi_*}\left[\left(\frac{D^{\mathcal{N}}(R_{\mathcal{Z}}) - \mathbb{E}_{\phi_*}[D^{\mathcal{N}}(R_{\mathcal{Z}})|\mathcal{Z}]}{n} - \frac{S^{\mathcal{M}}(R_{\mathcal{Z}}) - \mathbb{E}_{\phi_*}[S^{\mathcal{M}}(R_{\mathcal{Z}})|\mathcal{Z}]}{n} + \frac{X^{\mathcal{N}}_{\phi^*}(R_{\mathcal{Z}};\mathcal{Z})}{n}\right)^2 \,\middle|\, \mathcal{Z}\right]\right] \\
&\leqslant 9\max\left\{\mathbb{E}^n_{\phi_*}\left[\mathbb{E}^n_{\phi_*}\left[\left(\frac{D^{\mathcal{N}}(R_{\mathcal{Z}}) - \mathbb{E}^n_{\phi_*}[D^{\mathcal{N}}(R_{\mathcal{Z}})|\mathcal{Z}]}{n}\right)^2 \,\middle|\, \mathcal{Z}\right]\right],\right. \\
&\qquad \left.\mathbb{E}^n_{\phi_*}\left[\mathbb{E}^n_{\phi_*}\left[\left(\frac{S^{\mathcal{M}}(R_{\mathcal{Z}}) - \mathbb{E}^n_{\phi_*}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \mid \mathcal{Z}]}{n}\right)^2 \,\middle|\, \mathcal{Z}\right]\right], \mathbb{E}^n_{\phi_*}\left[\mathbb{E}^n_{\phi_*}\left[\left(\frac{X^{\mathcal{N}}_{\phi_*}(R_{\mathcal{Z}};\mathcal{Z})}{n}\right)^2\right] \,\middle|\, \mathcal{Z}\right]\right\}.
\end{aligned}$$

To conclude the proof we only need to show that there exists $\Delta < \infty$ such that each of the three terms in the max is smaller than Δ/n. For the first two terms, this follows immediately from Corollary 1, as the inequality holds for all realizations of $R_{\mathcal{Z}}$. Lemma 2 below shows that the third term, which equals $\mathbb{E}^n_{\phi_*}\left[\left(\frac{X^{\mathcal{N}}_{\phi_*}(R_{\mathcal{Z}};\mathcal{Z})}{n}\right)^2\right]$ is also smaller than $\frac{\Delta}{n}$. $\square$

Lemma 2. *Suppose the valuations and costs of the nN buyers and nM sellers are drawn according to the probability measure $\mathbb{P}^n_{\phi_*}$ and Assumptions 1, 2 and 3 hold, then there exists $\Delta < \infty$ such that:* $\mathbb{E}^n_{\phi_*}\left[\left(\frac{X^{\mathcal{N}}_{\phi_*}(R_{\mathcal{Z}};\mathcal{Z})}{n}\right)^2\right] \leqslant \frac{\Delta}{n}$.

Proof. We first need to establish two preliminary lemmas.

Lemma 3. *Suppose the valuations and costs of the nN buyers and nM sellers are drawn according to the probability measure $\mathbb{P}^n_{\phi_*}$ and Assumptions 1, 2 and 3 hold. Let $\mathcal{Z}$ be the event containing the information from the dropped-out traders. There exists $\Delta < \infty$ such that:*

$$\mathbb{E}^n_{\phi_*}\left[\left(\frac{X^{\mathcal{N}}_{\phi_*}(R_{\mathcal{Z}};\mathcal{Z}) - \mathbb{E}^n_{\phi(\mathcal{Z})}\left[D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}}) \mid \mathcal{Z}\right]}{n}\right)^2\right] \leqslant \frac{\Delta}{n}.$$

Proof. Note that:

$$\begin{aligned}
&\Big(X^{\mathcal{N}}_{\phi_*}(R_{\mathcal{Z}};\mathcal{Z})) - \mathbb{E}^n_{\phi(\mathcal{Z})}\big[D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}\big]\Big)^2 \\
&\quad= \Big(\mathbb{E}^n_{\phi_*}\big[D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}\big] - \mathbb{E}^n_{\phi(\mathcal{Z})}\big[D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}\big]\Big)^2 \\
&\quad\leqslant 2\Big(\mathbb{E}^n_{\phi_*}\big[D^{\mathcal{N}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}\big] - \mathbb{E}^n_{\phi(\mathcal{Z})}\big[D^{\mathcal{N}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}\big]\Big)^2 + 2\Big(\mathbb{E}^n_{\phi_*}\big[S^{\mathcal{M}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}\big] - \mathbb{E}^n_{\phi(\mathcal{Z})}\big[S^{\mathcal{M}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}\big]\Big)^2
\end{aligned} \tag{18}$$

Taking the expectation with respect to $\mathbb{P}^n_{\phi_*}$, by Assumption 3 there exists a $\zeta > 0$ such that:

$$\begin{aligned}
&\frac{1}{2\zeta}\cdot\mathbb{E}^n_{\phi_*}\left[\left(\frac{X^{\mathcal{N}}_{\phi_*}(R_{\mathcal{Z}};\mathcal{Z}) - \mathbb{E}^n_{\phi(\mathcal{Z})}\big[D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}\big]}{n}\right)^2\right] \\
&\leqslant \mathbb{E}^n_{\phi_*}\left[\int_0^{R_{\mathcal{Z}}}\left(\frac{\mathbb{E}^n_{\phi_*}\big[D^{\mathcal{N}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)\big] - \mathbb{E}^n_{\phi(\mathcal{Z})}\big[D^{\mathcal{N}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)\big]}{n}\right)^2 dt\right. \\
&\qquad\left.+\int_{R_{\mathcal{Z}}}^1\left(\frac{\mathbb{E}^n_{\phi_*}\big[S^{\mathcal{M}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)\big] - \mathbb{E}^n_{\phi(\mathcal{Z})}\big[S^{\mathcal{M}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)\big]}{n}\right)^2 dt\right] \\
&= \mathbb{E}^n_{\phi_*}\left[\int_0^{R_{\mathcal{Z}}}\left(\frac{\mathbb{E}^n_{\phi_*}\big[D^{\mathcal{N}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)\big] - D^{\mathcal{N}_{\mathcal{O}}(R_{\mathcal{Z}})}(t) + D^{\mathcal{N}_{\mathcal{O}}(R_{\mathcal{Z}})}(t) - \mathbb{E}^n_{\phi(\mathcal{Z})}\big[D^{\mathcal{N}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)\big]}{n}\right)^2 dt\right. \\
&\qquad\left.+\int_{R_{\mathcal{Z}}}^1\left(\frac{\mathbb{E}^n_{\phi_*}\big[S^{\mathcal{M}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)\big] - S^{\mathcal{M}_{\mathcal{O}}(R_{\mathcal{Z}})}(t) + S^{\mathcal{M}_{\mathcal{O}}(R_{\mathcal{Z}})}(t) - \mathbb{E}^n_{\phi(\mathcal{Z})}\big[S^{\mathcal{M}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)\big]}{n}\right)^2 dt\right] \\
&\leqslant 2\mathbb{E}^n_{\phi_*}\left[\int_0^{R_{\mathcal{Z}}}\left(\frac{\mathbb{E}^n_{\phi_*}\big[D^{\mathcal{N}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)\big] - D^{\mathcal{N}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)}{n}\right)^2 dt + \int_0^{R_{\mathcal{Z}}}\left(\frac{D^{\mathcal{N}_{\mathcal{O}}(R_{\mathcal{Z}})}(t) - \mathbb{E}^n_{\phi(\mathcal{Z})}\big[D^{\mathcal{N}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)\big]}{n}\right)^2 dt\right. \\
&\qquad\left.+\int_{R_{\mathcal{Z}}}^1\left(\frac{\mathbb{E}^n_{\phi_*}\big[S^{\mathcal{M}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)\big] - S^{\mathcal{M}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)}{n}\right)^2 dt + \int_{R_{\mathcal{Z}}}^1\left(\frac{S^{\mathcal{M}_{\mathcal{O}}(R_{\mathcal{Z}})}(t) - \mathbb{E}^n_{\phi(\mathcal{Z})}\big[S^{\mathcal{M}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)\big]}{n}\right)^2 dt\right] \\
&\leqslant 4\mathbb{E}^n_{\phi_*}\left[\int_0^{R_{\mathcal{Z}}}\left(\frac{\mathbb{E}^n_{\phi_*}\big[D^{\mathcal{N}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)\big] - D^{\mathcal{N}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)}{n}\right)^2 dt + \int_{R_{\mathcal{Z}}}^1\left(\frac{\mathbb{E}^n_{\phi_*}\big[S^{\mathcal{M}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)\big] - S^{\mathcal{M}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)}{n}\right)^2 dt\right]
\end{aligned}$$

where the first inequality follows from Assumption 3, the second from simple algebra, and the third from the definition of $\phi(\mathcal{Z})$ in (6) as the minimum distance estimation index.

Applying Corollary 1 concludes the proof of Lemma 3, as for some $\Delta > 0$ two terms in the square brackets on the right hand side are both less than $\frac{\Delta}{n\zeta}$ for all realization $R_{\mathcal{Z}}$. □

Lemma 4. *Suppose the valuation and costs of the nN buyers and nM sellers are drawn according to the probability measure $\mathbb{P}^n_{\phi_*}$ and Assumptions 1, 2 and 3 hold. Let $\mathcal{Z}$ be the event containing the information from the dropped-out traders. There exists $\Delta < \infty$ such that:*

$$\mathbb{E}^n_{\phi_*}\left[\left(\frac{\mathbb{E}^n_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}]}{n}\right)^2\right] \leqslant \frac{\Delta}{n}.$$

Proof. Recall that given an event $\mathcal{Z}$ estimated demand and estimated supply at $R_{\mathcal{Z}}$ are given by $\mathbb{E}_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}]$ and $\mathbb{E}_{\phi(\mathcal{Z})}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}]$. There are three cases, or set of events, to consider depending on whether estimated demand is greater, equal or smaller than estimated supply. The case of equality is trivial, as it obviously implies that the term in brackets in the inequality in the lemma is less than $\frac{\Delta}{n}$ for any $\Delta > 0$. The cases of estimated excess demand and estimated excess supply are mirror images of one another and we will thus only consider one of them.

Thus, take events $\mathcal{Z}$ for which $\mathbb{E}_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}] > \mathbb{E}_{\phi(\mathcal{Z})}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}]$, so that the last clock state of the DCA, the state when the reserve price is reached, is a buyers clock state. This implies that the state preceding the last clock state is either a double clock or a sellers' clock state and there was a sequence of clock prices along which the sellers' price decreased until it reached $R_{\mathcal{Z}}$ and the buyers' clock price stayed constant or increased and stopped at $R_{\mathcal{Z}} - \epsilon_B$. Along that price sequence conditional estimated supply must be at least as large as conditional estimated demand and the sequence must end with either a seller or a buyer dropping out of the DCA. Denote by $\mathcal{Z}_-$ the event that occurred before the state preceding the last; this is the event used to estimate demand and supply in the the second to last state. As we have argued, it must be $\mathbb{E}_{\phi(\mathcal{Z}_-)}[D^{\mathcal{N}}(R_{\mathcal{Z}} - \epsilon_B) \,|\, \mathcal{Z}_-] \leqslant \mathbb{E}_{\phi(\mathcal{Z}_-)}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}_-]$ and hence:

$$
\begin{aligned}
&\Big(\mathbb{E}_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}]\Big)^2 \qquad (19)\\
&\quad\leqslant \Bigg(\mathbb{E}_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}] - \mathbb{E}_{\phi(\mathcal{Z})}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}] + \mathbb{E}_{\phi(\mathcal{Z}_-)}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}_-] - \mathbb{E}_{\phi(\mathcal{Z}_-)}[D^{\mathcal{N}}(R_{\mathcal{Z}} - \epsilon_B) \,|\, \mathcal{Z}_-]\Bigg)^2\\
&\quad\leqslant \Big(\mathbb{E}_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}] \underbrace{-\mathbb{E}_{\phi_*}[D^{\mathcal{N}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}] + \mathbb{E}_{\phi_*}[D^{\mathcal{N}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}]}_{=0} - \underbrace{D^{\mathcal{N}}(R_{\mathcal{Z}}) + D^{\mathcal{N}}(R_{\mathcal{Z}})}_{=0}\\
&\quad\underbrace{-\mathbb{E}_{\phi_*}[D^{\mathcal{N}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}_-] + \mathbb{E}_{\phi_*}[D^{\mathcal{N}}(R_{\mathcal{Z}} - \epsilon_B) \,|\, \mathcal{Z}_-]}_{\mathbb{E}_{\phi_*}[D^{\mathcal{N}}(R_{\mathcal{Z}} - \epsilon_B) \,|\, \mathcal{Z}_-] \geqslant \mathbb{E}_{\phi_*}[D^{\mathcal{N}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}_-]} - \mathbb{E}_{\phi(\mathcal{Z}_-)}[D^{\mathcal{N}}(R_{\mathcal{Z}} - \epsilon_B) \,|\, \mathcal{Z}_-]\\
&\qquad - \mathbb{E}_{\phi(\mathcal{Z})}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}] \underbrace{+\mathbb{E}_{\phi_*}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}] - \mathbb{E}_{\phi_*}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}]}_{=0} \underbrace{+S^{\mathcal{M}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}})}_{=0}\\
&\qquad \underbrace{+\mathbb{E}_{\phi_*}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}_-] - \mathbb{E}_{\phi_*}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}_-]}_{=0} + \mathbb{E}_{\phi(\mathcal{Z}_-)}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}_-]\Big)^2\\
&\leqslant 8\Big(\mathbb{E}_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}] - \mathbb{E}_{\phi_*}[D^{\mathcal{N}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}]\Big)^2 + 8\Big(\mathbb{E}_{\phi_*}[D^{\mathcal{N}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}] - D^{\mathcal{N}}(R_{\mathcal{Z}})\Big)^2\\
&\quad + 8\Big(D^{\mathcal{N}}(R_{\mathcal{Z}}) - \mathbb{E}_{\phi_*}[D^{\mathcal{N}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}_-]\Big)^2 + 8\Big(\mathbb{E}_{\phi_*}[D^{\mathcal{N}}(R_{\mathcal{Z}} - \epsilon_B) \,|\, \mathcal{Z}_-] - \mathbb{E}_{\phi(\mathcal{Z}_-)}[D^{\mathcal{N}}(R_{\mathcal{Z}} - \epsilon_B) \,|\, \mathcal{Z}_-]\Big)^2
\end{aligned}
$$

$$+ 8\Big(\mathbb{E}_{\phi(\mathcal{Z})}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}] - \mathbb{E}_{\phi*}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}]\Big)^2 + 8\Big(\mathbb{E}_{\phi*}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}] - S^{\mathcal{M}}(R_{\mathcal{Z}})\Big)^2$$
$$+ 8\Big(S^{\mathcal{M}}(R_{\mathcal{Z}}) - \mathbb{E}_{\phi*}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}_-]\Big)^2 + 8\Big(\mathbb{E}_{\phi*}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}_-] - \mathbb{E}_{\phi(\mathcal{Z}_-)}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}_-]\Big)^2$$

Taking the expectation with respect to $\mathbb{P}^n_{\phi*}$, we obtain:

$$\begin{aligned}
&\frac{1}{8} \cdot \mathbb{E}_{\phi*}\bigg[\Big(\frac{\mathbb{E}_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}]}{n}\Big)^2\bigg] \\
&\quad \leqslant \mathbb{E}_{\phi*}\bigg[\Big(\frac{\mathbb{E}_{\phi*}[D^{\mathcal{N}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}] - D^{\mathcal{N}}(R_{\mathcal{Z}})}{n}\Big)^2\bigg] + \mathbb{E}_{\phi*}\bigg[\Big(\frac{D^{\mathcal{N}}(R_{\mathcal{Z}}) - \mathbb{E}_{\phi*}[D^{\mathcal{N}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}_-]}{n}\Big)^2\bigg] \\
&\quad + \mathbb{E}_{\phi*}\bigg[\Big(\frac{\mathbb{E}_{\phi*}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}] - S^{\mathcal{M}}(R_{\mathcal{Z}})}{n}\Big)^2\bigg] + \mathbb{E}_{\phi*}\bigg[\Big(\frac{S^{\mathcal{M}}(R_{\mathcal{Z}}) - \mathbb{E}_{\phi*}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}_-]}{n}\Big)^2\bigg] \\
&\quad + \frac{1}{n^2}\mathbb{E}_{\phi*}\bigg[\Big(\mathbb{E}_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}] - \mathbb{E}_{\phi*}[D^{\mathcal{N}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}]\Big)^2 + \Big(\mathbb{E}_{\phi(\mathcal{Z})}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}] - \mathbb{E}_{\phi*}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}]\Big)^2\bigg] \\
&\quad + \frac{1}{n^2}\mathbb{E}_{\phi*}\bigg[\Big(\mathbb{E}_{\phi*}[D^{\mathcal{N}}(R_{\mathcal{Z}} - \epsilon_B) \,|\, \mathcal{Z}_-] - \mathbb{E}_{\phi(\mathcal{Z}_-)}[D^{\mathcal{N}}(R_{\mathcal{Z}} - \epsilon_B) \,|\, \mathcal{Z}_-]\Big)^2 \\
&\qquad\qquad + \Big(\mathbb{E}_{\phi*}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}_-] - \mathbb{E}_{\phi(\mathcal{Z}_-)}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}_-]\Big)^2\bigg]
\end{aligned}$$

By Corollary 2 the first four terms on the right hand side of the last expression are bounded from above by Δ/n for some $\Delta < \infty$. The first of the remaining two terms is equal to half the right hand side of (18), while the second is equal to half the right hand side of (18) conditional on $\mathcal{Z}_-$ rather than $\mathcal{Z}$ and with the demand evaluated at $R_{\mathcal{Z}} - \epsilon_B$ instead of $R_{\mathcal{Z}}$. Following the same argument as in the proof of Lemma 3 we conclude that there exists a $\Delta < \infty$ such that Δ/n is an upper bound for the two terms. This conclude the proof of Lemma 4 since, as claimed above, the case of events with expected excess supply at $R_{\mathcal{Z}}$ (i.e., such that $\mathbb{E}_{\phi(\mathcal{Z})}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}] > \mathbb{E}_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}]$) can be dealt analogously to the case of expected excess demand we just considered. □

We now conclude the proof of Lemma 2. For all events $\mathcal{Z}$ is it is:

$$\begin{aligned}
\Big(\frac{X^{\mathcal{N}}_{\phi*}(R_{\mathcal{Z}}; \mathcal{Z})}{n}\Big)^2 &= \Big(\frac{X^{\mathcal{N}}_{\phi*}(R_{\mathcal{Z}}; \mathcal{Z}) - \mathbb{E}_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}]}{n} + \frac{\mathbb{E}_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}]}{n}\Big)^2 \\
&\leqslant 2\Big(\frac{X^{\mathcal{N}}_{\phi*}(R_{\mathcal{Z}}; \mathcal{Z}) - \mathbb{E}_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}]}{n}\Big)^2 + 2\Big(\frac{\mathbb{E}_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}]}{n}\Big)^2 \\
&\leqslant 4\max\bigg\{\Big(\frac{X^{\mathcal{N}}_{\phi*}(R_{\mathcal{Z}}; \mathcal{Z}) - \mathbb{E}_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}]}{n}\Big)^2, \Big(\frac{\mathbb{E}_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}]}{n}\Big)^2\bigg\}.
\end{aligned}$$

Taking the expectation $\mathbb{E}^n_{\phi*}$ on both sides of the inequality, Lemma 2 follows from Lemmas 3 and 4, stating that the expectation of each term in curly brackets is less than Δ/n for some $\Delta > 0$. □

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