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Lense-Thirring Precession – theoretical narrative

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We start with the premise that an *inertial* frame is defined as one that isn't accelerating in the usual detectable sense. General Relativity states that inertial frames are 'influenced and dragged by the distribution and flow of mass–energy in the universe', noting the relativistic equivalence of mass and energy [1]. This dragging of inertial frames is simply called frame dragging and is shown conceptually in Figure 1. Frame dragging also influences the flow of time around a spinning body.

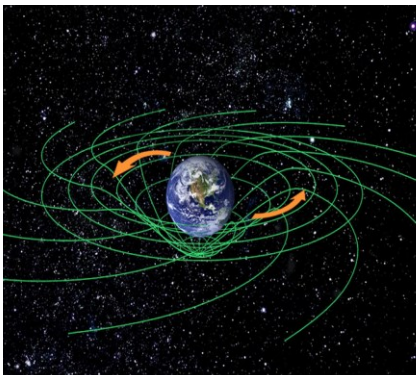

Figure 1. Distortion of space-time in the vicinity. Of the Earth due to Frame Dragging [1].

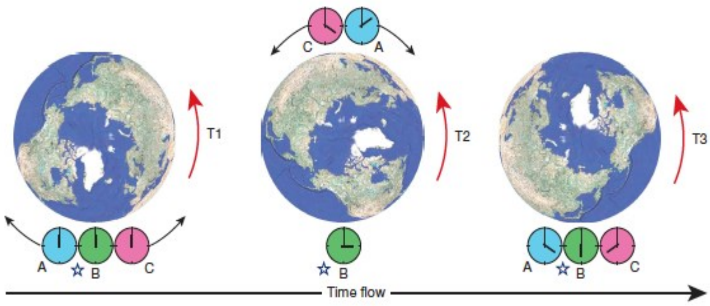

Figure 2. Time dilations for objects rotating around the Earth [1].

In Figure 2 the pink object is rotating prograde and takes longer to get back to the starting point, with respect to a distant fixed star, than the blue object which is rotating retrograde, assuming they are both on the same orbit. If we examine this more closely and consider twins flying arbitrarily slowly on exactly the same equatorial orbit but in opposite directions, then their age difference on meeting up again at the starting point will be of the order of 10^{-16} s. This is an example of the well-known twins paradox. A theory for frame-dragging was proposed by Lense and Thirring in 1918, in which inertial frames are dragged around a central *rotating* mass due to the effect of its gravity on the surrounding spacetime [2]. The rotation of the central mass twists the surrounding spacetime, and this perturbs the orbits of other masses nearby. This effect is known as Lense-Thirring precession, and henceforth as LT. The Earth's gravitational field is capable of generating frame dragging and this is generally considered to be demonstrable in three gravitomagnetic manifestations:

- by the precession of a gyroscope in orbit around the Earth,
- by the precession of orbital planes, where a mass orbiting the Earth constitutes a gyroscope whose orbital axis will precess,
- by the precession of the pericentre of the orbit of the test mass about the Earth.

GP-B measured the first two [3], and the LAGEOS satellites measured the second one only [4]. LAGEOS measured the LT drag of their orbital planes to ~0.031 arcsecs/year [1], which is ~ 8.611 * 10^{-6} °/year. This was subject to error due to uncertainty in the Earth's mass distribution, and there is still some debate about the true size of the error in LAGEOS's LT measurement but it mainly derived from the low eccentricity of the LAGEOS orbits and the difficulties in eliminating Earth multipoles. GP-B measured LT to ~ 0.039 arcsecs/year [1], which is 10.833 * 10^{-6} °/year. GP-B used *IM Pegasi HR 8703* as the guide star and operated on a circular polar orbit of 642 km altitude [1]. The spin axes of GP-B's gyroscopes drifted so the geodetic de Sitter precession [5] (due simply to the presence of the mass of Earth rather than its presence *and* its rotation) was only measured to a precision of 1.5%, which had a relatively significant knock-on effect on the measurement of LT. The relationships between the directions of the LT and the de Sitter precessions are orthogonal and are shown in Figure 3. The total relativistic precession on the body is therefore the vector sum of the LT and de Sitter precessions. Our main interest is the LT component. It is important to note that the relativistic frame dragging effect evidenced by LT precession is about ten million times smaller than, for example, the classical Newtonian effects operating on the plane of the LAGEOS orbits, requiring an 'enormously accurate treatment of background effects' [6]. We return to this later.

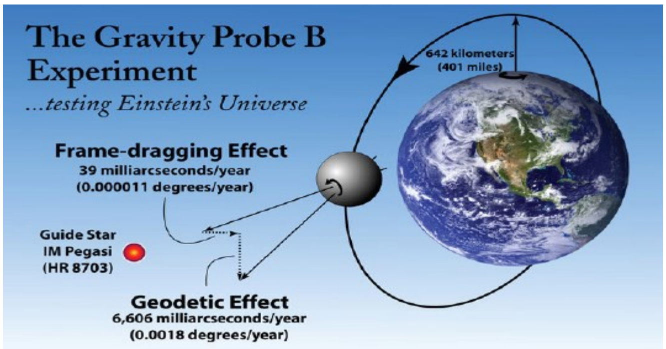

Figure 3. GP-B and the orthogonal relationships between the Lense-Thirring and the de Sitter geodetic precessions [1].

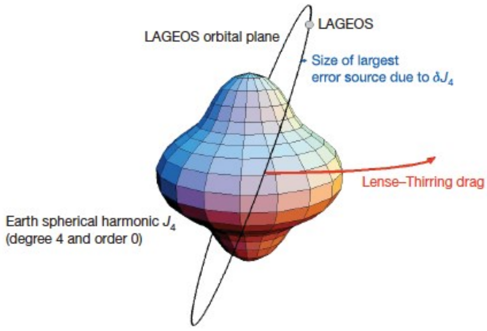

Figure 4. LAGEOS satellites and the effect on Lense-Thirring precession of the Earth's uneven mass distribution [1].

The analysis behind LT, in terms of (weak) gravitomagnetic effects on an accelerating mass, can be considered analogously with an accelerating charge producing a magnetic field. Specifically, the analogy is between the equations that govern the forces on a spinning electric charge with magnetic moment μ which moves through a magnetic field, and the forces of a spinning mass moving through the gravitational field of a rotating mass [1], and this analogy is made through Maxwell's equations which we return to later. Schwarschild [7] proposed an exact solution for space around a large non-rotating body and this solution is known as the Schwarschild metric and accounts for curved non-Euclidean space. This metric doesn't account for the rotation of the massive body but the Kerr metric [7] does, and GP-B measured LT to within 15% of the value predicted by the Kerr metric for Einstein's field equations. The full Kerr solution is complicated because of its highly nonlinear form but there is a useful simpler statement of it which assumes a slowly rotating body, and this is suitable as a background for LT analysis in the vicinity of the Earth. The complexity of the Kerr metric is largely due to the fact that spacetime is not a static background for physical processes and is dynamic and affected by any and all contributions to the energy-momentum tensor of the system of interest. This tensor is integral to Einstein's equations which describe the ten components of the metric, and then the metric is finally used to formulate the equations of motion of the system of interest. Fortunately, in exploring LT in the vicinity of the Earth, we are dealing with weak fields and non-relativistic velocities, so the full form of general relativity is not necessary and a linearised version of the theory is sufficient [8].

Spacetime is generally dynamic within the universe and there is no natural way of splitting it into space and time, but if we think of it as being stationary around the Earth then this simplifying stationarity can be used as a basis for thinking of it in terms of '3+1 slicing'. This means that the spacetime metric tensor $g_{\mu\nu}$ then decomposes naturally into constituent parts, and because of the prevailing conditions of weak gravity and non-relativistic (low) velocities this decomposition can be used to form the basis of a useful analogy with electromagnetism as expressed by Maxwell's equations, from which an expression for LT precession can eventually be obtained [8,9]. So, 'the formal analogy between weak-field low-velocity general relativity and Maxwellian electrodynamics is a simple way to illuminate a whole class of interesting physical phenomena dubbed gravitomagnetism, Lense-Thirring precession is one such example' [8].

We start with gravitational analogies for the electromagnetic scalar and vector potentials taken from the Kerr spacetime metric, stated in terms of the time-time and time-space components, where c is the speed of light,

$$\Phi = \frac{1}{2}(g_{00} - 1)c^2$$

$$A_i = g_{0i}c^2$$

(1)(2)

Now, taking Maxwell's equations in their usual form,

$$\nabla.\bar{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla.\bar{B} = 0$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$\nabla \times \bar{B} = \mu_0 \bar{J} + \mu_0 \varepsilon_0 \frac{\partial \bar{E}}{\partial t}$$

(3)-(6)

We then look at the physical context for Maxwell's equations. Equation (3) states that the quantity of electric field coming from a region of space is proportional to the total electric charge in that region of space. Equation (4) states that the magnetic field doesn't come or go but travels in a continuous loop, so a single magnetic pole or monopole can't exist in practice, according to Maxwell. Equation (5) says that the curl of the electric field is equal to the negative of the rate of change of the magnetic field. Changing the magnetic field alters the curl of the electric field, with the negative sign defining that they go in opposite directions. So, the curl of the electric field pushes electric charge round in a circle in the form of an electric current. Finally, equation (6) says that the curl of the magnetic field is proportional to the current density and a changing electric field. Defining terms precisely: $\bar{E}$ is the electric field, ρ is the electric charge density, ε_0 is the permittivity of free space, $\bar{B}$ is the magnetic field, μ_0 is the permeability of free space, and $\bar{J}$ is the current density.

We then bring in the gravitoelectric field $\bar{E}_G$ and the gravitomagnetic field $\bar{H}$ and it is well known that they are related to the potentials of equations (1) and (2) according to the simplifying Lorentz gauge [10], as follows,

$$\bar{E}_G = -\nabla\Phi - \frac{1}{4c}\frac{\partial \bar{A}}{\partial t}$$

$$\bar{H} = \nabla \times \bar{A}$$

(7)(8)

In the analogy given by [8], the electric field of Maxwell's equations $\bar{E}$ becomes the gravitoelectric field $\bar{E}_G$ and the magnetic field of Maxwell's equations $\bar{B}$ becomes the gravitomagnetic field $\bar{H}$. The electric charge density ρ becomes the mass density ρ_m. The charge current density $\bar{J}$ becomes the mass current density defined by $G\rho_m\bar{v}$, where G is Newton's gravitational constant and $\bar{v}$ is the velocity of the source mass. These substitutions are applied by means of the analogy in order to generate the gravitational analogue of Maxwell's electromagnetic equations,

$$\nabla . \bar{E}_G = -4\pi G\rho_m$$

$$\nabla . \bar{H} = 0$$

$$\nabla \times \bar{E}_G = 0$$

$$\nabla \times \bar{H} = 4\left[-4\pi G\frac{\rho_m \bar{v}}{c} + \frac{1}{c}\frac{\partial \bar{E}_G}{\partial t}\right]$$

(9)-(12)

Despite some structural similarities between the equations which emerge from the gravitational analogy (9)-(12) and Maxwell's equations themselves, equations (3)-(6), there are still some qualifiers and provisos to be made, [8] as follows:

- gravity is attractive, but electromagnetism is both attractive and repulsive (this difference leads to the minus signs in the RHS 'source terms' in equations (9) and (12),
- the gravitational tensor introduces the additional 4 in equation (12),
- the space-space components from the gravitational metric tensor correspond to curved space rather than Euclidean space. As we are only interested here in the effects of the Earth's rotation on an orbiting test mass then we can neglect the curvature of space and also those terms that are not gravitometric and of the order of $(\frac{v}{c})^2$.

Assuming that equations (9)-(12) can, in principle, be used to find the gravitoelectric and gravitomagnetic fields, the force on an orbiting test mass can be found from,

$$\bar{F} = m\bar{E}_G + \frac{m}{c}\bar{v} \times \bar{H}$$

(13)

from which we get,

$$m\frac{d\bar{v}}{dt} = -\frac{\alpha}{r^2}\bar{n} + \frac{m}{c}\bar{v} \times \bar{H}$$

(14)

and where $\bar{r} = r\bar{n}$ is the orbital radius vector of the test mass m and $\alpha = GMm$, where M is the mass of the Earth. We recall that $\bar{H}$ is the gravitomagnetic field due to the Earth's rotation and note that this can be found in principle from equations (9)-(12).

In general the magnetic moment is given by,

$$\bar{\mu} = \frac{1}{2c}\int [\bar{r} \times \bar{j}]dV$$

where $\bar{j}$ is the electric current density, and so the gravitational analogy leads to,

$$\bar{\mu}_G = -4G\frac{1}{2c}\int \rho_m\,[\bar{r} \times \bar{j}]dV = -2G\frac{\bar{S}}{c}$$

(15)

where $\bar{S} = \int \rho_m[\bar{r} \times \bar{j}]dV$, this being the rotating gravitating body's proper angular momentum. The conventional magnetic moment $\bar{\mu}$ creates a dipole magnetic field, given by,

$$\bar{B} = \frac{3\bar{n}(\bar{n} \cdot \bar{\mu}) - \bar{\mu}}{r^3}$$

So, inserting $\bar{\mu}_G$ instead of μ leads to an alternative form which now represents the Earth's gravitomagnetic field,

$$\bar{H} = \frac{2G}{c}\left[\frac{\bar{S} - 3\bar{n}(\bar{n} \cdot \bar{S})}{r^3}\right] \tag{16}$$

The abstract angular momentum for the large rotating body $\bar{S}$ can be replaced by the angular momentum specific to the Earth, defined as $\overline{L'}$ in [8], so we can extract the Earth's angular velocity as,

$$\bar{\Omega} = \frac{2G}{c^2 r^3}\overline{L'} \tag{17}$$

Therefore, the gravitomagnetic field which we derived in equation (16) can now be neatly restated in terms of the Earth's angular velocity, where $\bar{S} \equiv \overline{L'}$, noting that it is divided by the velocity of light in order to accommodate equation (17) correctly,

$$\frac{\bar{H}}{c} = \bar{\Omega} - 3\bar{n}(\bar{\Omega} \cdot \bar{n}) \tag{18}$$

In order to proceed to LT we need to revert to explicit angular momentum of the Earth, through equation (17) and then to rearrange to get the gravitomagnetic field in terms of fundamental quantities and in the conventional form, as follows,

$$\bar{H} = \frac{4G}{c}\left[\frac{\overline{L'}r^2 - 3\bar{r}(\overline{L'} \cdot \bar{r})}{2r^5}\right] \tag{19}$$

One can find the same result for $\bar{H}$ in [11] although the notation and the aggregation of constants is done differently there. Before we complete the analysis for the LT precessional term we state the general expression for the spin precession rate for LT from the Schiff formula statement of the LT metric [12], which is,

$$\bar{\Omega}_{Tot} = \bar{\Omega}_{Th} + \bar{\Omega}_{Geo} + \bar{\Omega}_{LT} \tag{20}$$

where $\bar{\Omega}_{Tot}$ is the total angular velocity measured, assuming an orbital test mass such as a satellite containing gyrospcopic measurement instruments. The right-hand side terms of equation (20) are the Thomas precession $\bar{\Omega}_{Th}$, the geodetic de Sitter precession $\bar{\Omega}_{Geo}$, and the LT precession $\bar{\Omega}_{LT}$. Concentrating on the LT precession, averaging over fast orbital motions [8] and persevering with their notation, we find [10] that LT is directly equal to,

$$\bar{\Omega}_{LT} = \frac{\bar{H}}{2c} \tag{21}$$

and so for a closely orbiting body we initially obtain from equation (19) the following for the averaged gravitomagnetic field at the poles,

$$\overline{H}_{poles} = \frac{4G}{c}\frac{\overline{L}'}{r^3} \tag{22}$$

and if we now move from a general closely orbiting body to a specific terrestrial location where there is a body elevated at h from the surface of the Earth (therefore at altitude R, where $R = r_E + h$, and r_E is the radius of the Earth at the location), then the LT precession is given by,

$$\Omega_{LT} = \frac{2G}{c^2R^3}[L'(\bar{z}\cdot\bar{r}] \tag{23}$$

The scalar angular momentum L' is given by $L' = I_{\oplus}\Omega_{\oplus}$ and considering the Earth initially as a non-oblate sphere, then $I_{\oplus} = \frac{2}{5}Mr_E{}^2$. But the actual radius of gyration of the Earth is 0.576 r_E [14], so the factor of $\frac{2}{5}$ becomes 0.576^2 which is 0.3316. Therefore $I_{\oplus} = 0.3316\,Mr_E{}^2$

From which we obtain,

$$\Omega_{LT} = \frac{0.6632\,GM\Omega_{\oplus}}{c^2R}cos\theta \tag{24}$$

where $\bar{z}\cdot\bar{r} = cos\theta$ and $R \approx r_E$ for h very small indeed (assuming that the bob is hanging a few cm above the ground). This result does not include the de Sitter precession and is purely the LT component. The angle θ is the colatitude which is the included angle between $\bar{z}$ and $\bar{r}$ (the spin axis of Earth and the local vertical axis at the location, respectively) so $\theta = \frac{\pi}{2} - \phi$, where ϕ is the latitude as measured from the equator.

Numerical data:

G = 6.67408*10^{-11} $m^3\,kg^{-1}\,s^{-2}$

M = 5.972*10^{24} kg

$\Omega_{\oplus}$ = 7.2921150*10^{-5} rad/s

c = 2.99792488*10^{8} m/s

R = 6356*10^{3} m at the NP

R = 6363.18*10^{3} m at Glasgow

ϕ = 1.5707963 rad at the NP

ϕ = 0.9750 rad at Glasgow

Numerical results - North Pole

Pippard [13] gives the LT precession as being 220 mas/year at the NP and the precession rate is $6*10^{-10}$ of $\Omega_{\oplus}$. No reference is given for the numbers so they are probably his own calculations. Ruggiero & Tartaglia [10] state the LT precession at the NP as being 281 mas/year, again probably their own calculation.

Using equation (24) and the above data we get Ω_{LT} = 219.5 mas/year at the NP and the precession rate is calculated to be $4.62733*10^{-10}$ of $\Omega_{\oplus}$.

The value calculated here for the polar LT is virtually equal to the value given by Pippard [13].

By changing both the latitude and the radius of the Earth to the values for the location of Glasgow the LT there is calculated to be Ω_{LT} = 181.5 mas/year.

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