Nikhef 2018-066
DESY 18–227
LTH 1191

December 2018

Five-loop contributions to low-N non-singlet anomalous dimensions in QCD

F. Herzog[a], S. Moch[b], B. Ruijl[c], T. Ueda[d], J.A.M. Vermaseren[a] and A. Vogt[e]

[a] *Nikhef Theory Group*
Science Park 105, 1098 XG Amsterdam, The Netherlands

[b] *II. Institute for Theoretical Physics, Hamburg University*
Luruper Chaussee 149, D-22761 Hamburg, Germany

[c] *Institute for Theoretical Physics, ETH Zürich*
Wolfgang-Pauli-Str. 27, 8093 Zürich, Switzerland

[d] *Department of Materials and Life Science, Seikei University*
3-3-1 Kichijoji Kitamachi, Musashino-shi, Tokyo 180-8633, Japan

[e] *Department of Mathematical Sciences, University of Liverpool*
Liverpool L69 3BX, United Kingdom

Abstract

We present the first calculations of next-to-next-to-next-to-next-to-leading order (N^4LO) contributions to anomalous dimensions of spin-N twist-2 operators in perturbative QCD. Specifically, we have obtained the respective non-singlet quark-quark anomalous dimensions at $N = 2$ and $N = 3$ to the fifth order in the strong coupling α_s. These results set the scale for the N^4LO contributions to the evolution of the non-singlet quark distributions of hadrons outside the small-x region, and facilitate a first approximate determination of the five-loop cusp anomalous dimension. While the N^4LO coefficients are larger than expected from the lower-order results, their inclusion stabilizes the perturbative expansions for three or more light flavours at a sub-percent accuracy for $\alpha_s < 0.3$.

The anomalous dimensions $\gamma_{\rm ik}(N)$ of spin-N twist-2 operators are important quantities in perturbative QCD. They are closely related, by an integer-N Mellin transform, to the splitting functions $P_{\rm ik}(x)$ that govern the scale dependence (evolution) of the parton densities of hadrons, and are hence directly relevant to the analysis of hard processes at the LHC. The coefficients $A_{\rm k}$ of the leading large-N term of the diagonal (i = k) anomalous dimensions in the standard $\overline{\rm MS}$ scheme [1–3]

$$\gamma_{\rm kk}(N) \;=\; A_{\rm k} \ln\widetilde{N} - B_{\rm k} + C_{\rm k} N^{-1} \ln\widetilde{N} - (D_{\rm k} - \frac{1}{2}A_{\rm k})N^{-1} + O(N^{-2}\ln^n \widetilde{N}) \tag{1}$$

(with $\ln\widetilde{N} = \ln N + \gamma_e$, where γ_e is the Euler-Mascheroni constant) are identical to the (lightlike) cusp anomalous dimensions [1], and thus relevant well beyond the evolution of parton distributions.

At present, the splitting functions are fully known to three loops (next-to-next-to-leading order, N^{2}LO), see refs. [4, 5] for the main unpolarized case and refs. [6, 7] for the helicity-dependent case. At four loops, the non-singlet quark-quark splitting functions have been determined analytically in the limit of a large number of colours n_c; the remaining terms are known with a high numerical accuracy except at momentum fractions $x \lesssim 10^{-2}$ [8]. The less advanced present status in the flavour-singlet sector beyond the leading large-n_f terms [9] has been summarized in ref. [10].

In this letter, we report on the first complete calculations of five-loop (N^{4}LO) twist-2 anomalous dimensions in QCD and its generalization to a general gauge group. Specifically, we have computed $\gamma_{\rm ns}^{+}(N{=}2)$ and $\gamma_{\rm ns}^{k}(N{=}3)$ for $k = -, {\rm v}$. The superscripts refer to the combinations of quark densities

$$q_{ab}^{\pm} \;=\; q_a \pm \bar{q}_a - (q_b \pm \bar{q}_b)\,, \quad q_{\rm v} \;=\; \textstyle\sum_{a=1}^{n_f} (q_a - \bar{q}_a) \tag{2}$$

and n_f represents the number of effectively massless flavours. These results set the scale for the N^{4}LO corrections to the evolution of the non-singlet quark distributions outside the small-x region. In particular they allow, together with $\gamma_{\rm ns}^{-,{\rm v}}(N{=}1) = 0$ and specific properties in the large-n_c limit, see below, first serious (if unavoidably rough) estimates of the five-loop cusp anomalous dimension.

In terms of operator definitions and renormalization, the present calculation is a direct generalization of ref. [8]. The computation of the required five-loop self-energy integrals is performed as in refs. [11, 12], i.e., we employ a recent implementation [13, 14] on the local R* operation [15–17] to reduce these to four-loop integrals that can be evaluated by the FORCER program [18]. All our symbolic manipulations are carried out using the latest version [19] of FORM [20,21]. The five-loop computation of $\gamma_{\rm ns}^{-,{\rm v}}(N = 3)$ require an effort comparable to that for the N^{4}LO corrections for Higgs decay to hadrons in the heavy top-quark limit [12], the hardest calculation performed before with the R* program of ref. [13]. A full extension to higher N is not realistic with the present setup.

Our notation for the twist-2 anomalous dimensions and their perturbative expansion is

$$\gamma_{\rm ns}^{\rm a}(N) \;=\; -P_{\rm ns}^{\rm a}(N) \;=\; \sum_{n=0} \gamma_{\rm ns}^{(n)\,{\rm a}}(N)\, a_{\rm s}^{n+1} \quad \text{with} \quad a_{\rm s} \;=\; \frac{\alpha_{\rm s}(\mu_f^2)}{4\pi}\,. \tag{3}$$

Here and in eqs. (4) – (8) below we identify the renormalization scale μ_r with the factorization scale μ_f. The expansion of $\gamma_{\rm ns}^{+}(N = 2)$ to the fourth order in $a_{\rm s}$ and the 4-loop contribution to $\gamma_{\rm ns}^{-,{\rm v}}(N = 3)$ have been written down in eqs. (B.1), (B.9) and (B.16) of ref. [8], see also refs. [22–25]. The lower orders of the latter can be found in appendix C of ref. [26] where, however, the normalization of the group factor $d_{abc}d^{abc}$ is larger by a factor of 16; see the discussion below eq. (30) in ref. [7].

Our new (except for the $C_F n_f^4$ terms which are identical to those obtained already in ref. [27]) five-loop contributions to these anomalous dimensions read

$$
\begin{aligned}
&\gamma_{\rm ns}^{(4)+}(N\!=\!2) \;= \\
&\quad C_F^5 \Big[\frac{9306376}{19683} - \frac{802784}{729}\,\zeta_3 - \frac{557440}{81}\,\zeta_5 + \frac{12544}{9}\,\zeta_3^2 + 8512\,\zeta_7 \Big] \\
&\quad - C_A C_F^4 \Big[\frac{81862744}{19683} - \frac{1600592}{243}\,\zeta_3 + \frac{59840}{81}\,\zeta_4 - \frac{142240}{27}\,\zeta_5 + 3072\,\zeta_3^2 - \frac{35200}{9}\,\zeta_6 + 19936\,\zeta_7 \Big] \\
&\quad + C_A^2 C_F^3 \Big[\frac{63340406}{6561} - \frac{1003192}{243}\,\zeta_3 - \frac{229472}{81}\,\zeta_4 + \frac{61696}{27}\,\zeta_5 + \frac{30976}{9}\,\zeta_3^2 - \frac{35200}{9}\,\zeta_6 + 15680\,\zeta_7 \Big] \\
&\quad - C_A^3 C_F^2 \Big[\frac{220224724}{19683} + \frac{4115536}{729}\,\zeta_3 - \frac{170968}{27}\,\zeta_4 - \frac{3640624}{243}\,\zeta_5 + \frac{70400}{27}\,\zeta_3^2 + \frac{123200}{27}\,\zeta_6 + \frac{331856}{27}\,\zeta_7 \Big] \\
&\quad + C_A^4 C_F \Big[\frac{266532611}{39366} + \frac{2588144}{729}\,\zeta_3 - \frac{221920}{81}\,\zeta_4 - \frac{3102208}{243}\,\zeta_5 + \frac{74912}{81}\,\zeta_3^2 + \frac{334400}{81}\,\zeta_6 + \frac{178976}{27}\,\zeta_7 \Big] \\
&\quad - \frac{d_{AA}^{(4)}}{N_A} C_F \Big[\frac{15344}{81} - \frac{12064}{27}\,\zeta_3 - \frac{704}{3}\,\zeta_4 + \frac{58400}{27}\,\zeta_5 - \frac{6016}{3}\,\zeta_3^2 - \frac{19040}{9}\,\zeta_7 \Big] \\
&\quad + \frac{d_{FA}^{(4)}}{N_F} C_F \Big[\frac{23968}{81} - \frac{733504}{81}\,\zeta_3 + \frac{176320}{81}\,\zeta_5 + \frac{6400}{3}\,\zeta_3^2 + \frac{77056}{9}\,\zeta_7 \Big] \\
&\quad - \frac{d_{FA}^{(4)}}{N_F} C_A \Big[\frac{82768}{81} - \frac{555520}{81}\,\zeta_3 + \frac{10912}{9}\,\zeta_4 - \frac{1292960}{81}\,\zeta_5 + \frac{84352}{27}\,\zeta_3^2 + \frac{140800}{27}\,\zeta_6 + 12768\,\zeta_7 \Big] \\
&\quad + n_f C_F^4 \Big[\frac{1824964}{19683} - \frac{463520}{243}\,\zeta_3 + \frac{21248}{81}\,\zeta_4 - \frac{16480}{81}\,\zeta_5 + \frac{6656}{9}\,\zeta_3^2 - \frac{6400}{9}\,\zeta_6 + \frac{8960}{3}\,\zeta_7 \Big] \\
&\quad - n_f C_A C_F^3 \Big[\frac{3375082}{6561} - \frac{420068}{243}\,\zeta_3 - \frac{48256}{81}\,\zeta_4 + \frac{458032}{81}\,\zeta_5 + \frac{3968}{3}\,\zeta_3^2 - \frac{8000}{3}\,\zeta_6 + \frac{4480}{3}\,\zeta_7 \Big] \\
&\quad + n_f C_A^2 C_F^2 \Big[\frac{15291499}{13122} + \frac{1561600}{243}\,\zeta_3 - \frac{114536}{27}\,\zeta_4 - \frac{252544}{243}\,\zeta_5 + \frac{24896}{27}\,\zeta_3^2 + \frac{13600}{27}\,\zeta_6 + \frac{11200}{27}\,\zeta_7 \Big] \\
&\quad - n_f C_A^3 C_F \Big[\frac{48846580}{19683} + \frac{4314308}{729}\,\zeta_3 - \frac{274768}{81}\,\zeta_4 - \frac{1389080}{243}\,\zeta_5 + \frac{27808}{81}\,\zeta_3^2 + \frac{184000}{81}\,\zeta_6 + \frac{39088}{27}\,\zeta_7 \Big] \\
&\quad + n_f \frac{d_{FA}^{(4)}}{N_F} \Big[\frac{22096}{27} + \frac{43712}{81}\,\zeta_3 - \frac{512}{9}\,\zeta_4 - \frac{217280}{81}\,\zeta_5 - \frac{25088}{27}\,\zeta_3^2 + \frac{25600}{27}\,\zeta_6 - 2464\,\zeta_7 \Big] \\
&\quad - n_f C_F \frac{d_{FF}^{(4)}}{N_F} \Big[\frac{170752}{81} - \frac{328832}{81}\,\zeta_3 + \frac{650240}{81}\,\zeta_5 - \frac{8192}{9}\,\zeta_3^2 - \frac{35840}{9}\,\zeta_7 \Big] \\
&\quad + n_f C_A \frac{d_{FF}^{(4)}}{N_F} \Big[\frac{207824}{81} + \frac{251392}{81}\,\zeta_3 - \frac{5632}{9}\,\zeta_4 - \frac{522880}{81}\,\zeta_5 + \frac{15872}{27}\,\zeta_3^2 + \frac{70400}{27}\,\zeta_6 - \frac{29120}{9}\,\zeta_7 \Big] \\
&\quad + n_f^2 C_F^3 \Big[\frac{1082297}{6561} - \frac{145792}{243}\,\zeta_3 + \frac{1072}{81}\,\zeta_4 + \frac{55552}{81}\,\zeta_5 + \frac{1792}{9}\,\zeta_3^2 - \frac{3200}{9}\,\zeta_6 \Big] \\
&\quad + n_f^2 C_A C_F^2 \Big[\frac{332254}{2187} - \frac{85016}{243}\,\zeta_3 + \frac{20752}{27}\,\zeta_4 - \frac{28544}{81}\,\zeta_5 - \frac{13952}{27}\,\zeta_3^2 + \frac{1600}{27}\,\zeta_6 \Big] \\
&\quad + n_f^2 C_A^2 C_F \Big[\frac{631400}{6561} + \frac{214268}{243}\,\zeta_3 - 784\,\zeta_4 - \frac{53344}{243}\,\zeta_5 + \frac{25472}{81}\,\zeta_3^2 + \frac{22400}{81}\,\zeta_6 \Big] \\
&\quad - n_f^2 \frac{d_{FF}^{(4)}}{N_F} \Big[\frac{43744}{81} - \frac{35648}{81}\,\zeta_3 - \frac{1792}{9}\,\zeta_4 - \frac{52480}{81}\,\zeta_5 + \frac{2048}{27}\,\zeta_3^2 + \frac{12800}{27}\,\zeta_6 \Big] \\
&\quad + n_f^3 C_F^2 \Big[\frac{265510}{19683} + \frac{11872}{729}\,\zeta_3 - \frac{128}{3}\,\zeta_4 + \frac{512}{27}\,\zeta_5 \Big] \\
&\quad + n_f^3 C_A C_F \Big[\frac{168677}{19683} + \frac{11872}{729}\,\zeta_3 + \frac{2752}{81}\,\zeta_4 - \frac{4096}{81}\,\zeta_5 \Big] - n_f^4 C_F \Big[\frac{5504}{19683} + \frac{1024}{729}\,\zeta_3 - \frac{128}{81}\,\zeta_4 \Big] \,, \qquad (4)
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{\rm ns}^{(4)-}(N\!=\!3) \;=\;& \\
& C_F^5 \left[\frac{81472935625}{80621568} + \frac{99382175}{23328}\zeta_3 - \frac{3395975}{162}\zeta_5 - \frac{9650}{9}\zeta_3^2 + \frac{34685}{2}\zeta_7 \right] \\
& - C_A C_F^4 \left[\frac{286028134219}{80621568} - \frac{23916529}{7776}\zeta_3 - 4490\,\zeta_3^2 + \frac{134090}{81}\zeta_4 - \frac{2468075}{108}\zeta_5 - \frac{55000}{9}\zeta_6 + \frac{155155}{4}\zeta_7 \right] \\
& + C_A^2 C_F^3 \left[\frac{20173099267}{3359232} - \frac{15401281}{864}\zeta_3 + \frac{732787}{1296}\zeta_4 + \frac{1972075}{216}\zeta_5 - \frac{63830}{9}\zeta_3^2 - \frac{79750}{9}\zeta_6 + \frac{139895}{4}\zeta_7 \right] \\
& - C_A^3 C_F^2 \left[\frac{166662991819}{20155392} - \frac{36397493}{2916}\zeta_3 - \frac{103763}{54}\zeta_4 + \frac{30994565}{3888}\zeta_5 - \frac{133990}{27}\zeta_3^2 - \frac{72875}{54}\zeta_6 + \frac{2127335}{108}\zeta_7 \right] \\
& + C_A^4 C_F \left[\frac{75932079965}{10077696} - \frac{27693563}{23328}\zeta_3 - \frac{1791229}{1296}\zeta_4 - \frac{9417425}{1944}\zeta_5 - \frac{96700}{81}\zeta_3^2 + \frac{163625}{81}\zeta_6 + \frac{199640}{27}\zeta_7 \right] \\
& - \frac{d_{AA}^{(4)}}{N_A} C_F \left[\frac{81725}{162} - \frac{33505}{18}\zeta_3 - \frac{1100}{3}\zeta_4 + \frac{52025}{18}\zeta_5 - \frac{7000}{3}\zeta_3^2 - \frac{48125}{36}\zeta_7 \right] \\
& - \frac{d_{FA}^{(4)}}{N_F} C_F \left[\frac{231575}{36} + \frac{6351445}{324}\zeta_3 - \frac{2927225}{162}\zeta_5 + \frac{23210}{3}\zeta_3^2 - \frac{200410}{9}\zeta_7 \right] \\
& + \frac{d_{FA}^{(4)}}{N_F} C_A \left[\frac{165871}{54} + \frac{1816625}{162}\zeta_3 - \frac{41800}{9}\zeta_4 - \frac{4456145}{162}\zeta_5 + \frac{196880}{27}\zeta_3^2 + \frac{200750}{27}\zeta_6 - \frac{7525}{4}\zeta_7 \right] \\
& + n_f C_F^4 \left[\frac{1776521549}{40310784} - \frac{1332919}{486}\zeta_3 + \frac{5000}{9}\zeta_3^2 + \frac{33290}{81}\zeta_4 - \frac{30325}{81}\zeta_5 - \frac{10000}{9}\zeta_6 + \frac{14000}{3}\zeta_7 \right] \\
& - n_f C_A C_F^3 \left[\frac{3737356319}{3359232} - \frac{2327111}{432}\zeta_3 + \frac{1280}{3}\zeta_3^2 + \frac{262069}{648}\zeta_4 + \frac{1693715}{162}\zeta_5 - \frac{14000}{3}\zeta_6 + \frac{7000}{3}\zeta_7 \right] \\
& + n_f C_A^2 C_F^2 \left[\frac{5637513931}{3359232} + \frac{2711207}{486}\zeta_3 - \frac{5020}{27}\zeta_3^2 - \frac{457499}{108}\zeta_4 + \frac{508820}{243}\zeta_5 - \frac{20375}{27}\zeta_6 + \frac{50155}{108}\zeta_7 \right] \\
& - n_f C_A^3 C_F \left[\frac{8766012215}{2519424} + \frac{45697231}{5832}\zeta_3 + \frac{1195}{81}\zeta_3^2 - \frac{2848403}{648}\zeta_4 - \frac{1808870}{243}\zeta_5 + \frac{222250}{81}\zeta_6 + \frac{250915}{108}\zeta_7 \right] \\
& - n_f C_F \frac{d_{FF}^{(4)}}{N_F} \left[\frac{24385}{27} - \frac{334010}{81}\zeta_3 - \frac{8480}{9}\zeta_3^2 + \frac{1622600}{81}\zeta_5 - \frac{135380}{9}\zeta_7 \right] \\
& + n_f \frac{d_{FA}^{(4)}}{N_F} \left[\frac{297889}{162} + \frac{154970}{81}\zeta_3 - \frac{62600}{27}\zeta_3^2 + \frac{3700}{9}\zeta_4 - \frac{122780}{81}\zeta_5 - \frac{36500}{27}\zeta_6 - 910\,\zeta_7 \right] \\
& + n_f C_A \frac{d_{FF}^{(4)}}{N_F} \left[\frac{241835}{162} + \frac{333487}{81}\zeta_3 + \frac{30560}{27}\zeta_3^2 - 10780/9\,\zeta_4 - \frac{316900}{81}\zeta_5 + \frac{110000}{27}\zeta_6 - \frac{71960}{9}\zeta_7 \right] \\
& + n_f^2 C_F^3 \left[\frac{512848319}{1679616} - \frac{57109}{54}\zeta_3 + \frac{2800}{9}\zeta_3^2 + \frac{9118}{81}\zeta_4 + \frac{86440}{81}\zeta_5 - \frac{5000}{9}\zeta_6 \right] \\
& + n_f^2 C_A C_F^2 \left[\frac{1080083}{5832} - \frac{296729}{972}\zeta_3 - \frac{21800}{27}\zeta_3^2 + \frac{56327}{54}\zeta_4 - \frac{42860}{81}\zeta_5 + \frac{2500}{27}\zeta_6 \right] \\
& + n_f^2 C_A^2 C_F \left[\frac{61747877}{419904} + \frac{2496811}{1944}\zeta_3 + \frac{39800}{81}\zeta_3^2 - \frac{3503}{3}\zeta_4 - \frac{88990}{243}\zeta_5 + \frac{35000}{81}\zeta_6 \right] \\
& - n_f^2 \frac{d_{FF}^{(4)}}{N_F} \left[\frac{19435}{27} - \frac{53366}{81}\zeta_3 + \frac{3200}{27}\zeta_3^2 - \frac{3160}{9}\zeta_4 - \frac{70000}{81}\zeta_5 + \frac{20000}{27}\zeta_6 \right] \\
& + n_f^3 C_F^2 \left[\frac{28758139}{1259712} + \frac{21673}{729}\zeta_3 - \frac{610}{9}\zeta_4 + \frac{800}{27}\zeta_5 \right] \\
& + n_f^3 C_A C_F \left[\frac{13729181}{1259712} + \frac{14947}{729}\zeta_3 + \frac{4390}{81}\zeta_4 - \frac{6400}{81}\zeta_5 \right] - n_f^4 C_F \left[\frac{259993}{629856} + \frac{1660}{729}\zeta_3 - \frac{200}{81}\zeta_4 \right]
\end{aligned}
\qquad (5)
$$

and

$$
\begin{aligned}
\gamma_{\rm ns}^{(4)\rm v}(N\!=\!3) \;&=\; \gamma_{\rm ns}^{(4)-}(N\!=\!3) \\
&+ n_f \frac{d_{abc}d^{abc}}{N_F} \Big\{ C_F^2 \Big[\frac{79906955}{46656} + \frac{246955}{54}\zeta_3 - \frac{504550}{81}\zeta_5 \Big] \\
&\quad - C_A C_F \Big[\frac{9797321}{3888} - \frac{475655}{54}\zeta_3 + \frac{17600}{9}\zeta_4 + \frac{516950}{81}\zeta_5 - \frac{500}{9}\zeta_3^2 + \frac{2800}{9}\zeta_7 \Big] \\
&\quad + C_A^2 \Big[\frac{166535}{486} - \frac{1783913}{324}\zeta_3 + \frac{5555}{9}\zeta_4 + \frac{507515}{81}\zeta_5 - \frac{2035}{27}\zeta_3^2 - \frac{5500}{27}\zeta_6 - \frac{2765}{18}\zeta_7 \Big] \\
&\quad + n_f C_A \Big[\frac{285985}{3888} + \frac{41954}{81}\zeta_3 + \frac{160}{27}\zeta_3^2 - \frac{1010}{9}\zeta_4 - \frac{56480}{81}\zeta_5 + \frac{1000}{27}\zeta_6 \Big] \\
&\quad + n_f C_F \Big[\frac{1098323}{3888} - \frac{49720}{81}\zeta_3 + \frac{3200}{9}\zeta_4 \Big] - n_f^2 \Big[\frac{21823}{1944} \Big] \Big\} \,. \qquad (6)
\end{aligned}
$$

Here N_A and N_F are the dimensions of the adjoint and fermion representation, with $N_A = 8$ and $N_F = 3$ in QCD, where the quadratic, cubic and quartic group invariants take the values $C_A = 3$, $C_F = 4/3$, $d_{abc}d^{abc} = 5/6$ and $d_{AA}^{(4)} \equiv d_A^{abcd}d_A^{abcd} = 135$, $d_{FA}^{(4)} = 15/2$, $d_{FF}^{(4)} = 5/12$, see ref. [28].

The terms with even-n values of Riemann's ζ-function in eqs. (4) – (6) provide a partial check that was not yet known at the time of ref. [12]. Consistent with the 'no-π^2 theorem' for Euclidean physical quantities [29–32], the ζ_6 terms cancel when the $\overline{\rm MS}$ anomalous dimensions are combined with the corresponding coefficient functions [33–35] to physical evolution kernels for the structure functions F_2 at $N = 2$ and F_3 at $N = 3$ in deep-inelastic scattering (the required transformation can by found in eqs. (2.7) – (2.9) of ref. [36]). The ζ_4 terms are removed by an additional transformation to a renormalization scheme in which the N^{4}LO beta function [11,37–39] does not include ζ_4-terms, such as the MiniMOM scheme in the Landau gauge [40–42] or the scheme introduced in ref. [43].

Combining eqs. (4) and (5) with the lower-order results leads to the numerical QCD expansions

$$
\begin{aligned}
\gamma_{\rm ns}^{+}(N\!=\!2,n_f\!=\!0) &= 0.2829\,\alpha_s(1 + 1.0187\,\alpha_s + 1.5307\,\alpha_s^2 + 2.3617\,\alpha_s^3 + 4.520\,\alpha_s^4 + \ldots)\,, \\
&\ldots \\
\gamma_{\rm ns}^{+}(N\!=\!2,n_f\!=\!3) &= 0.2829\,\alpha_s(1 + 0.8695\,\alpha_s + 0.7980\,\alpha_s^2 + 0.9258\,\alpha_s^3 + 1.781\,\alpha_s^4 + \ldots)\,, \\
\gamma_{\rm ns}^{+}(N\!=\!2,n_f\!=\!4) &= 0.2829\,\alpha_s(1 + 0.7987\,\alpha_s + 0.5451\,\alpha_s^2 + 0.5215\,\alpha_s^3 + 1.223\,\alpha_s^4 + \ldots)\,, \\
\gamma_{\rm ns}^{+}(N\!=\!2,n_f\!=\!5) &= 0.2829\,\alpha_s(1 + 0.7280\,\alpha_s + 0.2877\,\alpha_s^2 + 0.1571\,\alpha_s^3 + 0.849\,\alpha_s^4 + \ldots) \qquad (7)
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma_{\rm ns}^{-}(N\!=\!3,n_f\!=\!0) &= 0.4421\,\alpha_s(1 + 1.0153\,\alpha_s + 1.4190\,\alpha_s^2 + 2.0954\,\alpha_s^3 + 3.954\,\alpha_s^4 + \ldots)\,, \\
&\ldots \\
\gamma_{\rm ns}^{-}(N\!=\!3,n_f\!=\!3) &= 0.4421\,\alpha_s(1 + 0.7952\,\alpha_s + 0.7183\,\alpha_s^2 + 0.7607\,\alpha_s^3 + 1.508\,\alpha_s^4 + \ldots)\,, \\
\gamma_{\rm ns}^{-}(N\!=\!3,n_f\!=\!4) &= 0.4421\,\alpha_s(1 + 0.7218\,\alpha_s + 0.4767\,\alpha_s^2 + 0.3921\,\alpha_s^3 + 1.031\,\alpha_s^4 + \ldots)\,, \\
\gamma_{\rm ns}^{-}(N\!=\!3,n_f\!=\!5) &= 0.4421\,\alpha_s(1 + 0.6484\,\alpha_s + 0.2310\,\alpha_s^2 + 0.0645\,\alpha_s^3 + 0.727\,\alpha_s^4 + \ldots) \qquad (8)
\end{aligned}
$$

in powers of the strong coupling constant α_s. Here we have included $n_f = 0$ besides the physically relevant values, since it provides useful information about the behaviour of the perturbation series. The new N^{4}LO coefficients in eqs. (7) and (8) are larger than one may have expected from the N^{2}LO and N^{3}LO contributions.

It is interesting in this context to consider the effect of the quartic group invariants. For example, the $n_f = 0$ coefficients in eqs. (7) and (8) at N^{3}LO and N^{4}LO can be decomposed as

$$\begin{aligned} 2.3617 &= 2.0878 + 0.1096\, d_{FA}^{(4)}/n_c \\ 4.520 &= 3.552 \ - 0.0430\, d_{FA}^{(4)}/n_c + 0.0510\, d_{AA}^{(4)}/N_a \end{aligned} \tag{9}$$

and

$$\begin{aligned} 2.0954 &= 2.0624 + 0.0132\, d_{FA}^{(4)}/n_c \\ 3.954 &= 3.371 \ - 0.0171\, d_{FA}^{(4)}/n_c + 0.0371\, d_{AA}^{(4)}/N_a \end{aligned} \tag{10}$$

with $d_{FA}^{(4)}/n_c = 2.5$ and $d_{AA}^{(4)}/N_a = 16.875$, see, e.g., appendix C of ref. [42]. Without the rather large contributions of $d_{AA}^{(4)}$, which enters $\gamma_{\rm ns}$ at N^{4}LO for the first time, the series would look much more benign with consecutive ratios of 1.4 – 1.6 between the N^{4}LO, N^{3}LO, N^{2}LO and NLO coefficients. This sizeable $d_{AA}^{(4)}$ contribution ($\sim n_c^2 + 36$) also implies that the leading large-n_c contribution provides a less good approximation at N^{4}LO, at least for low N, than at the previous orders.

The generalization of the expansion coefficients in eq. (3) to $L \equiv \ln(\mu_r^2/\mu_f^2) \neq 0$ is given by [36]

$$\begin{aligned} \gamma_{\rm ns}^{(0)}(L) &= \gamma_{\rm ns}^{(0)} \,, \\ \gamma_{\rm ns}^{(1)}(L) &= \gamma_{\rm ns}^{(1)} + \beta_0 L \gamma_{\rm ns}^{(0)} \,, \\ \gamma_{\rm ns}^{(2)}(L) &= \gamma_{\rm ns}^{(2)} + 2\beta_0 L \gamma_{\rm ns}^{(1)} + \left(\beta_1 L + \beta_0^2 L^2\right) \gamma_{\rm ns}^{(0)} \\ \gamma_{\rm ns}^{(3)}(L) &= \gamma_{\rm ns}^{(3)} + 3\beta_0 L \gamma_{\rm ns}^{(2)} + \left(2\beta_1 L + 3\beta_0^2 L^2\right) \gamma_{\rm ns}^{(1)} + \left(\beta_2 L + \frac{5}{2}\beta_1\beta_0 L^2 + \beta_0^3 L^3\right) \gamma_{\rm ns}^{(0)} \,, \\ \gamma_{\rm ns}^{(4)}(L) &= \gamma_{\rm ns}^{(4)} + 4\beta_0 L \gamma_{\rm ns}^{(3)} + \left(3\beta_1 L + 6\beta_0^2 L^2\right) \gamma_{\rm ns}^{(2)} + \left(2\beta_2 L + 7\beta_1\beta_0 L^2 + 4\beta_0^3 L^3\right) \gamma_{\rm ns}^{(1)} \,, \\ &\quad + \left(\beta_3 L + 3\beta_2\beta_0 L^2 + \frac{3}{2}\beta_1^2 L^2 + \frac{13}{3}\beta_1\beta_0^2 L^3 + \beta_0^4 L^4\right) \gamma_{\rm ns}^{(0)} \end{aligned} \tag{11}$$

to N^{4}LO, where we have suppressed the superscript 'a' of eq. (3). $\beta_{0,1,2,3}$ are the $\overline{\rm MS}$ coefficients of the beta function up to N^{3}LO [44,45] with $\beta_0 = 11 - 2/3\, n_f$, $\beta_1 = 102 - 38/3\, n_f$ etc in QCD.

The numerical impact of the higher-order contributions to the anomalous dimensions $\gamma_{\rm ns}^{\pm}$ on the evolution of the $N = 2$ and $N = 3$ moments of the respective parton distributions (2) are illustrated in fig. 1. At $\alpha_{\rm s}(\mu_f^2) = 0.2$ and $n_f = 4$, the N^{4}LO corrections are about 0.15% at the default choice $\mu_r = \mu_f$ of the renormalization scale, roughly half the size of their N^{3}LO counterparts. Varying μ_r up and down by a factor of 2 one arrives at a band with a full width of about 0.7%. The N^{3}LO and N^{4}LO corrections are about twice as large at a lower scale with $n_f = 3$ and $\alpha_{\rm s}(\mu_f^2) = 0.25$.

In order to assess the implications of the above results beyond $N = 2$ and $N = 3$, and in particular for the five-loop cusp anomalous dimension, it is useful to consider the N-dependence of $\gamma_{\rm ns}^{\pm}(N)$ at lower orders and the large-n_c limit. In the left part of fig. 2, moments (3) of the NLO, N^{2}LO and N^{3}LO splitting functions $P_{\rm ns}^{\pm}(x)$ and their common large-n_c (Ln$_{\rm c}$) limit are displayed for $n_f = 3$ in a manner that facilitates a direct comparison with the size of the corresponding cusp anomalous dimensions, defined by $A_{\rm q} = A_1 a_{\rm s} + A_2 a_{\rm s}^2 + \ldots$, in eq. (1).

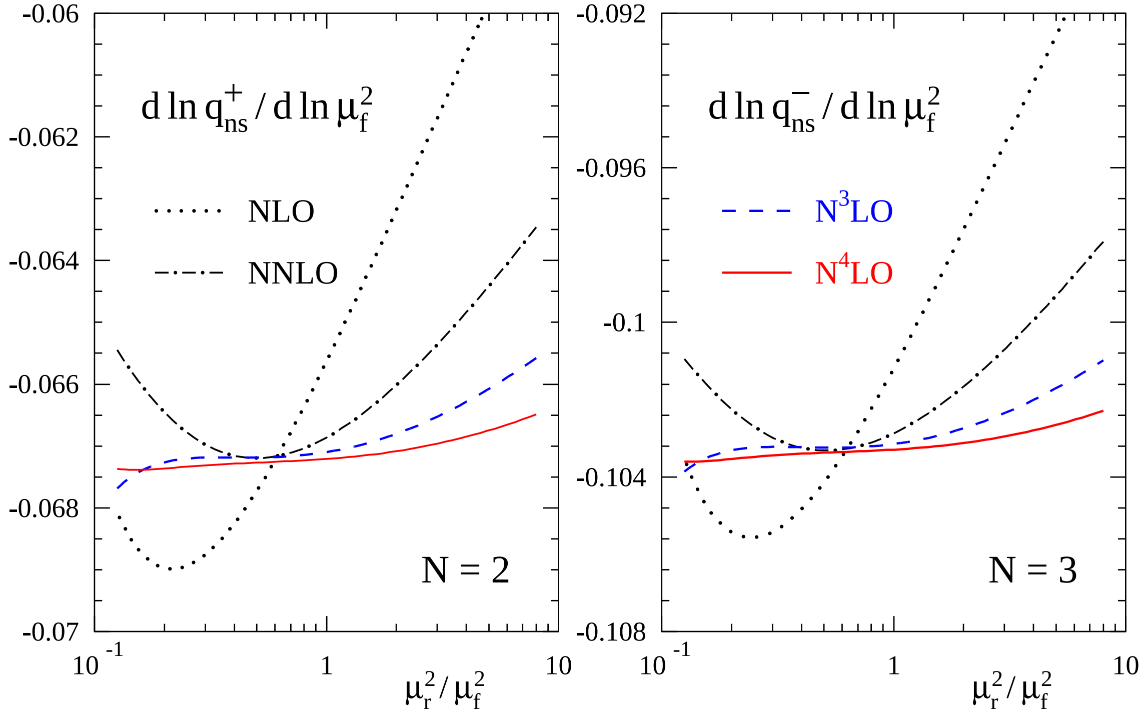

Figure 1: The renormalization-scale dependence of the logarithmic factorization-scale derivatives of the quark distributions $q_{\rm ns}^{+}$ at $N = 2$ and $q_{\rm ns}^{-}$ at $N = 3$ at a standard reference point with $\alpha_{\rm s}(\mu_f^2) = 0.2$ and $n_f = 4$.

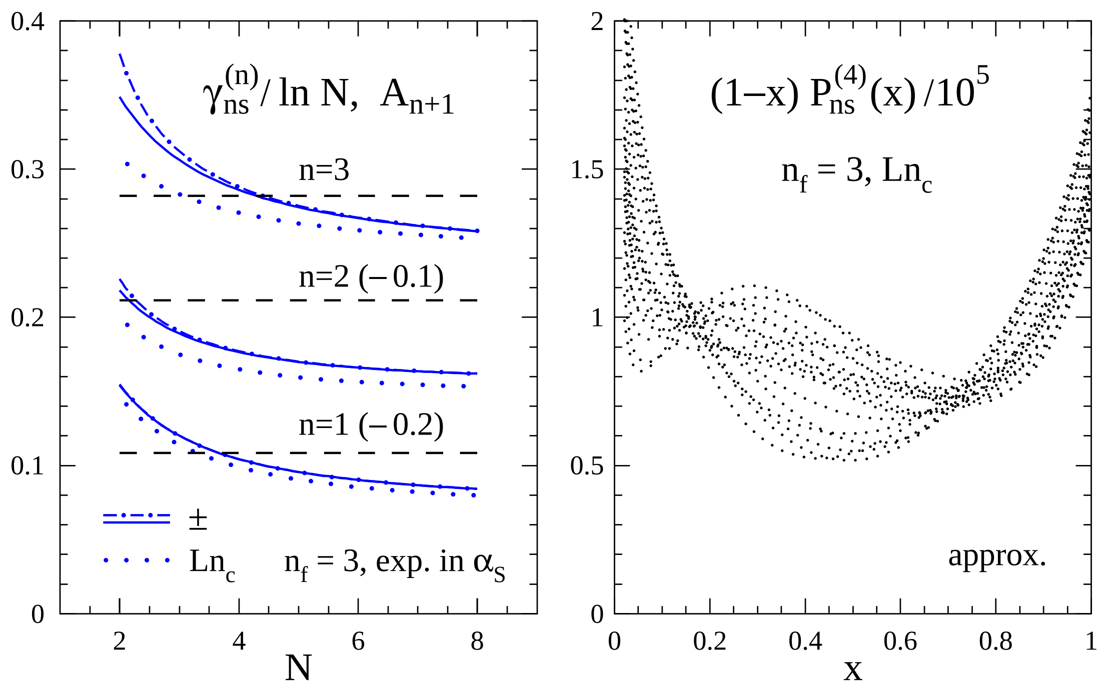

Figure 2: Left: non-singlet anomalous dimensions and their generalization to non-even/odd N at $2 \leq N \leq 8$. The quantities $\gamma_{\rm ns}^{(n)\pm}(N)/\ln N$ for $n_f = 3$ are compared to their common large-n_c (Ln$_c$) limits and their limits for $N \to \infty$ (shown as straight lines), the $(n{+}1)$-loop cusp anomalous dimensions A_{n+1} for $n = 1, 2, 3$. Right: 20 trial functions incorporating the present integer-N and endpoint constraints on the 5-loop Ln$_c$ splitting functions at $n_f = 3$. The resulting uncertainty band for A_5 in the large-n_c limit can be read off at $x = 1$.

At all orders known so far, $\gamma_{\rm ns}^{(n)-}(N)/\ln N$ at $N=3$ deviate from A_{n+1} by less than 8% for $n_f=3$. The relative deviations are even smaller at $n_f=0$, but larger at larger n_f due to cancellations between the n_f-dependent and n_f-independent contributions. However, the corresponding absolute deviations at $n_f=4$ and $n_f=5$ are comparable to those at $n_f=3$.

These results suggest that our above five-loop results can be used for a first estimate of the 5-loop cusp anomalous dimension. The situation is complicated somewhat by the large low-N contribution of the new colour structure $d_{AA}^{(4)}/N_A$ which may or may not persist to the large-N limit. Treating the size of this contribution as an additional uncertainty, we arrive at the predictions

$$A_5 \;=\; (1.7\pm 0.5\,,\; 1.1\pm 0.5\,,\; 0.7\pm 0.5)\cdot 10^5 \quad \text{for} \quad n_f \;=\; 3\,,4\,,5\;. \tag{12}$$

Together with the lower-order results [4, 8] these lead to the QCD expansions

$$\begin{aligned}
A_{\rm q}(n_f\!=\!3) &= 0.42441\,\alpha_{\rm s}\,(1+0.7266\,\alpha_{\rm s}+0.7341\,\alpha_{\rm s}^2+0.665\,\alpha_{\rm s}^3+(1.3\pm 0.4)\alpha_{\rm s}^4+\ldots)\;,\\
A_{\rm q}(n_f\!=\!4) &= 0.42441\,\alpha_{\rm s}\,(1+0.6382\,\alpha_{\rm s}+0.5100\,\alpha_{\rm s}^2+0.317\,\alpha_{\rm s}^3+(0.8\pm 0.4)\alpha_{\rm s}^4+\ldots)\;,\\
A_{\rm q}(n_f\!=\!5) &= 0.42441\,\alpha_{\rm s}\,(1+0.5497\,\alpha_{\rm s}+0.2840\,\alpha_{\rm s}^2+0.013\,\alpha_{\rm s}^3+(0.5\pm 0.4)\alpha_{\rm s}^4+\ldots)
\end{aligned} \tag{13}$$

for the physically relevant values of n_f. Here and in fig. 2 also the N^{3}LO results are approximate; their uncertainties are however irrelevant and amount to $2\cdot 10^{-4}$ for the coefficients in eq. (13).

A more direct determination is possible for the leading large-n_c contribution of A_5. In this limit $\gamma_{\rm ns}^{+}=\gamma_{\rm ns}^{-}$, thus the results at $N=1$, $N=2$ and $N=3$ refer to the same function. Furthermore, as already noted below eq. (3.11) in ref. [8], the large-n_c five-loop coefficients of $C_{\rm q}$ and $D_{\rm q}$ in eq. (1) can be predicted from known coefficients using [3, 8]

$$C_{\rm q} \;=\; (A_{\rm q})^2 \;\;,\quad D_{\rm q} \;=\; A_{\rm q}\cdot(B_{\rm q}-\beta(a_{\rm s})/a_{\rm s})\;\;, \tag{14}$$

where $\beta(a_{\rm s})=-\beta_0\,a_{\rm s}^2-\beta_1\,a_{\rm s}^3-\ldots$. Finally the coefficients of all Ln$_{\rm c}$ small-x logarithms at five loops, $\ln^{\ell}x$ with $\ell=1,\ldots,8$, can be predicted from the results of ref. [8] by solving [46]

$$\gamma_{\rm ns}(N,a_{\rm s})\cdot(\gamma_{\rm ns}(N,a_{\rm s})+N-\beta(a_{\rm s})/a_{\rm s}) \;=\; O(1)\;\;. \tag{15}$$

Together, these endpoint constraints imply that the function $P_{\rm ns}^{(4)}(x)$ is known in the large-n_c limit up to the large-x coefficients A_5 and B_5 of $1/(1-x)_+$ and $\delta(1-x)$, respectively, and a smooth function that approaches a constant for $x\to 0$ and vanishes for $x\to 1$.

Under these circumstances, the three available N-values are sufficient, just, for a first approximate reconstruction of $P_{\rm ns}^{(4)}(x)$: a sufficient number, here 20, of one-parameter smooth functions are chosen, and A_5, B_5 and this parameter are determined from the available three moments for each of these choices. The ensuing spread of the values of $P_{\rm ns}^{(4)}(x)$ indicates the remaining uncertainty of this function. The results are shown in the right part of fig. 2 for $n_f=3$ quark flavours. Corresponding procedures (all of which are, of course, mathematically non-rigorous) have been successfully employed to three-loop and four-loop quantities in the past, usually with (many) more calculated moments but weaker endpoint constraints, see, e.g., refs. [47, 48] and ref. [8]. We have checked the above setup by applying it at N^{3}LO, where a comparison with the exact results is possible.

In this manner we arrive at the five-loop $\mathrm{Ln_c}$ cusp anomalous dimensions

$$A_{5,\mathrm{L}} \;=\; (1.5 \pm 0.25\,,\, 0.8 \pm 0.2\,,\, 0.4 \pm 0.1) \cdot 10^5 \quad \text{for} \quad n_f \;=\; 3\,,4\,,5 \qquad (16)$$

and $A_{5,\mathrm{L}} = (4.7 \pm 0.6) \cdot 10^5$ for $n_f = 0$. Together with the lower-order results, which are here known exactly to N^{3}LO [8,49,50], these lead to the numerical expansions

$$\begin{aligned}
A_{\mathrm{q,L}}(n_f\!=\!3) &= 0.42441\,\alpha_{\mathrm{s}}\,(1 + 0.7266\,\alpha_{\mathrm{s}} + 0.7355\,\alpha_{\mathrm{s}}^2 + 0.706\,\alpha_{\mathrm{s}}^3 + (1.1 \pm 0.2)\,\alpha_{\mathrm{s}}^4 + \ldots)\,,\\
A_{\mathrm{q,L}}(n_f\!=\!4) &= 0.42441\,\alpha_{\mathrm{s}}\,(1 + 0.6382\,\alpha_{\mathrm{s}} + 0.5119\,\alpha_{\mathrm{s}}^2 + 0.355\,\alpha_{\mathrm{s}}^3 + (0.6 \pm 0.2)\,\alpha_{\mathrm{s}}^4 + \ldots)\,,\\
A_{\mathrm{q,L}}(n_f\!=\!5) &= 0.42441\,\alpha_{\mathrm{s}}\,(1 + 0.5497\,\alpha_{\mathrm{s}} + 0.2864\,\alpha_{\mathrm{s}}^2 + 0.047\,\alpha_{\mathrm{s}}^3 + (0.3 \pm 0.1)\,\alpha_{\mathrm{s}}^4 + \ldots)\,.
\end{aligned} \qquad (17)$$

These results differ from eq. (13) only from the N^{2}LO contributions which include a (small) term of the form $C_F^2 n_f$. The largest part to the more sizeable N^{3}LO difference is due to the (negative) $d_{FA}^{(4)}/n_c$ contribution. A difference between the N^{4}LO QCD and $\mathrm{Ln_c}$ results as shown by the central values in eqs. (13) and (17) would not be surprising in view of eqs. (9) and (10). However, the present uncertainties preclude any conclusions even about the sign of large-n_c suppressed contributions.

Up to N^{2}LO, the gluon cusp anomalous dimension is related to its quark counterpart by a simple 'Casimir scaling', $A_{\mathrm{g}}/A_{\mathrm{q}} = C_A/C_F = 2.25$ in QCD. This feature is broken at N^{3}LO by the contributions of the quartic group invariants [8,51,52], but appears to persist in a generalized form to N^{4}LO [53,54] that includes the above C_A/C_F relation for the $\mathrm{Ln_c}$ contributions. Assuming that the latter feature holds also at five loops, eq. (17) also provides a first result for the five-loop gluon cusp anomalous dimension $A_{\mathrm{g,L}}$. If a numerical estimate at N^{4}LO were required of A_{g} in QCD, we would recommend, for the time being, to use the last column in the main bracket of eq. (17) with the errors enhanced to ± 0.6 (twice the offset between the corresponding $\mathrm{Ln_c}$ and full QCD coefficients of A_{g} at N^{3}LO) together with the N^{3}LO results in eq. (4.4) of ref. [54].

To summarize, we have employed the implementation [13] of the local R* operation and the FORCER program [18] for the parametric reduction of massless self-energy integrals to extend previous calculations [8,22–25] of the anomalous dimensions $\gamma_{\mathrm{ns}}(N)$ of the lowest-N non-singlet twist-2 operators to the fifth order in the strong coupling constant α_{s}. While the coefficients of α_{s}^5 are larger than expected from the lower-order results, these N^{4}LO corrections stabilize the numerical results at a sub-percent level; a 1% correction is reached only at $\alpha_{\mathrm{s}} = 0.3$ for $n_f = 3$.

At least up to N^{3}LO, the anomalous dimensions $\gamma_{\mathrm{ns}}(N)$ can be written as $f(N)\ln N$, where the functions f depend rather weakly on N at $N \geq 3$. Assuming that this feature also holds at the present order, our results at $N = 2$ and $N = 3$ set the scale for the N^{4}LO corrections to the evolution of the non-singlet quark distributions outside the small-x region. Accordingly, we have provided first rough estimates of the large-N limit of $\gamma_{\mathrm{ns}}^{(4)}(N)/\ln N$, the five-loop quark cusp anomalous dimension A_5, for the physically relevant number of light flavours $n_f = 3$, 4 and 5 in QCD. A more direct approximate determination of A_5 has been presented in the limit of a large number of colours n_c, where the present lack of higher-N results is compensated, to a just sufficient extent, by constraints on the small-x and large-x limits of the corresponding non-singlet splitting functions [3,8,46].

A FORM file with our results in eqs. (4) – (6) and the corresponding lower-order coefficients can be obtained from the preprint server `https://arXiv.org` by downloading the source of this article. It is also available from the authors upon request.

Acknowledgements

This work has been supported by the Advanced Grant 320651, *HEPGAME*, of the *European Research Council* (ERC) and by the *Deutsche Forschungsgemeinschaft* (DFG) under grant number MO 1801/2-1. The research of F. H. is supported by the Vidi grant 680-47-551 of the Netherlands Organisation for Scientific Research (NWO). For a part of our computations we have used the `ulgqcd` cluster in Liverpool which was funded by the UK *Science & Technology Facilities Council* (STFC) grant ST/H008837/1.

References

[1] G.P. Korchemsky, Mod. Phys. Lett. A4 (1989) 1257

[2] S. Albino and R.D. Ball, Phys. Lett. B513 (2001) 93, hep-ph/0011133

[3] Yu.L. Dokshitzer, G. Marchesini and G.P. Salam, Phys. Lett. B634 (2006) 504, hep-ph/0511302

[4] S. Moch, J.A.M. Vermaseren and A. Vogt, Nucl. Phys. B688 (2004) 101, hep-ph/0403192

[5] A. Vogt, S. Moch and J.A.M. Vermaseren, Nucl. Phys. B691 (2004) 129, hep-ph/0404111

[6] S. Moch, J.A.M. Vermaseren and A. Vogt, Nucl. Phys. B889 (2014) 351, arXiv:1409.5131

[7] S. Moch, J.A.M. Vermaseren and A. Vogt, Phys. Lett. B748 (2015) 432, arXiv:1506.04517

[8] S. Moch, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, JHEP 10 (2017) 041, arXiv:1707.08315

[9] J. Davies, B. Ruijl, T. Ueda, J.A.M. Vermaseren, A. Vogt, Nucl. Phys. B915 (2017) 335, arXiv:1610.07477

[10] A. Vogt, F. Herzog, S. Moch, B. Ruijl, T. Ueda, and J. Vermaseren, PoS LL 2018 (2018) 050, arXiv:1808.08981

[11] F. Herzog, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, JHEP 02 (2017) 090, arXiv:1701.01404

[12] F. Herzog, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, JHEP 08 (2017) 113, arXiv:1707.01044

[13] F. Herzog and B. Ruijl, JHEP 05 (2017) 037, arXiv:1703.03776

[14] B. Ruijl, F. Herzog, T. Ueda, J.A.M. Vermaseren and A. Vogt, PoS RADCOR 2017 (2018) 011 and (shortened) J. Phys. Conf. Ser. 1085 (2018) no.5, 052006, arXiv:1801.06084

[15] K.G. Chetyrkin and F.V. Tkachov, Phys. Lett. 114B (1982) 340

[16] K.G. Chetyrkin and V.A. Smirnov, Phys. Lett. 144B (1984) 419

[17] K.G. Chetyrkin, Max-Plank-Institute preprint MPI-PH-PTH-13-91, available as arXiv:1701.08627

[18] B. Ruijl, T. Ueda and J.A.M. Vermaseren, arXiv:1704.06650

[19] B. Ruijl, T. Ueda and J.A.M. Vermaseren, arXiv:1707.06453

[20] J.A.M. Vermaseren, math-ph/0010025

[21] J. Kuipers, T. Ueda, J. Vermaseren and J. Vollinga, Comput. Phys. Commun. 184 (2013) 1453, arXiv:1203.6543

[22] P.A. Baikov and K.G. Chetyrkin, Nucl. Phys. Proc. Suppl. 160 (2006) 76

[23] V.N. Velizhanin, Nucl. Phys. B860 (2012) 288, arXiv:1112.3954

[24] V.N. Velizhanin, arXiv:1411.1331

[25] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, Nucl. Part. Phys. Proc. 261-262 (2015) 3, arXiv:1501.06739

[26] A. Retey and J.A.M. Vermaseren, Nucl. Phys. B604 (2001) 281, hep-ph/0007294

[27] J.A. Gracey, Phys. Lett. B322 (1994) 141, hep-ph/9401214

[28] T. van Ritbergen, A.N. Schellekens and J.A.M. Vermaseren, Int. J. Mod. Phys. A14 (1999) 41, hep-ph/9802376

[29] M. Jamin and R. Miravitllas, Phys. Lett. B779 (2018) 452, arXiv:1711.00787

[30] J. Davies and A. Vogt, Phys. Lett. B776 (2018) 189, arXiv:1711.05267

[31] P.A. Baikov and K.G. Chetyrkin, JHEP 06 (2018) 141, arXiv:1804.10088

[32] P.A. Baikov and K.G. Chetyrkin, PoS LL 2018 (2018) 008, arXiv:1808.00237

[33] J.A.M. Vermaseren, A. Vogt and S. Moch, Nucl. Phys. B724 (2005) 3, hep-ph/0504242

[34] S. Moch, J.A.M. Vermaseren, A. Vogt, Nucl. Phys. B813 (2009) 220, arXiv:0812.4168

[35] B. Ruijl, T. Ueda, J.A.M. Vermaseren, J. Davies and A. Vogt, PoS (LL2016) 071, arXiv:1605.08408

[36] W.L. van Neerven and A. Vogt, Nucl. Phys. B603 (2001) 42, hep-ph/0103123

[37] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, Phys. Rev. Lett. 118 (2017) 082002, arXiv:1606.08659

[38] T. Luthe, A. Maier, P. Marquard and Y. Schröder, JHEP 10 (2017) 166, arXiv:1709.07718

[39] K.G. Chetyrkin, G. Falcioni, F. Herzog and J.A.M. Vermaseren, JHEP 10 (2017) 179, arXiv:1709.08541

[40] L. von Smekal, K. Maltman and A. Sternbeck, Phys. Lett. B681 (2009) 336, arXiv:0903.1696

[41] J.A. Gracey, J. Phys. A46 (2013) 225403, arXiv:1304.5347

[42] B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, JHEP 06 (2017) 040, arXiv:1703.08532

[43] D. Boito, M. Jamin and R. Miravitllas, Phys. Rev. Lett. 117 (2016) 152001, arXiv:1606.06175

[44] T. van Ritbergen, J.A.M. Vermaseren and S.A. Larin, Phys. Lett. B400 (1997) 379, hep-ph/9701390

[45] M. Czakon, Nucl. Phys. B710 (2005) 485, hep-ph/0411261

[46] V.N. Velizhanin, Mod. Phys. Lett. A32 (2017) 1750213, arXiv:1412.7143v3

[47] W.L. van Neerven and A. Vogt, Phys. Lett. B490 (2000) 111, hep-ph/0007362

[48] H. Kawamura, N.A. Lo Presti, S. Moch and A. Vogt, Nucl. Phys. B864 (2012) 399, arXiv:1205.5727

[49] J. Henn, A.V. Smirnov, V.A. Smirnov and M. Steinhauser, JHEP 1605 (2016) 066, arXiv:1604.03126

[50] J. Henn, A.V. Smirnov, V.A. Smirnov, M. Steinhauser and R.N. Lee, JHEP 03 (2017) 139, arXiv:1612.04389

[51] R.H. Boels, T. Huber and G. Yang, Phys. Rev. Lett. 119 (2017) 201601, arXiv:1705.03444

[52] R.H. Boels, T. Huber and G. Yang, JHEP 01 (2018) 153, arXiv:1711.08449

[53] L.J. Dixon, JHEP 01 (2018) 075, arXiv:1712.07274

[54] S. Moch, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, Phys. Lett. B782 (2018) 627, arXiv:1805.09638