Finite size scaling and the effect of the gauge coupling in 12 flavor systems

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Finite size scaling is a powerful tool to study the critical properties of systems governed by one relevant operator, assuming all irrelevant operators have scaling dimensions much smaller then zero. This condition is likely not satisfied in many-fermion conformal systems where perturbation theory predicts a nearly-marginal irrelevant gauge coupling. In this work we carry out a new investigation of SU(3) lattice gauge theory with 12 fundamental flavors. Analyzing data at many different gauge couplings, our preliminary results indicate that a finite size scaling analysis that takes into account the effect of a nearly-marginal gauge coupling can resolve many of the inconsistencies observed previously in this system, leading to results consistent with conformal infrared dynamics and predicting a mass scaling anomalous around $\gamma_m = 0.25$.

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1. Introduction

Asymptotically free gauge theories with many fermionic degrees of freedom can exhibit unusual infrared properties at strong gauge coupling. Some develop a new conformal fixed point with possibly large anomalous dimensions. Others remain chirally broken but their dynamics may be approximately scale-invariant across a wide range of energies and could be candidates for Beyond-Standard Model physics. In either case there is interesting non-perturbative infrared dynamics worth studying. Lattice gauge calculations are particularly suitable to investigate these strongly-coupled systems, and in recent years significant computational resources have been devoted to this effort.

The SU(3) gauge model with $N_f = 12$ fundamental fermions is a controversial system. Several groups have studied the infrared properties of this model using different methods and different lattice actions, arriving at contradictory conclusions regarding its IR dynamics. (For a limited set of references see Refs. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] and the recent review Ref. [16].) Finite size scaling is one of the methods frequently used to study this system [5, 9, 11]. While 12-flavor data appear to obey finite size scaling, recent studies find that different physical quantities predict inconsistent scaling exponents, which suggests that it is not possible to consistently describe all the data assuming conformal dynamics [9, 11].

Finite size scaling techniques provide an effective tool to investigate models governed by a fixed point with only one relevant operator, especially if the irrelevant operators are strongly irrelevant, i.e., their scaling dimensions are much below zero. If this condition is not met, either very large volumes have to be used, or corrections to scaling have to be taken into account. Both perturbation theory and non-perturbative step scaling function calculations predict that in the 12-flavor systems the gauge coupling has very small scaling exponent, $-0.3 \lesssim y_0 \lesssim -0.1$ [17, 1]. In this paper we consider the possibility that some of the inconsistencies found in earlier investigations are due to this nearly-marginal gauge coupling.

In order to investigate the effects of a nearly-marginal irrelevant gauge operator, it is essential to study the system at many gauge coupling values. In this work we cover a wide range from a strong coupling near the onset of the "$\mathcal{S}^4$" lattice phase [6] to as weak coupling as our lattice
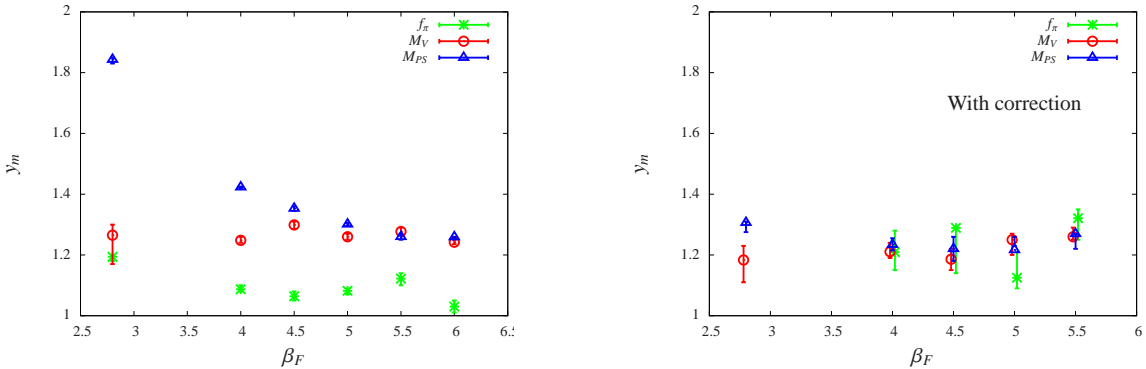
Figure 1: The scaling dimension y_m predicted by finite size scaling, as a function of the gauge coupling β_F for the pseudoscalar (blue triangles), vector (red circles) and f_π (green ×s). Left: fits including only the relevant mass operator. Right: fits including both the relevant operator and leading irrelevant corrections.

volumes allow. We find that finite size scaling using only the leading relevant operator predicts scaling exponents that depend both on the physical quantity considered as well as on the bare gauge coupling, as shown in the left panel of Fig. 1. When we include the corrections to scaling due to the nearly-marginal gauge coupling, our preliminary analysis predicts scaling exponents that are, within errors, independent of the gauge coupling and consistent for the pseudoscalar meson, vector meson, and f_π, as shown in the right panel of Fig. 1.

While we cannot prove that all physical quantities will scale consistently once corrections to scaling are taken into account – especially because these corrections might be more important to some quantities than to others – our results resolve some of the existing controversies of the 12-flavor system and reinforce the IR-conformal interpretation suggested by our earlier studies of the bare step scaling function [3], phase transitions [18] and Dirac eigenvalues [7]. Our finite size scaling results prefer a fairly small anomalous dimension, $\gamma_m^\star = y_m^\star - 1 \approx 0.25$. The statistical errors on $\gamma_m^\star$ are about 10%, with similar systematic uncertainties for the three quantities considered. At this point we cannot give a more precise error estimate, but note that this value is consistent with our findings for $\gamma_m^\star$ from the Dirac operator spectral density [7].

2. Numerical setup

In our numerical studies we use nHYP smeared staggered fermions with smearing parameters $(0.5, 0.5, 0.4)$ to ensure the numerical stability of simulations. Our gauge action contains fundamental and adjoint plaquette terms with $\beta_A/\beta_F = -0.25$ to avoid the potential scaling violation effects known to exist at positive adjoint plaquette coupling. In Ref. [6] we reported on the phase structure and other properties of this action.

In our previous studies we were able to run simulations in the $m = 0$ chiral limit with periodic spatial boundary conditions on volumes as large as $32^3 \times 64$ at gauge couplings up to and within the $\mathcal{S}^4$ phase [18]. In the present work we consider gauge couplings $\beta_F = 2.8, 4.0, 4.5, 5.0, 5.5$ and 6.0 on volumes $16^3 \times 32$, $20^3 \times 40$, $24^3 \times 48$ and $32^3 \times 64$. At the strongest gauge couplings we can also use $12^3 \times 24$ volumes. We choose the bare mass in the range $0.005 \leq m \leq 0.12$, requiring that the vector meson mass $M_V \lesssim 0.8$.

It is instructive to compare our spectrum data with the results published by the Lattice Higgs Collaboration (LHC) in Ref. [4]. We found (rather accidentally) that $\beta_F = 4.0$ in our action matches the LHC $\beta = 2.2$ stout-smeared spectrum very closely, as illustrated in the left panel of Fig. 2. The agreement of both the pseudoscalar and vector meson spectrum as functions of the bare mass suggests that the mass renormalization factors of the two actions are nearly identical. This is not very surprising, given that both actions employ smeared staggered fermions. We have not been able to find similar match with the results of the LatKMI Collaboration [11], who use the Highly Improved Staggered Quark action. The best estimate we can make is that our $\beta_F = 5.5, 6.0$ couplings are close to the $\beta = 3.7, 4.0$ of Ref. [11], but with different mass renormalization factors.

In the right panel of Fig. 2 we show the dimensionless ratio M_V/M_π at different gauge coupling values, as functions of the pseudoscalar mass. In a chirally broken system this ratio diverges as $1/M_\pi$ in the chiral limit and, at least in the scaling regime, different β values can be rescaled with the lattice spacing a to form a unique curve. In a conformal system the ratio should approach a
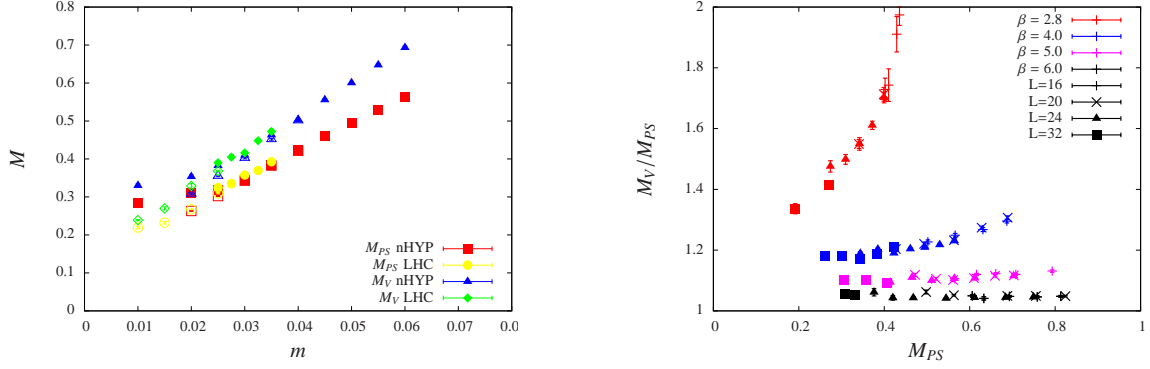
Figure 2: Left: The pseudoscalar and vector meson masses as the function of the bare fermion mass for our nHYP action at $\beta_F = 4.0$, and for the LHC action at $\beta = 2.2$, from Ref. [4]. Only $24^3 \times 48$ (filled symbols) and $32^3 \times 64$ (open symbols) data are included. Right: Ratio of vector and pseudoscalar meson masses as the function of the pseudoscalar mass at various gauge coupling values (nHYP data).

constant value in the chiral limit. At finite mass different β values could predict different ratios due to corrections from the irrelevant operators.

Our data show no indication of divergence of M_V/M_π in the chiral limit; in fact at stronger gauge couplings the ratio monotonically decreases with decreasing M_π. If the $N_f = 12$ flavor system were chirally broken, this indicates that none of our calculations are close enough to the chiral limit to probe spontaneous chiral symmetry breaking. The conformal scenario is more plausible, as the different gauge coupling data could be consistent with a ratio $M_V/M_\pi \approx 1.1$ in the chiral limit. We observe very strong mass dependence at the strongest coupling, $\beta_F = 2.8$, signaling large cut-off effects. This is not surprising as this coupling is near the edge of the $\mathcal{S}^4$ phase. The three other data sets at $\beta_F = 4.0$, 5.0 and 6.0 show considerably weaker mass dependence. At $\beta_F = 4.0$ the ratio still decreases as M_π decreases, at $\beta_F = 5.0$ it is approximately constant, and at $\beta_F = 6.0$ we see a slight increase towards the chiral limit. This qualitative change might indicate a conformal fixed point around $\beta_F^\star \sim 5.0$, in the scheme defined by this observable.

3. Finite size scaling

Finite size scaling is a well understood technique in statistical physics. Its derivation is easiest using renormalization group arguments and has been reviewed recently in connection with infrared conformal systems [19, 20]. Here we summarize only the steps relevant for the scaling of physical quantities "M_H" with mass (engineering) dimension $[M_H] = 1$.

For concreteness consider a system with one relevant operator, denoted by m, that has a scaling dimension $y_m > 0$. All other operators, denoted by g_i, are irrelevant with scaling exponents $y_i < 0$. Renormalization group arguments predict that in a finite spatial volume L^3, M_H depends only on specific combinations of the couplings, and can be written as

$$M_H = L^{-1} f\left(x, g_i m^{-y_i/y_m}\right),\qquad(3.1)$$

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where $x \equiv Lm^{1/y_m}$. In the critical $m \to 0$ limit, $g_i m^{-y_i/y_m} \to 0$ and we find the familiar finite-size scaling formula

$$M_H = L^{-1} f(x), \tag{3.2}$$

where $f(x)$ is an arbitrary but unique scaling function. It is important to note that the scaling function $f(x)$ depends on the observable M_H, but the exponent y_m in the scaling variable x is universal, characteristic of the corresponding fixed point.

If one of the irrelevant operators, let's say g_0, is nearly marginal with scaling exponent $y_0 \lesssim 0$, the term $g_0 m^{-y_0/y_m}$ can remain significant and has to be included in the scaling analysis. This leads to the modified finite-size scaling formula

$$M_H = L^{-1} f(x, g_0 m^\omega), \tag{3.3}$$

where $\omega \equiv -y_0/y_m \gtrsim 0$. The scaling function $f(x, g_0 m^\omega)$ is analytic even at the fixed point, and can be expanded as

$$LM_H = F(x) \left\{ 1 + g_0 m^\omega G(x) + \mathcal{O}\left(g_0^2 m^{2\omega}\right) \right\}. \tag{3.4}$$

The first term is the usual finite-size scaling expression; the second term accounts for leading corrections to scaling due to the nearly-marginal gauge coupling.

In the limit $x \to 0$, both $F(x)$ and $G(x)$ approach finite constants. In the infinite-volume limit, with small but fixed m, $F(x) \propto x$ while $G(x)$ remains finite. Our simulations cover a limited range $0.5 \lesssim x \lesssim 5$, over which we approximate $G(x)$ by a constant, $G(x) = c_G$, so that

$$\frac{LM_H}{1 + c_G g_0 m^\omega} = F(x). \tag{3.5}$$

We test the validity of this assumption by repeating our analyses using subsets of our data restricted to smaller ranges in x. Eq. 3.5 is very similar to the original Eq. 3.2 and can be fitted similarly. However, the analysis now involves three parameters: $c_0 \equiv c_G g_0$, $\omega = -y_0/y_m$ and y_m.

During the completion of this work Del Debbio and Zwicky released Ref. [21], where corrections to scaling due to irrelevant operators are discussed in detail, though only in infinite volume. In the appropriate limits our results agree with Ref. [21].

4. Finite size scaling fits

We begin by considering the relevant operator only, using the usual Eq. 3.2. The left panel of Fig. 3 shows the best curve collapse we found for the pseudoscalar mass at $\beta_F = 4.0$. With $y_m = 1.423$, different volumes form a unique curve for $M_\pi L \gtrsim 8$, but at smaller masses there is a clear mismatch between the different volumes (which does not improve with different values of y_m). To quantify the goodness of the curve collapse we fit the data with two independent quadratic polynomials, one at $x < x_0$ and the other at $x \geq x_0$. We minimize the χ^2 of this fit in terms of x_0 and y_m. The best fit as shown in the left panel of Fig. 3 has $\chi^2/\text{dof} = 6.3$. The left panel of Fig. 1 shows the results of similar analyses at other β_F values, as well as for the vector meson and f_π. The scaling exponents show significant variations between the three observables and as functions of β_F, suggesting that there is no consistent finite size scaling when using the form of Eq. 3.2.
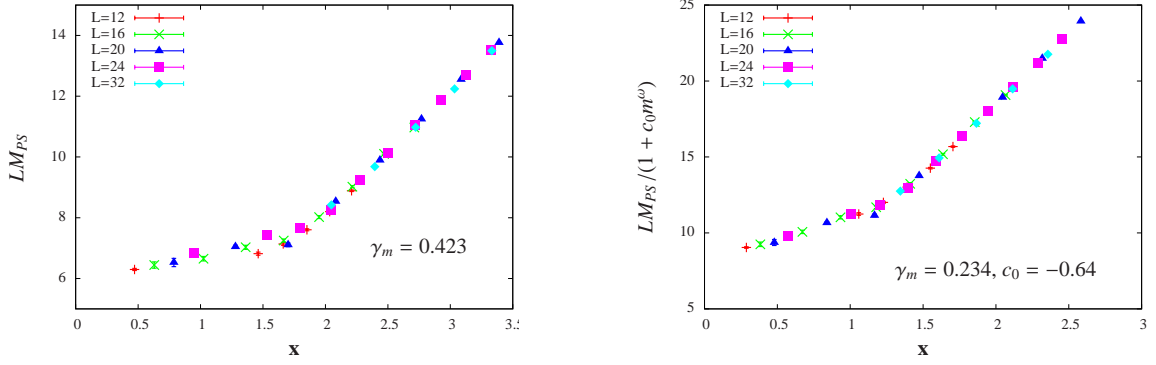
Figure 3: The best curve collapse fits for the pseudoscalar mass at $\beta_F = 4.0$. Both panels show $M_\pi L$ as the function of the scaling variable $x = m^{1/y_m}L$. Left: Fits considering only the leading relevant operator, using Eq. 3.2. Right: Fits taking into account corrections, using Eq. 3.5.

Next we take into account the leading corrections according to Eq. 3.5. We use the same two-polynomial form to fit the left side of Eq. 3.5 and minimize the χ^2 as the function of x_0, c_0 and y_m while keeping y_0 fixed in the range $-0.3 \lesssim y_0 \lesssim -0.1$ [17, 1]. We find very little dependence on y_0 within this range, with slight preference for $y_0 \approx -0.2$. The right panel of Fig. 3 shows the best curve collapse we found for the pseudoscalar mass at $\beta_F = 4.0$ using $y_0 = -0.2$. The corresponding scaling exponent is $y_m = 1.234$ with a correction term $c_0 = -0.64$ and $\chi^2/\text{dof} = 3.1$. We obtain consistent results from fitting only the small- or large-x regions, justifying our approximation of constant $G(x) = c_G$.

Repeating this analysis at other gauge couplings, and for the other two quantities considered, leads to the results in the right panel of Fig. 1, showing consistency between all three operators in the whole β_F range investigated. Unfortunately, the errors are significantly larger with the corrected fit, especially for f_π where the data constrain the correction coefficient c_0 only weakly. To address this we will attempt combined fits of all the data, with a universal y_m and scaling functions $F(x)$ that depend on the operator but not on the gauge coupling. The coefficients c_0 could depend on both the operators and the gauge coupling. This investigation is ongoing and will be presented in a forthcoming publication.

5. Conclusion

We have demonstrated that apparent inconsistencies in finite size scaling analyses of the $N_f = 12$ system can be resolved by considering the effect of the leading irrelevant gauge coupling, at least for f_π and the pseudoscalar and vector meson masses. We find that all three quantities, when considered independently, prefer an anomalous dimension $\gamma_m^\star = y_m^\star - 1 \approx 0.25$. By performing a combined fit to all data used in this work, we hope to strengthen our conclusion and obtain a robust prediction for $\gamma_m^\star$. It will also be important to consider other quantities, such as the string tension and mass of the lightest baryon, but at present we do not have these data available to analyze.

We expect that systems near the conformal boundary will generically possess a nearly-marginal gauge coupling. The initial results presented here suggest that this may have important effects that will need to be investigated in future studies of strongly-coupled many-flavor systems.

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Acknowledgments

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