'Balancing' the 'live' use of resources towards the introduction of the Iterative Numerical method

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This paper draws on the Knowledge Quartet (Rowland, Huckstep, & Thwaites, 2005) to analyse an introductory to the Iterative Numerical Method Year 13 lesson of a secondary mathematics teacher who uses a range of paper based and electronic resources including Autograph, a mathematics-education software. Data were collected during one lesson observation and a follow up interview with the teacher. Analysis identifies the different aspects of the Knowledge Quartet dimensions: foundation, transformation, connection and contingency, in relation to the introduction to the Iterative method and to the teaching of Year 13 students. Findings demonstrate how the teacher used students' contributions as resource for his teaching; how he balanced his use of resources; and how he created connections between these resources while he remained attentive to exam requirements.

Keywords: Knowledge Quartet, foundation, transformation, connection, contingency, resource.

Introduction

For some decades, it was thought that access to more teaching resources (e.g. textbooks, mathematics-education software) meant better teaching practice (Cohen, Raudenbush, & Ball, 2003). However, "more resources do not necessarily lead to better practice" (Adler, 2000, p. 206), and priority should be given to how such resources are used (Adler, 2000; Cohen et al., 2003). This is especially important when digital resources are used as these are thought to add to the complexity of teaching (Clark-Wilson & Noss, 2015), also due to the overwhelming number of resources available online nowadays. Teachers interact with these resources, choose from them and manage them; and "these interactions play a central role in the teacher's professional activity" (Gueudet, Buteau, Mesa, & Misfeldt, 2014, p. 141) and have great impact on students' learning (Carrillo, 2011). Resources influence and are influenced by teachers' professional knowledge, and a close look at the interactions between the two help identify opportunities to develop both (Rowland, 2013). The study presented here explores secondary mathematics teachers' ways of managing and balancing the different resources that influence their work, especially when they use a range of resources including mathematics-education software (Kayali & Biza, 2017). We are interested in how teachers use the available resources in relation to their teaching aims and how they materialise and manage their lessons in the light of the available resources. To reflect on such complex teaching situations, we draw on the *Knowledge Quartet* (Rowland et al., 2005) to analyse lesson observations and post observation interviews. Here, we present analysis of an introductory to the Iterative Numerical Method Year 13 lesson observation and a follow-up interview from one participant. With this analysis, we aim to investigate the resources used by this participant and the characteristics of his work with these resources by using the Knowledge Quartet. In the following two sections we offer an overview of what we consider as resource in this paper, and of the Knowledge Quartet. Then, we present a detailed analysis of selected parts of the lesson and teacher's reflections. We conclude the paper with a discussion on our findings.

Resources

Resources include artefacts, teaching materials, as well as social and cultural interactions that can be used or affect a teacher's teaching and teaching preparation work (Gueudet & Trouche, 2009). An artefact is an object that is designed and used for a specific purpose (Gueudet & Trouche, 2009). It could be a mathematical technique for solving a specific problem, a mathematics-education software or a tool like a pen. Teaching materials are textbooks, calculators, and any other materials designed for teaching mathematics (Adler, 2000). Social and cultural interactions are the interactions with the environment, students and colleagues, for example a student's feedback on an activity (Gueudet & Trouche, 2009). The term "resource" can also be "the verb re-source, to source again or differently" (Adler, 2000, p. 207). This implies that teachers interact with resources, manage them and reuse them to achieve their teaching aims (Gueudet, 2017). And "the effectiveness of resources for mathematical learning lies in their use, that is, in the classroom teaching and learning context" (Adler, 2000, p. 205). In this paper, we closely examine one teacher's use and interaction with resources, in the wider range defined above, using the Knowledge Quartet lens.

Knowledge Quartet

The Knowledge Quartet (Rowland, Huckstep, & Thwaites, 2005) was developed as a tool for analysing teachers' knowledge and beliefs. It can be used to trigger teachers', teacher educators' and researchers' reflections towards the development of mathematics teaching (Rowland, 2013). It is defined by four dimensions: *foundation, transformation, connection* and *contingency* (Rowland et al., 2005). The first dimension, foundation, represents teachers' knowledge and beliefs. This includes their knowledge about mathematics as well as about the teaching and learning of mathematics, and their beliefs about mathematics and its teaching and learning (Rowland, 2010). The codes listed under this dimension look at teachers': *adherence to textbook; awareness of purpose; concentration on procedures; identification of errors; overt subject knowledge; use of theoretical underpinning;* and *use of terminology*. The transformation dimension "concerns the ways that teachers make what they know accessible to learners and focuses in particular on their choice and use of representations and examples" (Thwaites, Jared, & Rowland, 2011, p. 227). Its codes cover aspects such as *choice of examples; choice of representation; demonstration;* and *use of instructional materials*. The connection dimension looks at teacher's choices in terms of their lesson plans, sequence of activities, and their connections between procedures and concepts where possible. Its codes concern teachers' *anticipation of complexity; decisions about sequencing; making connections between procedures; making connections between concepts*; and *recognition of conceptual appropriateness*. Finally, the contingency dimension examines how teachers respond to "unanticipated and unplanned events" (ibid, p. 227). This involves their *responses to unexpected student contributions; deviation from agenda; use of opportunities;* and *responses to the (un)availability of tools and resources* (ibid). The four dimensions of the quartet afford a focused look at the details of teachers' work in class which has great impact on students learning and achieving learning outcomes (Carrillo, 2011). In this paper we use these dimensions to investigate the characteristics of one teacher's use of resources.

Methodology

The study is conducted in East Anglia region of the UK and looks at upper secondary mathematics teachers' work when they employ mathematics-education software along with other resources. It provides qualitative findings established on an interpretative research methodology that values the participant's views and reflections and looks for meanings within the participant's environment (Merriam & Tisdell, 2016; Stake, 2010). Here, we discuss one video-recorded lesson observation (50-minute long) and the audio-recorded follow-up interview of one participant, George, with 15 years of teaching experience mostly in upper secondary education. George was teaching a mixed gender group of Year 13 students (17-18 years old), who were preparing for their school leaving examination. The lesson was on *Numerical Methods*. George started the lesson by introducing the chapter "Numerical Methods" from Wiseman and Searle (2005, pp. 118-135) including four methods: Change of Sign method, Iterative method, Mid-ordinate rule, and Simpson's rule by justifying why teaching this chapter is important. Then, he briefly introduced the four methods, and started explaining the Iterative method in detail using Autograph (a dynamic environment with visualising graphs affordances, see http://www.autograph-maths.com). George then showed his students some past exam questions about the Iterative method and went over their mark schemes. Finally, he asked his students to solve some textbook questions of his choice in the remaining time of the lesson. Later, George was interviewed (by the first author), and he reflected on his teaching of Iterative method and especially on points the first analysis had identified, such as his choices of resources and actions in the lesson (e.g. his aims, justification of his choices). An analysis of George's actions during the lesson and his responses in the interview was performed using the Knowledge Quartet. In the next section, we offer a detailed account of selected parts of the lesson observation, where his work with resources in the class was more visible through his communication with students, and relevant parts from the interview. These parts regard the introduction to Numerical methods and the demonstration of the Iterative method to the students also with the use of Autograph.

Introducing the Iterative method- A look through the Knowledge Quartet lens

Introducing "Numerical methods"

In his introduction to the chapter of "Numerical Methods" and its four methods: Change of Sign method, Iterative method, Mid-ordinate rule, and Simpson's rule; George justified its importance:

> The whole chapter is about things that you can't solve by doing the algebra to them, or just by integrating them, okay. And, it turns out that there are way more functions and things that you can't solve algebraically than there are ones that you can, ok. There are loads of horrible things to integrate that are too difficult to integrate, that it's just easier to do a numerical thing and kind of approximately, a bit like a trapezium rule

George reminded the students that they saw trapezium rule and Simpson's rule in a previous lesson. He explained that Simpson's rule and the mid-ordinate rule were about finding the area and extension of the trapezium rule, while the change of sign method and the Iterative method were about solving equations. Then, he briefly introduced the four methods, and mentioned that he was going to focus on the "Iterative method" and when this lesson's methods are good to use. This introduction reflected George's *awareness of purpose* and how he was addressing *connections* (e.g. numerical methods and trapezium rule). He tried to *connect concepts/procedures*, for example, to connect Simpson's rule and integration used in previous lessons to the methods he was planning to teach in this lesson.

Rearranging an equation – Why $x^2 + 4x + 1 = 0$?

After the introduction to the chapter, George started explaining the method, he mentioned that he needed to use an equation that cannot be factorised but can be solved using the quadratic formula. He spontaneously chose the quadratic equation $x^2 + 4x + 1 = 0$, and asked his students to suggest rearrangements of the equation to the form $x = \cdots$ (e.g. $x = \frac{-x^2}{4} - \frac{1}{4}$). After that, he asked them to find the equation roots using what they previously learned on how to solve quadratic equations or using calculators. Students used calculators and gave the approximate answers "-0.267" & "-3.732". George was spontaneous in his *choice of example*. He chose a quadratic equation with the aim that students would be able to solve it with the formula but not with factorisation; which would show them that the Iterative method worked. His choice was based on *awareness of purpose* which is to show the students that the Iterative method gives them a good approximation of the solutions. George also tried to *connect procedures* when he asked his students to create different rearrangements of the equation. Later, after finding the different rearrangements, George again tried to create a *connection* between the equation's solutions students found on their calculators and the Iterative method would "zoom in on the answers". He was always *responding to students' ideas*, commenting on when two rearrangements were the same and just written differently and on when a student completed the square instead of rearranging the equation. George was *demonstrating* about iteration to students while at the same time inviting them to contribute. These contributions were essential parts in the flow of his lesson, for example, when students suggested the rearrangements and calculated the quadratic solutions. We consider these *students' contributions* as a *resource* (Adler, 2000; Gueudet, 2017) that George triggered and used during the lesson, along with other resources.

The use of Autograph

After working with the students on finding the equation's different rearrangements and solutions, George's next step was to use Autograph (*instructional material*) in order to further explain about the Iterative method and show how the different rearrangements would work. He used Autograph to graph the functions $y = x$ and $y =$ the other side of the rearrangement (e.g. for $x = \frac{-x^2}{4} - \frac{1}{4}$ he graphed $y = x$ and $y = \frac{-x^2}{4} - \frac{1}{4}$). The first rearrangement he graphed on Autograph (Figure 1) was $x = \sqrt{-4x - 1}$ and it did not "work" as George said while pointing at the graphs on Autograph:

> Now, I can see that the blue one [$x = \sqrt{-4x - 1}$] does not quite, I think it almost touches, but not quite touches the red one [$y = x$], okay. So, they in fact don't cross. So, for that particular rearrangement sadly it's not going to work out. So, let's not do that one, let's pick a different one. Which one do you want?

We noticed here one *contingency* incident (*use of opportunity*: (un)anticipated outcome on the software) that happened when George entered the equations from one rearrangement on Autograph and noticed that the graphs of these equations did not cross, in this case he commented that some rearrangements did not work and asked the students to pick a different one. Again, a student's contribution here is a *resource* that George used to demonstrate ideas. In this case, one student picked

$x = \frac{-1}{x+4}$. George graphed $y = x$ and $y = \frac{-1}{x+4}$ on Autograph (Figure 2) and commented that the two graphs crossed this time at two points. Having noted that the two points are close to $x = -0.2$ and $x = -3$, George explained that the Iterative method is based on replacing the x in the denominator of this rearrangement by x_n and replacing the isolated x by x_{n+1}. Thus, he suggested the formula $x_{n+1} = \frac{-1}{x_n+4}$ to be used to complete the table in Figure 3 with a starting value $x_0 = 1$. This shows how George *used different representations* such as tables and graphs; and tried to create *connections between procedures* and also *between different resources* (student's contribution + Autograph).

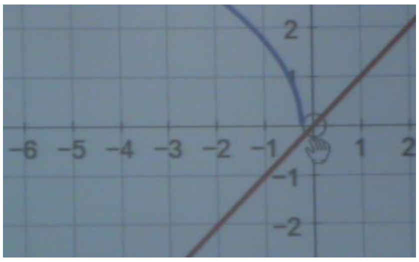

Figure 1: $y = x$ and $y = \sqrt{-4x-1}$ on Autograph

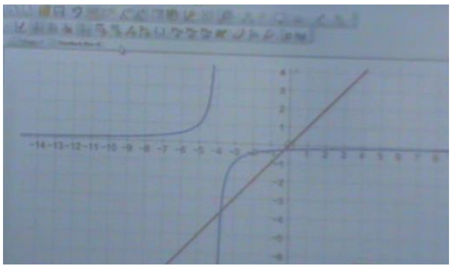

Figure 2: $y = x$ and $y = \frac{-1}{x+4}$ on Autograph

x_n	x_{n+1}
1	$-\frac{1}{5} = -0.2$
$-\frac{1}{5}$	$-\frac{5}{19} = -0.26315$
$-\frac{5}{19}$	$-\frac{19}{71} = -0.2676$
$-\frac{19}{71}$	-0.26769

Figure 3: Table completed using $x_{n+1} = \frac{-1}{x_n+4}$

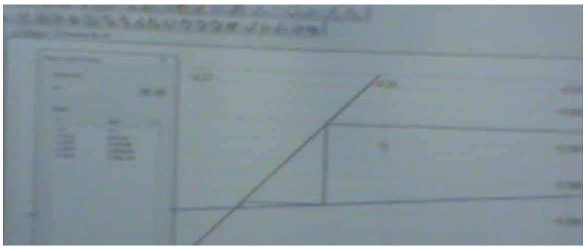

Figure 4: Autograph's Staircase diagram for $x = \frac{-1}{x+4}$

After showing the Iterative method procedure on the board, George decided to show how the Iterative method works on the graph by using Autograph (Figure 4):

> Look at this you started with number 1, okay. The number 1 went, which way did it go? It's gone down to the curve, across to the line and that was our 0.2 [sic]. And then, here look at that down to the curve across the line, down to the curve if I zoom in further oh missed it. Down to the curve and across to the line. Zooming in further, look at that! And we could keep on going down to the curve, across to the line zoom in zoom in zoom in there. It is, look at that, and the thing if I zoom out again looks like a staircase kind of. So, this is staircase diagram.

This use of Autograph showed teacher's *knowledge about the software* affordances and its use. It also showed an attempt to *connect different representations* (staircase diagram + formula). George then used $x_{n+1} = \frac{-1}{x_n+4}$ with different starting points for x_n on Autograph. When he used $x_0 = 10$ as a starting point, the software returned "overflow", namely no convergence, and he changed the point to $x_0 = -10$ without commenting on the outcome. He did not *use* this *opportunity* to explain what that meant during the lesson. One student asked what would happen if $x_n + 4 = 0$ in $x_{n+1} = \frac{-1}{x_n+4}$. At that point George used the graph on Autograph to show that this would not work, as he commented:

> You know like when you know your tan graph at 90 degrees, that 90 degrees on a tan graph is effectively saying we've got an infinitely tall triangle which wouldn't be a triangle at all, because it would be at 90 degrees. You're talking two parallel lines that's why tan of 90 doesn't work.

This is a *contingency* moment (*responding to students' ideas*) where George used Autograph as *instructional material* and *connected concepts* students had met previously (dividing by zero, tan of 90 degrees, parallel lines, asymptotes of the graph of the tangent line) to respond to student questions.

How to zoom on to a specific root?

Afterwards, George worked on different formula rearrangements and different starting points to show how some rearrangements "work" (when the two functions cross at two points) and some do not (when the two functions do not cross). He also wanted to show how a rearrangement that "works" can lead to one root or to different roots of the equation, when different starting values (x_0) are used. But, until this point, his work on different rearrangements with different starting points was leading to the same root. This made the students question how they could zoom to a specific root and how they could tell if they were going to zoom in on a certain root in the exam. George said "I don't know the answer to this" but he promised to comment on this question later during the lesson (not deviating from the *lesson's agenda*). He returned to this point later when he invited students to practice on past-exam questions by saying that exam questions would provide both rearrangement and initial point. His response reflected his attention to *exam requirements* and their importance to his teaching. After that conversation about how to zoom to specific roots, a student suggested the use of calculators to do the Iterative method. The teacher approved the student's use and method of use of calculators by commenting that it was a "really quick way". This is an example of him *responding to the availability of resources* (student's contribution + calculator). The use of calculators facilitated the zooming in on the other root, as this way was quicker students tried different starting points and found out that x=3 would lead to the second root (-3.732). George used x=3 on Autograph and showed the students how this number worked for the purpose of zooming in on to the other root. This is an example of *connections between resources* (student contribution +calculator +Autograph+ staircase diagram).

George's reflections

In the interview, George was invited to reflect on three aspects. First, about his spontaneous choice of equation, he said he preferred working "live" in the class over showing the students "a specific and special example". He added that he was happy to work slowly with different examples to show that not all examples work. George's spontaneous choice of equation reflected his confidence to *deviate from his agenda* which could create more contingent moments during the lesson. His purpose was to "show" the students that not all rearrangements "work out". Second, about what the "overflow" means on Autograph, he said it indicated that the Iterative method "instead of converging, it will diverge" at that point. During the lesson, George did not *use* the *opportunity* to explain the meaning of this outcome to the students. His focus was on getting a number and comparing this number to one of the equation roots students had found. Third, he was asked about the use of calculators. He said:

> That's what they need to do in the exam. So, the Autograph is just to explain, show, demonstrate it in a lesson. When they then move onto the calculators, that's them saying yeah they are happy with it, they understand it, and this is what they then need to do. This is how they do it in the exam. So, they have to use a calculator and they have to show how they use that, that's what they do on exam day today in fact.

George's comment on the use of calculators reflected again his considerations of *exam requirements* while at the same time his attention to *connections between resources*.

Discussion and conclusion

In this study we are interested in how teachers, in our case George, use the available resources in relation to their teaching aims and how they materialise and manage their lessons in the light of the available resources and we used the Knowledge Quartet lens to analyse teacher actions and justification of these actions. In our analysis, we noticed aspects of George's foundation knowledge including his awareness of purpose, use of terminology, identifying errors, concentration on procedures, his subject knowledge as well as his knowledge about the mathematics-education software (Autograph here) and exam requirements. When a student asked him a question he could not answer, he responded that students were not required to know the answer for the exam, so he could not answer the question because it was "beyond exam requirement". This indicates how important the exam requirements are in his teaching priorities. In his transformation of ideas, he was spontaneous in his choice of "live" examples and representations, and he seemed confident in these spontaneous choices. His use of instructional materials included the use of Autograph, textbook, past-exam papers and calculators. He used the textbook as source of exercises for classroom practice. He also asked for students' contributions and used these contributions as an essential resource in the flow of his teaching also in connection to other resources and instructional materials including Autograph. For him, students' contributions were predictable, as they were within a range of possible answers (like choosing one rearrangement of equation). He made decisions about sequencing the different parts of the lesson and frequently connected concepts and procedures. He was responding to students' ideas and sometimes using these as opportunities to create connections between concepts. In terms of contingency, George was responding to students' ideas and to the availability of tools and resources, and sometimes deviated slightly from his lesson's agenda without losing his overall aim. We distinguish the students' ideas under the contingency dimension from the students' contributions under the transformation dimension, as these are unpredictable students' ideas that George did not directly invite. While students' contributions are a deliberate result of George questioning students and engaging them in his demonstration of mathematical ideas, thus, we see them as a characteristic part of his teaching. As we consider students' contributions as resources used frequently by George, we suggest an extension of the code "*use of instructional materials*" that includes the use of resources. We recommend the code to be rephrased to "*use of resources*" to involve both instructional materials and students' contributions under the transformation dimension. We also saw George connecting between different resources he used (e.g. Autograph, textbook and student contributions). We recommend the code "*connection between resources*" to be added to the connections dimension. These suggestions are established on the episode included in this paper as well as other episodes with George. Overall, our standpoint is that the use of resources is intertwined with teachers' aims and actions (Gueudet & Trouche, 2009). The analysis of George's lessons, as well as of other lessons in this study, indicate that this use is a 'balancing' act on resources that become 'live' in their lessons. These resources go beyond textbooks or other digital (or not) materials and include students' contributions as well, which teachers use, balance and connect in their work.

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