Reconstruction of the Reflection Coefficient and Interface in Homogeneous Medium by Means of Gaussian Jets

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Abstract

This paper is devoted to the inverse problem of reconstruction of a shape of interface separating two homogeneous media in acoustic approximation from from the knowledge of the scattered field data. It is assumed that the infinitely smooth surface representing the interface is illuminated by an incident Gaussian jet described as a high-frequency non-stationary localized asymptotic solution (wave package). The parameters of the medium above the interface are known. Measuring the intensity of the reflected Gaussian jet along a horizontal line placed at some height above the interface gives the inverse data to solve the problem of reconstruction of a shape of interface as well as determination of velocity of wave propagation and density below the interface. In the paper we describe a corresponding algorithm of solving the inverse problem and demonstrate a few examples of its numerical testing.

1. Introduction

The reconstruction of surfaces of objects, diffraction gratings and interfaces separating layers in layered structures is well-known inverse problem in imaging of radar targets [1], optics [2], ground penetrating radar [3], non-destructive testing of surface control [4] and in many other branches of applied sciences, see e.g. [5]. One of the examples is the problem of reconstruction on interfaces inside Earth crust which to some extent may be approximated as a layered structure. In this paper, we study the inverse problem of reconstruction of a shape of interface separating two homogeneous media in acoustic approximation. It is assumed that the interface is an infinitely smooth surface. We offer a new algorithm of reconstruction of the shape of interface based on using a Gaussian jet (Gaussian beam). The algorithm is sufficiently simple, effective and robust comparing with the existing methods of solving this inverse problem [2], [6], [7], [8]. It depends less on the need for *a priori* information about the situation under investigation. This is due to unique properties of a Gaussian jet which is described as high-frequency asymptotic solution of the wave equation

[9], [15]. The Gaussian beam method has a wide bibliography of its application in optics[14], elasticity [10] and geophysics [11], [16]. What is more important that it is an asymptotically localized solution (see [12], [13]). It propagates as a wave packet along a fixed ray. Thus, it concentrates in the vicinity of asymptotically narrow tube containing this ray as a central and decay exponentially away from it. Moreover, it is a causal signal, it has an initial front, and then, its amplitude normally decays exponentially behind the front. Such an asymptotic solution effectively describes laser beams in optics. We assume that the upper side of the interface is illuminated by an incident Gaussian jet propagating through the upper layer. Measuring the intensity of the reflected Gaussian jet along a horizontal line placed at some height above the interface gives the inverse data to perform the reconstruction of the shape of interface. Because of the localization of a Gaussian jet, only an asymptotically small speckle is illuminated by an incident beam at the interface surface. The inverse data will be provided by a localized reflected beam, and thus, they will be localized in space and time. The spatial localization takes place in the asymptotically small vicinity of intersection of the reflected beam and the observation line. The localization in time is due to the presence of the initial front of the incident Gaussian jet. Thus, the inverse data will contain the information about the geometrical local properties of the illuminated speckle of the interface. In addition, analyzing the amplitude of the reflected beam, this algorithm is able to give the velocity of the wave propagation and density in the low medium.

It is worth noting that the algorithm works when the inverse data were collected near to caustics where the family of rays becomes singular. This is an important advantage of the described high-frequency approach to the inverse problem as the traditional stationary or space-time ray methods are not applicable near to caustics.

Although, in this paper we illustrate the power of the method in 2D case of interface between two homogeneous media, the description of algorithm may be easily generalizes both to 3d case and inhomogeneous media. This may be modified to a case of layered structures with finite number of interfaces separating inhomogeneous media. The algorithm was implemented into the computer code reconstructing an interface. We demonstrate a few numerical tests to illustrate the ability of the algorithm. The code proved to be quite fast and stable.

2. Reflection of Gaussian jet

In this section we briefly describe asymptotic solutions of the incident and reflected from interface Gaussian jets. Consider the following 2D problem of scattering from interface separating two homogeneous half-planes. The wave field satisfies the wave equation

$$u_{tt} - c^2 \Delta_{x,z} u = 0, \tag{1}$$

and the interface $z = z(x)$ continuity conditions

$$[u]\Big|_{z=z(x)} = 0, \qquad \left[\frac{1}{\rho}\frac{\partial u}{\partial n}\right]\Big|_{z=z(x)} = 0, \tag{2}$$

where c is the velocity of wave propagation, ρ is density, $\Delta_{x,z}$ is the Laplace operator with respect to x, z. Both c and ρ are constants, and

$$c = \begin{cases} c_-, & z < z(x), \\ c_+, & z > z(x), \end{cases} \qquad \rho = \begin{cases} \rho_-, & z < z(x), \\ \rho_+, & z > z(x). \end{cases}$$

The high frequency asymptotics of the total field in the upper half-plane is a sum of the incident and reflected wave

$$u(t, M) = u_i(t, M) + u_r(t, M), \quad z < z(x);$$

in the lower half-plane is the only transmitted wave

$$u(t, M) = u_t(t, M), \quad z > z(x).$$

The incident wave is the Gaussian jet given by (see [12])

$$u_i(t, M) = A(t - \frac{s_1}{c_-})\sqrt{\frac{c_-}{Q_1(s_1)}} \exp\left\{ip\Big(\theta_0(t - \frac{s_1}{c_-}) - \frac{1}{2}\theta_0'(t - \frac{s_1}{c_-})\Gamma_1(s_1)n_1^2\Big)\right\}\Big(1 + O(p^{-1/2})\Big), \tag{3}$$

where p is a large parameter (carrying frequency). We assume that the eikonal function $\theta_0(t)$ is given. For simplicity, in the numerical test below $\theta_0(t)$ is taken in the form

$$\theta_0 = -t.$$

It is very important that inequality $\theta_0'(t - \frac{s_1}{c_-}) < 0$ always holds true. The amplitude of the Gaussian jet is

$$A(t) = 4\alpha t e^{-\alpha t}, \quad t > 0, \qquad A(t) = 0, \quad t < 0.$$

so that the corresponding signal is causal. The description of Gaussian jet is based on usage of so-called Fermi coordinates s_1 and n_1 introduced through the formula

$$\mathbf{r_M} = \mathbf{r}(s_1) + \mathbf{e}(s_1)n_1,$$

where $\mathbf{r_M}$ is the radius-vector of the observation point M lying in the asymptotically small strip based on the central ray given by $\mathbf{r}(s_1)$ (see [12]). Here s_1 is the arc length measured along the central ray from a fixed point, $\mathbf{e}(s_1)$ is the unit vector orthogonal to the central ray, n_1 is the corresponding distance along $\mathbf{e}(s_1)$. In the homogeneous medium all rays are straight lines. The property of the Gaussian jet being asymptotically localized solution is due to the fact that the imaginary part of the function

$$\Gamma_1(s_1) = \frac{P_1(s_1)}{Q_1(s_1)} \tag{4}$$

is always positive and never vanishes [12]. The functions $Q_1(s_1)$ and $P_1(s_1)$ satisfy the so-called the system of ODE in variations

$$\dot{Q}_1 = c_- P_1, \qquad \dot{P}_1 = 0. \tag{5}$$

We choose its solutions in the form

$$P_1(s_1) = id, \qquad Q_1(s_1) = 1 + ic_- ds_1,$$

thus, ensuring that $\mathrm{Im}\Gamma_1(s_1) > 0$ and $Q_1(s_1) \neq 0$ (see [12]). Here d is a parameter which is responsible for the width of the Gaussian jet.

The reflected Gaussian jet asymptotics is given by

$$u_r(t, M) = A(t - \frac{s_1^* + s_2}{c_-})\sqrt{\frac{c_-}{Q_2(s_2)}}R$$

$$\exp\{ip(\theta_0(t - \frac{s_1^* + s_2}{c_-}) - \frac{1}{2}\theta_0'(t - \frac{s_1^* + s_2}{c_-})\Gamma_2(s_2)n_2^2\}(1 + O(p^{-1/2})), \tag{6}$$

where s_2 and n_2 are the arc length and normal distance corresponding to the reflected ray, s_1^* is the length of the incident ray measured at the specular point. The function $\Gamma_2(s_2)$ has the same form as $\Gamma_1(s_1)$

$$\Gamma_2(s_2) = \frac{P_2(s_2)}{Q_2(s_2)},$$

where

$$P_2(s_2) = -id - \frac{2k}{c_- \cos\varphi}(1 + c_- ids_1^*),$$

$$Q_2(s_2) = c_- s_2 \left[-d - \frac{2k}{c_- \cos\varphi}(1 + c_- i d s_1^*) \right] - 1 - c_- i d s_2^*$$

are solutions of (5) (see [12]), where k is the curvature of the interface at the specular point. The reflection coefficient is given by the formula

$$R = \frac{1 - \frac{\rho_- c_- \cos\psi}{\rho_+ c_+ \cos\varphi}}{1 + \frac{\rho_- c_- \cos\psi}{\rho_+ c_+ \cos\varphi}},$$

in which φ and ψ are the angles of incidence and refraction. They obey the Snell's law

$$\sin\psi = \frac{c_+}{c_-} \sin\varphi.$$

We omit here the presentation of the transmitted Gaussian jet asymptotics as we do not use it in the reconstruction procedure described in the next section.

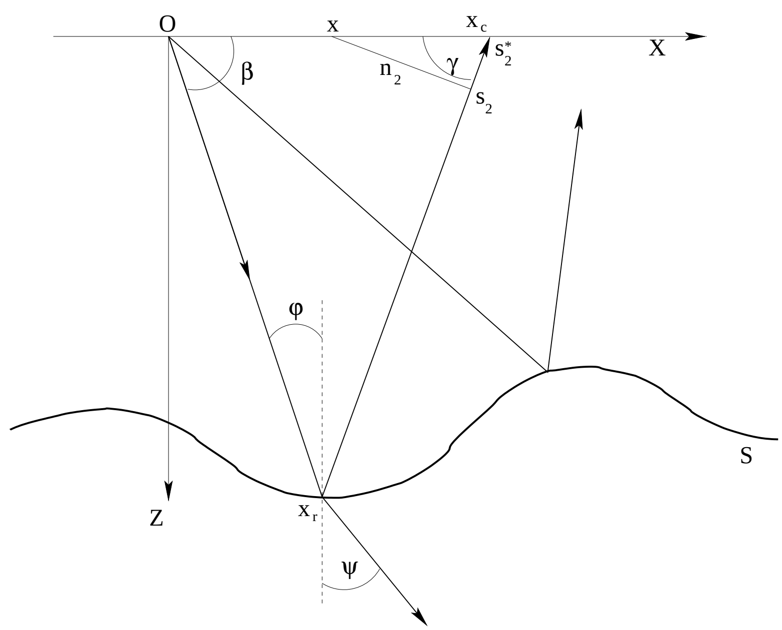


Figure 1: Refection of the Gaussian jet from the interface S

3 Determination of shape of interface by means of Gaussian jet

Consider the algorithm of reconstruction of the interface separating two homogeneous media. Measuring the intensity of the reflected Gaussian jet along a horizontal line placed at some height above the interface gives the inverse data to perform the reconstruction. Let the central ray of the reflected beam intersects the observation line $z = 0$ at the intersection point $(x_c, 0)$. Analyzing the the magnitude of the reflected jet at

$z = 0$,

$$|u_r|(x,t) = A(t - \frac{s_1^* + s_2}{c_-})\sqrt{\frac{c_-}{|Q_2(s_2)|}}|R|$$

$$\exp\{\frac{p}{2}\theta_0'(t - \frac{s_1^* + s_2}{c_-})\mathbf{Im}\Gamma_2(s_2)(\Delta x \sin\gamma)^2\}(1 + O(p^{-1/2})), \tag{7}$$

where

$$\mathbf{Im}\Gamma_2(s_2) = \frac{d}{\left|Q_2(s_2)\right|^2} = \frac{d}{\left(1 + \frac{2s_2 k}{\cos\varphi}\right)^2 + d^2 c_-^2\left(s_1^* + s_2 + \frac{2s_1^* s_2 k}{\cos\varphi}\right)^2},$$

for a fixed angle of incidence, we will show that we are able to reconstruct approximately the shape of the interface $z = z(x)$ in a small neighborhood of specular point as well as parameters c_+, ρ_+. Here, we introduced the horizontal distance between the observation and intersection points $\delta x = x - x)c$ and γ is the intersection angle, see Fig. 1. The procedure may represented as a sequence of the following steps:

1. Analysing data $|u_r|(x,t)$, we determine x_c, and the time of arriving of the maximum $|u_r|(x_c,t)$. Thus, we obtain the travel time $s_1^* + s_2^*$ of our pulse, where s_2^* is the length of the reflected ray between the specular and intersection points (see Fig. 1),

2. From the geometry of the ellipse with unknown semi-axis a and b we have

$$2a = s_1^* + s_2^*, \qquad 2c = x_c, \qquad b = \sqrt{a^2 - c^2}.$$

Thus, knowing the incidence angle β, we determine the coordinates of the specular point $x_r,\ z_r$ as well as $s_1^*,\ s_2^*,\ \gamma$ (see Fig. 1) and

$$c_1 = z'(x_r).$$

3. Evaluating $|u_r|(x,t)$ at x_c and at some different value of x such that $|x - x_c| = O(p^{-1/2})$, we obtain from (7) that

$$\frac{|u_r(x,t)|}{|u_r(x_c,t)|} = \exp\left[-\frac{pd(\Delta x \sin\gamma)^2}{2(y^2 + (dc_-(s_2^* + s_1^* y))^2)}\right](1 + O(p^{-1/2})),$$

where a new variable

$$y = 1 + \frac{2s_2^* k}{\cos\varphi}.$$

If the ration

$$\frac{|u_r(x,t)|}{|u_r(x_c,t)|}$$

is known from measurements, then, we come to an approximate quadratic equation for unknown y

$$y^2(1+(s_1^* dc_-)^2) + 2ys_1^* s_2^* d^2 c_-^2 + (s_2^* dc_-)^2 + \frac{2}{pd(\Delta x \sin\gamma)^2} \ln \left|\frac{|u_r(x,t)|}{|u_r(x_c,t)|}\right|^{-1} = 0.$$

We have two roots $y = -p \pm \sqrt{p^2+q}$ of the equation $y^2 + 2yp - q = 0$, for positive and negative curvature, correspondingly, where

$$p = \frac{s_1^* s_2^* d^2 c_-^2}{1+(s_1^* dc_-)^2}, \qquad q = -\frac{(s_2^* dc_-)^2 + \frac{2}{pd(\Delta x \sin\gamma)^2} \ln \left|\frac{|u_r(x,t)|}{|u_r(x_c,t)|}\right|^{-1}}{1+(s_1^* dc_-)^2}.$$

Determination of the sign of the curvature is based on the fact that spurious root y changes with d.

Having determined y, we are able calculate the curvature k at the specular point. Thus, we obtain

$$c_2 = z''(x_r) = k(1+c_1^2)^{3/2},$$

and the shape of the interface in a small neighborhood of specular point may be approximated by a quadratic spline

$$z = z_r + c_1(x - x_r) + \frac{1}{2}c_2(x - x_r)^2 + \dots .$$

4. Having determined curvature k, we are able to calculate $|Q_2(s_2^*)|$. Thus, measuring $|u_r|(x_c,t)|$, from (7) for various φ we obtain the corresponding values of $|R|$. Denote by A the ratio

$$A = \frac{\rho_- c_- \cos\psi}{\rho_+ c_+ \cos\varphi}.$$

Knowing $|R|$ $(0 < |R| < 1)$, we may calculate A as

$$A_1 = \frac{1-|R|}{1+|R|}, \qquad \frac{dA_1}{d|R|} < 0,$$

or

$$A_2 = \frac{1+|R|}{1-|R|}, \qquad \frac{dA_2}{d|R|} > 0,$$

however, we do not know which way is correct. First, we determine c_+. As the absolute value of the logarithmic derivatives of $A_{1,2}$ with respect to φ are the same, then, using the identity

$$\frac{d}{d\varphi} \ln A = \tan\varphi \left(1 - \frac{c_+^2}{c_-^2} \frac{\cos^2\varphi}{\cos^2\psi}\right),$$

we derive that

$$\cot\varphi \left|\frac{d}{d\varphi} \ln A\right| = \frac{\left|1 - \frac{c_+^2}{c_-^2}\right|}{1 - \frac{c_+^2}{c_-^2}\sin^2\varphi}.$$

Measuring $|u_r|(x_c, t)$ for two different $\varphi_{1,2}$, we obtain a known value for the ratio

$$\frac{1 - \frac{c_+^2}{c_-^2}\sin^2\varphi_1}{1 - \frac{c_+^2}{c_-^2}\sin^2\varphi_2},$$

which gives c_+, and correspondingly, $\psi_{1,2}$. Now, determination of sign of $\frac{dA}{d|R|}$ is possible from the identity

$$\frac{1}{A}\frac{dA}{d|R|} = \left(\frac{d|R|}{d\varphi}\right)^{-1}\tan\varphi\left(1 - \frac{c_+^2}{c_-^2}\frac{\cos^2\varphi}{\cos^2\psi}\right),$$

thus, allowing us to make right choice between A_1 and A_2 ($\frac{d|R|}{d\varphi}$ is taken from measurements). From correctly evaluated A, knowing c_+ we obtain ρ_+.

4. Numerical results

In this section we present the results of testing numerically the described above algorithm of reconstruction of the shape of interface. The algorithm of reconstruction was implemented into the computer code in the first part of which we generate the inverse data by solving the forward problem thus providing the data of the intensity of the reflected Gaussian jet on the horizontal line ($z = 0$), see Fig. 1. In the second part of the code by changing the direction of the incident Gaussian beam, and thus, obtaining a discreat set of specular points, we perform reconstruction of the interface surface locally for every small neighborhood of specular point by means of quadratic spline. Then, we assemble all these local quadratic approximations together, thus, providing approximation for the large part of the interface surface. For other large parts of the interface reconstruction we should translate horizontally the incident Gaussian jet to the right or left by finite steps, and repeat the described above procedure. In this way we will be able to reconstruct the entire part of the interface surface we need to know. The translation of the incident Gaussian beam is important as the reconstruction goes well for normal or oblique incidence of the beam with respect to the illuminated part of the surface thus avoiding grazing incidence when our asymptotic solution for the reflected jet is not valid.

The reconstructed interfaces in Fig. 2,3 are presented to compare with the corresponding original ones. Here the results are given for the 2D case with coordinates (z, x), and for the part of the linear shape

$$z(x) = 3.5 + 0.1x$$

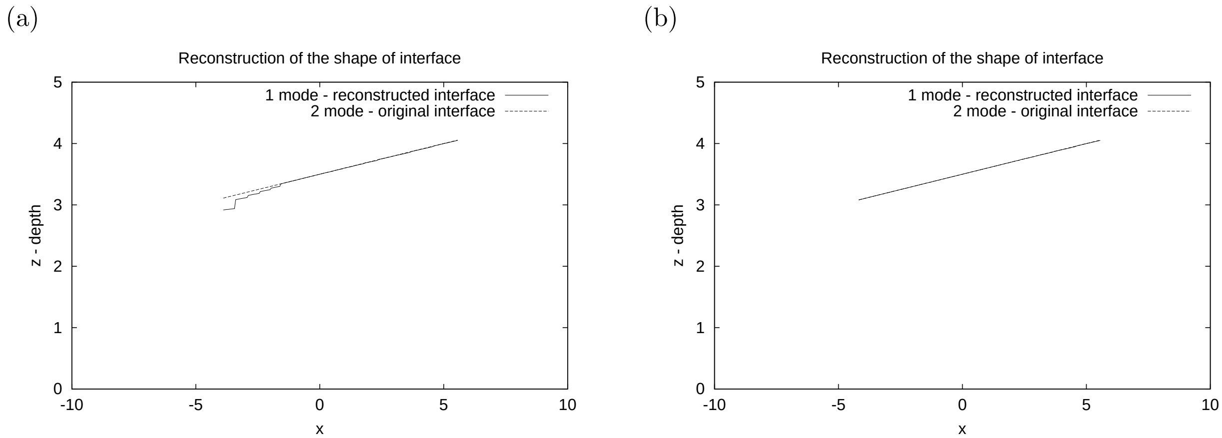


Figure 2: Reconstruction of the shape of interface with $p = 10Hz$ - (a) and $p = 50Hz$ - (b) against the original shape given by $z = 3.5 + 0.1x$.

with frequencies $p = 10, 50$ (see Fig. 2); and for the part of the interface

$$z(x) = 3.5 + 0.1x + 0.2\sin x + 0.04\sin 2x$$

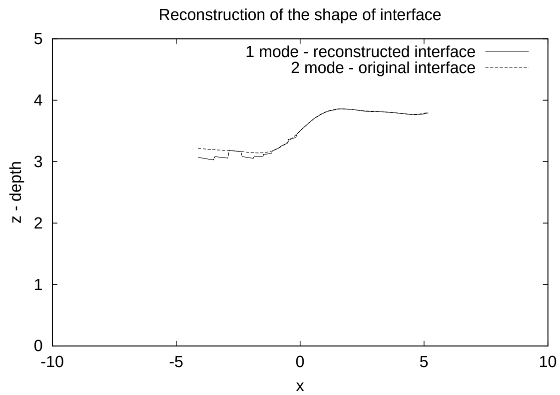
with frequencies $p = 50, 200$ (see Fig. 3). These graphs were computed with the following values of the parameters of the inverse problem $c_- = 2km/sec$, $c_+ = 2,5km/sec$, $\rho_+/\rho_- = 2$, $\alpha = 10$, $d_1 = 0.1$, $d_2 = 0.2$.

It may be seen that we have a good agreement between both types of data. A small discrepancy could be seen in Fig. 2(a) and Fig. 3(a) as the chosen value of the large parameter p becomes not sufficiently large for the corresponding interface to be reconstructed perfectly. This effect takes place as the Gaussian jet for these values of p is not sufficiently localized. So, the Fig. 2(b) and Fig. 3(b) show that by increasing the value of p we improve the quality of reconstruction of the interface.

5. Conclusion

We have described the new method of the reconstruction of the shape of interface in 2D case between two homogeneous media in acoustic approximation and developed a corresponding numerical algorithm. The demonstrated method of solving the inverse problem proved to be efficient in computer analysis. This method may be generalized to solve the corresponding inverse problem for 3D case and for inhomogeneous media. From acoustic approximation the method can be generalized for electrodynamic optics and theory

(a)



(b)

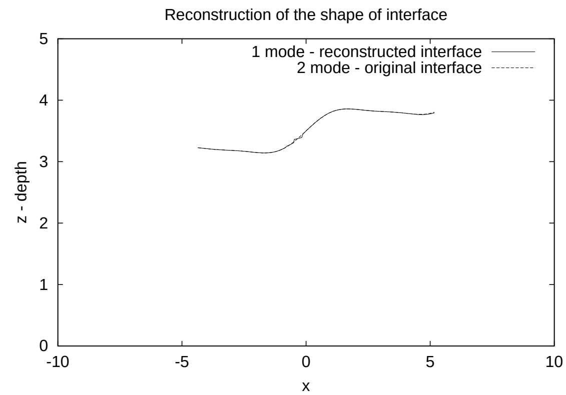


Figure 3: Reconstruction of the shape of interface with $p = 50Hz$ - (a) and $p = 200Hz$ - (b) against the original shape given by $z = 3.5 + 0.1x + 0.2\sin x + 0.04\sin 2x$.

of elasticity.

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