Unsteady undular bore transition in non-integrable dispersive wave dynamics

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Abstract

A complete set of conditions describing the transition across the dissipationless undular bore (dispersive shock) is derived for nonlinear weakly dispersive conservative systems that generally are not integrable by the inverse spectral transform method. In the absence of integrable structure, we postulate modulation description of the undular bore with the aid of the averaged Whitham equations complemented by the Gurevich-Pitaevskii type natural boundary conditions. The main assumption used is that of the hyperbolicity of the Whitham system. The undular bore transition conditions are obtained in a general form by finding a set of integrals available for the similarity reductions of the Whitham systems regardless of the existence of the Riemann invariants. The obtained set of conditions can be viewed as a "dispersive" replacement of the classical shock conditions and allow one to fit an *unsteady* undular bore into the solution of the ideal dispersionless equations. We apply the obtained general conditions to (integrable) Kaup-Boussinesq shallow-water system and to (non-integrable) system describing fully nonlinear ion-acoustic waves in collisionless plasma. A complete agreement with previous analytical and numerical solutions is demonstrated.

1 Introduction

Undular bores (dispersive shocks) represent regions of rapidly oscillating nonlinear wave structures generated in the breaking profiles of evolving large-scale perturbations in dispersive media. The phenomenon of the undular bore formation is quite ubiquitous and its physical contexts range from gravity water waves and bubbly fluid dynamics to space plasma physics and fibre optics. From the very beginning, one should distinguish between two different types of undular bores depending on the role of dissipation relative to that of nonlinearity and dispersion in the bore development and evolution.

Weakly dissipative undular bores despite their oscillatory structure exhibit global properties characteristic for classical, turbulent bores or shock waves: they have steady (though oscillatory in space) profile and constant width such that the speed of the bore propagation and the transition conditions can be derived within the frame of the classical theory of hyperbolic conservation laws (see [1] for instance). The qualitative theory of such undular bores has been first developed by Benjamin and Lighthill [2] for shallow-water waves and by Sagdeev [3] for rarefied plasma flows (where they are called collisionless shocks). The quantitative description of the weakly dissipative undular bores has been made in [4], [5], [6] on the basis of the unidirectional KdV-Burgers equation and in [7] using the integrable version of bi-directional Boussinesq equations modified by small viscous term.

The developed in [2] – [7] theory of steady undular bores, however, is valid only for the fully established regime when nonlinearity, dispersion and dissipation are in balance. Contrastingly, in the *dispersion-dominated* case, the traditional analysis of the mass, momentum and energy balance across the undular bore transition can not be applied (at least directly). The reason for that is that *the boundaries of the dissipationless undular bore diverge with time*, i.e instead of a single shock speed say c defined by the mass balance one has now two different speeds $s_1 > s_2$ determining motion of the undular bore boundaries. These speeds, however, can not be obtained without the analysis of the nonlinear oscillatory structure of the undular bore because in the conservative dynamics, dispersion not only dramatically modifies the fine structure of the flow in the bore region but also, along with nonlinearity, determines the undular bore location.

The predominantly dispersive dynamics of undular bores is of a considerable interest on its own and also, in many cases, can be viewed as an unsteady intermediate asymptotic in a general setting when the small dissipation is taken into account. Physical examples of such expanding undular bores include atmospheric undular bores (morning glory) [8], optical shocks in the long-distance optical communication systems [9], [10], and nonlinear oscillatory structures appearing in the profiles of powerful radio wave beams propagating in the lower Ionosphere [11].

In a weakly nonlinear case when the original conservative system can be approximated by one of the exactly integrable equations, the studies of the unsteady dispersive shock phenomenon have lead to the discovery of a whole new class of mathematical problems which can be broadly described as the singular semi-classical limits in integrable systems. Lax and Levermore [12] and Venakides [13] (see also review [14]) showed that the initial-value problem for the KdV equation reduces in the semi-classical limit to a certain natural boundary-value problem for the corresponding Whitham equations originally obtained by averaging over nonlinear wavepackets [15] [16]. The direct reformulation of the problem in terms of the modulation dynamics had been proposed much earlier by Gurevich and Pitaevskii [17] on the basis of a more universal, albeit more heuristic, reasoning and then has been put in the context with general theory of integrable equations of hydrodynamic type by Dubrovin an Novikov [18].

The modulation approach of Gurevich and Pitaevskii has been extended to other integrable equations such as nonlinear Schrödinger (NLS) equation ([19], [20], [10]), derivative NLS [21], Benjamin-Ono equation [22] and Kaup-Boussinesq system [23]. The characteristic feature of all cited works is that, owing to the integrable nature of the problems considered, the determination of the oscillatory zone boundaries along with any other global characteristics of the flow was naturally viewed as an intrinsic part of constructing the exact solution to the averaged Whitham equations.

The Whitham equations are known to inherit the integrability property from the original, dispersive system and can be represented in this case in a diagonal, Riemann form (see [18], [24] for instance). From this perspective, extracting the transition conditions from the already obtained global solution for a specific system would not have much sense. However, in many physically relevant cases the nonlinear wave propagation is described by non-integrable systems. The most typical example is provided by fully nonlinear dispersive waves, which are far less explored analytically (see for instance the discussion in [25]). In a non-integrable case, the Riemann form for the Whitham equations is normally not available, and in this context the possibility of determination of the undular bore transition conditions acquires an essentially different significance since it allows one to put the entire problem of the dissipationless undular bore dynamics in the classical hydrodynamic setting, when the shock wave is "fitted" into the solution of the Euler equations of ideal hydrodynamics. In the conservative weakly dispersive wave dynamics the role of the inviscid Euler equations is played by the dispersionless limit of the original nonlinear dispersive system (which may or may not coincide with the actual isentropic hydrodynamic system). In other words, such transition conditions would make it possible to to treat the unsteady undular bore problem in a self-consistent manner bypassing the construction of the full solution describing the undular bore region.

In this paper, we, by adopting the asymptotic "averaged" formulation of the problem established the integrable systems theory, derive a general set of conditions describing the undular bore transition from one constant state, say (ρ_2, u_2) as $x \to -\infty$ to another constant state (ρ_1, u_1) as $x \to +\infty$ such that the transition occurs in a finite space region. Our theory is based on several assumptions, which essentially represent natural extensions of the arguments of Gurevich and Pitaevskii formulated for the KdV undular bores in [17]. We notice that the original analytical Gurevich – Pitaevskii (GP) theory of the decay of an initial discontinuity in the KdV dynamics in fact does not rely on the existence of the Riemann invariants for the Whitham system: the Riemann form in the GP problem is only *a convenient tool for the effective integration* of the modulation equations. Existence of the Riemann invariants *per se* becomes crucial only when analysing the evolution of more general initial profiles (see for instance [26] and references therein). In the present study, we show that, in the case of the similarity (x/t) solutions one can extract the complete information about the location of the undular bore from a set of integrals available for the Whitham systems even in the absence of the Riemann invariants.

The obtained transition conditions include a relationship between the admissible values of the afore-mentioned constant states $\rho_{1,2}$, $u_{1,2}$ (this relationship has been proposed earlier by Gurevich and Meshcherkin [27] for "dissipationless shocks" in plasma using intuitive physical arguments) and two ordinary differential equations defining the boundaries of the expanding nonlinear oscillatory zone. These ordinary differential equations are obtained in a general form and are easily integrated in particular instances. The transition conditions are supplemented with a set of inequalities providing consistency of the whole construction. It should be emphasized that, although the proposed construction *exploits the formulation of the problem* established in the integrable systems theory it *in no way depends on the integrability* of the system under study.

To validate the assumptions behind the developed theory we consider two specific examples where the transition conditions can be extracted either from the available full modulation solution or from the results of direct numerical simulations. Our first example is the integrable version of the shallow water equations (the so-called Kaup-Boussinesq system) where the quantities of our interest are available from the full modulation solution obtained in [23]. The second example is the non-integrable system describing fully nonlinear flows in a two-temperature collisionless plasma. The decay of an initial discontinuity problem for this system has been studied numerically in [27]. In both cases our analytical results are in complete agreement with the results of previous anlytical/numerical studies. In a more broad context, such an agreement can be viewed as a strong indication of the asymptotical validity of the Whitham theory in non-integrable initial-value wave problems.

Some preliminary results of this study have been published in the Letter [28]. When this paper was at the reviewing stage, one of the authors (G.A.E.) has made a more comprehensive study of mathematical aspects of the proposed method [29].

2 General setting and formulation of the problem

We consider an important class of nonlinear dispersive systems describing bi-directional wave propagation. The class we are going to study can be broadly described as "2×2 strictly hyperbolic systems modified by a small amount of dispersion". In a symbolic form such a system can be represented as

$$\mathbf{U}_t = \mathbf{K}_4(\mathbf{U}, \partial_x \mathbf{U}, \partial^2_{xx}\mathbf{U}, \dots)\,, \tag{1}$$

where $\mathbf{U} = (\rho, u)^T$ is a 2-vector and $\mathbf{K}_4$ is a nonlinear vector operator such that the system has the fourth order with respect to spatial/mixed derivatives and also has a real-valued linear dispersion relation $\omega = \omega_0(k)$, where ω is the frequency and k is the wavenumber .

We shall assume that the system (1):

- admits a dispersionless limit which we introduce in the following way. We set new independent variables $X = \epsilon x$, $T = \epsilon t$, $\epsilon \ll 1$ in (1) and then tend $\epsilon \to 0$ assuming the X- and T- derivatives to be finite. Then for a broad class of systems (1) that can be characterised as *weakly dispersive* we obtain to leading order a hyperbolic quasilinear system which we shall call the dispersionless limit of the system (1). Let this limit have the form of the gas-dynamic Euler equations for the isentropic ideal gas

  $$\partial_T \rho + \partial_X(\rho u) = 0\,, \qquad \rho(\partial_T u + u\partial_X u) + \sigma^2(\rho)\partial_X \rho = 0, \tag{2}$$

  where ρ is the "density", u is the "velocity", and $\sigma(\rho)$ is the "sound speed" in the corresponding "gas dynamics". It should be emphasized that generally speaking the assumption about the gas dynamic "core" of the system (1) is just a convenient (and in many cases physically relevant) way to convey our ideas – actually there is no need to restrict oneself with this particular form of the dispersionless limit. The only property of the dispersionless limit system that we used in this paper is that this system can be represented in the Riemann form (which is always the case for 2×2 quasilinear systems). We also note that in terms of the linear dispersion relation the "weak dispersion" property implies $\omega_0 \sim k$ as $k \to 0$.

- possesses a four-parametric family of periodic travelling wave solutions

$$\rho(x,t) = \rho(\theta; A_1, \dots, A_4)\,, \quad \theta = kx - \omega t\,, \quad \rho(\theta + 2\pi) = \rho(\theta)\,, \quad u = u(\rho)\,, \qquad (3)$$

  where $A_1, \dots, A_4$ are constants of integration, $k = k(\mathbf{A})$, $\omega = \omega(\mathbf{A})$. It is convenient, along with "natural" integration constants A_j introduce the "physical" parametrisation of the periodic solution (3): $\{A_1, \dots, A_4\} \mapsto \{\bar{\rho}, \bar{u}, k, a\}$, where $\bar{\rho}$ and $\bar{u}$ are the mean (averaged over the family (3)) density and velocity and $a = \rho_{max} - \rho_{min}$ is the wave amplitude. The averaged values are defined in terms of A_j according to

$$\overline{F}(\mathbf{A}) = \frac{1}{2\pi} \int_0^{2\pi} F(\theta) d\theta\,, \qquad (4)$$

  where $F(\theta) \equiv F(\rho(\theta))$. The frequency ω is connected with the basis integrals of motion by the nonlinear dispersion relation which follows from the periodicity condition in Eq. (3) [41]. We also assume that solution (3) becomes sinusoidal as $a \to 0$ and turns into a localised solitary wave as $k \to 0$, i.e. exhibits the familiar "cnoidal wave - type" behaviour characteristic for weakly dispersive conservative systems.

- possesses four linearly independent conservation laws of the form

$$\partial_t P_j + \partial_x Q_j = 0\,. \qquad (5)$$

  (we will show later that only three of them are actually needed). For convenience of explanation we will assume $P_1 = \rho$, $P_2 = u$ although this identification is not essential.

The described class of nonlinear dispersive systems is quite broad and includes some known integrable models such as defocusing nonlinear Schrödinger equation and Kaup-Boussinesq system [30]. As physically important examples of non-integrable systems that possess the above general properties (including the isentropic gas dynamic form of the dispersionless limit) one can indicate the Green-Naghdi system for fully nonlinear shallow water gravity waves [31] which also describes nonlinear waves in bubbly fluids [32], the systems for nonlinear ion-acoustic and magnetoacoustic waves in collisionless plasma [33], [15], and many others. In this paper we shall not be concerned with the integrability properties of the system under consideration.

As a structurally representative example of a system possessing the above properties we present a (non-integrable) system describing propagation of fully nonlinear ion-acoustic waves in a two-temperature collisionless plasma (see [33] for instance):

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho u) = 0, \\ &\partial_t u + u\partial_x u + \partial_x \phi = 0, \\ &\partial^2_{xx}\phi = e^{\phi} - \rho\,, \end{aligned} \qquad (6)$$

where ρ is plasma density, u is plasma velocity and ϕ is electric potential (the system does not contain the time derivative of ϕ so ϕ is not a "real" dependent variable as concerns the 2×2 representation (1)).

We consider initial data for the system (1) in the form of a step for the variables ρ and u:

$$t = 0: \quad \rho = \rho_1, \quad u = u_1 \text{ for } x > 0; \quad \rho = \rho_2, \ u = u_2 \text{ for } x < 0\,, \tag{7}$$

where $\rho_{1,2}$ and $u_{1,2}$ are constants.

Analytical studies of the decay of an initial discontinuity problem for integrable dispersive wave equations (see for instance [20], [10], [23]) as well as direct numerical simulations for non-integrable systems [27] show that the asymptotic solution to the decay of an arbitrary initial discontinuity generally consists of three constant states separated by two expanding waves: centered rarefaction wave(s) and/or undular bore(s), which is quite natural taking into account the "two-wave" nature of the system (1). One of the possible decay patterns is shown in Fig. 1. The undular bore, when occurs, occupies asymptotically as $t \gg 1$ a linearly expanding region $s_2 t \le x \le s_1 t$ where s_1 and s_2 are the speeds of the leading and the trailing edges of the undular bore respectively. The undular bore has the following

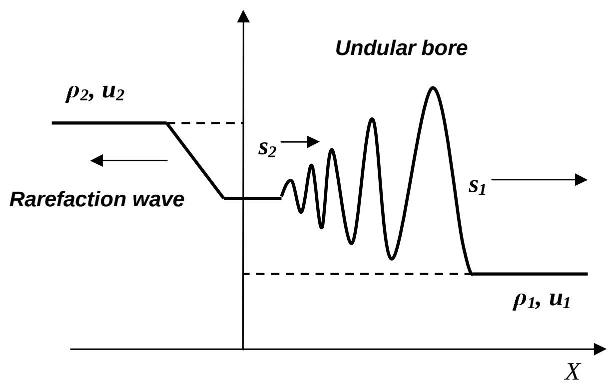

Figure 1: Typical density profile in the resolution of an initial discontinuity (dashed line) into an undular bore and a rarefaction wave in dispersive hydrodynamics

small-scale oscillatory structure : it represents a slowly varying nonlinear wave modulated in such a way that in the vicinity of the trailing edge the oscillations have small amplitude and the form close to sinusoidal, and they acquire the shape of successive solitary waves close to the opposite, leading edge. The typical spatial scale of the oscillations is determined by the characteristic dispersive length in the system.

Our aim in this work is to obtain a set of conditions describing the undular bore transition between two different constant states (ρ_1, u_1) and (ρ_2, u_2). For that we need

- to obtain the equation of the undular bore curve (ρ - u transition diagram) $\Phi(\rho_1, u_1, \rho_2, u_2) = 0$, which is to say to extract an admissible set of discontinuities resolving into *a single undular bore*.

- to determine the spatial location of the undular bore, i.e. to find the edge speeds $s_{1,2}$ in terms of the initial discontinuity parameters $\rho_1, u_1; \rho_2, u_2$

It is clear that the speeds $s_{1,2}$ cannot be consistently defined in frame of the original system (1) because of the presence of higher order spatial derivatives. These speeds, however, appear in a natural way in the asymptotic as $t \gg 1$ reformulation of the problem (1), (7), based on the quasi-linear Whitham equations [15].

3 Modulation description of undular bores and natural boundary conditions

We shall formulate the undular bore problem for a (generally) non-integrable system by adopting the resulting "averaged" setting from the theory of integrable systems, i.e. by direct application of the Whitham method to the system (1) in the conservative form (5) and then by postulating appropriate boundary conditions. There is, of course, a basic underlying assumption that the solution of the problem of our interest exists in some broad sense, which will be clear from what follows.

We invoke the reasoning established in the Lax - Levermore - Venakides theory [12], [13], [14] developed originally for the KdV equation, describing unidirectional wave propagation and more recently extended to the bi-directional defocusing NLS equation [34]. According to Lax-Levermore-Venakides results, the undular bore evolution is asymptotically described by the zero-dispersion limit of the initial value problem solution for the original dispersive wave equation ((1) in our case. Practically, it is achieved by introducing the slow variables $X = \epsilon x$, $T = \epsilon t$, $\epsilon \ll 1$ directly into the original equations and then, by tending ϵ to zero *on the solution*. The initial data in this approach are supposed to be large-scale in comparison with the characteristic dispersion length. The parameter ϵ then can be viewed as the dimensionless dispersion parameter of the system.

In the domains where the flow is smooth enough ($\Delta x \gg \epsilon$, where Δx is the characteristic spatial scale of the flow variations) the solution, asymptotically as $\epsilon \to 0$, converges *in a strong sense* to the solution of the dispersionless limit ($\epsilon \equiv 0$) of the original equations i.e. it satisfies (2). Contrastingly, in the oscillatory regions $\Delta x \sim \epsilon$ and the variations of the flow become more and more rapid as ϵ decreases. Here, as $\epsilon \to 0$ the solution manifests itself as a *distribution*, or a generalized solution. The corresponding limit then exists in a weak, i.e. averaged, sense (for non-integrable systems existence of weak limits is established numerically (see [14] and references therein)). Then by existence of the global solution one implies the continuous dependence of the mean values (moments) of the oscillatory quantities on the initial data.

Lax and Levermore [12] showed that the slow evolution of the mean values is governed by a system of quasilinear equations which turned to coincide with the so-called Whitham equations obtained earlier by averaging the nonlinear multiphase wavepackets [15], [16]. Venakides [35] showed that the higher order in ϵ of Lax-Levermore theory indeed leads to the description of the local waveform by the n-phase solution of the original system, where $n \geq 1$ (see also [14], [36]). The number of independent moments (and the order of the Whitham system) N coincides with the number of independent integrals of motion in the n-phase solution. For the initial conditions in the form of a step-like transition between two constant states, $n = 1$ (see [12], [37], [38]) and N coincides with the differential order of the original system with respect to the spatial/mixed derivatives i.e. $N = 4$ for the systems (1) studied in this paper.

The resulting, averaged formulation of the problem has been proposed (although without rigorous justification) much earlier by Gurevich and Pitaevskii (GP) [17] who complemented the Whitham system for the KdV equation with natural matching conditions at the free phase transition boundaries. We notice that GP formulation, although being consistent with the Lax-Levermore-Venakides theory, essentially does not rely on the integrability, and, therefore, could be extended to a more general class of nonlinear dispersive wave equations where spectral transform is not available. Our basic hypothesis in this work is that the GP formulation of the problem can be applied to the description of unsteady undular bores in conservative non-integrable systems belonging to the class (1) described in previous section within the domain of existence of the single-phase travelling wave solution (3).

Thus, the main premise of our construction is that the undular bore is locally described by the single-phase periodic travelling wave solution (3) of system (1). The travelling wave solution is parametrised by a certain number of integrals of motion (four in our case), which now are allowed to depend on slow slow variables $X = \epsilon x$, $T = \epsilon t$, $\epsilon \ll 1$, i.e. $\bar\rho = \bar\rho(X,T)$, $\bar u = \bar u(X,T)$, $k = k(X,T)$, $a = a(X,T)$. Then the evolution of these local integrals is governed as $\epsilon \to 0$ by the Whitham averaged equations [15]

$$\partial_T \bar P_j + \partial_X \bar Q_j = 0\,, \qquad j = 1, \ldots, 4\,, \tag{8}$$

where the averaging is done over the family (3) according to Eq. (4). The averaged equations, thus, describe slow modulations of the travelling wave solution. One more modulation equation is provided by the wave number conservation law

$$\partial_T k + \partial_X \omega = 0\,, \tag{9}$$

which is a compatibility condition in the formal perturbation procedure equivalent to the the Whitham method and can be used instead of any of the modulation equations (8). Its consistency with the system (8) can be usually verified directly [15]. Thus, to obtain a closed modulation system for the original dispersive system (1) one actually needs only three linearly independent conservation laws. If the original system (1) is completely integrable, then the averaged system (8) can be represented in the Riemann form, which makes it possible to effectively integrate the modulation equations (see [18] and references therein). In the case when the Riemann invariants for the Whitham equations are not available, the gas dynamic "core" nature of the original system (1) having the equations (2) as a dispersionless limit suggests a straightforward choice of the basis modulation variables: $\bar\rho$, $\bar u$, k, a. In the absence of the oscillations, $\bar\rho = \rho$, $\bar u = u$, $\overline{F(\rho,u)} = F(\rho,u)$ and the two first equations of the modulation system (8) describing evolution of $\bar\rho$ and $\bar u$ must become consistent with the dispersionless limit (2) of the original system (1). This degeneration can occur in two ways (see [29] for a more detailed explanation):

i) via the harmonic, vanishing amplitude wave limit $a \to 0$ (i.e. the oscillations become infinitesimally small);

ii) via the solitary wave limit, when $k \to 0$ (i.e. the oscillations become infinitely rare).

The first type of the transition is realised at the trailing edge of the undular bore, and the second one – at the leading edge (to be definite we assume here the negative sign of dispersion in the system, in the positive dispersion case the structure of the undular bore is reversed (see for instance [20])).

Thus in both harmonic and solitary wave limits, the Whitham equations must admit the *exact* reduction

$$\begin{aligned} &\partial_T \bar\rho + \partial_X(\bar\rho\bar u) = 0\,, \\ &\bar\rho(\partial_T \bar u + \bar u \partial_X \bar u) + \sigma^2(\bar\rho)\partial_X \bar\rho = 0\,, \\ &\partial_T k + \partial_X \omega = 0\,, \end{aligned} \tag{10}$$

where the wave number conservation law (9) should also be considered in the corresponding (harmonic or solitary wave) limit. We shall call the system (10) *the extended dispersionless limit* of the original system (1). To make it a closed system one should complement it with the dependence of the frequency ω on k, $\bar{\rho}$ and $\bar{u}$. The full travelling wave frequency ω is expressed in terms of the basis modulation variables by the nonlinear dispersion relation [15],

$$\omega = \omega(\bar{\rho}, \bar{u}, k, a) \,, \tag{11}$$

which follows from the 2π-periodicity condition for the solution (3). In the harmonic (linear) limit, when $a \to 0$ this relation assumes the form

$$\omega = \omega_0(k, \bar{\rho}, \bar{u}) \,, \tag{12}$$

which closes the system (10). The function $\omega_0(k, \bar{\rho}, \bar{u})$ can be obtained directly from the original system (1) by linearization against the slowly varying background $\bar{U}_j(X,T)$:

$$U_j \approx \bar{U}_j + b_j e^{i(kx-\omega t)} \,, \quad b_j(X,T) \ll 1 \tag{13}$$

and represents a usual linear dispersion relation where the constant background parameters say ρ_0, u_0 are replaced with the mean flow $\bar{\rho}(X,T)$, $\bar{u}(X,T)$. Owing to the two-wave nature of the problem under study, the linear dispersion relation (12) has two branches describing the waves propagating in the opposite directions (in the frame of reference moving with velocity $\bar{u}$). Hereafter, unless otherwise explicitly specified, we will consider the branch describing the waves moving to the right. Also, for sake of definiteness we assume $\partial^2\omega_0/\partial k^2 < 0$ (negative dispersion).

In the solitary wave limit, the finite speed of the wave propagation (hyperbolicity of the modulation system) implies the vanishing frequency: $\omega \to 0$ as $k \to 0$, while the phase velocity $\omega/k \to c_s(\bar{\rho}, \bar{u}, a)$ remains $\mathcal{O}(1)$. Thus, the solitary wave limit is a singular one for the wave number conservation law in (10). This limit will be considered in Section 5.2.

Now, for convenience we explicitly itemize the major implications of our hypothesis about the GP-type description of the undular bore in a non-integrable system (the details can be found in [29]).

- We assume the upper half-plane of the space-time of the asymptotic as $\epsilon \to 0$ solution to the initial value problem (1), (7) to be split into three domains : $\{(X, T > 0) : (-\infty, X_2(T)),\ [X_2(T), X_1(T)],\ (X_1(T), +\infty)\}$ in which the solution is governed by different equations (see Fig. 2):

  i) Outside the undular bore region, when $X \in (-\infty, X_2(T)) \cup (X_1(T), +\infty)$ the solution is governed by the dispersionless limit (2) of the original system (1).

  ii) Within the undular bore domain $X \in [X_2(T); X_1(T)]$, we replace the original system (1) with the modulation system (8).

- The solutions of the "inner" (Whitham (8)) and "outer" (Euler (2)) systems are then subject to matching conditions at the (free) boundaries of the undular bore $X_{1,2}(T)$. These edges are often referred to as the phase transition boundaries since the dispersionless limit system (2) can be viewed also as a result of the zero-phase averaging,

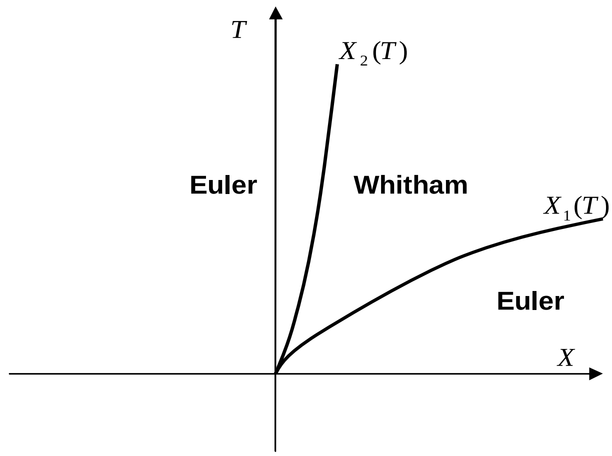

Figure 2: Splitting the upper (XT) half-plane in the generalised Gurevich – Pitaevskii problem

i.e. as the "zero-phase Whitham system". We require a "weak" continuity matching at the phase transition boundaries [39]:

$$X = X_2(T): \quad a = 0\,, \quad \bar{\rho} = \rho_E(X,T)\,, \quad \bar{u} = u_E(X,T)\,, \tag{14}$$

$$X = X_1(T): \quad k = 0\,, \quad \bar{\rho} = \rho_E(X,T)\,, \quad \bar{u} = u_E(X,T)\,. \tag{15}$$

Here $(\rho_E(X,T), u_E(X,T))$ stands for the solution of the dispersionless system (2) with the initial conditions appearing in the original formulation of the problem for the system (1). This solution is supposed to be known from the classical theory of quasi-linear hyperbolic systems of the second order (see for instance [40]). The conditions (14), (15) represent a natural extension of the GP conditions formulated for the KdV equation in [17], and then generalized for the defocusing NLS equation [19], [20] and the Kaup-Boussinesq system [23]. Of course, such a matching is only possible owing to the fact that the dispersionless system is consistent with the Whitham equations in the indicated limits.

- We assume hyperbolicity of the modulation system (8) for the solutions of our interest. In the context of the undular bore problem the hyperbolicity implies the modulational stability of the undular bore. Another implication of hyperbolicity is a possibility of using the classical characteristics method [40].

- In order to provide continuous matching of the solutions of two quasilinear hyperbolic systems of different order (the fourth order Whitham system and the second order Euler system in our case) the matching lines (the undular bore edges) $X = X_{1,2}(T)$ must *necessarily* be the double characteristics for the system with a higher order, i.e. for the Whitham system. These free boundaries are defined by the kinematic conditions

$$a = 0: \qquad \frac{dX_2}{dT} = \frac{\partial \omega_0}{\partial k}\,, \tag{16}$$

$$k = 0: \qquad \frac{dX_1}{dT} = \lim_{k \to 0} \frac{\omega}{k} \tag{17}$$

– consistent with the matching conditions (14), (15). Thus, the trailing edge moves with the group velocity of the linear wave packet generated at the rear end of the undular bore whereas the leading edge is identified with the position of the lead solitary wave. It is shown in [29] that these physically natural definitions of the undular bore edges indeed agree with the double characteristic velocities of the Whitham system.

The matching conditions (14),(15) complemented by the definitions of free boundaries (16), (17) completely determine the modulation solution $\{\bar{\rho}(X,T), \bar{u}(X,T), k(X,T), a(X,T)\}$ describing along with the travelling wave solution (53) the undular bore. One should note that the boundary values for k at the trailing edge and for a at the leading edge can not be specified arbitrarily due to the fact that the boundaries of the modulation solution are the *characteristics* of the Whitham system which imposes certain restrictions on Cauchy data prescription along these lines (see [40] for instance).

Since in our asymptotic formulation the governing equations (8), (2) are quasilinear, the solution of the decay of an initial discontinuity problem must depend on the self-similar variable $s = X/T$ alone. Hence the boundaries of the undular bore are the straight lines

$$X_{1,2} = s_{1,2}T\,, \tag{18}$$

and the matching conditions (14), (15) assume the form

$$s = s_2: \qquad a = 0, \quad \bar{\rho} = \rho_2\,, \quad \bar{u} = u_2\,, \tag{19}$$

$$s = s_1: \qquad k = 0, \quad \bar{\rho} = \rho_1\,, \quad \bar{u} = u_1\,. \tag{20}$$

Using definitions of the edges (16) – (18) we obtain the expressions for s_1, s_2 in terms of the characteristic modulation velocities:

$$s_2 = \left[\frac{\partial \omega_0}{\partial k}\right](k_2, \rho_2, u_2)\,, \tag{21}$$

where k_2 is the value of the wavenumber at the trailing edge, and

$$s_1 = \left[\lim_{k\to 0}\frac{\omega}{k}\right](a_1, \rho_1, u_1)\,, \tag{22}$$

where a_1 is the lead solitary wave amplitude.

Thus, to find the speeds $s_{1,2}$ of the undular bore edges as functions of the initial step parameters $\rho_{1,2}$, $u_{1,2}$ one has to

i) obtain the values of k_2 and a_1 in terms of $\rho_{1,2}$, $u_{1,2}$;

ii) evaluate the singular limit $\lim\limits_{k\to 0}\frac{\omega}{k}$.

These two issues will be addressed in Section 5.

In the conclusion of this section we note that the parameter ϵ formally introduced in the definition of the slow variables X and T appears naturally in the undular bore solution of the original system (1) (the existence of the solution is established in general case numerically) as a ratio of the characteristic scale of oscillations to the expanding oscillation zone width and, therefore, is proportional to t^{-1}. Thus, our definition of the undular bore edges is asymptotically accurate as $t \to \infty$.

4 Simple undular bore transition curve

4.1 Derivation of the simple undular bore transition curve

It is clear that since an arbitrary *four-parametric* discontinuity (7) for the system (1) resolves into two expanding waves (undular bore(s) and/or rarefaction wave(s)) separated by three constant states we need to impose some restriction on the initial parameters in order to produce a single undular bore. Finding this restriction is equivalent to extracting the family of the initial discontinuities resolving into a single undular bore propagating in a given direction (we associate the direction of the undular bore propagation with the direction of the corresponding linear characteristic). The condition to be found would have a form of a relationship

$$\Phi(\rho_1, \rho_2, u_1, u_2) = 0 \tag{23}$$

so the undular bore solution must be three-parametric.

Suppose that the initial discontinuity (7) satisfies the restriction (23) such that it resolves into a single undular bore propagating to the right. Using the asymptotic reformulation of the initial value problem for (1) as the generalised GP problem, i.e. a matching problem for two quasi-linear hyperbolic systems, the solution of our interest can be, in principle, constructed using characteristics. We note that due to (assumed) hyperbolicity (finite speed of propagation) the undular bore has zero width at $T = 0$. Therefore the line $T = 0$ of the characteristic XT- plane lies entirely in the "Euler", dispersionless part of the solution of the generalized GP problem (8), (14),(15).

We represent the Euler system (2) in the Riemann invariant form

$$\frac{\partial r_+}{\partial T} + V_+(r_+, r_-)\frac{\partial r_+}{\partial X} = 0\,, \qquad \frac{\partial r_-}{\partial T} + V_-(r_+, r_-)\frac{\partial r_-}{\partial X} = 0\,, \tag{24}$$

where

$$r_\pm(\rho, u) = u \pm \int_{\rho_0}^{\rho} \frac{\sigma(\rho')}{\rho'} d\rho'\,; \qquad V_\pm = u \pm \sigma(\rho)\,, \quad \rho_0 = constant. \tag{25}$$

The initial conditions (7) are rewritten in terms of the Riemann invariants as

$$r_\pm(X, 0) = r_\pm^{(2)} \text{ for } X < 0\,; \qquad r_\pm(X, 0) = r_\pm^{(1)} \text{ for } X < 0\,, \tag{26}$$

where $r_\pm^{(1)} = r_\pm(\rho_1, u_1)$, $r_\pm^{(2)} = r_\pm(\rho_2, u_2)$. We will assume that $\rho_2 > \rho_1$, $u_2 > u_1$, which in ideal hydrodynamics implies formation of the compression wave moving to the right (and it is the wave that, after the breaking, is resolved into the undular bore moving to the right). The latter inequality is taken for convenience and will be shown later to be redundant. In view of (26), (25) and the fact that $c(\rho) > 0$, $c'(\rho) > 0$ the assumed inequalities imply

$$r_+^{(2)} > r_+^{(1)}, \qquad V_+^{(2)} > V_+^{(1)}, \qquad V_-^{(1)} < u_1\,, \tag{27}$$

where $V_\pm^{(1)} = r_\pm(\rho_1, u_1)$, $V_\pm^{(2)} = r_\pm(\rho_2, u_2)$.

Now we note that although construction of the solution to the generalised GP problem in the upper XT-plane can be a very involved task for a non-integrable case, when the Riemann invariants for the modulation system (8) are not available; this problem can be easily solved in 'non-physical' domain $T < 0$. Indeed, there is a unique three-parametric solution to the gas dynamic system (24) in the lower XT- half-plane satisfying the initial conditions (26) and subject to inequalities (27). This solution is a centred expansion fan (in $-X$, $-T$ - coordinates) given by the expressions

$$r_- = r_0 = constant\,. \tag{28}$$

$$\begin{aligned} r_+ &= r_+^{(2)}, && X > a_1 T; \\ \frac{X}{T} &= V_+(r_+, r_0); && a_2 T \le X \le a_1 T; \\ r_+ &= r_+^{(1)}; && X < a_2 T. \end{aligned} \tag{29}$$

Here

$$a_{1,2} = V_+^{(1),(2)}\,, \qquad a_2 > a_1\,. \tag{30}$$

The solution (28) – (30) is characterised by three parameters $(r_+^{(1)}, r_+^{(2)}, d_-)$ and *it exists for all* $T < 0$.

It is clear from the construction above that solution (28) – (30) represents a continuation, along the characteristics to the lower XT- half-plane of the *full* simple undular bore solution to the generalised GP problem. Therefore, one can extract from it the global restriction (23) imposed on possible values of the hydrodynamic variables at the opposite sides of the simple undular bore. This restriction follows from (28) and has the form

$$r_-^{(1)} = r_-^{(2)}\,, \tag{31}$$

Using (25) we represent the sought dependence in the gas dynamics form (23)

$$u_2 - u_1 = \int_{\rho_1}^{\rho_2} \frac{\sigma(\rho)}{\rho} d\rho\,. \tag{32}$$

Given the state in front of the undular bore ρ_1, u_1 this relation yields all admissible states (ρ_2, u_2) behind it i.e. it represents the equation of the ρ-u diagram of the simple undular bore (simple undular bore curve). One can also see that since $\sigma(\rho) > 0$, the inequality $u_2 > u_1$ assumed in (27) now trivially follows from the condition $\rho_2 > \rho_1$ and can be omitted as redundant. Thus the condition (32) singles out the family of initial discontinuities consistent with the self-similar GP problem for (8).

One should note that the whole construction above is subject to the additional inequalities

$$V_-^{(2)} < s_2 < V_+^{(2)}\,, \quad V_+^{(1)} < s_1\,, \quad s_1 > s_2\,. \tag{33}$$

These inequalities provide validity of our original assumption (23) about the existence of the three-parametric solution. Indeed, the number of parameters characterizing the solution of the hyperbolic system in some domain is equal to the number of families of characteristics transferring given initial or boundary data into this domain (see for instance [42]). The inequalities (33) imply that only three of four families of characteristics (specifically, $dX/dT = V_+^{(2)}$, $dX/dT = V_+^{(1)}$, and $dX/dT = V_-^{(1)}$) transfer initial data (26) from the X-axis into the undular bore domain (see Fig. 3).

Now we discuss briefly the meaning of the obtained transition relation (32). One can see that it coincides with the relationship between any two pairs ρ, u in the simple wave solution

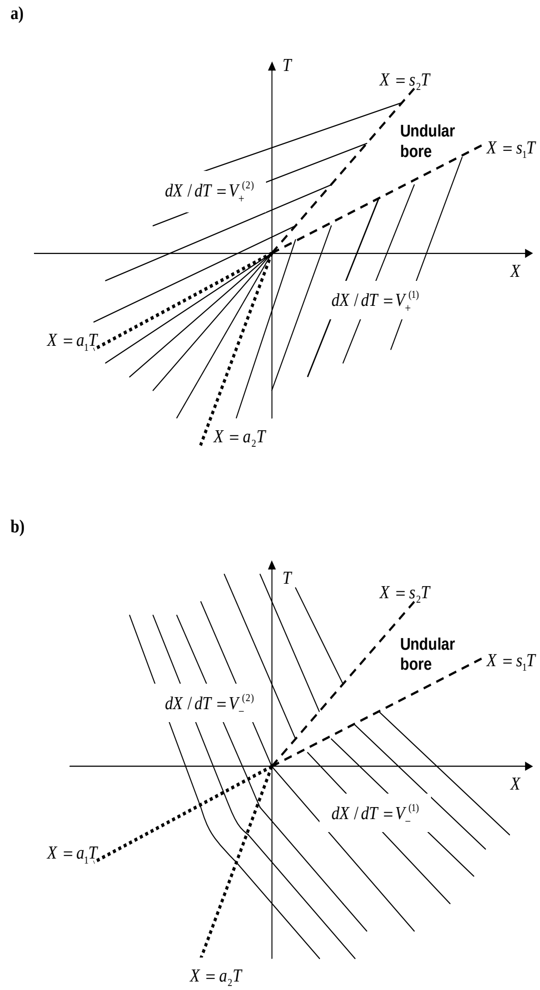

Figure 3: Qualitative behaviour of characteristics in the "simple-wave" decay in dispersive hydrodynamics. Dashed lines: the dispersive shock boundaries. Dotted lines: the "mirror" expansion fan boundaries. a) Family $dx/dt = V_+^{(1)}$ transfers the values of r_+, b) Family $dx/dt = V_-^{(2)}$ transfers the values of r_-.

of the isentropic gas dynamics (see for instance [41], [42]). A nontrivial fact is that, in view of (27) the corresponding wave is the simple wave of compression which breaks after a certain time interval. Contrastingly, the similarity solution of the Whitham equations satisfying the relationship (28) and describing the expanding undular bore does not break. This solution essentially represents a *compression fan*, which does not exist in usual hydrodynamics.

It is natural to call the undular bore satisfying the relationship (28) *a simple undular bore*. We emphasize that, according to the matching conditions (19), (20) the relationship (32) is only valid *for the boundary values* of the modulation parameters $\bar{\rho}$ and $\bar{u}$ and, of course, does not hold within the undular bore region.

In the solutions of the GP problem for integrable systems, the simple undular bore condition is a mere consequence of the constancy of one of the Riemann invariants of the Whitham system (see [20], [23] for instance). For instance, the simple undular undular bore resolution is, by definition, the case for the KdV dynamics where the dispersionless limit is governed by the Riemann-Hopf equation. In a non-integrable case, when the Riemann invariants are not available for the Whitham system, the condition (32) is not obvious at all. However, the obtained result can be put in an easy to understand framework using an intuitive argument that since the undular bores do not dissipate, it is "reasonable" to expect that the "external" Riemann invariant will be conserved across them (even if it does not have a meaning inside the bore region). "Hence" the simple wave behaviour. One should keep in mind that inequalities (33) are the necessary part of the transition conditions. In Section 6.1 an example will be presented where one of these inequalities is violated so that the single undular bore resolution becomes impossible for some domain of the initial data parameters even if the initial discontinuity satisfies relation (32).

We recall that the consideration above was concerned with the undular bores propagating to the right. For the left-propagating undular bores, an analogous construction leads to the relationship

$$r_+^{(1)} = r_+^{(2)}\,, \tag{34}$$

which extracts the discontinuities producing a single undular bore moving to the left. The corresponding simple undular bore relationship in terms of hydrodynamic variables assumes the form

$$u_2 - u_1 = -\int_{\rho_1}^{\rho_2} \frac{\sigma(\rho)}{\rho} d\rho\,. \tag{35}$$

The set of inequalities for this wave, analogous to (33) has the form

$$V_-^{(2)} < s_2^* < V_+^{(2)}\,, \quad V_-^{(2)} > s_1^*\,, \quad s_2^* > s_1^*\,. \tag{36}$$

The simple undular bore conditions in the form (32), (35) have been formulated for the first time in [27] using qualitative physical arguments and then interpreted in [39] in terms of "local Riemann invariant transport" through the Whitham zone. The previous studies, however, did not take into account the inequalities (33), (36), which represent an essential part of the transition conditions. We note in conclusion that in a more general case, when the dispersionless limit has the form other than isentropic gas dynamics, one should use the simple undular bore transition relations in the invariant form (31) or (34).

4.2 Comparison of the transition curves for steady and unsteady shallow water undular bores

To illustrate the difference between the steady (dissipative) and the expanding (dissipationless) undular bore transition relations we consider a particular example of the shallow water undular bores, which has been extensively discussed both in dissipative (see [41] and references therein) and dissipationless (see for instance [23] and references therein) contexts. We note that we use the term "shallow water" here in a broad, mathematical, sense which encompasses, for instance, some nonlinear optics applications. The inviscid dispersionless shallow water equations have the form

$$\partial_t h + \partial_x(hu), \quad \partial_t u + \partial_x(u^2/2 + h) = 0. \tag{37}$$

In the presence of additional small dissipation the *established* undular bore has constant width so that the jump relations follow from the balance of the mass and momentum across the transition (here we are only interested in the jump relations for h and u and assume that the "equation of state" $p = h^2/2$ does not change in the course of the undular bore propagation, which is the case, say for the shallow water gravity waves [41]):

$$-U[h] + [uh] = 0; \qquad -U[uh] + [u^2 h + h^2/2] = 0\,, \tag{38}$$

Here U is the bore speed and $[f] \equiv f_2 - f_1$. Eliminating U from (38) we obtain the transition relation for the right-propagating dissipative undular bore (a classical shock curve)

$$u_2 - u_1 = (h_2 - h_1)\sqrt{\frac{h_1 + h_2}{2h_1 h_2}}\,. \tag{39}$$

For the dissipationless undular bore we have from (32), (35), taking into account that for the shallow water waves $\sigma(h) = \sqrt{h}$,

$$u_2 - u_1 = 2(\sqrt{h_2} - \sqrt{h_1})\,, \tag{40}$$

which is a totally different transition relation.

For weak bores propagating to the right, $h_2 = 1 + \delta$, $\delta \ll 1$ we get from (39) and (40) the decompositions

$$u_2 = \delta - \frac{1}{4}\delta^2 + \frac{1}{8}\delta^3 + \dots\,, \qquad \text{steady undular bore} \tag{41}$$

$$u_2 = \delta - \frac{1}{4}\delta^2 + \frac{7}{32}\delta^3 + \dots\,, \qquad \text{unsteady undular bore} \tag{42}$$

i.e. the undular bore transition curve and the classical shock curve have high contact for small jumps which is to be expected in view of the invariant form of the undular bore transition curve (31). Indeed, it is well known that the Riemann invariants have the higher order jumps across weak shocks [41]. Thus, the distinction between the jump condition (39) and the simple undular bore relation (40) becomes noticeable only for finite-amplitude shallow water undular bores (which is not very relevant to the actual free-surface gravity shallow-water waves where the undular bores are known to exist only for $h_2/h_1 \le 1.28$ due to the turbulent viscosity [41]). One should emphasize, however, that it is the simple undular bore transition relation (40) (rather than jump condition (39)) that is consistent with the Whitham equations and allows for correct analytic determination of the undular bore location.

5 Location of the undular bore edges

In general terms, the self-similar reduction of the fourth-order Whitham system (8) describing the simple undular bore has three three independent integrals, say

$$F_j(\bar{\rho}, \bar{u}, k, a) = I_j\,; \quad j = 1, 2, 3\,, \tag{43}$$

I_j being constants. We note that this number of parameters is consistent with the restriction imposed by the simple undular bore relation (32) relating four initial parameters $\rho_{1,2}$, $u_{1,2}$ and leaving only tree of them independent. When the modulation system is diagonalisable, the functions $F_j(\bar{\rho}, \bar{u}, k, a)$ coincide with its Riemann invariants (see [17], [20], [23])). However, generally the Riemann structure is not available for the Whitham systems and obtaining these functions in an analytically amenable form can be a very involved task. At the same time, the zero-amplitude and zero-wavenumber reductions $\mathbf{F}(\bar{\rho}, \bar{u}, k, 0)$ and $\mathbf{F}(\bar{\rho}, \bar{u}, 0, a)$ of Eqs. (43) can be obtained indirectly by analysing much more simple extended dispersionless limit systems (10). Indeed, since Eqs. (10) represent an exact reduction of the full Whitham system, their possible integrals must be consistent with the corresponding reductions of Eqs. (43). These integrals will provide us with the missing information about the values of k at the trailing edge and a at the leading edge (see the end of Section 3) so that one could take advantage of the kinematic conditions (21), (22) to determine the speeds $s_{1,2}$ of the undular bore edges.

5.1 Trailing edge

We consider the exact reduction (10) of the modulation system (8) for $a = 0$:

$$\partial_T \bar{\rho} + \partial_X(\bar{\rho}\bar{u}) = 0\,, \qquad \bar{\rho}(\partial_T \bar{u} + \bar{u}\partial_X \bar{u}) + \sigma^2(\bar{\rho})\partial_X \bar{\rho} = 0\,, \tag{44}$$

$$\partial_T k + \partial_X \omega_0(k, \bar{\rho}, \bar{u}) = 0\,, \tag{45}$$

where $\omega = \omega_0(k, \bar{\rho}, \bar{u})$ is the linear dispersion relation, and will look for its integrals of the form

$$\Phi_1(\bar{u}, \bar{\rho}) = C_1\,, \qquad \Phi_2(k, \bar{\rho}) = C_2\,, \tag{46}$$

C_1, C_2 being constants. Eqs. (46) specify a zero-amplitude section of the integral family (43) in the four-dimensional space $\{\bar{\rho}, \bar{u}, k, a\}$ and provide constraints on admissible values of $\bar{u}$ and k at the trailing edge of the undular bore where $a = 0$. Thus they should yield dependencies $u_2(\rho_2)$ and $k(\rho_2)$ for the trailing edge given fixed $\bar{\rho} = \rho_1$, $\bar{u} = \rho_1$ at the leading edge. Also, the function $u_2(\rho_2)_{\rho_1, u_1}$ must agree with the simple undular bore curve (32).

Substitution $\bar{u} = \bar{u}(\bar{\rho})$ into (44) yields the standard simple-wave integrals of the Euler equations

$$\bar{u} = \pm \int \frac{\sigma(\bar{\rho})}{\bar{\rho}} d\bar{\rho}\,. \tag{47}$$

To provide consistency with the simple undular bore transition curve (32) one has to choose the plus sign in (47) and to fix an arbitrary constant C_1 in such a way that (47) becomes

$$\bar{u} = u_1 + \int_{\rho_1}^{\bar{\rho}} \frac{\sigma(\rho)}{\rho} d\rho\,. \tag{48}$$

The system (44), (45) then reduces to a 2×2 system,

$$\partial_T \bar{\rho} + V(\bar{\rho})\partial_X \bar{\rho} = 0\,, \qquad \partial_T k + \partial_X \Omega_0(k, \bar{\rho}) = 0\,, \tag{49}$$

where

$$V(\bar{\rho}) = V_+(\bar{\rho}, \bar{u}(\bar{\rho})) = \sigma(\bar{\rho}) + \int_{\rho_1}^{\bar{\rho}} \frac{\sigma(\rho)}{\rho} d\rho + u_1\,, \quad \Omega_0(k, \bar{\rho}) = \omega_0(k, \bar{\rho}, \bar{u}(\bar{\rho}))\,. \tag{50}$$

Now, substituting $k = k(\bar{\rho})$ into this system we arrive at the ordinary differential equation

$$\frac{dk}{d\bar{\rho}} = \frac{\partial\Omega_0/\partial\bar{\rho}}{V(\bar{\rho}) - \partial\Omega_0/\partial k} \tag{51}$$

with the initial condition $k(\rho_1) = 0$ following from the boundary condition (20). Indeed, the integral $k(\bar{\rho})$ holds for $a = 0$ and does not bear any restrictions on other variables, hence can be applied to $\bar{\rho} = \rho_1$ which, according to Eq. (20), implies $k = 0$. Integrating (51) one obtains $k(\bar{\rho})_{\rho_1,u_1}$. Then the self-similar coordinate (the speed) of the trailing edge is found according to (21) as

$$s_2 = \frac{\partial\Omega_0}{\partial k}(k_2, \rho_2)\,, \tag{52}$$

where $k_2 = k(\rho_2)_{\rho_1,u_1}$.

5.2 Leading edge

Now we consider the zero-wavenumber section of the integral curve family (43). The plan is to find the integrals of the reduced as $k \to 0$ Whitham system (10) analogous to those in the trailing edge case (see Eq. (46)) and then to evaluate the leading edge speed using Eq. (21). The complication, compared to the zero-amplitude case, is that, since both k and ω now vanish, one should consider a singular limiting transition for the wave number conservation law in the extended dispersionless system (10). To do that, it is convenient to introduce instead of $\{\bar{\rho}, \bar{u}, k, a\}$ a new system of the basis modulation variables in which, as we will see, the description of the leading edge will become essentially equivalent to that of the trailing edge.

First, we recall that the modulation variables locally represent a set of independent parameters specifying the travelling wave solution (3) of the original system (1). This solution is normally given by the ordinary differential equation of the form

$$(k\lambda_\theta)^2 = R(\lambda)\,, \qquad \theta = kx - \omega t\,, \qquad f(\theta + 2\pi) = f(\theta)\,, \tag{53}$$

where λ is usually one of the component of the two-vector $\mathbf{U}$; generally $\mathbf{U} = \mathbf{U}(\lambda)$. To be definite, we assume that the potential curve $R(\lambda)$ has at least three real roots $\lambda_1 < \lambda_2 < \lambda_3$ and a (single) minimum between λ_2 and λ_1 so that the 2π-periodic solution of (53) oscillates between the roots λ_2 and λ_3 (see Fig. 4a). The wavenumber and the mean values in this travelling wave are given by the integrals

$$k = \pi \left(\int\limits_{\lambda_2}^{\lambda_3} \frac{d\lambda}{\sqrt{R(\lambda)}} \right)^{-1}\,, \qquad \overline{F(\lambda)} = \frac{k}{\pi} \int\limits_{\lambda_2}^{\lambda_3} \frac{F(\lambda)d\lambda}{\sqrt{R(\lambda)}}\,. \tag{54}$$

We assume that the function $R(\lambda)$ has generic quadratic behaviour in the vicinity of its minimum:

$$R(\lambda) \approx (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda_3 - \lambda)G(\lambda)\,, \quad G(\lambda) = \mathcal{O}(1) \quad \text{for} \quad \left|\frac{\lambda - \lambda_1}{\lambda_3 - \lambda}\right| \ll 1\,. \tag{55}$$

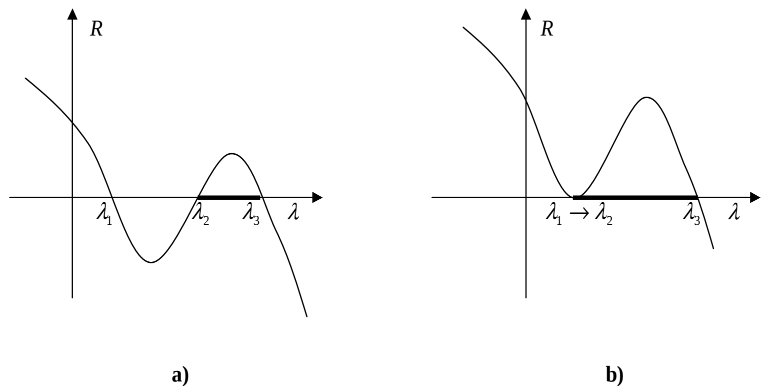

Figure 4: Potential curve $R(\lambda)$: a) general configuration; b) solitary wave configuration

Then the following general asymptotic behaviour of the modulation parameters can be inferred from Eq. (54) for nearly soliton configuration (Fig. 4b):

$$m' = \frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1} \ll 1 : \qquad k \sim \frac{1}{\ln(1/m')}\,, \quad \frac{\omega}{k} = c + \mathcal{O}(m')\,, \quad c = \mathcal{O}(1)\,. \tag{56}$$

The latter follows from 2π-periodicity (53). Also it is not difficult to show that in the limit

$$\lambda_2 \to \lambda_1 : \qquad \overline{F(\lambda)} \to F(\lambda_2)\,. \tag{57}$$

Now we introduce an alternative (conjugate) system of modulation variables which is more convenient when studying the solitary wave limit $\lambda_2 \to \lambda_1$ in the modulation system. We define the conjugate wavenumber and the conjugate mean value as

$$\tilde{k} = \pi \left(\int_{\lambda_1}^{\lambda_2} \frac{d\lambda}{\sqrt{-R(\lambda)}} \right)^{-1}, \qquad < F(\lambda) >= \frac{\tilde{k}}{\pi} \int_{\lambda_1}^{\lambda_2} \frac{F(\lambda) d\lambda}{\sqrt{-R(\lambda)}}\,. \tag{58}$$

As a matter of fact, any of these quantities can be taken as a modulation variable instead of any one from the set (54). The corresponding limiting behaviour for the conjugate variables are (cf. (56))

$$\lambda_2 \to \lambda_1 : \quad \tilde{k} \to \tilde{k}_s = O(1)\,, \qquad < F(\lambda) >\to F(\lambda_2)\,. \tag{59}$$

Comparing (57) and (59) one can see that in the limit considered

$$\lambda_2 \to \lambda_1 : \quad < F(\lambda) >\to \overline{F(\lambda)}\,. \tag{60}$$

We introduce a new equivalent set of basic modulation variables $\{\bar{\rho}, \bar{u}, k, a\} \mapsto \{\bar{\rho}, \bar{u}, \Lambda, \tilde{k}\}$, where $\Lambda = k/\tilde{k}$ and consider the solitary wave limit $k \to 0$ using these new variables in which we tend $\Lambda \to 0$. We will look for two integrals analogous to (46),

$$\Lambda = 0 : \qquad \Phi_1^*(\bar{u}, \bar{\rho}) = C_1^*\,, \qquad \Phi_2^*(\tilde{k}, \bar{\rho}) = C_2^*\,, \tag{61}$$

$C_{1,2}^*$ being constants.

First we note that the hydrodynamic part of the reduction as $\Lambda \to 0$ remains the same as in (10) i.e. ideal Euler hydrodynamics for $\bar{\rho}$ and $\bar{u}$. Then, using the arguments identical to those in the zero-amplitude case (see Section 5.1) we find the first integral in the form

$$\bar{u} = u_2 - \int_{\bar{\rho}}^{\rho_2} \frac{\sigma(\rho)}{\rho} d\rho \tag{62}$$

equivalent to (48) but parametrised by ρ_2, u_2 instead of ρ_1, u_1. Using Eq. (62) we further reduce the Euler system to the simple wave equation

$$\partial_T \bar{\rho} + V(\bar{\rho}) \partial_X \bar{\rho} = 0 \tag{63}$$

where

$$V(\bar{\rho}) = V_+(\bar{\rho}, \bar{u}(\bar{\rho})) = c(\bar{\rho}) - \int_{\bar{\rho}}^{\rho_2} \frac{c(\rho)}{\rho} d\rho + u_2\,, \tag{64}$$

which, in view of (32) coincides with Eq. (49).

The wave number conservation law in (10) requires a somewhat more detailed analysis. After the passage to new modulation variables it assumes the form

$$\Lambda(\partial_T \tilde{k} + \partial_X \tilde{\omega}) + \tilde{k}(\partial_T \Lambda + \frac{\tilde{\omega}}{\tilde{k}} \partial_X \Lambda) = 0\,, \tag{65}$$

where $\tilde{\omega}(\bar{\rho}, \bar{u}, \Lambda, \tilde{k}) = \omega/\Lambda = \tilde{k}c$. We also introduce a notation

$$\tilde{\omega}_s(\bar{\rho}, \bar{u}, \tilde{k}) \equiv \tilde{\omega}(\bar{\rho}, \bar{u}, 0, \tilde{k}) \tag{66}$$

for the conjugate frequency in the solitary wave limit. Making a decomposition for $\Lambda \ll 1$ in (65) and taking advantage of Eqs. (61), (63) to leading order in Λ we obtain an asymptotic equation

$$\partial_T \Lambda + \frac{\Omega_s}{\tilde{k}} \partial_X \Lambda + \frac{\Lambda}{\tilde{k}} \left\{ \frac{d\tilde{k}}{d\bar{\rho}} \left(\frac{\partial \Omega_s}{\partial \tilde{k}} - V(\bar{\rho}) \right) + \frac{\partial \Omega_s}{\partial \bar{\rho}} \right\} \partial_X \bar{\rho} = \mathcal{O}\left(\Lambda \partial_X \Lambda\right)\,, \tag{67}$$

where $\Omega_s(\tilde{k}, \bar{\rho}) = \tilde{\omega}_s(\bar{\rho}, \bar{u}(\bar{\rho}), \tilde{k})$ and the dependence $\tilde{k}(\bar{\rho})$ (the second integral in (61)) is unknown. Since the sought characteristic integral $\tilde{k} = \tilde{k}(\bar{\rho})$ holding for $\Lambda = 0$ must not depend on the way Λ tends to zero, Eq. (67) splits into two:

$$\partial_T \Lambda + \frac{\Omega_s}{\tilde{k}} \partial_X \Lambda = \mathcal{O}\left(\Lambda \partial_X \Lambda\right) \tag{68}$$

and

$$\frac{d\tilde{k}}{d\bar{\rho}} = \frac{\partial \Omega_s / \partial \bar{\rho}}{V(\bar{\rho}) - \partial \tilde{\Omega}_s / \partial \tilde{k}}\,. \tag{69}$$

To get the initial condition for ordinary differential equation (69) we reformulate the matching conditions (19), (20) in terms of new field variables $\bar{\rho}, \bar{u}, \Lambda, \tilde{k}$:

$$s = s_2: \qquad \tilde{k} = 0, \quad \bar{\rho} = \rho_2\,, \quad \bar{u} = u_2\,, \tag{70}$$

$$s = s_1: \qquad \Lambda = 0, \quad \bar{\rho} = \rho_1\,, \quad \bar{u} = u_1\,. \tag{71}$$

Now the initial condition for (69) follows from the boundary condition (70) and has the form $\tilde{k}(\rho_2) = 0$.

It follows from (68) that $\Lambda = constant$ for $dX/dT = \Omega_s/\tilde{k}$. Applying the boundary condition (71) we obtain the speed of the leading edge in terms of conjugate variables:

$$s_1 = \frac{\Omega_s(\tilde{k}_1, \rho_1)}{\tilde{k}_1}\,, \tag{72}$$

where $\tilde{k}_1 = \tilde{k}(\rho_1)$. Note that owing to the definition of Ω_s via the conjugate frequency $\tilde{\omega}$, the formula (72) is equivalent to our original expression (22) for s_1 in terms of the usual frequency ω and the wavenumber k. Expression (72), however, is much more attractive for actual calculations since it does not contain any singular limiting transitions.

One cannot help noticing that equations (51) and (69) describing relationships between the modulation variables in the linear and the solitary wave trains are identical in terms of the "effective" dispersion relations $\Omega_0(k, \bar{\rho})$ and $\Omega_s(\tilde{k}, \bar{\rho})$. The latter, however, is yet to be obtained.

To obtain the solitary wave dispersion relation $\tilde{\omega}_s(\tilde{k}, \bar{\rho}, \bar{u})$ we observe that expressions (58) can be viewed as analogs of (54) for the conjugate travelling wave given by the equation

$$(\tilde{k}\tilde{\lambda}_{\tilde{\theta}})^2 = -R(\tilde{\lambda})\,, \qquad \tilde{\theta} = \tilde{k}\tilde{x} - \tilde{\omega}\tilde{t}\,, \qquad \tilde{\lambda}(\tilde{\theta} + 2\pi) = \tilde{\lambda}(\tilde{\theta})\,, \tag{73}$$

where $\tilde{x}$, $\tilde{t}$ are the new independent variables. This travelling wave is determined by the same (but inverted) potential curve $R(\lambda)$ as in (53) so that the oscillations now occur between the roots λ_2 and λ_1. For problems associated with polynomial potential curves, the functions $\lambda(\theta)$ and $i\tilde{\lambda}(i\tilde{\theta})$ represent the same analytic function in the complex θ-plane, which is an elliptic function with the periods 2π and $2\pi i$ along the real and the imaginary axes.

The next observation is that the solitary wave limit $\lambda_2 \to \lambda_1$ in the original travelling wave (53) corresponds to the vanishing amplitude limit in the conjugate travelling wave equation (73) (see Fig. 4b) and therefore, $\tilde{\omega}_s$ and $\tilde{k}$ must satisfy the *linear dispersion relation* for the conjugate dispersive hydrodynamics system, which is obtained from the original system (1) by the change of variables $\tilde{x} = ix$, $\tilde{t} = it$,

$$-i\tilde{\mathbf{U}}_{\tilde{t}} = \mathbf{K}_4(\tilde{\mathbf{U}}, -i\partial_{\tilde{x}}\tilde{\mathbf{U}}, -\partial^2_{\tilde{x}\tilde{x}}\tilde{\mathbf{U}}, \dots)\,, \tag{74}$$

where $\tilde{\mathbf{U}} = (\tilde{\rho}, \tilde{u})^T$ and $\tilde{\mathbf{U}} = \tilde{\mathbf{U}}(\tilde{\lambda})$. The transformation above is equivalent to a mere change of the dispersion sign in the original system (1). The conjugate linear dispersion relation of our interest is obtained by linearizing the system (74) in the way similar to (13), i.e. against the slowly varying mean background $< \tilde{U}_j > (X, T)$

$$\tilde{U}_j \approx < \tilde{U}_j > + \tilde{a}_j e^{i(\tilde{k}\tilde{x} - \tilde{\omega}\tilde{t})}\,, \quad \tilde{a}_j << 1\,, \tag{75}$$

and has the form

$$\tilde{\omega} = \tilde{\omega}_s(\tilde{k}, < \tilde{\rho} >, < \tilde{u} >). \tag{76}$$

Since in both original and conjugate travelling wave solutions the relationships $\mathbf{U} = \mathbf{U}(\lambda)$ and $\tilde{\mathbf{U}} = \tilde{\mathbf{U}}(\tilde{\lambda})$ do not contain independent variables and the operator $\mathbf{K}_4$ is the same in (1) and (74), the functions $\mathbf{U}(\lambda)$ and $\tilde{\mathbf{U}}(\tilde{\lambda})$ must be identical i.e.

$$\mathbf{U}(z) = \tilde{\mathbf{U}}(z) \tag{77}$$

Then it follows from (60) that in the limit

$$\lambda_2 \to \lambda_1 : \qquad < \tilde{\rho} > \to \bar{\rho} \, , \quad < \tilde{u} > \to \bar{u} \, . \tag{78}$$

Therefore, the solitary wave dispersion relation has the form $\tilde{\omega} = \tilde{\omega}_s(\tilde{k}, \bar{\rho}, \bar{u})$ and can be obtained from the original linear dispersion relation (12) by the formal change

$$k \mapsto i\tilde{k} \, , \quad \omega \mapsto i\tilde{\omega} \, , \quad \omega_0 \mapsto i\tilde{\omega}_s . \tag{79}$$

In other words, $\tilde{\omega}_s(\tilde{k}, \bar{\rho}, \bar{u}) = -i\omega_0(i\tilde{k}, \bar{\rho}, \bar{u})$ and, consequently,

$$\Omega_s(\tilde{k}, \bar{\rho}) = -i\Omega_0(i\tilde{k}, \bar{\rho}) \, . \tag{80}$$

Thus, the relationship between the parameters of the lead solitary wave in the undular bore is determined by the original *linear dispersion relation*! Now, having $\Omega_s(\tilde{k}, \bar{\rho})$ we integrate (69) and then find s_1 with the aid of formula (72). For our consideration to be self-consistent the inequalities (33) should be satisfied, which ensures validity of the simple undular bore transition condition (32) used in the derivation of (51), (69).

Remark. We emphasize that all the obtained relationships between the original and the conjugate averaged variables essentially represent algebraic identities between integrals of the form (54) and (58) associated with given potential curve $R(\lambda)$ and do not imply any connection between their spatio-temporal dynamics in the original and the conjugate systems (1) and (74).

6 Validation of the developed theory

Since we have made several fundamental assumptions in the derivation of the undular bore transition conditions (see Section 2), the developed construction requires validation. It can be done by applying our resulting formulas to the nonlinear dispersive systems where the transition conditions can be extracted either from exact analytic solution (for the completely integrable systems) or from results of the direct numerical simulation (for non-integrable systems).

6.1 Integrable shallow water equations

We consider undular bores in the integrable version of the shallow-water equations derived by Kaup [30]. This system, which is often referred to as the Kaup-Boussinesq (KB) system, has the form

$$\partial_t h + \partial_x (hu) + \frac{1}{4}\partial^3_{xxx} u = 0, \quad \partial_t u + u\partial_x u + \partial_x h = 0. \tag{81}$$

Here h is total depth of the fluid and u is horisontal velocity averaged over depth. The decay of an initial discontinuity problem for the KB system has been studied in [23] where the exact solutions of the corresponding Whitham equations have been constructed for all possible values of the initial jump within the hyperbolicity region. We now construct the transition conditions for the KB undular bore using the general theory developed in the previous sections and compare them with the corresponding relationships extracted from the full modulation solution of [23]. For that, we need the dispersionless limit of (81), which is defined by the Riemann invariants and the characteristic velocities

$$r_\pm = \frac{u}{2} \pm \sqrt{h}\,, \qquad V_\pm = u \pm \sqrt{h}\,, \tag{82}$$

and the generalized linear dispersion relation (12), which for the KB system has the form

$$\omega_0 = k\bar{u} + k\bar{h}^{1/2}\left(1 - k^2/4\bar{h}\right)^{1/2} \tag{83}$$

(i) Simple undular bore transition curve

For the simple undular bore propagating to the right the transition relation (48), taking into account that for the KB system $\sigma(h) = \sqrt{h}$, assumes the form already discussed in Section 4.1

$$u_2 - u_1 = 2(\sqrt{h_2} - \sqrt{h_1}) \tag{84}$$

This relationship agrees with the full modulation solution of the KB system [23].

(ii) Undular bore location

(a) Trailing edge

For the evaluation of the trailing edge s_2 of the KB undular bore we need two functions $V(\bar{h})$ and $\Omega_0(k, \bar{h})$ entering the basic ODE (51) (with $\bar{\rho} \equiv \bar{h}$). These functions are obtained by setting the restriction (48) which in our case assumes the form

$$\bar{u} = 2\bar{h}^{1/2} + u_1 - 2{h_1}^{1/2}\,, \tag{85}$$

– into (82), (83). As a result, we obtain

$$V(\bar{h}) = V_+(\bar{h}, \bar{u}(\bar{h})) = 3\bar{h}^{1/2} + u_1 - 2{h_1}^{1/2}\,, \tag{86}$$

$$\Omega_0(k, \bar{h}) = \omega_0(k, \bar{h}, \bar{u}(\bar{h})) = k[2\bar{h}^{1/2} + \bar{h}^{1/2}(1 - k^2/4\bar{h})^{1/2} + u_1 - 2{h_1}^{1/2}]\,. \tag{87}$$

The ODE (51) for $k(\bar{h})$ after simple algebra assumes the form

$$\bar{h}\frac{d\alpha}{d\bar{h}} = -\frac{1}{2}(\alpha + 1)\,, \qquad \alpha(h_1) = 1\,, \tag{88}$$

where instead of $k(\bar{h})$ we have used a combination $\alpha(\bar{h}) = (1 - k^2/4\bar{h})^{1/2}$. Integrating (88) we find $k^2(\bar{h}) = 16h_1^{1/2}(\bar{h}^{1/2} - h_1^{1/2})$. Then the trailing edge is found from formula (52), which becomes in our case

$$s_2 = \frac{\partial\Omega_0}{\partial k}|_{\bar{h}=h_2} = u_1 + h_2^{1/2} - 4{h_1}^{1/2}\frac{h_2^{1/2} - h_1^{1/2}}{2h_1^{1/2} - h_2^{1/2}}\,, \tag{89}$$

(b) Leading edge

The solitary wave dispersion relation $\Omega_s(\tilde{k}, \bar{h})$ is obtained from the linear dispersion relation $\Omega_0(k, \bar{h})$ (87) by the change of variables $k \mapsto i\tilde{k}$, $\Omega_0 \mapsto i\Omega_s$:

$$\Omega_s(\tilde{k}, \bar{h}) = \tilde{k}[2\bar{h}^{1/2} + \bar{h}^{1/2}(1 + \tilde{k}^2/4\bar{h})^{1/2} + u_1 - 2{h_1}^{1/2}]\,, \tag{90}$$

while $V(\bar{h})$ remains the same as in Eq. (86). Substituting this into ODE (69) for $\tilde{k}(\bar{h})$ we arrive at

$$\bar{h}\frac{d\tilde{\alpha}}{d\bar{h}} = -\frac{1}{2}(\tilde{\alpha}+1)\,, \qquad \tilde{\alpha}(h_2) = 1\,, \tag{91}$$

where $\tilde{\alpha}(\bar{h}) = (1+\tilde{k}^2/4\bar{h})^{1/2}$. Integrating (91) we get $\tilde{k}^2(\bar{h}) = 16h_2^{1/2}(h_2^{1/2} - \bar{h}^{1/2})$. The leading edge is then found from formula (72),

$$s_1 = \frac{\tilde{\Omega}_s}{\tilde{k}}|_{\bar{h}=h_1} = u_2 + h_1^{1/2}\,. \tag{92}$$

To make a comparison of the trailing and the leading edge speeds obtained here with the corresponding values from the full modulation solution in [23] we have to set $u_2 = 0$, $h_2 = 1$, $h_1 = (c+1)^2/4$, where $-1 < c < 1$ into (84), (89), (92). Then we get

$$u_1 = c - 1\,, \qquad s_2 = 2c - \frac{1}{c}\,, \qquad s_1 = \frac{1+c}{2}\,, \tag{93}$$

which agrees with the results of [23].

(iii) Inequalities

Now, we have to check fulfillment of the inequalities (33) to verify consistency of our construction. In view of (25), (84) the inequalities assume the form

$$u_1 - 2{h_1}^{1/2} + h_2^{1/2} < s_2 < u_1 - 2{h_1}^{1/2} + 3h_2^{1/2}\,, \quad s_1 > 3{h_1}^{1/2} + u_2 - 2{h_2}^{1/2}\,, \quad s_1 > s_2\,. \tag{94}$$

One can see that the second and the third inequalities (94) are satisfied if $h_2 > h_1$. Now without loss of generality setting $u_2 = 0$, $h_2 = 1$, $h_1 = (c+1)^2/4$, $-1 < c < 1$ as in (93) we see that right-hand part of the first inequality (94) is satisfied unconditionally. However, the left-hand part of this inequality imposes a restriction on the initial data compatible with the above analysis. Specifically, if $0 < c < 1/2$ the mentioned inequality is violated and that means that the corresponding discontinuity cannot be resolved with a single undular bore. This, again, agrees with the full modulation solution of [23] where the region $0 < c < 1/2$ implies "interaction of the undular bore and the rarefaction wave", i.e. two waves are involved.

6.2 Fully nonlinear ion-acoustic collisionless shocks in plasma

As an example of effective description of the undular bore transition in a non-integrable system we apply the obtained conditions to the classical system (6) describing finite-amplitude ion-acoustic waves in a two-temperature ($T_e \gg T_i$) collisionless plasma (see for instance [33]). For convenience we reproduce it once more:

$$\begin{aligned} &\partial_t\rho + \partial_x(\rho u) = 0,\\ &\partial_t u + u\partial_x u + \partial_x\phi = 0,\\ &\partial^2_{xx}\phi = e^{\phi} - \rho\,. \end{aligned} \tag{95}$$

The system (95) satisfies all the necessary requirements described in Section 2 to be amenable to methods developed in this paper. Its four conservation laws can be found in [43]. In the dispersionless limit, $\phi = \ln \rho$, and, therefore, the sound speed $\sigma = 1$ in (2), while the linear dispersion relation has the form

$$\omega_0(k, \bar{\rho}, \bar{u}) = k[\bar{u} + (1 + k^2/\bar{\rho})^{-1/2}] \,. \tag{96}$$

The undular bore transition condition (32) then assumes the form $u_2 - u_1 = \ln(\rho_2/\rho_1)$. Without loss of generality we put $u_1 = 0$, $\rho_1 = 1$. Then the relationship (48) between $\bar{\rho}$ and $\bar{u}$ in the extended dispersionless limit system (44) – (45) becomes $\bar{u} = \ln \bar{\rho}$. As a result, we get all the necessary ingredients (49) for the basic ordinary differential equations (51), (69):

$$V(\bar{\rho}) = \ln \bar{\rho} + 1 \,, \qquad \Omega_0(k, \bar{\rho}) = k[\ln \bar{\rho} + (1 + k^2/\bar{\rho})^{-1/2}] \,. \tag{97}$$

The equation (51), after elementary transformations, assumes the form with separated variables

$$\bar{\rho}\frac{d\alpha}{d\bar{\rho}} = -\frac{(1+\alpha)^2\alpha}{2(1+\alpha+\alpha^2)} \,, \qquad \alpha(1) = 1 \,, \tag{98}$$

where $\alpha = (1 + k^2/\bar{\rho})^{-1/2}$. Integrating (98) we get

$$\ln \bar{\rho} + 2\ln\alpha + \frac{1-\alpha}{1+\alpha} = 0 \,. \tag{99}$$

Now, using (52) we obtain a simple implicit formula determining velocity of the trailing edge s_2 in terms of the density ratio across the collisionless shock $d = \rho_2/\rho_1 = \rho_2$,

$$\ln d + \frac{2}{3}\ln(s_2 - \ln d) = \frac{(s_2 - \ln d)^{1/3} - 1}{(s_2 - \ln d)^{1/3} + 1} \,. \tag{100}$$

The leading edge is handled in the same way. The solitary wave dispersion relation is obtained from Eq. (97) using the relationship (80) and has the form

$$\tilde{\Omega}_s(\tilde{k}, \bar{\rho}) = \tilde{k}[\ln \bar{\rho} + (1 - \tilde{k}^2/\bar{\rho})^{-1/2}] \,. \tag{101}$$

Then, integrating Eq. (69) we obtain $\tilde{k}_s(\bar{\rho})$ (it is also convenient to introduce $\tilde{\alpha} = (1 - \tilde{k}^2/\bar{\rho})^{-1/2}$ as an intermediate variable instead of $\tilde{k}$, cf.(98)). Substituting it into Eq. (72) we eventually get for the leading edge

$$\frac{1-s_1}{1+s_1} + 2\ln s_1 = \ln d \,. \tag{102}$$

One can easily verify that all inequalities (33) are satisfied provided $d > 1$. The curves $s_1(d)$ and $s_2(d)$ are presented in Fig. 5 and demonstrate complete agreement with results of direct numerical simulation of the decay of an initial discontinuity for the system (95) obtained in [27]. From theoretical point of view this agreement can be viewed a strong indication of validity of the modulation theory in non-integrable initial value problems where rigorous derivation of the Whitham asymptotics is not available.

The weakly nonlinear asymptotics of (100) and (102) for $\delta \equiv d - 1 \ll 1$:

$$s_2 \approx 1 - \delta \,, \qquad s_1 \approx 1 + \frac{2}{3}\delta \,, \tag{103}$$

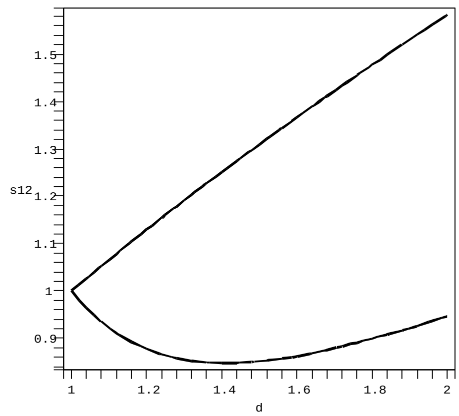

Figure 5: Speeds $s_{1,2}$ of the ion-acoustic undular bore edges versus density jump $d = \rho_2/\rho_1$ across the undular bore; upper curve s_1 – leading edge, lower curve s_2 – trailing edge

corresponds to the boundaries in the well-known exact modulation solution for the KdV equation found by Gurevich and Pitaevskii [17], which is another confirmation of validity of our approach. Curiously, as is clearly seen from Fig. 5, the fully nonlinear dynamics of the leading (solitary wave) edge s_1 is quite well approximated by the weakly nonlinear asymptotics (103) while the speed of the trailing (vanishing amplitude) edge s_2 demonstrates significant qualitative and quantitative deviations from its weakly nonlinear analog even for quite moderate values of the initial jump.

7 Conclusions

An asymptotic dynamics of undular bores in 2×2 hyperbolic systems modified by small dispersion has been studied. By postulating modulation description and the Gurevich - Pitaevskii type natural boundary conditions we have derived a complete set of transition conditions for the undular bore connecting two different constant states (ρ_1, u_1) and (ρ_2, u_2). The results can be summarized as follows. Let the dispersionless limit of the governing equations be represented in the Riemann form with the Riemann invariants $r_\pm(\rho, u)$ and the respective characteristic velocities $V_\pm(\rho, u)$ such that the $r_- = constant$ corresponds to the simple wave propagating to the right. Let the linear dispersion relation for the wave propagating on the background $\bar{\rho}$, $\bar{u}$ have the form $\omega = \omega_0(k, \bar{\rho}, \bar{u})$. Then the transition conditions for the undular bore propagating to the right include:

(i) A relationship between the admissible values of (ρ_1, u_1) and (ρ_2, u_2) that can be connected by a single undular bore

$$r_-(\rho_1, u_1) = r_-(\rho_2, u_2) \equiv r_0 \tag{104}$$

(ii) The speeds (self-similar co-ordinates) of the undular bore edges $s_{1,2}$ determined by the expressions

$$s_2 = \frac{\partial \Omega_0}{\partial k}(k_2, \rho_2)\,, \qquad s_1 = \frac{\Omega_s(\tilde{k}_1, \rho_1)}{\tilde{k}_1}\,, \tag{105}$$

where

$$\Omega_0(k,\bar\rho)=\omega_0(k,\bar\rho,\bar u(\bar\rho)) \qquad \Omega_s(\tilde k,\bar\rho)=-i\Omega_0(i\tilde k,\bar\rho)\,. \tag{106}$$

Here the dependence $\bar u=\bar u(\bar\rho)$ is given by

$$r_-(\bar\rho,\bar u)=r_0\,. \tag{107}$$

The parameters k_2, $\tilde k_1$ in Eqs. (105) are found as $k_2=k(\rho_2)$, $\tilde k_1=\tilde k(\rho_1)$, where the functions $k(\bar\rho)$ and $\tilde k(\bar\rho)$ are determined from the ordinary differential equations:

$$\frac{dk}{d\bar\rho}=\frac{\partial\Omega_0/\partial\bar\rho}{V(\bar\rho)-\partial\Omega_0/\partial k}\,, \qquad k(\rho_1)=0\,, \tag{108}$$

$$\frac{d\tilde k}{d\bar\rho}=\frac{\partial\Omega_s/\partial\bar\rho}{V(\bar\rho)-\partial\Omega_s/\partial\tilde k}\,, \qquad \tilde k(\rho_2)=0\,, \tag{109}$$

where $V(\bar\rho)=V_+(\bar\rho,\bar u(\bar\rho))$.

(iii) Inequalities providing consistency of (i) and (ii):

$$V_-(\rho_2,u_2)<s_2<V_+(\rho_2,u_2)\,, \quad V_+(\rho_1,u_1)<s_1\,, \quad s_1>s_2\,. \tag{110}$$

The construction (104) – (110) in no way relies on the integrability of the original nonlinear dispersive system (1).

To validate the assumptions underlying our analysis, the obtained general conditions have been applied to the integrable Kaup-Boussinesq system, where the full modulation solution for the undular bore is available from [23] and also to the non-intergrable system describing fully nonlinear ion-acoustic waves in collisionless plasma studied numerically in [27]. In both cases a complete agreement with previous analytical/numerical results is shown. The agreement in the latter (non-integrable) case can also be viewed as a strong indication of validity of the Whitham asymptotics in the initial-value problems for non-integrable dispersive wave systems at least for a certain class of initial data.

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References

[1] Lax, P.D. *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves.* Conference Board of the Mathematical Sciences, Regional Conference Series in Applied Mathematics, SIAM, Philadelphia (1973).

[2] T.B. Benjamin and M.J. Lighthill, Proc. Roy. Soc. A **254** (1954) 448

[3] R.Z. Sagdeev, in: M.A. Leontovich ed. *Reviews of Plasma Physics*, Volume 4, p. 20. Consultants Bureau. New York (1968)

[4] R.S. Johnson, J. Fluid Mech **42** (1970) 49.

[5] A.V. Gurevich and L.P. Pitaevskii, Sov Phys JETP **66** (1987) 490.

[6] V.V. Avilov, I.M. Krichever, and S.P. Novikov, Sov. Phys. Dokl., **32,** (1987) 564 .

[7] G.A. El, R.H.J. Grimshaw and A.M. Kamchatnov *Analytic model for a weakly dissipative shallow-water undular bore*, Chaos, v 15 (2005) (accepted).

[8] A.Porter and N.F. Smyth, J. Fluid Mech. **454** (2002) 1.

[9] S. Darmanyan, A. Kamchatnov, and F. Lederer, Phys. Rev. E58, No.4, R4120-23, (1998).

[10] Yu. Kodama, SIAM J. Appl. Math. **59** (1999) 2162.

[11] A.V. Gurevich, *Nonlinear phenomena in the Ionosphere*, Springer, New York (1978).

[12] P.D. Lax and C.D. Levermore I – III, Comm. Pure Appl. Math. **36** (1983) 253, 571, 809.

[13] S. Venakides, Comm. Pure Appl. Math. **38** (1985) 125.

[14] P.D. Lax, C.D. Levermore and S. Venakides, in: A.S. Focas and V.E. Zakharov eds. Important developments in soliton theory p. 205. Springer Ser. Nonlinear Dynam., Springer. Berlin (1994).

[15] G.B. Whitham, Proc.Roy.Soc. **A283** (1965) 238.

[16] H. Flaschka, G. Forest and D.W. McLaughlin, 1979 Comm. Pure Appl. Math. **33** (1979) 739.

[17] A.V. Gurevich and L.P. Pitaevsky, Sov.Phys.JETP **38** (1974) 291.

[18] Dubrovin, B.A. & Novikov, S.P. 1989 Russian Math. Surveys **44** (1989) 35.

[19] A.V. Gurevich, A.L. Krylov, Sov.Phys.JETP **65** (1987) 944

[20] G.A. El, V.V. Geogjaev, A.V. Gurevich and A.L. Krylov, Physica D **87** (1995) 186

[21] A.V. Gurevich, A.L. Krylov and G.A. El, Sov.Phys.JETP (1992) 957.

[22] M.C. Jorge, A.A. Minzoni and N.F. Smyth, Physica D **132** (1999) 1.

[23] G.A. El, R.H.J. Grimshaw, and M.V. Pavlov, Stud. Appl. Math. **106** (2001) 157.

[24] A.N. Kamchatnov, *Nonlinear periodic waves and their modulations – an introductory course*, World Scientific, Singapore (2000).

[25] W. Choi and R. Camassa, J. Fluid Mech. **396** (1999) 1.

[26] G.A. El, A.L. Krylov, S. Venakides, Comm Pure Appl Math **54** (2001) 1243.

[27] A.V. Gurevich, A.P. Meshcherkin, Sov. Phys. JETP **60** (1984) 732.

[28] G.A. El, V.V. Khodorovskii, A.V. Tyurina, Phys. Lett. A **318**(2003) 526.

[29] G.A. El, Resolution of a shock in hyperbolic systems modified by weak dispersion , to be published in focus issue of Chaos on *"Solitary waves in non-integrable systems"* (2005)(accepted). nlin.PS/0503010

[30] D.J. Kaup, Progr.Theor.Phys, **54** (1976) 396.

[31] A.E. Green, P.M. Naghdi, J.Fluid Mech. **78** (1976) 237.

[32] S.L. Gavrilyuk, V.M. Teshukov, Continuum Mech. Termodyn. **13** (2001) 365.

[33] V.I. Karpman, Nonlinear Waves in Dispersive Media, Pergamon, Oxford (1975).

[34] S. Jin, C.D. Levermore and D.W. McLaughlin, Comm. Pure Appl. Math **52** (1999) 512.

[35] S. Venakides, Comm. Pure Appl. Math. **43** (1990) 335.

[36] P. Deift, S. Venakides and X. Zhou, International Math. Research. Journ., no 4, (1997) 285.

[37] R.F. Bikbaev Funct Anal Appl **23** (1989) 257.

[38] V.V. Avilov and S.P. Novikov, Sov. Phys. Dokl., **32** (1987) 32

[39] A.V.Tyurina, G.A. El, Journ. Exp.Theor. Phys **88** (1999) 615.

[40] B.L. Rozhdestvensky and N.N. Yanenko *Systems of Quasilinear Equations and their Applications in Gas Dynamics* [in Russian]. Nauka, Moscow (1978).

[41] G.B. Whitham, *Linear and Nonlinear Waves*, Wiley, New York (1974).

[42] L.D. Landau and E.M. Lifshitz *Fluid Mechanics* Pergamon, Oxford (1987).

[43] A. V. Gurevich, A. L. Krylov, and G. A. El, Sov. Phys. JETP **71** (1990) 899.