Modulating pulse solutions for quasilinear wave equations

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Abstract

This paper presents an existence proof for symmetric modulating pulse solutions of a quasilinear wave equation. Modulating pulse solutions consist of a pulse-like envelope advancing in the laboratory frame and modulating an underlying wave-train; they are also referred to as 'moving breathers' since they are time-periodic in a moving frame of reference. The problem is formulated as an infinite-dimensional dynamical system with two stable, two unstable and infinitely many neutral directions. Using a partial normal form and a generalisation of local invariant-manifold theory to the quasilinear setting we prove the existence of modulating pulses on arbitrarily large, but finite domains in space and time.

1 Introduction

The following theory applies to a general class of quasilinear scalar wave equations with odd nonlinearity, in particular to certain models used in nonlinear optics, which are characterised by the fact that the focussing nonlinear Schrödinger equation appears as a modulation equation in the sense explained below. In order to keep the notation simple we will concentrate upon the prototype quasilinear wave equation of this kind, namely

$$\partial_t^2 u = \partial_x^2 u - u - u^2 \partial_x^2 u \tag{1}$$

on the infinite line $x \in \mathbb{R}$.

It is well known that on timescales of order $\mathcal{O}(1/\varepsilon^2)$ equation (1) has $\mathcal{O}(\varepsilon)$-amplitude solutions which are slow spatial and temporal modulations of an underlying wave train $\mathrm{e}^{\mathrm{i}(k_0 x - \omega_0 t)}$, where k_0 and ω_0 are related by the linear dispersion relation $\omega_0^2 = k_0^2 + 1$. Such solutions are described by the formula

$$u_A(x,t) = \varepsilon(A(X,T)\mathrm{e}^{\mathrm{i}(k_0 x - \omega_0 t)} + \mathrm{c.c.}) + \mathcal{O}(\varepsilon^2),$$

where $X = \varepsilon(x - c_{\mathrm{g}}' t)$, $T = \varepsilon^2 t$, $c_{\mathrm{g}}' = k_0/(1+k_0^2)^{1/2}$ is the linear group velocity and A satisfies the nonlinear Schrödinger equation

$$2\mathrm{i}\omega_0 \partial_T A + (1 - (c_{\mathrm{g}}')^2)\partial_X^2 A + 3k_0^2 |A|^2 A = 0 \tag{2}$$

(e.g. see Kalyakin [7], Kirrmann, Schneider & Mielke [10] and Schneider [14]).

Equation (2) possesses a three-parameter family of time-periodic solutions of the form

$$A(X,T) = B(X - X_0)\mathrm{e}^{-\mathrm{i}\gamma_0 T}\mathrm{e}^{\mathrm{i}\phi_0},$$

in which the real-valued function B satisfies the second-order ordinary differential equation

$$\partial_X^2 B = C_1 B - C_2 B^3, \tag{3}$$

where $C_1 = -2\gamma_0\omega_0/(1-(c_{\mathrm{g}}')^2)$ and $C_2 = 3k_0^2/(1-(c_{\mathrm{g}}')^2)$. For $\gamma_0 < 0$ and $\omega_0 > 0$ this equation has two homoclinic solutions

$$B_{\mathrm{pulse}}(X) = \pm\left(\frac{2C_1}{C_2}\right)^{1/2} \operatorname{sech}(C_1^{1/2} X)$$

which connect the origin with itself. We have therefore identified *modulating pulse solutions* of equation (1) which are described by the approximate formula

$$\begin{aligned} u_{\mathrm{pulse}} &= \varepsilon(B_{\mathrm{pulse}}(X)\mathrm{e}^{-\mathrm{i}\gamma_0 T}\mathrm{e}^{\mathrm{i}(k_0 x - \omega_0 t)} + \mathrm{c.c.}) \\ &= \varepsilon(B_{\mathrm{pulse}}(\varepsilon(x - c_{\mathrm{g}}' t))\mathrm{e}^{\mathrm{i}k_0(x - (c_{\mathrm{p}}' + \gamma_1\varepsilon^2)t)} + \mathrm{c.c.}) \end{aligned}$$

over timescales of order $\mathcal{O}(1/\varepsilon^2)$; here $c_{\mathrm{p}}' = (1+k_0^2)^{1/2}/k_0$ is the linear phase velocity and $\gamma_1 = \gamma_0/k_0$.

In this paper we consider whether equation (1) possesses modulating pulse solutions which exist on longer timescales than $\mathcal{O}(1/\varepsilon^2)$. We establish the following result.

Theorem 1.1 *Fix a positive integer n and positive real numbers k_0 and L. There exists $\varepsilon_0 > 0$ (depending upon n, k_0 and L) such that for all $\varepsilon \in (0, \varepsilon_0)$ we have an infinite-dimensional, continuous family of modulating pulse solutions to equation (1) of the form*

$$u(x,t) = v(x - c_{\mathrm{g}} t, x - c_{\mathrm{p}} t),$$

where v is $2\pi/k_0$-periodic in its second argument and

$$c_{\mathrm{p}} = c_{\mathrm{p}}' + \gamma_1 \varepsilon^2, \qquad c_{\mathrm{g}} = \frac{1}{c_{\mathrm{p}}}.$$

These solutions satisfy

$$v(\xi, y) = v(-\xi, -y), \qquad |v(\xi,y) - 2q(\xi,\varepsilon)\cos k_0 y| \le \varepsilon^{n+1}$$

for all $y \in \mathbb{R}$ and $\xi \in [-L/\varepsilon^n, L/\varepsilon^n]$, where

$$q(\xi,\varepsilon) = \varepsilon B_{\mathrm{pulse}}(\varepsilon\xi) + \mathcal{O}(\varepsilon^2)$$

and $\lim_{\xi\to\pm\infty} q(\xi,\varepsilon) = 0$.

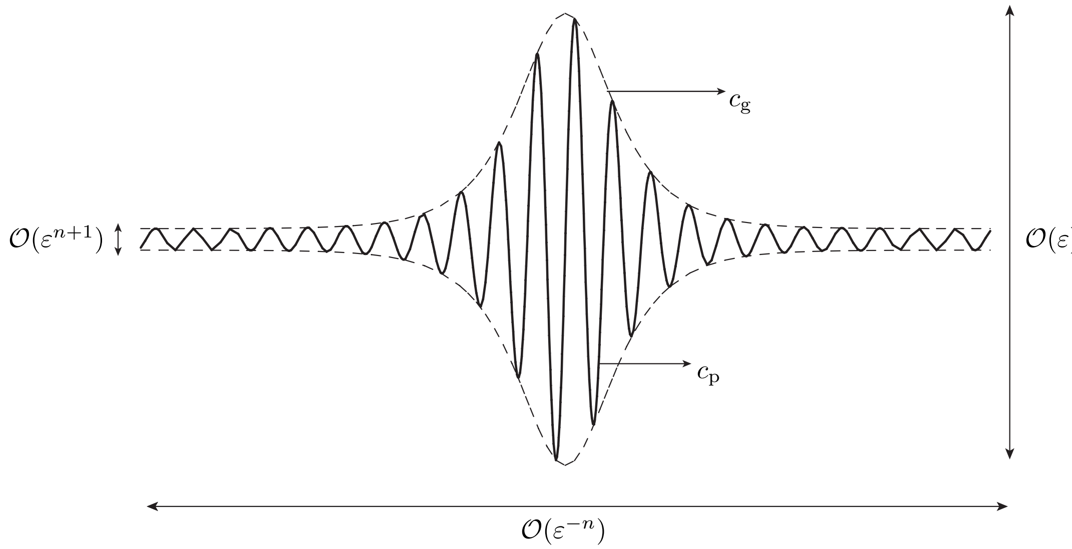

Figure 1: A modulating pulse solution guaranteed by Theorem 1.1.

The modulating pulse solutions guaranteed by Theorem 1.1 consist of a permanent pulse-like envelope with amplitude of order $\mathcal{O}(\varepsilon)$ which moves with constant speed c_{g} and modulates a periodic wave train moving with velocity c_{p}. A modulating pulse of this kind is shown in Figure 1. Notice that their existence is not established for $\xi \in \mathbb{R}$, but for $\xi \in [-L/\varepsilon^n, L/\varepsilon^n]$ (which is much larger than $\mathcal{O}(1/\varepsilon^2)$ for $n > 2$); this restriction is due to the lack of a sufficiently general global existence theory for quasilinear wave equations (see below).

Theorem 1.1 is proved by formulating the governing equation for $v(\xi, y)$ as an evolutionary problem in which the unbounded spatial coordinate ξ plays the role of time. This idea was introduced for nonlinear problems by Kirchgässner [9] and has become known as 'spatial dynamics'. The evolutionary problem is considered as an infinite-dimensional, reversible dynamical system in which the coordinates are the components of the Fourier-series expansion of $v(\cdot, y)$. The spectrum of the linearised system consists of infinitely many purely imaginary eigenvalues together with one positive and one negative real eigenvalue which are both of geometric multiplicity two and $\mathcal{O}(\varepsilon)$ in the bifurcation parameter. Our task is to find pulse-like solutions of this dynamical system.

The basic difficulty in constructing pulse-like solutions is readily appreciated by a heuristic argument for finite-dimensional dynamical systems. Consider an m-dimensional dynamical system with a large number of purely imaginary eigenvalues together with one positive and one negative real eigenvalue of geometric multiplicity two. In this situation the stable and unstable manifolds of the zero equilibrium are both two dimensional, and the existence of a homoclinic orbit (a pulse which tends to zero as $\xi \to \infty$) would imply an intersection of these manifolds in the m-dimensional phase space. Such an intersection, and hence the existence of a pulse, is therefore a nongeneric phenomenon for $m > 4$. A more promising approach is to relax the requirement that a pulse should decay to zero as $\xi \to \infty$ and accept as pulses those solutions whose amplitude at infinity is much less than at $\xi = 0$. Pulse solutions of this type can be found by considering intersections of an appropriately defined *global centre-stable manifold* with the *symmetric section.* The global centre-stable manifold W^{cs} in question comprises the initial data for solutions of the form $v(\xi) = q(\xi) + w(\xi)$, where q is an $\mathcal{O}(\varepsilon)$ homoclinic solution to an approximate dynamical system (for example a truncated normal form) and w is a solution which remains $\mathcal{O}(\varepsilon^{n+1})$ for all $\xi > 0$. The symmetric section Σ on the other hand consists of those points in phase space which are invariant under the reversibility operation; orbits passing through Σ are therefore necessarily symmetric with points of symmetry at their points of intersection with Σ. Intersections between $W^{\rm cs}$ and Σ are generic since $W^{\rm cs}$ has codimension two and Σ is a hyperplane of dimension $m/2$.

The situation in an infinite-dimensional setting for pulses which decay to zero as $\xi \to \infty$ is clearly considerably worse, since the two-dimensional stable and unstable manifolds are now required to intersect in an infinite-dimensional phase space, and in fact such an intersection typically only occurs in the presence of additional geometric structure such as complete integrability. This argument is reinforced by work of Denzler [4], who proved mathematically that a certain class of nonlinear wave equation has no homoclinic solution. The heuristic argument indicates that intersections of $W^{\rm cs}$ and Σ are still to be expected in infinite dimensions, although one now has the additional technical difficulty of constructing a global existence theory to facilitate a discussion of the relevant manifolds.

The above spatial dynamics approach to modulating pulses was introduced for semilinear wave equations such as

$$\partial_t^2 u = \partial_x^2 u - u + u^3$$

by Groves & Schneider [6], where a global centre-stable manifold $W^{\rm cs}$ (around the $\mathcal{O}(\varepsilon)$ homoclinic solution q identified by the nonlinear Schrödinger approximation) was shown to intersect the symmetric section Σ in a continuum of points. The centre-stable manifold is constructed by modifying the nonlinearities using a cut-off function to make them globally Lipschitz with a small Lipschitz constant; global solutions of the modified problem are then found by reformulating the dynamical system as an integral equation and applying semigroup theory and a contraction-mapping argument. These global solutions of the modified problem yield local solutions to the original problem, and in fact *a priori* estimates show that they remain within the critical neighbourhood of the origin defined by the cut-off function over an $\mathcal{O}(\varepsilon^n)$ timescale, so that they solve the original problem on this timescale. In this fashion we obtain a *local* centre-stable manifold $W^{\rm cs}_{\rm loc}(0)$, whose intersections with Σ yield symmetric modulating pulses which exist for $\xi \in [-L/\varepsilon^n, L/\varepsilon^n]$. The following supplemental argument is required to obtain a global centre-stable manifold $W^{\rm cs}$ from $W^{\rm cs}_{\rm loc}(0)$, or equivalently to show that any solution $q(\xi) + w(\xi)$ with $w(0) \in W^{\rm cs}$ satisfies $w(\xi) = \mathcal{O}(\varepsilon^{n+1})$ for all $\xi > 0$. Working with the modified system again, one can show that a solution of the form $q(\xi) + w(\xi)$, where $w(0) \in W^{\rm cs}$, converges exponentially to a solution $v(\xi)$ on the *centre manifold* $W^{\rm c}$, a graph in phase space upon which all solutions remain so long as they are $O(\varepsilon^{n+1})$. By construction $W^{\rm c}$ is globally invariant for the modified problem, but the existence of a Lyapunov function (the Hamiltonian function for the wave equation) shows that $W^{\rm c}$ is actually also globally invariant for the modified problem. Careful book-keeping shows that control by the Lyapunov function takes hold at a time smaller than L/ε, so that w is $\mathcal{O}(\varepsilon^{n+1})$ for all $\xi > 0$.

The technique used by Groves & Schneider [6] relies heavily upon semilinearity, in particular that global existence theory is available for globally Lipschitz nonlinearities with small Lipschitz constant. It is therefore not applicable to the present problem, whose spatial dynamics formulation consists of a quasilinear wave equation coupled with a dynamical system. Instead we directly apply an *iteration scheme* of the type suggested by Kato [8] to obtain a local existence theory over a timescale of $\mathcal{O}(\varepsilon^n)$ and construct a local centre-stable manifold $W^{\rm cs}_{\rm loc}$ by this route; intersections of this manifold with the symmetric section yield symmetric modulating pulses which exist for $\xi \in [-L/\varepsilon^n, L/\varepsilon^n]$. We therefore follow the strategy developed by Groves & Schneider [6] while replacing the organising centre of Hamiltonian structure in that reference by quasilinearity here. The spatial dynamics formulation is introduced in Section 2 and a suitable approximate system related to the nonlinear Schrödinger equation is derived by normal-form techniques in Section 3. The approximate system has a four-dimensional invariant subspace containing a symmetric homoclinic orbit q (Section 4) around which the local centre-stable manifold $W^{\mathrm{cs}}_{\mathrm{loc}}$ is constructed in Section 5. The proof of Theorem 1.1 is completed in Section 6, where it is shown that $W^{\mathrm{cs}}_{\mathrm{loc}}$ intersects the symmetric section in a continuum of points.

The result presented by Groves & Schneider [6] appears to be optimal in the sense that that modulating pulses exist for all $\xi \in \mathbb{R}$. It is however obtained by exploiting the following advantageous features of the spatial dynamics formulation of a semilinear wave equation.

(i) At the bifurcation point there is no spectrum outside the imaginary axis;

(ii) The nonlinearities contain no quadratic terms;

(iii) Results concerning invariant-manifold theory are readily generalised from finite-dimensional dynamical systems to semilinear evolutionary equations;

(iv) The norm induced by the restriction of the Hamiltonian function to the centre manifold is equivalent to that of the phase space.

Since many other physical problems without the above mathematical features also admit nonlinear Schr̈odinger equations as a modulation equations it is natural to seek existence theories for modulating pulse solutions for these problems. The result in the present paper represents a contribution to this programme in that it provides a satisfactory result when semilinearity is replaced by quasilinearity (so that feature (iii) is lost) but features (i) and (ii) are retained. Our result amounts to an existence theory for local invariant manifolds which is equivalent to that for semilinear wave equations when feature (iv) is lost.

The extension of our method to other problems in which further features from the above list are no longer present is planned as future research. Reduction theorems for quasilinear problems with respectively finite- and infinite-dimensional centre and hyperbolic parts have been given by Mielke [11] and Renardy [12], and it is likely that they generalise, in the semilinear setting, to reduction principles which indicate that problems whose centre and hyperbolic parts are both infinite-dimensional can be reduced to problems with feature (i). This approach may be helpful for problems such as lattice dynamics and Boussinesq equations. The paradigm for modulating pulses would however appear to be the two-dimensional water-wave problem: the cubic nonlinear Schrödinger equation has been identified as the relevant modulation equation (see Zakharov [15], Ablowitz & Segur [1] and Craig, Sulem & Sulem [3]), but the full problem admits none of the favourable properties listed above.

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2 Spatial dynamics formulation

We look for modulating pulse solutions of the nonlinear wave equation (1) of the form

$$u(x,t) = v(x - c_{\mathrm{g}}t, x - c_{\mathrm{p}}t) = v(\xi, y),$$

where v is periodic in y with period $2\pi/k_0$ for some $k_0 > 0$. Making this *Ansatz*, one arrives at the equation

$$(1-c_{\mathrm{g}}^2)\partial_\xi^2 v + 2(1-c_{\mathrm{g}}c_{\mathrm{p}})\partial_\xi\partial_y v + (1-c_{\mathrm{p}}^2)\partial_y^2 v - v - v^2\partial_\xi^2 v - 2v^2\partial_y\partial_\xi v - v^2\partial_y^2 v = 0.$$

It is convenient to choose

$$c_{\mathrm{p}} = c_{\mathrm{p}}' + \gamma_1\varepsilon^2, \qquad c_{\mathrm{g}} = 1/\mathrm{c}_p,$$

so that c_{p} is a small perturbation of the phase velocity c_{p}' of the linearised problem and the equation simplifies to

$$\partial_\xi^2 v + \frac{1-c_{\mathrm{p}}^2 - v^2}{1-c_{\mathrm{g}}^2 - v^2}\partial_y^2 v - \frac{1}{1-c_{\mathrm{g}}^2 - v^2}v - \frac{2}{1-c_{\mathrm{g}}^2 - v^2}v^2\partial_y\partial_\xi v = 0. \tag{4}$$

We study this equation, which is still a quasilinear wave equation, in the phase space

$$\mathcal{X}^s = \{(v, v_\xi) \in H^{s+1}_{\mathrm{per}}(0, 2\pi/k_0) \times H^s_{\mathrm{per}}(0, 2\pi/k_0)\}, \qquad s > 1,$$

so that the equation itself holds in $H^{s-1}_{\mathrm{per}}(0, 2\pi/k_0)$.

Equation (4) has two discrete symmetries which play an important role in the following theory. Firstly, it is *reversible*, that is invariant with respect to the transformation $\xi \mapsto -\xi$, $(v, \partial_\xi v) \mapsto S(v, \partial_\xi v)$, where the *reverser* S is defined by the formula

$$S(v(y), v_\xi(y)) = (v(-y), -v_\xi(-y)).$$

This symmetry has the consequence that $v(-\xi, -y)$ solves the equation whenever $v(\xi, y)$ is a solution. It also exhibits *antisymmetry in the dependent variable:* the equation is invariant under the transformation $(v, \partial_\xi v) \mapsto (-v, -\partial_\xi v)$.

We may express an element of $H^s_{\mathrm{per}}(0, 2\pi/k_0)$ as a Fourier series

$$v(y) = \sqrt{\frac{k_0}{\pi}}\sum_{m=1}^\infty \{v_{m,\mathrm{o}}\sin(k_0 m y) + v_{m,\mathrm{e}}\cos(k_0 m y)\}$$

and define projections $P_{m,\mathrm{o}}, P_{m,\mathrm{e}}, P_m : H^s_{\mathrm{per}}(0, 2\pi/k_0) \to H^s_{\mathrm{per}}(0, 2\pi/k_0)$ by the formulae

$$\begin{aligned} P_{m,\mathrm{o}}\left(\sqrt{\frac{k_0}{\pi}}\sum_{j=1}^\infty (v_{j,\mathrm{o}}(\xi)\sin(k_0 j y) + v_{j,\mathrm{e}}(\xi)\cos(k_0 j y))\right) &= \sqrt{\frac{k_0}{\pi}}v_{m,\mathrm{o}}(\xi)\sin(k_0 m y), \\ P_{m,\mathrm{e}}\left(\sqrt{\frac{k_0}{\pi}}\sum_{j=1}^\infty (v_{j,\mathrm{o}}(\xi)\sin(k_0 j y) + v_{j,\mathrm{e}}(\xi)\cos(k_0 j y))\right) &= \sqrt{\frac{k_0}{\pi}}v_{m,\mathrm{e}}(\xi)\cos(k_0 m y) \end{aligned}$$

with $P_m = P_{m,\mathrm{o}} + P_{m,\mathrm{e}}$. By extending the Fourier series coordinatewise to vector-valued functions we find that the phase space $\mathcal{X}^s$ decomposes into a direct sum $\oplus_{m\in\mathbb{N}_0} E_m$ of subspaces, where

$$E_m = E_{m,\mathrm{o}} \oplus E_{m,\mathrm{e}}, \qquad E_{m,\mathrm{o}} = \{(v_{m,\mathrm{o}}, \partial_\xi v_{m,\mathrm{o}})\}, \quad E_{m,\mathrm{e}} = \{(v_{m,\mathrm{e}}, \partial_\xi v_{m,\mathrm{e}})\}.$$

We may therefore write

$$\mathcal{X}^s = \ell^{s+1} \times \ell^s,$$

in which

$$\ell^t = \{v = \{v_m\}_{m\in\mathbb{N}} \mid \|v\|_t^2 := \sum_{m=1}^{\infty} m^{2t} |v_m|^2 < \infty\}, \qquad v_m = (v_{m,\mathrm{o}}, v_{m,\mathrm{e}}),$$

and $P_{m,\mathrm{o}}, P_{m,\mathrm{e}}, P_m$ also extend naturally to projections $\mathcal{X}^s \to \mathcal{X}^s$ which are denoted by the same symbols. Notice that P_m is infinitely smoothing due to its finite-dimensional range, so that

$$\|P_m v\|_{t_1} \le C_{m,s,t} \|v\|_{t_2}, \qquad t_1 \ge t_2;$$

the same smoothing property is enjoyed by $P_{m,\mathrm{o}}$ and $P_{m,\mathrm{e}}$. The action of the reverser S in the new coordinate system is readily confirmed to be

$$S(v_\mathrm{o}, v_\mathrm{e}, \partial_\xi v_\mathrm{o}, \partial_\xi v_\mathrm{e}) = (-v_\mathrm{o}, v_\mathrm{e}, \partial_\xi v_\mathrm{o}, -\partial_\xi v_\mathrm{e}),$$

where $v_\mathrm{o} = \{v_{m,\mathrm{o}}\}$, $v_\mathrm{e} = \{v_{m,\mathrm{e}}\}$. Note also that the periodicity in y combines with the translation invariance in this variable to give an $O(2)$ symmetry represented in the new coordinates by

$$\{(v_{m,\mathrm{o}}, v_{m,\mathrm{e}}, \partial_\xi v_{m,\mathrm{o}}, \partial_\xi v_{m,\mathrm{e}})\} \mapsto \{(R_{mk_0a}(v_{m,\mathrm{o}}, v_{m,\mathrm{e}}), R_{mk_0a}(\partial_\xi v_{m,\mathrm{o}}, \partial_\xi v_{m,\mathrm{e}}))\}, \qquad a \in \mathbb{R},$$

where R_θ is the 2×2 matrix representing a rotation through the angle θ.

The spectrum of the linearised system associated with (4) was calculated by Groves & Schneider [6]; we recall the complete result since extensive use is made of it in the following analysis. The mth Fourier component satisfies the ordinary differential equation

$$\partial_\xi^2 v_m = \frac{m^2 k_0^2 (1 - c_\mathrm{p}^2) + 1}{(1 - c_\mathrm{g}^2)} v_m, \tag{5}$$

and the associated eigenvalues $\lambda_{m,\varepsilon}$ of this equation are given by

$$\begin{aligned} \lambda_{m,\varepsilon}^2 &= \frac{m^2 k_0^2 (1 - c_\mathrm{p}^2) + 1}{(1 - c_\mathrm{g}^2)} \\ &= (k_0^2 + 1)(1 - m^2) - 2k_0 (1 + k_0^2)^{1/2} (k_0^2 + m^2) \gamma_1 \varepsilon^2 + \mathcal{O}(\varepsilon^4), \end{aligned}$$

in which the $\mathcal{O}(\varepsilon^4)$ estimate on the remainder term holds uniformly in m.

$\boldsymbol{m = 0}$: We have two simple, real eigenvalues $\pm\lambda_{0,\varepsilon} = \pm(1 + k_0^2)^{1/2} + \mathcal{O}(\varepsilon^2)$. The corresponding eigenvectors are given by

$$\begin{pmatrix} v \\ \partial_\xi v \end{pmatrix} = \begin{pmatrix} 1 \\ \pm\lambda_{1,\varepsilon} \end{pmatrix}.$$

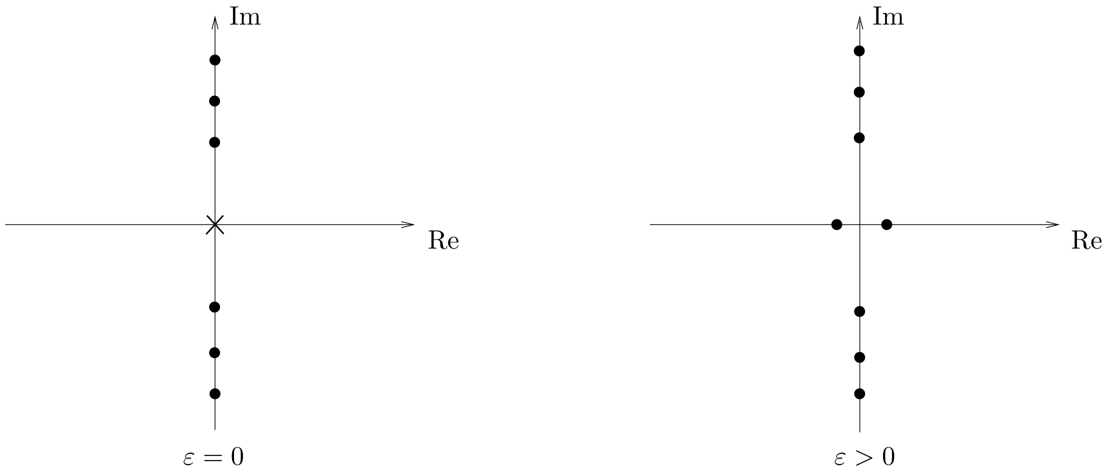

Figure 2: The spectrum of the linearised problem consists of infinitely many semisimple purely imaginary eigenvalues together with two Jordan blocks of length two at the origin for $\varepsilon = 0$ or two semisimple real positive and two semisimple real negative eigenvalues for $\varepsilon > 0$; all eigenvalues have geometric multiplicity two.

$\boldsymbol{m = 1}$: For $\varepsilon = 0$ we have a geometrically simple and algebraically double zero eigenvalue in $E_{1,\mathrm{o}}$. The eigenvector and associated generalized eigenvector are given by

$$\begin{pmatrix} v \\ \partial_\xi v \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(k_0 y), \qquad \begin{pmatrix} v \\ \partial_\xi v \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(k_0 y).$$

For $\varepsilon > 0$ we have two simple eigenvalues $\pm\lambda_{1,\varepsilon}$ which satisfy the equation $(\lambda_{1,\varepsilon})^2 = -2k_0\gamma_1\varepsilon^2(1+k_0^2)^{3/2} + \mathcal{O}(\varepsilon^4)$; they are therefore real if $\gamma_1 < 0$. The eigenvectors are

$$\begin{pmatrix} v \\ \partial_\xi v \end{pmatrix} = \begin{pmatrix} 1 \\ \pm\lambda_{1,\varepsilon} \end{pmatrix} \sin(k_0 y).$$

The same result holds in $E_{1,\mathrm{e}}$ with $\sin(k_0 y)$ replaced by $\cos(k_0 y)$.

$\boldsymbol{m > 1}$: We have two simple purely imaginary eigenvalues in $E_{m,\mathrm{o}}$ given by $\pm\lambda_{m,\varepsilon} = \pm\mathrm{i}(m^2 - 1)^{1/2}(k_0^2+1)^{1/2} + \mathcal{O}(\varepsilon^2)$. The eigenvectors are

$$\begin{pmatrix} v \\ \partial_\xi v \end{pmatrix} = \begin{pmatrix} 1 \\ \pm\lambda_{m,\varepsilon} \end{pmatrix} \sin(k_0 m y).$$

The same result holds in $E_{m,\mathrm{e}}$ with $\sin(k_0 y)$ replaced by $\cos(k_0 y)$.

Recall that equation (4) is invariant under the transformation $(v, \partial_\xi v) \mapsto (-v, -\partial_\xi v)$, and this symmetry allows us to look for solutions in which $v_{m,\mathrm{o}}$, $v_{m,\mathrm{e}}$ vanish for even values of m; we continue to use the previous notation with the understanding that m takes only odd values. In particular, this reduction has the effect of eliminating the subspace E_0. The eigenvalue picture is summarised in Figure 2; for $\varepsilon > 0$ we have a four-dimensional hyperbolic part $\mathcal{X}_\mathrm{h}^s = E_1$ of phase space together with an infinite-dimensional central part $\mathcal{X}_\mathrm{c}^s = \oplus_{m=1}^{\infty} E_{2m+1}$. Notice that P_1 is the projection onto the hyperbolic subspace $\mathcal{X}_\mathrm{h}^s$ along the central subspace $\mathcal{X}_\mathrm{c}^s$; in the theory below we therefore use the notation P_h for P_1, write $P_\mathrm{c} = I - P_\mathrm{h}$ and define $(z, \partial_\xi z) = P_\mathrm{h}(v, \partial_\xi v)$, $(w, \partial_\xi w) = P_\mathrm{c}(v, \partial_\xi v)$.

3 Normal-form theory

In this section we perform a sequence of changes of variable which simplifies the right-hand side of the quasilinear wave equation (4) and predicts the existence of pulse solutions. To this end we write (4) as

$$\partial_\xi^2 v + c_3^\varepsilon \partial_y^2 v + c_4^\varepsilon v + g_0^\varepsilon(v)\partial_y^2 v + g_1^\varepsilon(v) + g_2^\varepsilon(v)\partial_y\partial_\xi v = 0, \tag{6}$$

where

$$c_3^\varepsilon = \frac{1-c_\mathrm{p}^2}{1-c_\mathrm{g}^2}, \qquad c_4^\varepsilon = \frac{-1}{1-c_\mathrm{g}^2}$$

are negative constants and g_0^ε, g_1^ε, g_2^ε are functions defined by

$$\begin{aligned} c_3^\varepsilon + g_0^\varepsilon(v) &= (1-c_\mathrm{p}^2-v^2)/(1-c_\mathrm{g}^2-v^2), \\ c_4^\varepsilon v + g_1^\varepsilon(v) &= -v/(1-c_\mathrm{g}^2-v^2), \\ g_2^\varepsilon(v) &= -2v^2/(1-c_\mathrm{g}^2-v^2), \end{aligned}$$

so that $g_j^\varepsilon(0)=0$, $j=0,1,2$.

Equation (6) is equivalent to the coupled system

$$z'' - \lambda_{1,\varepsilon}^2 z + f^\varepsilon(z,z',w,w') = 0, \tag{7}$$

$$\begin{aligned} &\partial_\xi^2 w + c_3^\varepsilon \partial_y^2 w + c_4^\varepsilon w \\ &\quad + P_\mathrm{c}(g_3^\varepsilon(z,z',w)\partial_y^2 w) + g_4^\varepsilon(z,z',w,w') + P_\mathrm{c}(g_5^\varepsilon(z,z',w)\partial_y\partial_\xi w) + h_\varepsilon(z,z') = 0, \end{aligned} \tag{8}$$

in which

$$\begin{aligned} f^\varepsilon(z,z',w,w') &= P_\mathrm{h}[g_0^\varepsilon(z+w)\partial_y^2(z+w) + g_1^\varepsilon(z+w) + g_2^\varepsilon(z+w)\partial_y\partial_\xi(z+w)], \\ g_3^\varepsilon(z,z',w) &= g_0^\varepsilon(z+w), \\ g_4^\varepsilon(z,z',w,w') &= P_\mathrm{c}[(g_0^\varepsilon(z+w)-g_0^\varepsilon(z))\partial_y^2 z \\ &\qquad + g_1^\varepsilon(z+w) - g_1^\varepsilon(z) + (g_2^\varepsilon(z+w)-g_2^\varepsilon(z))\partial_y\partial_\xi z], \\ g_5^\varepsilon(z,z',w) &= g_2^\varepsilon(z+w), \\ h^\varepsilon(z,z') &= g_0^\varepsilon(z)\partial_y^2 z + g_1^\varepsilon(z) + g_2^\varepsilon(z)\partial_y\partial_\xi z \end{aligned}$$

satisfy the estimates

$$|f^\varepsilon(z,z',w,w')| = \mathcal{O}(\|(z,z',w,w')\|_{\mathcal{X}^s}^3),$$

$$\|g_3^\varepsilon(z,z',w)\|_s = \mathcal{O}(|(z,z')|^2 + \|w\|_s^2), \qquad \|g_5^\varepsilon(z,z',w)\|_s = \mathcal{O}(|(z,z')|^2 + \|w\|_s^2), \tag{9}$$

$$\|g_4^\varepsilon(z,z',w,w')\|_s = \mathcal{O}(\|(w,w')\|_{\mathcal{X}_\mathrm{c}^s}\|(z,z',w,w')\|_{\mathcal{X}^s}) \tag{10}$$

$$\|h^\varepsilon(z,z')\|_s = \mathcal{O}(|(z,z')|^3)$$

and the prime is a shorthand for ∂_ξ. (The stated structure of the nonlinearity in equation (8) is actually more general than that implied by the formulae defining g_3^ε, g_4^ε and g_5^ε; here we are anticipating changes of variable which preserve this more general structure.) Notice that (7) constitutes an ordinary differential equation in four-dimensional phase space while (8) is still a quasilinear wave equation.

In the special case that h^ε is identically zero, one finds that

$$\{(z, z', w, w') \mid (w, w') = (0, 0)\}$$

is an invariant subspace of (7), (8), the flow in which is described by the ordinary differential equation

$$z'' - \lambda_{1,\varepsilon}^2 z + f^\varepsilon(z, z', 0, 0) = 0.$$

A geometric phase-space analysis of the above equation shows that it has a pair of symmetric homoclinic orbits corresponding to the desired modulating pulse solution (see Section 4 below). A natural approach to the general case would be to construct a change of variables which eliminates h^ε, and a sequence of normal-form transformations which successively remove each term in the Taylor expansion of h^ε would be the classical method for the construction of such a change of variables. However, in general we cannot expect a scheme of this type to converge since it would imply a generic intersection of a pair of two-dimensional manifolds in an infinite-dimensional phase space (see the remarks in Section 1); it is however possible to remove terms in h^ε up to an arbitrarily high order using the normal-form method. The homoclinic orbits identified above are in this sense approximate modulating pulse solutions, and in Sections 5 and 6 we prove the persistence of these solutions in the full system (7), (8) on intervals of ξ of length $\mathcal{O}(\varepsilon^{-n})$, where n is approximately the square root of the order of the leading term in the Taylor expansion of h^ε.

Theorem 3.1 *For each $n \geq 2$ there is a near-identity and finite-dimensional change of coordinates which transforms the coupled system (7), (8) into*

$$z'' - \lambda_{1,\varepsilon}^2 z + \tilde{f}^\varepsilon(z, z', w, w') = 0, \tag{11}$$

$$\begin{aligned}&\partial_\xi^2 w + c_3^\varepsilon \partial_y^2 w + c_4^\varepsilon w \\ &\qquad + P_\mathrm{c}(\tilde{g}_3^\varepsilon(z, z', w)\partial_y^2 w) + \tilde{g}_4^\varepsilon(z, z', w, w') + P_\mathrm{c}(\tilde{g}_5^\varepsilon(z, z', w)\partial_y \partial_\xi w) + \tilde{h}_\varepsilon(z, z') = 0,\end{aligned} \tag{12}$$

in which the transformed nonlinearities $\tilde{f}^\varepsilon$, $\tilde{g}_3^\varepsilon$, $\tilde{g}_4^\varepsilon$, $\tilde{g}_5^\varepsilon$ satisfy the same estimates as respectively f^ε, g_3^ε, g_4^ε, g_5^ε and

$$\|\tilde{h}^\varepsilon(z, z')\|_s = \mathcal{O}(|(z, z')|^3 |(\varepsilon, z, z')|^{2n}).$$

The change of coordinates preserves the reversibility.

Proof. We proceed inductively by assuming that

$$\|h^\varepsilon(z, z')\|_s = \mathcal{O}(|(z, z')|^3 |(\varepsilon, z, z')|^{2p-2})$$

and constructing a near-identity transformation which removes the term $h_{2p+1}^\varepsilon(z, z')$ that is homogeneous of degree $2p+1$ in (ε, z, z') from h^ε. Observe that h_{2p+1}^ε is a mapping from E_1 to $\oplus_{m=1}^p E_{2m+1}$. This observation suggests using a finite-dimensional change of coordinates of the form

$$\tilde{w}_{2m+1,\mathrm{o}} = w_{2m+1,\mathrm{o}} + \sum c^{\mathrm{o}}_{2m+1,qijk\ell} \varepsilon^q z_\mathrm{o}^i z_\mathrm{o}'^j z_\mathrm{e}^k z_\mathrm{e}'^\ell, \tag{13}$$

$$\tilde{w}_{2m+1,\mathrm{e}} = w_{2m+1,\mathrm{e}} + \sum c^{\mathrm{e}}_{2m+1,qijk\ell} \varepsilon^q z_\mathrm{o}^i z_\mathrm{o}'^j z_\mathrm{e}^k z_\mathrm{e}'^\ell \tag{14}$$

for $m = 1, 2, \ldots, p$, where the sum is taken over the index set $\{(q, i, j, k, \ell) \mid q + i + j + k + \ell = 2p + 1, i + j + k + \ell \geq 3\}$, and

$$\tilde{w}_{j,\mathrm{o}} = w_{j,\mathrm{o}}, \quad \tilde{w}_{j,\mathrm{e}} = w_{j,\mathrm{e}}, \qquad j \neq 3, \ldots, 2p+1$$

to eliminate $h^{\varepsilon}_{2p+1}(z, z')$ from the equation.

Differentiating (13), (14) with respect to ξ and using (7) to eliminate second derivatives of z, we find that

$$\begin{aligned} \tilde{w}'_{2m+1,\mathrm{o}} &= w'_{2m+1,\mathrm{o}} + \sum c^{\mathrm{o}}_{2m+1,qijk\ell} \varepsilon^q [i z_{\mathrm{o}}^{i-1} z_{\mathrm{o}}'^{j+1} z_{\mathrm{e}}^k z_{\mathrm{e}}'^{\ell} + k z_{\mathrm{o}}^i z_{\mathrm{o}}'^j z_{\mathrm{e}}^{k-1} z_{\mathrm{e}}'^{\ell+1} \\ &\quad + \mathcal{O}(\|(z, z', w, w')\|^3_{\mathcal{X}^s} (\varepsilon^{2p} + \|(z, z', w, w')\|^{2p}_{\mathcal{X}^s})), \end{aligned} \tag{15}$$

together with the corresponding equation for $\tilde{w}'_{2m+1,\mathrm{e}}$. One can solve this system of equations using the implicit-function theorem to find that

$$\begin{aligned} w'_{2m+1,\mathrm{o}} &= \tilde{w}'_{2m+1,\mathrm{o}} + \mathcal{O}(\|(z, z', \tilde{w}, \tilde{w}')\|^3_{\mathcal{X}^s} (\varepsilon^{2p} + \|(z, z', \tilde{w}, \tilde{w}')\|^{2p}_{\mathcal{X}^s})), \\ w'_{2m+1,\mathrm{e}} &= \tilde{w}'_{2m+1,\mathrm{e}} + \mathcal{O}(\|(z, z', \tilde{w}, \tilde{w}')\|^3_{\mathcal{X}^s} (\varepsilon^{2p} + \|(z, z', \tilde{w}, \tilde{w}')\|^{2p}_{\mathcal{X}^s})). \end{aligned}$$

A further differentiation with respect to ξ yields

$$\begin{aligned} \tilde{w}''_{2m+1,\mathrm{o}} &= w''_{2m+1,\mathrm{o}} \\ &\quad + \sum c^{\mathrm{o}}_{2m+1,qijk\ell} \varepsilon^q [i(i-1) z_{\mathrm{o}}^{i-2} z_{\mathrm{o}}'^{j+2} z_{\mathrm{e}}^k z_{\mathrm{e}}'^{\ell} + k(k-1) z_{\mathrm{o}}^i z_{\mathrm{o}}'^j z_{\mathrm{e}}^{k-2} z_{\mathrm{e}}'^{\ell+2}] \\ &\quad + \sum 2 c^{\mathrm{o}}_{2m+1,qijk\ell} \varepsilon^q ik z_{\mathrm{o}}^{i-1} z_{\mathrm{o}}'^{j+1} z_{\mathrm{e}}^{k-1} z_{\mathrm{e}}'^{\ell+1} + \Phi^{\varepsilon}_{2m+1,\mathrm{o}}(z, z', w, w'), \end{aligned}$$

where

$$\begin{aligned} |\Phi^{\varepsilon}_{2m+1,\mathrm{o}}(z, z', w, w')| &= \mathcal{O}(\|(z, z', w, w')\|^3_{\mathcal{X}^s} (\varepsilon^{2p} + \|(z, z', w, w')\|^{2p}_{\mathcal{X}^s})) \\ &= \mathcal{O}(\|(z, z', \tilde{w}, \tilde{w}')\|^3_{\mathcal{X}^s} (\varepsilon^{2p} + \|(z, z', \tilde{w}, \tilde{w}')\|^{2p}_{\mathcal{X}^s})), \end{aligned}$$

together with a corresponding expression for $\tilde{w}''_{2m+1,\mathrm{e}}$; here we have used (7), (8) to eliminate second derivatives of z and w in the nonlinear terms and the smoothing property of f^{ε} to obtain the estimate on the remainder terms.

Observe that

$$\begin{aligned} & w''_{2m+1,\mathrm{o}} - \lambda^2_{2m+1,\varepsilon} w_{2\mathrm{m}+1,\mathrm{o}} + P_{2m+1,\mathrm{o}} h^{\varepsilon}_{2p+1}(z, z') \\ &= \tilde{w}''_{2m+1,\mathrm{o}} - \lambda^2_{2m+1,\varepsilon} \tilde{w}_{2m+1,\mathrm{o}} - \Phi^{\varepsilon}_{2m+1,\mathrm{o}}(z, z', w, w') \\ &\quad - \sum c^{\mathrm{o}}_{2m+1,qijk\ell} \varepsilon^q [-\lambda^2_{2m+1,0} z_{\mathrm{o}}^i z_{\mathrm{o}}'^j z_{\mathrm{e}}^k z_{\mathrm{e}}'^{\ell} + 2ik z_{\mathrm{o}}^{i-1} z_{\mathrm{o}}'^{j+1} z_{\mathrm{e}}^{k-1} z_{\mathrm{e}}'^{\ell+1}] \\ &\quad - \sum c^{\mathrm{o}}_{2m+1,qijk\ell} \varepsilon^q [i(i-1) z_{\mathrm{o}}^{i-2} z_{\mathrm{o}}'^{j+2} z_{\mathrm{e}}^k z_{\mathrm{e}}'^{\ell} + k(k-1) z_{\mathrm{o}}^i z_{\mathrm{o}}'^j z_{\mathrm{e}}^{k-2} z_{\mathrm{e}}'^{\ell+2}] \\ &\quad + \sum d^{\mathrm{o}}_{2m+1,qijk\ell} \varepsilon^q z_{\mathrm{o}}^i z_{\mathrm{o}}'^j z_{\mathrm{e}}^k z_{\mathrm{e}}'^{\ell}, \qquad m = 1, \ldots, p, \end{aligned} \tag{16}$$

where the notation

$$P_{2m+1,\mathrm{o}} h^{\varepsilon}_{2p+1}(z, z') = \sum d^{\mathrm{o}}_{2m+1,qijk\ell} \varepsilon^q z_{\mathrm{o}}^i z_{\mathrm{o}}'^j z_{\mathrm{e}}^k z_{\mathrm{e}}'^{\ell}$$

has been adopted, and of course

$$w''_{j,\mathrm{o}} - \lambda^2_{j,\varepsilon} w_{j,\mathrm{o}} + P_{j,\mathrm{o}} h^\varepsilon_{2p+1}(z,z') = \tilde{w}''_{j,\mathrm{o}} - \lambda^2_{j,\varepsilon}\tilde{w}_{j,\mathrm{o}}, \qquad j \neq 3,\ldots,p+1.$$

The following argument shows that it is always possible to choose the coefficients $c^{\mathrm{o}}_{2m+1,qijk\ell}$ to ensure cancellation of the sums on the right-hand side of equation (16). First notice that the special case in which $d^{\mathrm{o}}_{2m+1,qijk\ell}$ vanishes for $i+k \geq 2$, so that

$$\sum d^{\mathrm{o}}_{2m+1,qijk\ell}\varepsilon^q z^i_{\mathrm{o}} z'^j_{\mathrm{o}} z^k_{\mathrm{e}} z'^\ell_{\mathrm{e}} = \sum_{i+k\leq 1} d^{\mathrm{o}}_{2m+1,qijk\ell}\varepsilon^q z^i_{\mathrm{o}} z'^j_{\mathrm{o}} z^k_{\mathrm{e}} z'^\ell_{\mathrm{e}},$$

is clearly solved by choosing

$$c^{\mathrm{o}}_{2m+1,qijk\ell} = -\frac{1}{\lambda^2_{2m+1,0}} d^{\mathrm{o}}_{2m+1,qijk\ell}. \tag{17}$$

Next suppose that $d^{\mathrm{o}}_{2m+1,qijk\ell}$ vanishes for $i+k>\kappa$ for some $\kappa \geq 2$ (and possibly equal to $2p+1$). Choosing coefficients $c^{\mathrm{o},0}_{2m+1,qijk\ell}$ according to formula (17), we find that

$$\begin{aligned} & \sum c^{\mathrm{o},0}_{2m+1,qijk\ell}\varepsilon^q[-\lambda^2_{2m+1,0} z^i_{\mathrm{o}} z'^j_{\mathrm{o}} z^k_{\mathrm{e}} z'^\ell_{\mathrm{e}} + 2ik z^{i-1}_{\mathrm{o}} z'^{j+1}_{\mathrm{o}} z^{k-1}_{\mathrm{e}} z'^{\ell+1}_{\mathrm{e}}] \\ & \quad + \sum c^{\mathrm{o},0}_{2m+1,qijk\ell}\varepsilon^q[i(i-1) z^{i-2}_{\mathrm{o}} z'^{j+2}_{\mathrm{o}} z^k_{\mathrm{e}} z'^\ell_{\mathrm{e}} + k(k-1) z^i_{\mathrm{o}} z'^j_{\mathrm{o}} z^{k-2}_{\mathrm{e}} z'^{\ell+2}_{\mathrm{e}}] \\ & \quad - \sum d^{\mathrm{o}}_{2m+1,qijk\ell}\varepsilon^q z^i_{\mathrm{o}} z'^j_{\mathrm{o}} z^k_{\mathrm{e}} z'^\ell_{\mathrm{e}} \\ = & \sum d^{\mathrm{o},1}_{2m+1,qijk\ell}\varepsilon^q z^i_{\mathrm{o}} z'^j_{\mathrm{o}} z^k_{\mathrm{e}} z'^\ell_{\mathrm{e}}, \end{aligned}$$

where the coefficients $d^{\mathrm{o},1}_{2m+1,qijk\ell}$ vanish for $i+k>\kappa-2$. The next step is to repeat the argument with $d^{\mathrm{o}}_{2m+1,qijk\ell}$ replaced by $d^{\mathrm{o},1}_{2m+1,qijk\ell}$; the resulting coefficients $c^{\mathrm{o},1}_{2m+1,qijk\ell}$ either solve the new problem or yield further new coefficients $d^{\mathrm{o},2}_{2m+1,qijk\ell}$ which vanish for $i+k>\kappa-4$. This iteration scheme terminates after $[\kappa/2]$ steps, and the choice $c^{\mathrm{o}}_{2m+1,qijk\ell} = c^{\mathrm{o},0}_{2m+1,qijk\ell} + \cdots + c^{\mathrm{o},[\kappa/2]-1}_{2m+1,qijk\ell}$ solves the original problem. We apply the same method to the equation for $w_{2m+1,\mathrm{e}}$ to determine the coefficients $c^{\mathrm{e}}_{2m+1,qijk\ell}$.

Using the above choices of coefficients $c^{\mathrm{e}}_{2m+1,qijk\ell}$ and $c^{\mathrm{e}}_{2m+1,qijk\ell}$, one finds that our change of coordinates transforms (7), (8) into (11), (12) with

$$\tilde{f}^\varepsilon(z,z',\tilde{w},\tilde{w}') = f^\varepsilon(z,z',w,w'),$$

$$\tilde{g}^\varepsilon_3(z,z',\tilde{w}) = g^\varepsilon_3(z,z',w), \qquad \tilde{g}^\varepsilon_5(z,z',\tilde{w}) = g^\varepsilon_5(z,z',w)$$

and

$$\begin{aligned} & \tilde{g}^\varepsilon_4(z,z',\tilde{w},\tilde{w}') + \tilde{h}^\varepsilon(z,z') \\ & \quad = \quad g_3(z,z',w)\partial^2_y(w-\tilde{w}) + g^\varepsilon_5(z,z',w)\partial_\xi\partial_y(w-\tilde{w}) \\ & \qquad\qquad + g^\varepsilon_4(z,z',w,w') - \Phi^\varepsilon(z,z',w,w') + h^\varepsilon(z,z') - h^\varepsilon_{2p+1}(z,z'), \end{aligned}$$

in which the specification of $\tilde{g}^\varepsilon_4$ is completed by the requirement that it should vanish for $(\tilde{w},\tilde{w}') = (0,0)$ and by construction $\tilde{h}^\varepsilon_{2p+1}$ is identically zero. Careful book-keeping shows that the reversibility is preserved by each change of coordinates. The reversibility of the original system implies that $d^{\mathrm{o}}_{2m+1,qijk\ell}$ vanishes when $j+k$ is even (so that $i+\ell$ is odd) and $d^{\mathrm{e}}_{2m+1,qijk\ell}$ vanishes when $j+k$ is odd (so that $i+\ell$ is even). The coefficients $c_{2m+1,qijk\ell}$ have the same properties, and by examining the formulae (13)–(15), one finds that $(v(S(\tilde{v},\partial_\xi\tilde{v})),\partial_\xi v(S(\tilde{v},\partial_\xi\tilde{v}))) = S(v(\tilde{v},\partial_\xi\tilde{v}),\partial_\xi v(\tilde{v},\partial_\xi\tilde{v}))$, where $\tilde{v}=(z,\tilde{w})$, so that the transformed system is also reversible. □

4 Construction of an approximate modulating pulse

In this section we examine the ordinary differential equation

$$z'' - \lambda_{1,\varepsilon}^2 z + \tilde{f}^\varepsilon(z,z',0,0) = 0 \tag{18}$$

obtained by neglecting the small perturbation $\tilde{h}^\varepsilon$ and setting $(w,w')=(0,0)$ in equations (11), (12). We seek a homoclinic solution q^ε to (18), since such a solution serves as an approximation to a modulating pulse solution of the full system (11), (12). The argument is completed in Sections 5 and 6 below, which establish the persistence with respect to the perturbation $\tilde{h}^\varepsilon$ of a pulse-like solution near q^ε on intervals of length $\mathcal{O}(\varepsilon^{-n})$, where n is approximately the square root of the order of the leading term in the Taylor expansion of h^ε.

The real-valued functions z_{o} and z_{e} satisfy the coupled second-order differential equations

$$z_{\mathrm{o}}'' - \lambda_{1,\varepsilon}^2 z_{\mathrm{o}} + \tilde{f}_{\mathrm{o}}^\varepsilon(z_{\mathrm{o}},z_{\mathrm{e}},z_{\mathrm{o}}',z_{\mathrm{e}}') = 0, \tag{19}$$

$$z_{\mathrm{e}}'' - \lambda_{1,\varepsilon}^2 z_{\mathrm{e}} + \tilde{f}_{\mathrm{e}}^\varepsilon(z_{\mathrm{o}},z_{\mathrm{e}},z_{\mathrm{o}}',z_{\mathrm{e}}') = 0, \tag{20}$$

in which $\tilde{f}_{\mathrm{o}}^\varepsilon = P_{1,\mathrm{o}}\tilde{f}^\varepsilon|_{(w,w')=(0,0)}$ and $\tilde{f}_{\mathrm{e}}^\varepsilon = P_{1,\mathrm{e}}\tilde{f}^\varepsilon|_{(w,w')=(0,0)}$ are $\mathcal{O}(|(z_{\mathrm{o}},z_{\mathrm{e}},z_{\mathrm{o}}',z_{\mathrm{e}}')|^3)$. This system of equations inherits the reversibility and $O(2)$ symmetry of the full system; it is invariant under the transformation $\xi\mapsto-\xi$, $(z_{\mathrm{o}},z_{\mathrm{e}},z_{\mathrm{o}}',z_{\mathrm{e}}')\mapsto S_{\mathrm{h}}(z_{\mathrm{o}},z_{\mathrm{e}},z_{\mathrm{o}}',z_{\mathrm{e}}')$, where the reverser S_{h} is defined by

$$S_{\mathrm{h}}(z_{\mathrm{o}},z_{\mathrm{e}},z_{\mathrm{o}}',z_{\mathrm{e}}') = (-z_{\mathrm{o}},z_{\mathrm{e}},z_{\mathrm{o}}',-z_{\mathrm{e}}'),$$

and under the transformation

$$\begin{pmatrix} z_{\mathrm{o}} \\ z_{\mathrm{e}} \end{pmatrix} \mapsto R_{k_0 a}\begin{pmatrix} z_{\mathrm{o}} \\ z_{\mathrm{e}} \end{pmatrix}, \quad \begin{pmatrix} z_{\mathrm{o}}' \\ z_{\mathrm{e}}' \end{pmatrix} \mapsto R_{k_0 a}\begin{pmatrix} z_{\mathrm{o}}' \\ z_{\mathrm{e}}' \end{pmatrix}$$

for each $a\in[0,2\pi/k_0)$.

Introducing the scaled variables

$$\check{\xi}=\varepsilon\xi, \quad (z_{\mathrm{o}}(\xi),z_{\mathrm{e}}(\xi)) = \varepsilon(\check{z}_{\mathrm{o}}(\check{\xi}),\check{z}_{\mathrm{e}}(\check{\xi})),$$

one finds from (19), (20) that

$$\partial_{\check{\xi}}^2\check{z}_{\mathrm{o}} = \check{C}_1\check{z}_{\mathrm{o}} - \check{C}_2\check{z}_{\mathrm{o}}(\check{z}_{\mathrm{o}}^2+\check{z}_{\mathrm{e}}^2) + \mathcal{R}_{\mathrm{o}}^\varepsilon(\check{z}_{\mathrm{o}},\check{z}_{\mathrm{e}},\partial_{\check{\xi}}\check{z}_{\mathrm{o}},\partial_{\check{\xi}}\check{z}_{\mathrm{e}}), \tag{21}$$

$$\partial_{\check{\xi}}^2\check{z}_{\mathrm{e}} = \check{C}_1\check{z}_{\mathrm{e}} - \check{C}_2\check{z}_{\mathrm{e}}(\check{z}_{\mathrm{o}}^2+\check{z}_{\mathrm{e}}^2) + \mathcal{R}_{\mathrm{e}}^\varepsilon(\check{z}_{\mathrm{o}},\check{z}_{\mathrm{e}},\partial_{\check{\xi}}\check{z}_{\mathrm{o}},\partial_{\check{\xi}}\check{z}_{\mathrm{e}}), \tag{22}$$

in which

$$\check{C}_1 = -2k_0\gamma_1(1+k_0^2)^{3/2} > 0, \qquad \check{C}_2 = \frac{3k_0^3(1+k_0^2)}{4\pi} > 0$$

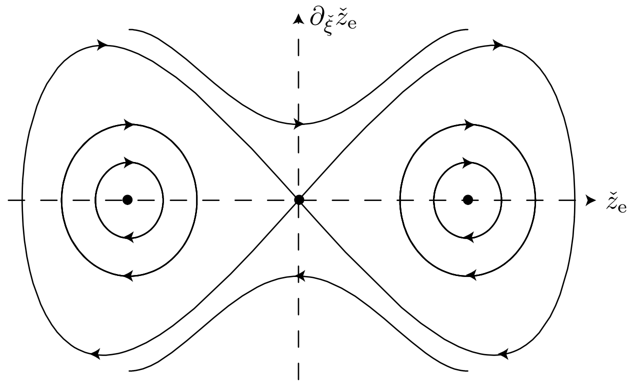

Figure 3: Dynamics in the $(\check{z}_{\mathrm{e}}, \partial_{\check{\xi}}\check{z}_{\mathrm{e}})$ *coordinate plane.*

and the remainder terms $\mathcal{R}_{\mathrm{o}}^{\varepsilon}$ and $\mathcal{R}_{\mathrm{e}}^{\varepsilon}$ are both $\mathcal{O}(\varepsilon^2)$ and respectively odd and even in $(\check{z}_{\mathrm{o}}, \partial_{\check{\xi}}\check{z}_{\mathrm{e}})$. In the limit $\varepsilon \to 0$ the system (21), (22) has the property that the $(\check{z}_{\mathrm{e}}, \partial_{\check{\xi}}\check{z}_{\mathrm{e}})$ coordinate plane is invariant; its phase portrait is shown in Figure 3. Notice in particular that there are two homoclinic orbits given by the explicit formulae

$$\check{q}^{\pm}(\check{\xi}) = \pm\left(\frac{2\check{C}_1}{\check{C}_2}\right)^{1/2} \operatorname{sech}(\check{C}_1^{1/2}\check{\xi}).$$

In fact each orbit in the four-dimensional phase space of the limiting equations is obtained from an orbit in the $(\check{z}_{\mathrm{e}}, \partial_{\check{\xi}}\check{z}_{\mathrm{e}})$ coordinate plane by a rotation $R_{k_0 a}$ for some $a \in (0, 2\pi/k_0)$ (so that each subspace $(R_{k_0 a}(0, \check{z}_{\mathrm{e}}), R_{k_0 a}(0, \partial_{\check{\xi}}\check{z}_{\mathrm{e}}))$, $a \in (0, 2\pi/k_0)$ is invariant).

The homoclinic solutions to the limiting form of (21), (22) in the $(\check{z}_{\mathrm{e}}, \partial_{\check{\xi}}\check{z}_{\mathrm{e}})$ coordinate plane are *reversible*, that is, they satisfy $S_{\mathrm{h}}\check{q}^{\pm}(-\xi) = \check{q}^{\pm}(\xi)$. This feature can be exploited to prove their persistence for small values of ε. The stable manifold of the zero equilibrium for the limiting equations is clearly given by $W_{\mathrm{s}}^{+}(0) \cup W_{\mathrm{s}}^{-}(0)$, where

$$\begin{aligned} W_{\mathrm{s}}^{+}(0) &= \{(R_{k_0 a}\check{q}^{+}(\check{\xi}), R_{k_0 a}\partial_{\check{\xi}}\check{q}^{+}(\check{\xi})) : a \in [0, 2\pi/k_0), \check{\xi} \in (-\infty, \delta]\}, \\ W_{\mathrm{s}}^{-}(0) &= \{(R_{k_0 a}\check{q}^{-}(\check{\xi}), R_{k_0 a}\partial_{\check{\xi}}\check{q}^{-}(\check{\xi})) : a \in [0, 2\pi/k_0), \check{\xi} \in [-\delta, \infty)\} \end{aligned}$$

and δ is a small positive number. Observe that $W_{\mathrm{s}}^{+}(0)$ intersects the symmetric section Fix $S_{\mathrm{h}} =$ span $\{(0,1,0,0),(0,0,1,0)\}$ at the point $P^{+} = (R_0\check{q}^{+}(0), R_0\partial_{\check{\xi}}\check{q}^{+}(0))$, and

$$\begin{aligned} & TW_{\mathrm{s}}^{+}(0)|_{P^+} \\ &= \text{span}\,\{\partial_a(R_{k_0 a}\check{q}^{+}(\check{\xi}), R_{k_0 a}\partial_{\check{\xi}}\check{q}^{+}(\check{\xi}))|_{(a,\check{\xi})=(0,0)}, \partial_{\check{\xi}}(R_{k_0 a}\check{q}^{+}(\check{\xi}), R_{k_0 a}\partial_{\check{\xi}}\check{q}^{+}(\check{\xi}))|_{(a,\check{\xi})=(0,0)}\} \\ &= \text{span}\,\{(1,0,0,0),(0,0,0,1)\}. \end{aligned}$$

It follows that $TW_{\mathrm{s}}^{+}(0)|_{P}^{+} + \text{Fix } S_{\mathrm{h}} = \mathbb{R}^4$, so that the intersection between $W_{\mathrm{s}}^{+}(0)$ and Fix S_{h} at P^{+} is transversal. The same argument shows that the intersection between $W_{\mathrm{s}}^{-}(0)$ and Fix S_{h} at $P^{-} = (R_0\check{q}^{-}(0), R_0\partial_{\check{\xi}}\check{q}^{-}(0))$ is also transversal. According to the stable manifold theorem (e.g. see Coddington & Levinson [2, Chapter 13]), the stable manifold $W_{\mathrm{s}}^{\varepsilon}(0)$ of the zero equilibrium for equations (21), (22) depends uniformly smoothly upon ε, and since the symmetric section is independent of ε it follows that $W_{\mathrm{s}}^{\varepsilon}(0)$ and Fix S_{h} intersect transversally in two points, one near

P^+, one near P^-, for sufficiently small positive values of ε. It follows that (21), (22) also has a pair of reversible homoclinic orbits. Lemma 4.1 below is obtained by returning to unscaled coordinates and using the fact that any solution on the stable manifold of (18) decays to zero at an exponential rate strictly less than $\lambda_{1,\varepsilon}$.

Lemma 4.1 *Equation (18) has a pair $(q^{\varepsilon\pm}, q^{\varepsilon\pm\prime})$ of reversible homoclinic orbits of the form*

$$\begin{pmatrix} q^\varepsilon(\xi) \\ q^{\varepsilon\prime}(\xi) \end{pmatrix} = \begin{pmatrix} \varepsilon\tilde{q}^\varepsilon(\varepsilon\xi) \\ \varepsilon^2\tilde{q}^{\varepsilon\prime}(\varepsilon\xi) \end{pmatrix},$$

where $\tilde{q}^\varepsilon$ is a smooth function with bounded derivatives. These homoclinic orbits satisfy

$$|q^\varepsilon(\xi)| \leq c\varepsilon \mathrm{e}^{-\varepsilon\theta|\xi|}, \quad |q^{\varepsilon\prime}(\xi)| \leq c\varepsilon^2 \mathrm{e}^{-\varepsilon\theta|\xi|}, \qquad \xi \in \mathbb{R} \tag{23}$$

for any $\theta \in (0, (-2k_0\gamma_1)^{1/2}(1+k_0^2)^{3/4})$.

5 The local centre-stable manifold

Let us now fix $n \in \mathbb{N}$ and suppose that the normal-form transformations described in Section 3 have been carried out to a sufficiently high order that $h(z, z') = \mathcal{O}(|(z, z')|^3|(\varepsilon, z, z')|^{2n})$. We proceed by constructing solutions $v = (z, w)$ of the transformed equations (11), (12) whose pointwise distance from the approximate pulse $v = (q, 0)$ identified in Section 4 remains $O(\varepsilon^{n+1})$ over the timescale $[0, L/\varepsilon^n]$ (see Figure 4); for notational simplicity we drop the tildes in equations (11), (12).

The first step is to formulate equation (11) as the first-order system

$$Z' = \Lambda^\varepsilon Z + F^\varepsilon(Z, w, w')$$

for the variable $Z = (z_\mathrm{o}, z_\mathrm{e}, z_\mathrm{o}', z_\mathrm{e}')$, in which

$$\Lambda^\varepsilon = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda_{0,\varepsilon}^2 & 0 & 0 & 0 \\ 0 & \lambda_{0,\varepsilon}^2 & 0 & 0 \end{pmatrix}, \qquad F^\varepsilon(Z, w, w') = \begin{pmatrix} 0 \\ 0 \\ -P_{1,\mathrm{o}} f^\varepsilon(z, z', w, w') \\ -P_{1,\mathrm{e}} f^\varepsilon(z, z', w, w') \end{pmatrix}.$$

Writing $Z = Q^\varepsilon + R$, where $Q^\varepsilon = (q_\mathrm{o}^\varepsilon, q_\mathrm{e}^\varepsilon, q_\mathrm{o}^{\varepsilon\prime}, q_\mathrm{e}^{\varepsilon\prime})$, one finds that

$$R' = \mathcal{L}^\varepsilon R + N^\varepsilon(R, w, w'), \tag{24}$$

where $\mathcal{L}^\varepsilon R = \Lambda^\varepsilon R + \mathrm{d}_1 F^\varepsilon[Q^\varepsilon, 0, 0](R)$ and $N^\varepsilon(R, w, w') = F^\varepsilon(Q^\varepsilon + R, w, w') - F^\varepsilon(R, w, w') - \mathrm{d}F^\varepsilon[Q^\varepsilon, 0, 0](R)$. With a slight abuse of notation, equation (12) becomes

$$\begin{aligned} &\partial_\xi^2 w + c_3^\varepsilon \partial_y^2 w + c_4^\varepsilon w + P_\mathrm{c}(g_3^\varepsilon(Q^\varepsilon + R, w)\partial_y^2 w) \\ &\qquad + g_4^\varepsilon(Q^\varepsilon + R, w, w') + P_\mathrm{c}(g_5^\varepsilon(Q^\varepsilon + R, w)\partial_y\partial_\xi w) + h^\varepsilon(Q^\varepsilon + R) = 0, \end{aligned} \tag{25}$$

and our task is to find solutions (R, w) of (24), (25) for which $|R(\xi)|$ and $\|(w(\xi), w'(\xi))\|_{\mathcal{X}_\mathrm{c}^s}$ are $\mathcal{O}(\varepsilon^{n+1})$ for $\xi \in [0, L/\varepsilon^n]$.

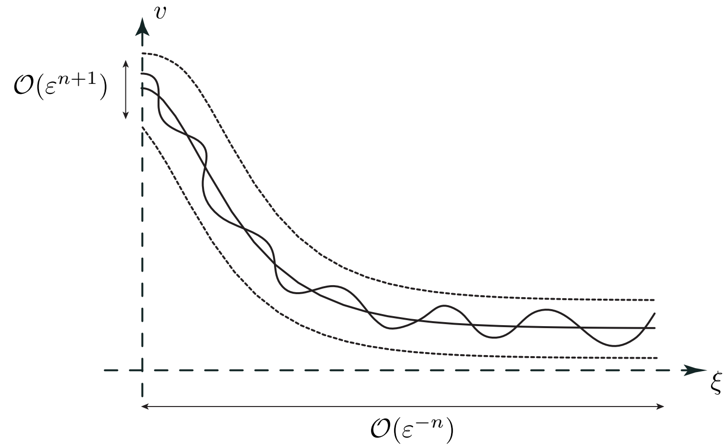

Figure 4: Solutions with initial data on the local centre-stable manifold $W^{\mathrm{cs}}_{\mathrm{loc}}$ remain within an $\mathcal{O}(\varepsilon^{n+1})$ neighbourhood of Q^ε on a timescale of $\mathcal{O}(\varepsilon^{-n})$.

The following result describes the stable and unstable directions associated with the time-dependent linear operator $\mathcal{L}^\varepsilon$; it is proved by noting that

$$\|\mathcal{L}^\varepsilon - \Lambda^\varepsilon\|_{\mathcal{X}^s_{\mathrm{h}} \to \mathcal{X}^s_{\mathrm{h}}} \le c\mathrm{e}^{-\varepsilon\theta|\xi|}, \qquad \xi \in \mathbb{R}$$

and using the method explained by Groves & Mielke [5, §4.3].

Lemma 5.1 *The equation*

$$\partial_\xi R = \mathcal{L}^\varepsilon R$$

has solutions $s_1(\xi)$, $s_2(\xi)$, $u_1(\xi)$, $u_2(\xi)$ on $[0,\infty)$ such that

$$|s_j(\xi)| \le c\mathrm{e}^{-\lambda_{1,\varepsilon}\xi}, \quad |u_j(\xi)| \le c\mathrm{e}^{\lambda_{1,\varepsilon}\xi}, \qquad j = 1,2, \quad \xi \in [0,\infty).$$

The dual basis $\{s_1^*(\xi), s_2^*(\xi), u_1^*(\xi), u_2^*(\xi)\}$ to $\{s_1(\xi), s_2(\xi), u_1(\xi), u_2(\xi)\}$ in $\mathcal{X}^s_{\mathrm{h}}$ satisfies

$$|s_j^*(\xi)| \le \frac{c}{\lambda_{1,\varepsilon}}\mathrm{e}^{\lambda_{1,\varepsilon}\xi}, \quad |u_j^*(\xi)| \le \frac{c}{\lambda_{1,\varepsilon}}\mathrm{e}^{-\lambda_{1,\varepsilon}\xi}, \qquad j = 1,2, \quad \xi \in [0,\infty).$$

We seek solutions of (24), (25) using a modification of a construction familiar in dynamical-systems theory. The *local centre-stable manifold* consists of the initial data for solutions of a dynamical system which exist for some time interval starting at $\xi = 0$. This manifold is constructed by modifying the nonlinearities using a cut-off function to make them globally Lipschitz with a small Lipschitz constant; global solutions (corresponding to local solutions of the original problem) can then be found by reformulating the dynamical system as an integral equation and applying a contraction-mapping argument. This technique of finding solutions is not applicable to the present problem, which consists of a quasilinear wave equation coupled with a dynamical system. Instead we directly apply the following *iteration scheme* to obtain a local existence theory from which the local centre-stable manifold can be constructed.

Choose real numbers $R^0_{s_1}, R^0_{s_2}$ whose magnitude is at most ε^{n+1} and $(w^0, w^{0\prime}) \in \mathcal{X}^s_{\mathrm{c}}$ such that $\|(w^0, w^{0\prime})\|_{\mathcal{X}^{s+1}_{\mathrm{c}}} \le \varepsilon^{n+1}$. The iteration scheme is initiated by letting $R_{(0)} = 0$ and $(w_{(0)}, w'_{(0)}) \in C([0, L/\varepsilon^n], \mathcal{X}^s_{\mathrm{c}})$ be the solution of the inhomogeneous linear wave equation

$$\partial_\xi^2 w_{(0)} + c_3^\varepsilon \partial_y^2 w_{(0)} + c_4^\varepsilon w_{(0)} + h^\varepsilon(Q^\varepsilon) = 0$$

with initial data $(w, w')|_{\xi=0} = (w^0, w^{0\prime})$, so that $\|(w_{(0)}, w'_{(0)})\|_{\mathcal{X}_c^s} \le c\varepsilon^{n+1}$. For $m = 0, 1, 2, \ldots$ we define $R_{(m+1)} \in C([0, L/\varepsilon^n], \mathbb{R}^4)$ by the formula

$$\begin{aligned} R_{(m+1)}(\xi) &= R^0_{s_1}\, s_1(\xi) + R^0_{s_2}\, s_2(\xi) \\ &\quad + \sum_{j=1}^{2} \int_0^{\xi} \langle N^\varepsilon_{(m)}(\tau), s_j^*(\tau)\rangle \,\mathrm{d}\tau\, s_j(\xi) - \sum_{j=1}^{2} \int_{\xi}^{L/\varepsilon^n} \langle N^\varepsilon_{(m)}(\tau), u_j^*(\tau)\rangle \,\mathrm{d}\tau\, u_j(\xi), \end{aligned} \tag{26}$$

and let $(w_{(m+1)}, w'_{(m+1)}) \in C([0, L/\varepsilon^n], \mathcal{X}_c^s)$ be the solution of the equation

$$\begin{aligned} &\partial_\xi^2 w_{(m+1)} + c_3^\varepsilon \partial_y^2 w_{(m+1)} + c_4^\varepsilon w_{(m+1)} \\ &\qquad + P_c(g^\varepsilon_{3(m)} \partial_y^2 w_{(m+1)}) + g^\varepsilon_{4(m)} + P_c(g^\varepsilon_{5(m)} \partial_y \partial_\xi w_{(m+1)}) + h^\varepsilon_{(m)} = 0, \end{aligned} \tag{27}$$

with initial data $(w_{(m+1)}, w'_{(m+1)})|_{\xi=0} = (w^0, w^{0\prime})$; here $N^\varepsilon_{(m)}$, $g^\varepsilon_{4(m)}$, $h^\varepsilon_{(m)}$ are abbreviations for respectively $N^\varepsilon(R_{(m)}, w_{(m)}, w'_{(m)})$, $g_4^\varepsilon(Q^\varepsilon + R_{(m)}, w_{(m)}, w'_{(m)})$, $h^\varepsilon(Q^\varepsilon + R_{(m)})$ and $g^\varepsilon_{j(m)}$ is an abbrevation for $g_j^\varepsilon(Q^\varepsilon + R_{(m)}, w_{(m)})$, $j = 3, 5$.

Lemma 5.2 *Suppose that $\|(w^0, w^{0\prime})\|_{\mathcal{X}_c^s} \le \varepsilon^{n+1}$. For each $L > 0$ the estimates*

$$\sup_{m\in\mathbb{N}} \sup_{\xi\in[0,L/\varepsilon^n]} |R_{(m)}(\xi)| \le c\varepsilon^{n+1}, \qquad \sup_{m\in\mathbb{N}} \sup_{\xi\in[0,L/\varepsilon^n]} \|(w_{(m)}(\xi), w'_{(m)}(\xi))\|_{\mathcal{X}_c^s} \le c\varepsilon^{n+1}.$$

hold for all sufficiently small values of ε.

Proof. It suffices to demonstrate that

$$\sup_{\xi\in[0,L/\varepsilon^n]} |R_{(m+1)}(\xi)| \le c\varepsilon^{n+1}, \qquad \sup_{\xi\in[0,L/\varepsilon^n]} \|(w_{(m+1)}(\xi), w'_{(m+1)}(\xi))\|_{\mathcal{X}_c^s} \le c\varepsilon^{n+1}$$

whenever

$$\sup_{\xi\in[0,L/\varepsilon^n]} |R_{(m)}(\xi)| \le \varepsilon^{n}, \qquad \sup_{\xi\in[0,L/\varepsilon^n]} \|(w_{(m)}(\xi), w'_{(m)}(\xi))\|_{\mathcal{X}_c^s} \le \varepsilon^{n};$$

the lemma follows inductively from this result.

Observe that

$$|N^\varepsilon_{(m)}| \le c(|R_{(m)}|^2 + \|(w_{(m)}, w'_{(m)})\|^2_{\mathcal{X}_c^s}) \le c\varepsilon^{2n},$$

from which it follows that

$$\begin{aligned} &|R_{(m+1)}(\xi)| \\ &\quad\le\ c\left(|R^0_{s_1}| + |R^0_{s_2}| + \frac{\varepsilon^{2n}}{\lambda_{1,\varepsilon}} \int_0^{\xi} \mathrm{e}^{\lambda_{1,\varepsilon}\tau}\,\mathrm{d}\tau\, \mathrm{e}^{-\lambda_{1,\varepsilon}\xi} + \frac{\varepsilon^{2n}}{\lambda_{1,\varepsilon}} \int_{\xi}^{L/\varepsilon^n} \mathrm{e}^{-\lambda_{1,\varepsilon}\tau}\,\mathrm{d}\tau\, \mathrm{e}^{\lambda_{1,\varepsilon}\xi}\right) \\ &\quad\le\ c\left(|R^0_{s_1}| + |R^0_{s_2}| + \frac{\varepsilon^{2n}}{\lambda^2_{1,\varepsilon}}\right) \\ &\quad\le\ c\varepsilon^{n+1} \end{aligned} \tag{28}$$

for $\xi \in [0, L/\varepsilon^n]$. The corresponding result for $w_{(m+1)}$ is obtained by applying energy estimates to equation (27); we differentiate this equation s times with respect to y, multiply the result by

$\partial_\xi \partial_y^s w_{(m+1)}$ and integrate with respect to y over one period. After an integration by parts the linear terms yield the relationship

$$\int (\partial_\xi \partial_y^s w_{(m+1)})(\partial_\xi^2 \partial_y^s w_{(m+1)} + c_3^\varepsilon \partial_y^{s+2} w_{(m+1)} + c_4^\varepsilon \partial_y^s w_{(m+1)}) \, \mathrm{d}y = \frac{1}{2} \partial_\xi \mathcal{E}_s(w_{(m+1)}),$$

where the *energy* $\mathcal{E}_s$ is defined by

$$\mathcal{E}_s(w) = \int \{ (\partial_\xi \partial_y^s w)^2 - c_3^\varepsilon (\partial_y^{s+1} w)^2 + c_4^\varepsilon (\partial_y^s w)^2 \} \, \mathrm{d}y.$$

Working in Fourier space, one finds that

$$\mathcal{E}_s(w) = \int (\partial_\xi \partial_y^s w)^2 \, \mathrm{d}y + \sum_{j=1}^{\infty} (-c_3^\varepsilon k_0^2 (2j+1)^2 + c_4^\varepsilon) k_0^{2s} (2j+1)^{2s} (|w_{2j+1,\mathrm{o}}|^2 + |w_{2j+1,\mathrm{e}}|^2),$$

and the facts that (w, w') has zero mean while $-c_3^\varepsilon k_0^2 (2j+1)^2 + c_4^\varepsilon = ((2j+1)^2 - 1)/(1 - c_\mathrm{g}^2)$ is positive for all j and behaves asymptotically like $(2j+1)^2$ imply that

$$\frac{1}{c} (\|w'\|_s^2 + \|w\|_{s+1}^2) \leq \mathcal{E}_s(w) \leq c (\|w'\|_s^2 + \|w\|_{s+1}^2),$$

so that $\mathcal{E}_s$ defines a norm equivalent to the usual norm on $\mathcal{X}_\mathrm{c}^s$.

Turning to the nonlinear terms, notice that

$$s_1 = \int (\partial_\xi \partial_y^s w_{(m+1)}) \partial_y^s g_{4(m)}^\varepsilon \, \mathrm{d}y$$

is *semilinear*, that is it can be estimated in terms of $\mathcal{E}_s(w_{(m+1)})$ and $\mathcal{E}_s(w_{(m)})$; in fact it obeys the estimate

$$\begin{aligned}
|s_1| &\leq \|\partial_\xi \partial_y^s w_{(m+1)}\|_0 \|\partial_y^s g_{4(m)}^\varepsilon\|_0 \\
&\leq c \mathcal{E}_s(w_{(m+1)})^{1/2} \|g_{4(m)}^\varepsilon\|_s \\
&\leq c \mathcal{E}_s(w_{(m+1)})^{1/2} \|(w_{(m)}, w'_{(m)})\|_{\mathcal{X}_\mathrm{c}^s} (\|(w_{(m)}, w'_{(m)})\|_{\mathcal{X}_\mathrm{c}^s}^2 + |Q^\varepsilon + R_{(m)}|^2) \\
&\leq c \mathcal{E}_s(w_{(m+1)})^{1/2} \mathcal{E}_s(w_{(m)})^{1/2} (\mathcal{E}_s(w_{(m)}) + |Q^\varepsilon + R_{(m)}|^2),
\end{aligned}$$

in which the estimate (10) has been used. Furthermore, we find that

$$\begin{aligned}
&\int (\partial_\xi \partial_y^s w_{(m+1)}) \partial_y^s (P_\mathrm{c}(g_{3(m)}^\varepsilon \partial_y^2 w_{(m+1)})) \, \mathrm{d}y \\
&= \int (\partial_\xi \partial_y^s w_{(m+1)}) \partial_y^s (g_{3(m)}^\varepsilon \partial_y^2 w_{(m+1)}) \, \mathrm{d}y + s_2 \\
&= \int (\partial_\xi \partial_y^s w_{(m+1)}) (g_{3(m)}^\varepsilon \partial_y^{s+2} w_{(m+1)}) \, \mathrm{d}y + s_2 + s_3 \\
&= -\int (\partial_\xi \partial_y^{s+1} w_{(m+1)}) (g_{3(m)}^\varepsilon \partial_y^{s+1} w_{(m+1)}) \, \mathrm{d}y + s_2 + s_3 + s_4 \\
&= -\frac{1}{2} \int (\partial_\xi (\partial_y^{s+1} w_{(m+1)})^2) g_{3(m)}^\varepsilon \, \mathrm{d}y + s_2 + s_3 + s_4 \\
&= -\frac{1}{2} \partial_\xi \int (\partial_y^{s+1} w_{(m+1)})^2 g_{3(m)}^\varepsilon \, \mathrm{d}y + s_2 + s_3 + s_4 + s_5,
\end{aligned}$$

where

$$\begin{aligned}
s_2 &= -\int (\partial_\xi \partial_y^s w_{(m+1)}) \partial_y^s (P_\mathrm{h}(g_{3(m)}^\varepsilon \partial_y^2 w_{(m+1)}))\, \mathrm{d}y, \\
s_3 &= \int (\partial_\xi \partial_y^s w_{(m+1)}) \left(\sum_{j=0}^{s-1} \begin{pmatrix} s \\ j \end{pmatrix} (\partial_y^{s-j} g_{3(m)}^\varepsilon)(\partial_y^{j+2} w_{(m+1)}) \right) \mathrm{d}y, \\
s_4 &= -\int (\partial_\xi \partial_y^s w_{(m+1)})(\partial_y g_{3(m)}^\varepsilon)(\partial_y^{s+1} w_{(m+1)})\, \mathrm{d}y, \\
s_5 &= \frac{1}{2} \int (\partial_y^{s+1} w_{(m+1)})^2 \partial_\xi g_{3(m)}^\varepsilon\, \mathrm{d}y;
\end{aligned}$$

these quantities can be estimated using the inequalities

$$\begin{aligned}
|s_2| &\leq \|\partial_\xi \partial_y^s w_{(m+1)}\|_0 \|P_\mathrm{h} g_{3(m)}^\varepsilon \partial_y^2 w_{(m+1)}\|_s \\
&\leq c\|\partial_\xi \partial_y^s w_{(m+1)}\|_0 \|w_{(m+1)}\|_s \|g_{3(m)}^\varepsilon\|_s \\
&\leq c\mathcal{E}_s(w_{(m+1)})(\mathcal{E}_s(w_{(m)}) + |Q^\varepsilon + R_{(m)}|^2),
\end{aligned}$$

$$\begin{aligned}
|s_3| &\leq c\Bigg(\sum_{j=1}^{s-1} \|\partial_y^{s-j} g_{3(m)}^\varepsilon\|_\infty \int |\partial_\xi \partial_y^s w_{(m+1)} \partial_y^{j+2} w_{(m+1)}|\, \mathrm{d}y \\
&\qquad + \|\partial_y^2 w_{(m+1)}\|_\infty \int |\partial_y^s g_{3(m)}^\varepsilon \partial_\xi \partial - y^s w_{(m+1)}|\, \mathrm{d}y \Bigg) \\
&\leq c\|g_{3(m)}^\varepsilon\|_s \|\partial_\xi w_{(m+1)}\|_s \|w_{(m+1)}\|_{s+1} \\
&\leq c\mathcal{E}_s(w_{(m+1)})(\mathcal{E}_s(w_{(m)}) + |Q^\varepsilon + R_{(m)}|^2),
\end{aligned}$$

$$\begin{aligned}
|s_4| &\leq \|\partial_y g_{3(m)}^\varepsilon\|_\infty \int |\partial_\xi \partial_y^s w_{(m+1)} \partial_y^{s+1} w_{(m+1)}|\, \mathrm{d}y \\
&\leq c\|g_{3(m)}^\varepsilon\|_s \|\partial_\xi w_{(m+1)}\|_s \|w_{(m+1)}\|_{s+1} \\
&\leq c\mathcal{E}_s(w_{(m+1)})(\mathcal{E}_s(w_{(m)}) + |Q^\varepsilon + R_{(m)}|^2),
\end{aligned}$$

$$\begin{aligned}
|s_5| &\leq \frac{1}{2} \|\partial_\xi g_{3(m)}^\varepsilon\|_\infty \int (\partial_y^{s+1} w_{(m+1)})^2\, \mathrm{d}y \\
&\leq c\mathcal{E}_s(w_{(m+1)}) \|\partial_\xi g_{3(m)}^\varepsilon\|_s \\
&\leq c\mathcal{E}_s(w_{(m+1)})(\mathcal{E}_s(w_{(m)}) + |Q^\varepsilon|^2 + |R_{(m)}|^2),
\end{aligned}$$

in which we have used the smoothing property of P_h, the estimates (9) and the calculation

$$\begin{aligned}
\partial_\xi g_{3(m)}^\varepsilon &= \mathrm{d}_1 g_3^\varepsilon [Q^\varepsilon + R_{(m)}, w](\partial_\xi Q^\varepsilon + R'_{(m)}) + \mathrm{d}_2[Q^\varepsilon + R_{(m)}, w](w') \\
&= \mathrm{d}_1 g_3^\varepsilon [Q^\varepsilon + R_{(m)}, w](N^\varepsilon(Q^\varepsilon, 0, 0) + \mathcal{L}^\varepsilon R_{(m)} + N_{(m)}^\varepsilon) + \mathrm{d}_2 g_3^\varepsilon [Q^\varepsilon + R_{(m)}, w](w').
\end{aligned}$$

We similarly find that

$$\begin{aligned}
&\int (\partial_\xi \partial_y^s w_{(m+1)}) \partial_y^s (P_{\mathrm{c}} g_{5(m)}^\varepsilon \partial_y \partial_\xi w_{(m+1)}) \,\mathrm{d}y \\
&\quad = \int (\partial_\xi \partial_y^s w_{(m+1)}) \partial_y^s (g_{5(m)}^\varepsilon \partial_y \partial_\xi w_{(m+1)}) \,\mathrm{d}y + s_6 \\
&\quad = \int (\partial_\xi \partial_y^s w_{(m+1)}) g_{5(m)}^\varepsilon (\partial_y^{s+1} \partial_\xi w_{(m+1)}) \,\mathrm{d}y + s_6 + s_7 \\
&\quad = \frac{1}{2} \int \partial_y ((\partial_\xi \partial_y^s w_{(m+1)})^2) g_{5(m)}^\varepsilon \,\mathrm{d}y + s_6 + s_7 \\
&\quad = s_6 + s_7 + s_8,
\end{aligned}$$

where

$$\begin{aligned}
s_6 &= -\int (\partial_\xi \partial_y^s w_{(m+1)}) \partial_y^s (P_{\mathrm{h}} (g_{5(m)}^\varepsilon \partial_y \partial_\xi w_{(m+1)})) \,\mathrm{d}y, \\
s_7 &= \int (\partial_\xi \partial_y^s w_{(m+1)}) \left(\sum_{j=0}^{s-1} \begin{pmatrix} s \\ j \end{pmatrix} (\partial_y^{s-j} g_{5(m)}^\varepsilon)(\partial_y^{j+1} \partial_\xi w_{(m+1)}) \right) dy, \\
s_8 &= -\frac{1}{2} \int (\partial_\xi \partial_y^s w_{(m+1)})^2 (\partial_y g_{5(m)}^\varepsilon) \,\mathrm{d}y
\end{aligned}$$

satisfy the estimate

$$|s_j| \leq c\mathcal{E}_s(w_{(m+1)})(\mathcal{E}_s(w_{(m)}) + |Q^\varepsilon + R_{(m)}|^2).$$

Finally, let us define

$$\mathcal{E}_s^{\mathrm{e}}(w) = \mathcal{E}_s(w) - \frac{1}{2} \int (\partial_y^{s+1} w)^2 g_{3(m)}^\varepsilon \,\mathrm{d}y$$

and note that

$$\frac{1}{c} \mathcal{E}_s(w) \leq \mathcal{E}_s^{\mathrm{e}}(w) \leq c\mathcal{E}_s(w) \tag{29}$$

since

$$\begin{aligned}
\left| \int (\partial_y^{s+1} w)^2 g_{3(m)}^\varepsilon \,\mathrm{d}y \right| &\leq \mathcal{E}_s(w) \| g_{3(m)}^\varepsilon \|_\infty \\
&\leq \mathcal{E}_s(w) \| g_{3(m)}^\varepsilon \|_s \\
&\leq c\mathcal{E}_s(w) \| w_{(m)} + Q^\varepsilon + R_{(m)} \|_s \\
&\leq c\varepsilon \mathcal{E}_s(w).
\end{aligned}$$

Altogether, we have that

$$\begin{aligned}
\partial_\xi \mathcal{E}_s^{\mathrm{e}}(w_{(m+1)}) \leq\; & c\mathcal{E}_s^{\mathrm{e}}(w_{(m+1)})^{1/2} \mathcal{E}_s(w_{(m)})^{1/2} (\mathcal{E}_s(w_{(m)}) + |Q|^2 + |R_{(m)}|^2) \\
& + c\mathcal{E}_s^{\mathrm{e}}(w_{(m+1)})(\mathcal{E}_s(w_{(m)}) + |Q^\varepsilon|^2 + |R_{(m)}|^2) + c\mathcal{E}_s^{\mathrm{e}}(w_{(m+1)})^{1/2} \|h^\varepsilon\|_s,
\end{aligned}$$

and the estimates $\mathcal{E}_s(w_{(m)})^{1/2} \leq \varepsilon^n$, $|R_{(m)}| \leq \varepsilon^n$, $|Q^\varepsilon| \leq c\varepsilon \mathrm{e}^{-\varepsilon\theta\xi}$ and

$$\|h^\varepsilon\|_s \leq c|Q^\varepsilon + R|^3 |(\varepsilon, Q^\varepsilon + R)|^{2n} \leq \varepsilon^{2n+3}$$

show that

$$\partial_\xi \mathcal{E}_s^{\mathrm{e}}(w_{(m+1)}) \le c(\varepsilon^{2n+3} + \varepsilon^{n+2}\mathrm{e}^{-\theta\varepsilon\xi})\mathcal{E}_s^{\mathrm{e}}(w_{(m+1)})^{1/2} + c(\varepsilon^{2n} + \varepsilon^2\mathrm{e}^{-\theta\varepsilon\xi})\mathcal{E}_s^{\mathrm{e}}(w_{(m+1)}).$$

It follows that

$$\begin{aligned}
&\mathcal{E}_s^{\mathrm{e}}(w_{(m+1)}(\xi)) \\
&\quad\le \mathcal{E}_s^{\mathrm{e}}(w_{(m+1)}(0)) + c\xi(\sup_{\tau\in[0,\xi]}\varepsilon^{2n}\mathcal{E}_s^{\mathrm{e}}(w_{(m+1)}(\tau)) + \sup_{\tau\in[0,\xi]}\varepsilon^{2n+3}\mathcal{E}_s^{\mathrm{e}}(w_{(m+1)}(\tau))^{1/2}) \\
&\qquad + c(\varepsilon\sup_{\tau\in[0,\xi]}\mathcal{E}_s^{\mathrm{e}}(w_{(m+1)}(\tau)) + \varepsilon^{n+1}\sup_{\tau\in[0,\xi]}\mathcal{E}_s^{\mathrm{e}}(w_{(m+1)}(\tau))^{1/2}) \\
&\quad\le \mathcal{E}_s^{\mathrm{e}}(w_{(m+1)}(0)) + c(\varepsilon\sup_{\tau\in[0,\xi]}\mathcal{E}_s^{\mathrm{e}}(w_{(m+1)}(\tau)) + \varepsilon^{n+1}\sup_{\tau\in[0,\xi]}\mathcal{E}_s^{\mathrm{e}}(w_{(m+1)}(\tau))^{1/2}) \\
&\quad\le \mathcal{E}_s^{\mathrm{e}}(w_{(m+1)}(0)) + c(\varepsilon+\delta)\sup_{\tau\in[0,\xi]}\mathcal{E}_s^{\mathrm{e}}(w_{(m+1)}(\tau)) + c\delta^{-1}\varepsilon^{2n+2}
\end{aligned}$$

for $\xi \le L/\varepsilon^n$, and choosing δ sufficiently small (independently of ε), we conclude that

$$\sup_{\tau\in[0,L/\varepsilon^n]}\mathcal{E}_s(w_{(m+1)}(\tau)) \le c(\mathcal{E}_s(w^0) + \varepsilon^{2n+2}) \le c\varepsilon^{2n+2},$$

where we have used (29) to replace $\mathcal{E}_s^{\mathrm{e}}$ with $\mathcal{E}_s$. □

Lemma 5.3 *Suppose that* $\|(w^0, w^{0\prime})\|_{\mathcal{X}_{\mathrm{c}}^{s+1}} \le \varepsilon^{n+1}$. *The iterates* $R_{(m)}$ *and* $w_{(m)}$ *satisfy*

$$\begin{aligned}
&\sup_{\xi\in[0,L/\varepsilon^n]}|\tilde{R}_{(m+1)}(\xi)| \\
&\qquad\le \frac{1}{2}\left(\sup_{\xi\in[0,L/\varepsilon^n]}|\tilde{R}_{(m)}(\xi)| + \sup_{\xi\in[0,L/\varepsilon^n]}\|(\tilde{w}_{(m)}(\xi), \tilde{w}_{(m)}'(\xi))\|_{\mathcal{X}_{\mathrm{c}}^s}\right), \\
&\sup_{\xi\in[0,L/\varepsilon^n]}\|(\tilde{w}_{(m+1)}(\xi), \tilde{w}_{(m+1)}'(\xi))\|_{\mathcal{X}_{\mathrm{c}}^s} \\
&\qquad\le \frac{1}{2}\left(\sup_{\xi\in[0,L/\varepsilon^n]}|\tilde{R}_{(m)}(\xi)| + \sup_{\xi\in[0,L/\varepsilon^n]}\|(\tilde{w}_{(m)}(\xi), \tilde{w}_{(m)}'(\xi))\|_{\mathcal{X}_{\mathrm{c}}^s}\right)
\end{aligned}$$

for each $m \in \mathbb{N}_0$, *where*

$$\tilde{R}_{(m+1)} = R_{(m+1)} - R_{(m)}, \qquad \tilde{w}_{(m+1)} = w_{(m+1)} - w_{(m)}.$$

Proof. To establish the first estimate we examine the equation

$$\begin{aligned}
\tilde{R}_{(m+1)}(\xi) = &\sum_{j=1}^{2}\int_0^\xi \langle (N_{(m)}^\varepsilon - N_{(m-1)}^\varepsilon)(\tau), s_j^*(\tau)\rangle\,\mathrm{d}\tau\, s_j(\xi) \\
&- \sum_{j=1}^{2}\int_\xi^{L/\varepsilon^n}\langle (N_{(m)}^\varepsilon - N_{(m-1)}^\varepsilon)(\tau), u_j^*(\tau)\rangle\,\mathrm{d}\tau\, u_j(\xi).
\end{aligned}$$

Observe that

$$|N^\varepsilon(R^1,w^1,w^{1\prime})-N^\varepsilon(R^2,w^2,w^{2\prime})|\leq c\varepsilon^n(|R^1-R^2|+\|(w^1,w^{1\prime})-(w^2,w^{2\prime})\|_{\mathcal{X}_{\rm c}^s})$$

for each $(R^1,w^1,w^{1\prime}),(R^2,w^2,w^{2\prime})\in\bar{B}_{\varepsilon^n}$, where

$$\bar{B}_{\varepsilon^n}=\{(R,w,w'):|R|\leq\varepsilon^n,\|(w,w')\|_{\mathcal{X}_{\rm c}^s}\leq\varepsilon^n\},$$

since $\mathrm{d}N^\varepsilon[R,w,w']$ is $\mathcal{O}(|R|+\|(w,w')\|_{\mathcal{X}_{\rm c}^s})=\mathcal{O}(\varepsilon^n)$ on $\bar{B}_{\varepsilon^n}$. It follows that

$$\begin{aligned}
&|\tilde{R}_{(m+1)}(\xi)|\\
&\quad\leq c\left(\frac{\varepsilon^n}{\lambda_{1,\varepsilon}}\int_0^\xi(|\tilde{R}_{(m)}(\xi)|+\|(\tilde{w}_{(m)}(\xi),\tilde{w}_{(m)}'(\xi))\|_{\mathcal{X}_{\rm c}^s})\mathrm{e}^{\lambda_{1,\varepsilon}\tau}\,\mathrm{d}\tau\,\mathrm{e}^{-\lambda_{1,\varepsilon}\xi}\right.\\
&\qquad\qquad\left.+\frac{\varepsilon^n}{\lambda_{1,\varepsilon}}\int_\xi^{L/\varepsilon^n}(|\tilde{R}_{(m)}(\xi)|+\|(\tilde{w}_{(m)}(\xi),\tilde{w}_{(m)}'(\xi))\|_{\mathcal{X}_{\rm c}^s})\mathrm{e}^{-\lambda_{1,\varepsilon}\tau}\,\mathrm{d}\tau\,\mathrm{e}^{\lambda_{1,\varepsilon}\xi}\right)\\
&\quad\leq c\frac{\varepsilon^n}{\lambda_{1,\varepsilon}^2}\sup_{\xi\in[0,L/\varepsilon^n]}(|\tilde{R}_{(m)}(\xi)|+\|(\tilde{w}_{(m)}(\xi),\tilde{w}_{(m)}'(\xi))\|_{\mathcal{X}_{\rm c}^s})
\end{aligned}$$

for $\xi\in[0,L/\varepsilon^n]$.

Similarly, the second estimate is obtained by studying the equation

$$\begin{aligned}
&\partial_\xi^2\tilde{w}_{(m+1)}+c_3^\varepsilon\partial_y^2\tilde{w}_{(m+1)}+c_4^\varepsilon\tilde{w}_{(m+1)}+P_{\rm c}(g_{3(m)}^\varepsilon\partial_y^2\tilde{w}_{(m+1)})+P_{\rm c}(g_{5(m)}^\varepsilon\partial_y\partial_\xi\tilde{w}_{(m+1)})\\
&\qquad+P_{\rm c}((g_{3(m)}^\varepsilon-g_{3(m-1)}^\varepsilon)\partial_y^2w_{(m)})+P_{\rm c}((g_{5(m)}^\varepsilon-g_{5(m-1)}^\varepsilon)\partial_y\partial_\xi w_{(m)})\\
&\qquad+g_{4(m)}^\varepsilon-g_{4(m-1)}^\varepsilon+h_{(m)}^\varepsilon-h_{(m-1)}^\varepsilon=0
\end{aligned}\tag{30}$$

and using the additional estimates

$$\begin{aligned}
\|g_{4(m)}^\varepsilon-g_{4(m-1)}^\varepsilon\|_s&\leq c(\varepsilon^{2n}+\varepsilon^2\mathrm{e}^{-\theta\varepsilon})(\mathcal{E}_s(\tilde{w}_{(m)})^{1/2}+|\tilde{R}_{(m)}|),\\
\|g_{j(m)}^\varepsilon-g_{j(m-1)}^\varepsilon\|_s&\leq c(\varepsilon^n+\varepsilon\mathrm{e}^{-\theta\varepsilon})(\mathcal{E}_s(\tilde{w}_{(m)})^{1/2}+|\tilde{R}_{(m)}|),\qquad j=3,5
\end{aligned}$$

and

$$\|h_{(m)}^\varepsilon-h_{(m-1)}^\varepsilon\|_s\leq c\varepsilon^{2n+2}|\tilde{R}_{(m)}|,$$

which are obtained from the facts that $\mathrm{d}g_4^\varepsilon[Q^\varepsilon+R,w,w']$ is $\mathcal{O}(|Q^\varepsilon|^2+|R|^2+\|(w,w')\|_{\mathcal{X}_{\rm c}^s}^2)$, $\mathrm{d}g_j^\varepsilon[Q^\varepsilon+R,w]$ is $\mathcal{O}(|Q^\varepsilon|+|R|+\|w\|_s)$ for $j=3,5$ and $\mathrm{d}h^\varepsilon[Q^\varepsilon+R]$ is $\mathcal{O}(|Q^\varepsilon+R|^2|(\varepsilon,Q^\varepsilon+R)|^{2n})$. We proceed by applying the operator ∂_y^s to (30), multiplying by $\partial_\xi\partial_y^s\tilde{w}_{(m+1)}$ and integrating over one period; arguing using the strategy explained in the previous lemma, we find that

$$\begin{aligned}
&\partial_\xi\mathcal{E}_s^{\rm e}(\tilde{w}_{(m+1)})\\
&\quad\leq c(\mathcal{E}_s(w_{(m)})+|Q^\varepsilon|^2+|R_{(m)}|^2)\mathcal{E}_s^{\rm e}(\tilde{w}_{(m+1)})\\
&\qquad+c\mathcal{E}_s^{\rm e}(\tilde{w}_{(m+1)})^{1/2}\mathcal{E}_{s+1}(w_{(m)})^{1/2}(\varepsilon^n+\varepsilon\mathrm{e}^{-\theta\varepsilon\xi})(\mathcal{E}_s(\tilde{w}_{(m)})^{1/2}+|\tilde{R}_m|)\\
&\qquad+c\mathcal{E}_s^{\rm e}(\tilde{w}_{(m+1)})^{1/2}(\varepsilon^{2n}+\varepsilon^2\mathrm{e}^{-\theta\varepsilon\xi})(\mathcal{E}_s(\tilde{w}_{(m)})^{1/2}+|\tilde{R}_m|)+c\varepsilon^{2n+2}|\tilde{R}_{(m)}|\mathcal{E}_s^{\rm e}(\tilde{w}_{(m+1)})^{1/2}\\
&\quad\leq c(\varepsilon^{2n}+\varepsilon^2\mathrm{e}^{-\theta\varepsilon\xi})\mathcal{E}_s^{\rm e}(\tilde{w}_{(m+1)})\\
&\qquad+c(\varepsilon^{2n}+\varepsilon^2\mathrm{e}^{-\theta\varepsilon\xi})\mathcal{E}_s^{\rm e}(\tilde{w}_{(m+1)})^{1/2}(\mathcal{E}_s(\tilde{w}_{(m)})^{1/2}+|\tilde{R}_m|)+c\varepsilon^{2n+2}|\tilde{R}_{(m)}|\mathcal{E}_s^{\rm e}(\tilde{w}_{(m+1)})^{1/2},
\end{aligned}$$

where we have used the estimate $\mathcal{E}_{s+1}(w_{(m)}) \leq \varepsilon^{2n}$, which is obtained by repeating Lemma 5.2 with s replaced by $s+1$ (and requires the stronger condition $\|(w^0, w^{0\prime})\|_{\mathcal{X}_c^{s+1}} \leq \varepsilon^{n+1}$). It follows that

$$
\begin{aligned}
&\mathcal{E}_s^{\mathrm{e}}(\tilde{w}_{(m+1)}(\xi)) \\
&\leq \; c\xi \left(\sup_{\tau\in[0,\xi]} \varepsilon^{2n}\mathcal{E}_s^{\mathrm{e}}(\tilde{w}_{(m+1)}(\tau)) + \sup_{\tau\in[0,\xi]} \varepsilon^{2n}\mathcal{E}_s^{\mathrm{e}}(\tilde{w}_{(m+1)}(\tau))^{1/2}(\mathcal{E}_s(\tilde{w}_{(m)}(\tau))^{1/2} + |\tilde{R}_{(m)}(\tau)|) \right) \\
&\qquad + c \left(\sup_{\tau\in[0,\xi]} \varepsilon\mathcal{E}_s^{\mathrm{e}}(\tilde{w}_{(m+1)}(\tau)) + \sup_{\tau\in[0,\xi]} \varepsilon\mathcal{E}_s^{\mathrm{e}}(\tilde{w}_{(m+1)}(\tau))^{1/2}(\mathcal{E}_s(\tilde{w}_{(m)}(\tau))^{1/2} + |\tilde{R}_{(m)}(\tau)|) \right) \\
&\qquad + c\xi \sup_{\tau\in[0,\xi]} \varepsilon^{2n+2}|\tilde{R}_{(m)}(\tau)|\mathcal{E}_s^{\mathrm{e}}(\tilde{w}_{(m+1)}(\tau))^{1/2} \\
&\leq \; c\varepsilon \sup_{\tau\in[0,\xi]} \mathcal{E}_s^{\mathrm{e}}(\tilde{w}_{(m+1)}(\tau)) + c\varepsilon \sup_{\tau\in[0,\xi]} (\mathcal{E}_s(\tilde{w}_{(m)}(\tau)) + |\tilde{R}_{(m)}(\tau)|^2)
\end{aligned}
$$

for $\xi \leq L/\varepsilon^n$ and hence that

$$
\sup_{\tau\in[0,L/\varepsilon^n]} \mathcal{E}_s(\tilde{w}_{(m+1)}(\tau)) \leq c\varepsilon \sup_{\tau\in[0,L/\varepsilon^n]} (\mathcal{E}_s(\tilde{w}_{(m)}(\tau)) + |\tilde{R}_{(m)}(\tau)|^2). \qquad \square
$$

The following convergence result is a direct consequence of the above lemmata.

Theorem 5.4 *For each $R_{s_1}^0$, $R_{s_2}^0$ and $(w^0, w^{0\prime})$ with*

$$
|R_{s_1}^0| \leq \varepsilon^{n+1}, \quad |R_{s_2}^0| \leq \varepsilon^{n+1}, \quad \|(w^0, w^{0\prime})\|_{\mathcal{X}_c^s} \leq \varepsilon^{n+1}
$$

the sequence $(R_{(m)}, w_{(m)}, w'_{(m)})_{m\in\mathbb{N}_0}$ converges in $C([0, L/\varepsilon^n], \mathcal{X}^s)$ to a limit $(R_\star, w_\star, w'_\star)$ which satisfies the estimate

$$
\sup_{\xi\in[0,L/\varepsilon^n]} \|(R_\star(\xi), w_\star(\xi), w'_\star(\xi))\|_{\mathcal{X}^s} \leq c\varepsilon^{n+1}
$$

and solves equations (24), (25).

We now use the above results to define a local centre-stable manifold at time $\xi = 0$ for the nonautonomous equations (24), (25). According to Lemmata 5.2 and 5.3 the solutions defining this manifold are available under the hypothesis that $\|(w^0, w^{0\prime})\|_{\mathcal{X}_c^{s+1}} \leq \varepsilon^{n+1}$; to ensure its differentiability one however requires the stronger hypothesis that $\|(w^0, w^{0\prime})\|_{\mathcal{X}_c^{s+3}} \leq \varepsilon^{n+1}$ (see Section 6 below), and we therefore make this hypothesis from the outset.

Definition 5.5 *The set of points*

$$
W_{\mathrm{loc}}^{\mathrm{cs}} = \bigcup \{(R_\star(0), w_\star(0), w'_\star(0))\},
$$

in which the union is taken over the set of $R_{s_1}^0$, $R_{s_2}^0$ and $(w^0, w^{0\prime})$ such that

$$
|R_{s_1}^0| \leq \varepsilon^{n+1}, \quad |R_{s_2}^0| \leq \varepsilon^{n+1}, \quad \|(w^0, w^{0\prime})\|_{\mathcal{X}_c^{s+3}} \leq \varepsilon^{n+1},
$$

is called the local centre-stable manifold for solutions to (24), (25) at time $\xi = 0$.

6 Construction of symmetric modulating pulses

In this section we identify solutions $(R_\star, w_\star)$ to equations (24), (25) on the interval $[0, L/\varepsilon^n]$ whose initial data $(R_\star(0), w_\star(0), w_\star'(0))$ lies on $W^{\mathrm{cs}}_{\mathrm{loc}}$ and which can be extended to solutions that remain $\mathcal{O}(\varepsilon^{n+1})$ on $[-L/\varepsilon^n, L/\varepsilon^n]$. The idea is to exploit the reversibility of equations (24), (25) (see Section 2); in particular, solutions with the property that $(R_\star(0), w_\star(0), w_\star'(0))$ lies on the *symmetric section*

$$\Sigma := \operatorname{Fix} S = \mathcal{X} \cap \{(v_{\mathrm{o}}, v_{\mathrm{e}}') = (0,0)\}$$

can be extended to symmetric solutions on $[-L/\varepsilon^n, L/\varepsilon^n]$. Because $(w_\star(0), w_\star'(0)) = (w^0, w^{0\prime})$ we have that $(w_\star(0), w_\star'(0)) \in \Sigma_{\mathrm{c}}$ whenever $(w^0, w^{0\prime}) \in \Sigma_{\mathrm{c}}$ and our task is reduced to that of finding a criterion on $(R^0_{s_1}, R^0_{s_2})$ which guarantees that $R_\star(0) \in \Sigma_{\mathrm{h}}$.

The next step is to introduce an artificial parameter by replacing h^ε in equations (24), (25) by ρh^ε; the construction of $W^{\mathrm{loc}}_{\mathrm{cs}}$ undertaken in Section 5 above clearly remains valid for all values of $\rho \in [0,1]$. Observe that $\rho = 1$ yields the original equations while $\rho = 0$ yields the system considered in Section 4, in which $(w, w') = (0,0)$ is an invariant subspace containing the homoclinic solution Q^ε (generated by the solution $(R, w) = (0,0)$ in the present coordinates). We consider a solution $(R_\star, w_\star)$ with $(R_\star(0), w_\star(0), w_\star'(0)) \in W^{\mathrm{cs}}_{\mathrm{loc}}$ as a function of $R^0_{s_1}$,$R^0_{s_2}$ which depends upon $\rho \in \mathbb{R}$ and $(w^0, w^{0\prime}) \in \Sigma_{\mathrm{c}}$ as parameters (with $\rho \in [0,1]$, $\|(w^0, w^{0\prime})\|_{\mathcal{X}^s_{\mathrm{c}}} \leq \varepsilon^{n+3}$) and therefore write $(R_\star, w_\star)$ as $(R_{\rho,w^0,w^{0\prime}}, w_{\rho,w^0,w^{0\prime}})(R^0_{s_1}, R^0_{s_2})$ in the following analysis. Notice that $(R_{\rho,w^0,w^{0\prime}}, w_{\rho,w^0,w^{0\prime}}, w'_{\rho,w^0,w^{0\prime}})(R^0_{s_1}, R^0_{s_2})|_{\xi=0} \in \Sigma$ whenever $(R^0_{s_1}, R^0_{s_2})$ is a solution of the equation

$$J_{\rho,w^0,w^{0\prime}}(R^0_{s_1}, R^0_{s_2}) = 0, \tag{31}$$

where $J_{\rho,w^0,w^{0\prime}} : \bar{B}_{\varepsilon^{n+1}}(0) \subset \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$J_{\rho,w^0,w^{0\prime}}(R^0_{s_1}, R^0_{s_2}) = (I - S_{\mathrm{h}}) R_{\rho,w^0,w^{0\prime}}(R^0_{s_1}, R^0_{s_2})|_{\xi=0}.$$

(The right-hand side of this equation is a vector in $\mathcal{X}^s_{\mathrm{h}}$ with only two nonzero entries, namely its z_{o} and z_{e}' components, and is therefore identified with a pair of real numbers.) Equation (31) has the solution $(R^0_{s_1}, R^0_{s_2}) = (0,0)$ at $(\rho, w^0, w^{0\prime}) = (0,0,0)$ since the unique solution of (24), (25) with $(\rho, w^0, w^{0\prime}) = (0,0,0)$ is $(R, w) = (0,0)$. We therefore seek a solution of (24), (25) near this known solution for parameter values $(\rho, w^0, w^{0\prime})$ near $(0,0,0)$, and it seems natural to apply the implicit-function theorem; notice, however, that we are forced to work from first principles (by applying the contraction mapping principle) since we require precise information concerning the parameter-dependence of the solutions, in particular that the solution exists for values of ρ up to one.

In order to carry out the above programme it is necessary to show that $J_{\rho,w^0,w^{0\prime}}$ is differentiable with respect to $R^0_{s_1}$, $R^0_{s_2}$ and obtain some estimates on its derivatives. We therefore need to show that the solutions $(R_{\rho,w^0,w^{0\prime}}, w_{\rho,w^0,w^{0\prime}})$ described above are differentiable with respect to $R^0_{s_1}$, $R^0_{s_2}$ and obtain some estimates on their derivatives. To this end we formally differentiate equations (24), (25) with respect to $R^0_{s_1}$ and use a dot to denote $\partial_{R^0_{s_1}}$; we treat the resulting linear equations for $\dot{R}$, $\dot{w}$ with the iteration scheme

$$
\begin{aligned}
\dot{R}_{(m+1)}(\xi) &= \tilde{R}^0_{s_1}\, s_1(\xi) \\
&\quad + \sum_{j=1}^{2} \int_0^{\xi} \langle \mathrm{d}N^{\varepsilon}(R,w,w')(\dot{R}_{(m)}, \dot{w}_{(m)}, \dot{w}'_{(m)})(\tau), s_j^*(\tau)\rangle \,\mathrm{d}\tau\, s_j(\xi) \\
&\quad - \sum_{j=1}^{2} \int_{\xi}^{L/\varepsilon^n} \langle \mathrm{d}N^{\varepsilon}(R,w,w')(\dot{R}_{(m)}, \dot{w}_{(m)}, \dot{w}'_{(m)})(\tau), u_j^*(\tau)\rangle \,\mathrm{d}\tau\, u_j(\xi),
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
&\partial_\xi^2 \dot{w}_{(m+1)} + c_3^{\varepsilon}\partial_y^2 \dot{w}_{(m+1)} + c_4^{\varepsilon}\dot{w}_{(m+1)} \\
&\quad + P_{\mathrm{c}}(g_3^{\varepsilon}(Q^{\varepsilon}+R,w)\partial_y^2 \dot{w}_{(m+1)}) + P_{\mathrm{c}}(g_5^{\varepsilon}(Q^{\varepsilon}+R,w)\partial_y\partial_\xi \dot{w}_{(m+1)}) \\
&\quad + P_{\mathrm{c}}(\mathrm{d}g_3^{\varepsilon}[Q^{\varepsilon}+R,w](\dot{R}_{(m)},\dot{w}_{(m)})\partial_y^2 w) + P_{\mathrm{c}}(\mathrm{d}g_5^{\varepsilon}[Q^{\varepsilon}+R,w](\dot{R}_{(m)},\dot{w}_{(m)})\partial_y\partial_\xi w) \\
&\quad + \mathrm{d}g_4^{\varepsilon}[Q^{\varepsilon}+R,w,w'](\dot{R}_{(m)},\dot{w}_{(m)},\dot{w}'_{(m)}) \\
&\quad + \rho\,\mathrm{d}h^{\varepsilon}[Q^{\varepsilon}+R](\dot{R}_{(m)}) = 0.
\end{aligned} \tag{33}
$$

Let us now choose (R,w,w') which satisfy $|R(\xi)|$, $\|(w(\xi),w'(\xi))\|_{\mathcal{X}_{\mathrm{c}}^{s+2}} \leq \varepsilon^n$ for $\xi \in [0,L/\varepsilon^n]$, take $\dot{R}_{(0)} = 0$, $\dot{w}_{(0)} = 0$, and for $m = 0,1,2,\ldots$ define $\dot{R}_{(m+1)} \in C([0,L/\varepsilon^n],\mathbb{R}^4)$ by the formula (32) and let $(\dot{w}_{(m+1)},\dot{w}'_{(m+1)}) \in C([0,L/\varepsilon^n],\mathcal{X}_{\mathrm{c}}^s)$ be the solution of (33) with initial data $(\dot{w},\dot{w}')|_{\xi=0} = (0,0)$.

Lemma 6.1

(i) The estimates

$$
\sup_{\xi\in[0,L/\varepsilon^n]} |\bar{R}_{(m+1)}(\xi)| \leq \frac{1}{2}\left(\sup_{\xi\in[0,L/\varepsilon^n]} |\bar{R}_{(m)}(\xi)| + \sup_{\xi\in[0,L/\varepsilon^n]} \|(\bar{w}_{(m)}(\xi),\bar{w}'_{(m)}(\xi))\|_{\mathcal{X}_{\mathrm{c}}^s}\right), \qquad m\in\mathbb{N}_0,
$$

$$
\sup_{\xi\in[0,L/\varepsilon^n]} \|(\bar{w}_{(m+1)}(\xi),\bar{w}'_{(m+1)}(\xi))\|_{\mathcal{X}_{\mathrm{c}}^s} \leq \frac{1}{2}\left(\sup_{\xi\in[0,L/\varepsilon^n]} |\bar{R}_{(m)}(\xi)| + \sup_{\xi\in[0,L/\varepsilon^n]} \|(\bar{w}_{(m)}(\xi),\bar{w}'_{(m)}(\xi))\|_{\mathcal{X}_{\mathrm{c}}^s}\right), \qquad m\in\mathbb{N}_0,
$$

hold uniformly over the set of (R,w,w') which satisfy $|R(\xi)|$, $\|(w(\xi),w'(\xi))\|_{\mathcal{X}_{\mathrm{c}}^{s+1}} \leq \varepsilon^n$ for $\xi\in[0,L/\varepsilon^n]$, where $\bar{R}_{(m+1)} = \dot{R}_{(m+1)} - \dot{R}_{(m)}$, $\bar{w}_{(m+1)} = \dot{w}_{(m+1)} - \dot{w}_{(m)}$.

(ii) Suppose additionally that $\|(w(\xi),w'(\xi))\|_{\mathcal{X}_{\mathrm{c}}^{s+2}} \leq \varepsilon^n$ for $\xi\in[0,L/\varepsilon^n]$. For each fixed value of $(\dot{R}_{(m)},\dot{w}_{(m)},\dot{w}'_{(m)})$ the iterate $(\dot{R}_{(m+1)},\dot{w}_{(m+1)},\dot{w}'_{(m+1)})$ depends Lipschitz-continuously on (R,w,w') in the topology of $\mathcal{X}^s$; the Lipschitz constant is a linear function of $\|(\dot{R}_{(m)},\dot{w}_{(m)},\dot{w}'_{(m)})\|_{C([0,L/\varepsilon^n],\mathcal{X}^s)}$.

Proof. The first assertion is established by adapting the proof of Lemma 5.3. Estimating $|\mathrm{d}N^{\varepsilon}_{(m)}[R,w,w']|$ as $\mathcal{O}(\varepsilon^n)$, we find that

$$
\begin{aligned}
&|\bar R_{(m+1)}(\xi)|\\
&\quad\leq c\left(\frac{\varepsilon^n}{\lambda_{1,\varepsilon}}\int_0^\xi(|\bar R_{(m)}(\tau)|+\|(\bar w_{(m)}(\tau),\bar w'_{(m)}(\tau))\|_{\mathcal{X}_{\mathrm{c}}^s})\mathrm{e}^{\lambda_{1,\varepsilon}\tau}\,\mathrm{d}\tau\,\mathrm{e}^{-\lambda_{1,\varepsilon}\xi}\right.\\
&\qquad\qquad\left.+\frac{\varepsilon^n}{\lambda_{1,\varepsilon}}\int_\xi^{L/\varepsilon^n}(|\bar R_{(m)}(\tau)|+\|(\bar w_{(m)}(\tau),\bar w'_{(m)}(\tau))\|_{\mathcal{X}_{\mathrm{c}}^s})\mathrm{e}^{-\lambda_{1,\varepsilon}\tau}\,\mathrm{d}\tau\,\mathrm{e}^{\lambda_{1,\varepsilon}\xi}\right)
\end{aligned}
$$

for $\xi\leq L/\varepsilon^n$ and hence

$$
\sup_{\xi\in[0,L/\varepsilon^n]}|\bar R_{(m+1)}(\xi)|\leq c\frac{\varepsilon^n}{\lambda_{1,\varepsilon}^2}\sup_{\xi\in[0,L/\varepsilon^n]}(|\bar R(\xi)|+\|(\bar w(\xi),\bar w'(\xi))\|_{\mathcal{X}_{\mathrm{c}}^s}).
$$

Similarly, applying the operator $\partial_\xi\partial_y^s\tilde w_{(m+1)}\partial_y^s$ to the equation for $\bar w_{(m+1)}$, integrating over one period and estimating $\mathrm{d}g_4^\varepsilon[Q^\varepsilon+R,w,w']$ as $\mathcal{O}(|Q^\varepsilon|^2+|R|^2+\|(w,w')\|_{\mathcal{X}_{\mathrm{c}}^s}^2)$, $\mathrm{d}g_j^\varepsilon[Q^\varepsilon+R,w]$ as $\mathcal{O}(|Q^\varepsilon|+|R|+\|w\|_s)$ for $j=3,5$ and $\mathrm{d}h^\varepsilon[Q^\varepsilon+R]$ as $\mathcal{O}(|Q^\varepsilon+R|^2|(\varepsilon,Q^\varepsilon+R^{2n})|)$, one obtains the formula

$$
\begin{aligned}
&\partial_\xi\mathcal{E}_s^{\mathrm{e}}(\bar w_{(m+1)})\\
&\quad\leq\ (\varepsilon^{2n}+\varepsilon^2\mathrm{e}^{-\theta\varepsilon\xi})\mathcal{E}_s^{\mathrm{e}}(\bar w_{(m+1)})\\
&\qquad\ +\mathcal{E}_s^{\mathrm{e}}(\bar w_{(m+1)})^{1/2}\mathcal{E}_{s+1}(w)^{1/2}(\varepsilon^n+\varepsilon\mathrm{e}^{-\theta\varepsilon\xi})(\mathcal{E}_s(\bar w_{(m)})^{1/2}+|\bar R_m|)\\
&\qquad\ +\mathcal{E}_s^{\mathrm{e}}(\bar w_{(m+1)})^{1/2}(\varepsilon^{2n}+\varepsilon^2\mathrm{e}^{\theta\varepsilon\xi})(\mathcal{E}_s(\bar w_{(m)})^{1/2}+|\bar R_m|)+c\rho\varepsilon^{2n+2}|\bar R_{(m)}|\mathcal{E}_s^{\mathrm{e}}(\bar w_{(m+1)})^{1/2}\\
&\quad\leq\ c(\varepsilon^{2n}+\varepsilon^2\mathrm{e}^{-\theta\varepsilon\xi})\mathcal{E}_s^{\mathrm{e}}(\bar w_{(m+1)})\\
&\qquad\ +c(\varepsilon^{2n}+\varepsilon^2\mathrm{e}^{-\theta\varepsilon\xi})\mathcal{E}_s^{\mathrm{e}}(\bar w_{(m+1)})^{1/2}(\mathcal{E}_s(\bar w_{(m)})^{1/2}+|\bar R_m|)+c\varepsilon^{2n+2}|\bar R_{(m)}|\mathcal{E}_s^{\mathrm{e}}(\bar w_{(m+1)})^{1/2};
\end{aligned}
$$

the argument used in the last step of Lemma 5.3 shows that

$$
\sup_{\tau\in[0,L/\varepsilon^n]}\mathcal{E}_s(\bar w_{(m+1)}(\tau))\leq c\varepsilon\sup_{\tau\in[0,L/\varepsilon^n]}(\mathcal{E}_s(\bar w_{(m)}(\tau))+|\bar R_{(m)}(\tau)|^2).
$$

Turning to the second assertion, note that

$$
\begin{aligned}
\hat R(\xi)=&\sum_{j=1}^2\int_0^\xi\langle(\mathrm{d}N_1^\varepsilon-\mathrm{d}N_2^\varepsilon)(\dot R_{(m)},\dot w_{(m)},\dot w'_{(m)})(\tau),s_j^*(\tau)\rangle\,\mathrm{d}\tau\,s_j(\xi)\\
&-\sum_{j=1}^2\int_\xi^{L/\varepsilon^n}\langle(\mathrm{d}N_1^\varepsilon-\mathrm{d}N_2^\varepsilon)(\dot R_{(m)},\dot w_{(m)},\dot w'_{(m)})(\tau),u_j^*(\tau)\rangle\,\mathrm{d}\tau\,u_j(\xi),
\end{aligned}
$$

$$
\begin{aligned}
&\partial_\xi^2\hat w_{(m+1)}+c_3^\varepsilon\partial_y^2\hat w_{(m+1)}+c_4^\varepsilon\hat w_{(m+1)}+P_{\mathrm{c}}(g_{3,1}^\varepsilon\partial_y^2\hat w_{(m+1)})+P_{\mathrm{c}}(g_{5,1}^\varepsilon\partial_y\partial_\xi\hat w_{(m+1)})\\
&\qquad+P_{\mathrm{c}}((g_{3,1}^\varepsilon-g_{3,2}^\varepsilon)\partial_y^2\dot w_{(m)})+P_{\mathrm{c}}((g_{5,1}^\varepsilon-g_{5,2}^\varepsilon)\partial_y\partial_\xi\dot w_{(m)})\\
&\qquad+P_{\mathrm{c}}((\mathrm{d}g_{3,1}^\varepsilon-\mathrm{d}g_{3,2}^\varepsilon)\partial_y^2w_2)+P_{\mathrm{c}}((\mathrm{d}g_{5,1}^\varepsilon-\mathrm{d}g_{5,2}^\varepsilon)\partial_y\partial_\xi w_2)\\
&\qquad+P_{\mathrm{c}}(\mathrm{d}g_{3,1}^\varepsilon\partial_y^2\tilde w)+P_{\mathrm{c}}(\mathrm{d}g_{5,1}^\varepsilon\partial_y\partial_\xi\tilde w)+\mathrm{d}g_{4,1}^\varepsilon-\mathrm{d}g_{4,2}^\varepsilon+\rho\mathrm{d}h_1^\varepsilon-\rho\mathrm{d}h_2^\varepsilon=0,
\end{aligned}
$$

where $(\hat R,\hat w)$ is the difference in the value of $(R_{(m+1)},w_{(m+1)})$ for $(R,w)=(R_1,w_1)$ and $(R,w)=(R_2,w_2)$, $\mathrm{d}N_j^\varepsilon$ is an abbreviation for $\mathrm{d}N_j^\varepsilon[R_1,w_1,w_1'](\dot R_{(m)},\dot w_{(m)},\dot w'_{(m)})$, $j=1,2$

(similar abbreviations have been used for the other nonlinearities) and $(\tilde{R},\tilde{w}) = (R_1 - R_2, w_1 - w_2)$. The usual arguments show that

$$
\begin{aligned}
&|\hat{R}(\xi)| \\
&\quad \leq c\left(\frac{1}{\lambda_{1,\varepsilon}}\int_0^\xi \|(\dot{R}_{(m)},\dot{w}_{(m)},\dot{w}'_{(m)})(\tau)\|_{\mathcal{X}^s}\|(\tilde{R}_{(m)},\tilde{w}_{(m)},\tilde{w}'_{(m)})(\tau)\|_{\mathcal{X}^s}\mathrm{e}^{\lambda_{1,\varepsilon}\tau}\,\mathrm{d}\tau\,\mathrm{e}^{-\lambda_{1,\varepsilon}\xi}\right. \\
&\qquad\qquad \left. +\frac{1}{\lambda_{1,\varepsilon}}\int_\xi^{L/\varepsilon^n} \|(\dot{R}_{(m)},\dot{w}_{(m)},\dot{w}'_{(m)})(\tau)\|_{\mathcal{X}^s}\|(\tilde{R}_{(m)},\tilde{w}_{(m)},\tilde{w}'_{(m)})(\tau)\|_{\mathcal{X}^s}\mathrm{e}^{-\lambda_{1,\varepsilon}\tau}\,\mathrm{d}\tau\,\mathrm{e}^{\lambda_{1,\varepsilon}\xi}\right)
\end{aligned}
$$

for $\xi \leq L/\varepsilon^n$, whence

$$
\sup_{\xi\in[0,L/\varepsilon^n]}|\hat{R}(\xi)| \leq c_\varepsilon \sup_{\xi\in[0,L/\varepsilon^n]}\|(\dot{R}_{(m)},\dot{w}_{(m)},\dot{w}'_{(m)})(\xi)\|_{\mathcal{X}^s}\sup_{\xi\in[0,L/\varepsilon^n]}\|(\tilde{R},\tilde{w},\tilde{w}')(\xi)\|_{\mathcal{X}^s}
$$

and

$$
\begin{aligned}
&\partial_\xi \mathcal{E}_s^{\mathrm{e}}(\hat{w}_{(m+1)}) \\
&\quad\leq\ \ \mathcal{E}_s(\hat{w}_{(m+1)}(\varepsilon^n+\varepsilon\mathrm{e}^{-\theta\varepsilon\xi})(\mathcal{E}_s(\dot{w}_{(m)})^{1/2}+|\dot{R}_{(m)}|) \\
&\qquad\quad + c\mathcal{E}_s^{\mathrm{e}}(\hat{w}_{(m+1)})^{1/2}\mathcal{E}_{s+1}(\dot{w}_{(m)})^{1/2}(\varepsilon^n+\varepsilon\mathrm{e}^{-\theta\varepsilon\xi})(\mathcal{E}_s(\tilde{w})^{1/2}+|\tilde{R}|) \\
&\qquad\quad + c\mathcal{E}_s^{\mathrm{e}}(\hat{w}_{(m+1)})^{1/2}\mathcal{E}_{s+1}(w_2)^{1/2}(\mathcal{E}_s(\dot{w}_{(m)})^{1/2}+|\dot{R}_{(m)}|)(\mathcal{E}_s(\tilde{w})^{1/2}+|\tilde{R}|) \\
&\qquad\quad + c\mathcal{E}_s^{\mathrm{e}}(\hat{w}_{(m+1)})^{1/2}\mathcal{E}_{s+1}(\tilde{w})^{1/2}(\varepsilon^n+\varepsilon\mathrm{e}^{-\theta\varepsilon\xi})(\mathcal{E}_s(\dot{w}_{(m)})^{1/2}+|\dot{R}_{(m)}|) \\
&\qquad\quad + c\mathcal{E}_s^{\mathrm{e}}(\hat{w}_{(m+1)})^{1/2}(\varepsilon^n+\varepsilon\mathrm{e}^{-\theta\varepsilon\xi})(\mathcal{E}_s(\tilde{w})^{1/2}+|\tilde{R}|)(\mathcal{E}_s(\dot{w}_{(m)})^{1/2}+|\dot{R}_{(m)}|) \\
&\qquad\quad + c\varepsilon^{2n+1}\mathcal{E}_s^{\mathrm{e}}(\hat{w}_{(m+1)})^{1/2}(\mathcal{E}_s(\tilde{w})^{1/2}+|\tilde{R}|),
\end{aligned}
$$

where we have used the first assertion with s replaced by $s+1$ (which requires the stronger condition $\|(w(\xi),w'(\xi))\|_{\mathcal{X}_c^{s+2}} \leq \varepsilon^n$ for $\xi\in[0,L/\varepsilon^n]$), whence

$$
\begin{aligned}
&\sup_{\tau\in[0,L/\varepsilon^n]}\mathcal{E}_s(\hat{w}_{(m+1)}(\tau)) \\
&\qquad \leq c_\varepsilon \sup_{\xi\in[0,L/\varepsilon^n]}\|(\dot{R}_{(m)},\dot{w}_{(m)},\dot{w}'_{(m)})(\xi)\|_{\mathcal{X}^s}\sup_{\tau\in[0,L/\varepsilon^n]}(\mathcal{E}_s(\tilde{w}_{(m)}(\tau))+|\tilde{R}_{(m)}(\tau)|^2).
\end{aligned}
$$

□

Corollary 6.2 *Any solution $(R_\star,w_\star)$ to equations (24), (25) whose initial data lies on $W^{\mathrm{cs}}_{\mathrm{loc}}$ is differentiable in the topology of $\mathcal{X}^s$ with respect to $R^0_{s_1}$ and $R^0_{s_2}$.*

Proof. Let T_1 and T_2 be the operators which map respectively $(R_{(m)},w_{(m)})$ to $(R_{(m+1)},w_{(m+1)})$ and $(\dot{R}_{(m)},\dot{w}_{(m)})$ to $(\dot{R}_{(m+1)},\dot{w}_{(m+1)})$ in the iteration schemes (26), (27) and (32), (33), which may therefore be written as

$$
(R_{(m+1)},w_{(m+1)}) = T_1(R_{(m)},w_{(m)}),
$$

$$
(\dot{R}_{(m+1)},\dot{w}_{(m+1)}) = T_2((R,w),(\dot{R}_{(m)},\dot{w}_{(m)})).
$$

Consider the new iteration scheme

$$(\dot{R}_{(m+1)}, \dot{w}_{(m+1)}) = T_2((R_{(m)}, w_{(m)}), (\dot{R}_{(m)}, \dot{w}_{(m)}))$$

with initial data $\dot{R}_{(0)} = 0$, $\dot{w}_{(0)} = 0$, which is obtained by differentiating (26), (27) with respect to $R^0_{s_1}$ and using the dot to denote $\partial_{R^0_{s_1}}$. Let us write this iteration scheme as

$$(\dot{R}_{(m+1)}, \dot{w}_{(m+1)}) = T_2((R_\star, w_\star), (\dot{R}_{(m)}, \dot{w}_{(m)})) + \alpha_{(m)},$$

where

$$\alpha_{(m)} = T_2((R_{(m)}, w_{(m)}), (\dot{R}_{(m)}, \dot{w}_{(m)})) - T_2((R_\star, w_\star), (\dot{R}_{(m)}, \dot{w}_{(m)})).$$

It follows from Lemma 6.1(i) that the further iteration scheme defined by $T_2((R_\star, w_\star), \cdot)$ alone converges in $C([0, L/\varepsilon^n], \mathcal{X}^s)$ to a limit $(\dot{R}_\star, \dot{w}_\star, \dot{w}'_\star)$ whenever $\|(w_\star(\xi), w'_\star(\xi))\|_{\mathcal{X}_c^{s+1}} \leq \varepsilon^n$ for $\xi \in [0, L/\varepsilon^n]$ and from Lemma 6.1(ii) that

$$\|\alpha_{(m)}\|_{C([0,L/\varepsilon^n],\mathcal{X}^s)} = o(1)\|(\dot{R}_{(m)}, \dot{w}_{(m)}, \dot{w}'_{(m)})\|_{C([0,L/\varepsilon],\mathcal{X}^s)}$$

whenever $\|(w_{(m)}(\xi), w'_{(m)}(\xi))\|_{\mathcal{X}_c^{s+2}}, \|(w_\star(\xi), w'_\star(\xi))\|_{\mathcal{X}_c^{s+2}} \leq \varepsilon^n$ for $\xi \in [0, L/\varepsilon^n]$; the hypothesis $\|(w^0, w^{0\prime})\|_{\mathcal{X}_c^{s+3}} \leq \varepsilon^{n+1}$ in the definition of $W^{\mathrm{cs}}_{\mathrm{loc}}$ guarantees that these conditions are met (see Lemmata 5.2 and 5.3). Elementary arguments show that $(\dot{R}_{(m)}, \dot{w}_{(m)}, \dot{w}'_{(m)})_{m\in\mathbb{N}_0}$ converges in $C([0, L/\varepsilon^n], \mathcal{X}^s)$ to $(\dot{R}_\star, \dot{w}_\star, \dot{w}'_\star)$. By construction, one has that $(\dot{R}_{(m)}, \dot{w}_{(m)}, \dot{w}'_{(m)}) = (\partial_{R^0_{s_1}} R_{(m)}, \partial_{R^0_{s_1}} w_{(m)}, \partial_{R^0_{s_1}} w'_{(m)})$ for each $m \in \mathbb{N}_0$, and a familiar uniform continuity argument asserts that

$$(\dot{R}_\star, \dot{w}_\star, \dot{w}'_\star) = (\partial_{R^0_{s_1}} R_\star, \partial_{R^0_{s_1}} w_\star, \partial_{R^0_{s_1}} w'_\star). \qquad \square$$

We now turn to the requisite estimates on the derivative of $J_{\rho,w^0,w^{0\prime}}$.

Proposition 6.3

(i) The operator $\mathrm{d}J_{0,0,0}[0,0] : \mathbb{R}^2 \to \mathbb{R}^2$ *is a bijection and*

$$|\mathrm{d}J_{0,0,0}[0,0]^{-1}| \leq \frac{c}{\lambda_{1,\varepsilon}}. \tag{34}$$

(ii) The operator $\mathrm{d}J_{\rho,w^0,w^{0\prime}}[R^0_{s_1}, R^0_{s_2}] : \mathbb{R}^2 \to \mathbb{R}^2$ *satisfies the estimate*

$$|\mathrm{d}J_{\rho,w^0,w^{0\prime}}[R^0_{s_1}, R^0_{s_2}] - \mathrm{d}J_{0,0,0}[0,0]| \leq c\frac{\varepsilon^n}{\lambda^2_{1,\varepsilon}}. \tag{35}$$

Proof. Clearly

$$\partial_j J_{0,0,0}[0,0] = (I - S_{\mathrm{h}})\partial_{R^0_{s_j}} R_{0,0,0}(0,0)|_{\xi=0}, \qquad j = 1, 2,$$

and

$$\partial_{R^0_{s_j}} R_{0,0,0}(0,0)|_{\xi=0} = \tilde{R}^0_{s_j} s_j(0), \qquad j = 1, 2,$$

so that

$$\mathrm{d}J_{0,0,0}[0,0](\tilde{R}^0_{s_1},\tilde{R}^0_{s_2}) = \tilde{R}^0_{s_1}(I-S_\mathrm{h})s_1(0) + \tilde{R}^0_{s_2}(I-S_\mathrm{h})s_2(0).$$

Taking the inner product of this equation with $(I+S_\mathrm{h})s_j^*(0)$ and using the fact that $S_\mathrm{h}:\mathcal{X}_\mathrm{h}\to\mathcal{X}_\mathrm{h}$ is a self-adjoint involution, one finds that

$$\tilde{R}^0_{s_j} = \frac{1}{2}\langle \mathrm{d}J_{0,0,0}[0,0](\tilde{R}^0_{s_1},\tilde{R}^0_{s_2}),(I+S_\mathrm{h})s_j^*(0)\rangle, \qquad j=1,2,$$

from which the first assertion is a direct consequence.

Define $R_1 = R_{\rho,w^0,w^{0\prime}}(R^0_{s_1},R^0_{s_2})$, $\dot{R}_1 = \mathrm{d}R_{\rho,w^0,w^{0\prime}}[R^0_{s_1},R_{s_2}]$, $\dot{R}_2 = \mathrm{d}R_{0,0,0}[0,0]$, and note that $R_{0,0,0}$ is identically zero. By construction we have that

$$\begin{aligned}
&(\dot{R}_1-\dot{R}_2)(\xi)\\
&= \sum_{j=1}^2 \int_0^\xi \langle(\partial_1 N_1^\varepsilon \dot{R}_1 + \partial_2 N_1^\varepsilon \dot{w}_1 + \partial_3 N_1^\varepsilon \partial_\xi \dot{w}_1)(\tau), s_j^*(\tau)\rangle\,\mathrm{d}\tau\, s_j(\xi)\\
&\quad -\sum_{j=1}^2 \int_\xi^{L/\varepsilon^n} \langle(\partial_1 N_1^\varepsilon \dot{R}_1 + \partial_2 N_1^\varepsilon \dot{w}_1 + \partial_3 N_1^\varepsilon \partial_\xi \dot{w}_1)(\tau), u_j^*(\tau)\rangle\,\mathrm{d}\tau\, u_j(\xi),
\end{aligned}$$

whence

$$\sup_{\xi\in[0,L/\varepsilon^n]} |(\dot{R}_1-\dot{R}_2)(\xi)| \leq c\frac{\varepsilon^n}{\lambda_{1,\varepsilon}^2} \sup_{\xi\in[0,L/\varepsilon^n]} (|\dot{R}_1(\xi)| + \|(\dot{w}_1(\xi),\dot{w}_1'(\xi))\|_{\mathcal{X}_c^s}),$$

and this inequality implies the second assertion. □

We now study the solution set of the equation

$$J_{\rho,w^0,w^{0\prime}}(R^0_{s_1},R^0_{s_2}) = 0$$

near the known solution $(R^0_{s_1},R^0_{s_2}) = (0,0)$ at $(\rho,w^0,w^{0\prime}) = (0,0,0)$ by writing it as

$$(R^0_{s_1},R^0_{s_2}) = (R^0_{s_1},R^0_{s_2}) - \mathrm{d}J_{0,0,0}[0,0]^{-1} J_{\rho,w^0,w^{0\prime}}(R^0_{s_1},R^0_{s_2}) \tag{36}$$

and examining this fixed point problem. According to a standard argument in nonlinear analysis the fixed-point problem (36) has a unique solution $(R^0_{s_1},R^0_{s_2}) = (R^0_{s_1},R^0_{s_2})(\rho,w^0,w^{0\prime})$ in $\bar{B}_\eta(0)\subset\mathbb{R}^2$ whenever

$$\begin{aligned}
|\mathrm{d}J_{0,0,0}[0,0]^{-1}||J_{\rho,w^0,w^{0\prime}}(0,0)| &\leq \frac{\eta}{2},\\
|\mathrm{d}J_{0,0}[0,0]^{-1}||\mathrm{d}J_{\rho,w^0,w^{0\prime}}[R^0_{s_1},R^0_{s_2}] - \mathrm{d}J_{0,0}[0,0]| &\leq \frac{1}{2}, \qquad (R^0_{s_1},R^0_{s_2})\in\bar{B}_\eta(0).
\end{aligned}$$

The estimates (34), (35) and

$$\begin{aligned}
|J_{\rho,w^0,w^{0\prime}}(0,0)| &\leq c|R_{\rho,w^0,w^{0\prime}}(R^0_{s_1},R^0_{s_2})|_{\xi=0}|\\
&\leq c\frac{\varepsilon^{2n}}{\lambda_{1,\varepsilon}^2}
\end{aligned}$$

(see formula (28)) show that we can take $\eta = \varepsilon^{n+1}$.

We have therefore constructed a family of symmetric solutions $(R_{w^0,w^{0\prime}}, w_{w^0,w^{0\prime}})$ to (24), (25) on $[-L/\varepsilon^n, L/\varepsilon^n]$ which are parameterised by $(w^0, w^{0\prime}) \in \Sigma_\mathrm{c}$ with $\|(w^0, w^{0\prime})\|_{\mathcal{X}_\mathrm{c}^s} \leq \varepsilon^{n+3}$ and satisfy $\|(R_{w^0,w^{0\prime}}(\xi), w_{w^0,w^{0\prime}}(\xi), w'_{w^0,w^{0\prime}}(\xi))\|_{\mathcal{X}^s} \leq c\varepsilon^{n+1}$ for each $\xi \in [-L/\varepsilon^n, L/\varepsilon^n]$. The formula

$$Z_{w^0,w^{0\prime}}(\xi) = Q^\varepsilon(\xi) + R_{w^0,w^{0\prime}}(\xi), \qquad \xi \in [-L/\varepsilon^n, L/\varepsilon^n],$$

where $Z = (z_\mathrm{o}, z_\mathrm{e}, z'_\mathrm{o}, z'_\mathrm{e})$, yields a family of pulse-like solutions to the coupled system (11), (12) which was obtained from the original spatial dynamics formulation of the problem by the normal-form theory in Section 3. These solutions are parameterised by $(w^0, w^{0\prime})$, and although all w_o and w'_e components of $(w^0, w^{0\prime})$ vanish because $(w^0, w^{0\prime}) \in \Sigma_\mathrm{c}$ there still exists a continuum of solutions parameterised by the w_e and w'_o components of $(w^0, w^{0\prime})$. Finally note that $Q^\varepsilon(0)$, $R_{w^0,w^{0\prime}}(0)$ lie in Σ_h, so that their z_o and z'_e components vanish. Tracing the coordinate transformations back to the original variable $v(\xi, y)$, we find that $v(-\xi, y) = v(-\xi, y)$ for these pulse-like solutions, which are therefore indeed symmetric. These remarks complete the proof of the existence result given in Section 1 (Theorem 1.1).

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