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Inverse Problem of Velocity Reconstruction in Weakly Lateral Heterogeneous Half-space

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Abstract - A wave propagation generated by a boundary source into a weakly lateral heterogeneous medium (WLHM) occupying a half-space is considered in the acoustic approximation. WLHM means that the velocity of the wave propagation depends weakly on the horizontal coordinates in comparison with the strong dependence on the vertical coordinate z. We consider the problem of the reconstruction of the velocity inside the half-space from the knowledge of the medium response measured at z=0. We obtain a recurrent system of 1D inverse problems to find the first two terms in the decomposition in the velocity profile $c(z, \epsilon x)$ with respect to the small parameter ϵ, the ratio of the horizontal and vertical gradients of the velocity. In the zero-order approximation we derive a system of non-linear Volterra integral equations for the refraction index. Next, the first-order approximation is determined as a solution of a coupled linear system of Volterra-type integral equations. We demostrate the effectiveness of the approach in numerical applications in the 2D case.

1. Introduction

The inverse problems of geo-exploration imply investigation of domains inside the earth crust which contain gas, oil or other minerals. A large part of the earth crust may be approximated by a layered structure, that is may be represented as a stack of layers separated with interfaces. Within every layer the properties (velocity of the wave propagation, density, and so on) depend smoothly on z and slightly vary along the horizontal coordinates and have jump discontinuities while crossing an interface. In this paper we concentrate on the reconstruction of a smooth part of the velocity in the acoustic approximation in an inhomogeneous half-space without interfaces.

Applying a boundary point source at the ground surface $z = 0$, we analyze a response of the medium at $z = 0$ which provides the data for the inverse problem. We study this problem assuming that the velocity of the wave propagation depends weakly on the horizontal coordinates, $x = (x_1, x_2)$ comparing to the strong dependence on the vertical coordinate, z, i.e. we deal with a weakly lateral heterogeneous medium (WLHM). Thus, we have a small parameter ϵ characterizing the ratio of the horizontal and vertical gradients of $c(z, x)$. Introducing the refraction index,

$$n(z, x) = \frac{1}{c(z, x)},$$

we assume

$$n^2(z, \epsilon x) \sim \sum_{m \geq 0} \epsilon^m p_{m_1,m_2}(z) x_1^{m_1} x_2^{m_2}, \qquad m = m_1 + m_2,$$

where $m_{1,2}$ are integers. The importance of WLHM is now well-understood in theoretical and mathematical geophysics and there are currently numerous results on the wave propagation in WLHM (see [1], [2], [3]). However, to the best of our knowledge, the only paper on inverse problems making use of this type of dependence of velocity on the vertical and horizontal coordinates is [4] based on quite a different approach to the one of the paper (see also [5], where $c = 1$ and there is a potential term $q(z, \epsilon x)$).

We develop a perturbation-type algorithm of the reconstruction of the main coefficients $p_{0,0}$, $p_{1,0}$ and $p_{0,1}$. We note that the method can be extended to higher-order approximations. We, however, do not do this not only for extra technicalities, but also taking into the account the reality of geophysical measurements. As the method reduces the inverse problem to a recurrent system of 1D inverse problems, it provides, for multi-dimensional inverse problems occurring in important practical applications, a technique which inherits some practically useful properties of the one-dimensional (1D) inverse problems, e.g. robustness and rapid convergence of corresponding numerical algorithms.

The zero-order approximation, $p_{0,0}$ is found from the solution of a 1D inverse problem for the wave equation in a layered medium which is well studied, see e.g. [6]. At this stage we employ a method of coupled non-linear Volterra-type integral equations similar to the method developed by Blagovestchenskii [5] , [7] , [8]. The method developed in [5] , [7] , [8] is based on the use of polynomial moments with respect to $x = (x_1, x_2)$. We suggest another approach based on the Fourier transform of the wave equation with respect to these coordinates. This makes possible to utilize the dependence of the resulting problems on ξ, with ξ being a dual to x. The next order terms of the refraction index are obtained as solutions to some linear coupled Volterra integral equations. We develop a numerical algorithm based on these integral equations which has proven to be quite robust and effective. Having said so, we note that the integral equations for higher order unknowns contain higher and higher order derivatives of the previously found terms, thus increasing ill-posedness of the inverse problem. This is hardly surprising taking into the account the well-known strong ill-posedness of the multi-dimensional inverse problems [9], [10]. What is, however, interesting is that, within the model considered in this paper, there is just a gradual increase of instability adding two derivations at each stage in the reconstruction algorithm.

The plan of the paper is as follows. In the next section we give a rigorous formulation of the problem and provide a general outline of the perturbation scheme to solve it for WLHM. In section 3 the method of coupled non-linear Volterra integral equations to reconstruct the leading term of the refraction index is described. In section 4 we derive a coupled system of linear Volterra integral equations to reconstruct the first order term of the refraction index. Finally, in section 5 we test the method numerically for the 2D case.

2. Formulation of the problem

Let us consider the wave propagation into the inhomogeneous half-space described by the wave equation

$$n^2(z, \epsilon x) u_{tt} - \Delta_{z,x} u = 0, \qquad x = (x_1, x_2), \tag{1}$$

where n is the refraction index, and ϵ is a small parameter ($0 < \epsilon \ll 1$) which characterizes the ratio of the horizontal and vertical gradients of n. Thus, for $x \sim O(1)$

$$n^2(z, \epsilon x) = n_0^2(z) + \epsilon < x, \bar{n}(z) > + O(\epsilon^2), \quad \bar{n}(z) = (n_1(z), n_2(z)), \tag{2}$$

where $<,>$ means a scalar product, and $n_0^2(z) = p_{0,0}(z)$, $n_1 = p_{1,0}$, $n_2 = p_{0,1}$. We assume that $u = 0$ for $t \le 0$, and the boundary source is given by

$$u\Big|_{z=0} = \frac{\delta(x) f(t)}{\sqrt{n_0(0)}}, \quad f(t) = \delta(t) \quad \text{or} \quad f(t) = \theta(t), \tag{3}$$

where $\delta(x)$ and $\theta(t)$ are the δ-function and the Heaviside function. We seek the solution to (1-3) in the form

$$u(z, x, t) = \frac{1}{4\pi^2} \int_{R^2} e^{-i<\xi, x>} U(z, \xi, t) d\xi, \qquad U(z, \xi, t) \sim \sum_{m=0} \epsilon^m i^m U^{(m)}(z, \xi, t), \qquad \xi = (\xi_1, \xi_2), \tag{4}$$

where all functions $U^{(m)}(z, \xi, t)$ are real. It may be shown that $U^{(m)}(z, \xi, t)$ are even, with respect to ξ, for even m and odd for odd m.

Our goal is to reconstruct the refraction index, namely, the functions $n_0(z)$, $n_{1,2}(z)$, from the knowledge of the leading terms of

$$\frac{\partial u}{\partial z}\Big|_{z=0} = R(x, t, \epsilon), \qquad R(x, t, \epsilon) \sim \sum_{m=0} \epsilon^m R_m(x, t), \qquad 0 < t < 2T. \tag{5}$$

Note that $R_m(x, t)$ are even, with respect to x, for even m, and odd for odd m.

Decompositions (2), (5) give rise to the recurrent system of problems for $U^{(m)}$. In particular, the zero-order problem is

$$n_0^2(z)U_{tt}^{(0)} - U_{zz}^{(0)} + |\xi|^2 U^{(0)} = 0, \qquad |\xi|^2 = \xi_1^2 + \xi_2^2, \qquad U^{(0)}\Big|_{z=0} = \frac{f(t)}{\sqrt{n_0(0)}}, \tag{6}$$

with the inverse data of the form

$$U_z^{(0)}\Big|_{z=0} = r_0(t,\xi) = \int_{R^2} \cos{(<\xi, x>)} R_0(z,x,t)dx.$$

The first-order problem is

$$n_0^2(z)U_{tt}^{(1)} - U_{zz}^{(1)} + |\xi|^2 U^{(1)} = <\bar{n}, \nabla_\xi U_{tt}^{(0)}>, \qquad U^{(1)}\Big|_{z=0} = 0, \tag{7}$$

with the inverse data of the form

$$U_z^{(1)}\Big|_{z=0} = r_1(t,\xi) = \int_{R^2} \sin{(<\xi, x>)} R_1(x,t)dx.$$

In this paper we confine ourselves to the reconstruction of only n_0, $\bar{n}$. The reconstruction of the higher order terms is, in principal, possible using the same ideas as for $\bar{n}$, although is more technically involved. Moreover, in practical applications in geophysics the measured data make possible to find the inverse data only for n_0, $\bar{n}$. Clearly, in real measurements R is not given as a power series. However, as R_0 is even and R_1 is odd, we have

$$r_0(t,\xi) = \int_{R^2} \cos{(<\xi, x>)} R(x,t)dx + O(\epsilon^2), \qquad r_1(t,\xi) = \int_{R^2} \sin{(<\xi, x>)} R(x,t)dx + O(\epsilon^2).$$

3 Algorithm solving IP in the zero-order approximation

In this section we describe an algorithm solving the inverse problem for the zero-order approximation $n_0(z)$. This is a modification of the approach first developed by Blagovestchenskii [5]. The crucial step is the derivation of a non-linear Volterra-type system of three integral equations allowing to solve the inverse problem. Let y be the travel time variable

$$y = \int_0^z n_0(z)dz, \tag{8}$$

so that

$$U_{tt}^{(0)} - \frac{1}{n_0}\frac{\partial}{\partial y}\left(n_0 U_y^{(0)}\right) + \frac{|\xi|^2}{n_0^2} U^{(0)} = 0. \tag{9}$$

Taking $f(t) = \delta(t)$ in (3) and introducing

$$\psi_1(y,t,\xi) = \sqrt{n_0} U^{(0)}, \qquad \psi_2(y,t,\xi) = \frac{\partial \psi_1}{\partial t} + \frac{\partial \psi_1}{\partial y}, \tag{10}$$

we reduce the second order PDE for $U^{(0)}$ to a system of two first order PDE's

$$\left\{ \begin{array}{lcl} \psi_{1t} + \psi_{1y} & = & \psi_2 \\ \psi_{2t} - \psi_{2y} & = & q(y)\psi_1, \end{array} \right. \tag{11}$$

where

$$q(y,\xi) = -\frac{(\sqrt{n_0})''}{\sqrt{n_0}} - \frac{|\xi|^2}{n_0^2}. \tag{12}$$

Integrating these equations along characteristics $\tau = \eta + t - y$ and $\tau = t + y - \eta$, we obtain for $t > y$

$$\begin{cases} \psi_1(y,t) &= \int\limits_0^y \psi_2(\eta, t+\eta-y)d\eta, \\ \psi_2(y,t) &= -\int\limits_0^y q(\eta)\psi_1(\eta, t-\eta+y)d\eta + g(t+y), \end{cases} \tag{13}$$

where

$$g(t) = \delta'(t) + \frac{n_0'(0)}{2n_0(0)}\delta(t) + \frac{r_0(t,\xi)}{\sqrt{n_0(0)}} \tag{14}$$

is bounded as $t \to 0$ due to the cancellation of $\delta'(t)$ and $\delta(t)$ singularities in rhs of (14).

The system (13) is not complete to solve the corresponding inverse problem as it has three unknown functions $\psi_1(y,t,\xi)$, $\psi_2(y,t,\xi)$ and $q(y,\xi)$ and just two equations. The derivation of the third equation is based on the singularity analysis for $U^{(0)}$. Using the propagative wave expansion,

$$U^{(0)}(y,t) \sim \sum_{m\geq 0} f_m(t-y)U_m^{(0)}(y), \qquad f_m(t) = \frac{t_+^m}{\Gamma(m+1)},$$

we see that

$$U^{(0)}(y,t) = \frac{\delta(t-y)}{\sqrt{n_0(y)}} + \frac{\theta(t-y)}{2\sqrt{n_0(y)}}\int\limits_0^y q(\eta,\xi)d\eta + \dots ,$$

i.e.,

$$\psi_2(y,t) = \frac{1}{2}\theta(t-y)q(y,\xi) + \dots \ .$$

This implies, due to the second equation in (13), that

$$q(y) = -2\int\limits_0^y q(\eta)\psi_1(\eta, 2y-\eta)d\eta + 2g(2y). \tag{15}$$

Clearly, (13) and (15) form a closed system of nonlinear Volterra-type equations to determine $q(y,\xi)$, for $0 < y < T$, from $g(t,\xi)$ known, for $0 < t < 2T$. Having $q(y,\xi_1)$ and $q(y,\xi_2)$, we recover

$$n_0(y) = \sqrt{\frac{|\xi|_2^2 - |\xi|_1^2}{q(y,\xi_1) - q(y,\xi_2)}}.$$

Making use of (8), we find $n_0(z)$.

4 Algorithm solving IP for the first-order approximation

Here we present a reconstruction algorithm to find $\bar{n}(z)$. We take $f(t) = \theta(t)$ in (3). Then, in the travel time coordinates,

$$U_{tt}^{(1)} - \frac{1}{n_0}\frac{\partial}{\partial y}\left(n_0 U_y^{(1)}\right) + \frac{|\xi|^2}{n_0^2}U^{(1)} = \frac{1}{n_0^2} < \bar{n}, \nabla_\xi U_{tt}^{(0)} >, \tag{16}$$

$$U^{(1)}\Big|_{y=0} = 0, \qquad U_y^{(1)}\Big|_{y=0} = \frac{\hat{r}_1(\xi,t)}{n_0(0)}, \qquad \hat{r}_1(\xi,t) = \int\limits_0^t r_1(t,\xi)dt.$$

Using the propagative wave expansion, we see that

$$U^{(1)}(y,t) = \theta(t-y)A_0(y) + \dots \ , \tag{17}$$

where

$$A_0(y) = \frac{1}{\sqrt{n_0(y)}}\int\limits_0^y < \bar{n}(\eta), \xi > p(\eta)d\eta, \qquad p(y) = -\frac{1}{2n_0^2(y)}\int\limits_0^y \frac{d\eta}{n_0^2(\eta)}.$$

Denote by $G(y,\eta,t)$ the Green's function, satisfying

$$G_{tt} - \frac{1}{n_0}\frac{\partial}{\partial y}\left(n_0 G_y\right) + \frac{\xi^2}{n_0^2}G = \delta(t)\delta(y-\eta), \qquad G = 0 \ \text{ if } \ y < \eta, \tag{18}$$

i.e. we consider the Green's function "causal" with respect to y. Note that having recovered $n_0(z)$, $G(y,\eta,t)$ is in our disposal. Then,

$$U^{(1)}(y,y+0) = \int_0^y d\eta \int_\eta^{2y-\eta} d\tau G(y,\eta,y-\tau)\frac{<\bar{n},\nabla_\xi U^{(0)}_{\tau\tau}>}{n_0^2(\eta)} - \frac{1}{n_0(0)}\int_0^{2y} d\tau G(y,0,y-\tau)\hat{r}_1(\tau,\xi). \tag{19}$$

Introducing $\bar{\varphi}(y) = (\varphi_1(y), \varphi_2(y)) = \bar{n}(y)p(y)$ and employing the expansion (17), we obtain the following equation

$$<\bar{\varphi}(y),\xi>= 2\int_0^y d\eta G_1(y,\eta,\eta-y)\frac{<\bar{\varphi}(\eta),\nabla_\xi U^{(0)}_{\tau\tau}(\eta,\tau,\xi)>}{n_0^2(\eta)p(\eta)} - \frac{2}{n_0(0)}G_1(y,0,-y)\hat{r}_1(2y,\xi)$$

$$+\int_0^y d\eta \int_\eta^{2y-\eta} d\tau G_2(y,\eta,y-\tau)\frac{<\bar{\varphi}(\eta),\nabla_\xi U^{(0)}_{\tau\tau}(\eta,\tau)>}{n_0^2(\eta)p(\eta)} - \frac{1}{n_0(0)}\int_0^{2y} d\tau G_2(y,0,y-\tau)\hat{r}_1(\tau,\xi). \tag{20}$$

where

$$G_1(y,\eta,t) = \sqrt{n_0(y)}G(y,\eta,t), \qquad G_2(y,\eta,t) = G_{1t}(y,\eta,t) + G_{1y}(y,\eta,t).$$

Taking $\xi_1 = (a,0)$ and $\xi_2 = (0,a)$, where $a = \text{const} \neq 0$, we obtain a pair of Volterra-type integral equations for $\varphi_1(y)$ and $\varphi_2(y)$. Note that due to (13) in the rhs of (20)

$$\lim_{\eta\to 0}\nabla_\xi U^{(0)}_{\tau\tau}(\eta,\tau) = 0,$$

so that the integral equations are not singular.

5. Numerical results

In this section we apply the above method to solving numerically the described inverse problem. Numerical results were obtained for the 2D case problem with coordinates (z,x), and

$$n^2 = n_0^2(z) + \epsilon x n_1(z).$$

The algorithms described above to solve the zero and first-order problems of finding $n_0(z)$ and $n_1(z)$ were implemented into a computer code. To generate the inverse data, $r_0(\xi,t)$ and $r_1(\xi,t)$, we use a finite-difference method to solve the initial boundary value problems (6) and (7). In Fig. 1-2 (the first group of results) the chosen profiles of $n_0(z)$ and $n_1(z)$ are given by trigonometric polynomials

$$n_0(z) = 1 + 0.27\sin z - 0.025\sin 2z,$$

$$n_1(z) = 0.2\cos 1.5z + 0.03\cos 3z - 0.21\sin 1.5z - 0.04\sin 3z.$$

The reconstructed profiles in Fig.1-2 are presented to compare with the corresponding original profiles. It may be seen that we have a good agreement between both types of data. The step in the discretized problem, δy is taken to be 0.08. In Fig. 3-5 (the second group of results), the chosen profiles are not periodic but highly oscillating

$$n_0(z) = 1 + 0.25\sin 1.5z - 0.05\sin 7.5z + 0.07z,$$

$$n_1(z) = -0.2 + 0.2\cos 1.5z + 0.03\cos 3z - 0.21\sin 1.5z - 0.04\sin 3z + 0.1z - 0.001z^2.$$

In Fig. 3-4 we present numerical results for the zero-order problem for various values of $\delta y = 0.08$ and $\delta y = 0.02$ to demonstrate that good agreement may be achieved by making the discretization finer. In Fig.

5 we took $\delta y = 0.02$ to find n_1. A small discrepancy may be seen in Fig.5 as the function $n_0(z)$ changes sharply and oscillates comparing to the previous case. The values $\xi = 0.1$ and $T = 8$ were chosen for all graphs.

The method has shown to be quite stable, fast and accurate. When solving Volterra-type integral equations, both non-linear and linear, the iteration processes need just a few iterations (for all graphs the number of iterations was chosen 10). Numerous computer experiments have shown that for a better accuracy and fast convergence of the iteration process it is reasonable to use for the chosen profiles the segment $|\xi| < 0.5$. It is worth noting that the parameter $|\xi|$, the maximum depth T and the maximum of $n_0''(z)$ are interconnected. For example, for the larger values of T and the maximum of $n_0''(z)$, while computing the profiles we were forced to take smaller values of $|\xi|$. Moreover, due to the non-linearity of (13), (15), when we increase T and/or $n_0''(z)$ and $|\xi|$, a blow up effect can occur, i.e. the iterations stop to converge. This may be remedied by a variant of the layer-stripping (for more details, including stability estimates, see e.g. [11]). Clearly, an error in the reconstruction of $n_0(z)$ affects $n_1(z)$ (compare the Fig. 2 based the Fig. 1, and the Fig. 5 based on the Fig. 4).

In applications to geophysics, the unity of the refraction index corresponds to the average speed of the wave propagation - c =3km/sec. Thus, the dimensionless depth coordinate z must be multiplied by $3km$, so in these graphs the maximum depth $z = 6$ corresponds around 16km.

6. Conclusion

We have described a new method of the reconstruction of the velocity of the wave propagation in WLHM and developed a corresponding numerical algorithm. The demonstrated method of solving the inverse problem proved to be efficient in computer analysis. This method may be generalized to solve the inverse problem of simultaneous reconstruction of heterogeneous density and velocity.

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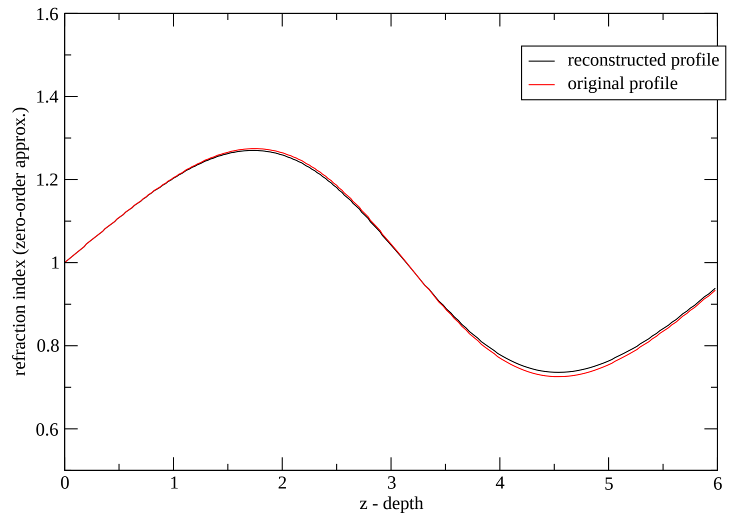

Figure 1: Numerical values of the refraction index n_0 against original profile (the first group of data with $\delta y = 0.08$).

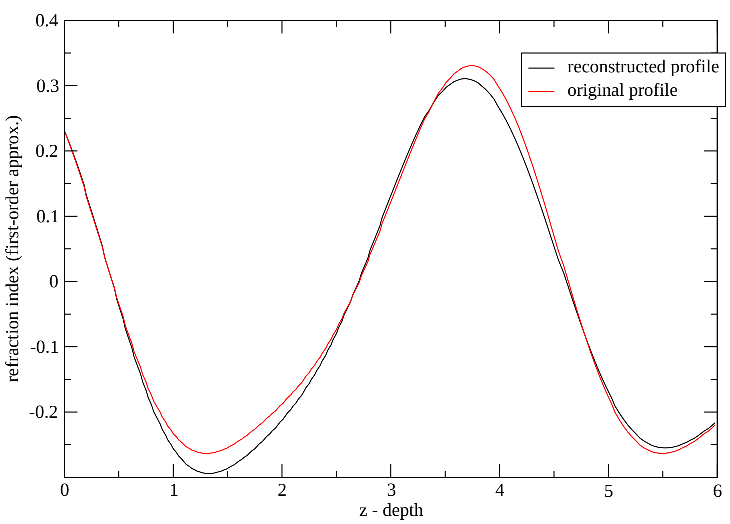

Figure 2: Numerical values of the refraction index n_1 against original profile (the first group of data with $\delta y = 0.08$).

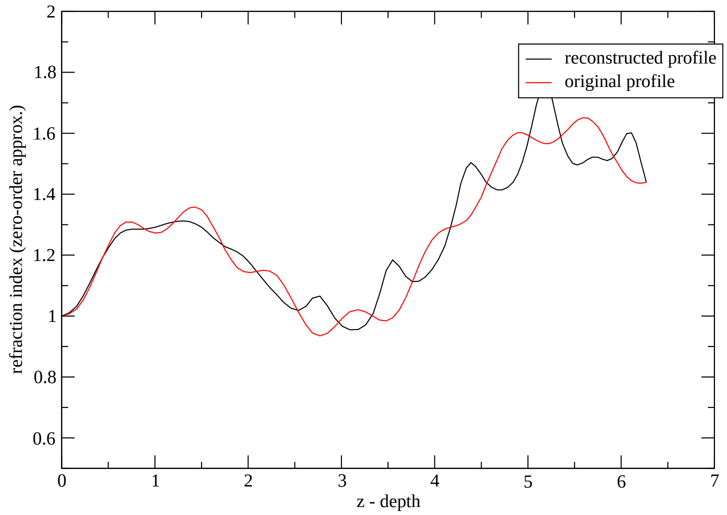

Figure 3: Numerical values of the refraction index n_0 against original profile (the second group of data with $\delta y = 0.08$).

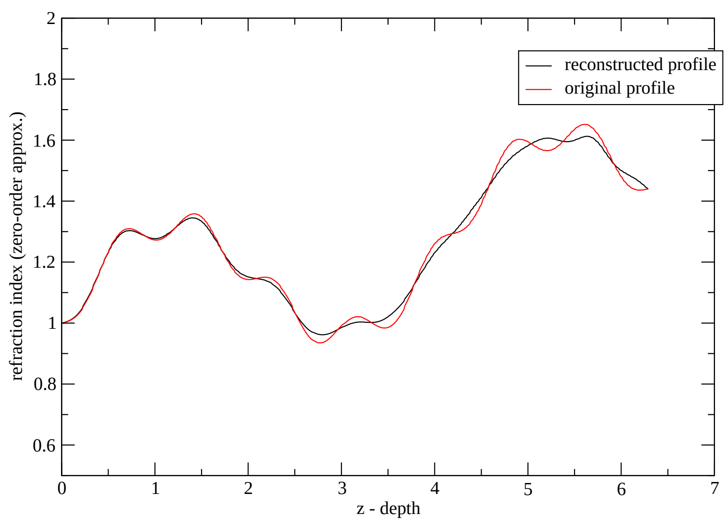

Figure 4: Numerical values of the refraction index n_0 against original profile (the second group of data with $\delta y = 0.02$).

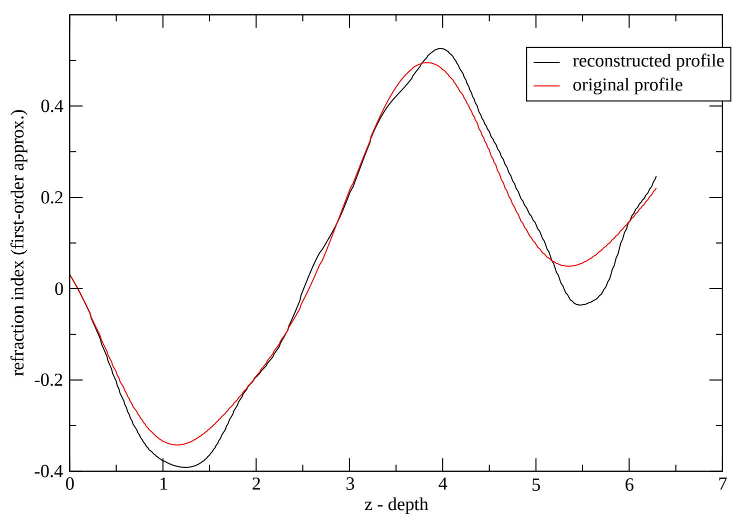

Figure 5: Numerical values of the refraction index n_1 against original profile (the second group of data with $\delta y = 0.02$).