Equations aux dérivées partielles / *Partial Differential Equations*

Transverse instability of gravity-capillary line solitary water waves

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Abstract. The *gravity-capillary water-wave problem* concerns the irrotational flow of a perfect fluid in a domain bounded below by a rigid bottom and above by a free surface under the influence of gravity and surface tension. In the case of large surface tension the system has a travelling line solitary-wave solution for which the free surface has a localised profile in the direction of propagation and is homogeneous in the transverse direction. In this note we show that this line solitary wave is linearly unstable under spatially inhomogeneous perturbations which are periodic in the direction transverse to propagation.

Instabilité d'ondes solitaires de gravité-capillarité

Résumé. *Le* problème des vagues *concerne l'écoulement irrotationnel d'un fluide parfait, de densité constante, dans un domaine dont le bord supérieur est une surface libre et le bord inférieur est un plan, sous l'action de la pesanteur et de la tension superficielle. Dans le cas où la tension superficielle est grande, le système possède une onde solitaire progressive, unidimensionnelle, dont la surface libre a un profil localisé dans la direction de propagation et homogène dans la direction transverse. Dans cette note, on montre que cette onde solitaire est linéairement instable par rapport à des perturbations périodiques dans la direction transverse à la propagation.*

Version française abrégée.

On considère une couche tridimensionnelle de fluide parfait, incompressible, de densité constante ρ, soumise à l'action de la pesanteur et de la tension superficielle, qui occupe un domaine $D_\eta = \{(x,Y,z) \;:\; x,z \in \mathbb{R},\; Y \in (0, h+\eta(x,z,t))\}$, où h désigne l'épaisseur au repos et $Y = h + \eta(x,z,t)$ décrit la surface libre. L'écoulement est supposé irrotationnel et Φ désigne le potentiel des vitesses. L'analyse est effectuée dans un référentiel en translation uniforme, de vitesse $-c$, dans la direction x. On introduit des variables sans dimension en prenant h comme échelle de longueur et c comme échelle de vitesse. Les équations du mouvement sont les équations d'Euler (1)–(4), qui dépendent de deux paramètres: α, l'inverse du nombre de Froude au carré et β, le nombre de Weber.

Dans le cas $\beta > 1/3$, c'est-à-dire à une tension superficielle grande, le système possède une onde solitaire unidimensionnelle $(\eta^*_\epsilon(x), \phi^*_\epsilon(x,Y))$ pour $\alpha = 1+\epsilon^2$ et ϵ petit [1]. Cette solution est spectralement stable par rapport à des perturbations de longueur d'onde moyenne qui sont homogènes en z [5]. Dans cette note, on montre que cette onde solitaire est instable par rapport à des perturbations non homogènes en z. Plus précisement, on démontre que la linéarisation de (1)–(4) autour de l'onde solitaire possède des solutions bornées en espace, périodiques en z et localisées en x, qui croissent exponentiellement en temps t. Cette instabilité a été prédite sur la base des équations modèles dans [6] et [2].

En utilisant la méthode développée dans [4], on écrit le système (1)–(4) sous forme de système Hamiltonien de dimension infinie, réversible, dans lequel z est la variable évolutive. L'onde solitaire unidimensionnelle $(\eta_\epsilon^*(x), \phi_\epsilon^*(x,Y))$ de (1)–(4) correspond à un équilibre réversible, homogène en z et indépendant de t, des équations de Hamilton (5)–(6). Cette onde solitaire est *instable par rapport à des perturbations transverses* si le problème linéarisé autour de l'onde solitaire a des solutions non triviales de la forme $U(x,t,y,z) = \mathrm{e}^{\sigma t} U_\sigma(x,y,z)$ avec $\sigma \in \mathbb{C}$, $\mathrm{Re}\,\sigma > 0$ et U_σ bornée.

Un changement de variables adéquat (Lemme 1) permet d'écrire le système linéarisé sous forme de système dynamique réversible

$$W_z = A_\epsilon W + N_\epsilon(\sigma) W,$$

dans l'espace des phases $Y_s = H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R} \times (0,1)) \times H^s(\mathbb{R} \times (0,1))$, $s \in (0, 1/2)$, où A_ϵ, $N_\epsilon(\sigma)$ sont des opérateurs linéaires de domaines denses, $N_\epsilon(\sigma)$ est analytique en σ et vérifie $N_\epsilon(0) = 0$. On démontre que ce problème a des solutions périodiques en z, pour tout $\sigma > 0$ petit, et on conclut que l'onde solitaire est instable par rapport à des perturbations transverses périodiques. La preuve repose sur un résultat sur le spectre de A_ϵ (Lemme 2) qui montre que A_ϵ a deux valeurs propres isolées, imaginaires pures $\pm \mathrm{i}\epsilon^2 k_\epsilon \neq 0$ et un argument utilisant le théorème des fonctions implicites (Théorème 3).

Cette instabilité de l'onde solitaire entraîne une bifurcation de type "rupture de dimension" dans le problème stationnaire, non linéaire. L'onde solitaire unidimensionnelle pert son homogénéité en z et des modulations stationnaires transverses périodiques apparaissent. Ce phénomène non linéaire est étudié dans [3].

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1 The hydrodynamic problem

We consider a three-dimensional inviscid fluid layer of mean depth h and constant density ρ upon which nonlinear surface waves travel from left to right with velocity c. Let (x, Y, z) denote the usual Cartesian coordinates. The fluid occupies the domain $D_\eta = \{(x, Y, z) \,:\, x, z \in \mathbb{R},\ Y \in (0, h + \eta(x,z,t))\}$, where $\eta > -h$ is a function of the spatial coordinates x, z and of time t, and $Y = h + \eta(x,z,t)$ describes the free surface. Assume the forces of gravity and surface tension are present, and denote by g the acceleration due to gravity and by σ the coefficient of surface tension. The flow is supposed to be irrotational and is therefore described by an Eulerian velocity potential ϕ. We introduce dimensionless variables by choosing the characteristic length to be h and the characteristic velocity to be c. The mathematical problem in a moving coordinate system is to solve Laplace's equation

$$\phi_{xx} + \phi_{YY} + \phi_{zz} = 0 \qquad \text{for } 0 < Y < 1 + \eta \tag{1}$$

with boundary conditions

$$\phi_Y = 0 \qquad \text{on } Y = 0, \tag{2}$$

$$\phi_Y = \eta_t + \eta_x + \eta_x\phi_x + \eta_z\phi_z \qquad \text{on } Y = 1+\eta, \tag{3}$$

$$\phi_t + \phi_x + \frac{1}{2}\left(\phi_x^2 + \phi_Y^2 + \phi_z^2\right) + \alpha\eta - \beta\,\mathcal{K} = 0 \qquad \text{on } Y = 1+\eta, \tag{4}$$

in which

$$\mathcal{K} = \left(\frac{\eta_x}{\sqrt{1+\eta_x^2+\eta_z^2}}\right)_x + \left(\frac{\eta_z}{\sqrt{1+\eta_x^2+\eta_z^2}}\right)_z$$

is twice the mean curvature of the free surface, the dimensionless numbers $\alpha = gh/c^2$ and $\beta = \sigma/\rho h c^2$ are respectively the inverse square of the Froude number and the Weber number and t is supposed to lie in an arbitrarily large interval $[-T, T]$.

For $\beta > 1/3$, that is for large surface tension, the system has a steady line solitary-wave solution $(\eta_\epsilon^*(x), \phi_\epsilon^*(x, Y))$ for $\alpha = 1 + \epsilon^2$ and sufficiently small ϵ (e.g. see [1]). This solution is spectrally stable under finite-wavelength perturbations which are homogeneous in z [5]. In this note we show that the solution is unstable under spatially inhomogeneous perturbations. More precisely, we prove that the linearisation of (1)-(4) about this solitary wave has spatially bounded solutions, periodic in z and localised in x, which grow exponentially in time t. This instability was predicted on the basis of a model equation for long waves in [6] and [2].

Using the method developed in [4], we formulate the mathematical problem (1)-(4) as an infinite-dimensional reversible Hamiltonian system in which the spatial coordinate z is the time-like variable and x, y, t are space-like variables; the difficulty due to the variable domain D_η is overcome by using the new vertical coordinate $y = Y/(1+\eta)$ and modified velocity potential $\Phi(x, y, z, t) = \phi(x, Y, z, t)$. The relevant Hamiltonian system $(X_{s,\delta}, \Omega, H)$ consists of the manifold $X_{s,\delta} = H_\delta^{s+1}(\mathbb{R}) \times H_\delta^s(\mathbb{R}) \times H_\delta^{s+1}(\mathbb{R}\times(0,1)) \times H_\delta^s(\mathbb{R}\times(0,1))$ with $s \in (0, 1/2)$, $\delta > 1/2$, where

$$H_\delta^s(\mathbb{R}) = \{u = \sqrt{\frac{1}{2T}}\sum_{m\in\mathbb{Z}} u_m(x,y)\mathrm{e}^{\pi i m t/T} \mid u_m \in H^s(\mathbb{R}), \|u\|_{s,\delta}^2 = \sum_{m\in\mathbb{Z}}(1+|m|^2)^{2\delta}\|u_m\|_s^2 < \infty\}$$

and $H_\delta^s(\mathbb{R}\times(0,1))$ is defined similarly, the constant symplectic 2-form Ω on $X_{s,\delta}$ given by

$$\Omega\left((\eta_1,\omega_1,\Phi_1,\xi_1),(\eta_2,\omega_2,\Phi_2,\xi_2)\right) = \int_{-T}^{T}\int_{\mathbb{R}}(\omega_2\eta_1 - \eta_2\omega_1)\,\mathrm{d}x\,\mathrm{d}t + \int_{-T}^{T}\int_0^1\int_{\mathbb{R}}(\xi_2\Phi_1 - \Phi_2\xi_1)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$

and the Hamiltonian function H, which is defined upon the manifold domain $N_{s,\delta} = \{(\eta,\omega,\Phi,\xi) \in X_{s+1,\delta} : |W(x,t)| < \beta,\ \eta(x,t) > -1 \text{ for all } x \in \mathbb{R}, t \in [-T,T]\}$ of $X_{s,\delta}$ by the formula

$$H(\eta,\omega,\Phi,\xi) = \int_{-T}^{T}\int_{\mathbb{R}}\left\{-\frac{1}{2}\alpha\eta^2 + \beta - (\beta^2 - W^2)^{1/2}(1+\eta_x^2)^{1/2}\right\}\mathrm{d}x\,\mathrm{d}t$$
$$+\int_{-T}^{T}\int_0^1\int_{\mathbb{R}}\left\{(\eta_t+\eta_x)y\Phi_y - (1+\eta)(\Phi_t+\Phi_x) - \frac{1+\eta}{2}\left(\Phi_x - \frac{y\eta_x\Phi_y}{1+\eta}\right)^2 + \frac{\xi^2 - \Phi_y^2}{2(1+\eta)}\right\}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.$$

Hamilton's equations for the Hamiltonian system (X_s, Ω, H) are given by

$$u_z = Du_t + F(u) \tag{5}$$

with boundary conditions

$$\Phi_y = b(u)_t + g(u), \quad \text{on } y = 0, 1. \tag{6}$$

Here $u=(\eta,\omega,\Phi,\xi)$, $Du=(0,\Phi|_{y=1},0,0)$, $F(u)=(f_1(u),f_2(u),f_3(u),f_4(u))$ with

$$
\begin{aligned}
f_1(u) &= W\left(\frac{1+\eta_x^2}{\beta^2-W^2}\right)^{1/2}, \qquad W=\omega+\int_0^1\frac{y\Phi_y\xi}{1+\eta}\,\mathrm{d}y,\\
f_2(u) &= \int_0^1\left\{\frac{\xi^2-\Phi_y^2}{2(1+\eta)^2}+\frac{1}{2}\left(\Phi_x+\frac{y\Phi_y\eta_x}{1+\eta}\right)\left(\Phi_x-\frac{y\Phi_y\eta_x}{1+\eta}\right)+\left[y\Phi_y\left(\Phi_x-\frac{y\Phi_y\eta_x}{1+\eta}\right)\right]_x\right\}\mathrm{d}y\\
&\quad+\alpha\eta-\left[\eta_x\left(\frac{\beta^2-W^2}{1+\eta_x^2}\right)^{1/2}\right]_x+\frac{W}{(1+\eta)^2}\left(\frac{1+\eta_x^2}{\beta^2-W^2}\right)^{1/2}\int_0^1 y\Phi_y\xi\,\mathrm{d}y+\Phi_x|_{y=1},\\
f_3(u) &= \frac{\xi}{1+\eta}+\frac{y\Phi_yW}{1+\eta}\left(\frac{1+\eta_x^2}{\beta^2-W^2}\right)^{1/2},\\
f_4(u) &= -\frac{\Phi_{yy}}{1+\eta}-[(1+\eta)\Phi_x-y\eta_x\Phi_y]_x+\left[y\eta_x\left(\Phi_x-\frac{y\Phi_y\eta_x}{1+\eta}\right)\right]_y+\frac{(y\xi)_yW}{1+\eta}\left(\frac{1+\eta_x^2}{\beta^2-W^2}\right)^{1/2},
\end{aligned}
$$

and

$$
b(u)=y\eta, \qquad g(u)=y(1+\Phi_x)\eta_x+\frac{\eta\Phi_y}{1+\eta}-\frac{y\eta_x^2\Phi_y}{1+\eta}+\frac{y\xi W}{1+\eta}\left(\frac{1+\eta_x^2}{\beta^2-W^2}\right)^{1/2}.
$$

Notice that $D:X_{s,\delta+1}\to X_{s,\delta+1}$ and $b:H^{s+1}_{\delta+1}(\mathbb{R})\to H^{s+1}_{\delta+1}(\mathbb{R})$ are bounded linear operators and $g:U\to H^{s+1}_\delta(\mathbb{R}\times(0,1))$, $F:U\to X_{s,\delta}$ are smooth nonlinear mappings of a neighbourhood $U\subset N_{s,\delta}$ of the origin in $X_{s+1,\delta}$. Equation (5) can therefore be understood as a quasilinear evolution equation in the phase space $X_{s,\delta}$; the domain of the densely-defined vector field on its right-hand side is $U\cap(H^{s+1}_{\delta+1}(\mathbb{R})\times H^s_\delta(\mathbb{R})\times H^{s+1}_{\delta+1}(\mathbb{R}\times(0,1))\times H^s_\delta(\mathbb{R}\times(0,1))$. Note further that (5) is reversible, that is invariant under the transformation $z\mapsto -z$, $(\eta,\omega,\Phi,\xi)\mapsto S(\eta,\omega,\Phi,\xi)$, where the reverser $S:X_{s,\delta}\to X_{s,\delta}$ is defined by $S(\eta,\omega,\Phi,\xi)=(\eta,-\omega,\Phi,-\xi)$.

The line solitary-wave solution $(\eta^*_\epsilon(x),\phi^*_\epsilon(x,Y))$ of (1)–(4) for $\beta>1/3$, $\alpha=1+\epsilon^2$ corresponds to the t-independent, z-homogeneous, reversible equilibrium $u^*_\epsilon=(\eta^*_\epsilon,0,\Phi^*_\epsilon,0)$ of (5), (6), where

$$
\begin{aligned}
\eta^*_\epsilon(x) &= -\epsilon^2\mathrm{sech}^2\left(\frac{\epsilon x}{2(\beta-1/3)^{1/2}}\right)+\epsilon^4R_\eta(\epsilon x),\\
\Phi^*_\epsilon(x,y) &= -\epsilon^2\int_0^x\mathrm{sech}^2\left(\frac{\epsilon s}{2(\beta-1/3)^{1/2}}\right)\mathrm{d}s+\epsilon^3R_\Phi(\epsilon x,y),
\end{aligned}
$$

and $R_\eta\in C^k_{\mathrm{b,u}}(\mathbb{R})$, $R_\Phi\in C^k_{\mathrm{b,u}}(\mathbb{R}\times(0,1))$ for any $k\geq 0$.

2 Transverse instability of line solitary waves

Linearising (5), (6) about the line solitary wave u^*_ϵ, we obtain the equation

$$U_z=DU_t+F'(u^*_\epsilon)U, \tag{7}$$

with boundary conditions

$$\Phi_y=b'(u^*_\epsilon)U_t+g'(u^*_\epsilon)U \qquad \text{on } y=0,1. \tag{8}$$

Definition. The line solitary wave u^*_ϵ is said to be *transversely unstable* if the linearised problem (7), (8) has a nonzero solution of the form $U(x,t,y,z)=\mathrm{e}^{\sigma t}U_\sigma(x,y,z)$ for some $\sigma\in\mathbb{C}$ with $\mathrm{Re}\,\sigma>0$ and $U_\sigma\in C_\mathrm{b}(\mathbb{R},Y_{s+1})$, where $Y_s=H^{s+1}(\mathbb{R})\times H^s(\mathbb{R})\times H^{s+1}(\mathbb{R}\times(0,1))\times H^s(\mathbb{R}\times(0,1))$.

Set $U(x,t,y,z)=\mathrm{e}^{\sigma t}V(x,y,z)$, so that V satisfies the equation

$$V_z=\sigma DV+F'(u^*_\epsilon)V \tag{9}$$

with boundary conditions

$$\Phi_y = \sigma b'(u_\epsilon^*)V + g'(u_\epsilon^*)V \qquad \text{on } y = 0, 1. \tag{10}$$

We begin by showing that, for any sufficiently small $\sigma \in \mathbb{R}$ (in particular for small $\sigma > 0$), this problem has bounded, z-periodic solutions which belong to $C_\mathrm{b}^k(\mathbb{R}, Y_{s+1})$ for arbitrary $k \geq 1$, so that u_ϵ^* is transversely unstable with respect to perturbations of this type.

We introduce a change of variable which converts the boundary conditions (10) into a standard form. Specifically, we replace Φ by $\Gamma = \Phi + \chi_y$, where χ is the unique solution of the elliptic boundary-value problem

$$\begin{aligned} -\chi_{xx} - \chi_{yy} &= \sigma b'(u_\epsilon^*)V + g'(u_\epsilon^*)V \qquad && \text{in } \mathbb{R} \times (0,1), \\ \chi &= 0 && \text{on } y = 0, 1. \end{aligned}$$

Lemma 1. *The formula $G_{\sigma,\epsilon}(\eta, \omega, \Phi, \xi) = (\eta, \omega, \Gamma, \xi)$ defines linear isomorphisms $Y_s \to Y_s$ and $Y_{s+1} \to Y_{s+1}$. These operators depend analytically upon σ and smoothly upon ϵ and the same is true of their inverses.*

Set $W = G_{\sigma,\epsilon}V = (\eta, \omega, \Gamma, \xi)$, so that (9) becomes

$$W_z = G_{\sigma,\epsilon}\left(\sigma D + F'(u_\epsilon^*)\right) G_{\sigma,\epsilon}^{-1} W = A_\epsilon W + N_\epsilon(\sigma) W, \tag{11}$$

where $A_\epsilon = G_{0,\epsilon} F'(u_\epsilon^*) G_{0,\epsilon}^{-1}$, with boundary conditions

$$\Gamma_y = 0 \qquad \text{on } y = 0, 1. \tag{12}$$

We treat (11) as a reversible evolution equation in the phase space Y_s in which $A_\epsilon, N_\epsilon(\sigma) : \mathcal{D} \subset Y_s \to Y_s$ are densely defined linear operators with domain $\mathcal{D} = \{(\eta, \omega, \Gamma, \xi) \in Y_{s+1} : \Gamma|_{y=0} = \Gamma|_{y=1} = 0\}$; note that the linear operator $N_\epsilon(\sigma) : \mathcal{D} \subset Y_s \to Y_s$ depends analytically upon σ and satisfies $N_\epsilon(0) = 0$. The reverser $\tilde{S} : Y_s \to Y_s$ is defined by $\tilde{S}(\eta, \omega, \Gamma, \xi) = (\eta, -\omega, \Gamma, -\xi)$.

Lemma 2. *Take $s \in [0, 1/2)$. The operator $A_\epsilon : \mathcal{D} \subset Y_s \to Y_s$ has two isolated, simple purely imaginary eigenvalues $\pm \mathrm{i}\epsilon^2 k_\epsilon$ with corresponding eigenvalues v_ϵ and $\bar{v}_\epsilon = \tilde{S} v_\epsilon$. The quantity $\Upsilon(\bar{v}_\epsilon, v_\epsilon)$ is a nonzero purely imaginary number and*

$$\mathcal{R}(A_\epsilon - \mathrm{i}\epsilon^2 k_\epsilon I) = \{V \in Y_s : \Upsilon(V, \bar{v}_\epsilon) = 0\}, \quad \mathcal{R}(A_\epsilon + \mathrm{i}\epsilon^2 k_\epsilon I) = \{V \in Y_s : \Upsilon(V, v_\epsilon) = 0\};$$

here $\Upsilon(V_1, V_2) = \tilde{\Omega}(G_{0,\epsilon}^{-1} V_1, G_{0,\epsilon}^{-1} V_2)$ and

$$\tilde{\Omega}\left((\eta_1, \omega_1, \Phi_1, \xi_1), (\eta_2, \omega_2, \Phi_2, \xi_2)\right) = \int_\mathbb{R} (\omega_2 \eta_1 - \eta_2 \omega_1) \, \mathrm{d}x + \int_0^1 \int_\mathbb{R} (\xi_2 \Phi_1 - \Phi_2 \xi_1) \, \mathrm{d}x \, \mathrm{d}y.$$

The proofs of Lemmata 1 and 2 are found in [3]. We now give our main result.

Theorem 3. *Take $s \in [0, 1/2)$. There exists $\sigma_\epsilon^* > 0$ such that for any $\sigma \in \mathbb{R}$ with $|\sigma| < \sigma_\epsilon^*$ and any complex number $a \in \mathbb{C}$ the system (11) has a periodic solution of the form*

$$W(z) = (a v_\epsilon + w_\sigma) \mathrm{e}^{\mathrm{i}(\epsilon^2 k_\epsilon + \mu_\sigma) z} + (\bar{a} \bar{v}_\epsilon + \bar{w}_\sigma) \mathrm{e}^{-\mathrm{i}(\epsilon^2 k_\epsilon + \mu_\sigma) z}$$

for some $w_\sigma \in \mathcal{D}$ and $\mu_\sigma \in \mathbb{R}$.

Proof. Set $s = (\epsilon^2 k_\epsilon + \mu)z$. We look for solutions of (11) of the form $W = u\mathrm{e}^{\mathrm{i}s} + \bar{u}\mathrm{e}^{-\mathrm{i}s}$ with $u \in \mathcal{D}$. Decompose $u = av_\epsilon + w$, $a \in \mathbb{C}$ using the spectral projection onto the eigenspace of A_ϵ corresponding to the simple eigenvalue $\mathrm{i}\epsilon^2 k_\epsilon$, that is

$$av_\epsilon = \frac{1}{2\pi\mathrm{i}} \int_\gamma (\zeta I - A_\epsilon)^{-1} u \,\mathrm{d}\zeta,$$

in which γ is a simple closed contour in the resolvent set of A_ϵ which encloses $\mathrm{i}\epsilon^2 k_\epsilon$ and no other point of the spectrum of A_ϵ. Equation (11) yields the equation

$$\left(A_\epsilon - \mathrm{i}\epsilon^2 k_\epsilon I\right) w = (\mathrm{i}\mu - N_\epsilon(\sigma))\,(av_\epsilon + w) \tag{13}$$

for w together with its complex conjugate for $\bar{w}$. The right-hand side of (13) lies in the range of $A_\epsilon - \mathrm{i}\epsilon^2 k_\epsilon I$ if and only if

$$\Upsilon\left((\mathrm{i}\mu - N_\epsilon(\sigma))\,(av_\epsilon + w), \bar{v}_\epsilon\right) = 0. \tag{14}$$

An application of the implicit-function theorem shows that

$$\left(A_\epsilon - \mathrm{i}\epsilon^2 k_\epsilon I\right) w = (\mathrm{i}\mu - N_\epsilon(\sigma))\,(av_\epsilon + w) - \frac{1}{\Upsilon(v_\epsilon, \bar{v}_\epsilon)} \Upsilon\left((\mathrm{i}\mu - N_\epsilon(\sigma))\,(av_\epsilon + w), \bar{v}_\epsilon\right) v_\epsilon$$

has a unique solution $w = \mathcal{W}(\mu, \sigma)a$ for sufficiently small μ and σ. Substituting this solution into the solvability condition (14), we obtain the equation

$$\mathrm{i}\mu a + \mathrm{i}h(\sigma, \mu)a = 0, \tag{15}$$

where $h : \mathcal{U} \to \mathbb{C}$ is an analytic function defined on a neighbourhood $\mathcal{U}$ of the origin in $\mathbb{R} \times \mathbb{R}$ which satisfies $h(0, \mu) = 0$. The reversibility of (11) implies that h is a real function, and the implicit function theorem shows that (15) has a unique solution μ_σ for sufficiently small σ. □

Corollary 4. *The solitary wave u_ϵ^* is transversely unstable with respect to periodic perturbations.*

Remark. A related nonlinear phenomenon is studied in [3], in which the line solitary wave u_ϵ^* undergoes a *dimension-breaking bifurcation* generating a one-parameter family of spatially three-dimensional solutions of (1)–(4) close to u_ϵ^* which have a solitary-wave profile in x and are periodic in z (*periodically modulated solitary waves*).

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References

[1] C. J. Amick and K. Kirchgässner, A theory of solitary water-waves in the presence of surface tension. *Arch. Rat. Mech. Anal.* **105** (1989), 1–49.

[2] J. C. Alexander, R. L. Pego and R. L. Sachs, On the transverse instability of solitary waves in the Kadomtsev-Petviashvili equation. *Phys. Lett. A* **226** (1997), 187–192.

[3] M. D. Groves, M. Haragus and S. M. Sun, A dimension-breaking phenomenon in the theory of steady gravity-capillary water waves. *Phil. Trans. Roy. Soc. Lond. A*, submitted.

[4] M. D. Groves and A. Mielke, A spatial dynamics approach to three-dimensional gravity-capillary steady water waves. *Proc. Roy. Soc. Edin. A* **131** (2001), 83–136.

[5] M. Haragus and A. Scheel, Finite-wavelength stability of capillary-gravity solitary waves. *Commun. Math. Phys.*, submitted.

[6] B. B. Kadomtsev and V. I. Petviashvili, On the stability of solitary waves in weakly dispersing media. *Sov. Phys. Dokl.* **15** (1970), 539–541.

Titre courant. Instability of line solitary waves